Torsion theories and wide subcategories in truncated derived categories

Théories de torsion et sous-catégories vastes dans les catégories dérivées tronquées

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Mots clés: Objets bousculants, classes de torsion, sémibriques, catégories vastes, nombres de Fuss-Catalan, treillis cambrien

Résumé: Soit Λ une algèbre de dimension finie sur un corps. D'après les résultats de la théorie de τ-basculement, on sait que les objets bousculants à 2 termes de Λ et les collections simplistes à 2 termes sont en bijection avec les classes de torsion fonctoriellement finies, les sémibriques finies à gauches, et les catégories vastes finies à gauches dans mod Λ, ainsi que les classes de cotorsion complètes dans $\mathsf{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Dans cette thèse, nous introduisons les catégories des modules étendues et les notions de classes de torsion positives, sémibriques et catégories vastes dans ces catégories-là. Nous montrons une bijection entre les objects bousculants à d termes, les classes de torsion positives et fonctoriellement finies, les sémibriques finies à gauches, les catégories vastes finies à gauches, et les classes de cotorsion complètes et héréditaires. Nous proposons également un modèle géométrique pour les algèbres aimables, tel que les objets bousculants à d termes dans ces algèbres correspondent à certaines collections d'arcs sur une surface marquée avec une dissection admissible.

Title: Torsion theories and wide subcategories in truncated derived categories

Keywords: Silting objects, torsion classes, semibricks, wide subcategories, Fuss-Catalan numbers, Cambrian lattices

Abstract: For a finite-dimensional algebra Λ over a field, it is known that 2-term silting objects and 2-term simple-minded collections are in bijection with several objects in the module category, namely, functorially finite torsion classes, left-finite semibricks, and left-finite wide subcategories. It is also known that they are in bijection with complete cotorsion classes in the extriangulated category $\mathsf{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. In this thesis, we introduce positive torsion classes, semibricks, and wide subcategories in certain extriangulated categories called the extended module categories. We then generalise the above results to d-term silting objects and d-term simple-minded collections by showing bijections with functorially finite, positive torsion classes, left-finite semibricks, and left-finite wide subcategories in extended module categories, as well as with complete, hereditary cotorsion classes in $\mathsf{K}^{[-d+1,0]}(\operatorname{proj}\Lambda)$. We also provide a geometric model for gentle algebras such that d-term silting objects in these algebras correspond to certain collections of arcs on a marked surface with an admissible dissection.

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Contents

Introduction

Dans ce chapitre, nous présentons un bref historique des résultats qui ont conduit au sujet de cette thèse et, puis, nous proposons un résumé de nos résultats.

Notation

Nous utiliserons Λ pour désigner une algèbre de dimension finie sur un corps K. Toutes les sous-catégories sont supposées être pleines, stables par isomorphismes et contenir l'objet zéro. Pour les morphismes $f : X \to Y$ et $g : Y \to Z$, nous désignons par gf la composition $X \xrightarrow{f} Y \xrightarrow{g} Z$. Pour toute catégorie $\mathcal{D}$, nous utilisons la notation $X \in \mathcal{D}$ pour indiquer que X est un objet de $\mathcal{D}$, et $\mathcal{C} \subseteq \mathcal{D}$ pour indiquer que $\mathcal{C}$ est une sous-catégorie de $\mathcal{D}$. Pour tout $X \in \mathcal{D}$, on utilise $\operatorname{add} X$ pour désigner l'enveloppe additive de X dans $\mathcal{D}$, c'est-à-dire la sous-catégorie pleine de $\mathcal{D}$ contenant tous les facteurs directs des sommes directes finies de copies de X. Pour deux sous-catégories $\mathcal{X}$ et $\mathcal{Y}$ de $\mathcal{D}$, on utilise $\operatorname{Hom}(\mathcal{X}, \mathcal{Y}) = 0$ pour signifier que $\operatorname{Hom}(X, Y) = 0$ pour tout $X \in \mathcal{X}$ et $Y \in \mathcal{Y}$. Pour tout $\mathcal{C} \subseteq \mathcal{D}$, un morphisme $f : X \to C$ dans $\mathcal{D}$ est appelé une *approximation à gauche de* $\mathcal{C}$ de X si $C \in \mathcal{C}$ et si, pour tout morphisme $h : X \to C'$ avec $C' \in \mathcal{C}$, il existe un morphisme $g : C \to C'$ tel que $h = gf$. Une sous-catégorie $\mathcal{C} \subseteq \mathcal{D}$ est dite *covariamment finie* si tout objet $X \in \mathcal{D}$ admet une approximation à gauche par $\mathcal{C}$. On peut définir de manière duale les notions d'approximations à droites de $\mathcal{C}$ et de finitude contravariante. Une sous-catégorie $\mathcal{C} \subseteq \mathcal{D}$ est dite *fonctionnellement finie* si elle est à la fois contravariamment finie et covariamment finie.

Le foncteur de décalage dans une catégorie triangulée $\mathcal{D}$ est noté $[1]$ ou Σ. Dans un triangle

$$X \to Y \to Z \to X[1],$$

dans $\mathcal{D}$, on omet souvent la dernière flèche et le dernier terme lorsqu'ils ne sont pas pertinents pour la discussion. Pour un morphisme f dans $\mathcal{D}$, $C(f)$ désignera le cône de f. Pour deux sous-catégories $\mathcal{X}$ et $\mathcal{Y}$ de $\mathcal{D}$, on désigne par $\mathcal{X} * \mathcal{Y}$ la sous-catégorie de $\mathcal{D}$ constituée des objets Z tels qu'il existe un triangle

$$X \to Z \to Y$$

avec $X \in \mathcal{X}$ et $Y \in \mathcal{Y}$. Notons que l'opération $*$ est associative d'après l'axiome d'octaédre. Une sous-catégorie $\mathcal{X}$ de $\mathcal{D}$ est dite *stable par extensions* si $\mathcal{X} * \mathcal{X} \subseteq \mathcal{X}$. Une sous-catégorie $\mathcal{X}$ de $\mathcal{D}$ est dite *épaisse* si elle est une sous-catégorie triangulée pleine stable par les facteurs directs. Pour un objet ou une sous-catégorie $\mathcal{X}$ d'une catégorie triangulée $\mathcal{D}$, nous utiliserons $\operatorname{thick} \mathcal{X}$ pour désigner la plus petite sous-catégorie épaisse de $\mathcal{D}$ contenant $\mathcal{X}$.

Pour une catégorie abélienne $\mathcal{A}$, on désigne par $\mathcal{D}(\mathcal{A})$, $\mathcal{D}^-(\mathcal{A})$ et $\mathcal{D}^b(\mathcal{A})$ respectivement la catégorie dérivée, la catégorie dérivée bornée par le haut et la catégorie dérivée bornée de $\mathcal{A}$. Toutes ces catégories sont des catégories triangulées dont le foncteur de décalage est donné par le décalage d'un complexe vers la gauche. Nous définissons les sous-catégories

$$\begin{aligned}\mathcal{D}^{\leq n}(\mathcal{A}) &:= \{X \in \mathcal{D}^b(\mathcal{A}) \mid \mathsf{H}^i(X) = 0 \text{ pour tout } i > n\},\\ \mathcal{D}^{\geq m}(\mathcal{A}) &:= \{X \in \mathcal{D}^b(\mathcal{A}) \mid \mathsf{H}^i(X) = 0 \text{ pour tout } i < m\},\\ \mathcal{D}^{[m,n]}(\mathcal{A}) &:= \mathcal{D}^{\leq n}(\mathcal{A}) \cap \mathcal{D}^{\geq m}(\mathcal{A}).\end{aligned}$$

et les équipons du foncteur $\mathbb{E}(X, Y) := \mathsf{Hom}(X, Y[1])$. Cela donne une structure extriangulée à ces catégories, car ce sont des sous-catégories stables par extensions d'une catégorie triangulée [79].

Pour une sous-catégorie $\mathcal{C}$ d'une catégorie extriangulée $\mathcal{D}$, nous définissons

$$\mathcal{C}^{\perp} = \{Z \in \mathcal{D} \mid \mathsf{Hom}(\mathcal{C}, Z) = 0\}$$

$$\mathcal{C}^{\perp_1} = \{Z \in \mathcal{D} \mid \mathbb{E}(\mathcal{C}, Z) = 0\}$$

De manière duale, nous définissons ${}^{\perp}\mathcal{C}$, ${}^{\perp_1}\mathcal{C}$.

Pour tout objet X dans une catégorie de Krull-Schmidt, on désignera par $|X|$ le nombre de classes d'isomorphisme des facteurs indécomposables de X. X sera dit *basique* s'il est une somme directe d'objets indécomposables non-isomorphes.

On désigne par $\mathsf{mod}\,\Lambda$ (resp. $\mathsf{Mod}\Lambda$) la catégorie des modules de droite sur Λ finiment engendrés (resp. de tous les modules de droite sur Λ), par $\mathsf{proj}\,\Lambda$ (resp. $\mathsf{inj}\,\Lambda$) la catégorie des modules de droite sur Λ projectifs (resp. injectifs) et finiment engendrés, et par $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$ (resp. $\mathsf{K}^b(\mathsf{inj}\,\Lambda)$) la catégorie d'homotopie des complexes bornés de modules projectifs (resp. injectifs) et finiment engendrés sur Λ. $D = \mathsf{Hom}_K(-, K) : \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Lambda^{op}$ désignera la K-dualité standard. Notons que la catégorie dérivée bornée $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ est équivalente à $\mathsf{K}^{b,-}(\mathsf{proj}\,\Lambda)$, la catégorie d'homotopie des complexes bornés à droite de projectifs finiment engendrés sur Λ avec une cohomologie bornée. $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ et $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$ sont toutes deux des catégories triangulées avec le foncteur de décalage donné par le décalage d'un complexe vers la gauche.

Pour des entiers m et n, on définit les sous-catégories

$$\begin{aligned}\mathsf{K}^{\geq m}(\mathsf{proj}\,\Lambda) &:= \{X \in \mathsf{K}^b(\mathsf{proj}\,\Lambda) \mid \mathsf{Hom}(X, \Lambda[> -m]) = 0\}\\ \mathsf{K}^{\leq m}(\mathsf{proj}\,\Lambda) &:= \{X \in \mathsf{K}^b(\mathsf{proj}\,\Lambda) \mid \mathsf{Hom}(\Lambda[< -m], X) = 0\}\\ \mathsf{K}^{[m,n]}(\mathsf{proj}\,\Lambda) &:= \mathsf{K}^{\geq m}(\mathsf{proj}\,\Lambda) \cap K^{\leq n}(\mathsf{proj}\,\Lambda),\end{aligned}$$

et appelons-les les *catégories d'homotopie tronquées* de Λ. Étant des sous-catégories stables par extensions de la catégorie triangulée $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$, ces catégories héritent de structures extriangulées avec

$$\mathbb{E}(X, Y) := \mathsf{Hom}(X, Y[1])$$

au sens de [79].

Fixons $d \geq 1$. Soit $P^\bullet = (\cdots \to P^{-2} \xrightarrow{\delta^{-2}} P^{-1} \xrightarrow{\delta^{-1}} P^0 \xrightarrow{\delta^0} P^1 \xrightarrow{\delta^1} P^2 \to \cdots)$ un complexe avec $P^i \in \mathsf{proj}\,\Lambda$. Rappelons que les « troncations stupides » de $P^\bullet$ sont définies comme suit :

$$\begin{aligned}t_{\leq d-1}P^\bullet &:= (\cdots \to P^{d-2} \to P^{d-1} \to 0 \to 0 \to 0 \to \cdots) \in \mathsf{C}^b(\mathsf{proj}\,\Lambda)\\ t_{\geq d}P^\bullet &:= (\cdots \to 0 \to 0 \to 0 \to P^d \to P^{d+1} \to \cdots) \in \mathsf{C}^b(\mathsf{proj}\,\Lambda),\end{aligned}$$

et il existe un triangle $t_{\geq d}P^\bullet \to P^\bullet \to t_{\leq d-1}P^\bullet$ dans $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$.

Pour un module $M \in \mathsf{mod}\,\Lambda$, $\mathsf{Fac}(M)$ (resp. $\mathsf{Sub}(M)$) désigne la sous-catégorie pleine des modules facteurs (resp. sous-modules) des sommes directes finies de copies de M. Enfin, $\mathsf{pdim}\,M$ désigne la dimension projective de M. M est dit *fidèle* si l'annihilateur de M est trivial, et M est dit *sincère* s'il n'existe aucun idempotent non nul $e \in \Lambda$ qui annihile M.

Par souci de simplicité, nous énoncerons la plupart des résultats de ce chapitre pour les algèbres de dimension finie sur des corps algébriquement clos, bien que bon nombre d'entre eux s'appliquent dans des contextes plus généraux. Nous invitons le lecteur à consulter les références indiquées pour les énoncés précis. En particulier, dans les chapitres suivants, nous ne supposerons pas que le corps soit algébriquement clos.

Travaux récents et résultats principales

L'objectif général de cette thèse est d'établir des généralisations à d termes des bijections apparaissant dans le cadre de la théorie des τ-basculement. Plus précisément, nous cherchons à trouver des catégories $\mathcal{C}_{d,\Lambda}$ et des notions appropriées de classes de torsion, de semi-briques et de sous-catégories vastes dans celles-ci afin d'obtenir des bijections avec des objets bousculants à d termes dans $\mathsf{K}^b(\operatorname{proj}\Lambda)$, de telle sorte que pour $d = 1$, nous retrouvions la catégorie des modules et les définitions classiques.

Une approche dans ce sens a été proposée dans [11, 12], où les auteurs examinent la catégorie des modules et considèrent $\mathcal{C}_{d,\Lambda}$ comme une sous-catégorie $\mathcal{M}$ de $\operatorname{mod}\Lambda$ qui est une sous-catégorie d-amas-basculant, et étudient les d-classes de torsion dans $\mathcal{M}$ (voir la section 2.4). Ils fournissent un application injective des d-classes de torsion fonctoriellement finies dans $\mathcal{M}$ vers des complexes bousculants à $(d+1)$ termes. Cependant, cette application n'est pas surjective.

Dans ce travail, nous adoptons l'approche inverse et élargissons la catégorie des modules en la considérant comme le cœur de la t-structure standard. Plus précisément, nous considérons des sous-catégories de la forme

$$d\text{-}\operatorname{mod}\Lambda := \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$$

qui sont constituées d'objets de la catégorie dérivée bornée $\mathcal{D}^b(\operatorname{mod}\Lambda)$ dont la cohomologie est concentrée dans l'interval $[-d+1,0]$. Ces sous-catégories sont appelées catégories de modules étendues ou catégories dérivées tronquées. Elles apparaissent également dans [59], où la version tronquée suivante du théorème 5.2 a été obtenue.

Theorem ([59, Théorème 3.3 et Corollaire 3.4]) *Il existe des bijections entre*

- *(1') les objets bousculants basiques à $(d+1)$ termes dans $\mathsf{K}^b(\operatorname{proj}\Lambda)$,*
- *(2') les t-structures bornées $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ sur $\mathcal{D}^b(\operatorname{mod}\Lambda)$ satisfaisant $\mathcal{D}^{\leq -d}(\operatorname{mod}\Lambda) \subseteq \mathcal{C}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}(\operatorname{mod}\Lambda)$ avec des cœurs de longueur finie,*
- *(3') les cœurs bornées de longueur finie appartenant à $\mathcal{D}^{[-d,0]}(\operatorname{mod}\Lambda)$, et*
- *(4') les collections simplistes à $(d+1)$ termes dans $\mathcal{D}^b(\operatorname{mod}\Lambda)$.*

Nous formulerons nos définitions et nos résultats dans un cadre plus général, en considérant $d\text{-}\operatorname{mod}\Lambda$ comme un d-cœur étendu au sens indiqué ci-dessous.

Soit $\mathcal{D}$ une catégorie triangulée. Supposons que $\mathcal{D}$ admette une t-structure bornée $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ [21]. Cela implique notamment que $\mathcal{D}$ est idempotente complète [72]. Soit $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ le cœur de cette t-structure. Pour tout entier n et m, définissons

$$\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n],\ \mathcal{D}^{\geq m} := \mathcal{D}^{\geq 0}[-m] \text{ et } \mathcal{D}^{[m,n]} := \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m}.$$

Nous désignons par

$$\sigma_{\leq n} \colon \mathcal{D} \to \mathcal{D}^{\leq n} \text{ et } \sigma_{\geq m} \colon \mathcal{D} \to \mathcal{D}^{\geq m}$$

les foncteurs de troncations adjoints aux inclusions $\mathcal{D}^{\leq n} \hookrightarrow \mathcal{D}$ et $\mathcal{D}^{\geq m} \hookrightarrow \mathcal{D}$, respectivement. Pour chaque $X \in \mathcal{D}$, on a un triangle canonique

$$\sigma_{\leq n} X \to X \to \sigma_{\geq n+1} X.$$

Fixons $d \geq 1$. On appellera $\mathcal{D}^{[-d+1,0]}$ le *d-cœur étendu* de la t-structure. Notons que $\mathcal{D}^{[0,0]} = \mathcal{H}$, et, en général,

$$\mathcal{D}^{[-d+1,0]} = \mathcal{H}[d-1] * \cdots * \mathcal{H}[1] * \mathcal{H}.$$

Puisque $\mathcal{D}^{[-d+1,0]}$ est une sous-catégorie stable par extensions d'une catégorie triangulée, elle hérite d'une structure extriangulée avec $\mathbb{E}(X,Y) = \operatorname{Hom}(X, Y[1])$ pour $X, Y \in \mathcal{D}^{[-d+1,0]}$. De plus, c'est une catégorie extriangulée avec des premières extensions négatives au sens de [1], avec $\mathbb{E}^{-1}(X,Y) := \operatorname{Hom}(X, Y[-1])$.

Dans le premier chapitre, nous aborderons trois notions différentes de classes de torsion dans $\mathcal{D}^{[-d+1,0]}$. Celles-ci se réduiront aux trois définitions équivalentes des classes de torsion dans mod Λ pour la t-structure standard et $d = 1$, mais ne seront pas équivalentes en général. Néanmoins, nous montrerons plusieurs résultats reliant ces trois définitions, qui joueront toutes un rôle important dans le développement de la théorie.

Dans le deuxième chapitre, nous commençons par généraliser un résultat classique d'Auslander pour obtenir l'équivalence de catégories suivante.

Proposition *Le foncteur de troncation*

$$\sigma_{\geq -d+1} : \mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda) \to \mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$$

induit une équivalence de catégories K-linéaires entre $\frac{\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda)}{\mathsf{add}\,\Lambda[d]}$ *et* $\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$*.*

Nous montrons ensuite que ce foncteur de troncation induit des bijections entre les paires de cotorsion dans $\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda)$ et les paires de torsion dans $\mathcal{D}^{[-d+1,0]}(\mathsf{proj}\,\Lambda)$, généralisant ainsi le résultat de Garcia. Nous utilisons ensuite des restrictions des bijections de König-Yang pour établir des bijections entre les objets bousculants à $d+1$ termes dans $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$, les paires de cotorsion complètes et héréditaires dans $\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda)$, et les paires de torsion positives et fonctoriellement finies dans $\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$, généralisant ainsi les résultats d'Adachi, Iyama et Reiten, ainsi que ceux de Pauksztello et Zvonareva.

Au chapitre 3, nous définissons les semi-briques et les sous-catégories vastes dans $\mathcal{D}^{[-d+1,0]}$. Nous démontrons l'existence d'une bijection entre les semi-briques et les sous-catégories vastes de longueur finie, généralisant ainsi le résultat de Ringel. Le principal outil utilisé à cette fin est la bijection correspondante pour les catégories exactes, démontrée par Enomoto dans [37]. Nous donnons également un moyen d'associer une sous-catégorie vaste à toute sous-catégorie stable par d-facteurs et extensions en utilisant son "cœur exact". Nous définissons ensuite les semi-briques (resp. sous-catégories vastes) finies à gauche dans $\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$ comme des semi-briques (resp. sous-catégories vastes) telles que la plus petite classe de torsion positive les contenant soit fonctoriellement finie. Nous donnons ensuite des bijections entre les collections simplistes à $(d+1)$ termes, les semi-briques finies à gauche, les sous-catégories vastes finies à gauche et les paires de torsion positive et fonctoriellement finies dans $\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$, généralisant ainsi les résultats de Marks et Šťovíček, ainsi que ceux d'Asai. Nous prouvons également certaines propriétés concernant les mutations des collections simplistes à $(d+1)$ termes et des objets bousculants à $(d+1)$ termes.

Le dernier chapitre traite d'un modèle géométrique pour les objets bousculants à d termes dans le cadre des algèbres aimables.

Introduction

In this chapter, we give a brief historical account of results leading to the development of τ-tilting theory, the starting point of this dissertation. We start by surveying relevant results in tilting theory and their connections with torsion theory. We then mention some results in cluster-tilting theory, moving on to τ-tilting theory. Finally, we provide a summary of our results.

1 Notation

We will use Λ to denote a finite-dimensional algebra over a field K. All subcategories are assumed to be full, closed under isomorphisms, and to contain the zero object. For morphisms $f : X \to Y$ and $g : Y \to Z$, we denote by gf the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$. For any category $\mathcal{D}$, we use the notation $X \in \mathcal{D}$ to denote that X is an object of $\mathcal{D}$, and $\mathcal{C} \subseteq \mathcal{D}$ to denote that $\mathcal{C}$ is a subcategory of $\mathcal{D}$. For any $X \in \mathcal{D}$, we use add X to denote the additive hull of X in $\mathcal{D}$, i.e., the full subcategory of $\mathcal{D}$ containing all direct summands of finite direct sums of copies of X. For any two subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{D}$, we use $\mathrm{Hom}(\mathcal{X}, \mathcal{Y}) = 0$ to mean that $\mathrm{Hom}(X, Y) = 0$ for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. For any $\mathcal{C} \subseteq \mathcal{D}$, a morphism $f : X \to C$ in $\mathcal{D}$ is called a *left $\mathcal{C}$-approximation* of X if $C \in \mathcal{C}$ and for any morphism $h : X \to C'$ with $C' \in \mathcal{C}$, there exists a morphism $g : C \to C'$ such that $h = gf$. A subcategory $\mathcal{C} \subseteq \mathcal{D}$ is called *covariantly finite* if any object $X \in \mathcal{D}$ admits a left $\mathcal{C}$-approximation. One can define dually the notions of *right $\mathcal{C}$-approximations* and *contravariantly finite*. A subcategory $\mathcal{C} \subseteq \mathcal{D}$ is called *functorially finite* if it is both contravariantly finite and covariantly finite.

The shift functor in a triangulated category $\mathcal{D}$ is denoted by $[1]$ or Σ. In a triangle

$$X \to Y \to Z \to X[1],$$

in $\mathcal{D}$, we often omit the last arrow and term when they are irrelevant to the discussion. For a morphism f in $\mathcal{D}$, $C(f)$ will denote the cone of f. For any two subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{D}$, we denote by $\mathcal{X} * \mathcal{Y}$ the subcategory of $\mathcal{D}$ consisting of objects Z such that there is a triangle

$$X \to Z \to Y$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Note that the operation $*$ is associative by the octahedral axiom. A subcategory $\mathcal{X}$ of $\mathcal{D}$ is called *closed under extensions* if $\mathcal{X} * \mathcal{X} \subseteq \mathcal{X}$. A subcategory $\mathcal{X}$ of $\mathcal{D}$ is called *thick* if it is a full triangulated subcategory closed under summands. For an object or a subcategory $\mathcal{X}$ of a triangulated category $\mathcal{D}$, we will use thick $\mathcal{X}$ to denote the smallest thick subcategory of $\mathcal{D}$ containing $\mathcal{X}$.

For an abelian category $\mathcal{A}$, we denote by $\mathcal{D}(\mathcal{A})$, $\mathcal{D}^-(\mathcal{A})$, and $\mathcal{D}^b(\mathcal{A})$ the full, bounded above, and bounded derived category of $\mathcal{A}$ respectively. All of these are triangulated categories with the shift functor given by shifting a complex to the left. We define subcategories

$$\begin{aligned}
\mathcal{D}^{\leq n}(\mathcal{A}) &:= \{X \in \mathcal{D}^b(\mathcal{A}) \mid \mathrm{H}^i(X) = 0 \text{ for all } i > n\}, \\
\mathcal{D}^{\geq m}(\mathcal{A}) &:= \{X \in \mathcal{D}^b(\mathcal{A}) \mid \mathrm{H}^i(X) = 0 \text{ for all } i < m\}, \\
\mathcal{D}^{[m,n]}(\mathcal{A}) &:= \mathcal{D}^{\leq n}(\mathcal{A}) \cap \mathcal{D}^{\geq m}(\mathcal{A}).
\end{aligned}$$

and equip them with the functor $\mathbb{E}(X,Y) := \mathsf{Hom}(X,Y[1])$. This gives an extriangulated structure on these categories as they are extension closed subcategories of a triangulated category [79].

For a subcategory $\mathcal{C}$ of an extriangulated category $\mathcal{D}$, we define

$$\mathcal{C}^{\perp} = \{Z \in \mathcal{D} \mid \mathsf{Hom}(\mathcal{C}, Z) = 0\}$$

$$\mathcal{C}^{\perp_1} = \{Z \in \mathcal{D} \mid \mathbb{E}(\mathcal{C}, Z) = 0\}$$

Dually, we define ${}^{\perp}\mathcal{C}$, ${}^{\perp_1}\mathcal{C}$.

For any object X in a Krull-Schmidt category, we will denote by $|X|$ the number of isomorphism classes of indecomposable summands of X. X will be called *basic* if it is a direct sum of non-isomorphic indecomposable objects.

We denote by $\mathsf{mod}\,\Lambda$ (resp. $\mathsf{Mod}\Lambda$) the category of finitely generated (resp. all) right Λ-modules, by $\mathsf{proj}\,\Lambda$ (resp. $\mathsf{inj}\,\Lambda$) the category of finitely generated projective (resp. injective) right Λ-modules, and by $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$ (resp. $\mathsf{K}^b(\mathsf{inj}\,\Lambda)$) the homotopy category of bounded complexes of finitely generated projectives (resp. injectives) over Λ. $D = \mathsf{Hom}_K(-,K) : \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Lambda^{op}$ will denote the standard K-duality. Note that the bounded derived category $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ is equivalent to $\mathsf{K}^{b,-}(\mathsf{proj}\,\Lambda)$, the homotopy category of right bounded complexes of finitely generated projectives over Λ with bounded cohomology. Both $\mathcal{D}^b(\mathsf{mod}\,\Lambda)$ and $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$ are triangulated categories with the shift functor given by shifting a complex to the left.

For integers m and n, we define subcategories

$$\begin{aligned} \mathsf{K}^{\geq m}(\mathsf{proj}\,\Lambda) &:= \{X \in \mathsf{K}^b(\mathsf{proj}\,\Lambda) \mid \mathsf{Hom}(X, \Lambda[> -m]) = 0\} \\ \mathsf{K}^{\leq m}(\mathsf{proj}\,\Lambda) &:= \{X \in \mathsf{K}^b(\mathsf{proj}\,\Lambda) \mid \mathsf{Hom}(\Lambda[< -m], X) = 0\} \\ \mathsf{K}^{[m,n]}(\mathsf{proj}\,\Lambda) &:= \mathsf{K}^{\geq m}(\mathsf{proj}\,\Lambda) \cap K^{\leq n}(\mathsf{proj}\,\Lambda), \end{aligned}$$

and call them the *truncated homotopy categories* of Λ. Being extension closed subcategories of the triangulated category $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$, these categories inherit extriangulated structures with

$$\mathbb{E}(X,Y) := \mathsf{Hom}(X, Y[1])$$

in the sense of [79].

Fix $d \geq 1$. Let $P^\bullet = (\cdots \to P^{-2} \xrightarrow{\delta^{-2}} P^{-1} \xrightarrow{\delta^{-1}} P^0 \xrightarrow{\delta^0} P^1 \xrightarrow{\delta^1} P^2 \to \cdots)$ be a complex with $P^i \in \mathsf{proj}\,\Lambda$. Recall that the 'stupid truncations' of $P^\bullet$ are defined as

$$\begin{aligned} t_{\leq d-1}P^\bullet &:= (\cdots \to P^{d-2} \to P^{d-1} \to 0 \to 0 \to 0 \to \cdots) \in \mathsf{C}^b(\mathsf{proj}\,\Lambda) \\ t_{\geq d}P^\bullet &:= (\cdots \to 0 \to 0 \to 0 \to P^d \to P^{d+1} \to \cdots) \in \mathsf{C}^b(\mathsf{proj}\,\Lambda), \end{aligned}$$

and there is a triangle $t_{\geq d}P^\bullet \to P^\bullet \to t_{\leq d-1}P^\bullet$ in $\mathsf{K}^b(\mathsf{proj}\,\Lambda)$.

For a module $M \in \mathsf{mod}\,\Lambda$, $\mathsf{Fac}(M)$ (resp. $\mathsf{Sub}(M)$) denotes the full subcategory of factor modules (resp. submodules) of finite direct sums of copies of M. Finally, $\mathsf{pdim}\,M$ denotes the projective dimension of M. M is called *faithful* if the annihilator of M is trivial, and M is called *sincere* if there is no non-zero idempotent $e \in \Lambda$ that annihilates M.

For simplicity, we will state most of the results in this chapter for finite-dimensional algebras over algebraically closed fields, although a lot of them hold in more general settings. We invite the reader to consult the particular references for the precise statements. In particular, in the subsequent chapters, we will not assume the field to be algebraically closed.

2 Tilting theory

The first shadow of tilting theory in the study of representation theory of algebras appears in the notion of 'image' functors (later called reflection functors) introduced by Bernstein, Gel'fand, and Ponomarev in [23], where they are used to give a bijection between the indecomposable modules of KQ and KQ', where Q and

Q' are finite, connected quivers related by reflection at a source or a sink. Subsequently, it was shown in [13] that these functors are equivalent to functors of the form $\mathrm{Hom}(T,-)$ for some special module $T \in KQ$ (later called *an APR-tilting module*). This construction was then generalised by Brenner and Butler in [26], where the term *tilting* was used for the first time and an axiomatic definition of tilting modules was provided. This set of axioms was relaxed by Happel and Ringel in [49], where the following definition was used to prove the subsequent theorem, which is a slight reformulation and generalisation of the *Brenner-Butler tilting theorem*.

Definition 2.1 Let Λ be a finite-dimensional K-algebra. A module $T \in \mathrm{mod}(\Lambda)$ is called a *tilting module* if

1. $\mathrm{pdim}\, T \leq 1$;
2. $\mathrm{Ext}^1(T,T) = 0$;
3. There is an exact sequence $0 \to \Lambda \to T' \to T'' \to 0$ with $T', T'' \in \mathrm{add}\, T$.

Dually, a module T is called *cotilting* if DT is a tilting module over Λ^{op}. Given a tilting module T, there is an associated *torsion pair* $(\mathcal{T}(T), \mathcal{F}(T))$ in $\mathrm{mod}\,\Lambda$ with $\mathcal{T}(T) = \mathrm{Fac}(T)$ and $\mathcal{F}(T) = T^{\perp}$ ([49, § 1.4]). The notion of torsion pairs was introduced by Dickson in [36] for (subcomplete) abelian categories as a generalisation of torsion abelian groups. We recall the definition below before stating the tilting theorem.

Definition 2.2 Let $\mathcal{A}$ be an abelian category. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ is called a *torsion pair* if

1. $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$;
2. For all $X \in \mathcal{A}$, there exists a short exact sequence
$$0 \to T \to X \to F \to 0$$
with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

In this case, $\mathcal{T}$ is called a *torsion class* and $\mathcal{F}$ a *torsion-free class*.

One can show that for such a pair $(\mathcal{T}, \mathcal{F})$, $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. In fact, if $\mathcal{A}$ is a length category, then this is equivalent to the definition. Another equivalent characterisation of torsion classes if $\mathcal{A}$ is a length category says that a subcategory $\mathcal{T} \subseteq \mathcal{A}$ is a torsion class if and only if it is closed under factors and extensions. Dually, $\mathcal{F} \subseteq \mathcal{A}$ is a torsion-free class if and only if it is closed under subobjects and extensions. This immediately gives us that, if $\mathcal{A}$ is a length category, the poset of torsion pairs in $\mathcal{A}$ ordered by inclusion of torsion classes is a lattice. We will use this fact often for $\mathcal{A} = \mathrm{mod}\,\Lambda$.

As mentioned before, a standard example of a torsion pair is given by the pair of torsion and torsion-free groups in the category of finitely-generated abelian groups.

Theorem 2.3 ([49]) *Let $T \in \mathrm{mod}\,\Lambda$ be a tilting module and define $\Lambda' := \mathrm{End}_{\Lambda}(T)$. Then T is a left Λ'-tilting module and $\Lambda \cong \mathrm{End}^{op}_{\Lambda'}(T)$. Let*
$$\mathcal{T}' := \{N \in \mathrm{mod}\,\Lambda' \mid N \otimes_{\Lambda'} T = 0\}$$
$$\mathcal{F}' := \{N \in \mathrm{mod}\,\Lambda' \mid \mathrm{Tor}_1^{\Lambda'}(N,T) = 0\}.$$

Then

1. *$(\mathcal{T}', \mathcal{F}')$ is a torsion pair in $\mathrm{mod}\,\Lambda'$ such that $(D\mathcal{F}', D\mathcal{T}')$ is the torsion pair $(\mathcal{T}({}_{\Lambda'}T), \mathcal{F}({}_{\Lambda'}T))$ induced by the left Λ'-module T in $\mathrm{mod}\,\Lambda'^{op}$.*
2. *The functors*
$$F = \mathrm{Hom}(T,-) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda'$$
$$F' = \mathrm{Ext}^1(T,-) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda'$$
restrict to equivalences between $\mathcal{T}(T)$ and $\mathcal{F}'$, and $\mathcal{F}(T)$ and $\mathcal{T}'$ respectively. Moreover, $\mathcal{T}(T) = \{M \in \mathrm{mod}\,\Lambda \mid F'(M) = 0\}$.

Additionally, Happel and Ringel also showed that if Λ is hereditary, then T is a tilting module if and only if $\mathrm{Ext}^1(T,T) = 0$ and $|T| = |\Lambda|$. This was generalised to all finite-dimensional algebras by Bongartz in [25], where he showed a stronger statement: If $T \in \mathrm{mod}\,\Lambda$ satisfies the first two conditions for a tilting module, then T is a direct summand of a tilting module. With this, he showed that such a module is a tilting module if and only if $|T| = |\Lambda|$. Motivated by this characterisation, Happel introduced the notion of *tilting sets* in the mesh categories of Dynkin quivers in [46] and, later, in the root categories of arbitrary acyclic quivers [44].

In [77], Miyashita introduced the notion of *generalised tilting modules* (of finite projective dimension) as follows.

Definition 2.4 A module $T \in \mathrm{mod}(\Lambda)$ is called a *generalised tilting module* if

1. $\mathrm{pdim}\, T < \infty$;
2. $\mathrm{Ext}^i(T,T) = 0$ for all $i > 0$;
3. there is an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to \cdots \to T_n \to 0$ with $T_i \in \mathrm{add}\, T$.

He showed a version of the tilting theorem for these modules establishing equivalences between certain subcategories of $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Lambda'$, which were not necessarily torsion or torsion-free classes (in general, there is no torsion pair associated to such modules). In some modern literature, this is taken to be the definition of tilting modules, although we will stick to our terminology for the rest of this text. In [96], Wakamatsu also considered tilting modules of infinite projective dimension.

It was first realized by Happel [44] that the above equivalences are, in fact, restrictions of equivalences between the derived categories. More precisely, he showed that if Λ is of finite global dimension, then for any generalised tilting module T, the functor $R\,\mathrm{Hom}(T,-) : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(\Lambda')$ is a triangle equivalence. This was taken forward in [32] by Cline, Parshall, and Scott, where they started the general study of Morita theory of derived categories, i.e., of the question when are the derived categories of two rings triangle equivalent. They considered generalised tilting modules over arbitrary rings and showed that they induce triangle equivalences on derived categories. They also showed a partial converse to this result: certain type of equivalences of $\mathcal{D}^b(\Lambda)$ and $\mathcal{D}^b(\Lambda')$ are always induced from generalised tilting modules.

In [66], Keller and Vossieck used t-structures (more precisely, aisles of t-structures) to study equivalences of derived categories. The notion of t-structures was introduced in [21] to study abelian subcategories of triangulated categories and has been extensively useful in various fields of mathematics.

Definition 2.5 Let $\mathcal{D}$ be a triangulated category. A pair of full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of $\mathcal{D}$ is called a *t-structure* if

1. $\mathrm{Hom}(X, Y[-1]) = 0$ for all $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 0}$;
2. $\mathcal{D}^{\leq 0}[>0] \subseteq \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}[<0] \subseteq \mathcal{D}^{\geq 0}$;
3. for all $D \in \mathcal{D}$, there exists a triangle
$$D' \to D \to D'' \to D'[1]$$
in $\mathcal{D}$ with $D' \in \mathcal{D}^{\leq 0}$ and $D''[1] \in \mathcal{D}^{\geq 0}$.

In this case, $\mathcal{D}^{\leq 0}$ is called the *aisle* of the t-structure and $\mathcal{D}^{\geq 0}$ the *coaisle*. A t-structure is called *bounded* if for every object $X \in \mathcal{D}$, there exists an integer $n > 0$ such that $X[n] \in \mathcal{D}^{\leq 0}$ and $X[-n] \in \mathcal{D}^{\geq 0}$.

For a t-structure as above, the category $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called its *heart*, and it was shown in [21] that it is an abelian exact subcategory of $\mathcal{D}$. A heart is called *bounded* if it arises from a bounded t-structure (we refer to [27] for an equivalent characterization of bounded hearts without using the notion of t-structures). A typical example of a t-structure is given by

$$(\mathcal{D}^{\leq 0}(\mathrm{mod}\,\Lambda), \mathcal{D}^{\geq 0}(\mathrm{mod}\,\Lambda))$$

on $\mathcal{D}^b(\operatorname{mod}\Lambda)$. We will call this the *standard t-structure*.

Following Happel, in [66], the authors introduced a notion of tilting sets for arbitrary finite-dimensional algebras and showed that for a Dynkin quiver Δ, the faithful aisles in $\mathcal{D}^b(k\Delta)$ are in bijection with complete tilting sets in $\mathcal{D}^b(k\Delta)$. Generalising this, they introduced the notion of *silting sets* to give a bijection with separated aisles. An important property of silting objects was proven: for $\Delta = A_n$, every silting set can be completed to a maximal one, while this was not true for tilting sets.

The search for a complete Morita theory of derived categories culminated in the seminal paper of Rickard [87], where he provided the following definition of a tilting complex and proved the subsequent theorem.

Definition 2.6 Let Λ be an arbitrary ring. An object $T \in \mathsf{K}^b(\operatorname{proj}\Lambda)$ is called a *tilting complex* if

1. $\operatorname{Hom}(T, T[i]) = 0$ for all $i \neq 0$;
2. $\operatorname{add}(T)$ generates $\mathsf{K}^b(\operatorname{proj}\Lambda)$ as a triangulated category.

Theorem 2.7 *Let Λ and Γ be two rings. Then $\mathcal{D}^-(\operatorname{Mod}\Lambda)$ is triangle equivalent to $\mathcal{D}^-(\operatorname{Mod}\Gamma)$ if and only if $\Gamma \cong \operatorname{End}(T)$ for T a tilting complex for Λ. If Λ is right coherent, then this holds if and only if $\mathcal{D}^b(\operatorname{mod}\Lambda)$ and $\mathcal{D}^b(\operatorname{mod}\Gamma)$ are triangle equivalent.*

For a detailed survey of the future impact of Rickard's work, we refer the reader to [61].

In parallel to this development of derived Morita theory, tilting theory also found connections in the study of homologically finite subcategories (a collective term for covariantly, contravariantly, or functorially finite subcategories), as started by Auslander and Smalø in [18] in order to study subcategories of $\operatorname{mod}\Lambda$ which have left/right almost split morphisms or almost split sequences. An important example of functorially finite subcategories was given by subcategories of the form $\operatorname{Fac}(M)$ or $\operatorname{Sub}(M)$ for some $M \in \operatorname{mod}\Lambda$ [18]. It was shown a that a functorially finite subcategory closed under extensions admits almost split sequences. This raised the question when subcategories of the form $\operatorname{Fac}(M)$ or $\operatorname{Sub}(M)$ are closed under extensions. Using [36, Theorem 2.3], this is equivalent to asking when is $\operatorname{Fac}(M)$ a torsion class or $\operatorname{Sub}(M)$ a torsion-free class. As seen by the tilting theorem, this is true for $\operatorname{Fac}(M)$ if M is a tilting module. It was shown in [17] that $\operatorname{Fac}(M)$ is a torsion class if and only if $\operatorname{Hom}(M, \tau M) = 0$ (around thirty years later, this result would form the basis of τ-tilting theory introduced by Adachi, Iyama, and Reiten). In [16], the authors show an explicit connection with tilting theory where they use the Ext-injectives in $\operatorname{Sub}(M)$, under the assumptions that it is closed under extensions and M is faithful, to get a tilting module. This result quickly sparked significant interest in the study of relationships between tilting and torsion theory. In [53], Hoshino studied tilting modules for which the torsion pair $(\mathcal{T}', \mathcal{F}')$ is Theorem 2.3 is split (this was shown to be the case if Λ is hereditary in [49]). In [54], he generalised the construction in [16] to associate a tilting module to more general torsion pairs. It was independently shown in [93] and [9] that a torsion pair $(\mathcal{T}, \mathcal{F})$ is induced from a tilting module if and only if $\mathcal{T}$ contains all injectives and $\mathcal{T} = \operatorname{Fac}(M)$ for some $M \in \operatorname{mod}\Lambda$ which is if and only if $\mathcal{T}$ contains all injectives and $\mathcal{F} = \operatorname{Sub}(N)$ for some $N \in \operatorname{mod}\Lambda$. It was also shown in [93] that for any torsion pair $(\mathcal{T}, \mathcal{F})$, $\mathcal{T}$ is functorially finite if and only if $\mathcal{F}$ is which is if and only if $\mathcal{T} = \operatorname{Fac}(M)$ which is if and only if $\mathcal{F} = \operatorname{Sub}(N)$.

Continuing this study of homologically finite subcategories via tilting theory, in [14], the authors observed that, under some conditions, contravariantly and covariantly finite subcategories occur in pairs and can be obtained one from the other by the vanishing of Ext^1. This can be formally encoded in the notion of cotorsion pairs which was introduced by Salce in [92], and can be seen as an analogue of torsion pairs (vanishing of Ext^1 instead of Hom).

Definition 2.8 Let $\mathcal{X}, \mathcal{Y}$ be two full subcategories of $\operatorname{mod}\Lambda$. Then $(\mathcal{X}, \mathcal{Y})$ is called a *cotorsion pair* if

$$\mathcal{X} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1(X, \mathcal{Y}) = 0\},$$

$$\mathcal{Y} = \{Y \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1(\mathcal{X}, Y) = 0\}.$$

In this case, $\mathcal{X}$ is called the *cotorsion-free class* and $\mathcal{Y}$ the *cotorsion class*.

A cotorsion pair is called *complete* if $\mathcal{X}$ is contravariantly finite, or equivalently $\mathcal{Y}$ is covariantly finite.

It was shown that generalised tilting/cotilting modules give rise to certain complete cotorsion pairs and the following correspondence was obtained [85, 15]. Recall that a subcategory of mod Λ is called *(co)resolving* if it is closed under extensions, summands, (cokernels of monomorphisms) kernels of epimorphisms, and contains all the (injective) projective modules. Moreover, for a subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ we denote by $\check{\mathcal{Y}}$ the subcategory of mod Λ consisting of objects X for which there is an exact sequence $0 \to X \to C_0 \to C_1 \to \cdots \to C_n \to 0$, with C_i in $\mathcal{C}$. Dually, the objects of the subcategory $\hat{\mathcal{Y}}$ are the Y in mod Λ for which there is an exact sequence $0 \to C_n \to \cdots \to C_1 \to C_0 \to Y \to 0$, with C_i in $\mathcal{C}$.

Theorem 2.9 ([85])

1. *There is a bijection between basic generalised tilting modules T and complete cotorsion pairs $(\mathcal{X}, \mathcal{Y})$ in* mod Λ *with $\mathcal{Y}$ coresolving and $\check{\mathcal{Y}} =$* mod Λ.
2. *There is a one-one correspondence between basic generalised cotilting modules U and complete cotorsion pairs $(\mathcal{X}, \mathcal{Y})$ in* mod Λ *with $\mathcal{X}$ resolving and $\hat{\mathcal{X}} =$* mod Λ.
3. *If* gl. dim.$(\Lambda) < \infty$*, the generalised tilting and cotilting modules coincide.*

Going back to the tilting theorem, we note that it gives us a way to transform a torsion pair obtained from a tilting module, to another torsion pair in a new abelian category. In [48], Happel, Reiten, and Smalø showed that this can be done for any torsion pair in any abelian category $\mathcal{A}$. This phenomenon is now called the *Happel-Reiten-Smalø tilt*. They showed this by establishing a connection between torsion pairs in $\mathcal{A}$ and t-structures in $\mathcal{D}^b(\mathcal{A})$.

Theorem 2.10 *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$. Then $(\mathcal{D}^{\leq -1}(\mathcal{A}) * \mathcal{T}, \mathcal{F}[1] * \mathcal{D}^{\geq 0}(\mathcal{A}))$ is a t-structure on $\mathcal{D}^b(\mathcal{A})$. Let $\mathcal{B}$ be the heart of this t-structure. Then $(\mathcal{F}[1], \mathcal{T})$ is a torsion pair in $\mathcal{B}$.*

They also formally generalised the notion of tilting/cotilting modules to tilting/cotilting objects in arbitrary abelian categories, and developed an analogous theory. This close relation between torsion pairs and t-structures was explored fully in [22], where the authors reinterpreted the notion of t-structures as a 'torsion theory' for triangulated categories and proved parallel properties. In analogy with Rickard's definition of tilting complexes in derived categories of rings, they defined tilting objects in triangulated categories. Like tilting objects in abelian categories give torsion pairs, tilting objects in triangulated categories (with coproducts) give rise to t-structures. This was used to give a generalisation and alternative proof of Rickard's theorem. These results were used in [97] to give the following correspondence theorem.

Say that a t-structure $(\mathcal{X}', \mathcal{Y}')$ is *intermediate* with respect to another t-structure $(\mathcal{X}, \mathcal{Y})$ if $\mathcal{X}[1] \subseteq \mathcal{X}' \subseteq \mathcal{X}$.

Proposition 2.11 *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{X}, \mathcal{Y})$ with heart $\mathcal{H}$. Then there is a bijection between torsion pairs in $\mathcal{H}$ and intermediate t-structures with respect to $(\mathcal{X}, \mathcal{Y})$.*

3 Mutations and cluster-tilting theory

As mentioned before, Bongartz showed that any module M satisfying pdim $M \leq 1$ and $\operatorname{Ext}^1(M, M) = 0$ can be completed to a tilting module, and gave an explicit way to construct such a completion. This is now called the *Bongartz completion* of M. This raises a natural question about the other completions such a module can admit. Say that a module $M \in \operatorname{mod}\Lambda$ is an *almost complete tilting module* if pdim $M \leq 1$, $\operatorname{Ext}^1(M, M) = 0$, and $|M| = |\Lambda| - 1$. For Λ hereditary, it was shown in [89] and [95] that any almost complete tilting module admits at most two completions. This was generalised to all finite-dimensional algebras in [88]. It was also shown that if two such completions exist, say $M \oplus X$ and $M \oplus Y$ with $X \not\cong Y$ indecomposables, then they fit into an exact sequence

$$0 \to X \to M' \to Y \to 0$$

with $M' \in \operatorname{add} M$. Furthermore, the authors also defined a quiver and a partial order on the set of tilting modules of Λ.

Definition 3.1 Let K_Λ be the quiver whose vertices are basic tilting modules over Λ and there is an arrow from M to M' if $M = T \oplus X$, $M' = T \oplus X'$ with $X \not\simeq X'$ indecomposable, and M the Bongartz completion of T.

It was shown that this quiver has no oriented cycles, and hence can be used to define a partial order on the set of basic tilting modules. There is yet another partial order on this set given by $M \leq M'$ if and only if Fac$(M) \supseteq$ Fac(M'). In general, this order is different from the previous one. However, the authors asked if their Hasse diagrams coincide. In [51], Happel and Unger studied these two orders for generalised tilting modules and showed that their Hasse diagrams coincide. These order-theoretic properties of generalised tilting modules were further studied in [33].

It was shown in [50] that if Λ is hereditary, then an almost complete tilting module M admits exactly two completions if and only if M is faithful (first done in [95] for hereditary wild algebras of rank 3.). This was then generalised to all finite-dimensional algebras in [45].

The above results naturally provide us with a notion of tilting 'mutation': taking a tilting module, removing an indecomposable summand, and replacing it to get a new tilting module. As mentioned, such a mutation is not always possible: it is if and only if the almost complete module obtained by removing the indecomposable summand is faithful.

With the introduction of cluster algebras in the early 2000s, a lot of work was done to connect them with representation theory. In [75], Marsh, Reineke, and Zelevinsky showed that tilting theory can be used to study the generalized associahedron, an object closely related to the cluster algebras of simply laced Dynkin quivers. In [31], Caldero, Chapoton, and Schiffler introduced the category of diagonals using triangulations of a polygon for categorification of cluster algebras of type A_n. Around the same time appeared the seminal work of Buan, Marsh, Reineke, Reiten, and Todorov [29], where *cluster categories* for hereditary algebras were introduced as an algebraic model for the combinatorics of the associated cluster algebra. It was shown that the tilting theory in the cluster category $\mathcal{C}_H$ of a hereditary algebra H can serve as a 'mutation completion' of the tilting theory of H. They defined a basic object $T \in \mathcal{C}_H$ to be tilting if $\mathrm{Ext}^1_{\mathcal{C}_H}(T, T) = 0$ and if T has a maximal number of non-isomorphic indecomposable summands with respect to this property. It was then shown that any almost complete tilting object T' in $\mathcal{C}_H$ (later called *a cluster-tilting object*) could be completed to a tilting object in exactly two ways. Moreover, the two complements M and M^* are related by the following triangle in $\mathcal{C}_H$, with f being the minimal right add T'-approximation of T' in $\mathcal{C}_H$.

$$M^* \xrightarrow{g} T \to M \to M^*[1]$$

As mentioned before, yet another property of tilting mutation is that the operation of mutation corresponds to minimal inclusion of the corresponding torsion classes. Even though cluster-tilting theory provided a suitable mutation completion, such an interpretation of the mutation structure was missing. To overcome this, in [4], the authors developed a systematic theory of silting subcategories/objects of triangulated categories along with their mutations. We recall these definitions below (for a detailed survey of recent developments in silting theory, please refer to [6]).

Definition 3.2 Let $\mathcal{T}$ be a triangulated category. An additive subcategory $\mathcal{M} \subseteq \mathcal{T}$ is called *presilting* if Hom$(\mathcal{M}, \mathcal{M}[> 0]) = 0$. It is called *silting* if, additionally, $\mathcal{T} =$ thick $\mathcal{M}$. An object $M \in \mathcal{T}$ is called *presilting* (resp. *silting*) if add M is *presilting* (resp. silting).

It was shown that if $\mathcal{T}$ is a Krull-Schmidt triangulated category, then the number of indecomposable summands of any basic silting object is the same, which is the rank of the Grothendieck group of $\mathcal{T}$. In this work, we will mostly be interested in the silting objects in K^b(proj Λ) for a finite-dimensional algebra Λ. Let silt Λ denote the set of isomorphism classes of basic silting objects in K^b(proj Λ). We present the definition of mutation in this setting.

Definition 3.3 ([4]) Let $P = \oplus_{i=1}^n P_i \in$ silt Λ. The *left mutation* of P at P_i is defined as

$$\mu_i^+(P) := \left(\bigoplus_{j \neq i} P_j \right) \oplus P_i',$$

where P_i' is the cone of a minimal left $\left(\operatorname{add}\bigoplus_{j\neq i}P_j\right)$-approximation of P_i. The *right mutation* $\mu_i^-(P)$ is defined dually using the cocone of a minimal right approximation.

Aihara and Iyama showed that silting mutation is always possible and that it exhibits tilting mutation as a special case. Additionally, they introduced a partial order on the set of silting objects, generalising the partial order on tilting modules, whose cover relations correspond to left mutations.

For $P, Q \in \operatorname{silt}\Lambda$, set

$$P \leq Q :\iff \operatorname{Hom}(Q, P[i]) = 0 \text{ for all } i > 0.$$

Theorem 3.4 ([4, Theorem 2.11, Theorem 2.31, Theorem 2.35])

1. *The above relation $\leq$ gives a partial order on* $\operatorname{silt}\Lambda$.
2. *For $P \in \operatorname{silt}\Lambda$, both $\mu_i^+(P)$ and $\mu_i^-(P)$ are again silting objects. Moreover, they satisfy the cover relation $\mu_i^-(P) \gtrdot P \gtrdot \mu_i^+(P)$, where $x \gtrdot y$ means that $x > y$ and there exists no z with $x > z > y$.*

Finally, for a triangulated category $\mathcal{T}$ with coproducts, they showed a bijection between silting objects and certain t-structures on $\mathcal{T}$.

A closely related notion to silting subcategories is that of co-t-structures, a kind of dual to t-structures. In [55], the authors studied t-structures induced by certain compact rigid objects (i.e. objects X such that $\operatorname{Hom}(X, X[>0]) = 0$) in triangulated categories. As an analogue, in [83], Pauksztello studied structures induced by compact corigid (i.e. objects X such that $\operatorname{Hom}(X, X[<0]) = 0$) objects. These were called co-t-structures and were defined as follows.

Definition 3.5 A *co-t-structure* on a triangulated category $\mathcal{D}$ is a pair $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ of full subcategories such that

1. $\mathcal{D}_{\geq 0}$, $\mathcal{D}_{\leq 0}$ are closed under direct sums and summands;
2. $\mathcal{D}_{\geq 0}[-1] \subseteq \mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq 0}[1] \subseteq \mathcal{D}_{\leq 0}$;
3. $\operatorname{Hom}(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0}[1]) = 0$;
4. For any object $X \in \mathcal{D}$ there exists a triangle

   $$A[-1] \to X \to B \to A$$

   with $A \in \mathcal{D}_{\geq 0}$ and $B \in \mathcal{D}_{\leq 0}$.

It is said to be *bounded* if

$$\bigcup_{n\in\mathbb{Z}} \mathcal{D}_{\geq 0}[n] = \mathcal{D} = \bigcup_{n\in\mathbb{Z}} \mathcal{D}_{\leq 0}[n].$$

The intersection $S := \mathcal{D}_{\geq 0} \cap \mathcal{D}_{\leq 0}$ is called the *coheart* of the co-t-structure, and $C = S * S[1]$ is called the *extended coheart*.

A co-t-structure (A', B') is said to be *intermediate* with respect to another co-t-structure (A, B) if $A \subseteq A' \subseteq A[1]$. A silting subcategory of $\mathcal{D}$ is said to be *2-term* with respect to (A, B) if it lies in the extended coheart of (A, B).

These objects were independently introduced in [24] where they called *weight structures*. For a finite-dimensional algebra, a typical example of a co-t-structure is given by

$$(\mathsf{K}^{\geq 0}(\operatorname{proj}\Lambda), \mathsf{K}^{\leq 0}(\operatorname{proj}\Lambda))$$

on $\mathsf{K}^b(\operatorname{proj}\Lambda)$. This is a bounded co-$t$-structure with coheart $\operatorname{proj}\Lambda$. We will call this the *standard co-t-structure*. Note that a 2-term silting object in $\mathsf{K}^b(\operatorname{proj}\Lambda)$ is precisely the additive generator of a 2-term silting subcategory with respect to the standard co-t-structure.

It was shown in [76] that the coheart of a bounded co-t-structure on $\mathcal{D}$ is a silting subcategory of $\mathcal{D}$ and that this map gives a bijection between bounded co-t-structures and silting subcategories of $\mathcal{D}$.

Both t-structures and co-t-structures form examples of torsion theories on triangulated categories in the sense of [58]. For a survey on parallelisms between t-structures and co-t-structures, please refer to [63].

The above-mentioned completion of mutation using silting theory by Aihara and Iyama comes at a cost: an almost complete silting object can have several completions in general. Say that an object $P \in$ silt Λ is d-term if P lies in $\mathsf{K}^{[-d+1,0]}(\operatorname{proj}\Lambda)$, i.e., it is quasi-isomorphic to a complex of projectives concentrated in degrees $-d+1, -d+2, \cdots, 0$. We will denote the subposet of d-term silting objects in $\mathsf{K}^b(\operatorname{proj}\Lambda)$ by d-silt Λ. In [35], Derksen and Fei showed that if we restrict to 2-silt Λ, then any almost complete silting object can be completed in exactly 2 ways, suggesting a close relationship with tilting and cluster-tilting mutation. However, an interpretation of this mutation in terms of inclusion of torsion classes was missing. All of these requirements were simultaneously satisfied by the theory of τ-tilting introduced in the seminal work of Adachi, Iyama, and Reiten [2].

4 τ-tilting theory

In [57], Ingalls and Thomas defined a module $M \in KQ$ to be *support tilting* if it is tilting as an $(\Lambda/\langle e\rangle)$-module for some idempotent $e \in KQ$. They showed that these modules are in bijection with cluster-tilting objects in the cluster category $\mathcal{C}_{KQ}$. Using this bijection, we can conclude that an almost complete support tilting module over KQ can be completed in exactly two ways. However, this is not true for arbitrary finite-dimensional algebras, as there may be sincere modules which are not faithful. This is rectified by considering *support τ-tilting modules* as defined in [2].

Definition 4.1 1. A module $M \in \operatorname{mod}\Lambda$ is called *τ-tilting* if $\operatorname{Hom}_\Lambda(M, \tau M) = 0$ and $|M| = |\Lambda|$.

2. A module $M \in \operatorname{mod}\Lambda$ is called *support τ-tilting* if there exists an idempotent e of Λ such that M is a τ-tilting $(\Lambda/\langle e\rangle)$-module.

An equivalent way of looking at a support τ-tilting module is as a *support τ-tilting pair*, (P, M) satisfying $\operatorname{Hom}(M, \tau M) = 0$, $\operatorname{Hom}(P, M) = 0$, and $|P| + |M| = |\Lambda|$. We then define an *almost complete support τ-tilting pair* as a pair such that $\operatorname{Hom}(M, \tau M) = 0$, $\operatorname{Hom}(P, M) = 0$, and $|P| + |M| = |\Lambda| - 1$. The following results show that the above definition satisfies all the requirements that we were looking for.

Theorem 4.2 *Any basic almost complete support τ-tilting pair can be completed to a support τ-tilting pair in exactly two ways.*

In this case, the two completions are called *mutations* of each other.

Theorem 4.3 *There are bijections between the following sets:*

1. *basic support τ-tilting pairs over Λ (up to isomorphism);*
2. *functorially finite torsion pairs in* $\operatorname{mod}\Lambda$*;*
3. 2-silt Λ.

The set of (functorially finite) torsion pairs has a natural partial order given by the inclusion of torsion classes. The above bijection can then be used to transfer this partial order to the set of basic support τ-tilting pairs of Λ. The authors show that, like silting mutation, mutation of τ-tilting pairs corresponds to cover relations in this partial order. They also show that the bijection between 2-term silting objects and support τ-tilting pairs is an isomorphism of posets.

In [28], Brüstle and Yang showed that 2-term silting objects over Λ are also in bijection with intermediate t-structures with length heart (with respect to the standard t-structure) and 2-term simple-minded collections in $\mathcal{D}^b(\text{mod}\,\Lambda)$ (see Definition 5.1). In [60], Iyama, Jørgensen, and Yang developed a general theory of τ-tilting for an arbitrary Hom-finite, Krull-Schmidt triangulated category $\mathcal{D}$, extending all the bijections of [2]. Additionally, they showed that 2-term silting subcategories are in bijection with intermediate co-t-structures (with respect to some fixed bounded co-t-structure on $\mathcal{D}$) (see Definition 3.5).

Another interesting class of categories closely related to torsion classes/pairs is that of *wide subcategories*.

Definition 4.4 A full subcategory $W \subseteq \text{mod}\,\Lambda$ is called *wide* if it is closed under kernels, cokernels, and extensions.

Note that the above definition is equivalent to saying that W is an exact abelian subcategory of mod Λ closed under extensions, i.e., it is an abelian subcategory closed under extensions such that a sequence in W is exact if and only if it is exact in mod Λ. These subcategories had already appeared in [91] as filtrations of 'orthogonal points' in mod Λ. The word 'wide' was first used in [56] where wide subcategories of Mod$\mathcal{R}$ were shown to correspond to thick subcategories of $\mathcal{D}(\mathcal{R})$ for some nice commutative ring $\mathcal{R}$. They were also shown to be related to torsion theories in Mod$\mathcal{R}$. For finite-dimensional path algebras KQ, the relation between wide subcategories and torsion classes was made explicit in [57], where it was shown that finitely generated torsion classes in mod KQ are in bijection with finitely generated wide subcategories in mod KQ. This was generalised to all finite-dimensional algebras by Marks and Šťovíček in [74], where it was shown that functorially finite torsion classes in mod Λ are in bijection with functorially finite wide subcategories $\mathcal{W}$ in mod Λ for which the smallest torsion class containing $\mathcal{W}$ is functorial finite. It was shown later by Asai in [8] that any wide subcategory for which the smallest torsion class containing $\mathcal{W}$ is functorially finite (called a *left-finite wide subcategory*) is necessarily functorially finite, relaxing the previous conditions slightly.

As mentioned above, wide subcategories appeared as filtrations of 'orthogonal points' in [91]. Ringel defined a *point* to be a module $M \in \text{mod}\,\Lambda$ whose endomorphism ring is a division ring, seeing it as a generalisation of a simple module. Later, he referred to these modules as *bricks* in [90]. Two bricks M, N are said to be *orthogonal* if $\text{Hom}(M, N) = 0 = \text{Hom}(N, M)$. A collection of orthogonal bricks is called a *semibrick*. Taking forward the work of Marks and Šťovíček, in [8] Asai studied relations between semibricks and τ-tilting theory. He defined a semibrick $\mathcal{S}$ to be *left-finite* (resp. *right-finite*) if the smallest torsion class (resp. torsion-free class) containing $\mathcal{S}$ is functorial finite. He then constructed explicit bijections between support τ-tilting modules, left-finite semibricks, right-finite semibricks, and 2-term simple-minded collections.

As seen before, tilting modules were shown to be in correspondence with certain cotorsion pairs in mod Λ in the work of Auslander and Reiten. One then wonders what cotorsion pairs correspond to τ-tilting modules. For this, one needs to look at cotorsion pairs in a slightly bigger category, $\mathsf{K}^{[-1,0]}(\text{proj}\,\Lambda)$. In [84], Pauksztello and Zvonareva developed an analogue of HRS tilting for t-structures (see 2.11) for co-t-structures in arbitrary triangulated categories. Since the extended coheart of a co-t-structure is an extension closed subcategory of a triangulated category, it is extriangulated, and thus, admits the notion of a cotorsion pair ([79], Definition 2.1.10). They showed a bijection between intermediate co-t-structures and complete cotorsion pairs in the extended coheart of a fixed co-t-structure. They then used this to prove that the 0-th cohomology functor H^0 gives a bijection between complete cotorsion pairs in $\mathsf{K}^{[-1,0]}(\text{proj}\,\Lambda)$ and functorially finite torsion pairs in mod Λ. This was generalised by Garcia in [38] to show that the functor H^0 actually gives a bijection between all cotorsion pairs in $\mathsf{K}^{[-1,0]}(\text{proj}\,\Lambda)$ and all torsion pairs in mod Λ.

For general extriangulated categories, a bijection between bounded, complete, hereditary cotorsion pairs and silting subcategories was provided in [3]. Viewing 2-term silting objects in $\mathsf{K}^{[-1,0]}(\text{proj}\,\Lambda)$ as analogues of support τ-tilting modules in mod Λ, and cotorsion pairs in $\mathsf{K}^{[-1,0]}(\text{proj}\,\Lambda)$ as analogues of torsion pairs in mod Λ, in [39], Garcia showed that the analogue of wide subcategories in mod Λ are thick subcategories in $\mathsf{K}^{[-1,0]}(\text{proj}\,\Lambda)$. This notion appeared in [78] where a subcategory $\mathcal{N}$ of an extriangulated category $\mathcal{C}$ is called *thick* if it is closed under isomorphisms and summands, and if for any conflation $A \to B \to C$ in $\mathcal{C}$, whenever any two of $A,\ B,\ C$ belong to $\mathcal{N}$, so does the third. Garcia showed that left-finite wide subcategories in mod Λ are in bijection with thick subcategories in $\mathsf{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ with enough injectives. A summary of these bijections is shown in Figure 1, where the intermediate t-structures and the intermediate co-t-structures are defined with respect to the standard t-structure and standard co-t-structure respectively.

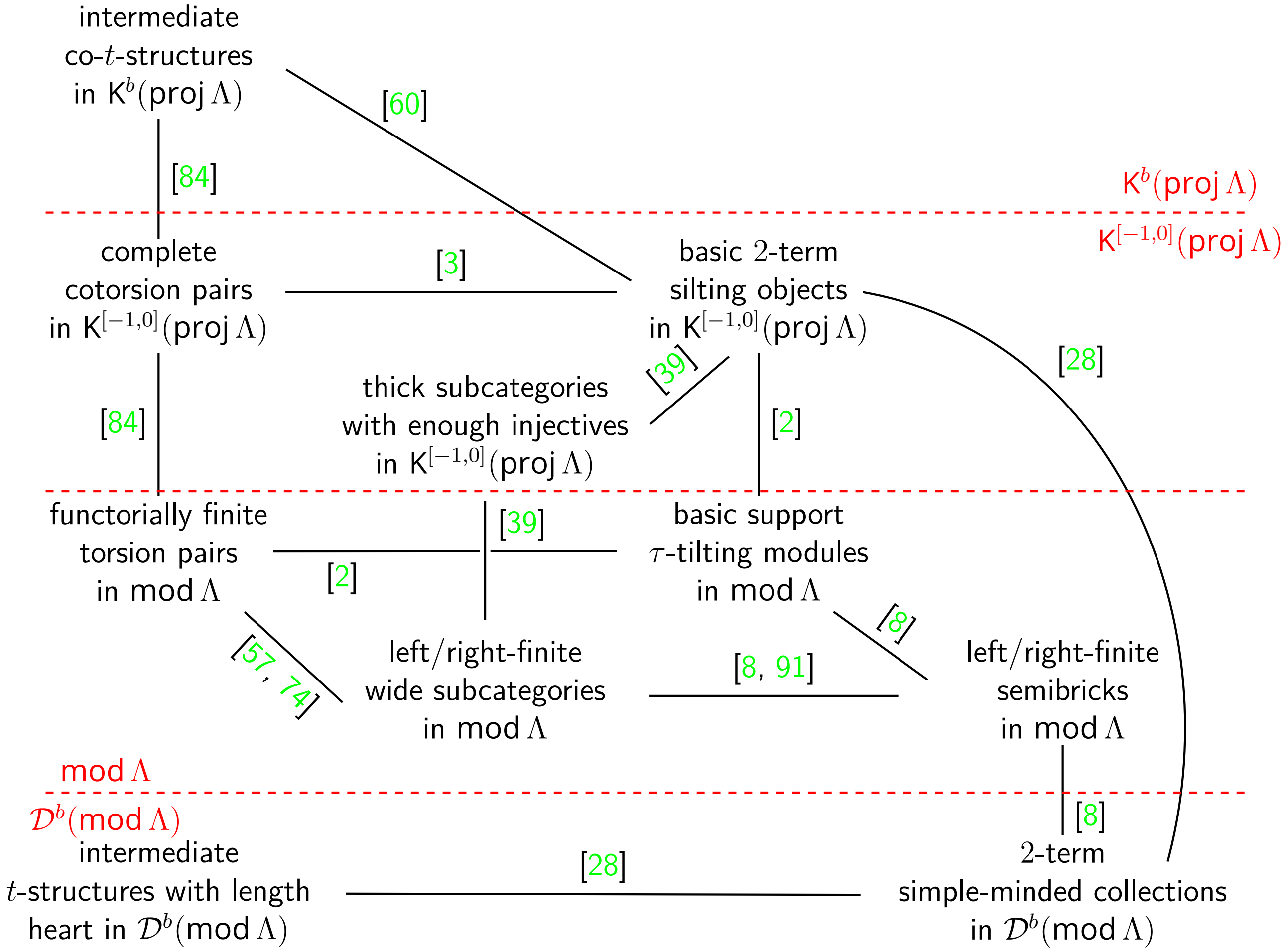


Figure 1: Bijections for $d = 1$.

5 Bijections in the derived setting

Some of the above-mentioned bijections can be viewed as (2-term) restrictions of bijections of objects in $\mathsf{K}^b(\operatorname{proj}\Lambda)$ and $\mathcal{D}^b(\operatorname{mod}\Lambda)$, namely, t-structures, co-t-structures, silting objects, and simple-minded collections. We start by recalling the definition of simple-minded collections.

Definition 5.1 ([86]) Let $\mathcal{D}$ be a triangulated category. A collection of objects $\mathcal{X}$ in $\mathcal{D}$ is called *simple-minded* if the following conditions hold for all $X, Y \in \mathcal{X}$.

(a) End(X) is a division algebra and Hom$(X, Y) = 0$ for $X \neq Y$;

(b) Hom$(X, Y[i]) = 0$ for all $i < 0$;

(c) thick $\mathcal{X} = \mathcal{D}^b(\operatorname{mod}\Lambda)$.

A simple-minded collection $\mathcal{X}$ in $\mathcal{D}^b(\operatorname{mod}\Lambda)$ is called $(d+1)$*-term* if $\mathcal{X} \subseteq \mathcal{D}^{[-d,0]}(\operatorname{mod}\Lambda)$.

Simple-minded collections are analogues of simple modules in mod Λ and were first studied in [86] in the context of symmetric algebras. Rickard showed that for a simple-minded collection $\mathcal{X}$ in $\mathcal{D}^b(\operatorname{mod}\Lambda)$ of a symmetric algebra Λ, there exists a symmetric algebra Γ derived equivalent to Λ such that the derived equivalence takes $\mathcal{X}$ to the set of simples in Γ.

A closely related variant is the notion of *simple-minded systems* introduced in [68] for studying equivalences of stable module categories. Spherical collections studied in algebraic geometry are also examples of simple-minded collections.

In [80], it was shown that any simple-minded collection $\mathcal{X}$ (called *cohomologically Schurian* there) in $\mathcal{D}^b(\text{mod}\,\Lambda)$ gives a t-structure on $\mathcal{D}^b(\text{mod}\,\Lambda)$ with $\mathcal{X}$ being the set of simples in its heart. It was shown explicitly by Keller and Nicolás in [65] that this map gives a bijection between simple-minded collections (up to certain equivalences) in $\mathcal{D}_{fd}(A)$ and bounded t-structures on $\mathcal{D}_{fd}(A)$ whose heart is length and contains finitely many simples, where A is a homologically smooth non-positive dg-algebra and $\mathcal{D}_{fd}(A)$ is the derived category of dg-modules with finite total dimension of homology. This was shown to be the case for the bounded derived categories of all finite-dimensional algebras in [69], where the authors also developed a general theory of mutations of simple-minded collections in Hom-finite, Krull Schmidt triangulated categories, generalising the mutations of spherical collections [70] and of simple-minded collections in derived categories of acyclic quivers [67]. We recall the definition of mutation of simple-minded collections in $\mathcal{D}^b(\text{mod}\,\Lambda)$, where it always exists.

Let $\mathcal{X}$ be a simple-minded collection in $\mathcal{D}^b(\text{mod}\,\Lambda)$. By [69, Corollary 5.5], the cardinality of $\mathcal{X}$ equals the rank of Λ. The *left mutation* of $\mathcal{X}$ at X_i is defined as

$$\mu_i^+(\mathcal{X}) = \{X_j'\}_{j\neq i} \cup \{X_i[1]\},$$

where X_j' is the cone of a minimal left $W(X_i)$-approximation of $X_j[-1]$, and $W(X_i)$ is the extension closure of X_i in $\mathcal{D}^b(\text{mod}\,\Lambda)$. The *right mutation* $\mu_i^-(\mathcal{X})$ is defined dually. Both $\mu_i^+(\mathcal{X})$ $\mu_i^-(\mathcal{X})$ are again simple-minded collections.

The following theorem of König and Yang, which will be used often in this work, compiles the above-mentioned results.

Theorem 5.2 ([69, Theorems 6.1 and 7.12]) *There are bijections between the following sets:*

(1) bounded co-t-structures on $\mathsf{K}^b(\text{proj}\,\Lambda)$*;*

(2) basic silting objects in $\mathsf{K}^b(\text{proj}\,\Lambda)$*;*

(3) bounded t-structures on $\mathcal{D}^b(\text{mod}\,\Lambda)$ with length hearts;

(4) length bounded hearts in $\mathcal{D}^b(\text{mod}\,\Lambda)$;

(5) simple-minded collections in $\mathcal{D}^b(\text{mod}\,\Lambda)$.

These bijections commute with left and right mutations. Specifically, a co-t-structure is mapped to an additive generator of its coheart. A silting object P corresponds to the t-structure $(\mathcal{D}^{\leq 0}(P), \mathcal{D}^{\geq 0}(P))$ where

$$\mathcal{D}^{\leq 0}(P) = \{X \in \mathcal{D}^b(\text{mod}\,\Lambda) \mid \text{Hom}(P, X[i]) = 0 \textit{ for all } i > 0\},$$

$$\mathcal{D}^{\geq 0}(P) = \{X \in \mathcal{D}^b(\text{mod}\,\Lambda) \mid \text{Hom}(P, X[i]) = 0 \textit{ for all } i < 0\}.$$

A t-structure is mapped to its heart, and a heart is mapped to its set of isoclasses of simple objects.

The co-t-structure corresponding to a silting object P is given by $(\mathcal{X}_P, \mathcal{Y}_P)$ with

$$\mathcal{X}_P := \textit{the smallest full subcategory of } \mathsf{K}^b(\text{proj}\,\Lambda) \textit{ containing } \{\Sigma^m M \mid m \leq 0\}$$
$$\textit{and closed under extensions and summands}$$

$$\mathcal{Y}_P := \textit{the smallest full subcategory of } \mathsf{K}^b(\text{proj}\,\Lambda) \textit{ containing } \{\Sigma^m M \mid m \geq 0\}$$
$$\textit{and closed under extensions and summands}$$

6 Recent work and main results

The broad goal of this dissertation is to establish d-term generalisations of the bijections appearing in the context of τ-tilting theory. Specifically, we aim to find categories $\mathcal{C}_{d,\Lambda}$ and appropriate notions of torsion classes, semibricks, and wide subcategories in them to get bijections with d-term silting objects in $\mathsf{K}^b(\operatorname{proj}\Lambda)$, such that for $d=1$, we recover the module category and the classical definitions.

An approach in this direction was provided in [11, 12], where the authors look inside the module category and take $\mathcal{C}_{d,\Lambda}$ to be a d-cluster-tilting subcategory $\mathcal{M}$ of $\operatorname{mod}\Lambda$ and consider d-torsion classes in $\mathcal{M}$ (see Section 2.4). They provide an injective map from functorially finite d-torsion classes in $\mathcal{M}$ to $(d+1)$-term silting complexes. However, this map fails to be surjective.

In this work, we take the opposite approach, and enlarge the module category viewing it as the heart of the standard t-structure. More precisely, we consider subcategories of the form

$$d\text{-}\operatorname{mod}\Lambda := \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$$

which consist of objects in the bounded derived category $\mathcal{D}^b(\operatorname{mod}\Lambda)$ whose cohomology is concentrated in degrees from $-d+1$ to 0. These subcategories are referred to as extended module categories or truncated derived categories. They also appeared in [59], where the following truncated version of Theorem 5.2 was obtained.

Theorem 6.1 ([59, Theorem 3.3 and Corollary 3.4]) *There are bijections between*

(1') basic $(d+1)$-term silting objects in $\mathsf{K}^b(\operatorname{proj}\Lambda)$,

(2') bounded t-structures $(\mathcal{C}^{\leq 0},\mathcal{C}^{\geq 0})$ on $\mathcal{D}^b(\operatorname{mod}\Lambda)$ with length hearts satisfying $\mathcal{D}^{\leq -d}(\operatorname{mod}\Lambda)\subseteq\mathcal{C}^{\leq 0}\subseteq\mathcal{D}^{\leq 0}(\operatorname{mod}\Lambda)$,

(3') length bounded hearts lying in $\mathcal{D}^{[-d,0]}(\operatorname{mod}\Lambda)$, and

(4') $(d+1)$-term simple-minded collections in $\mathcal{D}^b(\operatorname{mod}\Lambda)$.

We will formulate our definitions and results in a more general setting, viewing $d\text{-}\operatorname{mod}\Lambda$ as a d-extended heart in the sense below.

Let $\mathcal{D}$ be a triangulated category. Assume that $\mathcal{D}$ admits a bounded t-structure $(\mathcal{D}^{\leq 0},\mathcal{D}^{\geq 0})$ [21]. This implies, in particular, that $\mathcal{D}$ is idempotent complete [72]. Let $\mathcal{H}=\mathcal{D}^{\leq 0}\cap\mathcal{D}^{\geq 0}$ be the heart of this t-structure. For any integers n and m, define

$$\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n],\ \mathcal{D}^{\geq m} := \mathcal{D}^{\geq 0}[-m] \text{ and } \mathcal{D}^{[m,n]} := \mathcal{D}^{\leq n}\cap\mathcal{D}^{\geq m}.$$

We denote by

$$\sigma_{\leq n}\colon \mathcal{D}\to\mathcal{D}^{\leq n} \text{ and } \sigma_{\geq m}\colon \mathcal{D}\to\mathcal{D}^{\geq m}$$

the truncation functors adjoint to the inclusions $\mathcal{D}^{\leq n}\hookrightarrow\mathcal{D}$ and $\mathcal{D}^{\geq m}\hookrightarrow\mathcal{D}$, respectively. For each $X\in\mathcal{D}$, we have a canonical triangle

$$\sigma_{\leq n}X\to X\to\sigma_{\geq n+1}X.$$

Fix $d\geq 1$. We will call $\mathcal{D}^{[-d+1,0]}$ the *d-extended heart* of the t-structure. Note that $\mathcal{D}^{[0,0]}=\mathcal{H}$, and, in general,

$$\mathcal{D}^{[-d+1,0]}=\mathcal{H}[d-1]*\cdots*\mathcal{H}[1]*\mathcal{H}.$$

Since $\mathcal{D}^{[-d+1,0]}$ is an extension closed subcategory of a triangulated category, it inherits an extriangulated structure with $\mathbb{E}(X,Y)=\operatorname{Hom}(X,Y[1])$ for $X,Y\in\mathcal{D}^{[-d+1,0]}$. Moreover, it is an extriangulated category with negative first extensions in the sense of [1], with $\mathbb{E}^{-1}(X,Y):=\operatorname{Hom}(X,Y[-1])$.

In the first chapter, we will discuss three different notions of torsion classes in $\mathcal{D}^{[-d+1,0]}$. These will specialise to the three equivalent definitions of torsion classes in $\operatorname{mod}\Lambda$ for the standard t-structure and

$d = 1$, but will be nonequivalent in general. Nonetheless, we will show several results relating the three definitions and they will all play an important role in the development of the theory.

The first definition comes from the generalisation of Definition 2.2 to extriangulated categories with negative first extensions by Adachi, Enomoto, and Tsukamoto.

Definition 6.2 ([1, Definition 3.1]) An s-*torsion pair* $(\mathcal{T}, \mathcal{F})$ in $\mathcal{D}^{[-d+1,0]}$ is a pair of subcategories such that

1. $\mathcal{D}^{[-d+1,0]} = \mathcal{T} * \mathcal{F}$;
2. $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$;
3. $\mathrm{Hom}(\mathcal{T}, \mathcal{F}[-1]) = 0$.

Here, $\mathcal{T}$ is called an s-*torsion class*, and $\mathcal{F}$ an s-*torsion-free class*.

Unfortunately, it is not known if s-torsion classes are closed under intersections, a fact we would like in order to generalise the lattice structure of torsion classes in $\mathrm{mod}\,\Lambda$ and also to make sense of certain constructions related to semibricks and wide subcategories. For this, we consider the following definition of (positive) torsion pairs.

Definition 6.3 A pair of subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{D}^{[-d+1,0]}$ is called a *torsion pair* if $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. A torsion pair is called *positive* if

$$\mathrm{Hom}(\mathcal{T}, \mathcal{F}[j]) = 0 \text{ for all integers } j \leq 0.$$

In a (positive) torsion pair $(\mathcal{T}, \mathcal{F})$, the subcategory $\mathcal{T}$ is called a *(positive) torsion class* and $\mathcal{F}$ a *(positive) torsion-free class*.

The above definition of torsion pairs also appeared in [59] for triangulated categories. It will be shown in Theorem 1.2.2 and Corollary 1.2.4 that (positive) torsion/torsion-free classes are closed under intersections, giving us the lattice structure we were looking for.

Finally, generalising the notion of factors in abelian categories, there is a notion of d-factors in $\mathcal{D}^{[-d+1,0]}$ introduced by Zhou [99], which allows us to consider d-*FAE closed subcategories*, i.e., subcategories closed under d-factors and extensions.

Definition 6.4 ([99, Definition 1.13]) Let $\mathcal{X} \subseteq \mathcal{D}^{[-d+1,0]}$ and let m be a positive integer. An object $Z \in \mathcal{D}^{[-d+1,0]}$ is called an m-*factor* of $\mathcal{X}$ if there exist m triangles

$$Z_i \to X_i \to Z_{i-1},\ 1 \leq i \leq m,$$

such that $Z_0 = Z$, $Z_1, \ldots, Z_m \in \mathcal{D}^{[-d+1,0]}$, and $X_1, \ldots, X_m \in \mathcal{X}$. We denote by $\mathrm{Fac}_m(\mathcal{X})$ the full subcategory of $\mathcal{D}^{[-d+1,0]}$ consisting of m-factors of $\mathcal{X}$. For convenience, we set $\mathrm{Fac}_0(\mathcal{X}) = \mathcal{D}^{[-d+1,0]}$. We say that $\mathcal{X}$ is *closed under* d-*factors* if $\mathrm{Fac}_d(\mathcal{X}) \subseteq \mathcal{X}$.

Dually, we define m-*subobjects* of $\mathcal{X}$ and denote the corresponding subcategory by $\mathrm{Sub}_m(\mathcal{X})$. We say that $\mathcal{X}$ is *closed under* d-*subobjects* if $\mathrm{Sub}_d(\mathcal{X}) \subseteq \mathcal{X}$.

A subcategory $\mathcal{X}$ of $\mathcal{D}^{[-d+1,0]}$ is called d-*FAE closed* (resp. d-*SAE closed*) if it is closed under d-factors (resp. d-subobjects) and extensions.

As is clear from the definition, d-FAE/d-SAE closed subcategories are also closed under intersections. However, we would like to point out that although (positive) torsion (resp. s-torsion) classes and (positive) torsion-free (resp. s-torsion-free) classes occur in pairs, i.e., there is a bijective correspondence between the two, there is no such correspondence between d-FAE and d-SAE closed subcategories.

Figure 2 illustrates the inclusions and equalities among the three types of torsion-like classes of $\mathcal{D}^{[-d+1,0]}$ discussed above. Here, the equalities in the first two rows hold under the assumption that $\mathcal{D}$ is K-linear, Hom-finite, and Krull-Schmidt. When $d = 1$ and $\mathcal{D} = \mathcal{D}^b(\mathrm{mod}\,\Lambda)$ equipped with the standard t-structure,

$$\begin{array}{ccccc}
\left\{\begin{array}{c}\text{functorially finite}\\ s\text{-torsion classes}\end{array}\right\} & = & \left\{\begin{array}{c}\text{functorially finite}\\ \text{positive torsion classes}\end{array}\right\} & = & \left\{\begin{array}{c}\text{functorially finite}\\ d\text{-FAE closed subcategories}\end{array}\right\}\\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$}\\
\{s\text{-torsion classes}\} & = & \left\{\begin{array}{c}\text{contravariantly finite}\\ \text{positive torsion classes}\end{array}\right\} & = & \left\{\begin{array}{c}\text{contravariantly finite}\\ d\text{-FAE closed subcategories}\end{array}\right\}\\
\parallel & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$}\\
\{s\text{-torsion classes}\} & \subseteq & \{\text{positive torsion classes}\} & \subseteq & \{d\text{-FAE closed subcategories}\}
\end{array}$$

Figure 2: Inclusions and equalities among various torsion-like classes

the inclusions in the last row also become equalities, since in this case $\mathcal{D}^{[-d+1,0]} \simeq \operatorname{mod}\Lambda$. For general d, in the d-extended module category $d\text{-}\operatorname{mod}\Lambda$, the first inclusion in the last row is proper (see Example 2.3.3), while it is currently unknown whether the second inclusion is proper or not.

In the second chapter, we start by generalising a classical result of Auslander to get the following equivalence of categories.

Proposition 6.5 *The truncation functor*

$$\sigma_{\geq -d+1} : \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda) \to \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$$

induces an equivalence of K-linear categories between $\frac{\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)}{\operatorname{add}\Lambda[d]}$ *and* $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$.

We then show that this truncation functor induces bijections between cotorsion pairs in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$ and torsion pairs in $\mathcal{D}^{[-d+1,0]}(\operatorname{proj}\Lambda)$, generalising the result of Garcia. We then use restrictions of the König-Yang bijections to establish bijections between $d+1$-term silting objects in $\mathsf{K}^b(\operatorname{proj}\Lambda)$, complete, hereditary cotorsion pairs in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$, and functorially finite, positive torsion pairs in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$, generalising results of Adachi, Iyama, and Reiten, and Pauksztello and Zvonareva.

The extended module categories $d\text{-}\operatorname{mod}\Lambda$ were also investigated by Zhou in [99], where he established an Auslander-Reiten theory on these extriangulated categories. This structure involves an Auslander-Reiten translation denoted by $\tau_{[d]}$. This allowed him to introduce the notion of a $\tau_{[d]}$-tilting pair in $d\text{-}\operatorname{mod}\Lambda$ and he showed that such pairs are in bijection with $(d+1)$-term silting objects in $\mathsf{K}^b(\operatorname{proj}\Lambda)$ establishing a '$\tau_{[d]}$-tilting theory'. This also allowed him to express the torsion pair associated to a silting object P as

$$(\operatorname{Fac}_d(\sigma_{\geq -d+1}(P)), \operatorname{Sub}_d(\sigma_{\geq -d+1}(\nu P[-1])))$$

showing that the torsion pair only depends on the extriangulated structure of $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$, and does not require working with the full derived category. For general extended hearts, he shows a bijection between s-torsion pairs in $\mathcal{D}^{[-d+1,0]}$ and certain t-structures on $\mathcal{D}$. This allows him to establish a generalised Happel-Reiten-Smalø tilting for extended hearts: an s-torsion pair in $\mathcal{D}^{[-d+1,0]}$ gives an s-torsion pair in the d-extended heart of the associated t-structure.

In Chapter 3, we define semibricks and wide subcategories in $\mathcal{D}^{[-d+1,0]}$. We show a bijection between semibricks and length wide subcategories, generalising the result of Ringel. The main tool used for this is the corresponding bijection for exact categories proved by Enomoto in [37]. We also give a way to associate a wide subcategory to any d-FAE closed subcategory using its 'exact heart'. We then go on to define left-finite semibricks (resp. wide subcategories) in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$ as semibricks (resp. wide subcategories) such that the smallest positive torsion class containing them is functorially finite. We then give bijections between $(d+1)$-term simple-minded collections, left-finite semibricks, left-finite wide subcategories, and functorially finite, positive torsion pairs in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$, generalising results of Marks and Šťovíček, and Asai. We also prove certain properties about the mutations of $(d+1)$-term simple-minded collections and $(d+1)$-term silting objects.

A summary of the above bijections is shown in Figure 3. The results in the first three chapters are based on works [41] and [42].

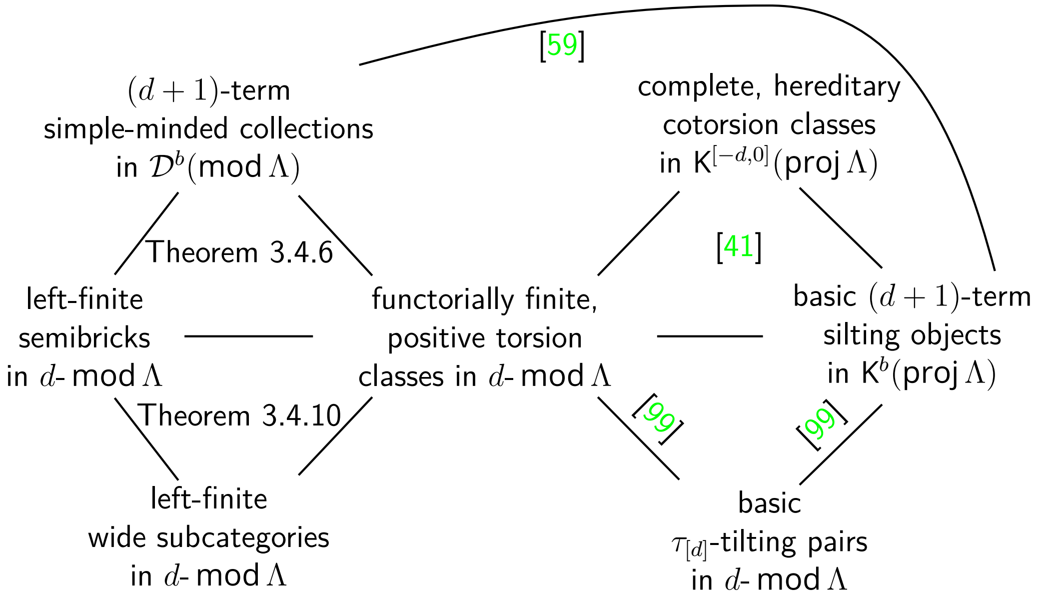


Figure 3: Bijections for arbitrary d.

The last chapter deals with a geometric model for d-term silting objects for gentle algebras. It is based on the work [40].

In recent years, (graded) gentle algebras have come to gather a lot of attention because of their ubiquity in several areas of mathematics. These include cluster theory, where they appear as cluster-tilted algebras and Jacobian algebras associated to triangulations of marked surfaces [10, 71], as well as geometry, where they appear as endomorphism rings of formal generators of some partially wrapped Fukaya categories [43]. A geometric model for the module categories of all gentle algebras was provided in [19], where the authors realised them as tiling algebras associated to partial triangulations of unpunctured surfaces with marked points on the boundary. A geometric model for the m-cluster category of type A_{n-1} was provided in [20] using diagonals of a regular polygon. In [81], the authors provided a complete model for the bounded derived category of a gentle algebra, which encoded information such as the indecomposable objects, morphisms, mapping cones, Auslander-Reiten translation for perfect objects, and Auslander–Reiten triangles. A model for studying support τ-tilting modules, or equivalently 2-term silting complexes, of all locally gentle algebras was provided in [82]. For an acyclic quiver Q, it was shown in [67, Theorem 5.14] that the hearts of $\mathcal{D}^b(\text{mod}\,kQ)$ lying between the canonical heart $\mathcal{H}_Q = \text{mod}\,kQ$ and $\mathcal{H}_Q[d-1]$, and reachable from $\mathcal{H}_Q$ are in bijection with basic d-term silting complexes as well as d-cluster tilting sets.

In the last chapter, we modify the model of [81] to give a model for the bounded derived category of a gentle algebra that encodes the indecomposable objects and their positive extensions as (ungraded) arcs on a surface and their intersections (Proposition 4.2.13). We can then restrict this model to study the truncated homotopy categories $\mathsf{K}^{[-d+1,0]}(\text{proj}\,\Lambda)$ of d-term complexes of projectives for a gentle algebra Λ. Since the model encodes positive extensions, it is particularly suited for the study of d-term silting complexes generalizing the case of 2-term silting complexes studied in [82] (Corollary 4.2.17, Remark 4.2.19). Applying the model to the case of linearly oriented A_n quiver, we recover the result that the number of d-term silting complexes in $\mathsf{K}^b(\text{proj}\,kA_n)$ is given by the Pfaff-Fuss-Catalan number C^d_{n+1} (§ 4.3). This can be obtained using the bijection between d-cluster tilting objects in type A_n and $d+2$-angulations of an $nd+2$-gon obtained from [20]. A more combinatorial proof can also be found in [94].

Chapter 1

Torsion-like classes in extended hearts

In this chapter, we will introduce three notions of torsion classes, namely d-FAE closed subcategories (where d-FAE stands for d-factors and extensions), positive torsion classes, and s-torsion classes, that arise as natural generalisations of torsion classes in module categories. We will explore their individual properties and see how they are related to each other. We will also mention the dual notions of d-SAE closed subcategories, positive torsion-free classes, and s-torsion-free classes. These results will then be used in the subsequent chapters to prove the main theorems. We will work in the general setting of the d-extended heart $\mathcal{D}^{[-d+1,0]}$ of a bounded t-structure on a triangulated category $\mathcal{D}$. Some results in Section 1.2 are generalisations of results in [41] from extended module categories to arbitrary extended hearts, while the remaining results of the chapter are contained in [42] and [41].

1.1 d-FAE and d-SAE closed subcategories

In this section, we study subcategories of $\mathcal{D}^{[-d+1,0]}$ defined by specific closure properties. We begin by recalling the following notions, which provide a natural generalization of the classical concepts of factor objects and subobjects in abelian categories to a higher setting.

Definition 1.1.1 ([99, Definition 1.13]) Let $\mathcal{X} \subseteq \mathcal{D}^{[-d+1,0]}$ and let m be a positive integer. An object $Z \in \mathcal{D}^{[-d+1,0]}$ is called an *m-factor* of $\mathcal{X}$ if there exist m triangles

$$Z_i \to X_i \to Z_{i-1},\ 1 \leq i \leq m,$$

such that $Z_0 = Z$, $Z_1, \ldots, Z_m \in \mathcal{D}^{[-d+1,0]}$, and $X_1, \ldots, X_m \in \mathcal{X}$. We denote by $\mathsf{Fac}_m(\mathcal{X})$ the full subcategory of $\mathcal{D}^{[-d+1,0]}$ consisting of m-factors of $\mathcal{X}$. For convenience, we set $\mathsf{Fac}_0(\mathcal{X}) = \mathcal{D}^{[-d+1,0]}$. We say that $\mathcal{X}$ is *closed under d-factors* if $\mathsf{Fac}_d(\mathcal{X}) \subseteq \mathcal{X}$.

Dually, we define *m-subobjects* of $\mathcal{X}$ and denote the corresponding subcategory by $\mathsf{Sub}_m(\mathcal{X})$. We say that $\mathcal{X}$ is *closed under d-subobjects* if $\mathsf{Sub}_d(\mathcal{X}) \subseteq \mathcal{X}$.

Note that we have inclusions

$$\mathcal{X} \subseteq \cdots \subseteq \mathsf{Fac}_2(\mathcal{X}) \subseteq \mathsf{Fac}_1(\mathcal{X}) \subseteq \mathsf{Fac}_0(\mathcal{X}) = \mathcal{D}^{[-d+1,0]},$$

and

$$\mathcal{X} \subseteq \cdots \subseteq \mathsf{Sub}_2(\mathcal{X}) \subseteq \mathsf{Sub}_1(\mathcal{X}) \subseteq \mathsf{Sub}_0(\mathcal{X}) = \mathcal{D}^{[-d+1,0]}.$$

Remark 1.1.2 ([99, Example 1.14]) Let $Y \in \mathcal{D}^{[-d+1,0]}$, and fix a non-negative integer l such that $Y[l] \in \mathcal{D}^{[-d+1,0]}$. Let $\mathcal{X} \subseteq \mathcal{D}^{[-d+1,0]}$, and fix a non-negative integer m such that $Y \in \mathsf{Fac}_m(\mathcal{X})$. Then $Y[l] \in \mathsf{Fac}_{m+l}(\mathcal{X})$.

The following construction of d-factors will be useful.

Lemma 1.1.3 *Let $\mathcal{X} \subseteq \mathcal{D}^{[-d+1,0]}$. For any $X \in \mathcal{X}$ and any integer $1 \leq j \leq d-1$, we have*

$$\sigma_{\geq -d+1+j}(X) \in \mathsf{Fac}_{d-j+1}(\mathcal{X}) \quad \textit{and} \quad \sigma_{\geq -d+1}(X[j]) \in \mathsf{Fac}_{d+1}(\mathcal{X}).$$

Proof. Since $X \in \mathcal{X} \subseteq \mathcal{D}^{[-d+1,0]}$, we have

$$\sigma_{\leq -d+j}(X) = (\sigma_{\leq 0}(X[-d+j]))[d-j],$$

by the definition of truncation and shift. It then follows from Remark 1.1.2 that

$$\sigma_{\leq -d+j}(X) \in \mathsf{Fac}_{d-j}(\mathcal{X}).$$

Consider the truncation triangle of X

$$\sigma_{\leq -d+j}(X) \to X \to \sigma_{\geq -d+j+1}(X).$$

It follows that

$$\sigma_{\geq -d+1+j}(X) \in \mathsf{Fac}_{d-j+1}(\mathcal{X}).$$

Applying Remark 1.1.2 once again, we conclude that

$$\sigma_{\geq -d+1}(X[j]) = (\sigma_{\geq -d+1+j}(X))[j] \in \mathsf{Fac}_{d+1}(\mathcal{X}).$$

□

We record the following basic properties of subcategories that are closed under d-factors.

Lemma 1.1.4 *Let $\mathcal{X} \subseteq \mathcal{D}^{[-d+1,0]}$ be closed under d-factors. Then:*

(1) $\mathcal{X}$ is closed under direct summands.

(2) For any morphism $f\colon X \to Y$ in $\mathcal{X}$, the truncation $\sigma_{\geq -d+1}(C(f)) \in \mathcal{X}$.

Proof. For (1), let $X = Y \oplus Z \in \mathcal{X}$. Then there are triangles

$$Z \to X \to Y,$$
$$Y \to X \to Z.$$

Iterating these triangles, we see that $Y \in \mathsf{Fac}_d(\mathcal{X}) \subseteq \mathcal{X}$.

For (2), from the triangle $X \xrightarrow{f} Y \xrightarrow{g} C(f)$, we have $C(f) \in \mathcal{D}^{[-d,0]}$. Consider the truncation triangle of $C(f)$:

$$\sigma_{\leq -d}(C(f)) \to C(f) \xrightarrow{h} \sigma_{\geq -d+1}(C(f)),$$

where $\sigma_{\leq -d}(C(f)) \in \mathcal{D}^{[-d,-d]} = \mathcal{H}[d]$ and $\sigma_{\geq -d+1}(C(f)) \in \mathcal{D}^{[-d+1,0]}$. Applying the octahedral axiom to the composition $h \circ g$, we obtain a diagram of triangles

$$\begin{tikzcd}
 & & \sigma_{\geq -d+1}(C(f))[-1] \arrow[r, equal] \arrow[d] & \sigma_{\geq -d+1}(C(f))[-1] \arrow[d] \\
\sigma_{\leq -d}(C(f))[-1] \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d] & \sigma_{\leq -d}(C(f)) \arrow[d] \\
\sigma_{\leq -d}(C(f))[-1] \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] \arrow[d, "h\circ g"] & C(f) \arrow[d, "h"] \\
 & & \sigma_{\geq -d+1}(C(f)) \arrow[r, equal] & \sigma_{\geq -d+1}(C(f))
\end{tikzcd}$$

From the triangle in the second row and the triangle in the third column, we obtain

$$Z \in (X * \sigma_{\leq -d}(C(f))) \cap (\sigma_{\geq -d+1}(C(f))[-1] * Y) \subseteq \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq -d+1} = \mathcal{D}^{[-d+1,0]}.$$

Since $\sigma_{\leq -d}(C(f))[-1] \in \mathcal{H}[d-1]$, by Remark 1.1.2, we have that $\sigma_{\leq -d}(C(f))[-1] \in \mathsf{Fac}_{d-1}(\mathcal{X})$. The triangle in the second row then yields $Z \in \mathsf{Fac}_d(\mathcal{X})$. Consequently, the triangle in the third column shows that

$$\sigma_{\geq -d+1}(C(f)) \in \mathsf{Fac}_{d+1}(\mathcal{X}) \subseteq \mathsf{Fac}_d(\mathcal{X}) \subseteq \mathcal{X}.$$

□

We now introduce the first notion of torsion-like classes, namely d-FAE closed subcategories, together with their dual notion of d-SAE closed subcategories.

Definition 1.1.5 A subcategory $\mathcal{X}$ of $\mathcal{D}^{[-d+1,0]}$ is called *d-FAE closed* (resp. *d-SAE closed*) if it is closed under d-factors (resp. d-subobjects) and extensions.

Recall that a subcategory $\mathcal{C}$ of $\mathcal{D}$ is called *suspended* (resp. *cosuspended*) if it is closed under extensions and positive (resp. negative) shifts.

Proposition 1.1.6 *There is a bijection between*

- *the set of d-FAE closed subcategories of $\mathcal{D}^{[-d+1,0]}$, and*
- *the set of suspended subcategories $\mathcal{C}$ of $\mathcal{D}$ satisfying $\mathcal{D}^{\leq -d} \subseteq \mathcal{C} \subseteq \mathcal{D}^{\leq 0}$,*

given by

$$\mathcal{X} \mapsto \mathcal{D}^{\leq -d} * \mathcal{X},$$

with inverse

$$\mathcal{C} \mapsto \mathcal{C} \cap \mathcal{D}^{[-d+1,0]}.$$

Dually, there is a bijection between

- *the set of d-SAE closed subcategories of $\mathcal{D}^{[-d+1,0]}$, and*
- *the set of cosuspended subcategories $\mathcal{C}$ of $\mathcal{D}$ satisfying $\mathcal{D}^{\geq 0} \subseteq \mathcal{C} \subseteq \mathcal{D}^{\geq -d}$,*

given by

$$\mathcal{X} \mapsto \mathcal{X}[1] * \mathcal{D}^{\geq 0},$$

with inverse

$$\mathcal{C} \mapsto \mathcal{C}[-1] \cap \mathcal{D}^{[-d+1,0]}.$$

Proof. We only prove the first bijection, since the second one can be showed dually.

We first show that for any suspended subcategory $\mathcal{C}$ of $\mathcal{D}$ satisfying $\mathcal{D}^{\leq -d} \subseteq \mathcal{C} \subseteq \mathcal{D}^{\leq 0}$, the subcategory $\mathcal{C} \cap \mathcal{D}^{[-d+1,0]}$ of $\mathcal{D}^{[-d+1,0]}$ is d-FAE closed. Since both $\mathcal{C}$ and $\mathcal{D}^{[-d+1,0]}$ are closed under extensions, their intersection $\mathcal{C} \cap \mathcal{D}^{[-d+1,0]}$ is also closed under extensions. Let $Z \in \mathsf{Fac}_d(\mathcal{C} \cap \mathcal{D}^{[-d+1,0]})$. By definition, we have

$$Z \in (\mathcal{C} \cap \mathcal{D}^{[-d+1,0]}) * (\mathcal{C} \cap \mathcal{D}^{[-d+1,0]})[1] * \cdots * (\mathcal{C} \cap \mathcal{D}^{[-d+1,0]})[d-1] * \mathcal{D}^{[-d+1,0]}[d].$$

Since $\mathcal{C}$ is closed under positive shifts, $(\mathcal{C} \cap \mathcal{D}^{[-d+1,0]})[i] \subseteq \mathcal{C}$ for all $i \geq 0$. Note also that $\mathcal{D}^{[-d+1,0]}[d] \subseteq \mathcal{D}^{\leq -d} \subseteq \mathcal{C}$. It follows that $Z \in \mathcal{C}$ because $\mathcal{C}$ is closed under extensions. Therefore, $\mathcal{C} \cap \mathcal{D}^{[-d+1,0]}$ is closed under d-factors.

We next show that for any d-FAE closed subcategory $\mathcal{X}$ of $\mathcal{D}^{[-d+1,0]}$, the subcategory $\mathcal{D}^{\leq -d} * \mathcal{X}$ of $\mathcal{D}$ is suspended. Note that $\mathcal{D}^{\leq -d} \subseteq \mathcal{D}^{\leq -d} * \mathcal{X} \subseteq \mathcal{D}^{\leq 0}$. To show that $\mathcal{D}^{\leq -d} * \mathcal{X}$ is closed under extensions, it is equivalent to prove

$$(\mathcal{D}^{\leq -d} * \mathcal{X}) * (\mathcal{D}^{\leq -d} * \mathcal{X}) \subseteq \mathcal{D}^{\leq -d} * \mathcal{X}.$$

Since both $\mathcal{D}^{\le -d}$ and $\mathcal{X}$ are closed under extensions, it is enough to verify

$$\mathcal{X} * \mathcal{D}^{\le -d} \subseteq \mathcal{D}^{\le -d} * \mathcal{X}.$$

Let $Z \in \mathcal{X} * \mathcal{D}^{\le -d}$. Then here exists a triangle

$$X \to Z \to Y$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{D}^{\le -d}$. By the octahedral axiom, we obtain the following commutative diagram of triangles:

$$\begin{tikzcd}
 & \sigma_{\ge -d+1}(Z)[-1] \arrow[r, equal] \arrow[d] & \sigma_{\ge -d+1}(Z)[-1] \arrow[d] & \\
Y[-1] \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d] & \sigma_{\le -d}(Z) \arrow[r] \arrow[d] & Y \arrow[d, equal] \\
Y[-1] \arrow[r] & X \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Y \\
 & \sigma_{\ge -d+1}(Z) \arrow[r, equal] & \sigma_{\ge -d+1}(Z) &
\end{tikzcd}$$

By the triangle in the second row, we have

$$V \in Y[-1] * \sigma_{\le -d}(Z) \subseteq \mathcal{D}^{\le -d+1} * \mathcal{D}^{\le -d} \subseteq \mathcal{D}^{\le -d+1},$$

and by the triangle in the second column,

$$V \in \sigma_{\ge -d+1}(Z)[-1] * X \subseteq \mathcal{D}^{\ge -d+2} * \mathcal{D}^{\ge -d+1} \subseteq \mathcal{D}^{\ge -d+1}.$$

Consequently, $V \in \mathcal{D}^{[-d+1,-d+1]} = \mathcal{H}[d-1]$, which implies $V \in \mathsf{Fac}_{d-1}(\mathcal{X})$ by Remark 1.1.2. Hence, by the triangle in the second column, we obtain $\sigma_{\ge -d+1}(Z) \in \mathsf{Fac}_{d}(\mathcal{X}) \subseteq \mathcal{X}$. It follows that $Z \in \mathcal{D}^{\le -d} * \mathcal{X}$, as desired.

Now, let $Z \in \mathcal{D}^{\le -d} * \mathcal{X}$. Then $\sigma_{\ge -d+1}(Z) \in \mathcal{X}$. By Lemma 1.1.3, we have

$$\sigma_{\ge -d+1}(Z[1]) = \sigma_{\ge -d+1}(\sigma_{\ge -d+1}(Z)[1]) \in \mathcal{X}.$$

Consider the triangle obtained by truncating $Z[1]$:

$$\sigma_{\le -d}(Z[1]) \to Z[1] \to \sigma_{\ge -d+1}(Z[1]).$$

Since $\sigma_{\le -d}(Z[1]) \in \mathcal{D}^{\le -d}$ and $\sigma_{\ge -d+1}(Z[1]) \in \mathcal{X}$, it follows that $Z[1] \in \mathcal{D}^{\le -d} * \mathcal{X}$. Therefore, $\mathcal{D}^{\le -d} * \mathcal{X}$ is closed under positive shifts, and hence suspended.

Finally, we show that these two maps are mutually inverse. For any $\mathcal{X} \subseteq \mathcal{D}^{[-d+1,0]}$, it is straightforward to see, by taking truncation, that

$$(\mathcal{D}^{\le -d} * \mathcal{X}) \cap \mathcal{D}^{[-d+1,0]} = \mathcal{X}.$$

Hence, it remains to show that

$$\mathcal{D}^{\le -d} * (\mathcal{C} \cap \mathcal{D}^{[-d+1,0]}) = \mathcal{C}$$

for any suspended subcategory $\mathcal{C}$ of $\mathcal{D}$ satisfying $\mathcal{D}^{\le -d} \subseteq \mathcal{C} \subseteq \mathcal{D}^{\le 0}$. Since both $\mathcal{D}^{\le -d}$ and $\mathcal{C} \cap \mathcal{D}^{[-d+1,0]}$ are contained in $\mathcal{C}$, and $\mathcal{C}$ is closed under extensions, we have

$$\mathcal{D}^{\le -d} * (\mathcal{C} \cap \mathcal{D}^{[-d+1,0]}) \subseteq \mathcal{C}.$$

Now, let $Z \in \mathcal{C} \subseteq \mathcal{D}^{\le 0}$. By taking truncation, there exists a triangle

$$\sigma_{\le -d}(Z) \to Z \to \sigma_{\ge -d+1}(Z) \to \sigma_{\le -d}(Z)[1],$$

where $\sigma_{\geq -d+1}(Z) \in \mathcal{D}^{[-d+1,0]}$. Since $\sigma_{\leq -d}(Z) \in \mathcal{D}^{\leq -d} \subseteq \mathcal{C}$ and $\mathcal{C}$ is closed under positive shifts, we have $(\sigma_{\leq -d}(Z))\,[1] \in \mathcal{C}$. Moreover, since $\mathcal{C}$ is closed under extensions, it follows that

$$\sigma_{\geq -d+1}(Z) \in Z * (\sigma_{\leq -d}(Z))\,[1] \subseteq \mathcal{C},$$

Hence, $\sigma_{\geq -d+1}(Z) \in \mathcal{C} \cap \mathcal{D}^{[-d+1,0]}$, and consequently $Z \in \mathcal{D}^{\leq -d} * (\mathcal{C} \cap \mathcal{D}^{[-d+1,0]})$. This shows that $\mathcal{C} \subseteq \mathcal{D}^{\leq -d} * (\mathcal{C} \cap \mathcal{D}^{[-d+1,0]})$, and the proof is complete. □

Remark 1.1.7 For any d-FAE (resp. d-SAE) closed subcategory $\mathcal{X}$ of $\mathcal{D}^{[-d+1,0]}$, the full subcategory $\mathcal{D}^{\leq -d} * \mathcal{X}$ (resp. $\mathcal{X}[1] * \mathcal{D}^{\geq 0}$) is closed under direct summands. To see this, suppose that $X \oplus Y \in \mathcal{D}^{\leq -d} * \mathcal{X}$. Then

$$\sigma_{\geq -d+1}(X) \oplus \sigma_{\geq -d+1}(Y) \cong \sigma_{\geq -d+1}(X \oplus Y) \in \mathcal{X}.$$

By Lemma 1.1.4(1), it follows that $\sigma_{\geq -d+1}(X) \in \mathcal{X}$. Hence,

$$X \in \sigma_{\leq -d}(X) * \sigma_{\geq -d+1}(X) \subseteq \mathcal{D}^{\leq -d} * \mathcal{X}.$$

Combining this with Proposition 1.1.6, we conclude that any suspended (resp. cosuspended) category $\mathcal{C}$ of $\mathcal{D}$ satisfying $\mathcal{D}^{\leq -d} \subseteq \mathcal{C} \subseteq \mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0} \subseteq \mathcal{C} \subseteq \mathcal{D}^{\geq -d}$) is closed under direct summands.

Remark 1.1.8 It is straightforward to verify that d-FAE closed subcategories are closed under intersections. Consequently, for any subcategory $\mathcal{Y}$ of $\mathcal{D}^{[-d+1,0]}$, there exists a smallest d-FAE closed subcategory containing $\mathcal{Y}$, denoted by $\phi(\mathcal{Y})$.

We have the following explicit description of the smallest d-FAE closed subcategory containing a given subcategory; this description will be used in Section 3.3. We define $\Phi^0(\mathcal{Y}) = \mathcal{Y}$, and for each $i \geq 1$ set

$$\Phi^i(\mathcal{Y}) = \mathsf{Fac}_d\left(\Phi^{i-1}(\mathcal{Y})\right) * \mathsf{Fac}_d\left(\Phi^{i-1}(\mathcal{Y})\right).$$

Note that for all $i \geq 1$, we have

$$\Phi^{i-1}(\mathcal{Y}) \subseteq \mathsf{Fac}_d\left(\Phi^{i-1}(\mathcal{Y})\right) \subseteq \Phi^i(\mathcal{Y}).$$

Lemma 1.1.9 *For any $\mathcal{Y} \subseteq \mathcal{D}^{[-d+1,0]}$, the smallest d-FAE closed subcategory $\phi(\mathcal{Y})$ containing $\mathcal{Y}$ is $\bigcup_{i\geq 0} \Phi^i(\mathcal{Y})$.*

Proof. Let $\mathcal{B} := \bigcup_{i\geq 0} \Phi^i(\mathcal{Y})$. By definition, $\mathcal{B} \subseteq \phi(\mathcal{Y})$. To prove the reverse inclusion, it suffices to show that $\mathcal{B}$ is d-FAE closed.

Consider a sequence of triangles

$$X_j \to Z_j \to X_{j-1},\ 1 \leq j \leq d,$$

with $X_0, X_1, \ldots, X_d \in \mathcal{D}^{[-d+1,0]}$ and $Z_1, \ldots, Z_d \in \mathcal{B}$. Then there exists $s \geq 0$ such that $Z_j \in \Phi^s(\mathcal{Y})$ for all j. By definition, the resulting object X_0 obtained from this sequence of triangles belongs to $\mathsf{Fac}_d(\Phi^s(\mathcal{Y})) \subseteq \Phi^{s+1}(\mathcal{Y}) \subseteq \mathcal{B}$. Hence, $\mathcal{B}$ is closed under d-factors.

Finally, suppose there is a triangle $L \to M \to N$ with $L, N \in \mathcal{B}$. As before, there exists $s \geq 0$ such that $L, N \in \Phi^s(\mathcal{Y})$. It follows that

$$M \in \Phi^s(\mathcal{Y}) * \Phi^s(\mathcal{Y}) \subseteq \Phi^{s+1}(\mathcal{Y}) \subseteq \mathcal{B}.$$

Therefore, $\mathcal{B}$ is closed under extensions. □

1.2 Positive torsion/torsion-free classes

For any $\mathcal{X} \subseteq \mathcal{D}^{[-d+1,0]}$, we define the following two full subcategories of $\mathcal{D}^{[-d+1,0]}$:

$${}^{\perp}\mathcal{X} = \{Z \in \mathcal{D}^{[-d+1,0]} \mid \mathsf{Hom}(Z, \mathcal{X}) = 0\},$$

$$\mathcal{X}^{\perp} = \{Z \in \mathcal{D}^{[-d+1,0]} \mid \mathsf{Hom}(\mathcal{X}, Z) = 0\}.$$

The second notion of torsion-like classes is that of positive torsion classes.

Definition 1.2.1 A pair of subcategories $(\mathcal{T},\mathcal{F})$ of $\mathcal{D}^{[-d+1,0]}$ is called a *torsion pair* if $\mathcal{T}={}^{\perp}\mathcal{F}$ and $\mathcal{F}=\mathcal{T}^{\perp}$. A torsion pair is called *positive* if

$$\mathrm{Hom}(\mathcal{T},\mathcal{F}[j])=0 \text{ for all integers } j\leq 0.$$

In a (positive) torsion pair $(\mathcal{T},\mathcal{F})$, the subcategory $\mathcal{T}$ is called a *(positive) torsion class* and $\mathcal{F}$ a *(positive) torsion-free class*.

We will denote the poset of torsion classes in $\mathcal{D}^{[-d+1,0]}$ under inclusion by tors $\mathcal{D}^{[-d+1,0]}$. This is isomorphic to the poset of torsion pairs in $\mathcal{D}^{[-d+1,0]}$ under the order

$$(\mathcal{T}_1,\mathcal{F}_1)\preceq(\mathcal{T}_2,\mathcal{F}_2) \iff \mathcal{T}_1\subseteq\mathcal{T}_2 \iff \mathcal{F}_1\supseteq\mathcal{F}_2.$$

Say that a torsion pair is contravariantly finite, covariantly finite, or functorially finite if the associated torsion class is contravariantly finite, covariantly finite, or functorially finite respectively. We will denote the subposets of positive, contravariantly finite, covariantly finite, and functorially finite torsion classes/pairs by p-tors $\mathcal{D}^{[-d+1,0]}$, contr-tors $\mathcal{D}^{[-d+1,0]}$, cov-tors $\mathcal{D}^{[-d+1,0]}$, f-tors $\mathcal{D}^{[-d+1,0]}$, respectively. Finally, we will denote the intersection of x-tors $\mathcal{C}$ and y-tors $\mathcal{C}$ by (x, y)-tors $\mathcal{C}$, where x, y $\in$ {p, contr, cov, f, s}. As is known for the poset of torsion classes in mod Λ, we will show that tors $\mathcal{D}^{[-d+1,0]}$ and p-tors $\mathcal{D}^{[-d+1,0]}$ are, in fact, lattices.

Theorem 1.2.2 *The poset of torsion pairs/classes in $\mathcal{D}^{[-d+1,0]}$ is a lattice.*

Proof. Let $(\mathcal{T}_1,\mathcal{F}_1)$, $(\mathcal{T}_2,\mathcal{F}_2)$ be two torsion pairs. To show that their meet exists, it is enough to show that $(\mathcal{T}_1\cap\mathcal{T}_2,(\mathcal{T}_1\cap\mathcal{T}_2)^{\perp})$ is a torsion pair, i.e., $\mathcal{T}_1\cap\mathcal{T}_2={}^{\perp}((\mathcal{T}_1\cap\mathcal{T}_2)^{\perp})$. By definition, $\mathcal{T}_1\cap\mathcal{T}_2\subseteq{}^{\perp}((\mathcal{T}_1\cap\mathcal{T}_2)^{\perp})$. On the other hand, for $i=1,2$, since $\mathcal{T}_1\cap\mathcal{T}_2\subseteq\mathcal{T}_i$, we get that ${}^{\perp}((\mathcal{T}_1\cap\mathcal{T}_2)^{\perp})\subseteq{}^{\perp}(\mathcal{T}_i^{\perp})=\mathcal{T}_i$. Thus, ${}^{\perp}((\mathcal{T}_1\cap\mathcal{T}_2)^{\perp})\subseteq\mathcal{T}_1\cap\mathcal{T}_2$. Hence $\mathcal{T}_1\cap\mathcal{T}_2={}^{\perp}((\mathcal{T}_1\cap\mathcal{T}_2)^{\perp})$.

Dually, the join of $(\mathcal{T}_1,\mathcal{F}_1)$, $(\mathcal{T}_2,\mathcal{F}_2)$ is the torsion pair $({}^{\perp}(\mathcal{F}_1\cap\mathcal{F}_2),\mathcal{F}_1\cap\mathcal{F}_2)$. □

It follows directly from the definition that any torsion class and any torsion-free class is closed under extensions. The following lemma guarantees that any positive torsion (resp. torsion-free) class is d-FAE (resp. d-SAE) closed.

Lemma 1.2.3 *Let $(\mathcal{T},\mathcal{F})$ be a torsion pair in $\mathcal{D}^{[-d+1,0]}$. The following are equivalent.*

1. *$\mathcal{T}$ is a positive torsion class.*
2. *For all morphisms f in $\mathcal{T}$, the truncation $\sigma_{\geq -d+1}(C(f))$ belongs to $\mathcal{T}$.*
3. *$\mathcal{T}$ is d-FAE closed.*
4. *$\mathcal{F}$ is a positive torsion-free class.*
5. *For all morphisms g in $\mathcal{F}$, the truncation $\sigma_{\leq 0}(C(g)[-1])$ belongs to $\mathcal{F}$.*
6. *$\mathcal{F}$ is d-SAE closed.*
7. *Hom$(\mathcal{T},\mathcal{F}[-1])=0$.*

Proof. $(1\implies 4)$ Follows from the definition.
$(1\implies 7)$ Clear. $(7\implies 2)$ Let $f:X\to Y\in\mathcal{T}$. For $F\in\mathcal{F}$, the triangle $X\to Y\to C(f)$ induces the exact sequence

$$\mathbb{E}^{-1}(X,F)\to\mathrm{Hom}(C(f),F)\to\mathrm{Hom}(Y,F).$$

By assumption, $\mathbb{E}^{-1}(X,F)$ and Hom(Y,F) vanish, which gives that Hom$(C(f),F)=0$. Using the triangle $\sigma_{\leq -d}C(f)\to C(f)\to\sigma_{\geq -d+1}C(f)$, we get the exact sequence

$$\mathrm{Hom}(\sigma_{\leq -d}C(f)[1],F)\to\mathrm{Hom}(\sigma_{\geq -d+1}C(f),F)\to\mathrm{Hom}(C(f),F)=0.$$

Since $\sigma_{\leq -d}C(f)[1] \in \mathcal{D}^{\leq -d-1}$ and $F \in \mathcal{D}^{\geq -d+1}$, we get that $\operatorname{Hom}(\sigma_{\leq -d}C(f)[1], F) = 0$ which gives that $\operatorname{Hom}(\sigma_{\geq -d+1}C(f), F) = 0$. Thus, $\sigma_{\geq -d+1}C(f) \in {}^{\perp}\mathcal{F} = \mathcal{T}$.
$(2 \implies 1)$ Let $T \in \mathcal{T}$. Using the triangle $T \to 0 \to \Sigma T$, we get that $\sigma_{\geq -d+1}\Sigma T \in \mathcal{T}$. Thus, we obtain $\sigma_{\geq -d+1}\Sigma(\sigma_{\geq -d+1}\Sigma T) \in \mathcal{T}$. But this is the same as $\sigma_{\geq -d+1}\Sigma^2 T$. Continuing in this way, we conclude that $\sigma_{\geq -d+1}\Sigma^k T \in \mathcal{T}$ for all $k \geq 0$.

Using the triangle $\sigma_{\leq -d}\Sigma^k T \to \Sigma^k T \to \sigma_{\geq -d+1}\Sigma^k T$, for all $F \in \mathcal{F}$, we get the exact sequence

$$0 = \operatorname{Hom}(\sigma_{\geq -d+1}\Sigma^k T, F) \to \operatorname{Hom}(\Sigma^k T, F) \to \operatorname{Hom}(\sigma_{\leq -d}\Sigma^k T, F).$$

Since $\sigma_{\leq -d}\Sigma^k T \in \mathcal{D}^{\leq -d}$, we get that $\operatorname{Hom}(\sigma_{\leq -d}\Sigma^k T, F) = 0$. Thus $\mathbb{E}^{-k}(T, F) \cong \operatorname{Hom}(\Sigma^k T, F) = 0$ for all $k \geq 0$, and $\mathcal{T}$ is a positive torsion class.
The equivalence of (4), (5), and (7) can be shown in a dual manner.
Finally, the equivalence of (1), (3), (4), and (6) can be deduced from the arguments in the proof of [99, Proposition 1.18] or Lemma 1.1.4. ☐

Corollary 1.2.4 *The subposet* p-tors $\mathcal{D}^{[-d+1,0]}$ *of* tors $\mathcal{D}^{[-d+1,0]}$ *is closed under meets and joins, and hence, a sublattice.*

Proof. It follows from the above lemma that the intersection of two positive torsion/torsion-free classes is again a positive torsion/torsion-free class. ☐

Remark 1.2.5 As a consequence of the above corollary, we get that, for any subcategory $\mathcal{Y}$ of $\mathcal{D}^{[-d+1,0]}$, there exists a smallest positive torsion class containing $\mathcal{Y}$, given by the intersection of all positive torsion containing it. This will be denoted by $T(\mathcal{Y})$.

We now present several results that allow one to construct (positive) torsion and torsion-free classes from arbitrary subcategories; in particular, the smallest positive torsion class mentioned above can be constructed in a direct manner, rather than merely as the intersection of all positive torsion classes containing $\mathcal{Y}$.

Lemma 1.2.6 *Let $\mathcal{Y}$ be a full subcategory of $\mathcal{D}^{[-d+1,0]}$. Then the following hold.*

1. *$\mathcal{Y}^{\perp}$ is a torsion-free class.*
2. *${}^{\perp}\mathcal{Y}$ is a torsion class.*

Proof. We only prove (1), as the proof of (2) is completely analogous. By definition, we have $\mathcal{Y} \subseteq {}^{\perp}(\mathcal{Y}^{\perp})$. This immediately implies $\mathcal{Y}^{\perp} \supseteq ({}^{\perp}(\mathcal{Y}^{\perp}))^{\perp}$. Conversely, for any $Y \in \mathcal{Y}^{\perp}$ and $V \in {}^{\perp}(\mathcal{Y}^{\perp})$, we have $\operatorname{Hom}(V, Y) = 0$, which shows that $Y \in ({}^{\perp}(\mathcal{Y}^{\perp}))^{\perp}$. Hence, $\mathcal{Y}^{\perp} \subseteq ({}^{\perp}(\mathcal{Y}^{\perp}))^{\perp}$. Combining these two inclusions, we obtain $\mathcal{Y}^{\perp} = ({}^{\perp}(\mathcal{Y}^{\perp}))^{\perp}$, and hence $\mathcal{Y}^{\perp}$ is a torsion-free class. ☐

For $\mathcal{Y} \subseteq \mathcal{D}^{[-d+1,0]}$, we define the following subcategories of $\mathcal{D}^{[-d+1,0]}$:

$${}^{\perp_{\leq 0}}\mathcal{Y} = \{Z \in \mathcal{D}^{[-d+1,0]} \mid \operatorname{Hom}(Z, \mathcal{Y}[j]) = 0,\ \forall j \leq 0\},$$

$$\mathcal{Y}^{\perp_{\leq 0}} = \{Z \in \mathcal{D}^{[-d+1,0]} \mid \operatorname{Hom}(\mathcal{Y}, Z[j]) = 0,\ \forall j \leq 0\}.$$

Lemma 1.2.7 *For any subcategory $\mathcal{Y}$ of $\mathcal{D}^{[-d+1,0]}$, we have*

$$\mathcal{Y}^{\perp_{\leq 0}} = (\operatorname{Fac}_d \mathcal{Y})^{\perp_{\leq 0}} = (\operatorname{Fac}_d \mathcal{Y})^{\perp}, \tag{1.1}$$

$${}^{\perp_{\leq 0}}\mathcal{Y} = {}^{\perp_{\leq 0}}(\operatorname{Sub}_d \mathcal{Y}) = {}^{\perp}(\operatorname{Sub}_d \mathcal{Y}). \tag{1.2}$$

Proof. We only show the equalities in (1.1), since those in (1.2) can be proved similarly. By [99, Lemma 1.19], we have $\mathcal{Y}^{\perp_{\leq 0}} \subseteq (\mathsf{Fac}_d\,\mathcal{Y})^{\perp_{\leq 0}}$. It is clear that $(\mathsf{Fac}_d\,\mathcal{Y})^{\perp_{\leq 0}} \subseteq (\mathsf{Fac}_d\,\mathcal{Y})^{\perp}$. Thus, it suffices to show $(\mathsf{Fac}_d\,\mathcal{Y})^{\perp} \subseteq \mathcal{Y}^{\perp_{\leq 0}}$.

Let $X \in (\mathsf{Fac}_d\,\mathcal{Y})^{\perp}$. For any $Y \in \mathcal{Y}$ and any $j \geq 0$, consider the truncation triangle of $Y[j]$:

$$\sigma_{\leq -d}(Y[j]) \to Y[j] \to \sigma_{\geq -d+1}(Y[j]) \to \sigma_{\leq -d}(Y[j])[1].$$

Applying $\mathsf{Hom}(-, X)$ to this triangle yields an isomorphism

$$\mathsf{Hom}(Y[j], X) \cong \mathsf{Hom}(\sigma_{\geq -d+1}(Y[j]), X).$$

By Lemma 1.1.3, we have $\sigma_{\geq -d+1}(Y[j]) \in \mathsf{Fac}_{d+1}(\mathcal{Y}) \subseteq \mathsf{Fac}_d(\mathcal{Y})$, and hence

$$\mathsf{Hom}(\sigma_{\geq -d+1}(Y[j]), X) = 0.$$

It follows that $\mathsf{Hom}(Y[j], X) = 0$ for all $j \geq 0$, so $X \in \mathcal{Y}^{\perp_{\leq 0}}$. □

We obtain the following description of the smallest positive torsion class containing a given subcategory, which will be used in Sections 3.3 and 3.4.

Proposition 1.2.8 *For any subcategory $\mathcal{Y}$ of $\mathcal{D}^{[-d+1,0]}$, the following hold.*

(1) *The subcategory ${}^{\perp_{\leq 0}}\mathcal{Y}$ is a positive torsion class. Its corresponding torsion-free class $({}^{\perp_{\leq 0}}\mathcal{Y})^{\perp}$ is the smallest positive torsion-free class containing $\mathcal{Y}$.*

(2) *The subcategory $\mathcal{Y}^{\perp_{\leq 0}}$ is a positive torsion-free class. Its corresponding torsion class ${}^{\perp}(\mathcal{Y}^{\perp_{\leq 0}})$ is the smallest positive torsion class containing $\mathcal{Y}$.*

Proof. We only prove (1), while the proof of (2) is similar. By Lemma 1.2.7, we have ${}^{\perp_{\leq 0}}\mathcal{Y} = {}^{\perp}(\mathsf{Sub}_d\,\mathcal{Y})$, and hence Lemma 1.2.6 implies that ${}^{\perp_{\leq 0}}\mathcal{Y}$ is a torsion class.

Let $Z \in \mathsf{Fac}_d({}^{\perp_{\leq 0}}\mathcal{Y})$. By definition, there exist triangles

$$Z_i \to X_i \to Z_{i-1},\ 1 \leq i \leq d,$$

with $Z_0 = Z$, $Z_1, \ldots, Z_d \in \mathcal{D}^{[-d+1,0]}$, and $X_1, \ldots, X_d \in {}^{\perp_{\leq 0}}\mathcal{Y}$. For any $Y \in \mathcal{Y}$ and any $j \leq 0$, applying $\mathsf{Hom}(-, Y[j])$ to these triangles yields exact sequences for $1 \leq i \leq d$:

$$\mathsf{Hom}(X_i[1], Y[j]) \to \mathsf{Hom}(Z_i[1], Y[j]) \to \mathsf{Hom}(Z_{i-1}, Y[j]) \to \mathsf{Hom}(X_i, Y[j]).$$

Since $X_i \in {}^{\perp_{\leq 0}}\mathcal{Y}$, the first and fourth terms vanish. It follows that

$$\mathsf{Hom}(Z_i[1], Y[j]) \cong \mathsf{Hom}(Z_{i-1}, Y[j]).$$

Iterating this for $i = 1, \ldots, d$, we obtain

$$\mathsf{Hom}(Z, Y[j]) \cong \mathsf{Hom}(Z_d, Y[j-d]) = 0,$$

which shows that $Z \in {}^{\perp_{\leq 0}}\mathcal{Y}$. Therefore, ${}^{\perp_{\leq 0}}\mathcal{Y}$ is closed under d-factors. By Proposition 1.2.3, it follows that ${}^{\perp_{\leq 0}}\mathcal{Y}$ is a positive torsion class, and hence $({}^{\perp_{\leq 0}}\mathcal{Y}, ({}^{\perp_{\leq 0}}\mathcal{Y})^{\perp})$ is a positive torsion pair.

Finally, by definition, we have $\mathcal{Y} \subseteq ({}^{\perp_{\leq 0}}\mathcal{Y})^{\perp}$. If $\mathcal{F}$ is any positive torsion-free class containing $\mathcal{Y}$, then ${}^{\perp}\mathcal{F} \subseteq {}^{\perp_{\leq 0}}\mathcal{Y}$, which implies

$$({}^{\perp_{\leq 0}}\mathcal{Y})^{\perp} \subseteq ({}^{\perp}\mathcal{F})^{\perp} = \mathcal{F}.$$

Hence, $({}^{\perp_{\leq 0}}\mathcal{Y})^{\perp}$ is indeed the smallest positive torsion-free class containing $\mathcal{Y}$. □

Remark 1.2.9 If $\mathcal{Y}$ is already closed under d-factors, Lemma 1.2.7 yields the simplified identity $\mathcal{Y}^{\perp} = \mathcal{Y}^{\perp_{\leq 0}}$. Hence, by Proposition 1.2.8, $\mathcal{Y}^{\perp}$ is a positive torsion-free class, and its corresponding torsion class ${}^{\perp}(\mathcal{Y}^{\perp})$ is the smallest positive torsion class containing $\mathcal{Y}$.

1.3 s-torsion/torsion-free classes

The third notion of torsion-like classes is given by s-torsion classes, introduced in [1] for extriangulated categories with negative first extensions.

Definition 1.3.1 ([1, Definition 3.1]) An *s-torsion pair* $(\mathcal{T},\mathcal{F})$ in $\mathcal{D}^{[-d+1,0]}$ is a pair of subcategories such that

1. $\mathcal{D}^{[-d+1,0]} = \mathcal{T} * \mathcal{F}$;
2. $\mathrm{Hom}(\mathcal{T},\mathcal{F}) = 0$;
3. $\mathrm{Hom}(\mathcal{T},\mathcal{F}[-1]) = 0$.

Here, $\mathcal{T}$ is called an *s-torsion class*, and $\mathcal{F}$ an *s-torsion-free class*.

By [1, Proposition 3.2] and Lemma 1.2.3, every s-torsion pair is a positive torsion pair. Hence, each s-torsion pair is uniquely determined by its s-torsion class (or equivalently, by its s-torsion-free class). The following property relating positive torsion classes and s-torsion classes will be crucial in Section 3.4.

Lemma 1.3.2 *Let $\mathcal{T}$ be an s-torsion class and $\mathcal{T}'$ a positive torsion class in $\mathcal{D}^{[-d+1,0]}$. Suppose that $\mathcal{T}' \subseteq \mathcal{T}$ and $\mathcal{T} \cap \mathcal{T}'^{\perp} = 0$. Then $\mathcal{T} = \mathcal{T}'$.*

Proof. We aim to show that $\mathcal{T}'^{\perp} \subseteq \mathcal{T}^{\perp}$, which immediately implies $\mathcal{T} = \mathcal{T}'$. Let $X \in \mathcal{T}'^{\perp}$. Since $\mathcal{T}$ is an s-torsion class, there exists a triangle

$$X_1 \to X \to X_2$$

with $X_1 \in \mathcal{T}$ and $X_2 \in \mathcal{T}^{\perp} \subseteq \mathcal{T}'^{\perp}$. Since $\mathcal{T}'$ is a positive torsion class, by Lemma 1.2.3, $\mathcal{T}'^{\perp}$ is closed under truncation of cocones. Hence $X_1 \in \mathcal{T}'^{\perp}$. Together with the assumption $\mathcal{T} \cap \mathcal{T}'^{\perp} = 0$, this forces $X_1 = 0$. Therefore, $X \cong X_2 \in \mathcal{T}^{\perp}$, as desired. □

Every s-torsion (resp. s-torsion-free) class is d-FAE (resp. d-SAE) closed. The bijections in Proposition 1.1.6 restrict to a bijection between s-torsion pairs in $\mathcal{D}^{[-d+1,0]}$ and bounded t-structures lying between $\mathcal{D}^{\leq -d}$ and $\mathcal{D}^{\leq 0}$.

Proposition 1.3.3 *There is a bijection between*

- *the set of s-torsion pairs in $\mathcal{D}^{[-d+1,0]}$, and*
- *the set of bounded t-structures $(\mathcal{C}^{\leq 0},\mathcal{C}^{\geq 0})$ on $\mathcal{D}$ satisfying $\mathcal{D}^{\leq -d} \subseteq \mathcal{C}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$,*

given by

$$(\mathcal{T},\mathcal{F}) \mapsto (\mathcal{D}^{\leq -d} * \mathcal{T}, \mathcal{F}[1] * \mathcal{D}^{\geq 0})$$

with inverse

$$(\mathcal{C}^{\leq 0},\mathcal{C}^{\geq 0}) \mapsto (\mathcal{C}^{\leq 0} \cap \mathcal{D}^{[-d+1,0]}, \mathcal{C}^{\geq 1} \cap \mathcal{D}^{[-d+1,0]}).$$

Moreover, let $(\mathcal{T},\mathcal{F})$ be an s-torsion pair in $\mathcal{D}^{[-d+1,0]}$, and let $(\mathcal{C}^{\leq 0},\mathcal{C}^{\geq 0})$ be the corresponding t-structure on $\mathcal{D}$. Then the following hold.

(1) $(\mathcal{F}[d],\mathcal{T})$ is an s-torsion pair in the d-extended heart $\mathcal{C}^{[-d+1,0]}$ of $(\mathcal{C}^{\leq 0},\mathcal{C}^{\geq 0})$.

(2) The bounded t-structure on $\mathcal{D}$ corresponding to $(\mathcal{F}[d],\mathcal{T})$ is $(\mathcal{D}^{\leq -d},\mathcal{D}^{\geq -d})$.

(3) $\mathcal{T} = \mathcal{D}^{[-d+1,0]} \cap \mathcal{C}^{[-d+1,0]}$ and $\mathcal{F} = \mathcal{D}^{[-d+1,0]} \cap \big(\mathcal{C}^{[-d+1,0]}[-d]\big)$.

(4) For any $X \in \mathcal{C}^{[-d+1,0]}$, the truncation triangle of X

$$\sigma_{\leq -d}(X) \to X \to \sigma_{\geq -d+1}(X)$$

coincides with the canonical triangle of X with respect to the s-torsion pair $(\mathcal{F}[d],\mathcal{T})$, that is, $\sigma_{\leq -d}(X) \in \mathcal{F}[d]$ and $\sigma_{\geq -d+1}(X) \in \mathcal{T}$.

Proof. This proposition agrees with [99, Proposition 1.9 and Theorem 1.12], except for the assertions in (2) and (3). The assertion in (2) can be verified by a straightforward computation:

$$\mathcal{C}^{\leq -d} * \mathcal{F}[d] = (\mathcal{D}^{\leq -d} * \mathcal{T})[d] * \mathcal{F}[d] = \mathcal{D}^{\leq 0}[d] = \mathcal{D}^{\leq -d}.$$

For (3), it follows from (1) that $\mathcal{T} \subseteq \mathcal{D}^{[-d+1,0]} \cap \mathcal{C}^{[-d+1,0]}$. Conversely, let $X \in \mathcal{D}^{[-d+1,0]} \cap \mathcal{C}^{[-d+1,0]} = (\mathcal{T} * \mathcal{F}) \cap (\mathcal{F}[d] * \mathcal{T})$. Then there exists a triangle

$$Y \to X \xrightarrow{f} Z$$

with $Y \in \mathcal{T}$ and $Z \in \mathcal{F}$. Since $\operatorname{Hom}(\mathcal{F}[d], \mathcal{F}) = 0 = \operatorname{Hom}(\mathcal{T}, \mathcal{F})$, we must have $f = 0$. Hence X is a direct summand of Y, and therefore $X \in \mathcal{T}$. This proves the first equality in (3). The second one can be shown similarly.

For (4), take the canonical triangle of X with respect to $(\mathcal{F}[d], \mathcal{T})$

$$X' \to X \to X'',$$

with $X' \in \mathcal{F}[d]$ and $X'' \in \mathcal{T}$. It follows from (2) that $X' \in \mathcal{D}^{\leq d}$ and $X'' \in \mathcal{D}^{\geq -d+1}$. Therefore, this triangle coincides with the truncation triangle of X. ☐

By Theorem 5.2 in the introduction, the above bijection induces the following correspondence, where d-$\mathcal{H}'$ denotes $\mathcal{H}'[d-1] * \mathcal{H}'[d-2] * \cdots * \mathcal{H}'$.

Corollary 1.3.4 *There is a bijection between*

- *the set of s-torsion pairs in $\mathcal{D}^{[-d+1,0]}$, and*
- *the set of bounded hearts on $\mathcal{D}$ lying in $\mathcal{D}^{[-d,0]}$,*

given by

$$(\mathcal{T}, \mathcal{F}) \mapsto (\mathcal{D}^{\leq -d} * \mathcal{T}) \cap (\mathcal{F}[1] * \mathcal{D}^{\geq 0})$$

with inverse

$$\mathcal{H}' \mapsto (d\text{-}\mathcal{H}' \cap \mathcal{D}^{[-d+1,0]}, d\text{-}\mathcal{H}'[-d] \cap \mathcal{D}^{[-d+1,0]}).$$

The following proposition describes the equivalence between contravariantly finite d-FAE closed subcategories, contravariantly finite positive torsion classes, and s-torsion classes, under standard finiteness assumptions on $\mathcal{D}$.

Proposition 1.3.5 *Suppose that $\mathcal{D}$ is K-linear,* Hom*-finite, and Krull-Schmidt. Let $\mathcal{T} \subseteq \mathcal{D}^{[-d+1,0]}$ be d-FAE closed. Then the following are equivalent.*

(1) $\mathcal{T}$ is a contravariantly finite d-FAE closed subcategory.

(2) $\mathcal{T}$ is a contravariantly finite positive torsion class.

(3) $\mathcal{T}$ is an s-torsion class.

In particular, the notions of functorially finite d-FAE closed subcategories, functorially finite positive torsion classes, and functorially finite s-torsion classes coincide.

Proof. As noted before, every s-torsion class is a positive torsion class, and every positive torsion class is d-FAE closed. Moreover, by definition, every s-torsion class is automatically contravariantly finite. Hence, the implications "(2) $\Longrightarrow$ (1)" and "(3) $\Longrightarrow$ (2)" hold.

To show "(1) $\Longrightarrow$ (3)", that is, to verify that any contravariantly finite d-FAE closed subcategory $\mathcal{T}$ is an s-torsion class, we check the defining conditions in two steps.

Step 1: Hom$(\mathcal{T}, \mathcal{T}^{\perp}[-1]) = 0$. Let $X \in \mathcal{T}$ and $Y \in \mathcal{T}^{\perp}$. Consider the truncation triangle

$$\sigma_{\leq -d}(X[1]) \to X[1] \to \sigma_{\geq -d+1}(X[1])$$

Applying Hom$(-, Y)$ to this triangle yields an exact sequence

$$\mathrm{Hom}(\sigma_{\geq -d+1}(X[1]), Y) \to \mathrm{Hom}(X[1], Y) \to \mathrm{Hom}(\sigma_{\leq -d}(X[1]), Y).$$

By Lemma 1.1.3, we have $\sigma_{\geq -d+1}(X[1]) \in \mathcal{T}$. Since $Y \in \mathcal{T}^{\perp}$, the first and last terms vanish, which implies Hom$(X[1], Y) = 0$ and hence Hom$(X, Y[-1]) = 0$.

Step 2: $\mathcal{T} * \mathcal{T}^{\perp} = \mathcal{D}^{[-d+1,0]}$. Let $Z \in \mathcal{D}^{[-d+1,0]}$. Since $\mathcal{T}$ is contravariantly finite and $\mathcal{D}$ is K-linear, Hom-finite, and Krull-Schmidt, there exists a minimal right $\mathcal{T}$-approximation $f : T \to Z$ of Z. By the triangulated version of Wakamatsu's Lemma ([62, Lemma 2.1]), we have Hom$(\mathcal{T}, C(f)) = 0$. It remains to show that $C(f) \in \mathcal{D}^{[-d+1,0]}$, which will imply $C(f) \in \mathcal{T}^{\perp}$ and complete the proof.

By an argument similar to that in the proof of Lemma 1.1.4(2), we obtain the following commutative diagram of triangles:

$$\begin{tikzcd}
T \arrow[r] \arrow[d, equal] & T' \arrow[r] \arrow[d, "f'"] & \sigma_{\leq -d}(C(f)) \arrow[d] \\
T \arrow[r, "f"] & Z \arrow[r] \arrow[d] & C(f) \arrow[d] \\
 & \sigma_{\geq -d+1}(C(f)) \arrow[r, equal] & \sigma_{\geq -d+1}(C(f))
\end{tikzcd}$$

with $T' \in \mathrm{Fac}_d(\mathcal{T})$. Since $\mathcal{T}$ is closed under d-factors, we have $T' \in \mathcal{T}$. Moreover, since f is a right $\mathcal{T}$-approximation of Z, so is the induced morphism $f' : T' \to Z$. As f is right minimal, the morphism f' decomposes as a direct sum of f and a morphism of the form $V \to 0$. Hence, we have an isomorphism $\sigma_{\geq -d+1}(C(f)) \cong C(f) \oplus V[1]$, which implies that $C(f) \in \mathcal{D}^{[-d+1,0]}$. $\square$

Chapter 2

Silting objects, torsion classes, and cotorsion classes

In this chapter, we will use the previously defined notion of torsion classes to get bijections with $(d+1)$-term silting objects in $\mathsf{K}^b(\operatorname{proj}\Lambda)$. In order to do this, we will also need to study the interaction of torsion classes in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$ with cotorsion classes in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. The results in this chapter appeared in the work [41].

2.1 Bijection between torsion and cotorsion pairs

2.1.1 Truncation functors

The goal of this section is to generalize the relation between $K^{[-1,0]}(\operatorname{proj}\Lambda)$ and $\operatorname{mod}\Lambda$ to higher truncations of the homotopy category. Fix $d \geq 1$. Then we note that the truncation functor $\sigma_{\geq -d+1} : \mathcal{D}^b(\operatorname{mod}\Lambda) \to \mathcal{D}^b(\operatorname{mod}\Lambda)$ gives a restriction

$$\sigma_{\geq -d+1} : \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda) \to \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda).$$

The following proposition recovers, for $d = 1$, some known properties of the cohomology functor $\mathsf{H}^0 : \mathsf{K}^{[-1,0]}(\operatorname{proj}\Lambda) \to \operatorname{mod}\Lambda$.

Proposition 2.1.1 *The truncation functor*

$$\sigma_{\geq -d+1} : \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda) \to \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$$

induces an equivalence between the categories $\frac{\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)}{\operatorname{add}\Lambda[d]}$ *and* $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$.

Proof. We divide the proof into three steps.

Step 1. The functor $\sigma_{\geq -d+1}$ is essentially surjective on $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$: Let $M \in \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$. Then, using the fact that the inclusion of $\mathsf{K}^{b,-}(\operatorname{proj}\Lambda)$ into $\mathcal{D}^b(\operatorname{mod}\Lambda)$ is an equivalence, M is isomorphic to a complex of projectives bounded above, say, $(\cdots \to P^{-d} \to \cdots P^{-1} \to P^0 \to 0)$. Then $P' = (\cdots \to 0 \to P^{-d} \to \cdots \to P^0 \to 0) \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$ and the following map gives a quasi-isomorphism between M and $\sigma_{\geq -d+1}(P')$.

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-d-1} \arrow[r, "\delta^{-d-1}"] \arrow[d] & P^{-d} \arrow[r, "\delta^{-d}"] \arrow[d, "\pi"] & P^{-d+1} \arrow[r] \arrow[d, "id"] & \cdots \arrow[r] & P^0 \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & \frac{P^{-d}}{\mathsf{Ker}(\delta^{-d})} \arrow[r, "\overline{\delta^{-d}}"'] & P^{-d+1} \arrow[r] & \cdots \arrow[r] & P^0 \arrow[r] & 0
\end{tikzcd}$$

Step 2. The functor $\sigma_{\geq -d+1}$ is full: Let $X, Y \in K^{[-d,0]}(\operatorname{proj}\Lambda)$ and $f : \sigma_{\geq -d+1}X \to \sigma_{\geq -d+1}Y \in \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$. By definition,

$$\sigma_{\geq -d+1}X = 0 \to \frac{X^{-d}}{\mathrm{Ker}(\delta_X^{-d})} \xrightarrow{\overline{\delta_X^{-d}}} X^{-d+1} \to \cdots \to X^0 \to 0$$

and

$$\sigma_{\geq -d+1}Y = 0 \to \frac{Y^{-d}}{\mathrm{Ker}(\delta_Y^{-d})} \xrightarrow{\overline{\delta_Y^{-d}}} Y^{-d+1} \to \cdots \to Y^0 \to 0.$$

Let $\cdots \to P^{-d-2} \to P^{-d-1} \to X^{-d} \to 0$ and $\cdots \to Q^{-d-2} \to Q^{-d-1} \to Y^{-d} \to 0$ be projective resolutions of $\frac{X^{-d}}{\mathrm{Ker}(\delta_X^{-d})}$ and $\frac{Y^{-d}}{\mathrm{Ker}(\delta_Y^{-d})}$ respectively. Set $X' = \cdots \to P^{-d-1} \to X^{-d} \to X^{-d+1} \to \cdots \to X^0 \to 0$ and $Y' = \cdots \to Q^{-d-1} \to Y^{-d} \to Y^{-d+1} \to \cdots \to Y^0 \to 0$. Then there exist canonical quasi-isomorphisms $i_X : X' \to \sigma_{\geq -d+1}X$ and $i_Y : Y' \to \sigma_{\geq -d+1}Y$. Using the equivalence between $\mathrm{K}^{b,-}(\operatorname{proj}\Lambda)$ and $\mathcal{D}^b(\operatorname{mod}\Lambda)$ again, we get that $g := (i_Y)^{-1} f i_X \in \mathrm{K}^{b,-}(\operatorname{proj}\Lambda)$. Let $g' : X \to Y$ be the map shown below.

$$\begin{tikzcd}
0 \arrow[r] & X^{-d} \arrow[r] \arrow[d, "g^{-d}"] & X^{-d+1} \arrow[r] \arrow[d, "g^{-d+1}"] & \cdots \arrow[r] & X^0 \arrow[r] \arrow[d, "g^0"] & 0 \\
0 \arrow[r] & Y^{-d} \arrow[r] & Y^{-d+1} \arrow[r] & \cdots \arrow[r] & Y^0 \arrow[r] & 0
\end{tikzcd}$$

Then $\sigma_{\geq -d+1}(g') = \sigma_{\geq -d+1}(g) = \sigma_{\geq -d+1}(f) = f$.

Step 3. The kernel of $\sigma_{\geq -d+1}$ is the set of maps factoring through $\operatorname{add}\Lambda[d]$: Clearly, if a map factors through $\operatorname{add}\Lambda[d]$, then it lies in $\mathrm{Ker}(\sigma_{\geq -d+1})$. Let $f : X \to Y \in \mathrm{Ker}(\sigma_{\geq -d+1})$. As before, we can define $X', Y' \in \mathrm{K}^{b,-}(\operatorname{proj}\Lambda)$ with canonical quasi-isomorphisms $i_X : X' \to \sigma_{\geq -d+1}X$ and $i_Y : Y' \to \sigma_{\geq -d+1}Y$. Since a morphism of modules can be lifted to a morphism of their projective resolutions, we can find maps $g^i : P^i \to Q^i$ for $i \leq -d-1$ such that the following is a morphism h of complexes.

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-d-1} \arrow[r] \arrow[d, "g^{-d-1}"] & X^{-d} \arrow[r] \arrow[d, "f^{-d}"] & X^{-d+1} \arrow[r] \arrow[d, "f^{-d+1}"] & \cdots \arrow[r] & X^0 \arrow[r] \arrow[d, "f^0"] & 0 \\
\cdots \arrow[r] & Q^{-d-1} \arrow[r] & Y^{-d} \arrow[r] & Y^{-d+1} \arrow[r] & \cdots \arrow[r] & Y^0 \arrow[r] & 0
\end{tikzcd}$$

Again it can be checked that $\sigma_{\geq -d+1}(f) \circ i_X = i_Y \circ h$, which implies that $h = 0$. Thus, there exist maps $k^i : X^i \to Y^{i-1}$ such that the homotopy relation holds.

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-d-1} \arrow[r] \arrow[d, "g^{-d-1}"] & X^{-d} \arrow[r] \arrow[d, "f^{-d}"] \arrow[dl, dashed, "k^{-d}"'] & X^{-d+1} \arrow[r] \arrow[d, "f^{-d+1}"] \arrow[dl, dashed, "k^{-d+1}"'] & \cdots \arrow[r] \arrow[dl, dashed] & X^0 \arrow[r] \arrow[d, "f^0"] \arrow[dl, dashed] & 0 \\
\cdots \arrow[r] & Q^{-d-1} \arrow[r, "\delta_Y^{-d-1}"'] & Y^{-d} \arrow[r] & Y^{-d+1} \arrow[r] & \cdots \arrow[r] & Y^0 \arrow[r] & 0
\end{tikzcd}$$

Consider the following two maps.

$$\begin{tikzcd}
0 \arrow[r] & X^{-d} \arrow[r] \arrow[d, "id"] & X^{-d+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & X^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^{-d} \arrow[r] \arrow[d, "\delta_Y^{-d-1} \circ k^{-d}"'] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y^{-d} \arrow[r] & Y^{-d+1} \arrow[r] & \cdots \arrow[r] & Y^0 \arrow[r] & 0
\end{tikzcd}$$

The diagram below shows that the composition of the above two maps is equal to f in the homotopy category.

$$\begin{tikzcd}
0 \arrow[r] & X^{-d} \arrow[r] \arrow[d, "f^{-d}-\delta_Y^{-d-1}\circ k^{-d}"'] & X^{-d+1} \arrow[r] \arrow[d, "f^{-d+1}"] \arrow[dl, dashed, "k^{-d+1}"] & \cdots \arrow[r] \arrow[dl, dashed] & X^0 \arrow[r] \arrow[d, "f^0"] \arrow[dl, dashed] & 0 \\
0 \arrow[r] & Y^{-d} \arrow[r] & Y^{-d+1} \arrow[r] & \cdots \arrow[r] & Y^0 \arrow[r] & 0
\end{tikzcd}$$

Thus, f factors through add $\Lambda[d]$, and we get an induced equivalence

$$\sigma_{\geq -d+1} : \frac{\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)}{\operatorname{add}\Lambda[d]} \to \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda).$$

□

Our next goal is to show that we can lift conflations in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$ to conflations in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$ across this equivalence (we talk about the conflations coming from the respective extriangulated structures). We will need the following lemma about inflations in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$.

Lemma 2.1.2 *Let $f : X \to Y$ be a morphism in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$ such that f^{-d} is a split mono. Then f is an inflation.*

Proof. We need to show that $C(f) \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. By definition, $C(f)$ is given as

$$0 \to X^{-d} \xrightarrow{\begin{bmatrix}\delta_X^{-d} \\ f^{-d}\end{bmatrix}} X^{-d+1} \oplus Y^{-d} \to \cdots X^0 \oplus Y_1 \to Y^0.$$

Since f^{-d} is a split mono, so is $\begin{bmatrix}\delta_X^{-d} \\ f^{-d}\end{bmatrix}$. Thus, $C(f)$ is isomorphic to

$$0 \to X^{-d} \xrightarrow{\begin{bmatrix}1_{X^{-d}} \\ 0\end{bmatrix}} X^{-d} \oplus Z \to \cdots X^0 \oplus Y_1 \to Y^0$$

for some $Z \in \operatorname{proj}\Lambda$. Thus $C(f) =$

$$\begin{tikzcd}
0 \arrow[r] & X^{-d} \arrow[r, "1_{X^{-d}}"] & X^{-d} \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

$$\oplus$$

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & Z \arrow[r] & \cdots \arrow[r] & X^0 \oplus Y^1 \arrow[r] & Y^0
\end{tikzcd}$$

Hence, it is quasi-isomorphic to

$$(Z \to \cdots \to X^0 \oplus Y^1 \to Y^0) \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda).$$

□

Proposition 2.1.3 *Let $\epsilon : X \xrightarrow{f} Y \xrightarrow{g} Z$ be a conflation in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$. Then there exists a conflation $\epsilon' : X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$ in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$ such that $\sigma_{\geq -d+1}\epsilon' \cong \epsilon$.*

Proof. Using the equivalence between $\mathsf{K}^{b,-}(\operatorname{proj}\Lambda)$ and $\mathcal{D}^b(\operatorname{mod}\Lambda)$, we can assume that f is of the following form:

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-d} \arrow[r] \arrow[d, "f^{-d}"'] & P^{-d+1} \arrow[r] \arrow[d, "f^{-d+1}"] & \cdots \arrow[r] & P^0 \arrow[r] \arrow[d, "f^0"] & 0 \\
\cdots \arrow[r] & Q^{-d} \arrow[r] & Q^{-d+1} \arrow[r] & \cdots \arrow[r] & Q^0 \arrow[r] & 0
\end{tikzcd}$$

This implies that $\epsilon \cong X \xrightarrow{f} Y \xrightarrow{g} C(f)$. Set

$$X' = t_{\geq -d}(X) = 0 \to P^{-d} \to P^{-d+1} \to \cdots \to P^0 \to 0,$$

and

$$Y'' = t_{\geq -d}(Y) = 0 \to Q^{-d} \to Q^{-d+1} \to \cdots \to Q^0 \to 0,$$

and $f'' = t_{\geq -d}(f)$. Set $Y' := Y'' \oplus P^{-d}[d]$. Let $i : X' \to P^{-d}[d]$ be the map that is identity in degree $-d$, and 0 in the others. Then the map $f' := \begin{bmatrix} f'' \\ i \end{bmatrix} : X' \to Y'$ is an inflation in $\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda)$ by Lemma 2.1.2 because it is a split mono in degree $-d$. We claim that $\sigma_{\geq -d+1}(X' \xrightarrow{f'} Y' \xrightarrow{g'} C(f')) \cong \epsilon$. Using the previous proposition, we know that $\sigma_{\geq -d+1}(f') \cong \sigma_{\geq -d+1}(f'') \cong f$. By definition,

$$\begin{aligned}
\sigma_{\geq -d+1} C(f') &= 0 \to \frac{P^{-d+1} \oplus Q^{-d} \oplus P^{-d}}{\mathsf{Ker}(\delta^{-d}_{C(f)}) \oplus P^{-d}} \to P^{-d+2} \oplus Q^{-d+1} \to \cdots Q^0 \to 0 \\
&\cong 0 \to \frac{P^{-d+1} \oplus Q^{-d}}{\mathsf{Ker}(\delta^{-d}_{C(f)})} \to P^{-d+2} \oplus Q^{-d+1} \to \cdots Q^0 \to 0,
\end{aligned}$$

where $\delta^{-d}_{C(f)}$ denotes the $-d$ boundary map of $C(f)$. Since $C(f) \in \mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$, the following map gives a quasi-isomorphism between $\sigma_{\geq -d+1} C(f')$ and $C(f)$.

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-d} \oplus Q^{-d-1} \arrow[r] \arrow[d] & P^{-d+1} \oplus Q^{-d} \arrow[r] \arrow[d, "\pi"] & \cdots \arrow[r] & Q^0 \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & \frac{P^{-d+1} \oplus Q^{-d}}{\mathsf{Ker}(\delta^{-d}_{C(f)})} \arrow[r] & \cdots \arrow[r] & Q^0 \arrow[r] & 0
\end{tikzcd}$$

Denoting the above map by $i_{C(f')}$, it can be easily checked that the following diagram commutes,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_{X'}"'] & Y \arrow[r, "g"] \arrow[d, "i_{Y'}"'] & C(f) \arrow[d, "i_{C(f')}"] \\
\sigma_{\geq -d+1} X' \arrow[r, "\sigma_{\geq -d+1} f'"'] & \sigma_{\geq -d+1} Y' \arrow[r, "\sigma_{\geq -d+1} g'"'] & \sigma_{\geq -d+1} C(f')
\end{tikzcd}$$

where $i_{X'}$ and $i_{Y'}$ are the canonical quasi-isomorphisms defined in the previous proposition.

Setting $\epsilon' = X' \xrightarrow{f'} Y' \to C(f')$, we get that $\sigma_{\geq -d+1} \epsilon' \cong \epsilon$. ☐

The following corollary gives the dual of the above two propositions for the homotopy category of injectives.

Corollary 2.1.4 1. *The truncation functor $\sigma_{\leq 0}$ induces an equivalence of categories between $\frac{\mathsf{K}^{[-d+1,1]}(\mathsf{inj}\,\Lambda)}{\mathsf{inj}\,\Lambda[-1]}$ and $\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$.*

2. *Given a conflation $\epsilon : X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$, there exists a conflation $\epsilon' : X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$ in $\mathsf{K}^{[-d+1,1]}(\mathsf{inj}\,\Lambda)$ such that $\sigma_{\leq 0} \epsilon' \cong \epsilon$.*

Our next goal is to characterize the extensions between any two objects in $\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda)$ in terms of the morphisms between their images under the truncation functors. This will be used in the next section to give a correspondence between cotorsion pairs in $\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda)$ and torsion pairs in $\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$.

Theorem 2.1.5 ([47, § I.4.6]) *Let $X, Y \in \mathsf{K}^b(\mathsf{proj}\,\Lambda)$. Then*

$$\mathsf{Hom}_{\mathsf{K}^b(\mathsf{proj}\,\Lambda)}(X, Y) \cong \mathsf{D}\,\mathsf{Hom}_{\mathcal{D}^b(\mathsf{mod}\,\Lambda)}(Y, \nu X),$$

where ν denotes the Nakayama functor.

Lemma 2.1.6 *Let $X, Y \in \mathsf{K}^{[-d,0]}(\operatorname{proj} \Lambda)$. Then*

$$\mathbb{E}(X,Y) \cong \mathsf{D}\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod}\Lambda)}(\sigma_{\geq -d+1}Y, \Sigma^{-1}\sigma_{\leq -1}\nu X).$$

Proof. Using Theorem 2.1.5, we get that

$$\mathbb{E}(X,Y) \cong \operatorname{Hom}_{\mathsf{K}^b(\operatorname{proj}\Lambda)}(X, \Sigma Y) \cong \mathsf{D}\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod}\Lambda)}(\Sigma Y, \nu X).$$

The triangle $\sigma_{\leq -d-1}\Sigma Y \to \Sigma Y \to \sigma_{\geq -d}\Sigma Y$ gives the exact sequence

$$\mathbb{E}^{-1}(\sigma_{\leq -d-1}\Sigma Y, \nu X) \to \operatorname{Hom}(\sigma_{\geq -d}\Sigma Y, \nu X) \to \operatorname{Hom}(\Sigma Y, \nu X) \to \operatorname{Hom}(\sigma_{\leq -d-1}\Sigma Y, \nu X).$$

By comparing the support of the cohomology of $\sigma_{\leq -d-1}\Sigma Y$ and νX, we see that $\mathbb{E}^{-1}(\sigma_{\leq -d-1}\Sigma Y, \nu X)$ and $\operatorname{Hom}(\sigma_{\leq -d-1}\Sigma Y, \nu X)$ vanish. Thus,

$$\operatorname{Hom}(\sigma_{\geq -d}\Sigma Y, \nu X) \cong \operatorname{Hom}(\Sigma Y, \nu X).$$

Now, the triangle $\sigma_{\leq -1}\nu X \to \nu X \to \sigma_{\geq 0}\nu X$ gives the exact sequence

$$\begin{tikzcd}
\mathbb{E}^{-1}(\sigma_{\geq -d}\Sigma Y, \sigma_{\geq 0}\nu X) \arrow[r] & \operatorname{Hom}(\sigma_{\geq -d}\Sigma Y, \sigma_{\leq -1}\nu X) \arrow[dl, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
\operatorname{Hom}(\sigma_{\geq -d}\Sigma Y, \nu X) \arrow[r] & \operatorname{Hom}(\sigma_{\geq -d}\Sigma Y, \sigma_{\geq 0}\nu X).
\end{tikzcd}$$

Again, we see that $\mathbb{E}^{-1}(\sigma_{\geq -d}\Sigma Y, \sigma_{\geq 0}\nu X)$ and $\operatorname{Hom}(\sigma_{\geq -d}\Sigma Y, \sigma_{\geq 0}\nu X)$ vanish. Thus,

$$\begin{aligned}
\operatorname{Hom}(\sigma_{\geq -d}\Sigma Y, \nu X) &\cong \operatorname{Hom}(\sigma_{\geq -d}\Sigma Y, \sigma_{\leq -1}\nu X) \\
&\cong \operatorname{Hom}(\Sigma\sigma_{\geq -d+1}Y, \sigma_{\leq -1}\nu X) \\
&\cong \operatorname{Hom}(\sigma_{\geq -d+1}Y, \Sigma^{-1}\sigma_{\leq -1}\nu X).
\end{aligned}$$

□

We need the following technical lemmas for the subsequent sections.

Lemma 2.1.7 *Let $f : X \to Y \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. Then*

$$\sigma_{\geq -d+1}C(f) \cong \sigma_{\geq -d+1}C(\sigma_{\geq -d+1}f).$$

Proof. Suppose f is of the following form.

$$\begin{tikzcd}
X^{-d} \arrow[r, "\delta_X^{-d}"] \arrow[d, "f^{-d}"] & \cdots \arrow[r] & X^{-1} \arrow[r, "\delta_X^{-1}"] \arrow[d, "f^{-1}"] & X^0 \arrow[d, "f^0"] \\
Y^{-d} \arrow[r, "\delta_Y^{-d}"'] & \cdots \arrow[r] & Y^{-1} \arrow[r, "\delta_Y^{-1}"'] & Y^0
\end{tikzcd}$$

Then, by definition, $\sigma_{\geq -d+1}C(f)$ is

$$\frac{X^{-d+1} \oplus Y^{-d}}{\operatorname{Ker}(\begin{bmatrix} -\delta_X^{-d+1} & 0 \\ f^{-d+1} & \delta_Y^{-d} \end{bmatrix})} \longrightarrow X^{-d+2} \oplus Y^{-d+1} \longrightarrow \cdots \longrightarrow X^0 \oplus Y^{-1} \longrightarrow Y^0.$$

On the other hand, $\sigma_{\geq -d+1}C(\sigma_{\geq -d+1}f)$ is given as

$$\frac{X^{-d+1} \oplus \frac{Y^{-d}}{\operatorname{Ker}(\delta_Y^{-d})}}{\operatorname{Ker}(\begin{bmatrix} -\delta_X^{-d+1} & 0 \\ f^{-d+1} & \overline{\delta_Y^{-d}} \end{bmatrix})} \longrightarrow X^{-d+2} \oplus Y^{-d+1} \longrightarrow \cdots \longrightarrow X^0 \oplus Y^{-1} \longrightarrow Y^0.$$

The following map gives an isomorphism between the two since the map $\overline{(x,y)} \mapsto \overline{(x,\overline{y})}$ is an isomorphism.

$$\begin{tikzcd}
\frac{X^{-d+1}\oplus Y^{-d}}{\text{Ker}(\begin{bmatrix} -\delta_X^{-d+1} & 0 \\ f^{-d+1} & \delta_Y^{-d} \end{bmatrix})} \arrow[r] \arrow[d, "\overline{(x,y)}\mapsto\overline{(x,\overline{y})}"] & X^{-d+2} \oplus Y^{-d+1} \arrow[r] \arrow[d, "id"] & \cdots \arrow[r] & X^0 \oplus Y^{-1} \arrow[r] \arrow[d, "id"] & Y^0 \arrow[d, "id"] \\
\frac{X^{-d+1}\oplus \frac{Y^{-d}}{\text{Ker}(\delta_Y^{-d})}}{\text{Ker}(\begin{bmatrix} -\delta_X^{-d+1} & 0 \\ f^{-d+1} & \overline{\delta_Y^{-d}} \end{bmatrix})} \arrow[r] & X^{-d+2} \oplus Y^{-d+1} \arrow[r] & \cdots \arrow[r] & X^0 \oplus Y^{-1} \arrow[r] & Y^0
\end{tikzcd}$$

□

Recall that a full subcategory $\mathcal{X}$ of a category $\mathcal{C}$ is called *contravariantly finite* (resp. *covariantly finite*) if for all $C \in \mathcal{C}$, there exists $f_C : X_C \to C$ (resp. $g_C : C \to X_C$) with $X_C \in \mathcal{X}$ such that for all $X \in \mathcal{X}$, $\text{Hom}(X, X_C) \xrightarrow{\text{Hom}(X,f_C)} \text{Hom}(X, C)$ (resp. $\text{Hom}(X_C, X) \xrightarrow{\text{Hom}(g_C,X)} \text{Hom}(C, X)$) is surjective.

Lemma 2.1.8 *Let $\mathcal{Y} \subseteq$ $\mathsf{K}^{[-d,0]}(\text{proj}\,\Lambda)$ be an additive subcategory containing* add $\Lambda[d]$*. Then $\mathcal{Y}$ is covariantly (resp. contravariantly) finite in* $\mathsf{K}^{[-d,0]}(\text{proj}\,\Lambda)$ *if and only if $\sigma_{\geq -d+1}\mathcal{Y}$ is covariantly (resp. contravariantly) finite in* $\mathcal{D}^{[-d+1,0]}(\text{proj}\,\Lambda)$.

Proof. We will only prove the case for covariant finiteness. The case for contravariant finiteness will be dual. ($\Longrightarrow$) Follows from the fact that the functor $\sigma_{\geq -d+1}$ is full and essentially surjective. ($\Longleftarrow$) Let

$Z \in \mathsf{K}^{[-d,0]}(\text{proj}\,\Lambda)$ and $f' : \sigma_{\geq -d+1}Z \to Y'$ be a left $\sigma_{\geq -d+1}\mathcal{Y}$-approximation of $\sigma_{\geq -d+1}Z$. Since $\sigma_{\geq -d+1}$ is full and essentially surjective, there exists $f : Z \to Y \in \mathsf{K}^{[-d,0]}(\text{proj}\,\Lambda)$ such that $\sigma_{\geq -d+1}f = f'$. Let $g : Z \to P$ be a left add $\Lambda[d]$-approximation of Z, which exists since add $\Lambda[d]$ is functorially finite. We claim that $\begin{bmatrix} f \\ g \end{bmatrix} : Z \to Y \oplus P$ is a left $\mathcal{Y}$-approximation of Z.

Let $f_1 : Z \to Y_1$ be a morphism with $Y_1 \in \mathcal{Y}$. Since f' is an approximation, there exists $g_1 : Y \to Y_1$ such that $f'\sigma_{\geq -d+1}g_1 = \sigma_{\geq -d+2}f_1$. Therefore, the map $g_1 f - f_1$ factors through some $P' \in$ add $\Lambda[d]$ as shown below.

$$\begin{tikzcd}
Z \arrow[rr, "g_1 f - f_1"] \arrow[dr, "h"'] & & Y_1 \\
& P' \arrow[ur, "k"'] &
\end{tikzcd}$$

Since $g : Z \to P$ is a left add $\Lambda[d]$-approximation of Z, there exists some $l : P \to P'$ such that $lg = h$. This gives the following commutative triangle and we are done.

$$\begin{tikzcd}
Z \arrow[r, "\begin{bmatrix} f \\ g \end{bmatrix}"] \arrow[dr, "f_1"'] & Y \oplus P \arrow[d, "\begin{bmatrix} g_1 & -kl \end{bmatrix}"] \\
& Y_1
\end{tikzcd}$$

□

Lemma 2.1.9 *For any morphism $f : A \to B \in$* $\mathsf{K}^{[-d,0]}(\text{proj}\,\Lambda)$*, there exists an inflation $f' : A \to B' \in$* $\mathsf{K}^{[-d,0]}(\text{proj}\,\Lambda)$ *such that $\sigma_{\geq -d+1}f \cong \sigma_{\geq -d+1}f'$.*

Proof. Let $f' : A \to A^{-d}[d] \oplus B$ be the following morphism.

$$\begin{tikzcd}
A^{-d} \arrow[r, "\delta_A^{-d}"] \arrow[d, swap, "\begin{bmatrix} id \\ f^{-d} \end{bmatrix}"] & A^{-d+1} \arrow[r] \arrow[d, "f^{-d}"] & \cdots \arrow[r] \arrow[d] & A^{-1} \arrow[r] \arrow[d, "f^{-1}"] & A^{0} \arrow[d, "f^{0}"] \\
A^{-d} \oplus B^{-d} \arrow[r, swap, "{[0 \ \ \delta_B^{-d}]}"] & B^{-d+1} \arrow[r] & \cdots \arrow[r] & B^{-1} \arrow[r] & B^{0}
\end{tikzcd}$$

Then, clearly, $\sigma_{\geq -d+1} f \cong \sigma_{\geq -d+1} f'$. Since f'^{-d} is a split mono, using Lemma 2.1.2, we get that f' is an inflation in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. □

2.1.2 From cotorsion pairs to torsion pairs

We will use the following definition of cotorsion pairs for an extriangulated category as appeared in [79]. We fix $\mathcal{C} := \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$.

Definition 2.1.10 Let $\mathcal{X}, \mathcal{Y}$ be full subcategories of $\mathcal{C}$.

1. The pair $(\mathcal{X}, \mathcal{Y})$ is called a *cotorsion pair* if
   (a) $\mathbb{E}(x, \mathcal{Y}) = 0$ if and only if $x \in \mathcal{X}$;
   (b) $\mathbb{E}(\mathcal{X}, y) = 0$ if and only if $y \in \mathcal{Y}$.
2. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called *hereditary* if $\mathbb{E}^k(\mathcal{X}, \mathcal{Y}) = 0$ for all $k \geq 2$.
3. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called *complete* if for each $c \in \mathcal{C}$
   (Ca) there exists a conflation $c \to y \to x$ with $x \in \mathcal{X}$ and $y \in \mathcal{Y}$;
   (Cb) there exists a conflation $y' \to x' \to c$ with $x' \in \mathcal{X}$ and $y' \in \mathcal{Y}$.

Remark 2.1.11 Using [79, Remark 4.4], to check if a pair of full subcategories $(\mathcal{X}, \mathcal{Y})$ is a complete cotorsion pair, it is enough to check the weaker conditions $\mathbb{E}(\mathcal{X}, \mathcal{Y}) = 0$, (Ca), and (Cb).

Remark 2.1.12 The definition of cotorsion pairs for extriangulated categories was first introduced in [79], and that of hereditary cotorsion pairs in [3]. However, we would like to point out that cotorsion pairs in the sense of [79, 3] are complete cotorsion pairs in our sense. Moreover, the authors in [3] also define the notion of s-cotorsion pairs which, in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$, turns out to be equivalent to the notion of hereditary cotorsion pairs (Lemma 2.1.13).

Following [92], for a cotorsion pair $(\mathcal{X}, \mathcal{Y})$, we will call $\mathcal{Y}$ a *cotorsion class*, and $\mathcal{X}$ a *cotorsion-free class*. We will call a cotorsion class *hereditary* (resp. *complete*) if the associated cotorsion pair is hereditary (resp. complete). We will denote the poset of cotorsion classes in $\mathcal{C}$ under inclusion by $\operatorname{cotors}\mathcal{C}$. This is isomorphic to the poset of cotorsion pairs in $\mathcal{C}$ under the order

$$(X_1, Y_1) \preceq (X_2, Y_2) :\iff X_1 \supseteq X_2 \iff Y_1 \subseteq Y_2.$$

Furthermore, we will denote the subposets of hereditary, complete, and complete and hereditary cotorsion classes/pairs by $\text{h-cotors}\,\mathcal{C}$, $\text{c-cotors}\,\mathcal{C}$, and $(\text{c},\text{h})\text{-cotors}\,\mathcal{C}$, respectively.

Lemma 2.1.13 *Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. Then the following are equivalent.*

1. *$(\mathcal{X}, \mathcal{Y})$ is a hereditary cotorsion pair.*
2. *$\mathbb{E}^2(\mathcal{X}, \mathcal{Y}) = 0$.*
3. *$\mathcal{X}$ is closed under cocones of deflations.*
4. *$\mathcal{Y}$ is closed under cones of inflations.*

Proof. ($1 \implies 2$) Clear.
($2 \implies 3$, $2 \implies 4$) Follows from [3, Lemma 3.2]. Their proof does not use the completeness axioms assumed by the authors.
($3 \implies 1$) Let $X \in \mathcal{X}$. Then there is a conflation $\Sigma^{-1}t_{\leq -1}X \to t_{\geq 0}X \to X$ with $t_{\geq 0}X, X \in \mathcal{X}$. Since $\mathcal{X}$ is closed under cocones, $t_{\leq 0}\Sigma^{-1}X \cong \Sigma^{-1}t_{\leq -1}X \in \mathcal{X}$. Repeating the above argument, we conclude that $t_{\leq 0}\Sigma^{-k}X \in \mathcal{X}$ for all $k \geq 1$.

Using the triangle $t_{\geq 1}\Sigma^{-k}X \to \Sigma^{-k}X \to t_{\leq 0}\Sigma^{-k}X$, we get that $\mathbb{E}^{k+1}(X, \mathcal{Y}) \cong \mathbb{E}(\Sigma^{-k}X, \mathcal{Y}) = 0$.
($4 \implies 1$) Dual to the above proof. □

The next theorem provides us with a way to link cotorsion pairs to torsion pairs using the truncation functor defined in the previous section. We will need the following generalization of Wakamatsu Lemma to extriangulated categories.

Lemma 2.1.14 ([73, Lemma 3.1]) *Let $\mathcal{A}$ be an extension closed subcategory of an extriangulated category $\mathcal{C}$ and $t \xrightarrow{f} a \to b$ be a conflation such that f is the minimal left $\mathcal{A}$-approximation of $t \in \mathcal{C}$. Then $b \in {}^{\perp_1}\mathcal{A}$.*

The following theorems 2.1.15 and 2.1.16 generalise the existing bijection between torsion classes in mod Λ and cotorsion classes in $\mathsf{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ ([84, Proposition 3.2], [39]).

Theorem 2.1.15 *Let $\mathcal{Y}$ be a cotorsion class in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. Then the truncation $\sigma_{\geq -d+1}\mathcal{Y}$ is a torsion class in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$. Moreover, if $\mathcal{Y}$ is hereditary (resp. complete, resp. hereditary and contravariantly finite), then $\sigma_{\geq -d+1}\mathcal{Y}$ is positive (resp. covariantly finite, resp. an s-torsion class).*

Proof. Set $\mathcal{Y}' := \sigma_{\geq -d+1}\mathcal{Y}$ and $\mathcal{X}' = (\sigma_{\geq -d+1}\mathcal{Y})^{\perp}$. We need to show that ${}^{\perp}\mathcal{X}' = \mathcal{Y}'$. By definition, $\mathcal{Y}' \subseteq {}^{\perp}\mathcal{X}'$. Let $Y' \in {}^{\perp}\mathcal{X}'$. Then there exists $Y \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$ such that $\sigma_{\geq -d+1}(Y) \cong Y'$. We want to show that $Y \in \mathcal{Y}$.

Note that, since $({}^{\perp_1}\mathcal{Y}, \mathcal{Y})$ is a cotorsion pair,

$$\begin{aligned} {}^{\perp_1}\mathcal{Y} &= \{X \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda) \mid \mathbb{E}(X, \mathcal{Y}) = 0\} \\ &= \{X \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda) \mid \operatorname{Hom}(\sigma_{\geq -d+1}\mathcal{Y}, \Sigma^{-1}\sigma_{\leq -1}\nu X) = 0\} \quad \text{(by Lemma 2.1.6)} \\ &= \{X \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda) \mid \Sigma^{-1}\sigma_{\leq -1}\nu X \in \mathcal{X}'\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{Y} &= \{Z \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda) \mid \mathbb{E}({}^{\perp_1}\mathcal{Y}, Z) = 0\} \\ &= \{Z \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda) \mid \operatorname{Hom}(\sigma_{\geq -d+1}Z, \Sigma^{-1}\sigma_{\leq -1}\nu\, {}^{\perp_1}\mathcal{Y}) = 0\} \quad \text{(by Lemma 2.1.6).} \end{aligned}$$

Since, for all $X \in {}^{\perp_1}\mathcal{Y}$, $\Sigma^{-1}\sigma_{\leq -1}\nu X \in \mathcal{X}'$, and $Y' \in {}^{\perp}\mathcal{X}'$, we get that $Y \in \mathcal{Y}$. Thus, $(\mathcal{Y}', \mathcal{X}')$ is a torsion pair in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$.

Now, suppose that $\mathcal{Y}$ is hereditary, which is equivalent to $\mathcal{Y}$ being stable under cones (Lemma 2.1.13). Let $g : Z \to Z' \in \mathcal{Y}'$. Then there exists $f : Y \to Y' \in \mathcal{Y}$ such that $\sigma_{\geq -d+1}f \cong g$. Using Lemma 2.1.9, we can assume that $C(f) \in \mathsf{K}^{[-d]}(\operatorname{proj}\Lambda)$. Since $\mathcal{Y}$ is closed under cones, $C(f) \in \mathcal{Y}$. Using Lemma 2.1.7, $\sigma_{\geq -d+1}C(g) \cong \sigma_{\geq -d+1}C(f) \in \mathcal{Y}'$. Using Lemma 1.2.3, we conclude that $\mathcal{Y}'$ is a positive torsion class.

Next, if $\mathcal{Y}$ is complete, then it is covariantly finite in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. This implies that $\mathcal{Y}'$ is covariantly finite in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$ (Lemma 2.1.8).

Finally, suppose $\mathcal{Y}$ is hereditary and contravariantly finite. Then $\mathcal{Y}'$ is positive and contravariantly finite in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$. Let $Z \in \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$ and $f : Y' \to Z$ a minimal right $\mathcal{Y}'$-approximation of Z. Consider the triangles $Y' \xrightarrow{f} Z \xrightarrow{g} C(f)$ and $\sigma_{\leq -d}C(f) \to C(f) \xrightarrow{\pi} \sigma_{\geq -d+1}C(f)$ in $\mathcal{D}^b(\operatorname{mod}\Lambda)$. Then $\mathrm{H}^0(g)$ is an epimorphism and $\mathrm{H}^0(\pi)$ is an isomorphism, which implies that $\mathrm{H}^0(\pi \circ g)$ is an epimorphism. Thus, the cocone of $\pi \circ g$, say Y, is in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$, and we get a conflation $Y \xrightarrow{f'} Z \xrightarrow{\pi \circ g} \sigma_{\geq -d+1}C(f)$ in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$. We will show that $Y \in \mathcal{Y}'$.

Using the octahedral axiom as shown below,

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, equal] & C(f) \arrow[r] \arrow[d, "\pi"] & \Sigma Y' \arrow[d, "\Sigma h"] \\
Z \arrow[r, "\pi\circ g"] & \sigma_{\geq -d+1}C(f) \arrow[r] \arrow[d] & \Sigma Y \arrow[d] \\
 & \Sigma\sigma_{\leq -d}C(f) \arrow[r, equal] & \Sigma\sigma_{\leq -d}C(f)
\end{tikzcd}$$

we get a triangle $Y' \xrightarrow{h} Y \to \sigma_{\leq -d}C(f)$. Let $F \in \mathcal{X}'$. Then we have an exact sequence $\mathsf{Hom}(\sigma_{\leq -d}C(f), F) \to \mathsf{Hom}(Y, F) \to \mathsf{Hom}(Y', F) = 0$. Since $F \in \mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$, $\mathsf{Hom}(\sigma_{\leq -d}C(f), F) = 0$. Thus, $\mathsf{Hom}(Y, F) = 0$ and $Y \in {}^{\perp}\mathcal{X}' = \mathcal{Y}'$.

Since $Y \in \mathcal{Y}'$ and f is a minimal right $\mathcal{Y}'$-approximation of Z, there exists a map $h' : Y \to Y'$ such that $fh' = f'$, which implies that $fh'h = f'h = f$. Since f is minimal, we get that $f' : Y \to Z \cong [0, f] : Y'' \oplus Y' \to Z$ for some $Y'' \in \mathcal{D}^b(\mathsf{mod}\,\Lambda)$. This implies that $C(f') \cong \sigma_{\geq -d+1}C(f) \cong C(f) \oplus \Sigma Y''$. This gives that $C(f) \in \mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$. Using Lemma 2.1.14 for the conflation $Y' \xrightarrow{f} Z \xrightarrow{g} C(f)$ in the extriangulated category $\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$, we get that $C(f) \in \mathcal{X}'$. Hence, $(\mathcal{Y}', \mathcal{X}')$ is an s-torsion pair in $\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$. ☐

The above theorem gives us a poset homomorphism Φ : cotors $\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda) \to$ tors $\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$.

2.1.3 The bijections

We now show that the above map Φ is a poset isomorphism.

Theorem 2.1.16 *The map* Φ : cotors $\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda) \to$ tors $\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$ *defined as* $\mathcal{Y} \mapsto \sigma_{\geq -d+1}\mathcal{Y}$ *is a poset isomorphism. Moreover, it restricts to an isomorphism between the following subposets.*

1. *The set of hereditary cotorsion classes and the set of positive torsion classes.*

2. *The set of complete cotorsion classes and the set of covariantly finite torsion classes.*

3. *The set of hereditary contravariantly finite cotorsion classes and the set of* s*-torsion classes.*

Proof. We claim that the poset homomorphism

$$\begin{aligned}\Psi : \text{tors}\,\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda) &\to \text{cotors}\,\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda)\\ \mathcal{T} &\mapsto \{Y \in \mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda) \mid \sigma_{\geq -d+1}Y \in \mathcal{T}\}\end{aligned}$$

is the inverse of Φ. Let $\mathcal{T} \in$ tors $\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$. Set $\mathcal{Y} = \{Y \in \mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda) \mid \sigma_{\geq -d+1}Y \in \mathcal{T}\}$, and $\mathcal{X} := {}^{\perp_1}\mathcal{Y}$.

Step 1. The map Ψ is well-defined: For this, we need to prove that $\mathcal{X}^{\perp_1} = \mathcal{Y}$. By definition, $\mathcal{Y} \subseteq \mathcal{X}^{\perp_1}$. Using Lemma 2.1.6, we know that $\mathcal{X} = \{X \in \mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda) \mid \Sigma^{-1}\sigma_{\leq -1}\nu X \in \mathcal{T}^{\perp}\}$, and $\mathcal{X}^{\perp_1} = \{Z \in \mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda) \mid \mathsf{Hom}(\sigma_{\geq -d+1}Z, \Sigma^{-1}\sigma_{\leq -1}\nu\mathcal{X}) = 0\}$. Let $Y \in \mathcal{X}^{\perp_1}$. In order to show that $Y \in \mathcal{Y}$, we need to show that $\sigma_{\geq -d+1}Y \in \mathcal{T} = {}^{\perp}(\mathcal{T}^{\perp})$. Let $F \in \mathcal{T}^{\perp}$. Then $\Sigma F \in \mathcal{D}^{[-d,-1]}(\mathsf{mod}\,\Lambda)$. Using Corollary 2.1.4, there exists $I \in \mathsf{K}^{[-d+1,1]}(\mathsf{inj}\,\Lambda)$ such that $\sigma_{\leq 0}I \cong F$. Thus, $\sigma_{\leq -1}\Sigma I \cong \Sigma F$. This implies that $\Sigma^{-1}\sigma_{\leq -1}\nu(\nu^{-1}\Sigma I) = F \in \mathcal{T}^{\perp}$. Thus, $\nu^{-1}\Sigma I \in \mathcal{X}$. Since $\mathcal{Y} \in \mathcal{X}^{\perp_1}$, $\mathsf{Hom}(\sigma_{\geq -d+1}Y, \Sigma^{-1}\sigma_{\leq -1}\nu(\nu^{-1}\Sigma I)) = \mathsf{Hom}(\sigma_{\geq -d+1}Y, F) = 0$. This implies that $\sigma_{\geq -d+1}Y \in \mathcal{T}$, and we are done.

Step 2. The maps Φ and Ψ are mutual inverses: This follows from the fact that $\sigma_{\geq -d+1}$ is full and essentially surjective with the kernel add $\Lambda[d]$ contained in any cotorsion class, and that $\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda)$ is a Hom-finite Krull-Schmidt category.

Step 3. The map Ψ sends positive torsion classes to hereditary cotorsion classes: Suppose $\mathcal{T}$ is positive. Then, using Lemma 1.2.3, for all morphisms $g \in \mathcal{T}$, $\sigma_{\geq -d+1}C(g) \in \mathcal{T}$. Let $y \xrightarrow{f} y' \to y''$ be a conflation in $\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda)$ with $y, y' \in \mathcal{Y}$. By definition, $\sigma_{\geq -d+1}y$, $\sigma_{\geq -d+1}y' \in \mathcal{T}$. Moreover, $\sigma_{\geq -d+1}(C(\sigma_{\geq -d+1}f)) \in \mathcal{T}$.

Thus, it follows from Lemma 2.1.7 that $\sigma_{\geq -d+1}y'' \cong \sigma_{\geq -d+1}(C(\sigma_{\geq -d+1}f)) \in \mathcal{T}$ and $y'' \in \mathcal{Y}$. This shows that $\mathcal{Y}$ is closed under cones, and hence, by Lemma 2.1.13, $\mathcal{Y}$ is a hereditary cotorsion class.

Step 4. The map Ψ sends covariantly finite torsion classes to complete cotorsion classes: Suppose that $\mathcal{T}$ is covariantly finite. Then using Lemma 2.1.8, $\mathcal{Y}$ is also covariantly finite. Using Step 1, we know that $\mathcal{X}^{\perp_1} = \mathcal{Y}$ which implies that $\mathcal{Y}$ is extension closed.

Let $C \in \mathsf{K}^{[-d+1,0]}(\operatorname{proj}\Lambda)$. Let $f: C \to Y$ be the minimal $\mathcal{Y}$-approximation of C. This gives a conflation $C \xrightarrow{f} Y \to C(f)$ in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. Since $\mathcal{Y}$ is closed under extensions using Lemma 2.1.14, we get that $C(f) \in {}^{\perp_1}\mathcal{Y} = \mathcal{X}$. Thus, for every $C \in \mathsf{K}^{[-d+1,0]}(\operatorname{proj}\Lambda)$, there exists a conflation $C \to Y \to C(f)$ with $Y \in \mathcal{Y}$ and $C(f) \in \mathcal{X}$.

Finally, let $C \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. We have triangles

$$\Sigma^{-1}t_{\leq -d}C \to t_{\geq -d+1}C \to C$$

and

$$\Sigma^{-1}t_{\leq -1}C \to t_{\geq 0}C \to C$$

with $\Sigma^{-1}t_{\leq -1}C, t_{\geq -d+1}C \in \mathsf{K}^{[-d+1,0]}(\operatorname{proj}\Lambda)$. Using the above argument, we know that there exist conflations $t_{\geq -d+1}C \to y \to x$ and $\Sigma^{-1}t_{\leq -1}C \to y' \to x'$ in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$ with $y, y' \in \mathcal{Y}$ and $x, x' \in \mathcal{X}$. Using the octahedral axiom for the following diagrams,

$$\begin{tikzcd}
 & \Sigma^{-1}x \arrow[r, equal] \arrow[d] & \Sigma^{-1}x \arrow[d] & \\
\Sigma^{-1}t_{\leq -d}C \arrow[r] \arrow[d, equal] & t_{\geq -d+1}C \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & t_{\leq -d}C \arrow[d, equal] \\
\Sigma^{-1}t_{\leq -d}C \arrow[r] & y \arrow[r] \arrow[d] & d \arrow[r] \arrow[d] & t_{\leq -d}C \\
 & x \arrow[r, equal] & x &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \Sigma^{-1}x' \arrow[r, equal] \arrow[d] & \Sigma^{-1}x' \arrow[d] & \\
\Sigma^{-1}C \arrow[r] \arrow[d, equal] & \Sigma^{-1}t_{\leq -1}C \arrow[r] \arrow[d] & t_{\geq 0}C \arrow[r] \arrow[d] & C \arrow[d, equal] \\
\Sigma^{-1}C \arrow[r] & y' \arrow[r] \arrow[d] & e \arrow[r] \arrow[d] & C \\
 & x' \arrow[r, equal] & x' &
\end{tikzcd}$$

we get conflations $C \to d \to x$ and $y' \to e \to C$. Since $y, t_{\leq -d}C \in \mathcal{Y}$, and $\mathcal{Y}$ is closed under extensions, we get that $d \in \mathcal{Y}$. Similarly, since $x', t_{\geq 0}C \in \mathcal{X}$, and $\mathcal{X}$ is closed under extensions, we get that $e \in \mathcal{X}$. Thus, using Remark 2.1.11, $(\mathcal{X}, \mathcal{Y})$ is a complete cotorsion pair.

Step 5. The map Ψ sends s-torsion classes to hereditary contravariantly finite cotorsion classes: Suppose $\mathcal{T}$ is an s-torsion class. Since an s-torsion class is positive and contravariantly finite, by Step 3 and Lemma 2.1.8, $\mathcal{Y}$ is hereditary and contravariantly finite in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. ☐

2.2 Silting objects, cotorsion pairs, and torsion pairs

The main goal of this section is to prove the following theorem.

Theorem 2.2.1 *The map Φ defined in Theorem 2.1.15 and the maps ψ and ψ' given in Definition 2.2.2 and Definition 2.2.5 restrict to give the following commutative triangle of poset isomorphisms.*

$$\begin{tikzcd}
(d+1)\text{-}\operatorname{silt}\Lambda \arrow[r, "\psi"] \arrow[dr, "\psi'"'] & (\mathsf{c},\mathsf{h})\text{-cotors}\,\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda) \arrow[d, "\Phi"] \\
 & (\mathsf{f},\mathsf{p})\text{-tors}\,\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)
\end{tikzcd} \tag{2.1}$$

2.2.1 Definition of ψ

Definition 2.2.2 Let M be a $(d+1)$-term silting object in $\mathsf{K}^b(\operatorname{proj}\Lambda)$. Define

$$\begin{aligned}
\mathcal{X}_M &:= \text{the smallest full subcategory of } \mathsf{K}^b(\operatorname{proj}\Lambda) \text{ containing } \{\Sigma^m M \mid m \leq 0\} \\
&\quad\ \text{and closed under extensions and summands,} \\
\mathcal{Y}_M &:= \text{the smallest full subcategory of } \mathsf{K}^b(\operatorname{proj}\Lambda) \text{ containing } \{\Sigma^m M \mid m \geq 0\} \\
&\quad\ \text{and closed under extensions and summands.}
\end{aligned}$$

Set $\psi(M) := (\mathcal{X}_M \cap \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda), \mathcal{Y}_M \cap \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda))$.

Using [69, Theorem 6.1], we know that the map $M \mapsto (\mathcal{X}_M, \mathcal{Y}_M) =: \Theta(M)$ gives a poset isomorphism from $\operatorname{silt}\mathsf{K}^b(\operatorname{proj}\Lambda)$ to the poset of bounded co-t-structures on $\mathsf{K}^b(\operatorname{proj}\Lambda)$. Moreover, the inverse of this map is given by taking the additive generator of the co-heart of a bounded co-t-structure. Set $\mathcal{X}'_M := \mathcal{X}_M \cap \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$ and $\mathcal{Y}'_M = \mathcal{Y}_M \cap \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$.

Lemma 2.2.3 *The map ψ given in Definition 2.2.2 induces an injective poset homomorphism $\psi : (d+1)\text{-}\operatorname{silt}\Lambda \to (\mathsf{c},\mathsf{h})\text{-cotors}\,\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$.*

Proof. We first show that ψ is well-defined. Since $(\mathcal{X}_M, \mathcal{Y}_M)$ is a co-t-structure on $\mathsf{K}^b(\operatorname{proj}\Lambda)$, $\mathbb{E}(\mathcal{X}'_M, \mathcal{Y}'_M) = 0$.

Moreover, since M is a $(d+1)$-term silting object, and $\mathsf{K}^{\leq 0}(\operatorname{proj}\Lambda)$ and $\mathsf{K}^{\geq -d}(\operatorname{proj}\Lambda)$ are full subcategories of $\mathsf{K}^b(\operatorname{proj}\Lambda)$ closed under extensions and summands containing $\{\Sigma^m M \mid m \geq 0\}$ and $\{\Sigma^m M \mid m \leq 0\}$ respectively, we get that $\mathcal{Y}_M \subseteq \mathsf{K}^{\leq 0}(\operatorname{proj}\Lambda)$ and $\mathcal{X}_M \subseteq \mathsf{K}^{\geq -d}(\operatorname{proj}\Lambda)$.

Let $C \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. Then there exists a triangle $T \xrightarrow{f} \Sigma C \xrightarrow{g} F$ with $T \in \mathcal{X}_M$ and $F \in \Sigma\mathcal{Y}_M$. Thus, $F \in \mathsf{K}^{\leq -1}(\operatorname{proj}\Lambda)$ and $T \in \mathsf{K}^{\geq -d}(\operatorname{proj}\Lambda)$. Since $\mathsf{K}^{\leq 0}(\operatorname{proj}\Lambda)$ is extension closed, using the triangle $\Sigma^{-1}F \to T \to \Sigma C$, we get that $T \in \mathsf{K}^{\leq 0}(\operatorname{proj}\Lambda)$. Similarly, using the triangle $\Sigma C \to F \to \Sigma T$, we get that $F \in \mathsf{K}^{\geq -d-1}(\operatorname{proj}\Lambda)$. Thus, we get a triangle $C \to \Sigma^{-1}F \to T$ with $\Sigma^{-1}F \in \mathcal{Y}'_M$ and $T \in \mathcal{X}'_M$.

Let $C \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. Then there exists a triangle $T \xrightarrow{f} C \xrightarrow{g} F$ with $T \in \mathcal{X}_M$ and $F \in \Sigma\mathcal{Y}_M$. Thus, $F \in \mathsf{K}^{\leq -1}(\operatorname{proj}\Lambda)$ and $T \in \mathsf{K}^{\geq -d}(\operatorname{proj}\Lambda)$. Since $\mathsf{K}^{\geq -d-1}(\operatorname{proj}\Lambda)$ is extension closed, using the triangle $X \to F \to \Sigma T$, we get that $F \in \mathsf{K}^{\geq -d-1}(\operatorname{proj}\Lambda)$. Thus, $\Sigma^{-1}F \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. Similarly, using the triangle $\Sigma^{-1}F \to T \to C$, we get that $T \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. Thus, we get a triangle $\Sigma^{-1}F \to T \to C$ with $\Sigma^{-1}F \in \mathcal{Y}'_M$ and $T \in \mathcal{X}'_M$.

Thus, using Remark 2.1.11, $(\mathcal{X}'_M, \mathcal{Y}'_M)$ is a complete cotorsion pair in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. Since $\mathcal{X}_M$ and $\mathcal{Y}_M$ are closed under cocones and cones respectively, so are $\mathcal{X}'_M$ and $\mathcal{Y}'_M$, which implies that $(\mathcal{X}'_M, \mathcal{Y}'_M)$ is hereditary by Lemma 2.1.13.

Now, suppose $M \leq N$ in $(d+1)$-silt Λ. Then $\mathcal{X}_M \supseteq \mathcal{X}_N$ which implies that $\mathcal{X}'_M \supseteq \mathcal{X}'_N$. Thus, ψ is a poset homomorphism.

We finally show that ψ is injective. Let $M, M' \in (d+1)\text{-}\operatorname{silt}\Lambda$ such that $\mathcal{Y}'_M = \mathcal{Y}'_{M'}$. Using the bijection between silting objects and co-t-structures [69, Theorem 6.1], it is enough to show that $\mathcal{Y}_M = \mathcal{Y}_{M'}$. Suppose not. Without loss of generality, let $Y \in \mathcal{Y}_M \setminus \mathcal{Y}_{M'}$. Since $\mathcal{Y}_M \subseteq \mathsf{K}^{\leq 0}(\operatorname{proj}\Lambda)$, $Y \in \mathsf{K}^{\leq 0}(\operatorname{proj}\Lambda)$. We have a triangle $\Sigma^{-1}t_{\leq -d-1}Y \to t_{\geq -d}Y \to Y$. Since $\mathcal{X}_M, \mathcal{X}_{M'} \subseteq \mathsf{K}^{\geq -d}(\operatorname{proj}\Lambda)$, we get that $\Sigma^{-1}t_{\leq -d-1}Y \in \mathcal{X}_M^{\perp_1} \cap \mathcal{X}_{M'}^{\perp_1} = \mathcal{Y}_M \cap \mathcal{Y}_{M'}$. Thus, $t_{\geq -d}Y \in \mathcal{Y}_M$, since $\mathcal{Y}_M$ is closed under extensions. Since $t_{\geq -d}Y \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$, we get that $t_{\geq -d}Y \in \mathcal{Y}'_M = \mathcal{Y}'_{M'}$. Since $\mathcal{Y}_{M'}$ is closed under cones, we get that $Y \in \mathcal{Y}_{M'}$, a contradiction. Hence $\mathcal{Y}_M = \mathcal{Y}_{M'}$ which implies that $M \cong M'$.

□

Theorem 2.2.4 *The map ψ is an isomorphism of posets.*

Proof. We will show this by constructing an inverse poset homomorphism χ : (c, h)-cotors $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda) \to (d+1)$- silt Λ of ψ.

Let $\mathcal{Y}'$ be a complete hereditary cotorsion class in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. Set $\mathcal{X}' = {}^{\perp_1}\mathcal{Y}'$. Define $C(\mathcal{X}')$ to be the smallest full subcategory of $\mathsf{K}^b(\operatorname{proj}\Lambda)$ closed under summands and extensions containing $\mathcal{X}'$ and $\mathsf{K}^{\geq 0}(\operatorname{proj}\Lambda)$ and $C(\mathcal{Y}')$ to be the smallest full subcategory of $\mathsf{K}^b(\operatorname{proj}\Lambda)$ closed under summands and extensions containing $\mathcal{Y}'$ and $\mathsf{K}^{\leq -d}(\operatorname{proj}\Lambda)$. Note that $C(\mathcal{X}') \subseteq \mathsf{K}^{\geq -d}(\operatorname{proj}\Lambda)$ and $C(\mathcal{Y}') \subseteq \mathsf{K}^{\leq 0}(\operatorname{proj}\Lambda)$. We will show that $(C(\mathcal{X}'), C(\mathcal{Y}'))$ is a bounded co-t-structure on $\mathsf{K}^b(\operatorname{proj}\Lambda)$ whose co-heart lies in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$, and set $\chi(\mathcal{Y}') := \Theta^{-1}(C(\mathcal{X}'), C(\mathcal{Y}'))$.

Using the fact that both $\mathbb{E}(\mathcal{X}', \mathcal{Y}')$ and $\mathbb{E}(\mathcal{X}', \mathsf{K}^{\leq -d}(\operatorname{proj}\Lambda))$ vanish, and that $\mathcal{X}'^{\perp_1}$ in $\mathsf{K}^b(\operatorname{proj}\Lambda)$ is closed under extensions and summands, we get that

$$\mathbb{E}(\mathcal{X}', C(\mathcal{Y}')) = 0.$$

Similarly, $\mathbb{E}(\mathsf{K}^{\geq 0}(\operatorname{proj}\Lambda), C(\mathcal{Y}')) = 0$. Thus, $\mathcal{X}', \mathsf{K}^{\geq 0}(\operatorname{proj}\Lambda) \subseteq {}^{\perp_1}C(\mathcal{Y}')$. Since ${}^{\perp_1}C(\mathcal{Y}')$ is also closed under extensions and summands, we get that

$$\mathbb{E}(C(\mathcal{X}'), C(\mathcal{Y}')) = 0.$$

We next show that $\Sigma^{-1}C(\mathcal{X}') \subseteq C(\mathcal{X}')$. Note that it is enough to show that $\Sigma^{-1}\mathcal{X}' \subseteq C(\mathcal{X}')$ and $\Sigma^{-1}\mathsf{K}^{\geq 0}(\operatorname{proj}\Lambda) \subseteq C(\mathcal{X}')$. Clearly,

$$\Sigma^{-1}\mathsf{K}^{\geq 0}(\operatorname{proj}\Lambda) = \mathsf{K}^{\geq 1}(\operatorname{proj}\Lambda) \subseteq \mathsf{K}^{\geq 0}(\operatorname{proj}\Lambda) \subseteq C(\mathcal{X}').$$

Let $X \in \mathcal{X}'$. We have the following triangle in $\mathsf{K}^b(\operatorname{proj}\Lambda)$.

$$t_{\geq 1}\Sigma^{-1}X \to \Sigma^{-1}X \to t_{\leq 0}\Sigma^{-1}X \to t_{\geq 0}X \to X$$

Since $X \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$, $t_{\geq 0}X \in {}^{\perp_1}\mathcal{Y}' = \mathcal{X}'$. Since $\mathcal{X}'$ is closed under cocones, we get that $t_{\leq 0}\Sigma^{-1}X \in \mathcal{X}'$. Since $\Sigma^{-1}X$ is an extension of $t_{\geq 1}\Sigma^{-1}X \in \mathsf{K}^{\geq 0}(\operatorname{proj}\Lambda)$ and $t_{\leq 0}\Sigma^{-1}X \in \mathcal{X}'$, we get that $\Sigma^{-1}X \in C(\mathcal{X}')$. Thus $\Sigma^{-1}C(\mathcal{X}') \subseteq C(\mathcal{X}')$. Dually, we can show that $\Sigma C(\mathcal{Y}') \subseteq C(\mathcal{Y}')$.

Finally, let $C \in \mathsf{K}^b(\operatorname{proj}\Lambda)$. Set $C' := t_{\geq -d}C$ and $C'' := t_{\leq 0}C'$. Since $t_{\leq 0}t_{\geq -d}C \in \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$, using the completeness of the cotorsion pair $(\mathcal{X}', \mathcal{Y}')$, we get that there exists a triangle $Y \to X \xrightarrow{f} t_{\leq 0}t_{\geq -d+1}C$ with $Y \in \mathcal{Y}'$ and $X \in \mathcal{X}'$.

Now, using the octahedral axiom for the triangles $X \xrightarrow{f} t_{\leq 0}t_{\geq -d}C \to \Sigma Y$, $t_{\leq 0}t_{\geq -d}C = t_{\geq -d}t_{\leq 0}C \xrightarrow{j} t_{\leq 0}C \to t_{\leq -d-1}t_{\leq 0}C = t_{\leq -d-1}C$, and $X \xrightarrow{j\circ f} t_{\leq 0}C \to C(j \circ f)$, we get a triangle $\Sigma Y \to C(j \circ f) \to t_{\leq -d-1}C$. Since $\Sigma Y, t_{\leq -d-1}C \in \Sigma C(\mathcal{Y}')$, we get that $C(j \circ f) \in \Sigma C(\mathcal{Y}')$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & t_{\leq 0}t_{\geq -d}C \arrow[r] \arrow[d, "j"] & \Sigma Y \arrow[d] \\
X \arrow[r, "j\circ f"'] & t_{\leq 0}C \arrow[r, "l'"'] \arrow[d] & C(j \circ f) \arrow[d] \\
 & t_{\leq -d-1}C \arrow[r, equal] & t_{\leq -d-1}C
\end{tikzcd}$$

Finally, using the octahedral axiom for the triangles $C \xrightarrow{l} t_{\leq 0}C \to \Sigma t_{\geq 1}C$, $t_{\leq 0}C \xrightarrow{l'} C(j \circ f) \to \Sigma X$, and $C \xrightarrow{l' \circ l} C(j \circ f) \to C(l' \circ l)$, we get a triangle $\Sigma t_{\geq 1}C' \to C(l' \circ l) \to \Sigma X$.

$$\begin{tikzcd}
C \arrow[r, "l"] \arrow[d, Rightarrow] & t_{\leq 0}C \arrow[r] \arrow[d, "l'"] & \Sigma t_{\geq 1}C \arrow[d] \\
C \arrow[r, "l'\circ l"'] & C(j\circ f) \arrow[r] \arrow[d] & C(l'\circ l) \arrow[d] \\
 & \Sigma X \arrow[r, Rightarrow] & \Sigma X
\end{tikzcd}$$

Since $\Sigma t_{\geq 1}C, \Sigma X \in \Sigma C(\mathcal{X}')$, we get that $C(l'\circ l) \in \Sigma C(\mathcal{X}')$. Thus, we have a triangle $\Sigma^{-1}C(l'\circ l) \to C \to C(j\circ f)$ with $\Sigma^{-1}C(l'\circ l) \in C(\mathcal{X}')$ and $C(j\circ f) \in \Sigma C(\mathcal{Y}')$. Therefore, $(C(\mathcal{X}'), C(\mathcal{Y}'))$ is a co-t-structure. Moreover, since $\mathsf{K}^{\geq 0}(\operatorname{proj}\Lambda) \subseteq C(\mathcal{X}')$, we get that $\bigcup_{i\in\mathbb{Z}} \Sigma^i C(\mathcal{X}') \supseteq \bigcup_{i\in\mathbb{Z}} \Sigma^i \mathsf{K}^{\geq 0}(\operatorname{proj}\Lambda) = \mathsf{K}^b(\operatorname{proj}\Lambda)$. Similarly, $\bigcup_{i\in\mathbb{Z}} \Sigma^i C(\mathcal{Y}') = \mathsf{K}^b(\operatorname{proj}\Lambda)$. Hence, $(C(\mathcal{X}'), C(\mathcal{Y}'))$ is a bounded co-t-structure on $\mathsf{K}^b(\operatorname{proj}\Lambda)$. Moreover, the co-heart $C(\mathcal{X}') \cap C(\mathcal{Y}')$ is contained in $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$, and hence its additive generator is a $(d+1)$-term silting object.

Note that if $\mathcal{Y}, \mathcal{Y}' \in$ (c, h)-cotors $\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$ such that $\mathcal{Y} \subseteq \mathcal{Y}'$, then $C(\mathcal{Y}) \subseteq C(\mathcal{Y}')$, which implies that $\Theta^{-1}(C({}^{\perp_1}\mathcal{Y}), C(\mathcal{Y})) \leq \Theta^{-1}(C({}^{\perp_1}\mathcal{Y}'), C(\mathcal{Y}'))$. Thus, χ is a poset homomorphism.

Since $\mathcal{X}' \subseteq C(\mathcal{X}') \cap \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$ and $\mathcal{Y}' \subseteq C(\mathcal{Y}') \cap \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$, we get that $\mathcal{Y}' = (\mathcal{X}')^{\perp_1} \supseteq C(\mathcal{X}')^{\perp_1} \cap \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda) = C(\mathcal{Y}') \cap \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. The last equality follows from the fact that for a co-t-structure $(\mathcal{A}, \mathcal{B})$, $\mathcal{A}^{\perp_1} = \mathcal{B}$ and $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ [83, Proposition 1.6]. Thus, $\mathcal{Y}' = C(\mathcal{Y}') \cap \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. Similarly, $\mathcal{X}' = C(\mathcal{X}') \cap \mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda)$. Thus, $\psi \circ \chi = \mathrm{Id}$. This implies that $\psi \circ \chi \circ \psi = \psi$. Since ψ is injective by Lemma 2.2.3, we get that $\chi \circ \psi = \mathrm{Id}$. Thus, χ is the inverse of ψ. □

2.2.2 Definition of ψ'

Definition 2.2.5 Let M be a $(d+1)$-term silting object in $\mathsf{K}^b(\operatorname{proj}\Lambda)$. Define

$$\begin{aligned}
\mathcal{U}_M &= \{N \in \mathcal{D}^b(\operatorname{mod}\Lambda) \mid \operatorname{Hom}(M, \Sigma^m N) = 0,\ \forall\, m > 0\}, \\
\mathcal{V}_M &= \{N \in \mathcal{D}^b(\operatorname{mod}\Lambda) \mid \operatorname{Hom}(M, \Sigma^m N) = 0,\ \forall\, m < 0\}.
\end{aligned}$$

Set $\psi'(M) := (\mathcal{U}_M \cap \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda), \Sigma^{-1}\mathcal{V}_M \cap \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda))$.

Using [69, Theorem 6.1] we know that the poset of equivalence classes of silting objects in $\mathsf{K}^b(\operatorname{proj}\Lambda)$ is isomorphic to the poset of bounded t-structures with length heart on $\mathcal{D}^b(\operatorname{mod}\Lambda)$ under the map $M \mapsto (\mathcal{U}_M, \Sigma^{-1}\mathcal{V}_M)$. Set $\mathcal{U}'_M := \mathcal{U}_M \cap \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$ and $\mathcal{V}'_M = \Sigma^{-1}\mathcal{V}_M \cap \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$.

Lemma 2.2.6 *The map $\psi' : (d+1)$-$\operatorname{silt}\Lambda \to$ p-tors $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$ given by $M \mapsto \mathcal{U}'_M$ is a poset homomorphism, with the image contained in the set of s-torsion classes. In particular, the image is contained in the set of contravariantly finite positive torsion classes.*

Proof. We start by showing that for $M \in (d+1)$-$\operatorname{silt}\Lambda$, $(\mathcal{U}'_M, \mathcal{V}'_M)$ is an s-torsion pair in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$.

Since $(\mathcal{U}_M, \Sigma^{-1}\mathcal{V}_M)$ is a t-structure in $\mathcal{D}^b(\operatorname{mod}\Lambda)$, $\operatorname{Hom}(\mathcal{U}'_M, \mathcal{V}'_M) = 0$. Moreover, since $\mathcal{U}_M$ is closed under positive shifts, $\mathbb{E}^{-1}(\mathcal{U}'_M, \mathcal{V}'_M) = 0$.

To prove condition (Sc), we claim that $\mathcal{D}^{\leq -d}(\operatorname{mod}\Lambda) \subseteq \mathcal{U}_M$. This is because if $N \in \mathcal{D}^{\leq -d}(\operatorname{mod}\Lambda)$, then $N \cong \sigma_{\leq -d}N$, which implies that

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod}\Lambda)}(\Sigma^{-m}M, N) &\cong \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod}\Lambda)}(\Sigma^{-m}M, \sigma_{\leq -d}N) \\
&\cong \operatorname{Hom}_{K(\operatorname{mod}\Lambda)}(\Sigma^{-m}M, \sigma_{\leq -d}N) \\
&= 0
\end{aligned}$$

for all $m > 0$, since M is a complex of projectives concentrated in $[-d, 0]$. Dually, we have that $\mathcal{D}^{\geq 1}(\operatorname{mod}\Lambda) \subseteq \Sigma^{-1}\mathcal{V}_M$.

Finally, let $Z \in \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$. Since $(\mathcal{U}_M, \Sigma^{-1}\mathcal{V}_M)$ is a t-structure in $\mathcal{D}^b(\operatorname{mod}\Lambda)$, there exists a triangle

$$U \xrightarrow{u} Z \xrightarrow{v} V$$

with $U \in \mathcal{U}_M$ and $V \in \Sigma^{-1}\mathcal{V}_M$. Note that we have a triangle

$$\sigma_{\leq 0}U \to U \to \sigma_{\geq 1}U.$$

Using the previous paragraph, we know that $\sigma_{\geq 1}U \in \Sigma^{-1}\mathcal{V}_M$, which implies that the map from $U \to \sigma_{\geq 1}U$ is 0. Thus, $\sigma_{\leq 0}U \cong U \oplus \Sigma^{-1}\sigma_{\geq 1}U$. This implies that $\sigma_{\geq 1}U \cong 0$ and $\sigma_{\leq 0}U \cong U$. Thus $U \in \mathcal{D}^{\leq 0}(\operatorname{mod}\Lambda)$. Using the triangle $Z \to V \to \Sigma U$ and the fact that $\mathcal{D}^{\leq 0}(\operatorname{mod}\Lambda)$ is closed under extensions, we get that $V \in \mathcal{D}^{\leq 0}(\operatorname{mod}\Lambda)$. Similarly, we have a triangle

$$\sigma_{\leq -d}V \to V \to \sigma_{\geq -d+1}V.$$

Since $\sigma_{\leq -d}V \in \mathcal{U}_M$, we get that the map from $\sigma_{\leq -d}V \to V$ is 0, which implies that $\sigma_{\geq -d+1}V \cong V \oplus \Sigma\sigma_{\leq -d}V$. This gives that $\sigma_{\leq -d}V = 0$ and $\sigma_{\geq -d+1}V \cong V$. Thus, $V \in \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$. Using the triangle $\Sigma^{-1}V \to U \to Z$ and the fact that $\mathcal{D}^{\geq -d+1}(\operatorname{mod}\Lambda)$ is closed under extensions, we get that $U \in \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$. Hence, $U \in \mathcal{U}'_M$ and $V \in \mathcal{V}'_M$. Thus, $(\mathcal{U}'_M, \mathcal{V}'_M)$ is an s-torsion pair. Using [1] and Proposition 1.3.5 we know that s-torsion classes are positive and contravariantly finite. Thus, the image of ψ' is contained in the set of contravariantly finite positive torsion classes.

Now, suppose $M \leq N$. Since $M \mapsto (\mathcal{U}_M, \Sigma^{-1}V_M)$ is a map of posets, $\mathcal{U}_M \subseteq \mathcal{U}_N$. And hence $\mathcal{U}'_M \subseteq \mathcal{U}'_N$. Thus, ψ' is a poset homomorphism. ☐

We now have the following triangle of maps.

$$\begin{tikzcd}
(d+1)\text{-}\operatorname{silt}\Lambda \arrow[r, "\psi"] \arrow[dr, "\psi'"'] & (\mathrm{c},\mathrm{h})\text{-cotors}\,\mathsf{K}^{[-d,0]}(\operatorname{proj}\Lambda) \arrow[d, "\Phi"] \\
& \text{p-tors}\,\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)
\end{tikzcd} \tag{2.2}$$

Proposition 2.2.7 *The above triangle commutes.*

Proof. We need to show that for all $M \in (d+1)\text{-}\operatorname{silt}\Lambda, \mathcal{U}'_M = \sigma_{\geq -d+2}\mathcal{Y}'_M$.

Let $N \in \mathcal{U}'_M$ and $\cdots \to N^{-2} \to N^{-1} \to N^0 \to 0$ be a complex in $\mathsf{K}^{b,-}(\operatorname{proj}\Lambda)$ quasi-isomorphic to N. Then $\sigma_{\geq -d+1}(N^{-d} \to \cdots \to N^0) \cong N$. We claim that $N^{-d} \to \cdots \to N^0 \in \mathcal{Y}_M$.

Consider the triangle $t_{\geq -d}N \to N \to t_{\leq -d-1}N$. Let $j \leq 0$. Then $\Sigma^j M \in \mathcal{X}_M$. We have the following exact sequence.

$$\operatorname{Hom}(\Sigma^j M, t_{\leq -d-1}N) \to \mathbb{E}(\Sigma^j M, t_{\geq -d}N) \to \mathbb{E}(\Sigma^j M, N)$$

Since $N \in \mathcal{U}'_M$, $\mathbb{E}(\Sigma^j M, N) \cong \operatorname{Hom}(M, \Sigma^{-j+1}N) = 0$. Moreover, since M is concentrated in degrees $-d, \cdots, 0$, $\operatorname{Hom}(\Sigma^j M, t_{\leq -d-1}N) = 0$. Therefore,

$$\mathbb{E}(\Sigma^j M, t_{\geq -d}N) = 0.$$

This implies that $\Sigma^j M \in {}^{\perp_1}(t_{\geq -d}N)$ for all $j \leq 0$. Since, ${}^{\perp_1}(t_{\geq -d}N)$ is closed under extensions and summands, we get that $\mathcal{X}_M \subseteq {}^{\perp_1}(t_{\geq -d}N)$. Thus, $t_{\geq -d}N \in \mathcal{X}_M^{\perp_1} = \mathcal{Y}_M$. Thus, $\mathcal{U}'_M \subseteq \sigma_{\geq -d+1}\mathcal{Y}'_M$.

For the other inclusion, we first claim that $\mathcal{Y}'_M = t_{\geq -d}\mathcal{Y}_M$. It is enough to show that $\mathcal{Y}_M$ is closed under $t_{\geq -d}$. Let $Y \in \mathcal{Y}_M$. Consider the triangle $t_{\geq -d}Y \to Y \to t_{\leq -d-1}Y$. Since $\mathcal{X}_M$ is contained in $\mathsf{K}^{\geq -d}(\operatorname{proj}\Lambda)$, we get that for all $X \in \mathcal{X}_M$, $\mathbb{E}(X, t_{\geq -d}Y) = 0$, which implies that $t_{\geq -d}Y \in \mathcal{X}_M^{\perp_1} = \mathcal{Y}_M$.

Thus, $\sigma_{\geq -d+1}\mathcal{Y}'_M = \sigma_{\geq -d+1}t_{\geq -d}\mathcal{Y}_M = \sigma_{\geq -d+1}\mathcal{Y}_M$.

Let $m > 0$ and $j \geq 0$. Using the triangle $\sigma_{\leq -d}\Sigma^j M \to \Sigma^j M \to \sigma_{\geq -d}\Sigma^j M$, we get the exact sequence

$$\operatorname{Hom}(\Sigma^{-m}M, \Sigma^j M) \to \operatorname{Hom}(\Sigma^{-m}M, \sigma_{\geq -d}\Sigma^j M) \to \mathbb{E}(\Sigma^{-m}M, \sigma_{\leq -d}\Sigma^j M).$$

Since M is a silting object, Hom$(\Sigma^{-m}M, \Sigma^j M) = 0$. Moreover, since M is concentrated in degrees $-d, \cdots, 0$ as a complex of projectives, $\mathbb{E}(\Sigma^{-m}M, \sigma_{\leq -d}\Sigma^j M) = 0$. Thus, Hom$(M, \Sigma^m \sigma_{\geq -d}\Sigma^j M) = 0$ for all $m > 0$, and $\sigma_{\geq -d}\Sigma^j M \in \mathcal{U}'_M$ for all $j \geq 0$.

Finally, to show that $\sigma_{\geq -d+1}\mathcal{Y}_M \subseteq \mathcal{U}'_M$, it is enough to show that the category

$$S := \{X \in \mathsf{K}^b(\mathsf{proj}\,\Lambda) \mid \mathsf{Hom}(M, \Sigma^m \sigma_{\geq -d+1}X) = 0 \;\forall\; m > 0\}$$

is closed under extensions and summands.

Clearly, it is closed under summands. Let $X \xrightarrow{f} Z \to Y$ be a triangle in K^b(proj Λ) with $X, Y \in S$. Set $C = C(\sigma_{\geq -d+1}f)$. Then $\sigma_{\geq -d+1}C \cong \sigma_{\geq -d+1}Y$. Let $m > 0$. Using the triangle $\sigma_{\leq -d}C \to C \to \sigma_{\geq -d+1}C$, we get the exact sequence

$$\mathsf{Hom}(M, \Sigma^m \sigma_{\leq -d}C) \to \mathsf{Hom}(M, \Sigma^m C) \to \mathsf{Hom}(M, \Sigma^m \sigma_{\geq -d+1}C),$$

such that Hom$(M, \Sigma^m \sigma_{\leq -d}C) =$ Hom$(M, \Sigma^m \sigma_{\geq -d+1}C) = 0$. Thus,

$$\mathsf{Hom}(\Sigma^{-m}M, C) \cong \mathsf{Hom}(M, \Sigma^m C) = 0.$$

Using the triangle $\sigma_{\geq -d+1}X \to \sigma_{\geq -d+1}Z \to C$ and the fact that $(\Sigma^{-m}M)^{\perp}$ is closed under extensions, we get that Hom$(M, \Sigma^m \sigma_{\geq -d+1}Z) = 0$ for all $m > 0$. Thus, $Z \in S$. ☐

2.2.3 Proof of Theorem 2.2.1

Proof. Using Lemma 2.2.6, we know that the image of ψ' is contained in the class of contravariantly finite torsion pairs. Using Proposition 2.2.7, this image is also contained in the image of Φ, which by Theorem 2.1.15 is contained in the class of covariantly finite torsion pairs. Thus, we get that the following commutative triangle is well-defined.

$$\begin{tikzcd}
(d+1)\text{-}\,\mathsf{silt}\,\Lambda \arrow[r, "\psi"] \arrow[dr, "\psi'"'] & \mathsf{(c,h)\text{-}cotors}\,\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda) \arrow[d, "\Phi"] \\
 & \mathsf{(f,p)\text{-}tors}\,\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)
\end{tikzcd}$$

Using Theorem 2.1.16, we know that the inverse Ψ : tors $\mathcal{D}^{[-d+1,0]}$(mod Λ) $\to$ cotors K$^{[-d,0]}$(proj Λ) of Φ sends functorially finite torsion pairs to complete hereditary cotorsion pairs. Hence,

$$\Phi : \mathsf{(c,h)\text{-}cotors}\,\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda) \to \mathsf{(f,p)\text{-}tors}\,\mathcal{D}^{[-d+1,0]}(\mathsf{mod}\,\Lambda)$$

is a poset isomorphism.

Thus, we conclude that all the maps in triangle (2.1) are isomorphisms of posets. ☐

Corollary 2.2.8 *Let $(\mathcal{T}, \mathcal{F})$ be a positive torsion pair in $\mathcal{D}^{[-d+1,0]}$(mod Λ). Then $\mathcal{T}$ is functorially finite if and only if $\mathcal{F}$ is.*

Proof. Using Theorem 2.2.1 and its dual, we know that $\mathcal{T}$ is functorially finite if and only if it $\mathcal{T} = \psi'(P)$ for P a $(d+1)$-term silting complex, which is true if and only if $\mathcal{F} = (\psi'(P))^{\perp}$ is functorially finite. ☐

Similar to Theorem 1.2.2, we can show that the poset of cotorsion pairs in any extriangulated category $\mathcal{C}$ is a lattice.

Theorem 2.2.9 *The poset of cotorsion pairs in $\mathcal{C}$ is a lattice.*

Proof. Let $(\mathcal{X}_1, \mathcal{Y}_1)$, $(\mathcal{X}_2, \mathcal{Y}_2)$ be two cotorsion pairs in $\mathcal{C}$. To show that their join exists, it is enough to show that $(\mathcal{X}_1 \cap \mathcal{X}_2, (\mathcal{X}_1 \cap \mathcal{X}_2)^{\perp_1})$ is a cotorsion pair, i.e., $\mathcal{X}_1 \cap \mathcal{X}_2 = {}^{\perp_1}((\mathcal{X}_1 \cap \mathcal{X}_2)^{\perp_1})$. By definition, $\mathcal{X}_1 \cap \mathcal{X}_2 \subseteq {}^{\perp_1}((\mathcal{X}_1 \cap \mathcal{X}_2)^{\perp_1})$. On the other hand, for $i = 1, 2$, since $\mathcal{X}_1 \cap \mathcal{X}_2 \subseteq \mathcal{X}_i$, we get that ${}^{\perp_1}((\mathcal{X}_1 \cap \mathcal{X}_2)^{\perp_1}) \subseteq {}^{\perp_1}(\mathcal{X}_i^{\perp_1}) = \mathcal{X}_i$. Thus, ${}^{\perp_1}((\mathcal{X}_1 \cap \mathcal{X}_2)^{\perp_1}) \subseteq \mathcal{X}_1 \cap \mathcal{X}_2$. Hence $\mathcal{X}_1 \cap \mathcal{X}_2 = {}^{\perp_1}((\mathcal{X}_1 \cap \mathcal{X}_2)^{\perp_1})$.

Dually, the meet of $(\mathcal{X}_1, \mathcal{Y}_1)$, $(\mathcal{X}_2, \mathcal{Y}_2)$ is the cotorsion pair $({}^{\perp_1}(\mathcal{Y}_1 \cap \mathcal{Y}_2), \mathcal{Y}_1 \cap \mathcal{Y}_2)$. ☐

Using the isomorphism between positive torsion pairs and hereditary cotorsoADD

Corollary 2.2.10 *The subposet* h-cotors $\mathsf{K}^{[-d,0]}(\mathsf{proj}\,\Lambda)$ *is a lattice.*

2.3 Examples

Example 2.3.1 Let $Q = 1 \to 2$ and $\Lambda = KQ$. Then the Hasse diagram of the poset 3-silt Λ is shown below.

$$\begin{tikzcd}
 & & P_1 \oplus P_2 & \\
 & S_1 \oplus P_1 \arrow[ur] & & \\
\Sigma S_1 \oplus P_1 \arrow[ur] & S_1 \oplus \Sigma P_2 \arrow[u] & & P_2 \oplus \Sigma P_1 \arrow[uul] \\
 & \Sigma P_2 \oplus \Sigma P_1 \arrow[u] \arrow[urr] & & \\
\Sigma^2 P_2 \oplus S_1 \arrow[uur] & \Sigma S_1 \oplus \Sigma P_1 \arrow[u] \arrow[uul] & & P_2 \oplus \Sigma^2 P_1 \arrow[uu] \\
 & \Sigma^2 P_2 \oplus \Sigma S_1 \arrow[u] \arrow[ul] & & \Sigma P_2 \oplus \Sigma^2 P_1 \arrow[u] \arrow[uull] \\
 & & \Sigma^2 P_2 \oplus \Sigma^2 P_1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Since $\mathcal{D}^{[-1,0]}(\mathsf{mod}\,\Lambda)$ has finitely many indecomposables, this is also the lattice of positive torsion classes in $\mathcal{D}^{[-1,0]}(\mathsf{mod}\,\Lambda)$. This example also shows that, unlike the case of (positive) torsion classes in a module category ([34, Theorem 3.1]), the lattice of positive torsion classes in $\mathcal{D}^{[-d,0]}(\mathsf{mod}\,\Lambda)$ is not necessarily semidistributive.

Example 2.3.2 Let Q be the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ and $I = \langle \alpha\beta, \beta\alpha \rangle$. Then the Hasse diagram of the poset 3- silt kQ/I is as follows.

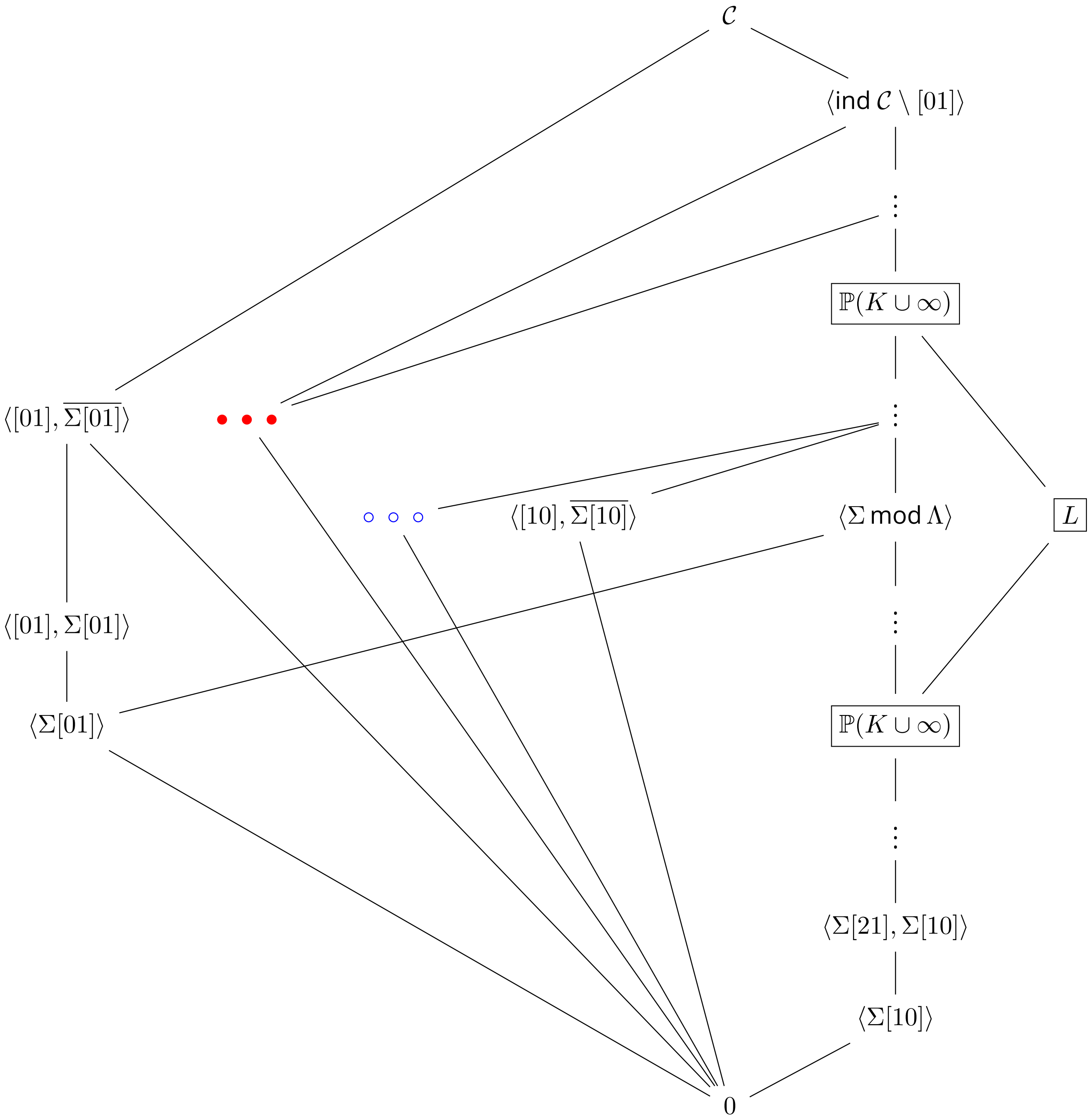
$$\begin{tikzcd}
 & & \begin{matrix}0\to0\to P_1\\ \oplus\\ 0\to0\to P_2\end{matrix} & & \\
 & \begin{matrix}0\to0\to P_1\\ \oplus\\ 0\to P_2\to P_1\end{matrix} \arrow[ur] & & \begin{matrix}0\to P_1\to P_2\\ \oplus\\ 0\to0\to P_2\end{matrix} \arrow[ul] & \\
\begin{matrix}P_2\to P_1\to0\\ \oplus\\ 0\to0\to P_1\end{matrix} \arrow[ur] & \begin{matrix}0\to P_2\to0\\ \oplus\\ 0\to P_2\to P_1\end{matrix} \arrow[u] & & \begin{matrix}0\to P_1\to0\\ \oplus\\ 0\to P_1\to P_2\end{matrix} \arrow[u] & \begin{matrix}P_1\to P_2\to0\\ \oplus\\ 0\to0\to P_2\end{matrix} \arrow[ul] \\
 & & \begin{matrix}0\to P_1\to0\\ \oplus\\ 0\to P_2\to0\end{matrix} \arrow[ull, dashed] \arrow[urr, dashed] & & \\
\begin{matrix}0\to P_1\to0\\ \oplus\\ P_2\to P_1\to0\end{matrix} \arrow[uu] & \begin{matrix}P_2\to0\to0\\ \oplus\\ 0\to P_2\to P_1\end{matrix} \arrow[uu] \arrow[ur] & & \begin{matrix}P_1\to0\to0\\ \oplus\\ 0\to P_1\to P_2\end{matrix} \arrow[uu] \arrow[ul] & \begin{matrix}0\to P_2\to0\\ \oplus\\ P_1\to P_2\to0\end{matrix} \arrow[uu] \\
 & \begin{matrix}P_2\to0\to0\\ \oplus\\ P_2\to P_1\to0\end{matrix} \arrow[ul] \arrow[u] & & \begin{matrix}P_1\to0\to0\\ \oplus\\ P_1\to P_2\to0\end{matrix} \arrow[u] \arrow[ur] & \\
 & & \begin{matrix}P_1\to0\to0\\ \oplus\\ P_2\to0\to0\end{matrix} \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

We also show that the class of s-torsion pairs can be strictly contained in the class of positive torsion pairs.

Example 2.3.3 Let $Q = \; 1 \rightrightarrows 2$ be the Kronecker quiver. Let $\mathcal{P}$ and $\mathcal{I}$ denote the preprojective and the postinjective component of the AR-quiver of KQ. The positive torsion classes in $\mathcal{C} = \mathcal{D}^{[-1,0]}(\text{mod}\,\Lambda)$ are given by the additive subcategories generated by the following sets.

1. Any final part of $\Sigma\mathcal{I}$.
2. Shift of a subset of the tubes and $\Sigma\mathcal{I}$.
3. Any final part of $\mathcal{I} \cup \Sigma\mathcal{P}$, the shift of all tubes, and $\Sigma\mathcal{I}$.
4. ΣS_2.
5. M and the final part of $\Sigma\mathcal{I}$ starting at ΣM, $M \in \mathcal{I}$.
6. Any subset S of the tubes, the shift of any subset S' containing S, and $\mathcal{I}$.
7. Any subset of the tubes, shifts of all tubes, and $\mathcal{I} \cup \Sigma\mathcal{P} \cup \Sigma\mathcal{I}$.
8. Any final subset of $\mathcal{P}$, all tubes and their shifts, $\Sigma\mathcal{P} \cup \mathcal{I} \cup \Sigma\mathcal{I}$.
9. $S_2, \Sigma S_2$.
10. M and the final part of $\Sigma\,\text{mod}\,\Lambda$ starting at ΣM, $M \in \mathcal{P}$.

The Hasse diagram of the lattice p-tors$\,\mathcal{C}$ is depicted in the following picture.



Here the ● points are indexed by modules in the preprojective component, and the ○ points are indexed by modules in the postinjective component. For an object $M \in \mathcal{P} \cup \mathcal{I}$, the notation $\overline{\Sigma M}$ is used to denote the set of objects in the final part of the AR quiver starting at ΣM. Finally, $\mathbb{P}(K \cup \infty)$ represents the boolean lattice of subsets of $K \cup \infty$, and L denotes the lattice of pairs of subsets (S, S') of $K \cup \infty$ with $S \subseteq S'$.

Note that the positive torsion class $\mathcal{T} = \text{add}\{\mathcal{S} \cup \Sigma\mathcal{S} \cup \Sigma\mathcal{I}\}$, where $\mathcal{S}$ is a proper subset of the tubes, is not an s-torsion class as the objects in $\mathcal{I}$ do not admit a right $\mathcal{T}$-approximation.

2.4 d-torsion classes in d-cluster tilting subcategories

In this section, we will recall the definitions of a d-cluster tilting subcategory $\mathcal{M}$ in $\text{mod}\,\Lambda$ and d-torsion classes in $\mathcal{M}$. These have been studied in [12] where the authors show the existence of an injective map from functorially finite d-torsion classes to $(d+1)$-term silting complexes.

Definition 2.4.1 A functorially finite subcategory $\mathcal{M} \subseteq \text{mod}\,\Lambda$ is called d-cluster tilting if

1. $\mathcal{M} = \{X \in \text{mod}\,\Lambda \mid \text{Ext}^i(X, \mathcal{M}) = 0 \;\forall\, i = 1, \cdots, d-1\}$

2. $\mathcal{M} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i(\mathcal{M}, X) = 0 \; \forall\, i = 1, \cdots, d-1\}$

Definition 2.4.2 Let $\mathcal{M} \subseteq \operatorname{mod}\Lambda$ be a d-cluster tilting subcategory. A subcategory $\mathcal{U} \subseteq \mathcal{M}$ is called a *d-torsion class* if for every $M \in \mathcal{M}$, there exists an exact sequence

$$0 \to U_{\mathcal{M}} \to M \to V_1 \to \cdots \to V_d \to 0$$

with $V_i \in \mathcal{M}$ for all $i = 1, \cdots, d$ such that

1. $U_M \in \mathcal{U}$;
2. For all $U \in U$, the sequence

   $$0 \to \operatorname{Hom}(U, V_1) \to \cdots \to \operatorname{Hom}(U, V_d) \to 0$$

   is exact.

Theorem 2.4.3 *[12, Theorem 8.1, Proposition 8.16] There exists an injective map*

$$\left\{\begin{array}{c}\textit{functorially finite}\\ d\textit{-torsion classes in } \mathcal{M}\end{array}\right\} \hookrightarrow \left\{\begin{array}{c}(d+1)\textit{-term silting}\\ \textit{complexes in } \mathsf{K}^b(\operatorname{proj}\Lambda)\end{array}\right\}$$

that sends $\mathcal{U} \mapsto P^{M_{\mathcal{U}}} \oplus P_{\mathcal{U}}[d]$, where $M_{\mathcal{U}}$ is the basic Ext^d*-projective generator of $\mathcal{U}$, $P^{M_{\mathcal{U}}}$ is its minimal projective d-presentation, and $P_{\mathcal{U}}$ is the maximal basic projective Λ-module such that* $\operatorname{Hom}(P_{\mathcal{U}}, \mathcal{U}) = 0$*.*

Moreover, the silting complex $P^{\bullet}_{\mathcal{U}} := P^{M_{\mathcal{U}}} \oplus P_{\mathcal{U}}[d]$ is inner acyclic*, i.e.,* $\operatorname{H}^i(P^{\bullet}_{\mathcal{U}}) = 0$ *for all $i \neq 0, -d$.*

Using the dual of Theorem 2.2.1, we know that $(d+1)$-term silting objects are in bijection with functorially finite, positive torsion-free classes in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$. We want to study the composition of the above inclusion with this bijection. We need the following lemma.

Lemma 2.4.4 *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)$. Then $\mathcal{F} \cap \operatorname{mod}\Lambda$ is a torsion-free class in $\operatorname{mod}\Lambda$, and $\mathcal{T} \cap (\operatorname{mod}\Lambda[d-1])$ is a torsion class in $\operatorname{mod}\Lambda[d-1]$.*

Proof. We only prove the torsion-free case. The torsion case is dual.

We claim that $\mathcal{F} \cap \operatorname{mod}\Lambda = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Hom}(\sigma_{\geq 0}\mathcal{T}, X) = 0\}$. Indeed, for $X \in \mathcal{F} \cap \operatorname{mod}\Lambda$ and $Y \in \mathcal{T}$, applying $\operatorname{Hom}(-, X)$ to the triangle

$$Y \to \sigma_{\geq 0}Y \to \Sigma\sigma_{\leq -1}Y,$$

we get the exact sequence $0 = \operatorname{Hom}(\Sigma\sigma_{\leq -1}Y, X) \to \operatorname{Hom}(\sigma_{\geq 0}Y, X) \to \operatorname{Hom}(X, Y) = 0$. Thus, we get that $\operatorname{Hom}(\sigma_{\geq 0}Y, X) = 0$. On the other hand, if there exists $X \in \operatorname{mod}\Lambda$ such that $\operatorname{Hom}(\sigma_{\geq 0}\mathcal{T}, X) = 0$, then $\operatorname{Hom}(\mathcal{T}, X) = 0$. This is because for $Y \in \mathcal{T}$, we have the triangle $\sigma_{\leq -1}Y \to Y \to \sigma_{\geq 0}Y$. Since $\operatorname{Hom}(\sigma_{\leq -1}Y, X) = 0 = \operatorname{Hom}(\sigma_{\geq 0}Y, X)$, we get that $\operatorname{Hom}(Y, X) = 0$. Thus, $X \in \mathcal{T}^{\perp} = \mathcal{F}$.

Thus, we showed that $\mathcal{F} \cap \operatorname{mod}\Lambda$ is the right orthogonal of some subcategory of $\operatorname{mod}\Lambda$. Hence, it is a torsion-free class. □

Proposition 2.4.5 *The following diagram commutes.*

$$\begin{tikzcd}
\left\{\begin{array}{c}\textit{functorially finite } d\textit{-torsion}\\ \textit{classes in } \mathcal{M}\end{array}\right\} \arrow[r, hook, "\mathcal{U} \mapsto P^{\bullet}_{\mathcal{U}}"] \arrow[d, "\mathcal{U} \mapsto \mathcal{U}^{\perp}"'] & \left\{\begin{array}{c}(d+1)\textit{-term silting}\\ \textit{complexes in } \mathsf{K}^b(\operatorname{proj}\Lambda)\end{array}\right\} \arrow[d, "\phi"', "\wr"] \\
\left\{\begin{array}{c}\textit{torsion-free classes in}\\ \operatorname{mod}\Lambda\end{array}\right\} & \left\{\begin{array}{c}\textit{functorially finite}\\ \textit{positive torsion-free}\\ \textit{classes in}\\ \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda)\end{array}\right\} \arrow[l, "\mathcal{F} \cap \operatorname{mod}\Lambda \hookleftarrow \mathcal{F}"']
\end{tikzcd}$$

Proof. Let $\mathcal{U}$ be a functorially finite d-torsion class in $\mathcal{M}$. Then $P^\bullet_{\mathcal{U}} = P^{M_{\mathcal{U}}} \oplus P_{\mathcal{U}}[d]$ By definition, $\phi(P^\bullet_{\mathcal{U}}) = \{X \in \mathcal{D}^{[-d+1,0]}(\operatorname{mod}\Lambda) \mid \operatorname{Hom}(P^\bullet_{\mathcal{U}}, \Sigma^i X) = 0 \,\forall\, i \leq 0\}$. We want to show that $\phi(P^\bullet_{\mathcal{U}}) \cap \operatorname{mod}\Lambda = \mathcal{U}^\perp$.

Let $X \in \mathcal{U}^\perp$. Since $X \in \operatorname{mod}\Lambda$, $\operatorname{Hom}(P_{\mathcal{U}}[d], X) = 0 = \operatorname{Hom}(P^{M_{\mathcal{U}}}, \Sigma^i X)$ for all $i < 0$. Note that $P^{M_{\mathcal{U}}} = t_{\geq -d} M^\bullet$, where $M^\bullet$ is a minimal projective resolution of $M_{\mathcal{U}}$. Using the triangle $t_{\geq -d}M^\bullet \to M^\bullet \to t_{\leq -d-1}M^\bullet$, since $\operatorname{Hom}(\Sigma^{-1} t_{\leq -d-1}M^\bullet, X) = 0 = \operatorname{Hom}(M^\bullet, X)$, we get that $\operatorname{Hom}(t_{\geq -d}M, X) = 0$. Thus $\operatorname{Hom}(P^\bullet_{\mathcal{U}}, X) = 0$, and $X \in \phi(P^\bullet_{\mathcal{U}}) \cap \operatorname{mod}\Lambda$.

Now suppose that $X \in \phi(P^\bullet_{\mathcal{U}}) \cap \operatorname{mod}\Lambda$ and $A \in \mathcal{U}$. Since $A \in \operatorname{mod}\Lambda$, there exists an epimorphism $g : \Lambda^n \to A$ for some $n \geq 1$. Using [12, Theorem 4.6], there exists a minimal $\mathcal{U}$-coresolution of Λ

$$0 \to \Lambda \xrightarrow{f} U^0 \to U^1 \to \cdots \to U^d \to 0$$

with U^i being Ext^d-projective in $\mathcal{U}$ for $i = 0, \cdots, d$. Since f is a left $\mathcal{U}$-approximation of Λ, there exists an epimorphism $h : (U^0)^n \to A$ such that the following triangle commutes.

$$\begin{tikzcd}
\Lambda^n \arrow[r, "f\oplus\cdots\oplus f"] \arrow[dr, two heads, "g"'] & (U^0)^n \arrow[d, dashed, two heads, "h"] \\
& A
\end{tikzcd}$$

Since $X \in \phi(P^\bullet_{\mathcal{U}}) \cap \operatorname{mod}\Lambda$, $\operatorname{Hom}(P^{M_{\mathcal{U}}}, X) = 0$. Again using the triangle $t_{\geq -d}M^\bullet \to M^\bullet \to t_{\leq -d-1}M^\bullet$, we get that $\operatorname{Hom}(M^\bullet, X) = 0$ since $\operatorname{Hom}(t_{\geq -d}M^\bullet, X) = 0 = \operatorname{Hom}(t_{\leq -d-1}M^\bullet, X)$. Since $M^\bullet \cong M_{\mathcal{U}}$ is the Ext^d-projective generator of $\mathcal{U}$, $U^0 \in \operatorname{add} M_{\mathcal{U}}$. Thus, $\operatorname{Hom}(U^0, X) = 0$. Since A is a quotient of $(U^0)^n$, we get that $\operatorname{Hom}(A, X) = 0$. Hence, $X \in \mathcal{U}^\perp$. □

Note that the torsion class $\mathcal{T}$ corresponding to $\mathcal{U}^\perp$ is the smallest torsion class in $\operatorname{mod}\Lambda$ containing $\mathcal{U}$, and hence satisfies $\mathcal{T} \cap \mathcal{M} = \mathcal{U}$ ([7, Theorem 1.1]).

Chapter 3

Semibricks and wide subcategories

In this chapter, we develop the notions of semibricks and wide subcategories in extended hearts. We then use these to give bijections between certain kinds of semibricks, wide subcategories, and simple-minded collections in $\mathcal{D}^b(\operatorname{mod}\Lambda)$. The results in this chapter appeared in [42], joint work with Yu Zhou.

The chapter is organised as follows. In section 3.1, we introduce semibricks and wide subcategories in the d-extended heart and prove that semibricks are in bijection with length wide subcategories. In Section 3.2, we provide a systematic way to associate a wide subcategory with any d-FAE closed subcategory by introducing the notion of the exact heart. Section 3.3 focuses on the properties of two specific maps, denoted by T and ϕ, which send a semibrick to its smallest positive torsion class and its smallest d-FAE closed subcategory, respectively. We also show that the original semibrick can be recovered from the exact hearts of these subcategories. In Section 3.4, we specialize to the case of extended module categories of finite-dimensional algebras. We define left-finite and right-finite semibricks as well as wide subcategories, and establish their bijections with functorially finite positive torsion classes and $(d+1)$-term simple-minded collections. We also demonstrate that these wide subcategories can be realized as module categories of certain finite-dimensional algebras. Finally, in Section 3.5, we provide a criterion to determine which mutations of a $(d+1)$-term silting complex remain $(d+1)$-term.

We start by recalling some basic terminology from the theory of exact categories. An exact category is a pair $(\mathcal{C},\mathcal{E})$ with $\mathcal{C}$ an additive category and $\mathcal{E}$ a collection of kernel-cokernel pairs in $\mathcal{C}$ satisfying certain axioms. Elements of $\mathcal{E}$ are called *conflations*, denoted by $A \rightarrowtail B \twoheadrightarrow C$. The first map of a conflation is called an *inflation* and the second a *deflation*. A morphism $f \in \mathcal{C}$ is called *admissible* if $f = g \circ h$ with h a deflation and g an inflation. For a subcategory $\mathcal{X}$ of $\mathcal{C}$, $\operatorname{Filt}(\mathcal{X})$ denotes the subcategory of objects $X \in \mathcal{C}$ such that there exists a sequence of inflations

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_n = X,$$

with the cokernel of $X_{i-1} \rightarrowtail X_i$ belonging to $\mathcal{X}$ for each i. Note that the category $\operatorname{Filt}(\mathcal{X})$ is precisely the extension closure of $\mathcal{X}$, the smallest subcategory of $\mathcal{C}$ containing $\mathcal{X}$ that is closed under extensions. A nonzero object $X \in \mathcal{C}$ is called *simple* if there is no conflation $L \rightarrowtail X \twoheadrightarrow M$ with $L, M \neq 0$. Note that this coincides with the classical definition for $\mathcal{C}$ being abelian. We denote the set of isoclasses of simple objects in $\mathcal{C}$ by $\operatorname{sim}\mathcal{C}$. An exact category $\mathcal{C}$ is called a *length* category if $\mathcal{C} = \operatorname{Filt}(\operatorname{sim}\mathcal{C})$. See [30] for a detailed exposition on exact categories.

3.1 Wide subcategories and semibricks

In this section, we generalize the notions of semibricks and wide subcategories, together with the correspondence between them, from abelian categories to the d-extended heart $\mathcal{D}^{[-d+1,0]}$.

Recall that a subcategory of an abelian category is said to be *wide* if it is closed under kernels, cokernels, and extensions. This notion was generalized to exact categories in [37].

Definition 3.1.1 ([37, Definition 2.4]) Let $\mathcal{C}$ be an exact category. A subcategory $\mathcal{W} \subseteq \mathcal{C}$ is called *wide* if the following conditions are satisfied:

1. $\mathcal{W}$ is closed under extensions, that is, for any conflation $L \rightarrowtail M \twoheadrightarrow N$ in $\mathcal{C}$, if L and N belong to $\mathcal{W}$, then so does M.
2. $\mathcal{W}$ is an abelian category.
3. The inclusion functor $\mathcal{W} \hookrightarrow \mathcal{C}$ is exact, that is, every short exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{W}$ is a conflation in $\mathcal{C}$.

It was shown in [37, Theorem 2.5] that there exists a bijection between the class of semibricks in $\mathcal{C}$ and the class of length wide subcategories of $\mathcal{C}$. This bijection is given by assigning to a semibrick its extension closure, while its inverse sends a wide subcategory to its collection of simple objects. This result generalizes the classical correspondence for abelian categories established in [91].

An exact category $\mathcal{C}$ is called *weakly idempotent complete* if for two composable morphisms f and g, whenever $g \circ f$ is an inflation (resp. a deflation), so is f (resp. g). Recall that a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is called *Serre* if for any conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{C}$, one has $Y \in \mathcal{B}$ if and only if both $X \in \mathcal{B}$ and $Z \in \mathcal{B}$.

The following proposition provides a canonical construction of a wide subcategory from any exact category by considering the collection of objects to which all incoming morphisms are admissible. This result is of fundamental importance as it provides the template for constructing wide subcategories within $\mathcal{D}^{[-d+1,0]}$ in the next section. While we expect this general result to be known, we were unable to find a proof in the literature and thus include one here for completeness.

Proposition 3.1.2 *Let $\mathcal{C}$ be a weakly idempotent complete exact category. Then the full subcategory*

$$\mathcal{C}' := \{X \in \mathcal{C} \mid \textit{any morphism in } \mathcal{C} \textit{ to } X \textit{ is admissible}\}$$

is a Serre and wide subcategory of $\mathcal{C}$.

Proof. We divide the proof into several steps.

(i) $\mathcal{C}'$ is closed under extensions. Let $X_1 \overset{n}{\rightarrowtail} X \overset{m}{\twoheadrightarrow} X_2$ be a conflation in $\mathcal{C}$ with $X_1, X_2 \in \mathcal{C}'$. Let $f : A \to X$ be a morphism in $\mathcal{C}$. Since $m \circ f$ is admissible, there exist a deflation $p : A \twoheadrightarrow C$ and an inflation $p' : C \rightarrowtail X_2$ such that $m \circ f = p' \circ p$. Let $i = \ker(p) : K \rightarrowtail A$. Then $m \circ f \circ i = 0$, which implies that there exists a morphism $a : K \to X_1$ such that $n \circ a = f \circ i$. See the following commutative diagram, where each row is a conflation.

$$\begin{tikzcd}
K \arrow[r, tail, "i"] \arrow[d, "a"'] & A \arrow[r, two heads, "p"] \arrow[d, "f"] & C \arrow[d, tail, "p'"] \\
X_1 \arrow[r, tail, "n"] & X \arrow[r, two heads, "m"] & X_2
\end{tikzcd}$$

Since $X_1 \in \mathcal{C}'$, the morphism a is admissible. Hence there exist a deflation $j_1 : K \twoheadrightarrow X'$ and an inflation $j_2 : X' \rightarrowtail X_1$ such that $a = j_2 \circ j_1$. Consider the pushout of the morphisms i and j_1, as shown below, where q is a deflation since it is the pushout of a deflation along an inflation (cf. the dual of [30, Proposition 2.15]):

$$\begin{tikzcd}
K \arrow[r, tail, "i"] \arrow[d, two heads, "j_1"'] \arrow[dr, phantom, "\ulcorner", very near end] & A \arrow[d, two heads, "q"] \arrow[ddr, bend left, "f"] & \\
X' \arrow[r, tail, "j"'] \arrow[drr, bend right, "n \circ j_2"'] & M \arrow[dr, dashed, "\exists !\, t"] & \\
 & & X
\end{tikzcd}$$

Since $f \circ i = n \circ a = n \circ j_2 \circ j_1$, the universal property of pushouts yields a unique morphism $t : M \to X$ such that

$$t \circ j = n \circ j_2 \quad \text{and} \quad t \circ q = f.$$

We claim that t is an inflation. Indeed, since $p \circ i = 0$, the universal property of pushouts yields a unique morphism $s : M \to C$ such that the following diagram commutes:

$$\begin{tikzcd}
K \arrow[r, tail, "i"] \arrow[d, two heads, "j_1"'] & A \arrow[d, two heads, "q"] \arrow[ddr, bend left, two heads, "p"] & \\
X' \arrow[r, tail, "j"'] \arrow[drr, bend right, "0"'] & M \arrow[dr, dashed, "\exists !\, s"] & \\
& & C
\end{tikzcd}$$

By [30, Remark 2.13], one has $s = \text{coker}(j)$, so the first row of the following diagram is a conflation:

$$\begin{tikzcd}
X' \arrow[r, tail, "j"] \arrow[d, tail, "j_2"'] & M \arrow[r, two heads, "s"] \arrow[d, "t"] & C \arrow[d, tail, "p'"] \\
X_1 \arrow[r, tail, "n"'] & X \arrow[r, two heads, "m"'] & X_2
\end{tikzcd}$$

The commutativity $p' \circ s = m \circ t$ follows from

$$p' \circ s \circ q = p' \circ p = m \circ f = m \circ t \circ q.$$

By [30, Corollary 3.2], this implies that t is an inflation. Hence $f = t \circ q$ is admissible, and therefore $X \in \mathcal{C}'$. Thus, $\mathcal{C}'$ is closed under extensions.

(ii) Let $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ be a conflation of $\mathcal{C}$. If $Y \in \mathcal{C}'$, then $X \in \mathcal{C}'$. For any morphism $k : A \to X$, the composite $f \circ k$ is admissible, so there exist a deflation $l : A \twoheadrightarrow B$ and an inflation $m : B \rightarrowtail Y$ such that $f \circ k = m \circ l$. Since $g \circ m \circ l = g \circ f \circ k = 0$, it follows that $g \circ m = 0$. See the following diagram.

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[dd, "k"'] & A \arrow[d, two heads, "l"] & \\
& B \arrow[d, tail, "m"] \arrow[dl, dashed, "s"'] & \\
X \arrow[r, tail, "f"'] & Y \arrow[r, two heads, "g"'] & Z
\end{tikzcd}$$

Thus there exists a morphism $s : B \to X$ such that $f \circ s = m$. Since $\mathcal{C}$ is weakly idempotent complete, s is an inflation. Moreover, $f \circ s \circ l = m \circ l = f \circ k$ implies $s \circ l = k$. Hence k is admissible, and therefore $X \in \mathcal{C}'$.

(iii) Let $Z \overset{h}{\rightarrowtail} X \overset{p}{\twoheadrightarrow} Y$ be a conflation of $\mathcal{C}$. If $X \in \mathcal{C}'$, then $Y \in \mathcal{C}'$. Given a morphism $a : A \to Y$, consider the pullback of morphisms p and a. By the dual of [30, Proposition 2.12], we obtain a commutative diagram

$$\begin{tikzcd}
Z \arrow[r, tail, "h'"] \arrow[d, equal] & P \arrow[r, two heads, "i"] \arrow[d, "d"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "a"] \\
Z \arrow[r, tail, "h"] & X \arrow[r, two heads, "p"] & Y
\end{tikzcd}$$

in which all rows are conflations, and the sequence

$$\begin{tikzcd}
P \arrow[r, tail, "\begin{bmatrix} i \\ -d \end{bmatrix}"] & A \oplus X \arrow[r, two heads, "\begin{bmatrix} a & p \end{bmatrix}"] & Y
\end{tikzcd}$$

is a conflation. Since d is admissible, there exist a deflation $b : P \twoheadrightarrow B$ and an inflation $c : B \rightarrowtail X$ such that $d = c \circ b$. Let $j = \text{ker}(b) : L \rightarrowtail P$ and $e = \text{coker}(c) : X \twoheadrightarrow D$. See the following diagram.

$$\begin{tikzcd}
 & L \arrow[d, tail, "j"] & & \\
Z \arrow[r, tail, "h'"] \arrow[dd, equal] & P \arrow[r, two heads, "i"] \arrow[d, two heads, "b"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[dd, "a"] \arrow[dr, dashed, "m"] & \\
 & B \arrow[d, tail, "c"] & & K \arrow[dl, dashed, tail, "k"] \\
Z \arrow[r, tail, "h"] & X \arrow[r, two heads, "p"] \arrow[d, two heads, "e"] & Y \arrow[dl, dashed, two heads, "l"] & \\
 & D & &
\end{tikzcd}$$

Then $e \circ h = e \circ c \circ b \circ h' = 0$ implies the existence of a morphism $l : Y \to D$ with

$$l \circ p = e.$$

Since $\mathcal{C}$ is weakly idempotent complete, l is a deflation. Let $k = \mathrm{ker}(l) : K \rightarrowtail Y$. Since $l \circ a \circ i = l \circ p \circ c \circ b = e \circ c \circ b = 0$, we have that

$$l \circ a = 0.$$

This yields a morphism $m : A \to K$ with $k \circ m = a$.

Consider the following commutative diagram

$$\begin{tikzcd}
L \arrow[r, "i \circ j"] \arrow[d, tail, "j"'] & A \arrow[r, "m"] \arrow[d, tail, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & K \arrow[d, tail, "k"] \\
P \arrow[r, tail, "\begin{bmatrix} i \\ -d \end{bmatrix}"] \arrow[d, two heads, "b"'] & A \oplus X \arrow[r, two heads, "\begin{bmatrix} a & p \end{bmatrix}"] \arrow[d, two heads, "\begin{bmatrix} 0 & 1 \end{bmatrix}"] & Y \arrow[d, two heads, "l"] \\
B \arrow[r, tail, "-c"'] & X \arrow[r, two heads, "e"'] & D
\end{tikzcd}$$

in which the second and third rows and all columns are conflations. Applying [30, Corollary 3.6], it follows that the first row is also a conflation, which implies that m is a deflation. Hence $a = k \circ m$ is admissible, and therefore $Y \in \mathcal{C}'$.

(iv) $\mathcal{C}'$ is abelian and the inclusion functor $\mathcal{C}' \to \mathcal{C}$ is exact. By [37, Proposition 3.1], an exact category in which every morphism is admissible is abelian, and its conflations coincide with short exact sequences. Since $\mathcal{C}'$ is closed under extensions in $\mathcal{C}$, it inherits an exact structure from $\mathcal{C}$. It therefore suffices to show that every morphism in $\mathcal{C}'$ is admissible in this induced exact structure.

Let $f : X \to Y$ be a morphism in $\mathcal{C}'$. Since f is admissible in $\mathcal{C}$, there exist a deflation $p : X \twoheadrightarrow Z$ in $\mathcal{C}$ and an inflation $i : Z \rightarrowtail Y$ in $\mathcal{C}$ such that $f = i \circ p$. The associated conflations

$$Z_1 \rightarrowtail X \overset{p}{\twoheadrightarrow} Z \text{ and } Z \overset{i}{\rightarrowtail} Y \twoheadrightarrow Z_2$$

have all terms in $\mathcal{C}'$ by closure under subobjects and quotients. Hence p and i are a deflation and an inflation in $\mathcal{C}'$, respectively. This shows that $\mathcal{C}'$ is abelian and that the inclusion functor $\mathcal{C}' \to \mathcal{C}$ is exact. □

In what follows, we extend the correspondence established in [91] in a different direction, namely to the extriangulated category $\mathcal{D}^{[-d+1,0]}$. We begin by introducing the notion of wide subcategories in $\mathcal{D}^{[-d+1,0]}$.

Definition 3.1.3 A subcategory $\mathcal{W} \subseteq \mathcal{D}^{[-d+1,0]}$ is said to be *wide* if it satisfies the following conditions:

(W1) $\mathcal{W}$ is closed under extensions;

(W2) $\mathrm{Hom}(\mathcal{W}, \mathcal{W}[i]) = 0$ for all $i < 0$;

(W3) the extriangulated structure of $\mathcal{D}^{[-d+1,0]}$ restricted to $\mathcal{W}$ is abelian.

Furthermore, a wide subcategory is called *length* if it is a length abelian category.

Remark 3.1.4 Condition (W1) implies that $\mathcal{W}$ inherits an extriangulated structure from $\mathcal{D}^{[-d+1,0]}$; see [79, Remark 2.18]. By [1, Corollary 2.7], condition (W2) then ensures that this inherited extriangulated structure is, in fact, an exact structure. Consequently, condition (W3) means that $\mathcal{W}$ is an abelian category such that, for any $X, Y, Z \in \mathcal{W}$, the sequence

$$X \to Y \to Z$$

is a short exact sequence if and only if it is a conflation in the exact category $\mathcal{W}$. Equivalently, $\mathcal{W}$ is a wide subcategory of itself in the sense of Definition 3.1.1.

Remark 3.1.5 Since any abelian category is idempotent complete, it follows that every wide subcategory of $\mathcal{D}^{[-d+1,0]}$ is closed under direct summands.

In the case where $d = 1$, $\mathcal{D}^{[0,0]} = \mathcal{H}$ is an abelian category, and our definition of wide subcategories coincides with the classical one. As mentioned above, the collection of simple objects in a wide subcategory of an abelian category forms a semibrick. Motivated by this observation, we introduce the following notion of semibricks in $\mathcal{D}^{[-d+1,0]}$, ensuring that the collection of simple objects in a wide subcategory remains a semibrick in this generalized setting.

Definition 3.1.6 A collection of objects $\mathcal{S}$ in $\mathcal{D}^{[-d+1,0]}$ is called a *semibrick* if it satisfies the following conditions:

1. End(S) is a division algebra for all $S \in \mathcal{S}$, that is, every S is a brick.
2. Hom$(\mathcal{S}, \mathcal{S}[i]) = 0$ for all $i < 0$.
3. Hom$(S, S') = 0$ for any distinct $S, S' \in \mathcal{S}$.

Note that when $d = 1$, the collection $\mathcal{S}$ lies in the heart $\mathcal{H}$, and thus condition (2) is automatically satisfied. In this case, Definition 3.1.6 recovers the classical notion of a semibrick in an abelian category (cf. [91]).

We consider semibricks up to isomorphism. Where appropriate, we identify a semibrick $\mathcal{S}$ with the full subcategory consisting of objects in $\mathcal{S} \cup \{0\}$.

For any wide subcategory $\mathcal{W}$ of $\mathcal{D}^{[-d+1,0]}$, since $\mathcal{W}$ is an abelian category, the collection sim $\mathcal{W}$ of its simple objects forms a semibrick. Thus, taking simple objects defines a map

$$\text{sim}\colon \{\text{wide subcategories of } \mathcal{D}^{[-d+1,0]}\} \to \{\text{semibricks in } \mathcal{D}^{[-d+1,0]}\}. \tag{3.1}$$

Conversely, for any semibrick $\mathcal{S}$ in $\mathcal{D}^{[-d+1,0]}$, let $W(\mathcal{S})$ denote its extension closure in $\mathcal{D}^{[-d+1,0]}$, that is, the smallest full subcategory of $\mathcal{D}^{[-d+1,0]}$ containing $\mathcal{S}$ and closed under extensions. Recall that when $d = 1$, the extension closure $W(\mathcal{S})$ coincides with Filt$(\mathcal{S})$, which is well known to be a wide subcategory of $\mathcal{H}$; see [91].

Proposition 3.1.7 *There is a bijection between the set of semibricks in $\mathcal{D}^{[-d+1,0]}$ and the set of length wide subcategories in $\mathcal{D}^{[-d+1,0]}$, given by the assignment*

$$\mathcal{S} \mapsto W(\mathcal{S}),$$

with inverse map given by

$$\mathcal{W} \mapsto \text{sim}\, \mathcal{W}.$$

Proof. Let $\mathcal{S}$ be a semibrick in $\mathcal{D}^{[-d+1,0]}$. By definition, $W(\mathcal{S})$ satisfies condition (W1). Since Hom$(\mathcal{S}, \mathcal{S}[i]) = 0$ for all $i < 0$, it follows by induction on the filtration length that Hom$(W(\mathcal{S}), W(\mathcal{S})[i]) = 0$ for all $i < 0$; thus, $W(\mathcal{S})$ satisfies condition (W2). Consequently, $W(\mathcal{S})$ inherits an exact structure where the conflations are precisely those extriangles in $\mathcal{D}^{[-d+1,0]}$ whose terms all lie in $W(\mathcal{S})$.

In this exact setting, the extension closure $W(\mathcal{S})$ coincides with Filt($\mathcal{S}$). By [37, Theorem 2.5], Filt($\mathcal{S}$) is a length wide subcategory of itself, with its simple objects given by $\mathcal{S}$. Therefore, $W(\mathcal{S})$ is a length wide subcategory of $\mathcal{D}^{[-d+1,0]}$, and the equality sim $W(\mathcal{S}) = \mathcal{S}$ holds.

Conversely, let $\mathcal{W}$ be a length wide subcategory in $\mathcal{D}^{[-d+1,0]}$. By Remark 3.1.4, $\mathcal{W}$ is a wide subcategory of itself. Hence, by [37, Theorem 2.5], we have $\mathcal{W} =$ Filt(sim $\mathcal{W}$). Since Filt(sim $\mathcal{W}$) $= W$(sim $\mathcal{W}$) by the argument above, this completes the proof. $\square$

In the case where $d = 1$, the bijection established above recovers the classical correspondence described in [91, § 1.2].

Example 3.1.8 Let Λ be the path algebra $K(1 \to 2)$ of a quiver of type A_2, and $d = 2$. The Auslander-Reiten quiver of 2- mod Λ is given as follows:

$$\begin{tikzcd}
 & P_1 \arrow[dr] & & P_2[1] \arrow[dr] & & I_1[1] \\
P_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] &
\end{tikzcd}$$

The poset of semibricks and their corresponding wide subcategories in 2- mod Λ is depicted in Figure 3.1. (The poset structure is inherited from the bijection with functorially finite, positive torsion classes; see Theorem 3.4.6.) In the figure, semibricks are indicated by circles, while the corresponding wide subcategories are given by the additive hull of indecomposable objects in the shaded regions. In this specific example, all wide subcategories shown are either semisimple or equivalent to mod Λ, and hence, are abelian categories. Moreover, these constitute all possible wide subcategories; that is, every wide subcategory in this case is length. However, this property does not hold in general; see Example 3.2.7 for a counterexample.

Consider the semibrick $\mathcal{S} = \{P_2, I_1[1]\}$, where $W(\mathcal{S}) =$ add $\mathcal{S}$ is a semisimple abelian category. When $W(\mathcal{S})$ is viewed as a subcategory of $\mathcal{D}^b$(mod Λ), the space Ext$^2(I_1[1], P_2)$ is non-zero. However, when $W(\mathcal{S})$ is considered as an abelian category in its own right, the internal extension group satisfies $\mathrm{Ext}^2_{\mathrm{Ab}}(I_1[1], P_2) = 0$. This illustrates that, in general, higher extensions within $W(\mathcal{S})$ may not coincide with those in the ambient derived category. Note that our definition only requires an equivalence of extriangulated categories, which ensures an isomorphism on Ext^1 but does not necessarily control higher-degree extensions.

While the correspondence between semibricks and length wide subcategories provides a classification of the simple building blocks of $\mathcal{D}^{[-d+1,0]}$, it is natural to ask how these wide subcategories fit into larger structural frameworks, such as those determined by finiteness or closure conditions. In the next section, we shift our focus to d-FAE closed subcategories and investigate their role in constructing wide subcategories, thereby further enriching the structural theory of the d-extended heart.

3.2 From d-FAE closed subcategories to wide subcategories

In this section, we describe a constructive method for obtaining wide subcategories from d-FAE closed subcategories. To this end, we first introduce the notion of an "exact heart" for such subcategories, which serves as a generalization of the classical hearts of t-structures.

Let $\mathcal{T}$ be a d-FAE closed subcategory of $\mathcal{D}^{[-d+1,0]}$. We associate with $\mathcal{T}$ a full subcategory $\mathcal{H}_{\mathcal{T}}$ of $\mathcal{D}$ defined by:

$$\mathcal{H}_{\mathcal{T}} := (\mathcal{D}^{\leq -d} * \mathcal{T}) \cap (\mathcal{T}^{\perp}[1] * \mathcal{D}^{\geq 0}) \subseteq \mathcal{D}^{[-d,0]}.$$

We refer to $\mathcal{H}_{\mathcal{T}}$ as the *exact heart* of $\mathcal{T}$.

Lemma 3.2.1 *Let $\mathcal{T}$ be a d-FAE closed subcategory of $\mathcal{D}^{[-d+1,0]}$. Then the following properties hold:*

(1) Hom$(\mathcal{D}^{\leq -d} * \mathcal{T}, (\mathcal{T}^{\perp}[1] * \mathcal{D}^{\geq 0})[j]) = 0$ *for all* $j < 0$.

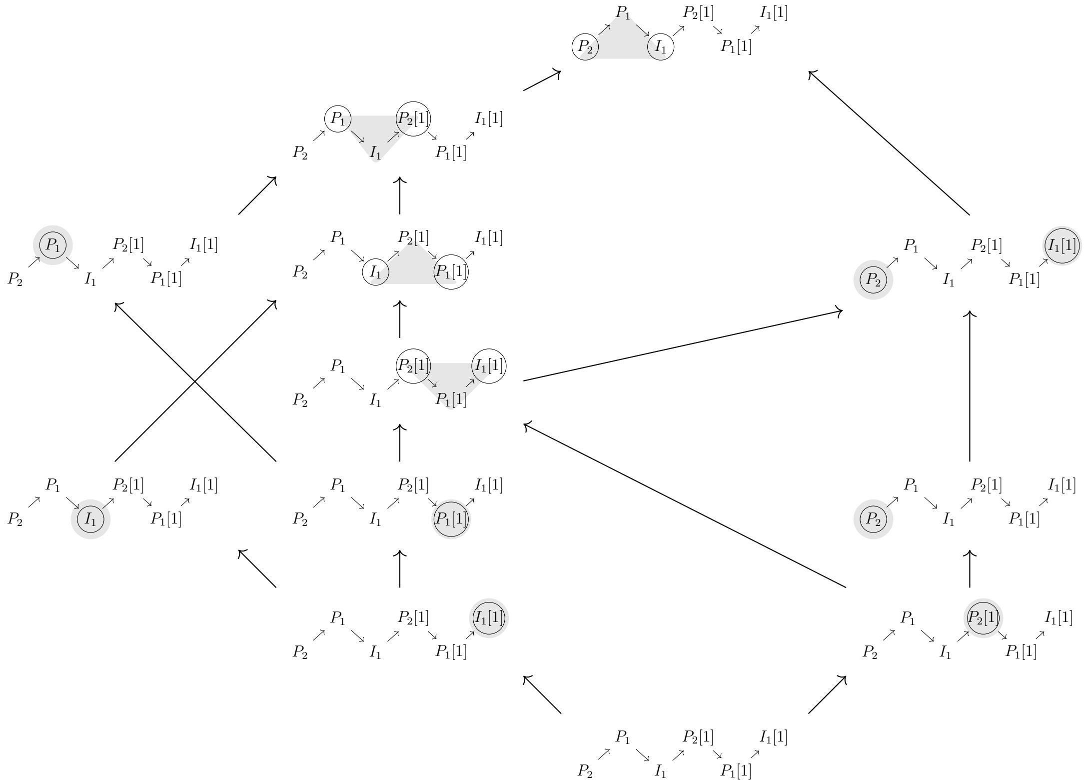


Figure 3.1: Poset of semibricks and length wide subcategories in 2-mod KA_2 (the order is given by the inclusion of the smallest torsion class containing the semibrick/wide subcategory)

(2) Hom$(\mathcal{H}_{\mathcal{T}}, \mathcal{H}_{\mathcal{T}}[j]) = 0$*, for all* $j < 0$*.*

(3) $\mathcal{H}_{\mathcal{T}}$ *is closed under extensions and direct summands.*

(4) $\mathcal{H}_{\mathcal{T}}$ *is an idempotent complete exact category.*

Proof. For (1), since $\mathcal{T}^{\perp}[1] * \mathcal{D}^{\geq 0} \subseteq \mathcal{D}^{\geq -d}$, we have Hom$(\mathcal{D}^{\leq -d}, (\mathcal{T}^{\perp}[1] * \mathcal{D}^{\geq 0})[j]) = 0$ for all $j < 0$. Thus, it suffices to show that

$$\text{Hom}\left(\mathcal{T}, (\mathcal{T}^{\perp}[1] * \mathcal{D}^{\geq 0})[j]\right) = 0, \quad \text{for all } j < 0.$$

By Proposition 1.1.6, the subcategory $\mathcal{T}^{\perp}[1] * \mathcal{D}^{\geq 0}$ is closed under positive shifts. Therefore, it is enough to verify the case $j = -1$. For any $X \in \mathcal{T}^{\perp}[1] * \mathcal{D}^{\geq 0}$, there exists a triangle

$$X_1[1] \to X \to X_2,$$

with $X_1 \in \mathcal{T}^{\perp}$ and $X_2 \in \mathcal{D}^{\geq 0}$. Applying the functor Hom$(\mathcal{T}, -[-1])$ to this triangle, we obtain the exact sequence

$$\text{Hom}(\mathcal{T}, X_1) \to \text{Hom}(\mathcal{T}, X[-1]) \to \text{Hom}(\mathcal{T}, X_2[-1]),$$

where the first term vanishes because $X_1 \in \mathcal{T}^{\perp}$, and the last term vanishes since $\mathcal{T} \subseteq \mathcal{D}^{[-d+1,0]}$ and $X_2[-1] \in \mathcal{D}^{\geq 1}$. This implies Hom$(\mathcal{T}, X[-1]) = 0$, completing the proof of (1). Property (2) then follows directly from (1).

For (3), Remark 1.2.9 ensures that $\mathcal{T}^{\perp}$ is a positive torsion-free class and is thus d-SAE closed. By Proposition 1.1.6, $\mathcal{D}^{\leq -d} * \mathcal{T}$ is a suspended subcategory and $\mathcal{T}^{\perp}[1] * \mathcal{D}^{\geq 0}$ is a cosuspended subcategory of $\mathcal{D}$. Consequently, both are closed under extensions. Furthermore, Remark 1.1.7 implies they are closed under direct summands. It follows that their intersection $\mathcal{H}_{\mathcal{T}}$ is also closed under extensions and direct summands.

For (4), by combining the extension closure from (3) with the vanishing of negative extensions established in (2), $\mathcal{H}_{\mathcal{T}}$ satisfies the necessary conditions to inherit an exact structure from the triangulated structure of $\mathcal{D}$. The proof then follows by an argument analogous to that in Remark 3.1.4. □

Remark 3.2.2 If $\mathcal{T}$ is an s-torsion class, its exact heart $\mathcal{H}_{\mathcal{T}}$ coincides with the heart of the bounded t-structure associated to the s-torsion pair $(\mathcal{T}, \mathcal{T}^{\perp})$; see Proposition 1.3.3. In this case, $\mathcal{H}_{\mathcal{T}}$ is an abelian category.

Since $\mathcal{T}$ is closed under extensions and direct summands, the intersection $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$ is a full subcategory of the exact category $\mathcal{H}_{\mathcal{T}}$ that remains closed under extensions and direct summands. It follows that $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$ is itself an idempotent complete exact category.

Definition 3.2.3 Let $\mathcal{T} \subseteq \mathcal{D}^{[-d+1,0]}$ be a d-FAE closed subcategory. We define a full subcategory $W_L(\mathcal{T})$ of $\mathcal{D}^{[-d+1,0]}$ as follows:

$$W_L(\mathcal{T}) := \{X \in \mathcal{T} \cap \mathcal{H}_{\mathcal{T}} \mid \text{every morphism in } \mathcal{T} \cap \mathcal{H}_{\mathcal{T}} \text{ to } X \text{ is admissible}\}.$$

Applying Proposition 3.1.1 to the exact category $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$ yields the main result of this section.

Proposition 3.2.4 *Let $\mathcal{T} \subseteq \mathcal{D}^{[-d+1,0]}$ be a d-FAE closed subcategory. Then $W_L(\mathcal{T})$ is a wide subcategory of $\mathcal{D}^{[-d+1,0]}$.*

Proof. Since $W_L(\mathcal{T}) \subseteq \mathcal{T} \cap \mathcal{H}_{\mathcal{T}} \subseteq \mathcal{H}_{\mathcal{T}}$, Lemma 3.2.1(2) ensures that

$$\mathrm{Hom}(W_L(\mathcal{T}), W_L(\mathcal{T})[i]) = 0 \quad \text{for all } i < 0.$$

As $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$ is an idempotent complete exact category, Proposition 3.1.2 implies that $W_L(\mathcal{T})$ is a wide subcategory of $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$. In particular, $W_L(\mathcal{T})$ is closed under extensions in $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$, and consequently in $\mathcal{D}^{[-d+1,0]}$. Furthermore, since $W_L(\mathcal{T})$ is a wide subcategory of itself (viewed as an exact category), it satisfies the requirements of Definition 3.1.1. Therefore, $W_L(\mathcal{T})$ is a wide subcategory of $\mathcal{D}^{[-d+1,0]}$. □

Consequently, the assignment $\mathcal{T} \mapsto W_L(\mathcal{T})$ defines a map

$$W_L \colon \{d\text{-FAE closed subcategories of } \mathcal{D}^{[-d+1,0]}\} \to \{\text{wide subcategories of } \mathcal{D}^{[-d+1,0]}\}. \tag{3.2}$$

Remark 3.2.5 We show that the assignment W_L generalizes the map α of [57, 74]. In the case where $d = 1$, $\mathcal{D}^{[-d+1,0]} = \mathcal{H}$ is an abelian category, and $\mathcal{T}$ is a full subcategory of $\mathcal{H}$ closed under extensions and factors. Since $\mathcal{T} \subseteq \mathcal{H}_{\mathcal{T}}$, we have $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}} = \mathcal{T}$. In [57, § 2.3] (see also [64, Exercise 8.23] as mentioned in [74, § 3]), the authors associated with $\mathcal{T}$ a wide subcategory $\alpha(\mathcal{T})$ of $\mathcal{H}$ defined by:

$$\alpha(\mathcal{T}) := \{X \in \mathcal{T} \mid \forall\, (g : Y \to X) \in \mathcal{T},\ \mathrm{ker}(g) \in \mathcal{T}\}.$$

We claim that, in this case, $W_L(\mathcal{T}) = \alpha(\mathcal{T})$.

To see this, let $X \in W_L(\mathcal{T})$ and let $g : Y \to X$ be a morphism in $\mathcal{T}$. Since g is admissible in the exact category $\mathcal{T}$, it admits a factorization $g = g_2 \circ g_1$ corresponding to conflations in $\mathcal{T}$:

$$X_1 \rightarrowtail Y \overset{g_1}{\twoheadrightarrow} X_2 \quad \text{and} \quad X_2 \overset{g_2}{\rightarrowtail} X \twoheadrightarrow X_3,$$

with $X_1, X_2, X_3 \in \mathcal{T}$. Since $\mathcal{T} \subseteq \mathcal{H}$, these conflations are short exact sequences in $\mathcal{H}$. Consequently, $\mathrm{ker}(g) \cong \mathrm{ker}(g_1) = X_1 \in \mathcal{T}$, which proves that $X \in \alpha(\mathcal{T})$.

Conversely, let $X \in \alpha(\mathcal{T})$ and let $g \colon Y \to X$ be a morphism in $\mathcal{T}$. In the abelian category $\mathcal{H}$, we have the standard short exact sequences:

$$0 \to \mathrm{ker}(g) \to Y \to \mathrm{im}(g) \to 0 \quad \text{and} \quad 0 \to \mathrm{im}(g) \to X \to \mathrm{coker}(g) \to 0.$$

Since $\mathcal{T}$ is closed under factors, it follows that $\mathrm{im}(g), \mathrm{coker}(g) \in \mathcal{T}$. Moreover, the assumption $X \in \alpha(\mathcal{T})$ ensures that $\mathrm{ker}(g) \in \mathcal{T}$. Thus, both sequences are conflations in $\mathcal{T}$. This implies that g is an admissible morphism in $\mathcal{T}$, and therefore $X \in W_L(\mathcal{T})$.

Remark 3.2.6 The full subcategory $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$ admits the following alternative characterization:

$$\mathcal{T} \cap \mathcal{H}_{\mathcal{T}} = \{Y \in \mathcal{T} \mid \mathsf{Hom}(\mathcal{T}, Y[-1]) = 0\},$$

that depends only on the structure of negative extensions on $\mathcal{T}$, and does not require calculating the heart $\mathcal{H}_{\mathcal{T}}$.

The inclusion $\subseteq$ follows immediately from Lemma 3.2.1(1), which states that $\mathsf{Hom}(\mathcal{T}, \mathcal{H}_{\mathcal{T}}[j]) = 0$ for all $j < 0$. To verify the converse inclusion, let $Y \in \mathcal{T}$ such that $\mathsf{Hom}(\mathcal{T}, Y[-1]) = 0$. Consider the truncation triangle

$$\sigma_{\leq 0}(Y[-1]) \to Y[-1] \to \sigma_{\geq 1}(Y[-1]).$$

Applying $\mathsf{Hom}(\mathcal{T}, -)$ to this triangle and noting that $\mathsf{Hom}(\mathcal{T}, \sigma_{\geq 1}(Y[-1])[-1]) = 0$ (as $\sigma_{\geq 1}(Y[-1]) \in \mathcal{D}^{\geq 1}$ and $\mathcal{T} \subseteq \mathcal{D}^{\leq 0}$), we obtain $\mathsf{Hom}(\mathcal{T}, \sigma_{\leq 0}(Y[-1])) = 0$. This implies $\sigma_{\leq 0}(Y[-1]) \in \mathcal{T}^{\perp}$, and thus $Y \in \mathcal{T}^{\perp}[1] * \mathcal{D}^{\geq 0}$. Since Y is also in $\mathcal{T} \subseteq \mathcal{D}^{\leq -d} * \mathcal{T}$, we conclude $Y \in \mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$.

Example 3.2.7 Let Q be the Kronecker quiver

$$Q = \quad 1 \rightrightarrows 2$$

and $\Lambda = KQ$. Let $\mathcal{P}$, $\mathcal{R}$, and $\mathcal{I}$ denote the preprojective, regular, and postinjective components of the Auslander-Reiten quiver of KQ, respectively. Let $d = 2$. Then the additive hull of $\mathcal{R} \cup \mathcal{I} \cup \mathcal{P}[1]$ in 2-$\operatorname{mod} \Lambda$ is a wide subcategory that is not length.

Now, take an arbitrary $M \in \mathcal{I}$ and consider the positive torsion class

$$\mathcal{T} := \overline{M[1]},$$

defined as the additive hull of $M[1]$ and all objects to its right in the Auslander-Reiten quiver of 2-mod Λ (as indicated by the shaded region in Figure 3.2). Since $\overline{M[1]} \subseteq (\operatorname{mod} \Lambda)[1]$, it follows from Remark 3.2.6 that $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}} = \overline{M[1]}$.

We claim that $W_L(\mathcal{T}) = \operatorname{add} M[1]$. By Remark 3.1.5, $W_L(\mathcal{T})$ is closed under direct summands; thus, it suffices to show that $M[1]$ is the only indecomposable object in $W_L(\mathcal{T})$. For any indecomposable object $N \in \overline{M[1]}$ not isomorphic to $M[1]$, say

$$(\ K^{r+1} \rightrightarrows K^{r}\ )[1],$$

there exists an irreducible surjection in $\operatorname{mod} \Lambda$

$$f : (\ K^{r+2} \rightrightarrows K^{r+1}\ ) \to (\ K^{r+1} \rightrightarrows K^{r}\ )$$

Then $f[1]$ is a morphism in $\overline{M[1]}$. If $f[1]$ were admissible in the exact category $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$, it would admit a factorization $f[1] = h \circ g$, where g is a deflation and h is an inflation. Since $f[1]$ is irreducible, either g must be a section (making g an isomorphism) or h must be a retraction (making h an isomorphism). In the first case, $f[1]$ would be an inflation, implying its cone $C(f[1]) \in \mathcal{R}[2]$, which is not contained in $\overline{M[1]}$. In the second case, $f[1]$ would be a deflation, implying $C(f[1])[-1] \in \mathcal{R}[1]$, which is again not contained in $\overline{M[1]}$. Hence, $N \notin W_L(\mathcal{T})$, and we conclude that $W_L(\mathcal{T}) = \operatorname{add} M[1]$.

This calculation provides a concrete realization of the general theory; in particular, the result aligns with the classification established in Theorem 3.4.10, identifying $\overline{M[1]}$ as the positive torsion class $T(\{M[1]\})$.

We now investigate the simple objects in the wide subcategory $W_L(\mathcal{T})$, or equivalently, the composition of the maps W_L from (3.2) and sim from (3.1). For any d-FAE closed subcategory $\mathcal{T}$ of $\mathcal{D}^{[-d+1,0]}$, we denote by $\operatorname{sim} \mathcal{H}_{\mathcal{T}}$ the set of (isoclasses of) simple objects in the exact heart $\mathcal{H}_{\mathcal{T}}$.

Lemma 3.2.8 *Let $\mathcal{T} \subseteq \mathcal{D}^{[-d+1,0]}$ be a d-FAE closed subcategory. Then any simple object in the wide subcategory $W_L(\mathcal{T})$ is also a simple object in the exact heart $\mathcal{H}_{\mathcal{T}}$. In particular, we have*

$$\operatorname{sim} W_L(\mathcal{T}) \subseteq (\operatorname{sim} \mathcal{H}_{\mathcal{T}}) \cap \mathcal{D}^{[-d+1,0]}.$$

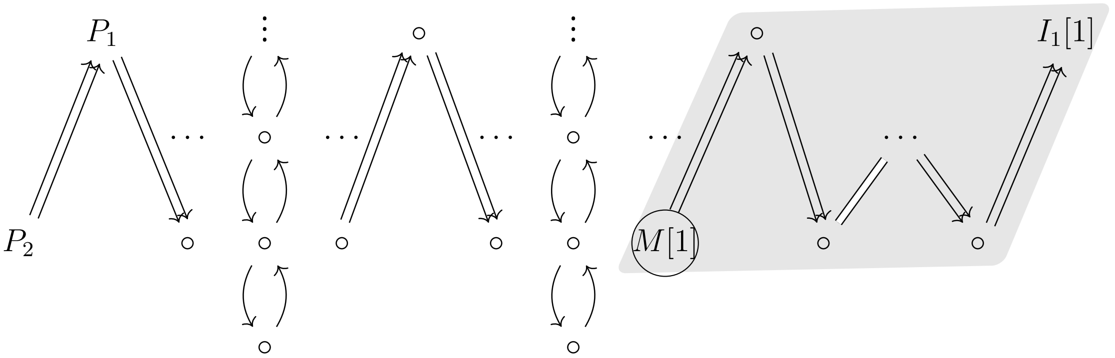


Figure 3.2: An example of $W_L(\mathcal{T})$ for a positive torsion class $\mathcal{T}$

Proof. Since $W_L(\mathcal{T}) \subseteq \mathcal{T} \cap \mathcal{H}_{\mathcal{T}} \subseteq \mathcal{D}^{[-d+1,0]}$, it suffices to prove the first statement. Let $S \in \operatorname{sim} W_L(\mathcal{T})$, and consider an arbitrary conflation

$$X \overset{f}{\rightarrowtail} S \twoheadrightarrow Y$$

in $\mathcal{H}_{\mathcal{T}}$. We must show that either $X = 0$ or $Y = 0$. Suppose $X \neq 0$. It follows from the above conflation that

$$X \in \mathcal{H}_{\mathcal{T}} \cap (Y[-1] * S) \subseteq \mathcal{D}^{[-d,0]} \cap \mathcal{D}^{[-d+1,1]} = \mathcal{D}^{[-d+1,0]}.$$

Moreover, as $X \in \mathcal{H}_{\mathcal{T}} \subseteq \mathcal{D}^{\leq -d} * \mathcal{T}$, we have $X \in \mathcal{T}$. Thus $X \in \mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$.

Since $S \in W_L(\mathcal{T})$, the morphism $f\colon X \to S$ is admissible in the exact category $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$. Thus, there exist conflations

$$X' \rightarrowtail X \overset{f_1}{\twoheadrightarrow} X'' \quad \text{and} \quad X'' \overset{f_2}{\rightarrowtail} S \twoheadrightarrow X'''$$

in $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$ such that $f = f_2 \circ f_1$. By Proposition 3.1.2, $W_L(\mathcal{T})$ is closed under subobjects and factors in $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$; therefore, both X'' and X''' belong to $W_L(\mathcal{T})$. Since S is simple in $W_L(\mathcal{T})$, we must have either $X'' = 0$ or $X''' = 0$.

The assumption $X \neq 0$ implies $f \neq 0$, which forces $f_2 \neq 0$ and $X'' \neq 0$. This implies $X''' = 0$, meaning the map $f_2\colon X'' \rightarrowtail S$ is an isomorphism. Consequently, f is equivalent to the deflation f_1 in $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$, with kernel $X' \cong Y[-1]$. However, since $X' \in \mathcal{T} \cap \mathcal{H}_{\mathcal{T}} \subseteq \mathcal{H}_{\mathcal{T}}$ and $Y \in \mathcal{H}_{\mathcal{T}}$, the condition $X' \cong Y[-1]$ is only possible if $Y = 0$, as $\operatorname{Hom}(\mathcal{H}_{\mathcal{T}}, \mathcal{H}_{\mathcal{T}}[-1]) = 0$. This completes the proof. ☐

So far, we have constructed a path from d-FAE closed subcategories to wide subcategories via the exact heart construction. Specifically, we have shown that the semibrick associated with such a wide subcategory is contained in the set of simple objects of the corresponding exact heart. To complete the structural correspondence, it is natural to investigate the inverse direction. In the following section, we shall study the construction that leads from semibricks back to the torsion-like classes introduced in Chapter 1, such as d-FAE closed subcategories and positive torsion classes. This development will culminate in the main classification theorems of this paper.

3.3 From semibricks to torsion-like classes

For module categories of finite-dimensional algebras, it was shown in [8, Lemma 2.8] that the assignment sending a semibrick $\mathcal{S}$ in $\operatorname{mod} \Lambda$ to the smallest torsion class containing $\mathcal{S}$ is injective. Recall from Chapter 1 that we introduced three notions of torsion-like classes in $\mathcal{D}^{[-d+1,0]}$. While these notions coincide with the classical concept of torsion classes in module categories, they are generally not equivalent for $d > 1$. Among these, d-FAE closed subcategories and positive torsion classes are closed under intersections.

Consequently, we can define the following two maps:

$$\begin{tikzcd} & \{\text{positive torsion classes in } \mathcal{D}^{[-d+1,0]}\} \\ \{\text{semibricks in } \mathcal{D}^{[-d+1,0]}\} \arrow[ur, "T"] \arrow[dr, "\phi"'] & \\ & \{d\text{-FAE closed subcategories in } \mathcal{D}^{[-d+1,0]}\} \end{tikzcd} \qquad (3.3)$$

These two maps send a semibrick $\mathcal{S}$ in $\mathcal{D}^{[-d+1,0]}$ to the smallest positive torsion class $T(\mathcal{S})$ containing $\mathcal{S}$, and the smallest d-FAE closed subcategory $\phi(\mathcal{S})$ containing $\mathcal{S}$, respectively. As in Section 3.1, we identify a semibrick $\mathcal{S}$ with the full subcategory consisting of its objects and their direct sums. Since every positive torsion class is d-FAE closed, the inclusion $\phi(\mathcal{S}) \subseteq T(\mathcal{S})$ holds.

We remark that if $\mathcal{D}$ is K-linear, Hom-finite, and Krull–Schmidt, and if the extended heart $\mathcal{D}^{[-d+1,0]}$ contains only finitely many indecomposable objects, then every subcategory of $\mathcal{D}^{[-d+1,0]}$ closed under direct sums and summands is functorially finite. In such cases, Proposition 1.3.5 implies that $\phi(\mathcal{S}) = T(\mathcal{S})$. However, whether this equality holds in general is not pursued in this paper.

In this section, we will prove that both ϕ and T are injective. Furthermore, we will show that the simple objects in the wide subcategory $W_L(\phi(\mathcal{S}))$, constructed from $\phi(\mathcal{S})$ as in Section 3.2, are precisely the elements of $\mathcal{S}$. Although it remains unclear whether a similar characterization holds for $T(\mathcal{S})$ in general, we will show in the subsequent section that $T(\mathcal{S})$ satisfies this property whenever $\mathcal{S}$ satisfies certain finiteness conditions, namely being left-finite. On this basis, we will establish bijections between left/right-finite semibricks, functorially finite torsion/torsion-free classes, and $(d+1)$-term simple-minded collections. Crucially, these bijections are naturally formulated in terms of T and cannot be recovered by ϕ alone, highlighting the indispensable role of positive torsion classes in the classification of simple-minded collections.

Let $\mathcal{S}$ be a semibrick in $\mathcal{D}^{[-d+1,0]}$. We denote by Tor$(\mathcal{S})$ the set of d-FAE closed subcategories $\mathcal{T}$ of $\mathcal{D}^{[-d+1,0]}$ that contain $\mathcal{S}$ and satisfy $\mathcal{T}^{\perp} = \mathcal{S}^{\perp_{\leq 0}}$.

Lemma 3.3.1 *Both $T(\mathcal{S})$ and $\phi(\mathcal{S})$ belong to* Tor$(\mathcal{S})$*. Moreover, for any $\mathcal{T} \in$* Tor$(\mathcal{S})$*, the inclusions $\phi(\mathcal{S}) \subseteq \mathcal{T} \subseteq T(\mathcal{S})$ hold.*

Proof. According to Proposition 1.2.8, $(T(\mathcal{S}), \mathcal{S}^{\perp_{\leq 0}})$ is a positive torsion pair. This implies $T(\mathcal{S})^{\perp} = \mathcal{S}^{\perp_{\leq 0}}$ and ${}^{\perp}(\mathcal{S}^{\perp_{\leq 0}}) = T(\mathcal{S})$. The first equality ensures that $T(\mathcal{S}) \in$ Tor$(\mathcal{S})$, while the second equality implies that any $\mathcal{T} \in$ Tor$(\mathcal{S})$, being a subcategory of ${}^{\perp}(\mathcal{T}^{\perp}) = {}^{\perp}(\mathcal{S}^{\perp_{\leq 0}})$, must be contained in $T(\mathcal{S})$.

Next, since $\phi(\mathcal{S}) \subseteq T(\mathcal{S})$, we have $\phi(\mathcal{S})^{\perp} \supseteq T(\mathcal{S})^{\perp} = \mathcal{S}^{\perp_{\leq 0}}$. Recall that $\phi(\mathcal{S})$ is d-FAE closed, so Remark 1.2.9 yields $\phi(\mathcal{S})^{\perp} = \phi(\mathcal{S})^{\perp_{\leq 0}}$. Furthermore, as $\mathcal{S} \subseteq \phi(\mathcal{S})$, it follows that $\mathcal{S}^{\perp_{\leq 0}} \supseteq \phi(\mathcal{S})^{\perp_{\leq 0}}$. Combining these inclusions, we obtain $\phi(\mathcal{S})^{\perp} = \mathcal{S}^{\perp_{\leq 0}}$, which means $\phi(\mathcal{S}) \in$ Tor$(\mathcal{S})$. Finally, for any $\mathcal{T} \in$ Tor$(\mathcal{S})$, since $\mathcal{T}$ is by definition a d-FAE closed subcategory containing $\mathcal{S}$, it must contain the smallest such subcategory $\phi(\mathcal{S})$. This establishes the inclusion $\phi(\mathcal{S}) \subseteq \mathcal{T}$. □

Recall from Section 3.2 that, for a d-FAE closed subcategory $\mathcal{T}$ of $\mathcal{D}^{[-d+1,0]}$, the exact heart of $\mathcal{T}$ is defined as

$$\mathcal{H}_{\mathcal{T}} := (\mathcal{D}^{\leq -d} * \mathcal{T}) \cap (\mathcal{T}^{\perp}[1] * \mathcal{D}^{\geq 0}).$$

We denote the set of (isoclasses of) simple objects in $\mathcal{H}_{\mathcal{T}}$ by sim $\mathcal{H}_{\mathcal{T}}$. The following proposition guarantees that we can recover the original semibrick $\mathcal{S}$ from the exact heart of any $\mathcal{T} \in$ Tor$(\mathcal{S})$.

Proposition 3.3.2 *Let $\mathcal{S}$ be a semibrick in $\mathcal{D}^{[-d+1,0]}$ and let $\mathcal{T} \in$* Tor$(\mathcal{S})$*. Then*

$$\mathcal{S} = (\text{sim}\, \mathcal{H}_{\mathcal{T}}) \cap \mathcal{D}^{[-d+1,0]}.$$

Proof. We divide the proof into three steps.

Step 1: $\boldsymbol{\mathcal{S} \subseteq \mathcal{H}_{\mathcal{T}}}$. Since $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{D}^{\leq -d} * \mathcal{T}$, it suffices to show $\mathcal{S} \subseteq T^{\perp}[1] * \mathcal{D}^{\geq 0}$. Let $X \in \mathcal{S}$. Applying the functor $\mathsf{Hom}(\mathcal{S}, -)$ to the truncation triangle

$$\sigma_{\leq -1}(X) \to X \to \sigma_{\geq 0}(X),$$

we obtain the exact sequence

$$\mathsf{Hom}(\mathcal{S}, (\sigma_{\geq 0}(X))[j]) \to \mathsf{Hom}(\mathcal{S}, (\sigma_{\leq -1}(X))[j+1]) \to \mathsf{Hom}(\mathcal{S}, X[j+1])).$$

When $j \leq -2$, the first term vanishes because $\mathcal{S} \subseteq \mathcal{D}^{[-d+1,0]}$ and $\sigma_{\geq 0}(X)[j] \in \mathcal{D}^{\geq 2}$, while the last term vanishes since $j+1 \leq -1$ and $\mathcal{S}$ is a semibrick. Consequently, the middle term also vanishes, which implies $(\sigma_{\leq -1}(X))[-1] \in \mathcal{S}^{\perp_{\leq 0}} = \mathcal{T}^{\perp}$. Thus, $X \in \sigma_{\leq -1}(X) * \sigma_{\geq 0}(X) \subseteq T^{\perp}[1] * \mathcal{D}^{\geq 0}$, and we conclude $\mathcal{S} \subseteq \mathcal{H}_{\mathcal{T}}$.

Step 2: $\boldsymbol{\mathcal{S} \subseteq}$ sim $\boldsymbol{\mathcal{H}_{\mathcal{T}}}$. Let $X \in \mathcal{S}$ and let $L \overset{f}{\rightarrowtail} X \twoheadrightarrow M$ be a conflation in $\mathcal{H}_{\mathcal{T}}$. We show that either $L = 0$ or $M = 0$. Suppose $L \neq 0$. Since $L \in \mathcal{H}_{\mathcal{T}} \subseteq \mathcal{D}^{\leq -d} * \mathcal{T}$, there exists a triangle $L_1 \xrightarrow{g} L \xrightarrow{h} L_2$ with $L_1 \in \mathcal{D}^{\leq -d}$ and $L_2 \in \mathcal{T}$. Since $X \in \mathcal{T}$ and $\mathsf{Hom}(\mathcal{D}^{\leq -d}, \mathcal{T}) = 0$, we have $f \circ g = 0$. Thus, f factors through h, yielding the following commutative diagram of triangles via the octahedral axiom:

$$\begin{tikzcd}
L_1 \arrow[r, equal] \arrow[d, "g'"'] & L_1 \arrow[d, "g"] & \\
M[-1] \arrow[r] \arrow[d] & L \arrow[r, "f"] \arrow[d, "h"] & X \arrow[d, equal] \\
W \arrow[r] & L_2 \arrow[r] & X
\end{tikzcd}$$

Given $M[-1] \in \mathcal{D}^{[-d+1,1]}$ and $L_1 \in \mathcal{D}^{\leq -d}$, we have $g' = 0$, which implies $g = 0$. Consequently, $L_2 \cong L_1[1] \oplus L$, which forces $L_1 = 0$ and $L \cong L_2 \in \mathcal{T}$. Since $L \neq 0$, $L \notin \mathcal{T}^{\perp} = \mathcal{S}^{\perp_{\leq 0}}$, and hence, there exists a non-zero morphism $f' : X' \to L[m]$ for some $X' \in \mathcal{S}$ and $m \leq 0$.

Since both X' and L belong to $\mathcal{H}_{\mathcal{T}}$, Lemma 3.2.1(2) implies $\mathsf{Hom}(\mathcal{H}_{\mathcal{T}}, \mathcal{H}_{\mathcal{T}}[m]) = 0$ for $m < 0$, forcing $m = 0$. Moreover, $f \circ f'$ is non-zero; otherwise, by the triangle in the second row of the diagram above, f' would factor through $M[-1] \in \mathcal{H}_{\mathcal{T}}[-1]$, contradicting $\mathsf{Hom}(\mathcal{H}_{\mathcal{T}}, \mathcal{H}_{\mathcal{T}}[-1]) = 0$ from Lemma 3.2.1(2). Since $\mathcal{S}$ is a semibrick and $X', X \in \mathcal{S}$, the non-zero morphism $f \circ f' \colon X' \to X$ must be an isomorphism. It follows that f is a retraction in the exact category $\mathcal{H}_{\mathcal{T}}$. As f is also an inflation (being part of a conflation), it must be an isomorphism, which implies $M = 0$. This proves $X \in \mathsf{sim}\,\mathcal{H}_{\mathcal{T}}$.

Step 3: $\boldsymbol{(}$sim $\boldsymbol{\mathcal{H}_{\mathcal{T}} \cap \mathcal{D}^{[-d+1,0]}) \setminus \mathcal{S} = \emptyset}$. Suppose there exists $X \in (\mathsf{sim}\,\mathcal{H}_{\mathcal{T}} \cap \mathcal{D}^{[-d+1,0]}) \setminus \mathcal{S}$. Then $X \in \mathcal{H}_{\mathcal{T}} \cap \mathcal{D}^{[-d+1,0]} \subseteq \mathcal{T}$. If there exists a non-zero $f \colon S \to X$ with $S \in \mathcal{S}$, then $C(f) \in \mathcal{H}_{\mathcal{T}}$ (proof below). This makes $S \xrightarrow{f} X \to C(f)$ a conflation in $\mathcal{H}_{\mathcal{T}}$. As X is simple, $C(f) = 0$ and f is an isomorphism, contradicting $X \notin \mathcal{S}$. If no such f exists, then $\mathsf{Hom}(\mathcal{S}, X) = 0$. Since $\mathcal{S} \subseteq \mathcal{H}_{\mathcal{T}}$ and $X \in \mathcal{H}_{\mathcal{T}}$, Lemma 3.2.1(2) implies that $\mathsf{Hom}(\mathcal{S}, X[i]) = 0$ for all $i < 0$. Consequently, $X \in \mathcal{S}^{\perp_{\leq 0}} = \mathcal{T}^{\perp}$. Since X is also an element of $\mathcal{T}$, we must have $X = 0$, which contradicts the fact that X is a simple object in $\mathcal{H}_{\mathcal{T}}$.

Finally, we prove $C(f) \in \mathcal{H}_{\mathcal{T}}$. Lemma 1.1.4(2) ensures that $\sigma_{\geq -d+1}(C(f)) \in \mathcal{T}$, which implies $C(f) \in \mathcal{D}^{\leq -d} * \mathcal{T}$. Let $S' \in \mathcal{S}$. Applying the functor $\mathsf{Hom}(S', -)$ to the triangle $S \xrightarrow{f} X \to C(f)$, we obtain the following exact sequence for any $j \leq 0$:

$$\mathsf{Hom}(S', X[j-1]) \to \mathsf{Hom}(S', C(f)[j-1]) \to \mathsf{Hom}(S', S[j]) \to \mathsf{Hom}(S', X[j]).$$

The first term vanishes since $S', X \in \mathcal{H}_{\mathcal{T}}$ and $j - 1 < 0$ (applying Lemma 3.2.1(2)).

- For $j = 0$: If $S' \neq S$, then $\mathsf{Hom}(S', S) = 0$ as $\mathcal{S}$ is a semibrick. If $S' = S$, the last map $\mathsf{Hom}(S', f)$ in the sequence is injective because $f \neq 0$ and S is a brick. In either case, we obtain $\mathsf{Hom}(S', C(f)[-1]) = 0$.

- For $j < 0$: The third term $\mathsf{Hom}(S', S[j])$ vanishes because $\mathcal{S}$ is a semibrick.

It follows that Hom$(S', C(f)[k]) = 0$ for all $k \leq -1$. We then apply Hom$(S', -)$ to the truncation triangle

$$\sigma_{\leq 0}(C(f)[-1]) \to C(f)[-1] \to \sigma_{\geq 1}(C(f)[-1]),$$

yielding the exact sequence

$$\mathrm{Hom}(S', \sigma_{\geq 1}(C(f)[-1])[j]) \to \mathrm{Hom}(S', \sigma_{\leq 0}(C(f)[-1])[j+1]) \to \mathrm{Hom}(S', C(f)[j]).$$

For all $j \leq 0$, the first term vanishes because $\sigma_{\geq 1}(C(f)[-1]) \in \mathcal{D}^{\geq 1}$ and $S' \in \mathcal{D}^{\leq 0}$, while the last term vanishes by our previous calculation. This implies that $\sigma_{\leq 0}(C(f)[-1]) \in \mathcal{S}^{\perp_{\leq 0}} = \mathcal{T}^{\perp}$. Consequently, $C(f)[-1] \in \mathcal{T}^{\perp} * \mathcal{D}^{\geq 1}$, or equivalently, $C(f) \in \mathcal{T}^{\perp}[1] * \mathcal{D}^{\geq 0}$. Combined with $C(f) \in \mathcal{D}^{\leq -d} * \mathcal{T}$, we conclude $C(f) \in \mathcal{H}_{\mathcal{T}}$. ☐

Applying Proposition 3.3.2 to the specific cases where $\mathcal{T} = T(\mathcal{S})$ and $\mathcal{T} = \phi(\mathcal{S})$, we obtain the following consequence.

Corollary 3.3.3 *The maps T and ϕ defined in* (3.3) *are both injective.*

Proof. Suppose $\mathcal{S}$ and $\mathcal{S}'$ are two semibricks in $\mathcal{D}^{[-d+1,0]}$ such that $T(\mathcal{S}) = T(\mathcal{S}')$. According to Lemma 3.3.1, we have $T(\mathcal{S}) \in \mathrm{Tor}(\mathcal{S})$ and $T(\mathcal{S}') \in \mathrm{Tor}(\mathcal{S}')$. Proposition 3.3.2 then implies that

$$\mathcal{S} = (\mathrm{sim}\, \mathcal{H}_{T(\mathcal{S})}) \cap \mathcal{D}^{[-d+1,0]} = (\mathrm{sim}\, \mathcal{H}_{T(\mathcal{S}')}) \cap \mathcal{D}^{[-d+1,0]} = \mathcal{S}'.$$

This establishes the injectivity of T. The injectivity of ϕ follows from an analogous argument. ☐

The following is another immediate consequence of Proposition 3.3.2, which will be used later.

Lemma 3.3.4 *Let $\mathcal{S}$ be a semibrick in $\mathcal{D}^{[-d+1,0]}$ and let $\mathcal{T} \in \mathrm{Tor}(\mathcal{S})$. Then*

$$\mathrm{Hom}(\mathcal{T}, \mathcal{S}[j]) = 0 \text{ for all } j < 0.$$

Proof. By Proposition 3.3.2, we have $\mathcal{S} \subseteq \mathcal{H}_{\mathcal{T}}$. The required vanishing follows directly from Lemma 3.2.1(1) and the fact that $\mathcal{T} \subseteq \mathcal{D}^{\leq -d} * \mathcal{T}$. ☐

Recall from Lemma 1.1.9 that there is an explicit description of the smallest d-FAE closed subcategory containing a given subcategory. From this, we can deduce the following property of $\phi(\mathcal{S})$. In the classical case, this recovers [8, Lemma 2.7(1)].

Lemma 3.3.5 *For any non-zero morphism $f\colon X \to S$ with $X \in \phi(\mathcal{S})$ and $S \in \mathcal{S}$, the cocone $C(f)[-1]$ belongs to $\phi(\mathcal{S})$.*

Proof. By Lemma 1.1.9, X belongs to $\Phi^s(\mathcal{S})$ for some $s \geq 0$, where $\Phi^0(\mathcal{S}) = \mathcal{S}$ and

$$\Phi^i(\mathcal{S}) = \mathrm{Fac}_d\left(\Phi^{i-1}(\mathcal{S})\right) * \mathrm{Fac}_d\left(\Phi^{i-1}(\mathcal{S})\right), \quad \text{for any } i > 0.$$

We proceed by induction on s to prove the result.

The base case $s = 0$ implies $X \in \mathcal{S}$. Since $\mathcal{S}$ is a semibrick, the non-zero morphism f is an isomorphism, hence $C(f)[-1] = 0 \in \phi(\mathcal{S})$. Now, assume the result holds for all $s \leq m$, and let $s = m + 1$.

We first establish a claim: for any non-zero morphism $g\colon Z \to S$ with $Z \in \mathrm{Fac}_d(\Phi^m(\mathcal{S}))$, the cocone $C(g)[-1]$ belongs to $\phi(\mathcal{S})$. By the definition of Fac$_d$, there exist d triangles

$$Z_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_{i-1} \xrightarrow{h_i} Z_i[1],\ 1 \leq i \leq d,$$

with $Z_0 = Z, Z_1, \ldots, Z_d \in \mathcal{D}^{[-d+1,0]}$, and $Y_1, \ldots, Y_d \in \Phi^m(\mathcal{S})$. We claim that the composition $g \circ g_1 : Y_1 \to S$ is non-zero. Otherwise, g would factor through h_1 as shown in the following diagram:

$$\begin{tikzcd}
Z_1 \arrow[r, "f_1"] & Y_1 \arrow[r, "g_1"] & Z_0 \arrow[r, "h_1"] \arrow[d, "g"] & Z_1[1] \arrow[dl, dashed, "\alpha_1"] \\
 & & S &
\end{tikzcd}$$

By Lemma 3.3.4, Hom$(Y_2, S[-1]) = 0$, which implies that the composition $\alpha_1[-1] \circ g_2 \colon Y_2 \to S[-1]$ vanishes. Thus, $\alpha_1[-1]$ factors through h_2:

$$\begin{tikzcd}
Z_2 \arrow[r, "f_2"] & Y_2 \arrow[r, "g_2"] & Z_1 \arrow[r, "h_2"] \arrow[d, "{\alpha_1[-1]}"'] & Z_2[1] \arrow[dl, dashed, "\alpha_2"] \\
 & & S[-1] &
\end{tikzcd}$$

Continuing this process, we obtain a factorization

$$g = \alpha_d[d-1] \circ h_d[d-1] \circ \cdots \circ h_2[1] \circ h_1,$$

where $\alpha_d : Z_d[1] \to S[-d+1]$ has to be zero because both Z_d and S belong to $\mathcal{D}^{[-d+1,0]}$. This implies $g = 0$, a contradiction. Thus, $g \circ g_1 \neq 0$. By the octahedral axiom, we have the following commutative diagram:

$$\begin{tikzcd}
Z_1 \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d] & C(g)[-1] \arrow[d] \\
Z_1 \arrow[r, "f_1"] & Y_1 \arrow[r, "g_1"] \arrow[d, "g \circ g_1"] & Z_0 \arrow[d, "g"] \\
 & S \arrow[r, equal] & S
\end{tikzcd}$$

By the inductive hypothesis applied to $Y_1 \in \Phi^m(\mathcal{S})$, the object $V = C(g \circ g_1)[-1]$ belongs to $\phi(\mathcal{S}) \subseteq \mathcal{D}^{[-d+1,0]}$. It follows that

$$C(g)[-1] \in (V * Z_1[1]) \cap (S[-1] * Z_0) \subseteq \mathcal{D}^{[-d+1,0]}.$$

The first row of the diagram then shows $C(g)[-1] \in \mathsf{Fac}_d(\phi(\mathcal{S})) \subseteq \phi(\mathcal{S})$, which proves the claim.

For the general case $X \in \Phi^{m+1}(\mathcal{S})$, there exists a triangle $X_1 \xrightarrow{\alpha} X \xrightarrow{\beta} X_2$ with $X_1, X_2 \in \mathsf{Fac}_d(\Phi^m(\mathcal{S}))$. We distinguish two cases:

(1) $f \circ \alpha \neq 0$: The octahedral axiom gives the following diagram:

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & C(f)[-1] \arrow[r] \arrow[d] & X_2 \arrow[d, equal] \\
X_1 \arrow[r, "\alpha"] \arrow[d, "f \circ \alpha \neq 0"'] & X \arrow[r, "\beta"] \arrow[d, "f"] & X_2 \\
S \arrow[r, equal] & S &
\end{tikzcd}$$

By the claim, $M = C(f \circ \alpha)[-1] \in \phi(\mathcal{S})$. Since $X_2 \in \phi(\mathcal{S})$, we conclude $C(f)[-1] \in \phi(\mathcal{S})$.

(2) $f \circ \alpha = 0$: Then f factors through β as $f = \gamma \circ \beta$ for some $\gamma \colon X_2 \to S$. The octahedral axiom yields:

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, equal] & C(f)[-1] \arrow[r] \arrow[d] & N \arrow[d] \\
X_1 \arrow[r, "\alpha"] & X \arrow[r, "\beta"] \arrow[d, "f"] & X_2 \arrow[d, "\gamma \neq 0"] \\
 & S \arrow[r, equal] & S
\end{tikzcd}$$

By the claim, $N = C(\gamma)[-1] \in \phi(\mathcal{S})$. Since $X_1 \in \phi(\mathcal{S})$, it follows that $C(f)[-1] \in \phi(\mathcal{S})$.

This completes the induction and the proof. ☐

For $T(\mathcal{S})$, an analogous statement holds only under a certain additional condition and with a different approach.

Lemma 3.3.6 *Let $\mathcal{S}$ be a semibrick in $\mathcal{D}^{[-d+1,0]}$ such that $T(\mathcal{S})$ is an s-torsion class. For any non-zero morphism $f : Y \to X$ with $Y \in T(\mathcal{S})$ and $X \in \mathcal{S}$, we have that $C(f)[-1] \in T(\mathcal{S})$.*

Proof. By Proposition 1.3.3, $T(\mathcal{S})$ is an s-torsion-free class in the d-extended heart

$$d\text{-}\mathcal{H}_{T(\mathcal{S})} = \mathcal{H}_{T(\mathcal{S})}[d-1] * \cdots * \mathcal{H}_{T(\mathcal{S})}[1] * \mathcal{H}_{T(\mathcal{S})}.$$

Thus, there exists a triangle

$$Y_1 \to Y \to Y_2 \to Y_1[1],$$

with $Y_1 \in \mathcal{H}_{T(\mathcal{S})}[d-1] * \cdots * \mathcal{H}_{T(\mathcal{S})}[1]$ and $Y_2 \in \mathcal{H}_{T(\mathcal{S})}$. Since by Proposition 3.3.2, $\mathcal{S} \subseteq \operatorname{sim} \mathcal{H}_{T(\mathcal{S})} \subseteq \mathcal{H}_{T(\mathcal{S})}$, the morphism space $\operatorname{Hom}(Y_1, S)$ vanishes. It follows that the morphism f factors through Y_2. Using the octahedral axiom, we get the following commutative diagram of triangles.

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d, equal] & C(f)[-1] \arrow[r] \arrow[d] & N \arrow[d] \\
Y_1 \arrow[r] & Y \arrow[r] \arrow[d, "f"] & Y_2 \arrow[d, "g"] \\
 & X \arrow[r, equal] & X
\end{tikzcd}$$

Since $f \neq 0$, so is g. Since $Y_2 \in \mathcal{H}_{T(\mathcal{S})}$ and $\mathcal{S} \subseteq \operatorname{sim} \mathcal{H}_{T(\mathcal{S})}$, the morphism g is an epimorphism in $\mathcal{H}_{T(\mathcal{S})}$, which, together with the triangle in the third column of the above diagram implies $N \in \mathcal{H}_{T(\mathcal{S})}$. Therefore, by the triangle in the first row, we have

$$C(f)[-1] \in Y_1 * N \subseteq \mathcal{H}_{T(\mathcal{S})}[d-1] * \cdots * \mathcal{H}_{T(\mathcal{S})}[1] * \mathcal{H}_{T(\mathcal{S})} = d\text{-}\mathcal{H}_{T(\mathcal{S})}.$$

Since $T(\mathcal{S})$ is an s-torsion-free class in $d\text{-}\mathcal{H}_{T(\mathcal{S})}$, due to $X, Y \in T(\mathcal{S})$, by Lemma 1.2.3, we get that $C(f)[-1] \in T(\mathcal{S})$. ☐

Recall from Section 3.2 that, for a d-FAE closed subcategory $\mathcal{T}$ of $\mathcal{D}^{[-d+1,0]}$, we introduced an associated wide subcategory:

$$W_L(\mathcal{T}) = \{X \in \mathcal{T} \cap \mathcal{H}_{\mathcal{T}} \mid \text{any morphism in } \mathcal{T} \cap \mathcal{H}_{\mathcal{T}} \text{ to } X \text{ is admissible}\}.$$

Proposition 3.3.7 *Let $\mathcal{S}$ be a semibrick in $\mathcal{D}^{[-d+1,0]}$. Then $\mathcal{S} = \operatorname{sim} W_L(\phi(\mathcal{S}))$. If $T(\mathcal{S})$ is an s-torsion class in $\mathcal{D}^{[-d+1,0]}$, then $\mathcal{S} = \operatorname{sim} W_L(T(\mathcal{S}))$.*

Proof. Let $\mathcal{T} = \phi(\mathcal{S})$ or $T(\mathcal{S})$. By Lemma 3.3.1, $\mathcal{T} \in \operatorname{Tor}(\mathcal{S})$. Applying Lemma 3.2.8 we have the inclusion:

$$\operatorname{sim} W_L(\mathcal{T}) \subseteq (\operatorname{sim} \mathcal{H}_{\mathcal{T}}) \cap \mathcal{D}^{[-d+1,0]}.$$

Applying Proposition 3.3.2 yields:

$$\mathcal{S} = (\operatorname{sim} \mathcal{H}_{\mathcal{T}}) \cap \mathcal{D}^{[-d+1,0]}. \tag{3.4}$$

Consequently, it suffices to show that $\mathcal{S} \subseteq \operatorname{sim} W_L(\mathcal{T})$.

From (3.4), we know that $\mathcal{S} \subseteq \mathcal{H}_{\mathcal{T}}$, which implies $\mathcal{S} \subseteq \mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$. To establish $\mathcal{S} \subseteq W_L(\mathcal{T})$, we must show that any morphism $f\colon Y \to S$ in the exact category $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$ with $S \in \mathcal{S}$ is admissible. Since the zero morphism is always admissible, we assume $f \neq 0$. In this case, Lemmas 3.3.5 and 3.3.6 ensure that $C(f)[-1] \in \mathcal{T}$. Applying Hom$(\mathcal{T}, -)$ to the triangle $Y \xrightarrow{f} S \to C(f)$, we obtain the exact sequence:

$$\text{Hom}(\mathcal{T}, S[-2]) \to \text{Hom}(\mathcal{T}, C(f)[-2]) \to \text{Hom}(\mathcal{T}, Y[-1]).$$

The first and last terms vanish by Lemma 3.2.1(2). Hence, Hom$(\mathcal{T}, C(f)[-2]) = 0$, and by the characterization in Remark 3.2.6, we conclude that $C(f)[-1] \in \mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$.

Thus, the sequence $C(f)[-1] \to Y \xrightarrow{f} S$ is a conflation in the exact category $\mathcal{T} \cap \mathcal{H}_{\mathcal{T}}$. In particular, f is a deflation and is therefore admissible. This proves $\mathcal{S} \subseteq W_L(\mathcal{T})$. Furthermore, since the elements of $\mathcal{S}$ are simple objects in $\mathcal{H}_{\mathcal{T}}$ by (3.4), they are necessarily simple in the subcategory $W_L(\mathcal{T})$ as well. ☐

The results obtained up to this point are summarized in Figure 3.3, with the outer triangle composing to identity, that is, sim $\circ W_L \circ \phi =$ id, as proved in Proposition 3.3.7.

$$\begin{tikzcd}
 & & \left\{\begin{matrix}\text{wide subcategories}\\ \text{of } \mathcal{D}^{[-d+1,0]}\end{matrix}\right\} \arrow[dll, "\text{sim}"'] \arrow[d, phantom, sloped, "\supseteq"] \\
\left\{\begin{matrix}\text{semibricks}\\ \text{in } \mathcal{D}^{[-d+1,0]}\end{matrix}\right\} \arrow[rr, shift left, "W", "\text{1-1}"'] \arrow[dr, hook, "T"] \arrow[ddr, hook, "\phi"'] & & \left\{\begin{matrix}\text{length wide}\\ \text{subcategories}\\ \text{of } \mathcal{D}^{[-d+1,0]}\end{matrix}\right\} \arrow[ll, shift left, "\text{sim}"] \\
 & \left\{\begin{matrix}\text{positive torsion}\\ \text{classes in } \mathcal{D}^{[-d+1,0]}\end{matrix}\right\} \arrow[d, phantom, sloped, "\subseteq"] & \\
 & \left\{\begin{matrix}d\text{-FAE closed}\\ \text{subcategories of } \mathcal{D}^{[-d+1,0]}\end{matrix}\right\} \arrow[uuur, bend right, "W_L"'] &
\end{tikzcd}$$

Figure 3.3: A summary of the maps

The map W_L, as defined in (3.2), provides a systematic way to produce wide subcategories. However, for a given d-FAE closed subcategory $\mathcal{T}$, the resulting $W_L(\mathcal{T})$ may not be a length abelian category in general. To establish a self-consistent framework within the set of length wide subcategories—which are in one-to-one correspondence with semibricks—we introduce the modified map

$$W_L^{len} = W \circ \text{sim} \circ W_L \colon \left\{\begin{matrix}d\text{-FAE closed}\\ \text{subcategories of } \mathcal{D}^{[-d+1,0]}\end{matrix}\right\} \to \left\{\begin{matrix}\text{wide subcategories}\\ \text{of } \mathcal{D}^{[-d+1,0]}\end{matrix}\right\}.$$

This composition “refines” the wide subcategory $W_L(\mathcal{T})$ by first extracting its simple objects and then taking their extension closure, thereby ensuring that the output remains within the class of length wide subcategories.

For any wide subcategory $\mathcal{W}$, similarly as for a semibrick, we denote by $\phi(\mathcal{W})$ (resp. $T(\mathcal{W})$) the smallest d-FAE closed subcategory (resp. positive torsion class) of $\mathcal{D}^{[-d+1,0]}$ containing $\mathcal{W}$. Propositions 3.1.7 and 3.3.7 give us the following consequence, which in the classical case restricts to [74, Proposition 3.3].

Corollary 3.3.8 *Let $\mathcal{W}$ be a length wide subcategory of $\mathcal{D}^{[-d+1,0]}$. Then $W_L^{len}(\phi(\mathcal{W})) = \mathcal{W}$. If $T(\mathcal{W})$ is an s-torsion class, then $W_L^{len}(T(\mathcal{W})) = \mathcal{W}$.*

Although Proposition 3.3.7 and Corollary 3.3.8 suggest that the map ϕ behaves more favorably than T, for arbitrary semibricks and length wide subcategories, we shall see in the following section that T becomes more advantageous upon restriction to certain subsets (see Theorems 3.4.6 and 3.4.10 for more details).

3.4 Left/right-finite semibricks and wide subcategories

For the remainder of this paper, we fix the triangulated category $\mathcal{D} = \mathcal{D}^b(\operatorname{mod}\Lambda)$, and the t-structure

$$(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) = (\mathcal{D}^{\leq 0}(\operatorname{mod}\Lambda), \mathcal{D}^{\geq 0}(\operatorname{mod}\Lambda)).$$

In this setting, the extended heart is given by $\mathcal{D}^{[-d+1,0]} = d\text{-}\operatorname{mod}\Lambda$. In this section, we define left/right-finite semibricks and wide subcategories in $d\text{-}\operatorname{mod}\Lambda$ and establish their bijections with functorially finite positive torsion pairs in $d\text{-}\operatorname{mod}\Lambda$ and $(d+1)$-term simple-minded collections in $\mathcal{D}^b(\operatorname{mod}\Lambda)$. We begin by restricting the bijection introduced in Proposition 1.3.3 as follows.

Proposition 3.4.1 *There is a bijection between*

- *the set of functorially finite positive torsion pairs in* $d\text{-}\operatorname{mod}\Lambda$*, and*
- *the set of bounded* t*-structures* $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ *on* $\mathcal{D}^b(\operatorname{mod}\Lambda)$ *with length heart satisfying* $\mathcal{D}^{\leq -d} \subseteq \mathcal{C}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$.

The map is given by

$$(\mathcal{T}, \mathcal{F}) \mapsto (\mathcal{D}^{\leq -d} * \mathcal{T}, \mathcal{F}[1] * \mathcal{D}^{\geq 0})$$

with the inverse map defined by

$$(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) \mapsto (\mathcal{C}^{\leq 0} \cap d\text{-}\operatorname{mod}\Lambda, \mathcal{C}^{\geq 1} \cap d\text{-}\operatorname{mod}\Lambda).$$

Proof. First, recall that functorially finite positive torsion pairs in $d\text{-}\operatorname{mod}\Lambda$ coincide with functorially finite s-torsion pairs, as shown in Proposition 1.3.5. According to Theorem 2.2.1, there exists a bijection between basic $(d+1)$-term silting objects in $\mathsf{K}^b(\operatorname{proj}\Lambda)$ and functorially finite positive torsion pairs in $d\text{-}\operatorname{mod}\Lambda$, which sends a silting object P to the pair $(\mathcal{T}(P), \mathcal{F}(P))$ defined by

$$\mathcal{T}(P) = \{X \in d\text{-}\operatorname{mod}\Lambda \mid \operatorname{Hom}(P, X[i]) = 0,\ \forall i > 0\},$$

$$\mathcal{F}(P) = \{X \in d\text{-}\operatorname{mod}\Lambda \mid \operatorname{Hom}(P, X[i]) = 0,\ \forall i \leq 0\}.$$

On the other hand, Theorem 5.2 in the introduction provides a bijection between basic $(d+1)$-term silting objects and bounded t-structures $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ on $\mathcal{D}^b(\operatorname{mod}\Lambda)$ with length hearts and satisfying $\mathcal{D}^{\leq -d} \subseteq \mathcal{C}^{\leq 0} \subseteq \mathcal{D}^{\leq 0}$, sending a silting object P to $(\mathcal{D}^{\leq 0}(P), \mathcal{D}^{\geq 0}(P))$ defined by

$$\mathcal{D}^{\leq 0}(P) = \{X \in \mathcal{D}^b(\operatorname{mod}\Lambda) \mid \operatorname{Hom}(P, X[i]) = 0 \quad \forall\, i > 0\},$$

$$\mathcal{D}^{\geq 0}(P) = \{X \in \mathcal{D}^b(\operatorname{mod}\Lambda) \mid \operatorname{Hom}(P, X[i]) = 0 \quad \forall\, i < 0\}.$$

Observing that $\mathcal{T}(P) = \mathcal{D}^{\leq 0}(P) \cap d\text{-}\operatorname{mod}\Lambda$ and $\mathcal{F}(P) = \mathcal{D}^{\geq 0}(P)[-1] \cap d\text{-}\operatorname{mod}\Lambda$, we obtain the desired bijection by combining these results and comparing them with Proposition 1.3.3. □

Let $\mathcal{T}$ be a functorially finite positive torsion class in $d\text{-}\operatorname{mod}\Lambda$. Its exact heart

$$\mathcal{H}_{\mathcal{T}} = (\mathcal{D}^{\leq -d} * \mathcal{T}) \cap (\mathcal{T}^{\perp}[1] * \mathcal{D}^{\geq 0}) \subseteq \mathcal{D}^{[-d,0]}$$

coincides with the heart of the corresponding bounded t-structure. Dually, for any functorially finite positive torsion-free class $\mathcal{F}$, its heart $\mathcal{H}^{\mathcal{F}}$ is defined as the heart of the corresponding bounded t-structure:

$$\mathcal{H}^{\mathcal{F}} := (\mathcal{D}^{\leq -d} * {}^{\perp}\mathcal{F}) \cap (\mathcal{F}[1] * \mathcal{D}^{\geq 0}) \subseteq \mathcal{D}^{[-d,0]}.$$

Thus, by Proposition 3.4.1 and Corollary 1.3.4, we obtain the following two bijections:

$$\begin{tikzcd}
\left\{\begin{matrix}\text{functorially finite positive}\\ \text{torsion classes in } d\text{-}\operatorname{mod}\Lambda\end{matrix}\right\} \arrow[dr, "\Phi_L"] & \\
& \left\{\begin{matrix}\text{length hearts of } \mathcal{D}^b(\operatorname{mod}\Lambda)\\ \text{lying in } \mathcal{D}^{[-d,0]}\end{matrix}\right\} \\
\left\{\begin{matrix}\text{functorially finite positive}\\ \text{torsion-free classes in } d\text{-}\operatorname{mod}\Lambda\end{matrix}\right\} \arrow[ur, "\Phi_R"'] &
\end{tikzcd} \tag{3.5}$$

These maps are given by $\Phi_L(\mathcal{T}) = \mathcal{H}_{\mathcal{T}}$ and $\Phi_R(\mathcal{F}) = \mathcal{H}^{\mathcal{F}}$, and their respective inverses are given by:

$$\Phi_L^{-1}(\mathcal{H}) = d\text{-}\mathcal{H} \cap d\text{-}\operatorname{mod}\Lambda \text{ and } \Phi_R^{-1}(\mathcal{H}) = d\text{-}\mathcal{H}[-d] \cap d\text{-}\operatorname{mod}\Lambda.$$

Recall that, according to Theorem 5.2, there exists a bijection

$$\operatorname{sim}\colon \left\{\begin{matrix}\text{length hearts of } \mathcal{D}^b(\operatorname{mod}\Lambda)\\ \text{lying in } \mathcal{D}^{[-d,0]}\end{matrix}\right\} \to \left\{\begin{matrix}(d+1)\text{-term simple-minded}\\ \text{collections in } \mathcal{D}^b(\operatorname{mod}\Lambda)\end{matrix}\right\}, \tag{3.6}$$

which sends a length heart $\mathcal{H}$ to the set $\operatorname{sim}\mathcal{H}$ of (isoclasses of) its simple objects.

3.4.1 From simple-minded collections to semibricks

Comparing Definitions 5.1 and 3.1.6, one observes that for any $(d+1)$-term simple-minded collection $\mathcal{X}$, the intersections $\mathcal{X} \cap d\text{-}\operatorname{mod}\Lambda$ and $\mathcal{X}[-1] \cap d\text{-}\operatorname{mod}\Lambda$ are semibricks in $d\text{-}\operatorname{mod}\Lambda$. Consequently, we define two maps

$$\Pi_L,\, \Pi_R\colon \left\{\begin{matrix}(d+1)\text{-term simple-minded}\\ \text{collections in } \mathcal{D}^b(\operatorname{mod}\Lambda)\end{matrix}\right\} \to \{\text{semibricks in } d\text{-}\operatorname{mod}\Lambda\}$$

given by $\Pi_L(\mathcal{X}) = \mathcal{X} \cap d\text{-}\operatorname{mod}\Lambda$ and $\Pi_R(\mathcal{X}) = \mathcal{X}[-1] \cap d\text{-}\operatorname{mod}\Lambda$, respectively. In the sequel, we will prove that these maps are injective and provide a precise characterization of their images.

A consequence of Proposition 3.4.1 is the following useful lemma.

Lemma 3.4.2 *Let $\mathcal{T}$ be a functorially finite positive torsion class in $d\text{-}\operatorname{mod}\Lambda$, and let $\mathcal{S}_{\mathcal{T}} = (\Pi_L \circ \operatorname{sim} \circ \Phi_L)(\mathcal{T})$ be the associated semibrick. Then $\mathcal{T} \cap \mathcal{S}_{\mathcal{T}}^{\perp_{\leq 0}} = 0$.*

Proof. Let $X \in \mathcal{T} \cap \mathcal{S}_{\mathcal{T}}^{\perp_{\leq 0}}$, and suppose $X \neq 0$. By Proposition 1.3.3(1), $\mathcal{T}$ is contained in the d-extended heart of $\mathcal{H}_{\mathcal{T}}$:

$$\mathcal{T} \subseteq d\text{-}\mathcal{H}_{\mathcal{T}} := \mathcal{H}_{\mathcal{T}}[d-1] * \mathcal{H}_{\mathcal{T}}[d-2] * \cdots * \mathcal{H}_{\mathcal{T}}.$$

Thus, there exists an integer $0 \leq l \leq d-1$ such that X admits a triangle

$$X'[l] \xrightarrow{h} X \to X'',$$

where $0 \neq X' \in \mathcal{H}_{\mathcal{T}}$ and $X'' \in \mathcal{H}_{\mathcal{T}}[l-1] * \cdots \mathcal{H}_{\mathcal{T}}[1] * \mathcal{H}_{\mathcal{T}}$. Since $\mathcal{H}_{\mathcal{T}}$ is length by Proposition 3.4.1, there exists a conflation in $\mathcal{H}_{\mathcal{T}}$:

$$S \overset{f}{\rightarrowtail} X' \twoheadrightarrow Y,$$

where $S \in \operatorname{sim}\mathcal{H}_{\mathcal{T}}$. Applying the octahedral axiom to the composition $g = h \circ f[l]$, we obtain the following diagram of triangles:

$$\begin{tikzcd}
S[l] \arrow[r, equal] \arrow[d, "{f[l]}"'] & S[l] \arrow[d, "g"] & \\
X'[l] \arrow[r, "h"] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[d] \\
Y[l] \arrow[r] & L \arrow[r] & X''
\end{tikzcd}$$

By the triangle in the third row of the diagram, we have

$$L \in Y[l] * X'' \subseteq \mathcal{H}_{\mathcal{T}}[l] * \cdots \mathcal{H}_{\mathcal{T}}[1] * \mathcal{H}_{\mathcal{T}} \subseteq \mathcal{D}^{\geq -d-l},$$

as $\mathcal{H}_{\mathcal{T}} \subseteq \mathcal{D}^{\geq -d}$. Then, the triangle in the second row implies:

$$S[l] \in L[-1] * X \subseteq \mathcal{D}^{\geq -d-l+1} * \mathcal{T} \subseteq \mathcal{D}^{\geq -d-l+1},$$

where we use the fact that $\mathcal{T} \subseteq \mathcal{D}^{\geq -d+1} \subseteq \mathcal{D}^{\geq -d-l+1}$. This implies $S \in \mathcal{D}^{\geq -d+1}$. Moreover, since $S \in \mathcal{H}_{\mathcal{T}} \subseteq \mathcal{D}^{\leq 0}$, we conclude that $S \in d\text{-}\operatorname{mod}\Lambda$, which implies $S \in \mathcal{S}_{\mathcal{T}}$.

Because $X \in \mathcal{S}_{\mathcal{T}}^{\perp_{\leq 0}}$ and $S \in \mathcal{S}_{\mathcal{T}}$, the morphism $g\colon S[l] \to X$ must be zero. This forces $f[l]$ to factor through $X''[-1]$. However, $S[l] \in \mathcal{H}_{\mathcal{T}}[l]$ and $X''[-1] \in \mathcal{H}_{\mathcal{T}}[l-2] * \cdots * \mathcal{H}_{\mathcal{T}} * \mathcal{H}_{\mathcal{T}}[-1]$. By the property of the heart, $\operatorname{Hom}(\mathcal{H}_{\mathcal{T}}, \mathcal{H}_{\mathcal{T}}[j]) = 0$ for $j < 0$. Thus $\operatorname{Hom}(S[l], X''[-1]) = 0$, which implies $f[l] = 0$ and hence $f = 0$. This contradicts the fact that S is a non-zero simple object. We conclude that $X = 0$. □

The following proposition states that, analogous to the classical case $d = 1$ [28, Remark 4.11], a $(d+1)$-term simple-minded collection $\mathcal{X}$ is precisely the union of $\Pi_L(\mathcal{X})$ and $\Pi_R(\mathcal{X})[1]$. However, unlike the classical case, this union is not necessarily disjoint. For instance, consider the simple-minded collection $\mathcal{X} = \{P_2, I_1[1]\}$ from Example 3.1.8, where $d = 2$. In this case, $\Pi_L(\mathcal{X}) = \mathcal{X}$ and $\Pi_R(\mathcal{X})[1] = \{I_1[1]\}$, illustrating the overlap.

Proposition 3.4.3 *Let $\mathcal{X}$ be a $(d+1)$-term simple-minded collection. Then*

$$\mathcal{X} = \Pi_L(\mathcal{X}) \cup \Pi_R(\mathcal{X})[1].$$

Proof. The inclusion $\Pi_L(\mathcal{X}) \cup \Pi_R(\mathcal{X})[1] \subseteq \mathcal{X}$ follows directly from the definitions of Π_L and Π_R. For the reverse inclusion, let $X \in \mathcal{X}$. Suppose $X \notin \Pi_L(\mathcal{X})$, which means $X \notin d\text{-}\operatorname{mod}\Lambda$. We aim to show that $X[-1] \in d\text{-}\operatorname{mod}\Lambda$.

Let $\mathcal{T}$ be the functorially finite positive torsion pair in $d\text{-}\operatorname{mod}\Lambda$ associated with $\mathcal{X}$, that is, $\mathcal{X} = \operatorname{sim}\mathcal{H}_{\mathcal{T}}$. Since $\mathcal{H}_{\mathcal{T}} \subseteq \mathcal{D}^{[-d,0]}$ and $X \notin d\text{-}\operatorname{mod}\Lambda = \mathcal{D}^{[-d+1,0]}$, the truncation of X with respect to the standard t-structure yields a triangle:

$$\sigma_{\leq -d}(X) \xrightarrow{f} X \to \sigma_{\geq -d+1}(X), \tag{3.7}$$

where $f \neq 0$ and $\sigma_{\leq -d}(X) \in \mathcal{D}^{[-d,-d]}$. By Proposition 1.3.3(4), we have $\sigma_{\leq -d}(X) \in \mathcal{T}^{\perp}[d]$ and $\sigma_{\geq -d+1}(X) \in \mathcal{T}$.

Recall from Proposition 1.3.3(1) that $\mathcal{T}^{\perp}[d] \subseteq \mathcal{H}_{\mathcal{T}}[d-1] * \mathcal{H}_{\mathcal{T}}[d-2] * \cdots * \mathcal{H}_{\mathcal{T}}$. Consequently, there exists a triangle

$$Z \xrightarrow{g} \sigma_{\leq -d}(X) \xrightarrow{h} Y$$

with $Z \in \mathcal{H}_{\mathcal{T}}[d-1] * \cdots * \mathcal{H}_{\mathcal{T}}[1]$ and $Y \in \mathcal{H}_{\mathcal{T}}$. Since $X \in \mathcal{H}_{\mathcal{T}}$, we have $\operatorname{Hom}(Z, X) = 0$, which implies $f \circ g = 0$. Hence, there exists a morphism $f' : Y \to X$ such that $f = f' \circ h$. Thus, by the octahedral axiom, we obtain the following commutative diagram:

$$\begin{tikzcd}
Z \arrow[r, equal] \arrow[d] & Z \arrow[d, "g"] & \\
(\sigma_{\geq -d+1}(X))[-1] \arrow[r] \arrow[d] & \sigma_{\leq -d}(X) \arrow[r, "f"] \arrow[d, "h"] & X \arrow[d, equal] \\
Y' \arrow[r] & Y \arrow[r, "f'"'] & X
\end{tikzcd}$$

Since $f \neq 0$, the induced map $f' \colon Y \to X$ is also non-zero. Given that X is a simple object in $\mathcal{H}_{\mathcal{T}}$ and $Y \in \mathcal{H}_{\mathcal{T}}$, the map f' must be an epimorphism in the abelian category $\mathcal{H}_{\mathcal{T}}$, which implies that the triangle in the bottom row is a short exact sequence in $\mathcal{H}_{\mathcal{T}}$. Thus, in particular, we have $Y' \in \mathcal{H}_{\mathcal{T}}$.

From the triangle in the left column, we have $\sigma_{\geq -d+1}(X) \in Z[1] * Y'[1]$. Since $Z[1] \subseteq \mathcal{H}_{\mathcal{T}}[d] * \cdots * \mathcal{H}_{\mathcal{T}}[2]$ and $Y'[1] \in \mathcal{H}_{\mathcal{T}}[1]$, we have

$$\sigma_{\geq -d+1}(X) \in \mathcal{H}_{\mathcal{T}}[d] * \cdots * \mathcal{H}_{\mathcal{T}}[1] = \mathcal{D}^{[-d,-1]}.$$

Therefore, the triangle (3.7) shows that $X \in \mathcal{D}^{[-d,-1]}$, which forces $X[-1] \in d\text{-}\operatorname{mod}\Lambda$. This completes the proof. □

Say that the left (resp. right) mutation of a $(d+1)$-term simple-minded collection $\mathcal{X}$ at an element X_i exists, if the resulting simple-minded collection $\mu_i^+(\mathcal{X})$ (resp. $\mu_i^-(\mathcal{X})$) is again $(d+1)$-term.

Proposition 3.4.4 *Let $\mathcal{X} = \{X_i\}_{i=1}^n$ be a $(d+1)$-term simple-minded collection.*

(1) The left mutation of $\mathcal{X}$ at X_i exists if and only if $X_i \in \Pi_L(\mathcal{X})$.

(2) The right mutation of $\mathcal{X}$ at X_i exists if and only if $X_i \in \Pi_R(\mathcal{X})[1]$.

Moreover, for any $1 \leq i \leq n$, at least one of the left and right mutations of X at X_i exists.

Proof. Suppose that the left mutation of $\mathcal{X}$ at X_i exists. By definition, all elements of the mutated collection $\mu_i^+(\mathcal{X})$ must lie in the extended heart $\mathcal{D}^{[-d,0]}$. In particular, $X_i[1] \in \mathcal{D}^{[-d,0]}$, which implies $X_i \in \mathcal{D}^{\geq -d+1}$. Since $X_i \in \mathcal{X} \subseteq \mathcal{D}^{\leq 0}$, it follows that $X_i \in d\text{-}\operatorname{mod}\Lambda$, and thus $X_i \in \Pi_L(\mathcal{X})$.

Conversely, suppose $X_i \in \Pi_L(\mathcal{X})$. The collection $\mu_i^+(\mathcal{X})$ consists of $\{X'_j\}_{j\neq i} \cup \{X_i[1]\}$. Since $X_i \in d\text{-}\operatorname{mod}\Lambda = \mathcal{D}^{[-d+1,0]}$, we have $X_i[1] \in \mathcal{D}^{[-d,1]} \subseteq \mathcal{D}^{[-d,0]}$. For each $j \neq i$, the object X'_j is defined by a triangle $Y_j \to X_j \to X'_j \to Y_j[1]$, where Y_j lies in the extension closure of X_i. Since both X_j and $X_i[1]$ are in $\mathcal{D}^{[-d,0]}$, it follows that $X'_j \in \mathcal{D}^{[-d,0]}$. Thus, $\mu_i^+(\mathcal{X})$ is a $(d+1)$-term simple-minded collection. The argument for the right mutation is analogous.

Finally, by Proposition 3.4.3, every X_i is either in $\Pi_L(\mathcal{X})$ or in $\Pi_R(\mathcal{X})[1]$. Therefore, at least one of the left mutation and the right mutation of $\mathcal{X}$ at X_i exists. □

3.4.2 Left/right-finite semibricks

Recall that for any semibrick $\mathcal{S}$ in $\mathcal{D}^{[-d+1,0]}$, we denote by $T(\mathcal{S})$ the smallest positive torsion class containing $\mathcal{S}$. Dually, we denote by $F(\mathcal{S})$ the smallest positive torsion-free class containing $\mathcal{S}$. Thus, T and F define maps from the set of semibricks in $d\text{-}\operatorname{mod}\Lambda$ to the set of positive torsion classes and positive torsion-free classes in $d\text{-}\operatorname{mod}\Lambda$, respectively. We are particularly interested in those semibricks $\mathcal{S}$ whose associated classes $T(\mathcal{S})$ or $F(\mathcal{S})$ satisfy the functorial finiteness condition.

Definition 3.4.5 A semibrick $\mathcal{S} \subseteq d\text{-}\operatorname{mod}\Lambda$ is called *left-finite* (resp. *right-finite*) if $T(\mathcal{S})$ (resp. $F(\mathcal{S})$) is functorially finite.

Our first main result is summarized in the following theorem.

Theorem 3.4.6 *There exists a commutative diagram of bijections as follows, such that each oriented digon composes to the identity.*

$$\begin{tikzcd}
\left\{\begin{matrix}\textit{left-finite semibricks}\\ \textit{in } d\text{-}\operatorname{mod}\Lambda\end{matrix}\right\} \arrow[dddd, "T"] & & \left\{\begin{matrix}\textit{right-finite semibricks}\\ \textit{in } d\text{-}\operatorname{mod}\Lambda\end{matrix}\right\} \arrow[dddd, "F"'] \\
& \left\{\begin{matrix}(d+1)\textit{-term simple-minded}\\ \textit{collections in } \mathcal{D}^b(\operatorname{mod}\Lambda)\end{matrix}\right\} \arrow[ul, "\mathcal{X}\cap d\text{-}\operatorname{mod}\Lambda \mapsfrom \mathcal{X}", "\Pi_L"'] \arrow[ur, "\mathcal{X}\mapsto \mathcal{X}[-1]\cap d\text{-}\operatorname{mod}\Lambda"', "\Pi_R"] & \\
& & \\
& \left\{\begin{matrix}\textit{length bounded hearts}\\ \textit{lying in } \mathcal{D}^{[-d,0]}(\operatorname{mod}\Lambda)\end{matrix}\right\} \arrow[uu, "\mathrm{sim}"] \arrow[dl, shift right, "d\text{-}\mathcal{H}\cap d\text{-}\operatorname{mod}\Lambda \mapsfrom \mathcal{H}"'] \arrow[dr, shift left, "\mathcal{H}\mapsto d\text{-}\mathcal{H}[-d]\cap d\text{-}\operatorname{mod}\Lambda"] & \\
\left\{\begin{matrix}\textit{functorially finite}\\ \textit{positive torsion}\\ \textit{classes in } d\text{-}\operatorname{mod}\Lambda\end{matrix}\right\} \arrow[ur, shift right, "\Phi_L\colon \mathcal{T}\mapsto\mathcal{H}_{\mathcal{T}}"'] \arrow[rr, shift left, "\mathcal{T}\mapsto\mathcal{T}^{\perp}"] & & \left\{\begin{matrix}\textit{functorially finite}\\ \textit{positive torsion-free}\\ \textit{classes in } d\text{-}\operatorname{mod}\Lambda\end{matrix}\right\} \arrow[ul, shift left, "\mathcal{H}^{\mathcal{F}}\mapsfrom\mathcal{F}\colon\Phi_R"] \arrow[ll, shift left, "{}^{\perp}\mathcal{F}\mapsfrom\mathcal{F}"]
\end{tikzcd}$$

Proof. The bijections Φ_L, Φ_R (and their inverses) and sim were established in (3.5) and (3.6) respectively. The lower triangles commute since, for any functorially finite positive torsion pair $(\mathcal{T}, \mathcal{F})$, we have the identity $\mathcal{H}_{\mathcal{T}} = \mathcal{H}^{\mathcal{F}}$ by construction.

Let $\mathcal{T}$ be a functorially finite positive torsion class in $d\text{-}\operatorname{mod}\Lambda$, and let $\mathcal{S}_{\mathcal{T}} = (\Pi_L \circ \operatorname{sim} \circ \Phi_L)(\mathcal{T})$. By definition, we have

$$\mathcal{S}_{\mathcal{T}} = \operatorname{sim}\mathcal{H}_{\mathcal{T}} \cap d\text{-}\operatorname{mod}\Lambda \subseteq \mathcal{H}_{\mathcal{T}} \cap d\text{-}\operatorname{mod}\Lambda \subseteq d\text{-}\mathcal{H}_{\mathcal{T}} \cap d\text{-}\operatorname{mod}\Lambda = \mathcal{T}.$$

Since $\mathcal{T}$ is a positive torsion class containing $\mathcal{S}_{\mathcal{T}}$, it follows that $T(\mathcal{S}_{\mathcal{T}}) \subseteq \mathcal{T}$. Combining Lemma 3.3.1 and Lemma 3.4.2, we obtain $\mathcal{T} \cap T(\mathcal{S}_{\mathcal{T}})^{\perp} = \mathcal{T} \cap \mathcal{S}_{\mathcal{T}}^{\perp_{\leq 0}} = 0$. Since $\mathcal{T}$ is an s-torsion class and $T(\mathcal{S}_{\mathcal{T}})$ is a positive torsion class in $d\text{-}\operatorname{mod}\Lambda$ with $T(\mathcal{S}_{\mathcal{T}}) \subseteq \mathcal{T}$, Lemma 1.3.2 implies $\mathcal{T} = T(\mathcal{S}_{\mathcal{T}})$. This proves the identity

$$T \circ \Pi_L \circ \operatorname{sim} \circ \Phi_L = \operatorname{id}.$$

Consequently, the restriction of the map T to the set of left-finite semibricks is surjective. Moreover, because sim and Φ_L are bijections, the map Π_L is injective and its image is contained within the set of left-finite semibricks. The identity above also ensures the commutativity of the left part of the diagram. By Corollary 3.3.3, T is injective, and thus its restriction is a bijection. Finally, this implies that the image of Π_L coincides exactly with the set of left-finite semibricks.

A dual argument applies to the right-hand side. ☐

The following result is an immediate consequence of Theorem 3.4.6 and [69, Corollary 5.5]. For the case $d = 1$, it recovers [8, Corollary 2.10].

Corollary 3.4.7 *Let $\mathcal{S}$ be a semibrick in $d\text{-}\operatorname{mod}\Lambda$. If $\mathcal{S}$ is either left-finite or right-finite, then the cardinality of $\mathcal{S}$ is at most the rank of Λ.*

3.4.3 Left/right-finite wide subcategories

Recall that for any subcategory $\mathcal{Y}$ of $d\text{-}\operatorname{mod}\Lambda$, we denote by $T(\mathcal{Y})$ (resp. $F(\mathcal{Y})$) the smallest positive torsion class (resp. torsion-free class) containing $\mathcal{Y}$.

Definition 3.4.8 A wide subcategory $\mathcal{W} \subseteq d\text{-}\operatorname{mod}\Lambda$ is called *left-finite* (resp. *right-finite*) if the smallest positive torsion class $T(\mathcal{W})$ (resp. torsion-free class $F(\mathcal{W})$) containing $\mathcal{W}$ is functorially finite.

Note that for any semibrick $\mathcal{S}$ in $d\text{-}\operatorname{mod}\Lambda$, we have the following identities

$$T(\mathcal{S}) = T(W(\mathcal{S})) \text{ and } F(\mathcal{S}) = F(W(\mathcal{S})), \tag{3.8}$$

where W is the bijection established in Proposition 3.1.7; specifically, $W(\mathcal{S})$ is the extension closure of $\mathcal{S}$. This implies that a semibrick $\mathcal{S}$ is left-finite (resp. right-finite) if and only if $W(\mathcal{S})$ is left-finite (resp. right-finite).

Proposition 3.4.9 *Any left-finite or right-finite wide subcategory of $d\text{-}\operatorname{mod}\Lambda$ is a length category.*

Proof. Let $\mathcal{W}$ be a left-finite wide subcategory. By definition, $\mathcal{T} := T(\mathcal{W})$ is a functorially finite positive torsion class. We claim that $\mathcal{W} \subseteq \mathcal{H}_{\mathcal{T}}$. If this claim holds, $\mathcal{W}$ becomes a wide subcategory of the length abelian category $\mathcal{H}_{\mathcal{T}}$ (as per Proposition 3.4.1), which directly implies that $\mathcal{W}$ is itself a length category.

To prove the claim, we follow a strategy similar to Step 1 of the proof of Proposition 3.3.2. First, the inclusion $\mathcal{W} \subseteq \mathcal{T} \subseteq \mathcal{D}^{\leq -d} * \mathcal{T}$ is immediate from the definition of $\mathcal{T}$. Next, let $X \in \mathcal{W}$. For the truncation triangle $\sigma_{\leq -1}(X) \to X \to \sigma_{\geq 0}(X) \to \sigma_{\leq -1}(X)[1]$, applying the functor $\operatorname{Hom}(\mathcal{W}, -)$ yields the following exact sequence for any $j \in \mathbb{Z}$:

$$\operatorname{Hom}(\mathcal{W}, (\sigma_{\geq 0}(X))[j]) \to \operatorname{Hom}(\mathcal{W}, (\sigma_{\leq -1}(X))[j+1]) \to \operatorname{Hom}(\mathcal{W}, X[j+1]).$$

For $j \leq -2$, the first term $\operatorname{Hom}(\mathcal{W}, \sigma_{\geq 0}(X)[j])$ vanishes because $\mathcal{W} \subseteq \mathcal{D}^{\leq 0}$ and $\sigma_{\geq 0}(X)[j] \in \mathcal{D}^{\geq -j} \subseteq \mathcal{D}^{\geq 2}$. The last term $\operatorname{Hom}(\mathcal{W}, X[j+1])$ also vanishes because $\mathcal{W}$ is a wide subcategory and $j+1 < 0$. This implies that $\operatorname{Hom}(\mathcal{W}, \sigma_{\leq -1}(X)[k]) = 0$ for all $k \leq -1$, or equivalently, $(\sigma_{\leq -1}(X))[-1] \in \mathcal{W}^{\perp_{\leq 0}}$.

Recall that $\mathcal{W}^{\perp_{\leq 0}} = \mathcal{T}^{\perp}$ by Proposition 1.2.8. Thus, $(\sigma_{\leq -1}(X))[-1] \in \mathcal{T}^{\perp}$, which means $\sigma_{\leq -1}(X) \in \mathcal{T}^{\perp}[1]$. Consequently,

$$X \in \sigma_{\leq -1}(X) * \sigma_{\geq 0}(X) \subseteq \mathcal{T}^{\perp}[1] * \mathcal{D}^{\geq 0}.$$

Combined with $\mathcal{W} \subseteq \mathcal{D}^{\leq -d} * \mathcal{T}$, we conclude $\mathcal{W} \subseteq \mathcal{H}_{\mathcal{T}}$. The case for right-finite wide subcategories is analogous. □

Recall from (3.2) that we defined a map W_L, which sends a d-FAE closed subcategory of $d\text{-}\operatorname{mod}\Lambda$ to a wide subcategory of $d\text{-}\operatorname{mod}\Lambda$. Our second main result establishes the following commutative diagram, compared with Figure 3.3.

Theorem 3.4.10 *The following is a commutative triangle of bijections.*

$$\begin{tikzcd}
\left\{\begin{array}{c}\textit{left-finite semibricks}\\ \textit{in } d\text{-}\operatorname{mod}\Lambda\end{array}\right\} \arrow[rr, "W"] \arrow[dr, "T"'] & & \left\{\begin{array}{c}\textit{left-finite wide}\\ \textit{subcategories in } d\text{-}\operatorname{mod}\Lambda\end{array}\right\} \\
& \left\{\begin{array}{c}\textit{functorially finite, positive}\\ \textit{torsion classes in } d\text{-}\operatorname{mod}\Lambda\end{array}\right\} \arrow[ur, "W_L"'] &
\end{tikzcd}$$

Proof. By Theorem 3.4.6, the map T restricts to a bijection between left-finite semibricks and functorially finite positive torsion classes. Moreover, Proposition 3.4.9 and the identities in (3.8) ensure that the bijection W from Proposition 3.1.7 restricts to a bijection between the set of left-finite (resp. right-finite) semibricks and the set of left-finite (resp. right-finite) wide subcategories.

For any left-finite semibrick $\mathcal{S}$, the torsion class $T(\mathcal{S})$ is functorially finite and positive by definition. According to Proposition 1.3.5, such a class is necessarily an s-torsion class. This allows us to apply Proposition 3.3.7, which yields the identity $\mathcal{S} = \operatorname{sim} W_L(T(\mathcal{S}))$. Since $W(\mathcal{S})$ is also characterized by its simple objects (i.e., $\mathcal{S} = \operatorname{sim} W(\mathcal{S})$), the commutativity of the triangle $W = W_L \circ T$ follows. Consequently, the map W_L restricts to the desired bijection as shown in the diagram. □

3.4.4 Realizing left/right-finite wide subcategories as module categories

In the classical case $d = 1$, left-finite wide subcategories are known to be equivalent to module categories of certain quotient algebras [8, Theorem 3.15]. In this subsection, we show that this property extends to the d-setting.

Let $P = \bigoplus_{i=1}^{n} P_i$ be a basic $(d+1)$-term silting complex, and let

$$\mathcal{H}(P) = \{X \in \mathcal{D}^b(\mathsf{mod}\,\Lambda) \mid \mathsf{Hom}(P, X[i]) = 0,\ \forall\, i \neq 0\} \tag{3.9}$$

be the heart of the corresponding t-structure (see Theorem 5.2). We recall the equivalence of categories from [69]:

$$\mathsf{Hom}(P, -) : \mathcal{H}(P) \to \mathsf{mod}\, C, \tag{3.10}$$

where $C := \mathsf{End}(P)$. Let $\mathcal{X} = \mathsf{sim}\,\mathcal{H}(P) = \{X_1, \dots, X_n\}$ be the corresponding $(d+1)$-term simple-minded collection, ordered such that $\mathsf{Hom}(P_i, X_j[l]) = 0$ for $i \neq j$ or $l \neq 0$.

By Proposition 3.4.3, the simple-minded collection $\mathcal{X}$ is the union of its left and (shift of) right semibrick parts. Without loss of generality, we may assume that the left-finite semibrick is

$$\mathcal{S}_L = \Pi_L(\mathcal{X}) = \{X_1, \dots, X_l, X_{l+1}, \dots, X_m\}$$

and the right-finite semibrick is

$$\mathcal{S}_R = \Pi_R(\mathcal{X}) = \{X_{l+1}[-1], \dots, X_m[-1], X_{m+1}[-1], \dots, X_n[-1]\},$$

where the indices reflect the possible overlap between $\Pi_L(\mathcal{X})$ and $\Pi_R(\mathcal{X})[1]$. Let $\mathcal{W} = W(\mathcal{S}_L)$ and $\mathcal{W}' = W(\mathcal{S}_R)$ be the corresponding wide subcategories in d-$\mathsf{mod}\,\Lambda$.

Let $f_i \in C$ be the primitive idempotent corresponding to P_i, and set $f = \sum_{i=m+1}^{n} f_i$. Dually, considering the Nakayama functor $\nu\colon \mathsf{K}^b(\mathsf{proj}\,\Lambda) \to \mathsf{K}^b(\mathsf{inj}\,\Lambda)$, the algebra C is isomorphic to $C' := \mathsf{End}(\nu P)$, and the functor

$$D\,\mathsf{Hom}(-, \nu P) : \mathcal{H}(P) \to \mathsf{mod}\, C'$$

is an equivalence. Let $f'_i \in C'$ be the primitive idempotent corresponding to νP_i, and set $f' = \sum_{i=1}^{l} f'_i$.

Theorem 3.4.11 *Any left-finite (resp. right-finite) wide subcategory of* d-$\mathsf{mod}\,\Lambda$ *is equivalent to the module category of a finite-dimensional algebra. More precisely:*

1. *The equivalence* $\mathsf{Hom}(P, -)$ *restricts to an equivalence*

$$\mathcal{W} \simeq \mathsf{mod}(C/\langle f \rangle).$$

2. *The equivalence* $D\,\mathsf{Hom}(-, \nu P)$ *restricts to an equivalence*

$$\mathcal{W}'[1] \simeq \mathsf{mod}(C'/\langle f' \rangle).$$

Proof. (1) The equivalence $\mathcal{H}(P) \simeq \mathsf{mod}\, C$ maps the wide subcategory $\mathcal{W}$ to the extension closure of the simple C-modules $\{\mathsf{Hom}(P, X_i) \mid i = 1, \dots, m\}$. In $\mathsf{mod}\, C$, such an extension closure is precisely the category of modules whose composition factors are among these m simples, which is equivalent to the category $\mathsf{mod}(C/\langle \sum_{i=m+1}^{n} f_i \rangle)$. The reader is referred to [8, Theorem 3.15] for further details. The proof of (2) follows dually. □

3.4.5 Cases when $T(\mathcal{S}) = \phi(\mathcal{S})$

As mentioned before, in general, we do not know if $T(\mathcal{S}) = \phi(\mathcal{S})$ for an arbitrary semibrick $\mathcal{S}$. We want to finish this section by collecting some cases in which this holds true for left-finite semibricks.

Let $\mathcal{S}$ be a left-finite semibrick in d-$\operatorname{mod}\Lambda$. Let $\mathcal{H}_{\mathcal{S}}$ be the length heart of the functorially finite positive torsion class $T(\mathcal{S})$.

Lemma 3.4.12 *Let $d = 2$ and $0 \neq X \in \mathcal{H}_{\mathcal{S}} \cap \operatorname{mod}\Lambda$. Then there exists a triangle*

$$Y \to X \to X'$$

with $Y[1] \in \operatorname{Fac}_2 \mathcal{S}$, $X' \in \mathcal{H}_{\mathcal{S}} \cap \operatorname{mod}\Lambda$, and $\operatorname{dim}(X') < \operatorname{dim}(X)$ in $\operatorname{mod}\Lambda$.

Proof. Since $\mathcal{H}_{\mathcal{S}}$ is length, there exists a triangle $S \xrightarrow{f} X \to M$ with $S \in \operatorname{sim}\mathcal{H}_{\mathcal{S}}$, $M \in \mathcal{H}_{\mathcal{S}}$, and $f \neq 0$. We have the following diagram of triangles.

$$\begin{tikzcd}
S \arrow[r, "g"] \arrow[d, equal] & Y \arrow[r] \arrow[d] & \sigma_{\leq -1}M \arrow[d] \\
S \arrow[r, "f"] & X \arrow[r] \arrow[d] & M \arrow[d] \\
 & \sigma_{\geq 0}M \arrow[r, equal] & \sigma_{\geq 0}M
\end{tikzcd}$$

Set $X' := \sigma_{\geq 0}M$. Moreover, we have $Y \in X'[-1] * X \subseteq \mathcal{D}^{[0,1]}(\operatorname{mod}\Lambda)$ and $Y \in S * (\sigma_{\leq -1}M)[-1] \subseteq \mathcal{D}^{[-2,0]}(\operatorname{mod}\Lambda)$. This implies that $Y \in \operatorname{mod}\Lambda$. Thus, $Y \to X \to X'$ is a short exact sequence in $\operatorname{mod}\Lambda$. Since $f \neq 0$, $Y \neq 0$, which implies that $\operatorname{dim}(X') < \operatorname{dim}(X)$. Moreover, $S \in (\sigma_{\leq -1}M)[-1] * Y \subseteq \mathcal{D}^{[-1,0]}(\operatorname{mod}\Lambda)$. This implies that $S \in \mathcal{S}$. The triangles

$$Y \to 0 \to Y[1]$$

$$(\sigma_{\leq -1}M)[-1] \to S \to Y$$

give that $Y[1] \in \operatorname{Fac}_2 \mathcal{S}$. Since $X' \in \operatorname{mod}\Lambda \subseteq \mathcal{H}_{\mathcal{S}} * \mathcal{H}_{\mathcal{S}}[-1] * \mathcal{H}_{\mathcal{S}}[-2]$, there is a triangle $X_1 \xrightarrow{h_1} X \xrightarrow{h_2} X_2$ with $X_1 \in \mathcal{H}_{\mathcal{S}}$ and $X_2 \in \mathcal{H}_{\mathcal{S}}[-1] * \mathcal{H}_{\mathcal{S}}[-2]$. We have the exact sequence

$$0 = \operatorname{Hom}((\sigma_{\leq -1}M)[1], X_2) \to \operatorname{Hom}(X', X_2) \to \operatorname{Hom}(M, X_2) = 0$$

where the first equality holds since $(\sigma_{\leq -1}M)[1] \in \mathcal{D}^{[-3,-2]}(\operatorname{mod}\Lambda)$ and $\mathcal{H}_{\mathcal{S}}[-1] * \mathcal{H}_{\mathcal{S}}[-2] \subseteq \mathcal{D}^{[-1,2]}(\operatorname{mod}\Lambda)$. Therefore, $h_2 = 0$, and $X' \in \mathcal{H}_{\mathcal{S}}$.

□

Lemma 3.4.13 *Let $d = 2$. Then $\mathcal{H}_{\mathcal{S}}[1] \cap \operatorname{mod}\Lambda[1] \subseteq \phi(\mathcal{S})$.*

Proof. Let $X \in \mathcal{H}_{\mathcal{S}}[1] \cap \operatorname{mod}\Lambda[1]$. We use induction on the dimension of $X[-1]$ in $\operatorname{mod}\Lambda$.

If $(\operatorname{dim} X[-1]) = 0$, $X = 0 \in \phi(\mathcal{S})$.

Suppose the result holds for all $Y \in \mathcal{H}_{\mathcal{S}}[1] \cap \operatorname{mod}\Lambda[1]$ with $\operatorname{dim} Y[-1] < \operatorname{dim} X[-1]$. By the above lemma, there exists a triangle $Z \to X \to X'$ with $Z \in \phi(\mathcal{S})$ and $X' \in \mathcal{H}_{\mathcal{S}}[1] \cap \operatorname{mod}\Lambda[1]$ with $\operatorname{dim} X'[-1] < \operatorname{dim} X[-1]$. Then $X' \in \phi(\mathcal{S})$. Since $\phi(\mathcal{S})$ is closed under extensions, $X \in \phi(\mathcal{S})$. □

Theorem 3.4.14 *For a left-finite semibrick $\mathcal{S} \subseteq d$-$\operatorname{mod}\Lambda$, $\phi(\mathcal{S}) = T(\mathcal{S})$ if one of the following happens:*

1. *$d \leq 2$;*

2. *Λ is hereditary;*

3. *d-$\operatorname{mod}\Lambda$ has finitely many indecomposable objects up to isomorphism.*

Proof. We already know that $\phi(\mathcal{S}) \subseteq T(\mathcal{S})$. The other inclusion is equivalent to showing that $\mathcal{D}^{\leq -d} * T(\mathcal{S}) \subseteq \mathcal{D}^{\leq -d} * \phi(\mathcal{S})$. Using [69, Theorem 6.1], we know that $\mathcal{D}^{\leq -d} * T(\mathcal{S})$, the aisle of the t-structure corresponding to $\mathcal{S}$, is the extension closure of $\{X_i[l] \mid i = 1, \cdots, r, l \geq 0\}$ in $\mathcal{D}^b(\text{mod}\,\Lambda)$, where $\{X_i\}_{i=1}^r$ is the set of simples in $\mathcal{H}_{\mathcal{S}}$. We know that $\mathcal{S} = \{X_i\}_{i=1}^r \cap d\text{-}\,\text{mod}\,\Lambda$. Proposition 1.1.6 now tells us that $\mathcal{D}^{\leq -d} * \phi(\mathcal{S})$ is closed under extensions and positive shifts. Therefore, $\mathcal{D}^{\leq -d} * T(\mathcal{S}) \subseteq \mathcal{D}^{\leq -d} * \phi(\mathcal{S})$ if and only if $\sigma_{\geq -d+1} X_i \in \phi(\mathcal{S})$ for all $i = 1, \cdots, r$. We show this case by case.

(2) □ is hereditary: Since each X_i is a brick, it is indecomposable. By [47], $X_i \cong Y[t]$ for some $Y \in \text{mod}\,\Lambda$ and $d \geq t \geq 0$. If $t \leq d-1$, then $X_i \in \mathcal{X} \subseteq \phi(\mathcal{S})$. If $t = m$, then $\sigma_{\geq -d+1} X_i = 0 \in \phi(\mathcal{S})$.

(1) d $=$ 2: Using Lemma 3.4.13, it is enough to show that for all $X_i \notin \mathcal{X}$, $\sigma_{\geq -1} X_i \in \mathcal{H}_{\mathcal{S}}[1] \cap \text{mod}\,\Lambda[1]$. Since $X_i \in \mathcal{D}^{[-2,0]}$, we have a triangle $\sigma_{\leq -2} X_i \xrightarrow{f} X_i \to \sigma_{\geq -1} X_i$ with $f \neq 0$ since $X_i \notin \mathcal{X}$. Let $P_{\mathcal{S}}$ be the $(d+1)$-term silting object corresponding to $T(\mathcal{S})$. Applying $\text{Hom}(P_{\mathcal{S}}, -)$ to this triangle, we get the exact sequence

$$\text{Hom}(P_{\mathcal{S}}, (\sigma_{\geq -1} X_i)[-1-j]) \to \text{Hom}(P_{\mathcal{S}}, (\sigma_{\leq -2} X_i)[-j]) \to \text{Hom}(P_{\mathcal{X}}, X_i[-j])$$

for all $j \in \mathbb{Z}$. Note that for $j \geq 1$, the first and the last term vanish since X_i is in the simple-minded collection corresponding to $P_{\mathcal{S}}$. This implies that $(\sigma_{\leq -2} X_i)[-2], (\sigma_{\leq -2} X_i)[-1] \in T(\mathcal{S})^{\perp}$. Hence, $\sigma_{\leq -2} X_i, (\sigma_{\leq -2} X_i)[1] \in T(\mathcal{S})^{\perp}[2] \subseteq \mathcal{H}_{\mathcal{S}}[1] * \mathcal{H}_{\mathcal{S}}$ by [99, Theorem 1.12]. This implies that $\sigma_{\leq -2} X_i \in \mathcal{H}_{\mathcal{S}}$. Thus, f is a surjective map in $\mathcal{H}_{\mathcal{S}}$ since f is non-zero and X_i is simple in $\mathcal{H}_{\mathcal{S}}$. This implies that $\sigma_{\geq -1} X_i \in \mathcal{H}_{\mathcal{S}}[1] \subseteq \mathcal{D}^{[-3,-1]}$. Hence, $\sigma_{\geq -1} X_i \in \mathcal{H}_{\mathcal{S}}[1] \cap \text{mod}\,\Lambda[1]$.

(3) $d = 1$ or d- mod Λ has finitely many indecomposables: In these cases, a subcategory is a positive torsion class if and only if it is closed under d-factors and extensions. Therefore, $\phi(\mathcal{S}) = T(\mathcal{S})$ for any semibrick $\mathcal{S}$. ☐

3.5 Mutations of $(d+1)$-term silting complexes

In this final section, we provide a criterion to determine which mutations of a $(d+1)$-term silting complex P remain $(d+1)$-term. We achieve this by establishing a relationship between the elements of the simple-minded collection associated with P and certain objects in the additive subcategory add P.

Throughout this section, let P be a basic $(d+1)$-term silting complex in $\mathsf{K}^b(\text{proj}\,\Lambda)$. We begin with the following technical lemma concerning the existence of a specific sequence of triangles.

Lemma 3.5.1 *There exist d triangles in $\mathsf{K}^b(\text{proj}\,\Lambda)$ of the form*

$$\begin{tikzcd}
Z_0 = \Lambda \arrow[r, "f_0"] & Q_0 \arrow[r, "g_1"] & Z_1 \\
Z_1 \arrow[r, "f_1"] & Q_1 \arrow[r, "g_2"] & Z_2 \\
 & \vdots & \\
Z_{d-1} \arrow[r, "f_{d-1}"] & Q_{d-1} \arrow[r, "g_d"] & Q_d = Z_d
\end{tikzcd} \tag{3.11}$$

such that for each $0 \leq j \leq d$, the following conditions hold:

1. *$Q_j \in$ add P;*

2. *f_j is a minimal left (add P)-approximation of Z_j, with $f_d = \text{id}_{Z_d}$;*

3. *$\text{Hom}(Z_j, P[l]) = 0$ for all $l \geq 1$;*

4. *$\text{Hom}(P, Z_j[l]) = 0$ for all $l \geq d - j + 1$.*

Proof. We construct the triangles by taking $f_j \colon Z_j \to Q_j$ to be a minimal left (add P)-approximation of Z_j and defining $Z_{j+1} := C(f_j)$ for $0 \leq j \leq d-1$. By this construction, (1) and (2) hold for all $0 \leq j \leq d-1$.

We proceed to show that conditions (3) and (4) hold by induction on j.

For the starting case $j = 0$, we have $Z_0 = \Lambda$. Since P is a $(d+1)$-term silting complex, it follows that $\mathrm{Hom}(Z_0, P[l]) = 0$ for all $l \geq 1$ and $\mathrm{Hom}(P, Z_0[l]) = 0$ for all $l \geq d+1$, satisfying the requirements.

Assume that (3) and (4) hold for all $j \leq t < d$. For $j = t+1$, applying the functor $\mathrm{Hom}(P, -)$ to the $(t+1)$-th triangle in (3.11) yields the exact sequence

$$\mathrm{Hom}(P, Q_t[l]) \to \mathrm{Hom}(P, Z_{t+1}[l]) \to \mathrm{Hom}(P, Z_t[l+1]).$$

By the induction hypothesis, the third term vanishes for all $l+1 \geq d-t+1$. Since P is silting and $Q_t \in \mathrm{add}\, P$, the first term vanishes for all $l > 0$. Thus, $\mathrm{Hom}(P, Z_{t+1}[l]) = 0$ for any $l \geq d-t$, which confirms (4) for $j = t+1$.

Next, applying $\mathrm{Hom}(-, P)$ to the same triangle, we obtain the exact sequence

$$\mathrm{Hom}(Q_t, P[l]) \to \mathrm{Hom}(Z_t, P[l]) \to \mathrm{Hom}(Z_{t+1}, P[l+1]) \to \mathrm{Hom}(Q_t, P[l+1]).$$

For $l \geq 1$, the second term vanishes for $l \geq 1$ by the induction hypothesis. For $l \geq 0$, the last term vanishes for $l \geq 0$ since P is silting. For $l = 0$, the first map is surjective because f_t is a left $(\mathrm{add}\, P)$-approximation. We conclude that $\mathrm{Hom}(Z_{t+1}, P[l+1]) = 0$ for all $l \geq 0$, which is equivalent to $\mathrm{Hom}(Z_{t+1}, P[l]) = 0$ for any $l \geq 1$. Thus, (3) holds for $j = t+1$.

Finally, we show that $Z_d \in \mathrm{add}\, P$. Since $Z_d \in \mathsf{K}^b(\mathrm{proj}\, \Lambda)$ and P is a silting complex, by [4, Proposition 2.23 (b)], there exists a triangle

$$X \xrightarrow{f} Z_d \xrightarrow{g} Y,$$

where $X \in P[-s] * \cdots * P$ and $Y \in P[1] * \cdots * P[s]$ for some $s > 0$. Applying (3) to Z_d, we have $\mathrm{Hom}(Z_d, P[l]) = 0$ for any $l \geq 1$, which implies $\mathrm{Hom}(Z_d, Y) = 0$. Thus, $g = 0$, and Z_d is a direct summand of X. Since $P[-s] * \cdots * P$ is closed under direct summands [58, Proposition 2.1], we have $Z_d \in P[-s] * \cdots * P$.

Consequently, there exists a triangle $V \xrightarrow{h} Z_d \to Z$ such that $Z \in \mathrm{add}\, P$ and $V \in P[-s] * \cdots * P[-1]$. Applying (4) to Z_d, we have $\mathrm{Hom}(P, Z_d[l]) = 0$ for all $l \geq 1$, which implies $\mathrm{Hom}(V, Z_d) = 0$. Thus $h = 0$, and we conclude that Z_d is a direct summand of Z and hence belongs to $\mathrm{add}\, P$. □

Let $h_j := f_j \circ g_j \colon Q_{j-1} \to Q_j$ for $1 \leq j \leq d$. Let $\mathcal{H}(P)$ be the heart of the bounded t-structure associated with P; see (3.9). For brevity, we denote $\mathrm{Hom}(X, Y)$ by (X, Y) in the following proofs.

Lemma 3.5.2 *For each* $X \in \mathrm{sim}\, \mathcal{H}(P)$*, we have* $\mathrm{Hom}(h_j, X) = 0$ *for all* $1 \leq j \leq d$*.*

Proof. Since f_{j-1} is a minimal left approximation, the map $g_j \in \mathrm{rad}(Q_{j-1}, Z_j)$, which implies that the composition h_j lies in $\mathrm{rad}(Q_{j-1}, Q_j)$. We consider the following commutative diagram:

$$\begin{tikzcd}
(Q_j, X) \arrow[r, "{(h_j, X)}"] \arrow[d, "\cong"'] & (Q_{j-1}, X) \arrow[d, "\cong"] \\
((P, Q_j), (P, X)) \arrow[r, "{((P, h_j), (P, X))}"'] & ((P, Q_{j-1}), (P, X))
\end{tikzcd}$$

The vertical isomorphisms are induced by the Yoneda lemma, as $X \in \mathcal{H}(P)$ and $Q_j, Q_{j-1} \in \mathrm{add}\, P$. Since $h_j \in \mathrm{rad}(Q_{j-1}, Q_j)$, the equivalence of categories

$$\mathrm{add}\, P \xrightarrow{(P,-)} \mathrm{proj}\, \mathrm{End}(P)$$

ensures that $(P, h_j) \in \mathrm{rad}((P, Q_{j-1}), (P, Q_j))$. Note that (P, Q_{j-1}) and (P, Q_j) are projective modules in $\mathrm{mod}\, \mathrm{End}(P)$. Consequently, the image of (P, h_j) is contained in $\mathrm{rad}(P, Q_j)$.

From the equivalence (3.10) that (P, X) is a simple module in $\mathrm{mod}\, \mathrm{End}(P)$. Any morphism from a projective module to a simple module must vanish on the radical of the projective module. Therefore, the morphism $((P, h_j), (P, X))$ must be zero, which implies $(h_j, X) = 0$ as required. □

Suppose $P = \bigoplus_{i=1}^{n} P_i$ with P_i indecomposable. Let $\mathcal{X} = \{X_i\}_{i=1}^{n} = \mathsf{sim}\,\mathcal{H}(P)$ be the associated simple-minded collection, ordered such that $(P_i, X_j[m]) = 0$ for $i \neq j$ or $m \neq 0$, and $(P_i, X_i) \cong \mathsf{End}(X_i)$. The following proposition demonstrates that for each $0 \leq j \leq d$, $\mathsf{H}^{-j}(X_i)$ is nonzero if and only if $P_i \in \mathsf{add}\,Q_j$.

Proposition 3.5.3 *For any $X \in \mathsf{sim}\,\mathcal{H}(P)$ and each $0 \leq j \leq d$, there is an isomorphism*

$$\mathsf{Hom}(\Lambda[j], X) \cong \mathsf{Hom}(Q_j, X).$$

Proof. **Claim 1: $(Z_j[l], X) = 0$ for any $l < 0$ and $0 \leq j \leq d$.** This is immediate for $j = d$ since $Z_d = Q_d \in \mathsf{add}\,P$ and $X \in \mathcal{H}(P)$. For $0 \leq j \leq d-1$, applying $(-, X)$ to the triangles in (3.11) yields the exact sequence

$$(Z_{j+1}[l], X) \to (Q_j[l], X) \to (Z_j[l], X) \to (Z_{j+1}[l-1], X) \to (Q_j[l-1], X), \quad l \in \mathbb{Z}. \tag{3.12}$$

For any $l < 0$, the terms involving Q_j vanish because $Q_j \in \mathsf{add}\,P$ and $X \in \mathcal{H}(P)$. Thus, we obtain the following isomorphism chain:

$$(Z_j[l], X) \cong (Z_{j+1}[l-1], X) \cong \cdots \cong (Q_d[j+l-d], X) \cong 0.$$

Claim 2: The map $(f_j, X) : (Q_j, X) \to (Z_j, X)$ is an isomorphism for $0 \leq j \leq d$, and the map $(g_j, X) : (Z_j, X) \to (Q_{j-1}, X)$ is zero for $1 \leq j \leq d$. We proceed by induction on j from d down to 0. For the starting case $j = d$, $f_d = \mathsf{id}_{Z_d}$ and $g_d = h_d$, so it is trivial that (f_j, X) is an isomorphism, and $(g_d, X) = 0$ follows directly from Lemma 3.5.2. Now, assume the statement holds for $j+1$ and consider $0 \leq j \leq d-1$. The exact sequence (3.12) for $l = 0$ is:

$$(Z_{j+1}, X) \xrightarrow{(g_{j+1}, X)} (Q_j, X) \xrightarrow{(f_j, X)} (Z_j, X) \to (Z_{j+1}[-1], X).$$

The last term vanishes by Claim 1, and the first map (g_{j+1}, X) is zero by the induction hypothesis. Thus, (f_j, X) is an isomorphism. Furthermore, for $j \geq 1$, Lemma 3.5.2 implies $(h_j, X) = (g_j, X) \circ (f_j, X) = 0$. Since (f_j, X) is an isomorphism, we must have $(g_j, X) = 0$.

Claim 3: $(Z_j[1], X) \cong (Z_{j+1}, X)$ for all $0 \leq j \leq d-1$. The exact sequence (3.12) for $l = 1$ is

$$(Q_j[1], X) \to (Z_j[1], X) \to (Z_{j+1}, X) \xrightarrow{(g_{j+1}, X)} (Q_j, X).$$

Since $(Q_j[1], X) = 0$ and $(g_{j+1}, X) = 0$ (from Claim 2), the middle map is an isomorphism.

Claim 4: $(\Lambda[j], X) \cong (Z_{j-1}[1], X)$ for all $1 \leq j \leq d$. Observing that for $l > 1$, the terms in the exact sequence (3.12) involving Q_j also vanish as $Q_j \in \mathsf{add}\,P$ and $X \in \mathcal{H}(P)$. Thus, we obtain the isomorphism chain:

$$(\Lambda[j], X) = (Z_0[j], X) \cong (Z_1[j-1], X) \cong \cdots \cong (Z_{j-1}[1], X).$$

Conclusion: By combining these claims, for all $1 \leq j \leq d$, we have:

$$(\Lambda[j], X) \overset{\text{C4}}{\cong} (Z_{j-1}[1], X) \overset{\text{C3}}{\cong} (Z_j, X) \overset{\text{C2}}{\cong} (Q_j, X).$$

For $j = 0$, the isomorphism $(\Lambda, X) = (Z_0, X) \cong (Q_0, X)$ follows directly from Claim 2. This completes the proof. ☐

Say that the left (resp. right) mutation of a $(d+1)$-term silting object P at a direct summand P_i exists if $\mu_i^+(P)$ (resp. $\mu_i^-(P)$) remains a $(d+1)$-term silting object. We obtain the following criterion for the existence of such mutations.

Corollary 3.5.4 *Let $P = \bigoplus_{i=1}^{n} P_i$ be a basic $(d+1)$-term silting complex with P_i indecomposable.*

1. *The left mutation of P at P_i exists if and only if $P_i \notin \mathsf{add}\,Q_d$.*

2. *The right mutation of P at P_i exists if and only if $P_i \notin \mathsf{add}\,Q_0$.*

Furthermore, for any indecomposable summand P_i, at least one of the left and right mutations of P at P_i exists.

Proof. Let $\mathcal{X} = \{X_i\}_{i=1}^n = \operatorname{sim}\mathcal{H}(P)$ be the simple-minded collection associated with P. By [69, Theorem 7.12], the left mutation of P at P_i exists if and only if the left mutation of $\mathcal{X}$ at X_i exists. According to Proposition 3.4.4, this is equivalent to $X_i \in \Pi_L(\mathcal{X})$, which means $X_i \in d\text{-}\operatorname{mod}\Lambda$.

By Proposition 3.5.3, we have $\operatorname{Hom}(\Lambda[d], X_i) \cong \operatorname{Hom}(Q_d, X_i)$. Since $X_i \in \mathcal{D}^{[-d,0]}$, the condition $X_i \in d\text{-}\operatorname{mod}\Lambda = \mathcal{D}^{[-d+1,0]}$ is equivalent to the vanishing of $\operatorname{Hom}(\Lambda[d], X_i)$. The isomorphism $\operatorname{Hom}(Q_d, X_i) \cong \operatorname{Hom}(P_i, X_i)^{\oplus m_i}$, where m_i is the multiplicity of P_i in Q_d, gives that this vanishes if and only if $P_i \notin \operatorname{add} Q_d$.

Analogously, the right mutation of P at P_i exists if and only if $X_i \in \Pi_R(\mathcal{X})[1] = \mathcal{D}^{[-d,-1]}$, which is equivalent to $\operatorname{Hom}(\Lambda, X_i) \cong \operatorname{Hom}(Q_0, X_i) = 0$. This occurs if and only if $P_i \notin \operatorname{add} Q_0$. The existence of at least one mutation follows directly from Proposition 3.4.4. □

Chapter 4

Gentle algebras

The results in this chapter appeared in [40].

4.1 Marked surfaces and admissible dissections

Throughout this chapter, K will be an algebraically closed field. We recall here the special definition of marked surfaces as used in [5], the admissible dissections of which are in bijection with gentle algebras.

Definition 4.1.1 ([5, Definition 3.7]) A **marked surface** is a triple (S, M, P), where

1. S is an oriented closed smooth surface with non-empty boundary ∂S;
2. $M = M_\circ \cup M_\bullet$ is a finite set of marked points on ∂S. The elements of $M_\circ$ and $M_\bullet$ will be represented by symbols $\circ$ and $\bullet$, respectively. They are required to alternate on each connected component of ∂S, and each such component is required to contain at least one marked point;
3. $P = P_\bullet$ is a finite set of marked points in the interior of S, called punctures. The elements of $P_\bullet$ will also be represented by $\bullet$.

Definition 4.1.2 ([5, Definition 3.8]) A $\circ$**-arc** (or $\bullet$**-arc**) is a smooth map γ from $[0,1]$ to $S \setminus P$ such that its endpoints $\gamma(0)$ and $\gamma(1)$ are in $M_\circ$ (or in $M_\bullet \cup P_\bullet$, respectively). The curve γ is required not to be contractible to a point in $M_\circ$ (or $M_\bullet \cup P_\bullet$, respectively).

We will usually consider arcs up to homotopy and inverses. Two arcs are said to **intersect** if any choice of homotopic representatives intersect.

Definition 4.1.3 ([5, Definition 3.9]) A collection of pairwise non-intersecting and pairwise different $\circ$-arcs $\{\gamma_1, \cdots, \gamma_r\}$ on the surface (S, M, P) is called **admissible** if the arcs $\{\gamma_1, \cdots, \gamma_r\}$ do not enclose a subsurface containing no punctures of $P_\bullet$ and with no boundary segment on its boundary. A maximal admissible collection of $\circ$-arcs is called an **admissible** $\circ$**-dissection**.

The notion of admissible $\bullet$-dissection is defined similarly with the assumption that $P_\circ = \emptyset$.
To any admissible $\circ$-dissection, we can associate a dual $\bullet$-dissection in the following sense.

Proposition 4.1.4 ([5, Proposition 3.13]) *Let (S, M, P) be a marked surface, and let Δ be an admissible $\circ$-dissection. There exists a unique admissible $\bullet$-dissection Δ^* (up to homotopy) such that each arc of Δ^* intersects exactly one arc of Δ.*

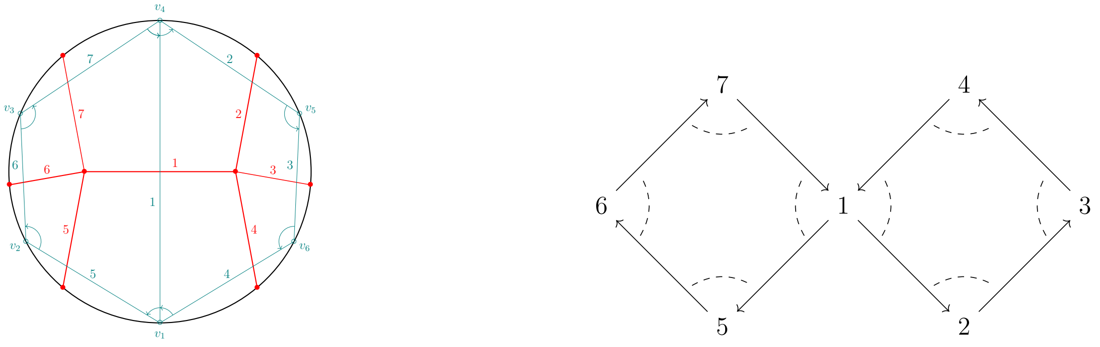
Definition 4.1.5 ([5, Definition 3.14]) The dissections Δ and Δ^* are called **dual dissections**.

4.2 Admissible dissections and gentle algebras

Definition 4.2.1 ([5, Definition 4.1]) Let Δ be an admissible $\circ$-dissection of a marked surface (S, M, P). The K-algebra $A(\Delta)$ is the quotient of the path algebra of the quiver $Q(\Delta)$ by the ideal $I(\Delta)$ defined as follows:

- the vertices of $Q(\Delta)$ are in bijection with the $\circ$-arcs in Δ;
- there is an arrow $i \to j$ in $Q(\Delta)$ whenever the $\circ$-arcs i and j meet at a marked point $\circ$, with i preceding j in the counter-clockwise order around $\circ$, and with no other arc coming to $\circ$ between i and j.
- the ideal $I(\Delta)$ is generated by the following relations: whenever i and j meet at a marked point as above, and the other end of j meets k at a marked point as above, then the composition of the corresponding arrows $i \to j$ and $j \to k$ is a relation.

Example 4.2.2 Let (S, M, P) be the marked surface in Figure 4.1a with Δ and Δ^* the depicted $\circ$ and $\bullet$-dissections respectively. Then $A(\Delta)$ is the path algebra of the quiver with relations shown in Figure 4.1b.



(a) (b)

Figure 4.1: A gentle algebra associated to a marked surface

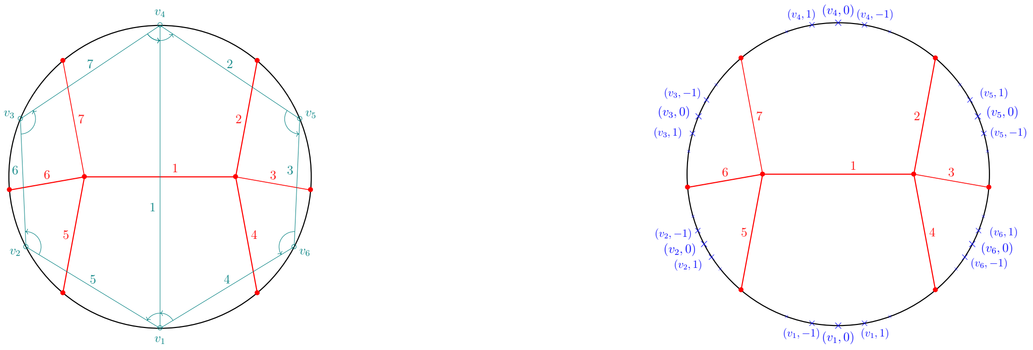
Theorem 4.2.3 ([5, Theorem 4.3]) *The assignment $((S, M, P), \Delta) \to A(\Delta)$ defines a bijection from the set of homeomorphism classes of marked surfaces (S, M, P) with an admissible dissection to the set of isomorphism classes of gentle algebras.*

Let Λ be a gentle algebra of rank n. Let $((S, M, P), \Delta)$ be the corresponding marked surface with an admissible dissection obtained from the previous theorem. We will give another model of the bounded derived category of Λ using this surface.

We start by choosing a labelling of the $\circ$-points in M. We replace each $\circ$-point v on a boundary component between two $\bullet$-points with a collection of $\mathbb{Z}$-indexed $\times$-points $T = \{T_{(i,v)} \mid i \in \mathbb{Z}\}$ arranged in descending order along the orientation of the component. Moreover, we do so in such a way that the set T has precisely two limit points given by the two $\bullet$-points. For a point $T_{(i,v)}$, we define $\epsilon(T_{(i,v)}) := i$ and $\sigma(T_{(i,v)}) := v$. We denote this new model by S_Λ. We will consider special arcs joining the $\times$-points which we call 'slaloms'.

Example 4.2.4 The above process transforms the marked surface in Figure 4.2a to the surface in Figure 4.2b.

Definition 4.2.5 A $\times$-arc is a smooth map γ from the interval $[0, 1]$ to $S \setminus P$ such that its endpoints $\gamma(0)$ and $\gamma(1)$ are $\times$-points. The curve γ is required not to be contractible to a $\times$-point.



(a) (b)

Figure 4.2: Replacing $\circ$-points with $\times$-points

Definition 4.2.6 ([5, Definition 4.4]) Let γ be a $\circ$-arc. Assume γ intersects the arcs of the dual $\bullet$-dissection Δ^* at a finite number of points, minimally and transversally. A **grading** on γ is a function

$$f : \gamma \cap \Delta^* \to \mathbb{Z}$$

satisfying the following: if p and q are in $\gamma \cap \Delta^*$ and q is the successor of p, then γ enters a polygon enclosed by $\bullet$-arcs of Δ^* via p and leaves it via q. If the $\circ$ point in this polygon is to the left of γ, then $f(q) = f(p) + 1$; otherwise, $f(q) = f(p) - 1$.

In this case, (γ, f) is called a **graded $\circ$-arc**.

Definition 4.2.7 Let γ be a $\times$-arc. We assume γ intersects the arcs of the dual $\bullet$-dissection Δ^* at a finite number of points, minimally and transversally. Define

$$f_\gamma : \gamma \cap \Delta^* \to \mathbb{Z}$$

as follows. The orientation of γ induces a total order on the points in $\gamma \cap \Delta^*$. Let p_0 be the initial point in this order. Then $f(p_0) := \epsilon(s(\gamma))$. Next, if p and q are in $\gamma \cap \Delta^*$ and q is the successor of p, then γ enters a polygon enclosed by $\bullet$-arcs of Δ^* via p and leaves it via q. If the $\times$-points in this polygon are to the left of γ, then $f(q) := f(p) + 1$; otherwise, $f(q) := f(p) - 1$.

A $\times$-arc γ is called a **slalom** if $f(q_0) = \epsilon(t(\gamma))$, where q_0 is the last point of $\gamma \cap \Delta^*$.

Example 4.2.8 Let γ_1 be the slalom in Figure 4.3a. Then the corresponding function f_{γ_1} is given as

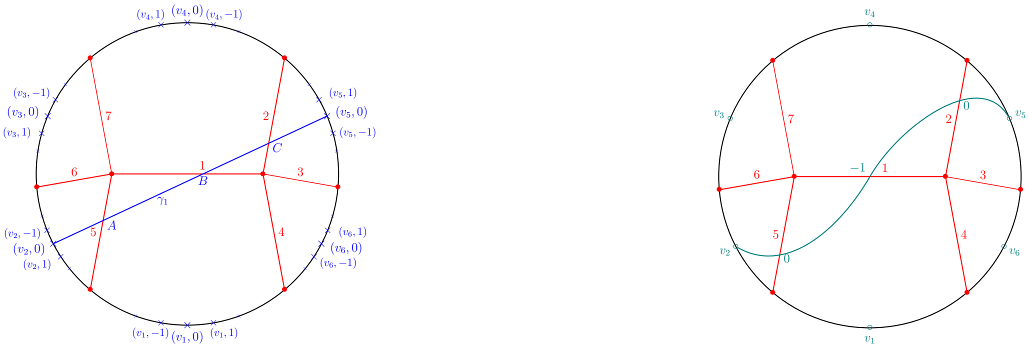
$$f_{\gamma_1}(A) = 0,\ f_{\gamma_1}(B) = -1,\ f_{\gamma_1}(C) = 0.$$

Let γ be a slalom connecting $T_{(i,v_1)}$ to $T_{(j,v_2)}$. We consider the homotopy class of γ where the homotopies are allowed to move the endpoints of γ along the boundary without crossing a $\bullet$-point. Then this homotopy class contains a unique (up to homotopy) $\circ$-arc connecting v_1 and v_2. We denote this $\circ$-arc by σ_γ. Moreover, the function f_γ naturally defines a grading on the arc σ_γ. We will denote the graded $\circ$-arc $(\sigma_\gamma, f_\gamma)$ as $\Sigma(\gamma)$.

Example 4.2.9 Let γ_1 be the slalom in Figure 4.3a. Then using the procedure described above, γ_1 is transformed into the graded arc $\Sigma(\gamma_1)$ shown in Figure 4.3b.

The map Σ defines a bijection between the set of homotopy classes of slaloms with the set of homotopy classes of graded $\circ$-arcs, as defined in [5, Definition 4.4], which are, in turn, in bijection with certain indecomposable objects of $\mathsf{K}^{-,b}(\operatorname{proj} \Lambda)$ called (finite) string objects. We denote by $P^\bullet_\gamma$ the object associated to $\Sigma(\gamma)$. For two graded $\circ$-arcs (μ_1, f_1) and (μ_2, f_2), we will use the description of $\operatorname{Hom}(P^\bullet_{(\mu_1,f_1)}, P^\bullet_{(\mu_2,f_2)})$ in terms of the 'graded intersection points' of (μ_1, f_1) and (μ_2, f_2), as given in [81].

Definition 4.2.10 Let γ_1 and γ_2 be two slaloms and p an intersection point of γ_1 and γ_2 lying in the interior of S. Then p is called **contractible** if (one of) the region(s) bounded by the following three segments is contractible and does not contain any $\bullet$-points on the boundary (Figure 4.4):



(a) (b)

Figure 4.3: Transforming slaloms to graded arcs

1. the part of γ_1 between p and the boundary.
2. the part of γ_2 between p and the boundary.
3. the part of the boundary between the endpoints of γ_1 and γ_2.

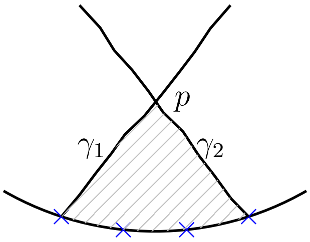


Figure 4.4: p is a contractible intersection point of γ_1 and γ_2

Otherwise, p is called **non-contractible**.

Note that both contractible and non-contractible intersection points are, by definition, in the interior of S.

Definition 4.2.11 Let (μ_1, f_1) and (μ_2, f_2) be two graded $\circ$-arcs intersecting at a point p on the boundary of S. Then the **degree** of the intersection point p is defined to be $f_1(p_1) - f_2(p_2)$, where p_1 and p_2 are the points closest to p in $\mu_1 \cap \Delta^*$ and $\mu_2 \cap \Delta^*$ respectively, and μ_2 is supposed to be clockwise to μ_1 around p (Figure 4.5).

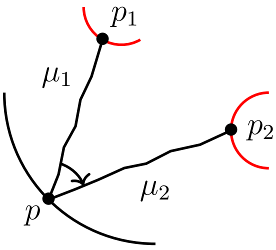


Figure 4.5: A boundary intersection point

Theorem 4.2.12 *Let γ_1 and γ_2 be two slaloms. Then we have the following:*

1. *There is a bijection between the set of interior intersection points of $\Sigma(\gamma_1)$ and $\Sigma(\gamma_2)$, and the set of non-contractible intersection points of γ_1 and γ_2.*
2. *There is a bijection between the set of boundary intersection points of $\Sigma(\gamma_1)$ and $\Sigma(\gamma_2)$ of positive degree, and the set of contractible intersection points of γ_1 and γ_2.*
3. *There is a bijection between the set of boundary intersection points of $\Sigma(\gamma_1)$ and $\Sigma(\gamma_2)$ of degree 0, and the boundary intersection points of γ_1 and γ_2.*

Proof. Let $(\mu_1, f_1) := \Sigma(\gamma_1)$ and $(\mu_2, f_2) := \Sigma(\gamma_2)$. Assume that we have chosen homotopy representatives such that μ_1 and μ_2 intersect minimally, and so do γ_1 and γ_2. Using the fact that μ_1 (resp. μ_2) is homotopic to γ_1 (resp. γ_2) where the homotopies are allowed to move the endpoints without crossing ●-points, and the topological fact that the number of minimal intersection points of two arcs is invariant under homotopy, we can conclude that the interior intersection points of μ_1 and μ_2 will correspond to some interior intersection points of γ_1 and γ_2. Moreover, these interior intersection points of γ_1 and γ_2 will be non-contractible because, otherwise, μ_1 and μ_2 do not intersect minimally (Figure 4.6). Thus, the interior intersection points of μ_1 and μ_2 will be in bijection with the non-contractible intersection points of γ_1 and γ_2.

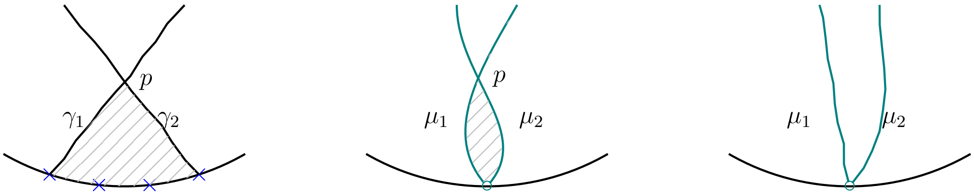


Figure 4.6: p has to be a non-contractible intersection point of γ_1 and γ_2

We now prove the three statements one by one.

1. Let p be an interior intersection point of $\Sigma(\gamma_1)$ and $\Sigma(\gamma_2)$. Then, as argued above, it will correspond to a non-contractible intersection point of γ_1 and γ_2.

   Conversely, if p is a non-contractible intersection point of γ_1 and γ_2, then it is still an interior intersection point of $\Sigma(\gamma_1)$ and $\Sigma(\gamma_2)$ when they intersect minimally.

2. Suppose p is a boundary intersection point of $\Sigma(\gamma_1)$ and $\Sigma(\gamma_2)$ of degree > 0. Then, without loss of generality, we can assume that the configuration around p is as shown in Figure 4.7a with $f_2(p_2) > f_1(p_1)$, where p_1 and p_2 are the points closest to p in $\mu_1 \cap \Delta^*$ and $\mu_2 \cap \Delta^*$ respectively. This implies that $\epsilon(q') > \epsilon(q)$, where q (resp. q') is the endpoint of γ_1 (resp. γ_2) such that $\sigma(q) = p$ (resp. $\sigma(q') = p$). Thus, the configuration of γ_1 and γ_2 is as shown in Figure 4.7b. Since μ_1 and μ_2 are homotopic to γ_1 and γ_2 respectively, and they intersect on the boundary, the point p' has to be a contractible intersection point of γ_1 and γ_2.

(a)

(b)

Figure 4.7: Transformation of a boundary intersection point of positive degree

Conversely, let p' be a contractible intersection point of γ_1 and γ_2. Then, without loss of generality, we can assume that γ_1 and γ_2 are as shown in Figure 4.7b. In particular, the part of the boundary between q and q' does not contain any $\bullet$-points, which implies that $\sigma(q) = \sigma(q')$, and hence $\Sigma(\gamma_1)$ and $\Sigma(\gamma_2)$ intersect on the boundary. Moreover, this intersection point has a positive degree because $f_2(p_2) = \epsilon(q') > \epsilon(q) = f_1(p_1)$.

3. Suppose p is a boundary intersection point of $\Sigma(\gamma_1)$ and $\Sigma(\gamma_2)$ of degree 0. Then we can assume that the configuration around p is as shown in Figure 4.7a such that $f_1(p_1) = f_2(p_2)$. This implies that $\epsilon(q) = \epsilon(q') = f_1(p_1)$, where q (resp. q') is the endpoint of γ_1 (resp. γ_2) such that $\sigma(q) = p$ (resp. $\sigma(q') = p$). Thus, $q = q'$ is a boundary intersection point of γ_1 and γ_2. Conversely, if q is a boundary intersection point of γ_1 and γ_2, then, by definition, $\Sigma(\gamma_1)$ and $\Sigma(\gamma_2)$ have an intersection point on the boundary of degree 0.

□

Proposition 4.2.13 *Two slaloms γ_1, γ_2 intersect in the interior of S if and only if either* $\mathrm{Ext}^i(P^\bullet_{\gamma_1}, P^\bullet_{\gamma_2}) \neq 0$ *for some $i > 0$ or* $\mathrm{Ext}^i(P^\bullet_{\gamma_2}, P^\bullet_{\gamma_1}) \neq 0$ *for some $i > 0$.*

Proof. Suppose γ_1 and γ_2 intersect in the interior of S at a point p. If p is a contractible intersection point, then using Theorem 4.2.12, it corresponds to a boundary intersection point of $\Sigma(\gamma_1)$ and $\Sigma(\gamma_2)$ of positive degree. This implies that either $\mathrm{Ext}^i(P^\bullet_{\gamma_1}, P^\bullet_{\gamma_2}) \neq 0$ for some $i > 0$ or $\mathrm{Ext}^i(P^\bullet_{\gamma_2}, P^\bullet_{\gamma_1}) \neq 0$ for some $i > 0$. On the other hand, if p is a non-contractible intersection point, then it corresponds to an interior intersection point of $\Sigma(\gamma_1)$ and $\Sigma(\gamma_2)$. Using [5, Lemma 5.5], we get that either $\mathrm{Ext}^i(P^\bullet_{\gamma_1}, P^\bullet_{\gamma_2}) \neq 0$ for some $i > 0$ or $\mathrm{Ext}^i(P^\bullet_{\gamma_2}, P^\bullet_{\gamma_1}) \neq 0$ for some $i > 0$.

Conversely, suppose γ_1 and γ_2 do not intersect in the interior of S. Then either they do not intersect at all or they intersect on the boundary. In both cases, $\Sigma(\gamma_1)$ and $\Sigma(\gamma_2)$ can only have zero or negative degree boundary intersection points using Theorem 4.2.12. This implies that $\mathrm{Ext}^i(P^\bullet_{\gamma_1}, P^\bullet_{\gamma_2}) = 0$ for all $i > 0$ and $\mathrm{Ext}^i(P^\bullet_{\gamma_2}, P^\bullet_{\gamma_1}) = 0$ for all $i > 0$. □

The above proposition helps us to give a characterization of presilting and silting objects in $\mathsf{K}^{-,b}(\mathrm{proj}\,\Lambda)$.

Definition 4.2.14 Say that a collection $\gamma_1, \gamma_2, \ldots, \gamma_m$ of slaloms is **mutually non-intersecting** if γ_i does not intersect γ_j in the interior of S for all $1 \leq i, j \leq m$.

Corollary 4.2.15 *There is a bijective correspondence between basic presilting objects in* $\mathsf{K}^{-,b}(\mathrm{proj}\,\Lambda)$ *and mutually non-intersecting collections of slaloms in S_Λ.*

Proof. Suppose $U = \oplus_{i=1}^m U_i$ is a basic presilting object in $\mathsf{K}^{-,b}(\mathrm{proj}\,\Lambda)$ with U_i indecomposable. Then, using [5, Lemma 5.4], we get that U_i is of the form $P^\bullet_{\gamma_i}$ for some slalom γ_i. Since U is presilting, we get that $\mathrm{Ext}^j(U_{i_1}, U_{i_2}) = 0$ for all $j > 0$ and for all $1 \leq i_1, i_2 \leq m$. Using the previous proposition, this gives that $\{\gamma_1, \cdots, \gamma_m\}$ is a mutually non-intersecting collection of slaloms in S_Λ.

Conversely, given a mutually non-intersecting collection of slaloms $\{\gamma_1, \cdots, \gamma_m\}$ in S_Λ, the previous proposition immediately implies that $U = \oplus_{i=1}^m P^\bullet_{\gamma_i}$ is a basic presilting object in $\mathsf{K}^{-,b}(\mathrm{proj}\,\Lambda)$. □

Using [5, Proposition 5.7], we know that for a gentle algebra Λ, a basic presilting object U is silting if and only if it has $n = |\Lambda|$ many indecomposable summands. This gives us the following corollary.

Corollary 4.2.16 *There is a bijection between basic silting objects in* $\mathsf{K}^{-,b}(\mathrm{proj}\,\Lambda)$ *and mutually non-intersecting collections of n slaloms in S_Λ.*

We can also restrict the above model to obtain a characterization of d-term presilting objects of Λ. To do this, we keep only the $\times$-points labelled from 0 to $-d+1$ between each pair of neighbouring $\bullet$-points. A slalom in the restricted model is a slalom γ in the original model for which $f_\gamma(\gamma \cap \Delta^*) \subseteq [-d+1, 0]$, as this ensures that $P^\bullet_\gamma$ is concentrated in $[-d+1, 0]$. We will denote this restricted model by S^d_Λ. It is easy to see that Theorem 4.2.12 and Proposition 4.2.13 still hold in this restricted model, which gives us the following corollary.

Corollary 4.2.17 *There is a bijection between basic d-term silting objects of Λ and mutually non-intersecting collections of n slaloms in S^d_Λ.*

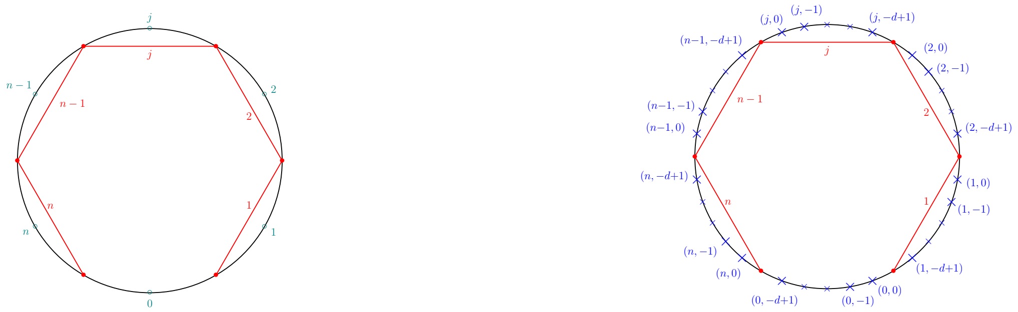
Remark 4.2.18 Note that in the full model S_Λ, for every finite arc starting at a $\times$-point, there exists a unique choice for the other endpoint which makes it a slalom. However, this choice might not exist in the restricted model S^d_Λ, because of the additional condition that $f_\gamma(\gamma \cap \Delta^*) \subseteq [-d+1, 0]$.

Remark 4.2.19 In [82], the authors introduced a model for basic support τ-tilting modules by considering blossom points on the boundary of a marked surface and slaloms connecting these points. Viewing these blossom points as $\times$-points, one precisely recovers the model S^2_Λ described above, which is consistent with the fact that basic support τ-tilting modules are in bijection with basic 2-term silting complexes [2].

4.3 Counting d-term silting objects in linearly oriented A_n

Let KA_n be the path algebra of the linearly oriented quiver of type A_n, that is, $KA_n = K(1 \to 2 \to \cdots \to n)$. In this section, we will give a recursive formula for the number of d-term silting objects in KA_n using the above corollary. This will recover the result of [94] that these are counted by the Pfaff-Fuss-Catalan numbers.

The marked surface $((S, M, P), \Delta)$ obtained from Theorem 4.2.3 for $\Lambda = KA_n$ is shown in Figure 4.8a. By replacing each $\circ$-point in this figure with d $\times$-points, we obtain the surface $S^d_{KA_n}$ defined above (Figure 4.8b). For simplicity, we will denote this by S^d_n later.



(a) (b)

Figure 4.8: Replacing $\circ$-points with d $\times$-points

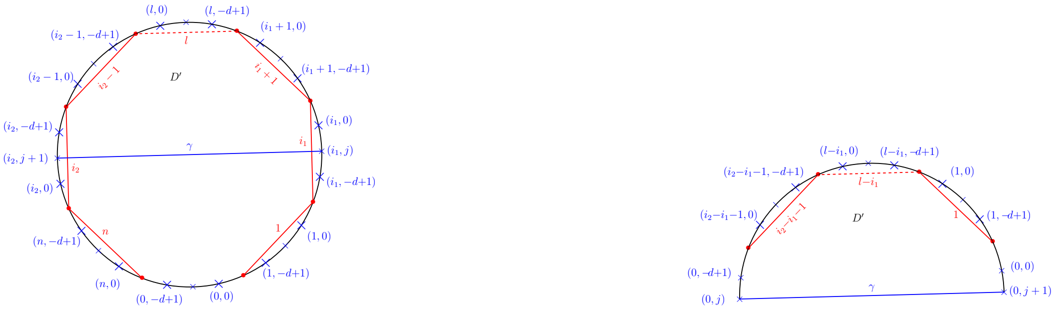
Let B^d_n denote the number of basic d-term silting objects in KA_n. Using Corollary 4.2.17, we get that B^d_n is also the number of collections of n mutually non-intersecting slaloms in Figure 4.8b. To calculate B^d_n, we first count the number of such collections containing some fixed slalom γ. This is done by cutting the disc along this slalom, and relabelling (one of) the parts thus obtained to get $S^d_{KA_m}$ for some $m < n$. This allows us to build a recursive formula for B^d_n. We explain the detailed process below.

Let Γ be a collection of n mutually non-intersecting slaloms. Such a collection will be maximal with respect to the property of mutual non-intersection as the number of indecomposable summands of any presilting object is less than or equal to n [98]. Our first claim is that in such a collection of n slaloms, at least one of $(0,0), (0,-1), (0,-2), \cdots (0,-d+1)$ has to be an endpoint of some slalom. This is because otherwise, we can add the slalom in Figure 4.9a to Γ to get a collection of $n+1$ mutually non-intersecting slaloms, contradicting the maximality of Γ.

Lemma 4.3.1 *Let γ be a slalom in S^d_n connecting (i_1, j) to $(i_2, j+1)$ with $0 < i_1 < i_2$ as shown in Figure 4.10a. Then a $\times$-arc lying in the disc D' is a slalom in S^d_n if and only if it is a slalom in $S^d_{i_2-i_1-1}$ obtained from D' by deleting the points $(i_1, 0)$ and $(i_2, -(d-1))$, and relabeling the other points as shown in Figure 4.10b.*

(a) (b)

Figure 4.9: (a) Slalom contradicting maximality (b) Configuration of d slaloms



(a) (b)

Figure 4.10: Cutting a disc along a slalom and relabeling

Proof. We first note that there are no slaloms in S_n^d starting at $(i_1, 0)$ and lying in D' because the endpoint of such a slalom will have to be of the form $(j, 1)$ which does not exist in the restricted model. Thus, we can remove $(i_1, 0)$. Moreover, since the only slaloms in D' starting at (i_1, m) for some $j \leq m \leq -1$ end at $(l, m+1)$ for some $i_1 + 1 \leq l \leq i_2 - 1$, removing the ●-arc i_1 and relabelling as in Figure 4.10b gives a bijection between the set of slaloms starting at these points. Similarly, there are no slaloms in S_n^d lying in D' and ending at $(i_2, -d+1)$ because the starting point of such a slalom will have to be of the form $(j, -d)$ which does not exist in the restricted model. Therefore, we can remove $(i_2, -d+1)$ as well. Moreover, since the only slaloms in D' ending at (i_2, m) for some $-d+2 \leq m \leq j+1$ start at $(l, m-1)$ for some $i_1 + 1 \leq l \leq i_2 - 1$, removing the ●-arc i_2, the point $(i_2, -d+1)$, and relabelling as in Figure 4.10b gives a bijection between the set of slaloms ending at these points. Combining this with the map that sends the slalom connecting (s_1, t) to $(s_2, t+1)$ for some $i_1 < s_1 < s_2 < i_2$ in D' to the slalom connecting $(s_1 - i_1, t)$ to $(s_2 - i_1, t+1)$ in the relabelled figure, we get the required bijection. ☐

Lemma 4.3.2 *Let γ be a slalom connecting $(0, j)$ to (i, j) in S_n^d as shown in Figure 4.11a. Then a ×-arc lying in the disc D' is a slalom in S_n^d if and only if it is a slalom in S_{n-i}^d obtained from D' by deleting $(i, 0)$ and relabelling the points as shown in Figure 4.11b.*

Proof. Since there are no slaloms in S_n^d starting at $(i, 0)$ and lying in D' (other than possibly γ when $j = 0$), we can remove it. Moreover, since the only slaloms in D' starting at (i, m) for some $j \leq m \leq -1$ end at $(l, m+1)$ for some $i + 1 \leq l \leq n$, removing the ●-arc i and relabelling as in Figure 4.11b gives a bijection between the set of slaloms starting at these points. Combining this with the map that sends the slalom connecting (s_1, t) to $(s_2, t+1)$ for some $i < s_1 < s_2 \leq n$ in D' to the slalom connecting $(s_1 - i, t)$ to $(s_2 - i, t+1)$ in the relabelled figure, and the slalom connecting (s, t) to $(0, t)$, for some $i < s_1 \leq n$ and $-d+1 \leq t \leq j$, in D' to the slalom connecting $(s - i, t)$ to $(0, t)$ in the relabelled figure, we get the required bijection. ☐

Corollary 4.3.3 *The number of mutually non-intersecting collections of $i_2 - i_1 - 1$ slaloms in S_n^d lying in D' is $B_{i_2-i_1-1}^d$.*

(a) (b)

Figure 4.11: Cutting a disc along a slalom and relabeling

Proof. It is easy to see that a collection of $i_2 - i_1 - 1$ slaloms in S_n^d lying in D' is mutually non-intersecting if and only if it is mutually non-intersecting in $S^d_{i_2-i_1-1}$ under the above bijection. Using Corollary 4.2.17, the number of such collections is equal to $B^d_{i_2-i_1-1}$. ☐

We can now divide the problem into two cases.

1. $(0,0)$ is the endpoint of a slalom in Γ: Let γ_d be the rightmost such slalom and $(i_{d-1},0)$ the other endpoint of γ_d. If $i_{d-1} = 1$, then all the other $n-1$ slaloms in Γ lie to the left of γ_d (Figure 4.13b). Using Lemma 4.3.2, we get that the number of such mutually non-intersecting collections of slaloms is B^d_{n-1}.

   Now suppose, $i_{d-1} > 1$. Since γ_d is the rightmost slalom connected to $(0,0)$, we need to block the slaloms connecting $(0,0)$ to some $(j,0)$ for $j < i_{d-1}$. This gives that at least one of $(1,-1), \cdots (1,-d+1)$ is the endpoint of some slalom in Γ. Suppose $(1,-d+1)$ is the endpoint of some slalom in Γ. Let γ_1 be the leftmost such slalom and $(i_1,-d+2)$ be the other endpoint of γ_1. To block the slalom connecting $(0,0)$ to $(i_1,0)$, we need that Γ contains a slalom connecting at least one of $(i_1,0), \cdots (i_1,-d+2)$ to (i_2,m) for some $i_2 > i_1$ and $0 \leq m \leq -d+2$. We now repeat the above argument with the assumption that at each step the point with the lowest index is the endpoint of some slalom in Γ, i.e., we choose slaloms $\gamma_2, \gamma_3, \ldots$ such that γ_l connects $(i_{l-1},-d+l)$ to $(i_l,-d+l+1)$ with $i_{l-1} < i_l$, and is the leftmost such slalom in Γ starting at $(i_{l-1},-d+l)$. Thus, we get a collection of d slaloms $\gamma_1, \cdots, \gamma_d$ in Γ, where γ_s connects $(i_{s-1},-d+s)$ to $(i_s,-d+s+1)$ for $1 \leq s \leq d-1$, arranged as shown in Figure 4.9b. Using Lemmas 4.3.1 and 4.3.2, each of the parts of the discs cut by these slaloms can be relabelled to obtain S^d_m for some $m < n$. The exact value of m for each of the parts is shown in Figure 4.12. Thus the number of collections of non-mutually intersecting n slaloms containing the above collection of d slaloms is given by the sum of the product of the number of collections of non-mutually intersecting $a_1, \cdots, a_d$ slaloms in $S^d_{KA_{i_1-2}}, S^d_{KA_{i_2-i_1-1}}, \cdots, S^d_{KA_{i_{d-1}-i_{d-2}-1}}, S^d_{KA_{n-i_{d-1}}}$ respectively. Now, since we

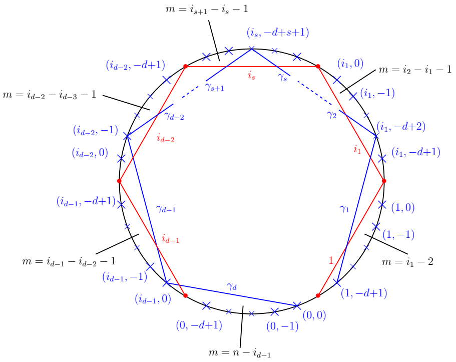
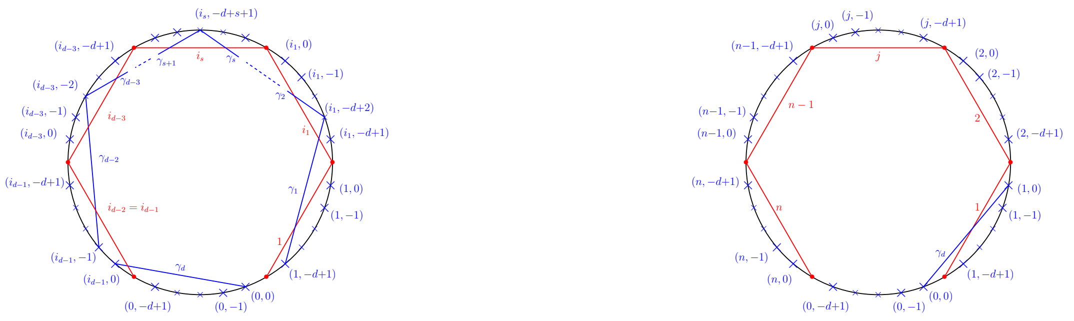


Figure 4.12: The bigger disc can be divided into d smaller pieces



(a) (b)

Figure 4.13: Illustrations for the cutting argument

know that $a_1 \leq i_1 - 2$, $a_l \leq i_l - i_{l-1} - 1$ for $2 \leq l \leq d-1$, $a_d \leq n - i_{d-1}$, and $\Sigma_{m=1}^{d} a_d = n - d$, we get that $a_1 = i_1 - 2$, $a_l = i_l - i_{l-1} - 1$ for $2 \leq l \leq d-1$, and $a_d = n - i_{d-1}$. This means that the above number is equal to the product $B^d_{i_1-2} B^d_{i_2-i_1-1} \cdots B^d_{i_{d-1}-i_{d-2}-1} B^d_{n-i_{d-1}}$. Taking into account all possible values of $i_1, \cdots i_{d-1}$, we get the sum

$$\Sigma_{i_{d-1}=2}^{n} \Sigma_{i_{d-2}=2}^{i_{d-1}-1} \cdots \Sigma_{i_1=2}^{i_2-1} B^d_{i_1-2} B^d_{i_2-i_1-1} \cdots B^d_{i_{d-1}-i_{d-2}-1} B^d_{n-i_{d-1}}.$$

Now in the penultimate step of the above argument, when one of $(i_{d-2}, 0)$ or $(i_{d-2}, -1)$ has to be connected to something in Γ, suppose $(i_{d-2}, -1)$ is not connected to anything. Then $(i_{d-2}, 0)$ has to be connected to something in Γ. The only such slalom possible is the one connecting $(i_{d-2}, 0)$ to $(0, 0)$. Since γ_d is the rightmost slalom starting at $(0, 0)$, we will get that $i_{d-2} = i_{d-1}$ (Figure 4.13a). Again, in this case, the parts of the disc cut by the $d-1$ slaloms can be relabelled by Lemmas 4.3.1 and 4.3.2 to get S^d_m for some $m < n$. And we will get that the number of collections Γ containing this set of $d-1$ slaloms is given by the product of $d-1$ terms $B^d_{i_1-2}, B^d_{i_2-i_1-1}, \cdots, B^d_{i-i_{d-3}-1}, B^d_{n-i}$. Repeating this argument at each step, we get that the total number of collections Γ in which $(0, 0)$ is connected to something is given by

$$\Sigma_{k=0}^{d-1} \binom{d-1}{k} \Sigma_{i_k=2}^{n} \Sigma_{i_{k-1}=2}^{i_k-1} \cdots \Sigma_{i_1=2}^{i_2-1} B^d_{i_1-2} B^d_{i_2-i_1-1} \cdots B^d_{i_k-i_{k-1}-1} B^d_{n-i_k},$$

where the sum over 0 elements is taken to be B^d_{n-1}, corresponding to the case in Figure 4.13b.

2. $(0, 0)$ is not the endpoint of a slalom in Γ: First suppose that $(0, -1)$ is connected to something. It can be easily seen that the above cutting and relabelling procedure works in this case as well, except that the maximum number of parts in which the disc is cut is now $d-1$. This gives that the number of collections Γ in this case is

$$\Sigma_{k=0}^{d-2} \binom{d-2}{k} \Sigma_{i_k=2}^{n} \Sigma_{i_{k-1}=2}^{i_k-1} \cdots \Sigma_{i_1=2}^{i_2-1} B^d_{i_1-2} B^d_{i_2-i_1-1} \cdots B^d_{i_k-i_{k-1}-1} B^d_{n-i_k}.$$

In the general case, let m be the smallest integer such that $(0, -m)$ is connected to something. Then the number of collections is given by

$$\Sigma_{k=0}^{d-1-m} \binom{d-1-m}{k} \Sigma_{i_k=2}^{n} \Sigma_{i_{k-1}=2}^{i_k-1} \cdots \Sigma_{i_1=2}^{i_2-1} B^d_{i_1-2} B^d_{i_2-i_1-1} \cdots B^d_{i_k-i_{k-1}-1} B^d_{n-i_k}.$$

Thus, we get that the total number of d-term silting objects in KA_n is given by the recursive formula

$$B^d_n = \Sigma_{k=0}^{d-1} \Sigma_{m=k}^{d-1} \binom{m}{k} \Sigma_{i_k=2}^{n} \Sigma_{i_{k-1}=2}^{i_k-1} \cdots \Sigma_{i_1=2}^{i_2-1} B^d_{i_1-2} B^d_{i_2-i_1-1} \cdots B^d_{i_k-i_{k-1}-1} B^d_{n-i_k},$$

where $B^d_0 = 1$, and $B^d_1 = d$.

Since the above recursive formula also counts the number of complete ordered d-ary trees with $n+1$ internal nodes, which are known to be counted by the Pfaff-Fuss-Catalan numbers [52], we recover the result that the number of d-term silting objects in KA_n is given by the Pfaff-Fuss-Catalan number C^d_{n+1}.

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