Hyperlogarithmic Functional Equations Associated with del Pezzo Surfaces

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Seminar on Number Theory and Automorphic Forms

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Logarithm

$$\mathbf{Log}(\boldsymbol{x}) - \mathbf{Log}(\boldsymbol{y}) + \mathbf{Log}\left(\boldsymbol{y}/\boldsymbol{x}\right) = 0$$

Logarithm

$$\mathbf{Log}(x) - \mathbf{Log}(y) + \mathbf{Log}\,(y/x) = 0$$

$$\vdots$$

Hyperlogarithm

$$\sum_{i=1}^{2160} \mathbf{HLog}^{6}\Big(U_i(x,y)\Big) = 0$$

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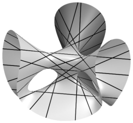

Hyperlogarithm

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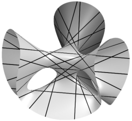

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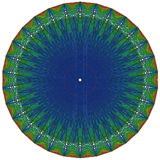

Plan

I Introduction

Polylogarithms & their functional identities

Theorem : $\mathbf{HLog}^w = 0$ for $w = 1, \ldots, 6$

II Ingredients and proof

Del Pezzo surfaces, Hyperlogarithms

Proof of $\mathbf{HLog}^w = 0$

Comparison

III An approach à la Gelfand-MacPherson

Face maps on $\boldsymbol{G}/\boldsymbol{P}$

The logarithm

- $\mathsf{Li}_1(z) = -\mathsf{Log}(1-z)$ $\qquad (z \in \mathbb{C})$

- **Series expansion :** $\mathsf{Li}_1(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k}$

- **Integral formula :**

$$\mathsf{Log}(z) \;=\; \int^{z} \frac{\mathrm{d}u}{u-0}$$

$$\mathsf{Li}_1(z) \;=\; -\int^{z} \frac{\mathrm{d}u}{u-1}$$

- **Monodromy :** $\mathcal{M}_0\left(\mathsf{Log}\right) \;=\; \mathsf{Log} + 2i\pi$

- **Cauchy's functional identity :**

$$\mathsf{Log}(\, x \,) - \mathsf{Log}(\, y \,) + \mathsf{Log}\left(\frac{y}{x}\right) = 0$$

The dilogarithm

- $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ $\qquad \left(|z| < 1 \right)$

- **Integral formula :** $\qquad \mathrm{Li}_2(z) = \mathbf{L}_{01}(z) = -\int^z \mathrm{Log}(1-u)\,\frac{du}{u-0}$

$$\mathbf{L}_{10}(z) = \int^z \mathrm{Log}(u-0)\,\frac{du}{1-u}$$

- **Monodromy :** $\qquad \mathcal{M}_1\left(\mathrm{Li}_2\right) = \mathrm{Li}_2 - 2i\pi\,\mathrm{Log}$

- **Abel's functional equation** $(\mathcal{A}b)$ $\qquad$ **[Spence 1809]**

$$\mathrm{Li}_2(x) - \mathrm{Li}_2(y) - \mathrm{Li}_2\left(\frac{x}{y}\right) - \mathrm{Li}_2\left(\frac{1-y}{1-x}\right) + \mathrm{Li}_2\left(\frac{x(1-y)}{y(1-x)}\right) = \mathrm{Log}(y)\,\mathrm{Log}\left(\frac{1-y}{1-x}\right) - \frac{\pi^2}{6}$$

The dilogarithm

- $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ $\qquad\qquad \big(\, |z| < 1 \, \big)$
- **Integral formula :** $$\mathrm{Li}_2(z) = \mathbf{L}_{01}(z) = -\int^z \mathbf{L}\mathrm{og}(1-u)\,\frac{du}{u-0}$$ $$\mathbf{L}_{10}(z) = \int^z \mathbf{L}\mathrm{og}(u-0)\,\frac{du}{1-u}$$
- **Monodromy :** $$\mathcal{M}_1\,(\mathrm{Li}_2) = \mathrm{Li}_2 - 2i\pi\,\mathbf{L}\mathrm{og}$$
- **Abel's functional identity** $(\mathcal{A}b)$ **[Spence 1809]**

$$\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\Big(\frac{x}{y}\Big) - \mathbf{R}\Big(\frac{1-y}{1-x}\Big) + \mathbf{R}\Big(\frac{x(1-y)}{y(1-x)}\Big) = 0$$

$$\mathbf{R}(x) = \frac{1}{2}\Big(\mathbf{L}_{01}(x) - \mathbf{L}_{10}(x)\Big)$$

The n-th polylogarithm $\mathbf{Li}_n$ for $n \geq 1$

- $\mathbf{Li}_n(z) \; = \; \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad\qquad\qquad\qquad \Big(\, |z| < 1 \,\Big)$

- **Integral formula :**

$$\mathbf{Li}_n(z) \; = \; \int^z \mathbf{Li}_{n-1}(u) \frac{du}{u}$$

$$\mathbf{Li}_n{}'(z) \; = \; \mathbf{Li}_{n-1}(z)/z$$

- **Monodromy :** $\qquad \mathcal{M}_1\,(\,\mathbf{Li}_n\,) \; = \; \mathbf{Li}_n - 2i\pi\,\frac{(\mathbf{Log})^{n-1}}{(n-1)!}$

- **Functional identities in one variable :**

$$\mathbf{Li}_n\Big(z^m\Big) \; = \; r^{m-1} \sum_{\omega^m=1} \mathbf{Li}_n(\omega\, z)$$

$$\mathbf{Li}_n(z) + (-1)^n \mathbf{Li}_n\Big(z^{-1}\Big) \; = \; -\frac{(2i\pi)^n}{n!}\, \mathbf{B}_n\left(\frac{\mathbf{Log}\, z}{2i\pi}\right)$$

The n-th polylogarithm $\mathbf{Li}_n$ for $n \geq 1$

- $\mathbf{Li}_n(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad\qquad \Big(\; |z| < 1 \;\Big)$

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- **Monodromy :** $\qquad \mathcal{M}_1\,(\,\mathbf{Li}_n\,) \;=\; \mathbf{Li}_n - 2i\pi\,\frac{(\mathbf{Log})^{n-1}}{(n-1)!}$

- **Functional identities in several variables ($\exists$?) :**

$$\sum_{i \in I} c_i\, \mathbf{Li}_n\big(\boldsymbol{U_i}\big) = \mathbf{Elem}_{<n}$$

$$\Big(\; I \;\text{ finite}\,,\; c_i \in \mathbb{Z}\,,\; \boldsymbol{U_i} \in \mathbb{Q}(x_1, \ldots, x_N) \;\Big)$$

The n-th polylogarithm $\mathbf{Li}_n$ for $n \geq 1$

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- **Functional identities in several variables ($\exists$?) :**

$$\sum_{i\in I} c_i \, \mathbf{Li}_n(\boldsymbol{U_i}) = \mathbf{Elem}_{<n} \quad \Longleftrightarrow \quad \sum_{i\in I} c_i \, \mathcal{L}_n(\boldsymbol{U_i}) = \mathbf{0}$$

$\Big(\, I$ finite, $c_i \in \mathbb{Z}$, $\boldsymbol{U_i} \in \mathbb{Q}(x_1, \ldots, x_N) \, \Big)$

Example : $\mathbf{Li}_3$

- $\mathbf{Li}_3(z) \;=\; \sum_{k=1}^{\infty} z^k/k^3 \;=\; \int^z \mathbf{Li}_2(u)\frac{du}{u}$

- **Spence-Kummer identity $(\mathcal{SK})$ (1809-1840) :**

$$2\,\mathbf{Li}_3(x) + 2\,\mathbf{Li}_3(y) - \mathbf{Li}_3\left(\frac{x}{y}\right) + 2\,\mathbf{Li}_3\left(\frac{1-x}{1-y}\right) + 2\,\mathbf{Li}_3\left(\frac{x(1-y)}{y(1-x)}\right) - \mathbf{Li}_3(xy)$$
$$+ 2\,\mathbf{Li}_3\left(-\frac{x(1-y)}{(1-x)}\right) + 2\,\mathbf{Li}_3\left(-\frac{(1-y)}{y(1-x)}\right) - \mathbf{Li}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right)$$
$$= 2\,\mathbf{Li}_3(1) - \mathbf{Log}\,(y)^2\mathbf{Log}\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3}\mathbf{Log}(y) + \frac{1}{3}\mathbf{Log}\,(y)^3$$

Example : Li_3

- $\mathrm{Li}_3(z) \;=\; \sum_{k=1}^{\infty} z^k/k^3 \;=\; \int^z \mathrm{Li}_2(u)\frac{du}{u}$

- **Spence-Kummer identity** $(\mathcal{SK})$ **(1809-1840) :**

$$\begin{aligned}2\,\mathrm{Li}_3(x) + 2\,\mathrm{Li}_3(y) - \mathrm{Li}_3\left(\frac{x}{y}\right) + 2\,\mathrm{Li}_3\left(\frac{1-x}{1-y}\right) + 2\,\mathrm{Li}_3\left(\frac{x(1-y)}{y(1-x)}\right) - \mathrm{Li}_3(xy)\\ + 2\,\mathrm{Li}_3\left(-\frac{x(1-y)}{(1-x)}\right) + 2\,\mathrm{Li}_3\left(-\frac{(1-y)}{y(1-x)}\right) - \mathrm{Li}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right)\\ = 2\,\mathrm{Li}_3(1) - \mathrm{Log}\,(y)^2\mathrm{Log}\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3}\mathrm{Log}(y) + \frac{1}{3}\mathrm{Log}\,(y)^3\end{aligned}$$

$$\begin{aligned}2\,\mathcal{L}_3(x) + 2\,\mathcal{L}_3(y) - \mathcal{L}_3\left(\frac{x}{y}\right) + 2\,\mathcal{L}_3\left(\frac{1-x}{1-y}\right) + 2\,\mathcal{L}_3\left(\frac{x(1-y)}{y(1-x)}\right) - \mathcal{L}_3(xy)\\ + 2\,\mathcal{L}_3\left(-\frac{x(1-y)}{(1-x)}\right) + 2\,\mathcal{L}_3\left(-\frac{(1-y)}{y(1-x)}\right) - \mathcal{L}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right) = 0\end{aligned}$$

$$\mathcal{L}_3(z) = \mathrm{Li}_3(z) - \mathrm{Li}_2(z)\,\mathrm{Log}|z| + \tfrac{1}{3}\,\mathrm{Li}_1(z)(\mathrm{Log}|z|)^2$$

Example : Li_4

- **Kummer's functional identity $\mathcal{K}(4)$ (1840) :**

$$
\begin{aligned}
&\mathcal{L}_4\Big(-\frac{x^2y\eta}{\zeta}\Big)+\mathcal{L}_4\Big(-\frac{y^2x\zeta}{\eta}\Big)+\mathcal{L}_4\Big(\frac{x^2y}{\eta^2\zeta}\Big)+\mathcal{L}_4\Big(\frac{y^2x}{\zeta^2\eta}\Big)\\
&-6\,\mathcal{L}_4\Big(xy\Big)-6\,\mathcal{L}_4\Big(\frac{xy}{\eta\zeta}\Big)-6\,\mathcal{L}_4\Big(-\frac{xy}{\eta}\Big)-6\,\mathcal{L}_4\Big(-\frac{xy}{\zeta}\Big)\\
&-3\,\mathcal{L}_4\Big(x\eta\Big)-3\,\mathcal{L}_4\Big(y\zeta\Big)-3\,\mathcal{L}_4\Big(\frac{x}{\eta}\Big)-3\,\mathcal{L}_4\Big(\frac{y}{\zeta}\Big)\\
&-3\,\mathcal{L}_4\Big(-\frac{x\eta}{\zeta}\Big)-3\,\mathcal{L}_4\Big(-\frac{y\zeta}{\eta}\Big)-3\,\mathcal{L}_4\Big(-\frac{x}{\eta\zeta}\Big)-3\,\mathcal{L}_4\Big(-\frac{y}{\eta\zeta}\Big)\\
&+6\,\mathcal{L}_4\Big(x\Big)+6\,\mathcal{L}_4(y)+6\,\mathcal{L}_4\Big(-\frac{x}{\zeta}\Big)+6\,\mathcal{L}_4\Big(-\frac{y}{\eta}\Big)=\mathbf{0}
\end{aligned}
$$

$$\Big(\ \zeta=1-x,\ \eta=1-y\ \Big)$$

- **Functional identities (FI) of polylogarithms Li_n :**
  - Hyperbolic geometry
  - Web geometry ($n \leq 3$)
  - K-theory of number fields ($n \leq 4$)
  - Periods (MZVs)
  - Particle physics ('*Scattering amplitudes*')
  - Mathematical physics ('*Y-systems*') ($n = 2$)
  - Cluster algebras ($n \leq 4$)
  - Mirror symmetry ('*Scattering diagrams*') ($n = 2$)

K-theory & polylogarithms

- $\boldsymbol{F}$ = number field $\rightsquigarrow$ $K_n(\boldsymbol{F}) = \begin{cases} n \text{ even} & \checkmark \text{ (Borel)} \\ n \text{ odd} & ? \end{cases}$

- **[Zagier]** $K_3(\boldsymbol{F}) \leftrightsquigarrow B_2(\boldsymbol{F})$ = Bloch group

$$B_2(\boldsymbol{F}) = \frac{\mathbb{Z}\Big[\ \boldsymbol{F}\backslash\{0,1\}\ \Big]}{\Big\langle\ [\,x\,]-[\,y\,]-\Big[\,\frac{x}{y}\,\Big]-\Big[\,\frac{1-y}{1-x}\,\Big]+\Big[\,\frac{x(1-y)}{y(1-x)}\,\Big]\ \Big|\ x,y\in\boldsymbol{F}\backslash\{0,1\}\,,\,x\neq y\ \Big\rangle}$$

$$\left(\mathbf{R}(x)-\mathbf{R}(y)-\mathbf{R}\Big(\frac{x}{y}\Big)-\mathbf{R}\Big(\frac{1-y}{1-x}\Big)+\mathbf{R}\Big(\frac{x(1-y)}{y(1-x)}\Big)=0\right)$$

- **Regulators (Borel)**

$$\mathcal{R}_2^B = \boldsymbol{\mathcal{L}}_2 : K_3(\boldsymbol{F}) \longrightarrow \mathbb{R}$$
$$\mathcal{R}_n^B = \boldsymbol{\mathcal{L}}_n : K_{2n-1}(\boldsymbol{F}) \longrightarrow \mathbb{R} \quad (\ n \geq 2\ )$$

- To understand the FI of $\boldsymbol{\mathcal{L}}_n$ $\rightarrow$
  - Desc° of $K_{2n-1}(\boldsymbol{F})$ by generators and relat°
  - Applications to "Zagier's Conjecture"

[Hain - MacPherson 1990] Higher logarithms

The dilogarithm has properties analogous to those of the logarithm. It has been widely believed, both in the nineteenth century and more recently, that these two functions should be the first two elements of an infinite sequence of higher logarithms which share analogous properties. To date, several sequences of such functions have been proposed, but no function beyond the dilogarithm in any of these sequences is known to possess all the desired properties.

(Similar questions were raised in **[Griffiths 2002]**, **[Gangl 2013]**, etc.)

[Goncharov-Rudenko 2018] Motivic correlator, cluster algebras...

Conclusion. *If* $n > 3$, ***the problem of writing explicitly functional equations for*** Li_n ***might not be the "right" problem****. It seems that when n is growing the functional equations become so complicated that one cannot write them down on a piece of paper.*

- **Main problems regarding polylogarithms :**
  - Find explicit **FI** for $\mathcal{L}_n$ $\big($ *e.g.* $\exists\ n \geq 8$? $\big)$
  - $\exists$ a sequence $(\boldsymbol{FI}_n)_{n\geq 1}$ of **FI** for the polylogarithms ?
  - Better understanding of polylogarithmic **FI**

- **In this talk, we give a series of hyperlogarithmic FIs :**
  - $\mathbf{HLog}^1 \Longleftrightarrow \Big(\mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}(x/y) = 0 \Big)$

    $\mathbf{HLog}^2 \Longleftrightarrow \Big(\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\big(\frac{x}{y}\big) - \mathbf{R}\big(\frac{1-y}{1-x}\big) + \mathbf{R}\big(\frac{x(1-y)}{y(1-x)}\big) = 0 \Big)$

    $\vdots$

    $\mathbf{HLog}^6$ $\Big($ weight 6 hyperlogarithmic **FI** $\Big)$
  - we have $\big(\mathbf{HLog}^w\big) : \sum_{i=1}^{\kappa} \boldsymbol{AH}_i^w\big(\phi_i\big) = 0 \quad (w = 1, \ldots, 6)$

A geometric view on Abel's identity

- $(\mathcal{A}b)$ $\quad \underset{\substack{\| \\ U_1}}{\mathrm{R}(x)} - \underset{\substack{\| \\ U_2}}{\mathrm{R}(y)} - \mathrm{R}\Big(\underset{\substack{\| \\ U_3}}{\frac{x}{y}}\Big) - \mathrm{R}\Big(\underset{\substack{\| \\ U_4}}{\frac{1-y}{1-x}}\Big) + \mathrm{R}\Big(\underset{\substack{\| \\ U_5}}{\frac{x(1-y)}{y(1-x)}}\Big) = 0$

- **Blow-up** $\quad \beta : X_4 = \mathrm{Bl}_{p_1,\ldots,p_4}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$

$$\begin{tikzcd} \mathbf{X}_4 \arrow[d, "\beta"'] \arrow[rr, bend left, "\phi_i"] & & \\ \mathbb{P}^2 \arrow[rr, dashed, "U_i"'] & & \mathbb{P}^1 \end{tikzcd}$$

$\phi_1, \ldots, \phi_5 : X_4 \longrightarrow \mathbb{P}^1$ **are the five fibrations by conics on the del Pezzo surface** X_4

- $(\mathcal{A}b) \Longleftrightarrow \quad \sum_{i=1}^{5} \epsilon_i \, \mathrm{R}(\phi_i) = 0 \quad$ (with $(\epsilon_i)_{i=1}^{5} \in \{\pm 1\}^5$)

Generalisation to del Pezzo surfaces

- $p_1, \ldots, p_r \in \mathbb{P}^2$: points in general position $\quad (r \in \{3, \ldots, 8\})$
- **Blow-up** $\beta_r : \boldsymbol{X}_r = \mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2 \quad (\boldsymbol{X}_r = \mathbf{dP}_{9-r})$

Prop : 1. $\exists$ *a finite number* $= \kappa_r$ *of conic fibrations on* $\boldsymbol{X}_r$ $\quad \phi_1, \ldots, \phi_\kappa : \boldsymbol{X}_r \longrightarrow \mathbb{P}^1$

2. *For any* i : $\boldsymbol{\Sigma}_i = \mathbf{SingVal}(\phi_i) \subset \mathbb{P}^1$ *has* $r-1$ *elements*

Def° : *The* **complete antisymmetric hyperlog. of weight $r-2$** : $\boldsymbol{AH}^{r-2}_{\boldsymbol{\Sigma}_i} : \widetilde{\mathbb{P}^1 \setminus \boldsymbol{\Sigma}_i} \longrightarrow \mathbb{C}$

Thm [Castravet-P.] $\exists \ (\epsilon_i)_{i=1}^{\kappa} \in \{ \pm 1 \}^{\kappa}$, $\pm$-unique, such that

$$\left(\mathbf{HLog}^{r-2}\right) \qquad \sum_{i=1}^{\kappa} \epsilon_i \, \boldsymbol{AH}^{r-2}_{\boldsymbol{\Sigma}_i}(\phi_i) = 0$$

$$\boxed{\Big(\mathbf{HLog}^{r-2}\Big) \qquad \sum_{i=1}^{\kappa} \epsilon_i\, \boldsymbol{AH}^{r-2}_{\boldsymbol{\Sigma_i}}(\phi_{\boldsymbol{i}}) = 0}$$

- One identity $\mathbf{HLog}^{r-2}$ for each del Pezzo $\mathbf{dP}_d = \boldsymbol{X}_r \ \big(d = 9 - r\big)$

[$d = 6$] $\mathbf{dP}_6$ is unique, $\boldsymbol{AH}^1_{\boldsymbol{\Sigma_i}} = \mathbf{Log}$ for every i

$$\Big(\mathbf{HLog}^1\Big) \ : \ \mathbf{Log}(\,\boldsymbol{x}\,) - \mathbf{Log}(\,\boldsymbol{y}\,) - \mathbf{Log}(\,\boldsymbol{x}/\boldsymbol{y}\,) = \mathbf{0}$$

[$d = 5$] $\mathbf{dP}_5$ is unique : $\boldsymbol{AH}^2_{\boldsymbol{\Sigma_i}} = \frac{1}{2}\Big(\mathbf{L}_{01} - \mathbf{L}_{10}\Big) = \mathbf{R}$ for every i

$$\Big(\mathbf{HLog}^2\Big) \ : \ \sum_{\boldsymbol{i=1}}^{\mathbf{5}} \boldsymbol{\epsilon_i}\, \mathbf{R}(\,\phi_{\boldsymbol{i}}\,) = \mathbf{0} \qquad (\boldsymbol{\mathcal{A}}\mathbf{b})$$

$$\left(\mathbf{HLog}^{r-2}\right) \qquad \sum_{i=1}^{\kappa} \epsilon_i \, \boldsymbol{AH}^{r-2}_{\boldsymbol{\Sigma_i}}(\boldsymbol{\phi_i}) = 0$$

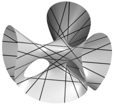
$\boldsymbol{[\,d = 4\,]}$ $\mathbf{dP}_4$ moduli $\infty^{\mathbf{2}}$ $\rightsquigarrow$ $\infty^{\mathbf{2}}$ identities $\mathbf{HLog}^3$

$$\boldsymbol{AH}^{\mathbf{3}}_{\mathbf{1}}(\,\boldsymbol{x}\,) + \boldsymbol{AH}^{\mathbf{3}}_{\mathbf{2}}\left(\frac{\mathbf{1}}{\boldsymbol{y}}\right) + \boldsymbol{AH}^{\mathbf{3}}_{\mathbf{3}}\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right) + \cdots$$

$$\cdots + \boldsymbol{AH}^{\mathbf{3}}_{\mathbf{9}}\left(\frac{\boldsymbol{y(x-b)}}{\boldsymbol{x(y-a)}}\right) + \boldsymbol{AH}^{\mathbf{3}}_{\mathbf{10}}\left(\frac{\boldsymbol{a\,(b-x)}}{\boldsymbol{by-ax}}\right) = \mathbf{0}$$

$\boldsymbol{[\,d = 3\,]}$ $\mathbf{dP}_3$ = cubic surface in $\mathbb{P}^3$ $\rightsquigarrow$ $\infty^{\mathbf{4}}$ identities $\mathbf{HLog}^4$

$$\sum_{\boldsymbol{i=1}}^{\mathbf{27}} \boldsymbol{AH}^{\mathbf{4}}_{\boldsymbol{i}}(\,\boldsymbol{\phi_i}\,) = \mathbf{0}$$



II Del Pezzo surfaces

- $\boldsymbol{S}$ del Pezzo $\Leftrightarrow$ $-\boldsymbol{K_S}$ ample $\longrightarrow$ $\boldsymbol{d} = (-\boldsymbol{K_S})^2 \in \{1, \dots, 9\}$

- $\mathbf{dP}_d = \mathbf{X}_r = \mathbf{Bl}_{p_1,\dots,p_r}(\mathbb{P}^2)$ $\quad$ $\mathbf{Pic}(\mathbf{dP}_d) = \mathbb{Z}\,\boldsymbol{h} \oplus \left(\bigoplus_{i=1}^r \mathbb{Z}\,\boldsymbol{\ell}_i\right)$

  $-\boldsymbol{K}_{\mathbf{dP}_d} = 3\boldsymbol{h} - \sum_{i=1}^r \boldsymbol{\ell}_i$ ample $\rightsquigarrow$ $\varphi_{|-\boldsymbol{K}|} : \mathbf{dP}_d \hookrightarrow \mathbb{P}^d$ embedding

- $\mathbf{Pic}(\mathbf{dP}_d) \supset \boldsymbol{K}^\perp = \langle \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_r \rangle$ $\quad$ $\boldsymbol{\rho}_i = \boldsymbol{\ell}_i - \boldsymbol{\ell}_{i+1} \quad i \leq r-1$

  $\boldsymbol{\rho}_r = \boldsymbol{h} - \sum_{i=1}^3 \boldsymbol{\ell}_i$

  $-(\cdot,\cdot) + \{\boldsymbol{\rho}_i\}_{i=1}^r$ $\rightsquigarrow$ Root system $\mathrm{E}_r \subset \mathbf{R}_r = \boldsymbol{K}^\perp \otimes \mathbb{R}$

- For any root $\boldsymbol{\rho}$: $\quad s_{\boldsymbol{\rho}} : \mathbf{R}_r \longrightarrow \mathbf{R}_r$ $\quad$ (orthog. reflection)

  $$d \longmapsto d + (d, \boldsymbol{\rho})\boldsymbol{\rho}$$

  $\boldsymbol{W}_r = \boldsymbol{W}(\mathrm{E}_r) = \langle s_{\rho_1}, \dots, s_{\rho_r} \rangle \subset \mathrm{O}(\mathbf{R}_r)$: Weyl group of type E_r

Del Pezzo surfaces

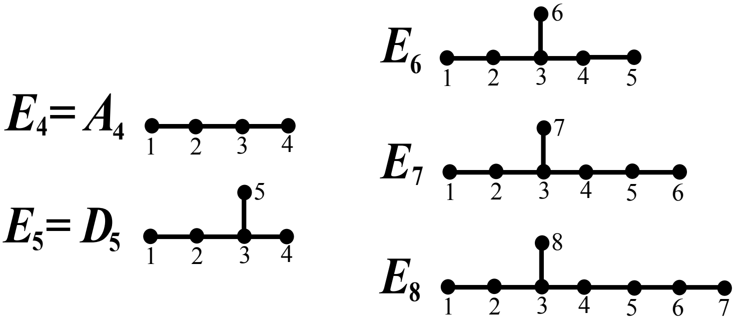


Figure – Dynkin diagram E_r $\big(k$ stands for ρ_k for any $k = 1, \ldots, r\big)$

Lines and conics on $X_r = \mathrm{dP}_d$ $\quad \big(d = 9 - r \big)$

- Lines $\quad \mathcal{L}_r = \Big\{ \ell \in \mathbf{Pic}(\boldsymbol{X}_r) \;\Big|\; \ell \cdot \boldsymbol{K} = \ell^2 = -1 \Big\} = \boldsymbol{W}_r \cdot \ell_r$

- Conics $\quad \mathcal{C}_r = \Big\{ \mathfrak{c} \in \mathbf{Pic}(\boldsymbol{X}_r) \;\Big|\; -\boldsymbol{K} \cdot \mathfrak{c} = 2 \,,\; \mathfrak{c}^2 = 0 \Big\} = \boldsymbol{W}_r \cdot \mathfrak{c}$

$\quad \Cup$

$\mathfrak{c} \;\longrightarrow\; |\,\mathfrak{c}\,| \simeq \mathbb{P}^1 \;\rightsquigarrow\;$ Conic fibration $\;\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1$

r	3	4	5	6	7	8
E_r	$A_2 \times A_1$	A_4	D_5	E_6	E_7	E_8
$W_r = W(E_r)$	$\mathfrak{S}_3 \times \mathfrak{S}_2$	$\mathfrak{S}_5$	$(\mathbf{Z}/2\mathbf{Z})^4 \ltimes \mathfrak{S}_5$	$W(E_6)$	$W(E_7)$	$W(E_8)$
$\omega_r = \lvert W_r \rvert$	12	5!	$2^4 \cdot 5!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
$l_r = \lvert \mathcal{L}_r \rvert$	6	10	16	27	56	240
$\kappa_r = \lvert \mathcal{K}_r \rvert$	3	5	10	27	126	2160

Reducible conics in $X_r = \mathrm{dP}_d$

- $\boldsymbol{L}_r = \cup_{\ell \in \boldsymbol{\mathcal{L}}_r} \ell \subset \boldsymbol{X}_r \quad \rightsquigarrow \quad \boldsymbol{U}_r = \boldsymbol{X}_r \setminus \boldsymbol{L}_r$

- $\mathcal{C}_r \ni \mathfrak{c} \quad \rightsquigarrow$ Conic fibration $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1$

$$\boldsymbol{\Sigma}_{\mathfrak{c}} = \mathbf{SingVal}(\phi_{\mathfrak{c}}) = \left\{ \sigma \in \mathbb{P}^1 \;\middle|\; \phi_{\mathfrak{c}}^{-1}(\sigma) \text{ non-irreducible } \right\}$$
$$= \left\{ \sigma_{\mathfrak{c}}^1 \, , \, \ldots \, , \, \sigma_{\mathfrak{c}}^{r-2} \, , \, \sigma_{\mathfrak{c}}^{r-1} = \infty \right\} \subset \mathbb{P}^1$$

- For $\sigma_{\mathfrak{c}}^i \in \boldsymbol{\Sigma}_{\mathfrak{c}} \; : \; \phi_{\mathfrak{c}}^{-1}(\sigma_{\mathfrak{c}}^i) = \ell_{\mathfrak{c}}^i + \widetilde{\ell}_{\mathfrak{c}}^i \qquad \left(\ell_{\mathfrak{c}}^i \, , \, \widetilde{\ell}_{\mathfrak{c}}^i \in \boldsymbol{\mathcal{L}}_r \right)$

- $\mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\mathbb{P}^1 \, , \, \Omega^1_{\mathbb{P}^1}(\mathrm{Log}\, \boldsymbol{\Sigma}_{\mathfrak{c}})\Big) = \left\langle \frac{dz}{z - \sigma_{\mathfrak{c}}^i} \right\rangle_{i=1}^{r-2} \simeq \mathbb{C}^{r-2}$

$\quad \wr|$

- $\mathbf{H}_{\mathfrak{c}} = \phi_{\mathfrak{c}}^*\Big(\mathcal{H}_{\mathfrak{c}}\Big) = \left\langle \frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}} - \sigma_{\mathfrak{c}}^i} \right\rangle_{i=1}^{r-2} \subset \mathbf{H}^0\Big(\boldsymbol{X}_r \, , \, \Omega^1_{\boldsymbol{X}_r}(\mathrm{Log}\, \boldsymbol{L}_r)\Big) = \mathbf{H}_{\boldsymbol{X}_r}$

Iterated integrals

- **Poincaré (1884), Lappo-Danilevski (1928), Chen (1973)**

- $\boldsymbol{Y}$ complex manifold

- $\mathbf{H} = \left\langle \omega_1 \,, \, \ldots \,, \, \omega_m \right\rangle \subset \mathbf{H}^0\left(\boldsymbol{Y}, \Omega^1_{\boldsymbol{Y}} \right) \; + \left[\begin{array}{c} d\omega_i = 0 \\ \omega_i \wedge \omega_j = 0 \end{array} \right]$

- **Ex :** $\phi : \boldsymbol{Y} \to \boldsymbol{C}$ and $\omega_i \in \phi^*\Big(\mathbf{H}^0\big(\boldsymbol{C}, \Omega^1_{\boldsymbol{C}} \big)\Big) \quad i = 1, \ldots, m$

- Base point $y \in \boldsymbol{Y}$, path $\gamma^x = \gamma^x_y : [0,1] \to \boldsymbol{Y}$ from y to x :
  - $\mathbb{I}_{\omega_i} \; : \; x \longmapsto \int_{\gamma^x} \omega_i \quad \rightsquigarrow \quad \mathbb{I}_{\omega_i} \in \mathcal{O}_y$
  - $\mathbb{I}_{\omega_j \omega_i} \; : \; x \longmapsto \int_{\gamma^x} \omega_j(u) \cdot \mathbb{I}_{\omega_i}(u) \quad \rightsquigarrow \quad \mathbb{I}_{\omega_j \omega_i} \in \mathcal{O}_y$
  - $\mathbb{I}_{\omega_k \omega_j \omega_i} \; : \; x \longmapsto \int_{\gamma^x} \omega_k(u) \cdot \mathbb{I}_{\omega_j \omega_i}(u) \quad \rightsquigarrow \quad \mathbb{I}_{\omega_k \omega_j \omega_i} \in \mathcal{O}_y$

Iterated integrals (polylogarithms)

- $$\mathbf{II}^w \;:\; \mathbf{H}^{\otimes w} \longrightarrow \mathcal{O}_y$$

$$\underline{\omega} = \omega_{i_1} \otimes \left(\otimes_{s=2}^{w} \omega_{i_s} \right) \longmapsto \left[\; \mathbb{I}_{\underline{\omega}} \;:\; z \mapsto \int_y^z \omega_{i_1} \, \mathbb{I}_{\omega_{i_2} \cdots \omega_{i_w}} \; \right]$$

- $\mathbf{II} : \left(\bigoplus_{w \geq 0} \mathbf{H}^{\otimes w} \, , \, ш \right) \longrightarrow \mathcal{O}_y$ injective morphism of complex algebras

- $\forall \, \underline{\omega}$: $\mathbb{I}_{\underline{\omega}} \in \mathcal{O}_y \cap \widetilde{\mathcal{O}}(\boldsymbol{Y})$ has unipotent monodromy $\longrightarrow$ Symbol $\boldsymbol{\mathcal{S}}(\mathbb{I}_{\underline{\omega}}) = \underline{\omega}$

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \boldsymbol{\Sigma}$ with $\boldsymbol{\Sigma} = \{ 0, 1, \infty \}$

$$\mathbf{H} = \left\langle \frac{dz}{z} \, , \, \frac{dz}{1-z} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 \, , \, \Omega^1_{\mathbb{P}^1}(\mathrm{Log}\, \boldsymbol{\Sigma}) \Big)$$

$$\mathbf{Li}_n = \mathbf{II}^n\Big(\left(\tfrac{dz}{z}\right)^{\otimes(n-1)} \otimes \left(\tfrac{dz}{1-z}\right) \Big)$$ (**'Polylogarithms'**)

Iterated integrals (hyperlogarithms)

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \boldsymbol{\Sigma}$ with $\boldsymbol{\Sigma} = \left\{ \sigma^1 , \ldots , \sigma^{r-2} , \sigma^{r-1} = \infty \right\}$

$$\mathbf{H} = \left\langle \tfrac{dz}{z-\sigma^1}, \ldots, \tfrac{dz}{z-\sigma^{r-2}} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 , \Omega^1_{\mathbb{P}^1}(\mathrm{Log}\, \boldsymbol{\Sigma}) \Big)$$

$$\mathbf{II}^n\Big(\big(\tfrac{dz}{z-\sigma^{i_1}}\big) \otimes \cdots \otimes \big(\tfrac{dz}{z-\sigma^{i_n}}\big) \Big)$$

'Hyperlogarithm'

- **Complete antisymmetric hyperlog of weight $r-2$ on $\mathbf{P}^1 \setminus \boldsymbol{\Sigma}$** :

$$\left(\frac{dz}{z-\sigma^{1}}\right) \otimes \cdots \otimes \left(\frac{dz}{z-\sigma^{r-2}}\right)$$

Iterated integrals (hyperlogarithms)

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \boldsymbol{\Sigma}$ with $\boldsymbol{\Sigma} = \left\{ \sigma^1 , \ldots , \sigma^{r-2} , \sigma^{r-1} = \infty \right\}$

$$\mathbf{H} = \left\langle \tfrac{dz}{z-\sigma^1}, \ldots, \tfrac{dz}{z-\sigma^{r-2}} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 , \Omega^1_{\mathbb{P}^1}(\mathrm{Log}\, \boldsymbol{\Sigma}) \Big)$$

$$\mathbf{II}^n\Big(\big(\tfrac{dz}{z-\sigma^{i_1}}\big) \otimes \cdots \otimes \big(\tfrac{dz}{z-\sigma^{i_n}}\big) \Big)$$

'Hyperlogarithm'

- **Complete antisymmetric hyperlog of weight $r-2$ on $\mathbf{P}^1 \setminus \boldsymbol{\Sigma}$:**

$$\mathbf{Asym}\left(\left(\frac{dz}{z-\sigma^{\mathbf{1}}}\right) \otimes \cdots \otimes \left(\frac{dz}{z-\sigma^{\boldsymbol{r-2}}}\right)\right)$$

Iterated integrals (hyperlogarithms)

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \boldsymbol{\Sigma}$ with $\boldsymbol{\Sigma} = \left\{ \sigma^1 , \ldots , \sigma^{r-2} , \sigma^{r-1} = \infty \right\}$

$$\mathbf{H} = \left\langle \tfrac{dz}{z-\sigma^1}, \ldots, \tfrac{dz}{z-\sigma^{r-2}} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 , \Omega^1_{\mathbb{P}^1}(\mathrm{Log}\,\boldsymbol{\Sigma}) \Big)$$

$$\mathbf{II}^n\Big(\big(\tfrac{dz}{z-\sigma^{i_1}}\big) \otimes \cdots \otimes \big(\tfrac{dz}{z-\sigma^{i_n}}\big) \Big)$$ **'Hyperlogarithm'**

- **Complete antisymmetric hyperlog of weight $r-2$ on $\mathbf{P}^1 \setminus \boldsymbol{\Sigma}$:**

$$\boldsymbol{AH}_{\boldsymbol{\Sigma}}^{r-2} = \mathbf{II}\left(\mathbf{Asym}\left(\left(\frac{dz}{z-\sigma^{\mathbf{1}}}\right) \otimes \cdots \otimes \left(\frac{dz}{z-\sigma^{\boldsymbol{r-2}}}\right) \right) \right)$$

Iterated integrals (hyperlogarithms)

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \boldsymbol{\Sigma}$ with $\boldsymbol{\Sigma} = \left\{ \sigma^1 , \ldots , \sigma^{r-2} , \sigma^{r-1} = \infty \right\}$

$$\mathbf{H} = \left\langle \frac{dz}{z-\sigma^1}, \ldots, \frac{dz}{z-\sigma^{r-2}} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 , \Omega^1_{\mathbb{P}^1}(\mathrm{Log}\, \boldsymbol{\Sigma}) \Big)$$

$$\mathbf{II}^n\Big(\big(\tfrac{dz}{z-\sigma^{i_1}}\big) \otimes \cdots \otimes \big(\tfrac{dz}{z-\sigma^{i_n}}\big) \Big) \qquad \text{'Hyperlogarithm'}$$

- **Complete antisymmetric hyperlog of weight $r-2$ on $\mathbf{P}^1 \setminus \boldsymbol{\Sigma}$:**

$$\boldsymbol{AH}^{r-2}_{\boldsymbol{\Sigma}} = \mathbf{II}\left(\mathrm{Asym}\left(\left(\frac{dz}{z-\sigma^{1}}\right) \otimes \cdots \otimes \left(\frac{dz}{z-\sigma^{r-2}}\right) \right) \right)$$

$$= \mathbf{II}\left(\frac{1}{(r-2)!} \sum_{\nu \in \mathfrak{S}_{r-2}} (-1)^\nu \left(\frac{dz}{z-\sigma^{\nu(1)}}\right) \otimes \cdots \otimes \left(\frac{dz}{z-\sigma^{\nu(r-2)}}\right) \right)$$

- **Ex :** $\boldsymbol{AH}^2_{\{0,1,\infty\}} = \frac{1}{2}\, \mathbf{II}^2\left(\frac{dz}{z} \otimes \frac{dz}{(1-z)} - \frac{dz}{(1-z)} \otimes \frac{dz}{z} \right) = \mathbf{R}$

Identity HLog^{r-2} : proof(s)

$$\left(\mathbf{HLog}^{r-2}\right) \ : \quad \sum_{\mathfrak{c}\in\mathcal{C}} \epsilon_{\mathfrak{c}}\, \boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\phi_{\mathfrak{c}}) = 0 \quad \text{with} \quad \boldsymbol{AH}_{\mathfrak{c}}^{r-2} = \boldsymbol{AH}_{\boldsymbol{\Sigma}_{\mathfrak{c}}}^{r-2}$$

- $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1 \supset \boldsymbol{\Sigma}_{\mathfrak{c}} = \{\, \sigma_{\mathfrak{c}}^i \,\}_{i=1}^{r-1}$ $\qquad \mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\Omega^1_{\mathbb{P}^1}\big(\,\mathrm{Log}\,\boldsymbol{\Sigma}_{\mathfrak{c}}\,\big)\Big)$

- $\phi_{\mathfrak{c}}^*\big(\,\mathcal{H}_{\mathfrak{c}}\,\big) = \mathbf{H}_{\mathfrak{c}} \subset \mathbf{H}_{\boldsymbol{X}_r} = \mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\,\boldsymbol{L}_r\,\big)\Big)$

- $\boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\phi_{\mathfrak{c}}) = \mathbf{II}\left(\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c}}^1}\right)\wedge\cdots\wedge\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c}}^{r-2}}\right)\right) \in \mathbf{II}^{r-2}\Big(\wedge^{r-2}\,\mathbf{H}_{\mathfrak{c}}\Big)$

  $\big\downarrow$ $\boldsymbol{\mathcal{S}}$ (symbol)

- $\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = \left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c}}^1}\right)\wedge\cdots\wedge\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c}}^{r-2}}\right) \ \in \wedge^{r-2}\mathbf{H}_{\mathfrak{c}} \subset \wedge^{r-2}\mathbf{H}_{\boldsymbol{X}_r}$

$$\left(\mathbf{HLog}^{r-2}\right) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \text{in} \quad \wedge^{r-2}\mathbf{H}_{X_r}$$

Identity HLog^{r-2} : proof(s)

Proofs of : $\mathbf{hlog}^{r-2} = \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}} \, \mathbf{\Omega}_{\mathfrak{c}}^{r-2} = 0$ **in** $\wedge^{r-2}\mathbf{H}_{X_r}$

$$\mathbf{H}_{\boldsymbol{X}_r} = \mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\operatorname{Log} \boldsymbol{L}_r\big)\Big) \xrightarrow{\;\oplus_\ell \mathbf{Res}_\ell\;} \mathbb{C}^{\mathcal{L}_r} \qquad \text{injective}$$

$$\mathbf{\Omega}_{\mathfrak{c}}^{r-2} \in \wedge^{r-2}\mathbf{H}_{X_r} \lhook\joinrel\longrightarrow \wedge^{r-2}\,\mathbb{C}^{\mathcal{L}_r} \curvearrowleft \boldsymbol{W}(\mathrm{E}_r)$$

[Pr1] One decomposes $\mathbf{hlog}^{r-2}$ in a natural basis of $\wedge^{r-2}\,\mathbb{C}^{\mathcal{L}_r}$

[Pr2] $\mathbf{sign}_r \hookrightarrow \oplus_{\mathfrak{c}}\big(\mathbf{H}_{\mathfrak{c}}\big)^{\wedge(r-2)} \longrightarrow \wedge^{r-2}\,\mathbb{C}^{\mathcal{L}_r} \quad \Big\langle \mathbf{sign}_r\,,\,\wedge^{r-2}\,\mathbb{C}^{\mathcal{L}_r}\Big\rangle = 0$

$$\mathbf{1} \longmapsto \big(\mathbf{\Omega}_{\mathfrak{c}}^{r-2}\big)_{\mathfrak{c}} \longmapsto \textstyle\sum_{\mathfrak{c}} \mathbf{\Omega}_{\mathfrak{c}}^{r-2}$$

(GAP3)

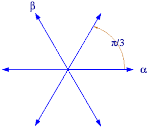
[Pr3] Explicit descrp° of lines $\longrightarrow$ $\mathbb{Z}^{\mathcal{L}_r} \simeq \mathbb{Z}^{|\mathcal{L}_r|}$

$+$ linear algebra / $\mathbb{Z}$ $\longrightarrow$ $\sum \epsilon_{\mathfrak{c}} \mathbf{\Omega}_{\mathfrak{c}}^{r-2} = 0$ (Maple)

Logarithm

$\mathbf{Log}(x) + \mathbf{Log}(y) - \mathbf{Log}\,(xy) = 0$

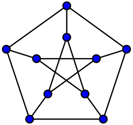
$$\mathrm{dP}_6 = \mathrm{Bl}_3(\mathbb{P}^2) \subset \mathbb{P}^6$$

$A_1 \times$ 

Dilogarithm

$\mathrm{R}(x) - \mathrm{R}(y) \cdots + \mathrm{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$

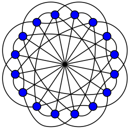
$$\mathrm{dP}_5 = \mathrm{Bl}_4(\mathbb{P}^2) \subset \mathbb{P}^5$$

A_4 

Weight 3 hyperlog

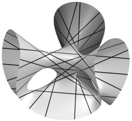
$\sum_{i=1}^{10} AH_i^3(U_i(x,y)) = 0$

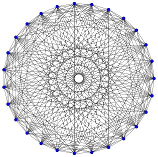
$$\mathrm{dP}_4 = Q_1 \cap Q_2 \subset \mathbb{P}^4$$

D_5 

Weight 4 hyperlog

$\sum_{i=1}^{27} AH_i^4(U_i(x,y)) = 0$

$\mathrm{dP}_3 =$ 

E_6 

Remarks

- A hyperlogarithm

$$\mathbf{HL}^{\zeta}_{s_1,\ldots,s_m}(z)=\int_{\zeta}^{z}\frac{\mathrm{d}u}{u-s_1}\otimes\frac{\mathrm{d}u}{u-s_2}\otimes\cdots\otimes\frac{\mathrm{d}u}{u-s_m}$$

is a multiple polylogarithm : we have

$$\mathbf{HL}^{\zeta}_{s_1,\ldots,s_m}(z)=\mathbf{L}\mathrm{i}_{1,\ldots,1}\left(\frac{z-\zeta}{s_1-\zeta},\frac{s_1-\zeta}{s_2-\zeta},\ldots,\frac{s_{m-1}-\zeta}{s_m-\zeta}\right)$$

where $$\mathbf{L}\mathrm{i}_{1,\ldots,1}(z_1,\ldots,z_m)=\sum_{n_1>n_2>\cdots>n_m\geq 1}\frac{z_1^{n_1}\cdots z_m^{n_m}}{n_1\cdots n_m}$$

- For $\boldsymbol{\Sigma}\subset\mathbf{P}^1$: $\mathbf{H}^0\Big(\mathbf{P}^1,\Omega^1_{\mathbf{P}^1}(\mathrm{Log}\,\boldsymbol{\Sigma})\Big)$ carries a natural $\mathbb{Z}$-structure (by requiring $\mathrm{Res}(\omega)\subset\mathbb{Z}$)

$\longrightarrow$ $\sum_{i=1}^{\kappa}\epsilon_i\,\boldsymbol{AH}^{r-2}_{\boldsymbol{\Sigma}_i}(\boldsymbol{\phi_i})=0$ is defined over $\mathbb{Z}$

Comparison

- $\boxed{\mathbf{HLog}^2}$ $$\mathbf{R}(x)-\mathbf{R}(y)-\mathbf{R}\left(\frac{x}{y}\right)-\mathbf{R}\left(\frac{1-y}{1-x}\right)+\mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right)=0$$

- $\boxed{\mathbf{HLog}^3}$ $$AH_1^3(x)+AH_2^3\left(\frac{1}{y}\right)+AH_3^3\left(\frac{y}{x}\right)+AH_4^3\left(\frac{x-y}{x-1}\right)$$ $$+AH_5^3\left(\frac{b(a-x)}{ay-bx}\right)+AH_6^3\left(\frac{P(x,y)}{(x-1)(y-b)}\right)+AH_7^3\left(\frac{(x-y)(y-b)}{y\,P(x,y)}\right)$$ $$+AH_8^3\left(\frac{x\,P(x,y)}{(x-y)(x-a)}\right)+AH_9^3\left(\frac{y(x-b)}{x(y-a)}\right)+AH_{10}^3\left(\frac{a\,(b-x)}{by-ax}\right)=0$$

- $\mathbf{HLog}^2$ and $\mathbf{HLog}^3$ are very similar but are not quite the same :
  - $\mathbf{HLog}^2$ unique **vs** there are ∞^2 $\mathbf{HLog}^3_{a,b}$
  - only $\mathbf{R}$ in $\mathbf{HLog}^2$ **vs** $\exists$ several AH_i^3 in $\mathbf{HLog}^3$
  - $W_{A_4}=\mathbf{Aut}(\mathrm{dP}_5)$ acts geometrically **vs** not the case for $W(\mathrm{E}_5)=W_{D_5}$

III Gelfand–MacPherson correspondence

- $\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$ on $\mathbf{dP}_5$

- Up to the natural isomorphism $\mathbf{dP}_5 \simeq \overline{\mathcal{M}}_{0,5}$:

$$\left\{ x\,,\ \ y\ ,\ x/y\ ,\ \ldots,\ \frac{x(1-y)}{y(1-x)} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Forgetful maps} \\ f_i : \mathcal{M}_{0,5} \to \mathcal{M}_{0,4} \end{array} \right\}$$

- **Gelfand-MacPherson** (over $\mathbf{k} = \mathbb{R}, \mathbb{C}$) **:**

$$\begin{tikzcd}
G_2^o(\mathbf{k}^5) \arrow[r, "F_i"] \arrow[d, "\bullet/H_4"] & G_2^o\left(\mathbf{k}^5/\langle e_i\rangle\right) \arrow[d, "\bullet/H_{3,i}"] \arrow[l, bend right, "a_i"'] \\
\mathcal{M}_{0,5}(\mathbf{k}) \arrow[r, "f_i"] & \mathcal{M}_{0,4}(\mathbf{k}) \simeq \mathbf{k} \setminus \{0,1\}
\end{tikzcd}$$

$\longleftarrow$ The vertical arrows are the quotient maps by the Cartan subgps
$(\mathbf{k}^*)^4 \simeq H_4 \subset \mathrm{GL}(\mathbf{k}^5)$
$(\mathbf{k}^*)^3 \simeq H_{3,i} \subset \mathrm{GL}(\mathbf{k}^5/\langle e_i\rangle)$

Gelfand-MacPherson construction of $(\mathcal{A}b)$

$$\begin{tikzcd}
\boldsymbol{G}_2^o(\mathbb{R}^5) \arrow[r, "F_i"] \arrow[d] & \boldsymbol{G}_2^o\left(\mathbb{R}^5/_{\langle e_i \rangle}\right) \arrow[d] \\
\mathcal{M}_{0,5} \arrow[r, "f_i"] & \mathcal{M}_{0,4} \simeq \mathbb{R} \setminus \{0,1\}
\end{tikzcd}$$

1. the differential $\mathrm{d}\mathbf{R}$ of Rogers dilogarithm is constructed on $\mathcal{M}_{0,4}$ by fiber integ° of a $\mathrm{SO}_5(\mathbb{R})$-invariant representative of the 1st Pontryagin class $\mathbf{P}_1 \in \mathbf{H}^4\Big(\boldsymbol{G}_2(\mathbb{R}^\bullet)\Big)$ of the tautological bundle

2. For $\xi \in \boldsymbol{G}_2^o(\mathbb{R}^5)$ the orbit $\boldsymbol{H}\,\xi$ is isom. to the hypersimplex $\boldsymbol{\Delta}_2^5$

3. Stokes theorem for fiber integ° + polytopal structure of the fibers :

$$0 = \sum_{i=1}^5 (-1)^i \, f_i^*(\mathrm{d}\mathbf{R}) \qquad (\mathcal{A}\mathbf{b})$$

Cox varieties

- **Question :** How can one obtain $\boldsymbol{G}_2(\mathbb{C}^5)$ from $\boldsymbol{X}_4 = \mathbf{dP}_5 = \overline{\boldsymbol{\mathcal{M}}}_{0,5}$?
  Answer : $\boldsymbol{G}_2(\mathbb{C}^5)$ is the projective **Cox variety** of $\mathbf{dP}_5$

- **Def°** : $\boldsymbol{S}$ = projective manifold such that $\mathbf{Pic}_{\mathbb{Z}}(\boldsymbol{S}) = \bigoplus_{i=0}^{r} \mathbb{Z}\,[E_i]$

$$\mathbf{Cox}(\boldsymbol{S}) = \bigoplus_{n_0,\ldots,n_r\in\mathbb{Z}} \mathbf{H}^0\Big(\boldsymbol{S}, \mathcal{O}_{\boldsymbol{S}}(n_0E_0 + n_1E_1 + \cdots + n_rE_r)\Big)$$

- $\mathbf{Cox}(\boldsymbol{S})$ of finite type (i.e. $\boldsymbol{S}$ is a "Mori Dream Space")

$$\begin{array}{ccc} \mathbf{Cox}(\boldsymbol{S}) & & \\ \parallel & & \\ \mathbb{C}[\Gamma_1,\ldots,\Gamma_m]/\boldsymbol{\mathcal{J}}_{\boldsymbol{S}} & \longrightarrow & \mathbf{A}(\boldsymbol{S}) = \mathrm{Spec}\Big(\mathbf{Cox}(\boldsymbol{S})\Big) \subset \mathbb{A}^m_{\Gamma} \end{array}$$

« Affine Cox variety » of $\boldsymbol{S}$

- **Fact** : Surface $\boldsymbol{S} \supset \ell$ with $\ell \simeq \mathbb{P}^1$ and $\ell^2 = -1$ $\Longrightarrow$ $\sigma_\ell \in \mathbf{H}^0(\mathcal{O}_{\boldsymbol{X}_r}(\ell)) \setminus \{0\}$ generator in $\mathbf{Cox}(\boldsymbol{S})$

Cox varieties of del Pezzo surfaces

Thm [Batyrev, Popov] For $r = 3, \ldots, 8$, one has

$$\mathbf{Cox}\Big(\mathbf{dP}_d = \boldsymbol{X_r} = \mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2)\Big) = \mathbb{C}\Big[\,\sigma_\ell \,\big|\, \ell \in \boldsymbol{\mathcal{L}_r}\,\Big]\big/\mathcal{J}_{\mathbf{dP_d}}$$

- $\mathbf{T}_{\mathrm{NS}} = \mathrm{Hom}_{\mathbb{Z}}\Big(\mathbf{Pic}_{\mathbb{Z}}(\boldsymbol{X}_r)\,,\,\mathbb{C}^*\Big) \circlearrowleft \mathbf{A}(\boldsymbol{X}_r) \quad \rightsquigarrow \quad \boldsymbol{X}_r = \mathbf{A}(\boldsymbol{X}_r)/\!\!/\,\mathbf{T}_{\mathrm{NS}}$

- $\big(-K_{\boldsymbol{X_r}}\,,\,\cdot\,\big)$ induces a $\mathbb{Z}$-grading on $\mathbf{Cox}(\boldsymbol{X_r})$

- **Def**$^\circ$: The **Projective Cox variety** of $\boldsymbol{X}_r$ is

$$\mathbf{P}(\boldsymbol{X}_r) = \mathrm{Proj}\Big(\mathbf{Cox}(\boldsymbol{X_r})\Big) \subset \mathbb{P}\Big(\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}\Big)$$

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$$\mathbf{Cox}\Big(\mathbf{dP}_d = \boldsymbol{X_r} = \mathbf{Bl}_{p_1,\dots,p_r}(\mathbb{P}^2)\Big) = \mathbb{C}\Big[\,\sigma_\ell \,\big|\, \ell \in \boldsymbol{\mathcal{L}_r}\,\Big]\big/\boldsymbol{\mathcal{J}}_{\mathbf{dP_d}}$$

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- $(-K_{\boldsymbol{X_r}}\,,\,\cdot\,)$ induces a $\mathbb{Z}$-grading on $\mathbf{Cox}(\boldsymbol{X_r})$

- **Def**$^\circ$: The **Projective Cox variety** of $\boldsymbol{X}_r$ is

$$\mathbf{P}(\boldsymbol{X}_r) = \mathrm{Proj}\Big(\mathbf{Cox}(\boldsymbol{X_r})\Big) \subset \mathbb{P}\Big(\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}\Big) \circlearrowleft \begin{array}{c} \mathcal{T}_{\mathrm{NS}} = \mathbf{T}_{\mathrm{NS}}/\mathbb{C}^* \\ \boldsymbol{W}(\mathrm{E}_r) \end{array}$$

Cox varieties of del Pezzo surfaces

Thm [Batyrev, Popov] For $r = 3, \dots, 8$, one has

$$\mathbf{Cox}\Big(\mathbf{dP}_d = \boldsymbol{X_r} = \mathbf{Bl}_{p_1,\dots,p_r}(\mathbb{P}^2)\Big) = \mathbb{C}\Big[\,\sigma_\ell \,\big|\, \ell \in \boldsymbol{\mathcal{L}_r}\,\Big]/\mathcal{J}_{\mathbf{dP_d}}$$

- $\mathbf{T}_{\mathrm{NS}} = \mathrm{Hom}_{\mathbb{Z}}\Big(\mathbf{Pic}_{\mathbb{Z}}(\boldsymbol{X}_r)\,,\,\mathbb{C}^*\Big) \circlearrowleft \mathbf{A}(\boldsymbol{X}_r) \quad \rightsquigarrow \quad \boldsymbol{X}_r = \mathbf{A}(\boldsymbol{X}_r)/\!/\,\mathbf{T}_{\mathrm{NS}}$

- $\big(-K_{\boldsymbol{X_r}}\,,\,\cdot\,\big)$ induces a $\mathbb{Z}$-grading on $\mathbf{Cox}(\boldsymbol{X_r})$

- **Def**$^\circ$: The **Projective Cox variety** of $\boldsymbol{X}_r$ is

$$\mathbf{P}(\boldsymbol{X}_r) = \mathrm{Proj}\Big(\mathbf{Cox}(\boldsymbol{X_r})\Big) \subset \mathbb{P}\Big(\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}\Big) \circlearrowleft \left.\begin{matrix} \mathcal{T}_{\mathrm{NS}} = \mathbf{T}_{\mathrm{NS}}/\mathbb{C}^* \\ \boldsymbol{W}(\mathrm{E}_r) \end{matrix}\right\} \rightsquigarrow \boldsymbol{G}(\mathrm{E}_r)$$

Cox varieties of del Pezzo surfaces

- $\mathbf{P}(\boldsymbol{X}_r) = \mathrm{Proj}\Big(\mathbf{Cox}(\boldsymbol{X}_r)\Big) \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r}) \circlearrowright \left.\begin{array}{l} \mathcal{T}_{\mathrm{NS}} = \mathbf{T}_{\mathrm{NS}}/\mathbb{C}^* \\ \boldsymbol{W}(\mathrm{E}_r) \end{array}\right\} \rightsquigarrow \boldsymbol{G}(\mathrm{E}_r)$

Thm [**Batyrev, Popov, Derenthal, Serganova-Skorobogatov**]

1. $\mathbb{C}^{\mathcal{L}_r}$ is a **minuscule** representation of $\boldsymbol{G}_r = \boldsymbol{G}(\mathrm{E}_r)$
2. There is a $\mathcal{T}_{\mathrm{NS}} = \boldsymbol{H}_r$-equivariant embedding
$$\mathbf{P}(\boldsymbol{X}_r) \hookrightarrow \boldsymbol{G}_r/\boldsymbol{P}_r \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r})$$
3. There is an embedding $f_{SS} : \boldsymbol{X}_r \hookrightarrow \boldsymbol{\mathcal{Y}}_r = (\boldsymbol{G}_r/\boldsymbol{P}_r)/\!\!/\boldsymbol{H}_r$ st
$$\begin{tikzcd} \mathbf{P}(\boldsymbol{X}_r) \arrow[r, hook] \arrow[d] & \boldsymbol{G}_r/\boldsymbol{P}_r \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r}) \arrow[d] \\ \mathbf{dP}_{9-r} = \boldsymbol{X}_r \arrow[r, hook, "f_{SS}"] & \boldsymbol{\mathcal{Y}}_r \end{tikzcd}$$
is commutative
4. There is an isomorphism $\boldsymbol{W}(\mathrm{E}_r) \simeq \mathbf{Aut}(\boldsymbol{\mathcal{Y}}_r)$

- $r = 4$, $\boldsymbol{A_4}$: $\mathbf{P}(\boldsymbol{X}_4) = \boldsymbol{G}_4/\boldsymbol{P}_4 = \boldsymbol{G}_2(\mathbb{C}^5) \hookrightarrow \mathbb{P}(\mathbb{C}^{\mathcal{L}_4}) \simeq \mathbb{P}^9$ (Plücker)

$$\begin{tikzcd}
& \mathbf{P}(\boldsymbol{X}_4) \arrow[r, equal] \arrow[d] & \boldsymbol{G}_2(\mathbb{C}^5) \arrow[r, dashed, "\boldsymbol{F_i}"] \arrow[d] & \boldsymbol{G}_2(\mathbb{C}^4_i) \arrow[d] \\
\mathbf{dP}_5 = & \boldsymbol{X}_4 \arrow[r, hook, "f_{SS}", "\sim"'] & \boldsymbol{\mathcal{Y}}_4 = \overline{\boldsymbol{\mathcal{M}}}_{0,5} \arrow[r, dashed, "\boldsymbol{f_i}"] & \overline{\boldsymbol{\mathcal{M}}}_{0,4} \simeq \mathbb{P}^1
\end{tikzcd}$$

$\longrightarrow$ *'Everything comes from the right square diagrams'*

- For $r = 4, \ldots, 7$ we have

$$\begin{tikzcd}
& \mathbf{P}(\boldsymbol{X}_r) \arrow[r, hook] \arrow[d] & \boldsymbol{G}_r/\boldsymbol{P}_r \arrow[d] \\
\mathbf{dP}_d = & \boldsymbol{X}_r \arrow[r, hook, "f_{SS}"] & \boldsymbol{\mathcal{Y}}_r
\end{tikzcd}$$

- **Question :** Can the pencils of conics on $\mathbf{dP}_d$ and the identity $\mathbf{HLog}^{r-2}$ be obtained from $\boldsymbol{G}_r/\boldsymbol{P}_r$ and $\boldsymbol{\mathcal{Y}}_r$?

- $r = 4$, $\boldsymbol{A_4} : \mathbf{P}(\boldsymbol{X}_4) = \boldsymbol{G}_4/\boldsymbol{P}_4 = \boldsymbol{G}_2(\mathbb{C}^5) \hookrightarrow \mathbb{P}(\mathbb{C}^{\mathcal{L}_4}) \simeq \mathbb{P}^9$ (Plücker)

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\mathbf{P}(\boldsymbol{X}_4) \arrow[rr, equal] \arrow[d] & & \boldsymbol{G}_2(\mathbb{C}^5) \arrow[r, dashed, "\boldsymbol{F_i}"] \arrow[d] & \boldsymbol{G}_2(\mathbb{C}^4_i) \arrow[d] \\
\mathbf{dP}_5 = \boldsymbol{X}_4 \arrow[rr, hook, "\boldsymbol{f_{SS}}", "\sim"'] & & \boldsymbol{\mathcal{Y}}_4 = \overline{\boldsymbol{\mathcal{M}}}_{0,5} \arrow[r, dashed, "\boldsymbol{f_i}"] & \overline{\boldsymbol{\mathcal{M}}}_{0,4} \simeq \mathbb{P}^1
\end{tikzcd}$$

$\longrightarrow$ *'Everything comes from the right square diagrams'*

Theorem. There are **face maps** Π_F, π_F giving rise to commutative diagrams

$$\begin{tikzcd}
\mathbf{P}(\boldsymbol{X}_r) \arrow[rr, hook] \arrow[d] & & \boldsymbol{G_r}/\boldsymbol{P_r} \arrow[r, dashed, "\Pi_F"] \arrow[d] & \mathbb{Q}_F^{2r-4} = \boldsymbol{G}_F/\boldsymbol{P}_F \arrow[d] \\
\mathbf{dP}_{9-r} = \boldsymbol{X}_r \arrow[rr, hook, "\boldsymbol{f_{SS}}"] & & \boldsymbol{\mathcal{Y}}_r \arrow[r, dashed, "\pi_F"] & \boldsymbol{\mathcal{Y}}_F \simeq \mathbb{P}^{r-3}
\end{tikzcd}$$

such that the $\pi_F \circ \boldsymbol{f_{SS}}$'s are precisely the pencils of conics on $\boldsymbol{X}_r$

Differential identities $\mathbf{HLOG}_{\mathcal{Y}_r}$

$$\begin{tikzcd}
 & \mathbf{P}(\boldsymbol{X}_r) \arrow[r, hook] \arrow[d] & \boldsymbol{G}_r/\boldsymbol{P}_r \arrow[r, dashed, "\Pi_F"] \arrow[d] & \mathbb{Q}_F^{2r-4} = \boldsymbol{G}_F/\boldsymbol{P}_F \arrow[d] \\
\mathbf{dP}_{9-r} = & \boldsymbol{X}_r \arrow[r, hook, "f_{SS}"] & \boldsymbol{\mathcal{Y}}_r \arrow[r, dashed, "\pi_F"] & \boldsymbol{\mathcal{Y}}_F \simeq \mathbb{P}^{r-3}
\end{tikzcd}$$

- Setting $\mathrm{U} = 1 + \sum_i u_i$, we define a multivalued $(r-3)$-form on $\mathbb{P}^{r-3}$

$$\boldsymbol{\Omega}_r = \mathbf{Asym}_{u_1,\ldots,u_{r-3},\mathrm{U}}\left(\mathrm{Log}(\mathrm{U})\left(\frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_{r-3}}{u_{r-3}}\right)\right)$$

Thm. **1.** The $\pi_F \circ \boldsymbol{f_{SS}}$'s give the pencils of conics on $\boldsymbol{X}_r$

2. We have $\mathbf{HLOG}_{\mathcal{Y}_r}$: $\sum_F \pi_F^*\left(\boldsymbol{\Omega}_r\right) = 0$

3. $\mathbf{HLOG}_{\mathcal{Y}_r} \rightsquigarrow \mathbf{HLog}^{r-2}_{\mathbf{dP}_{9-r}}$ for any $\mathbf{dP}_{9-r}$

Thank you very much for your attention

ご清聴ありがとうございました。