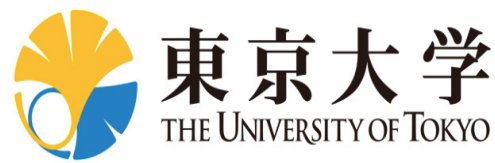
Lines, Curves, Surfaces, and Functional Equations

Luc PIRIO

FJ-LMI Day 2026

June 2, 2026



An informal and visual talk

- Families of curves give rise to geometric objects called **webs**.
- Somewhat unexpectedly, these webs are closely related to an interesting class of functional equations.

Surfaces carrying families of lines

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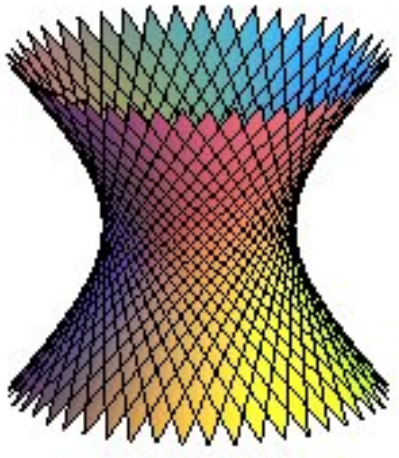
- **[Wren 1669] Generatio corporis cylindroidis hyperbolici ...**

the one-sheeted hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

carries two rulings by lines



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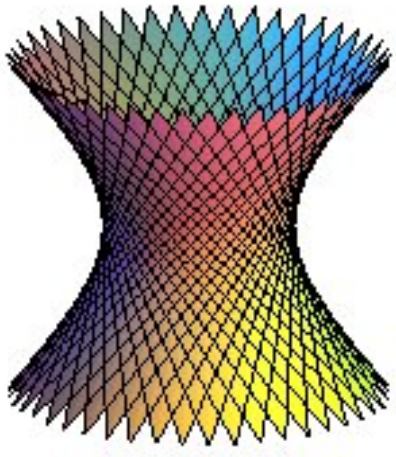

Kobe Port Tower

Canton Tower

Corp° Street Bridge

The quadric surface

- Let $\mathcal{Q} \subset \mathbb{P}^3$ be a (smooth complex) **quadric surface** :

$$\mathcal{Q} = \left\{ [x : y : z : t] \in \mathbb{P}^3 \mid \mathbf{q}(x, y, z, t) = 0 \right\}$$

where $\mathbf{q}$ is a homogeneous polynomial of degree 2

Theorem. **1.** $\mathcal{Q}$ is projectively isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^3$ by $|\mathcal{O}(1,1)|$

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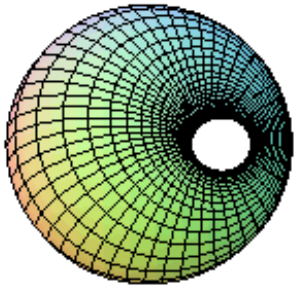
Theorem. Let $\boldsymbol{S}$ be a smooth analytic surface in $\mathbb{P}^3$

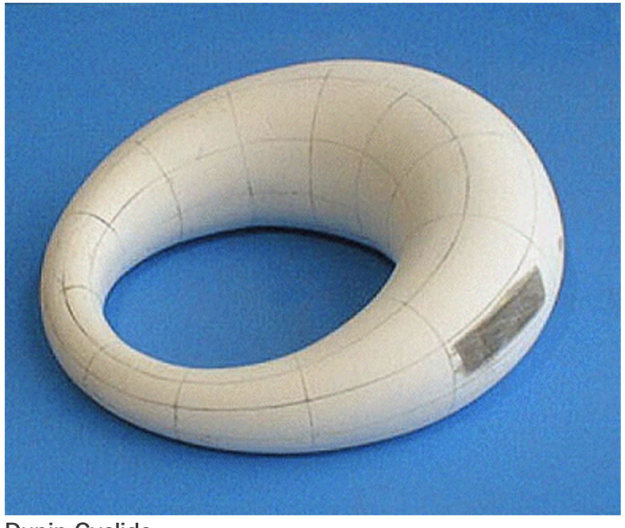
$\boldsymbol{S}$ carries two rulings by (segments of) lines $\Longrightarrow$ $\exists\ \mathcal{Q}$ quadric surface such that $\boldsymbol{S} \subset \mathcal{Q}$

Surfaces carrying families of 'simple' curves

- **[Dupin 1802]** *Discovery of* ***cyclides*** *i.e. surfaces* $\boldsymbol{S} \subset \mathbb{E}^3$ *whose lines of curvature are circles*



Cyclides



Dupin Cyclide

Properties of Cyclides

A cyclide of Dupin is obtained by an inversion of the standard torus. It is a quartic surface and has the property that the lines of curvature on this cyclide are circles or straight lines intersecting orthogonally. These lines are inscribed on the models.

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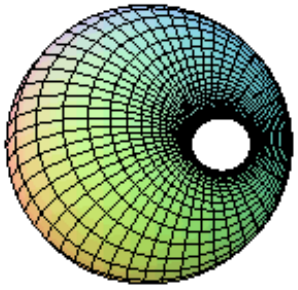

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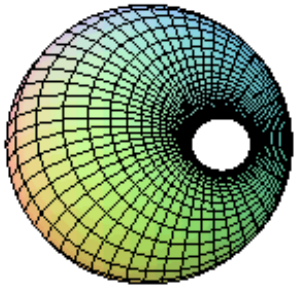

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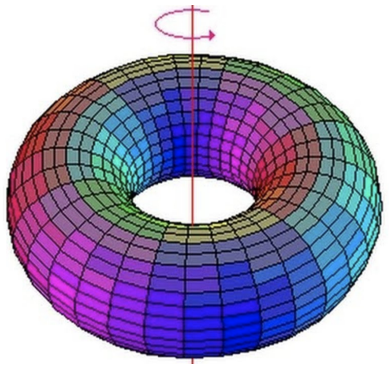

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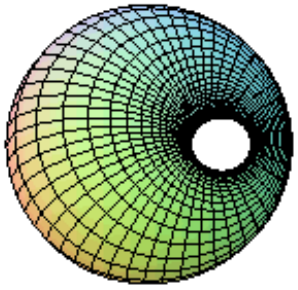

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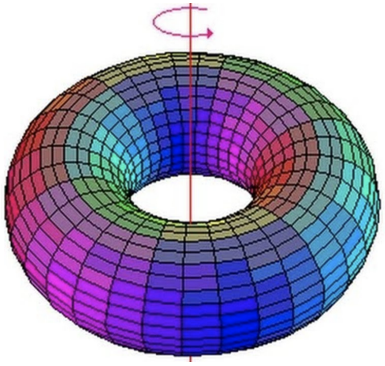
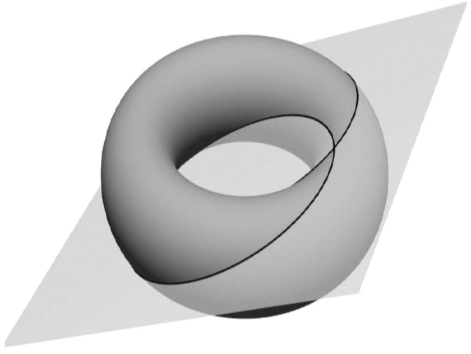
- **[Villarceau 1848]**

  A torus $\mathbf{T} \subset \mathbb{E}^3$ *carries four families of circles contained in it*





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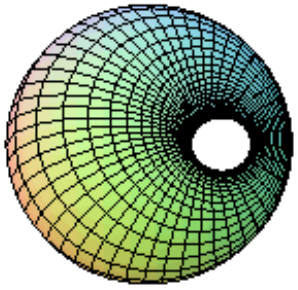

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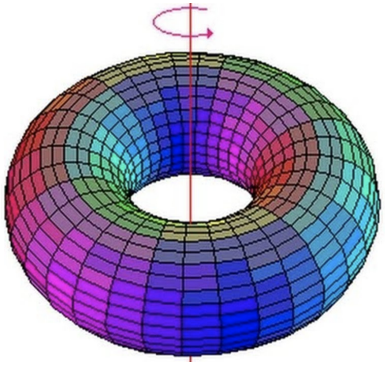

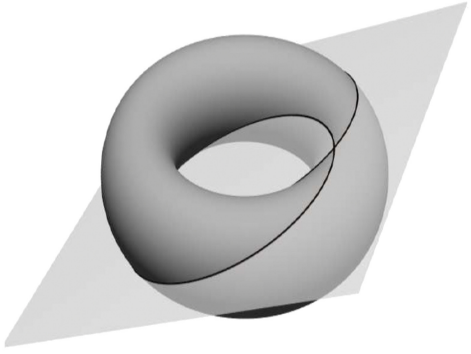

- **[Mannheim 1860]** *Through a generic point of a cyclide* $\mathcal{S} = \iota(\mathbf{T})$ *pass at least four circles contained in* $\mathcal{S}$

Webs

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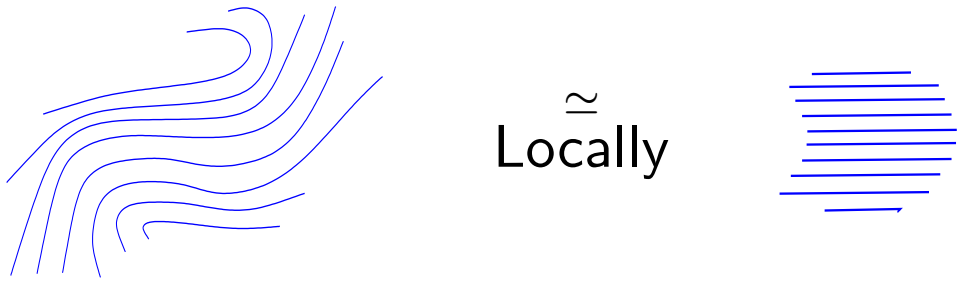

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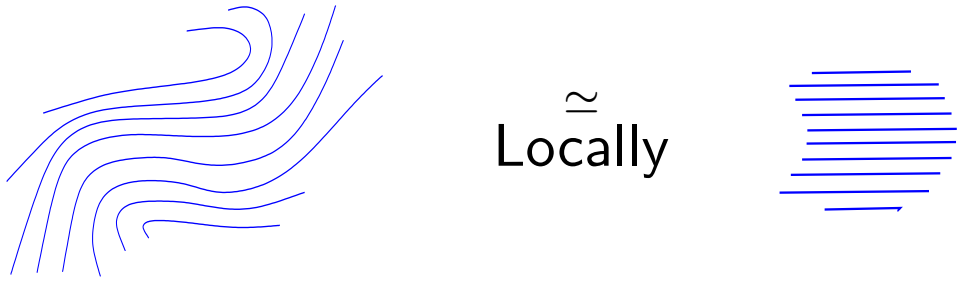

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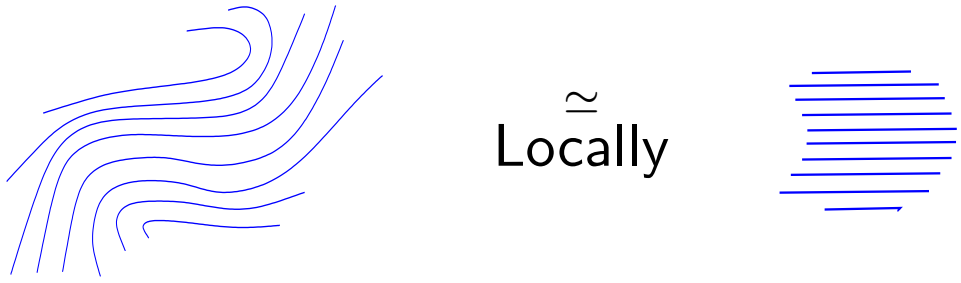

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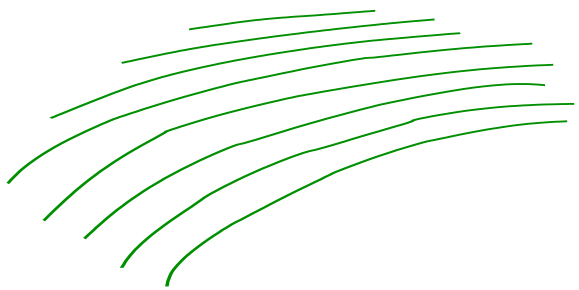

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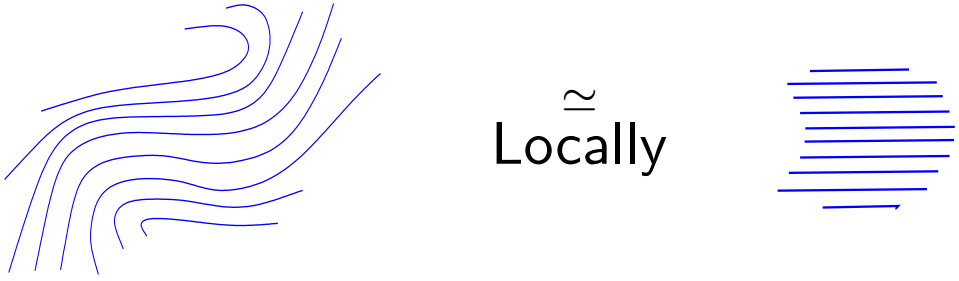

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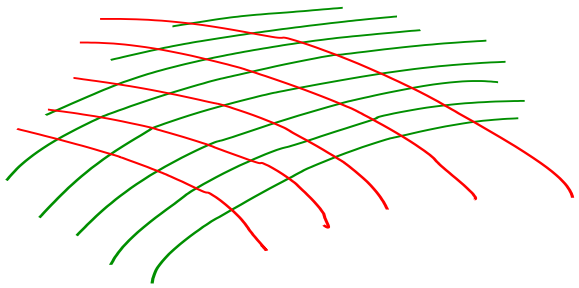

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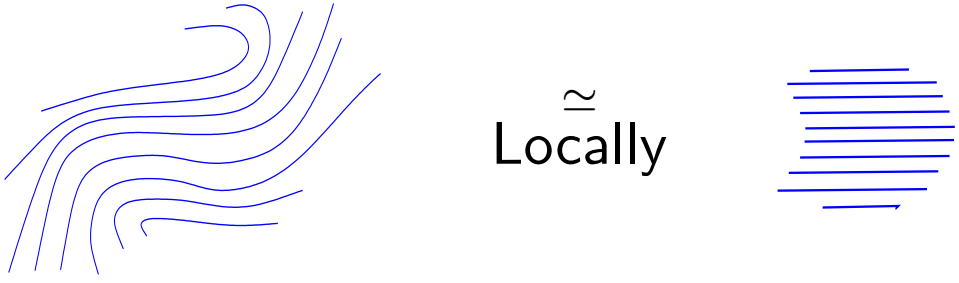

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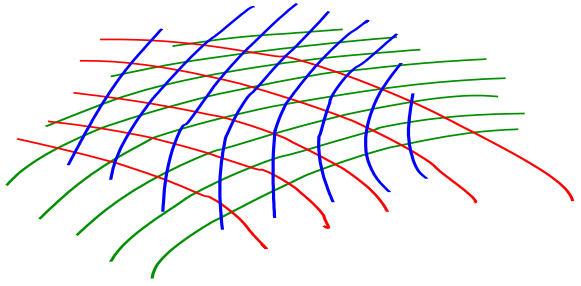

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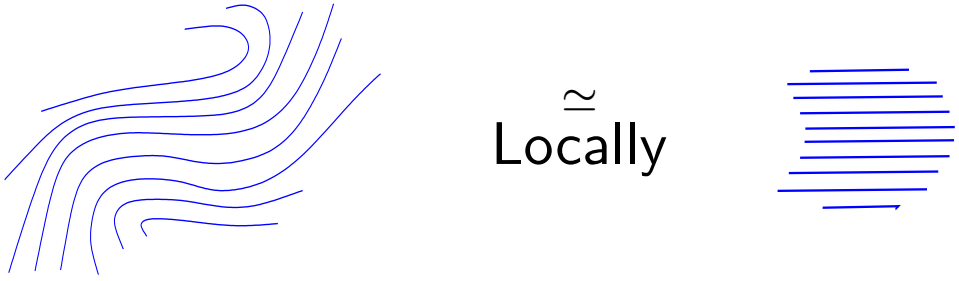

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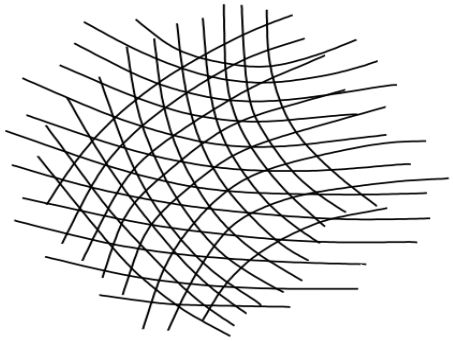

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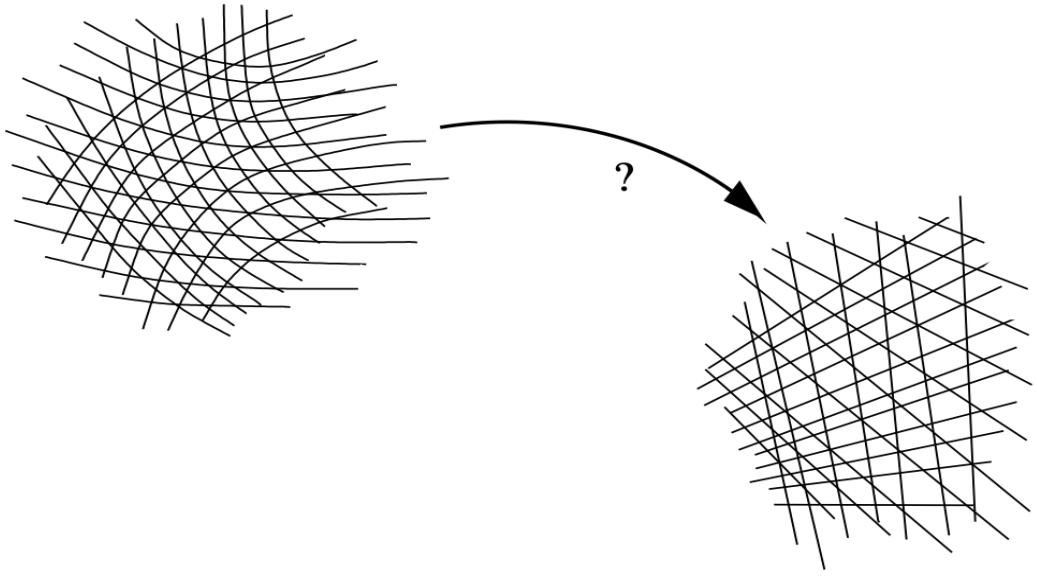

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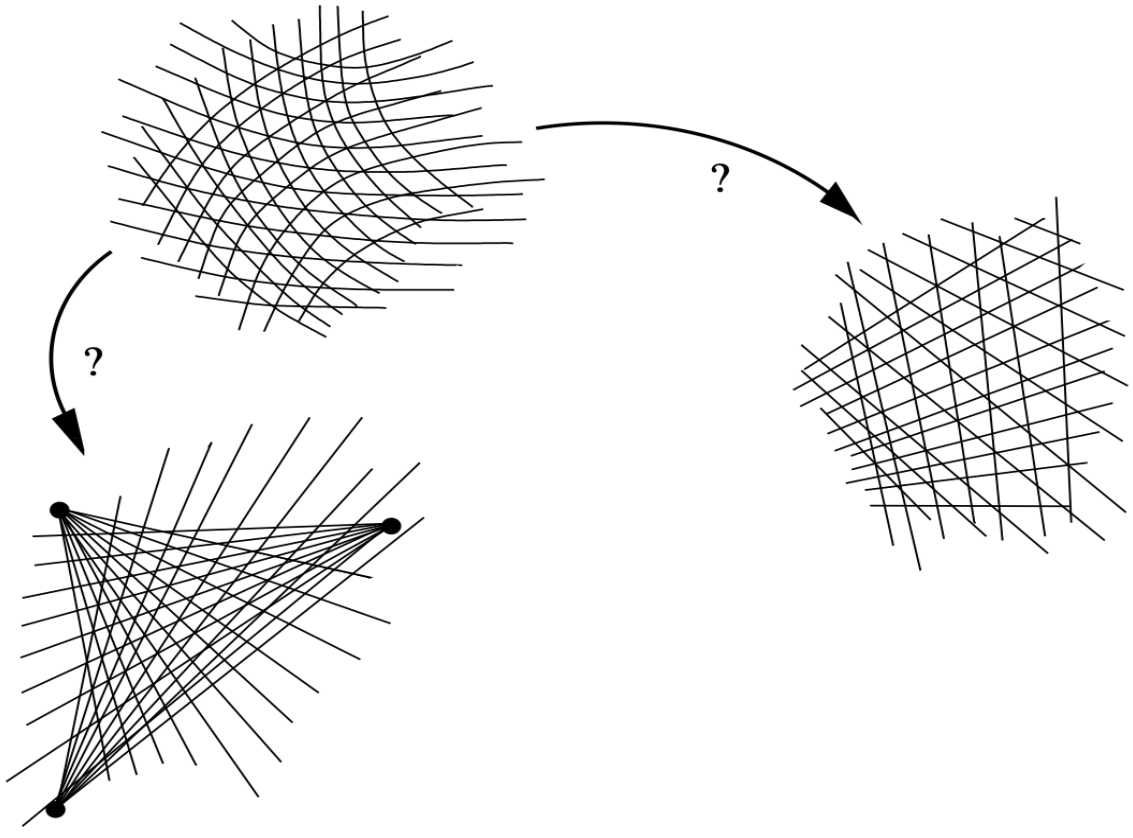

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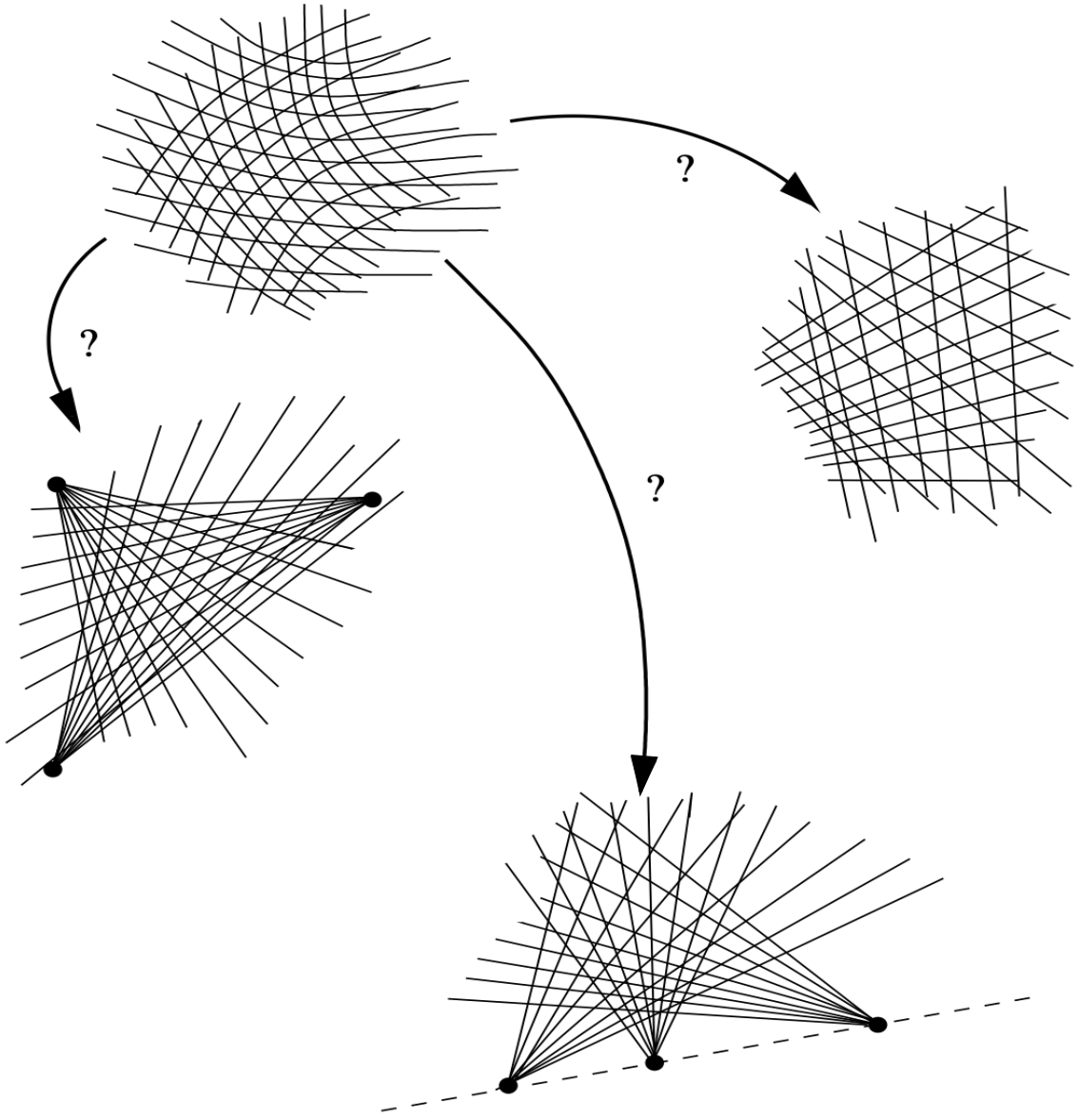
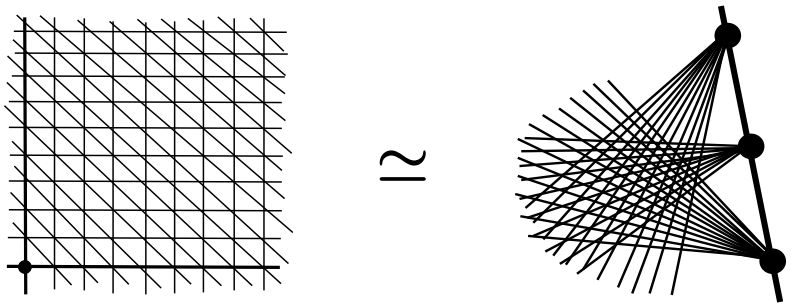


Examples of webs : Lie groups

- **Planar parallel 3-web :** $\mathcal{W}(x, y, x + y)$



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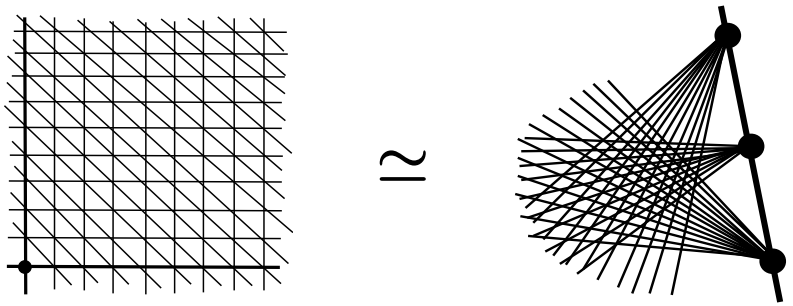

- A **Lie group** G comes with three natural maps :
  - the **projections onto the factors** $\pi_i : G \times G \to G$
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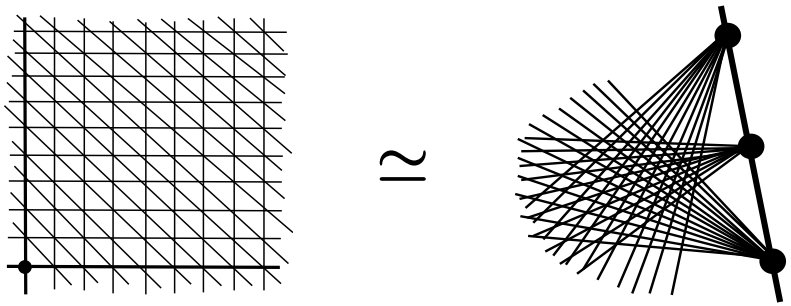

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$$\rightsquigarrow \mathcal{W}_G = \mathcal{W}(\pi_1\,,\, \pi_2\,,\, \mu\,) \text{ is a 3-web on } G^2$$

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- Projective duality gives rise to a classical correspondence :

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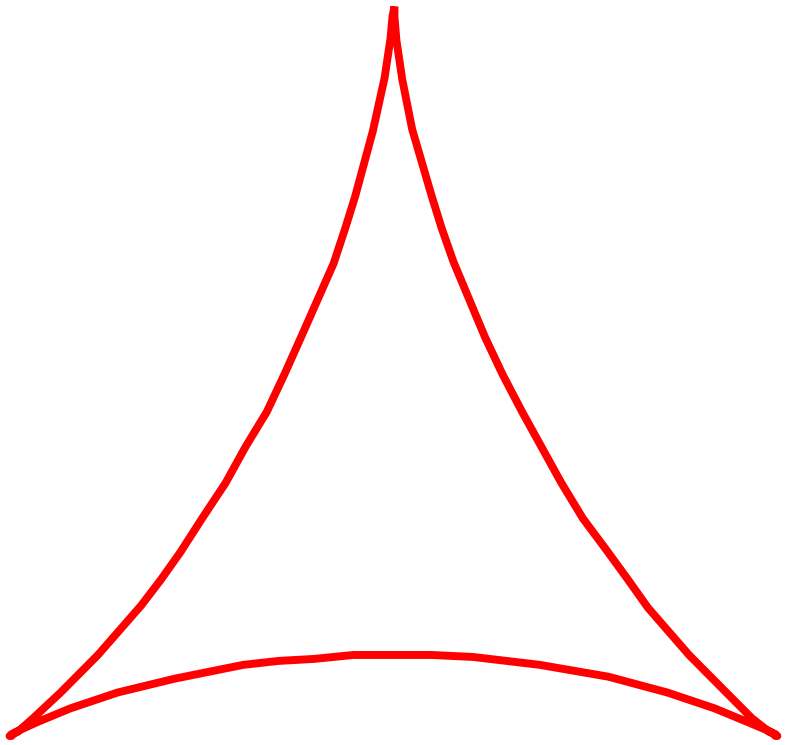

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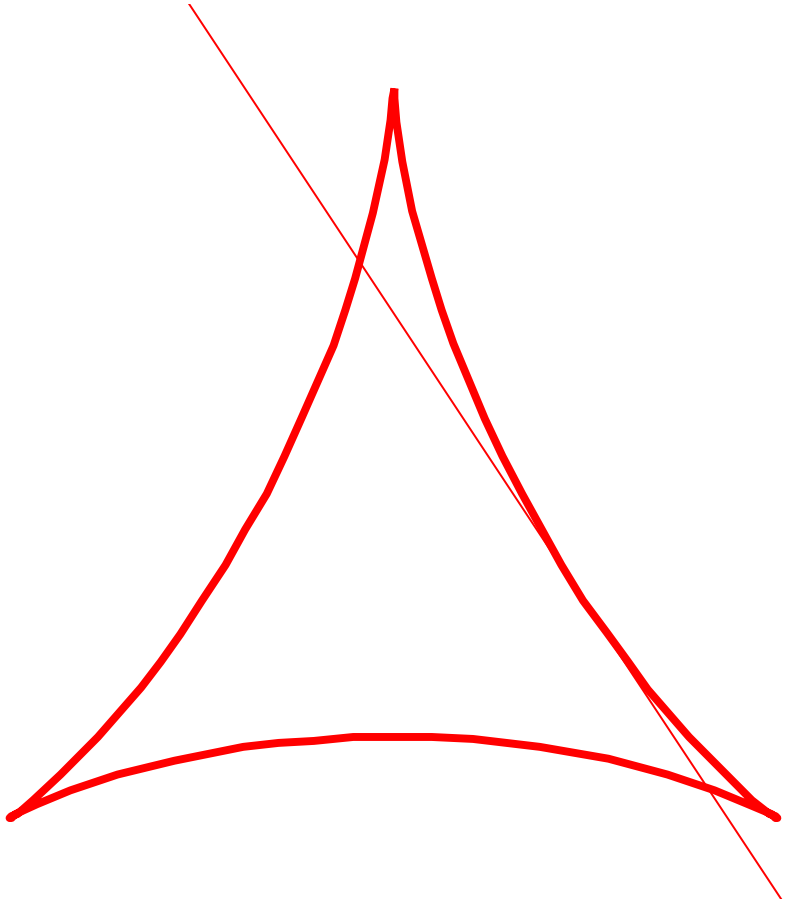

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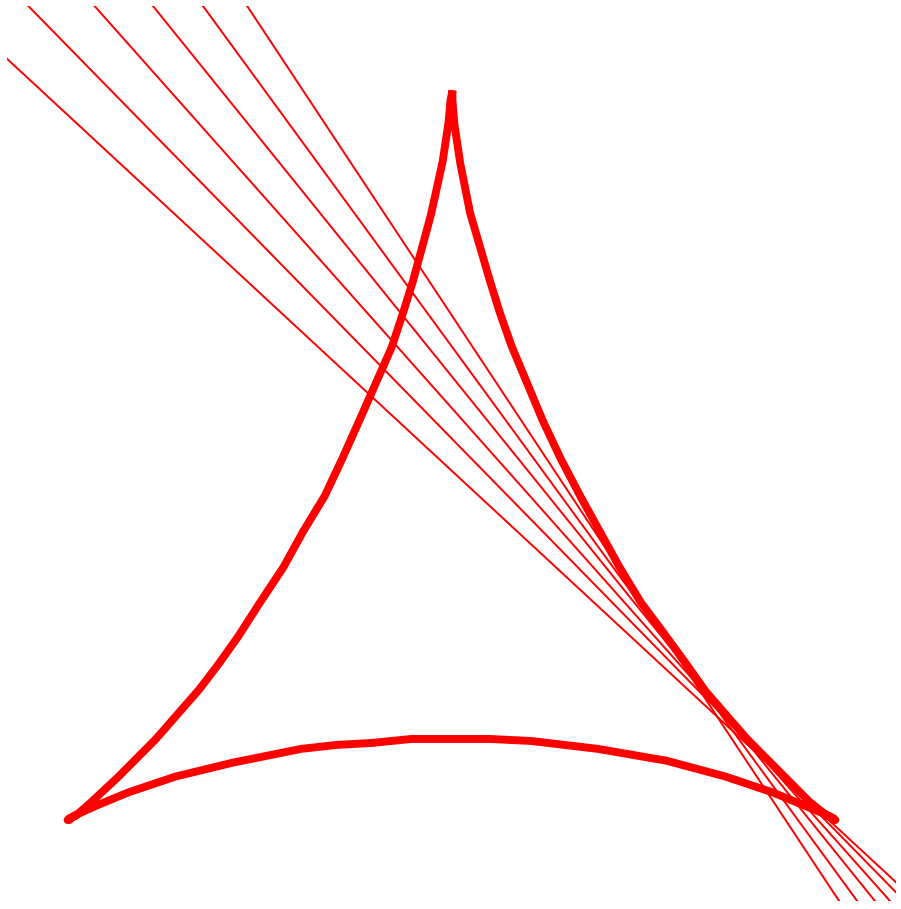
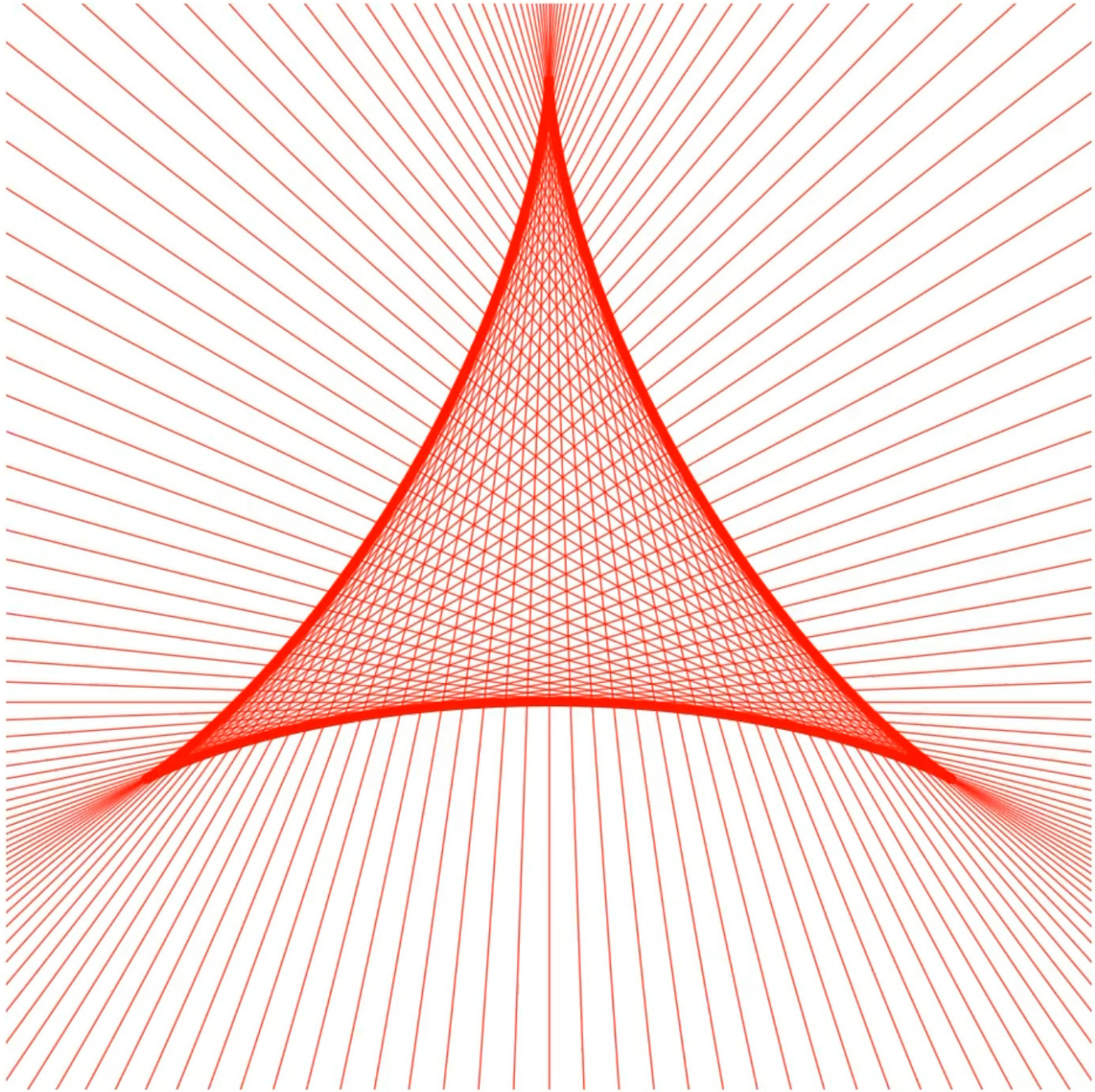
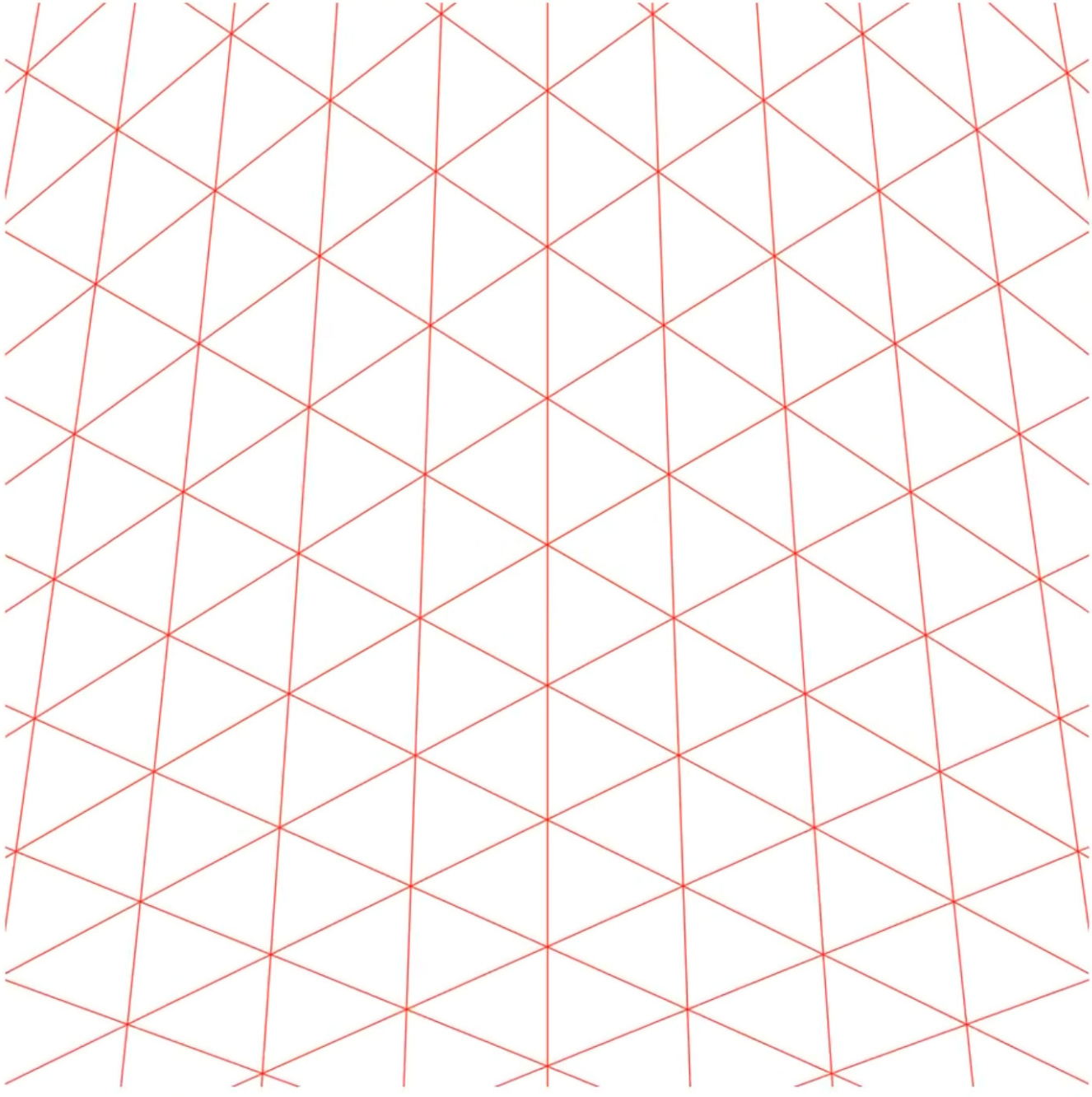
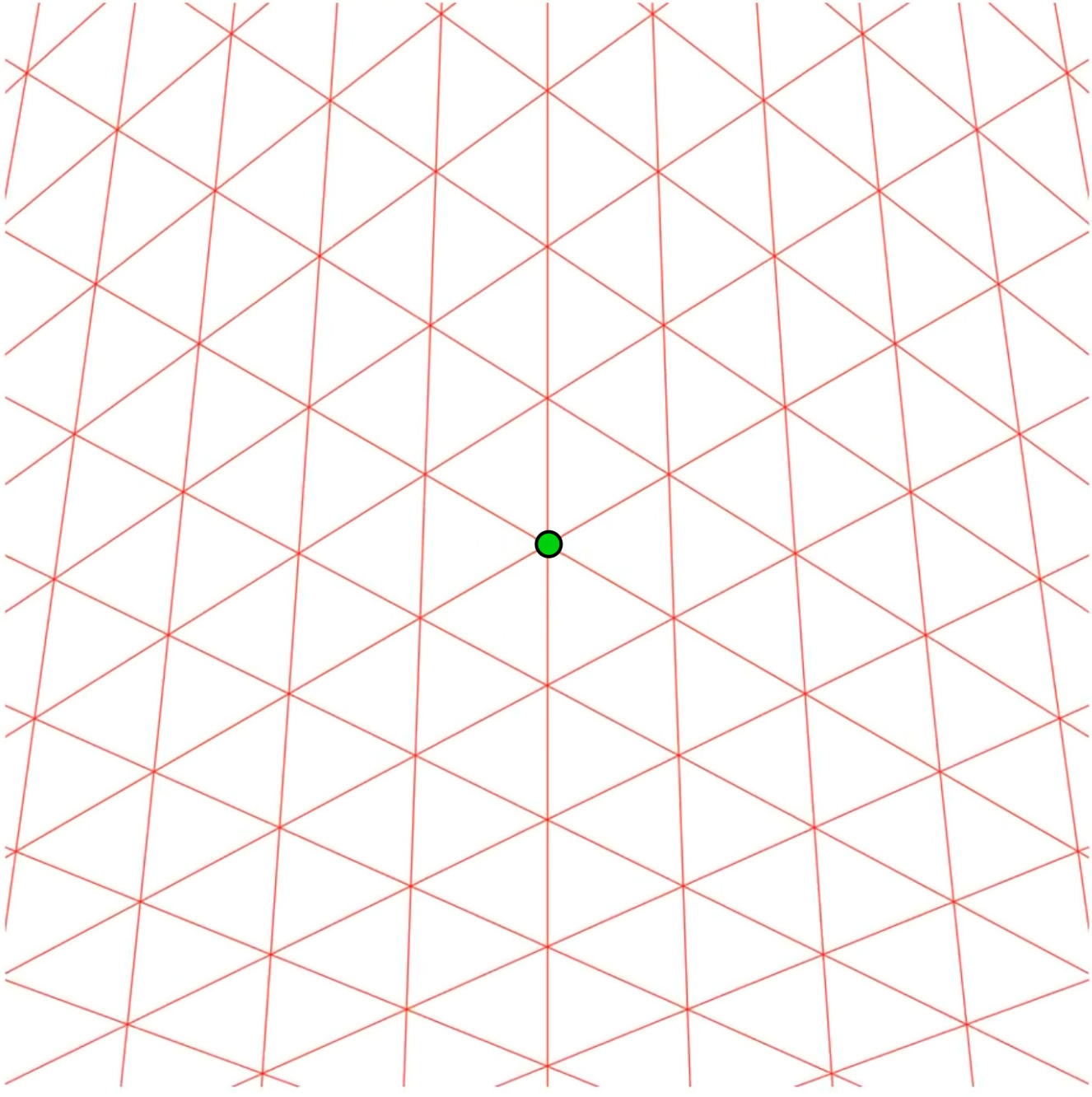
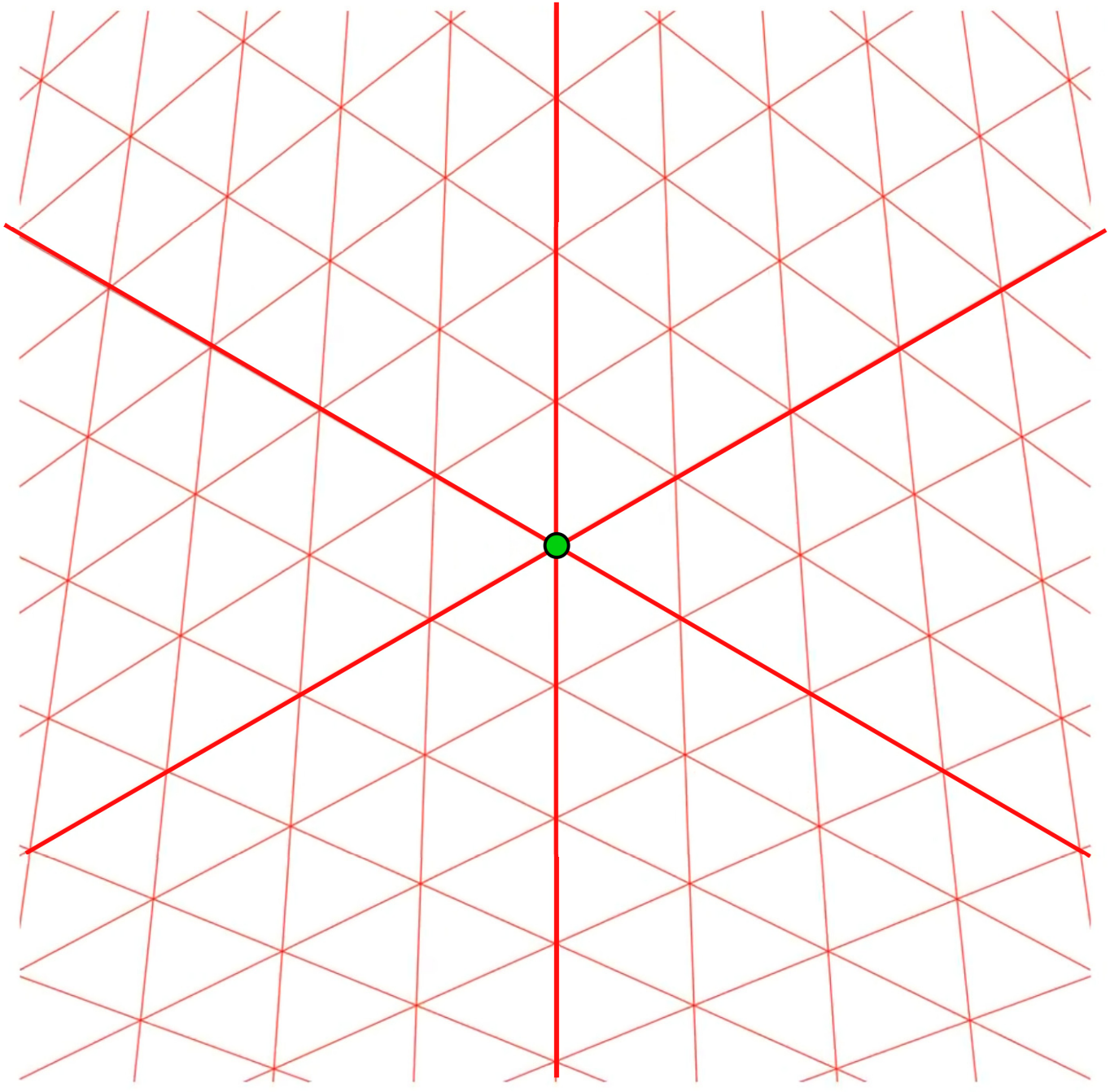


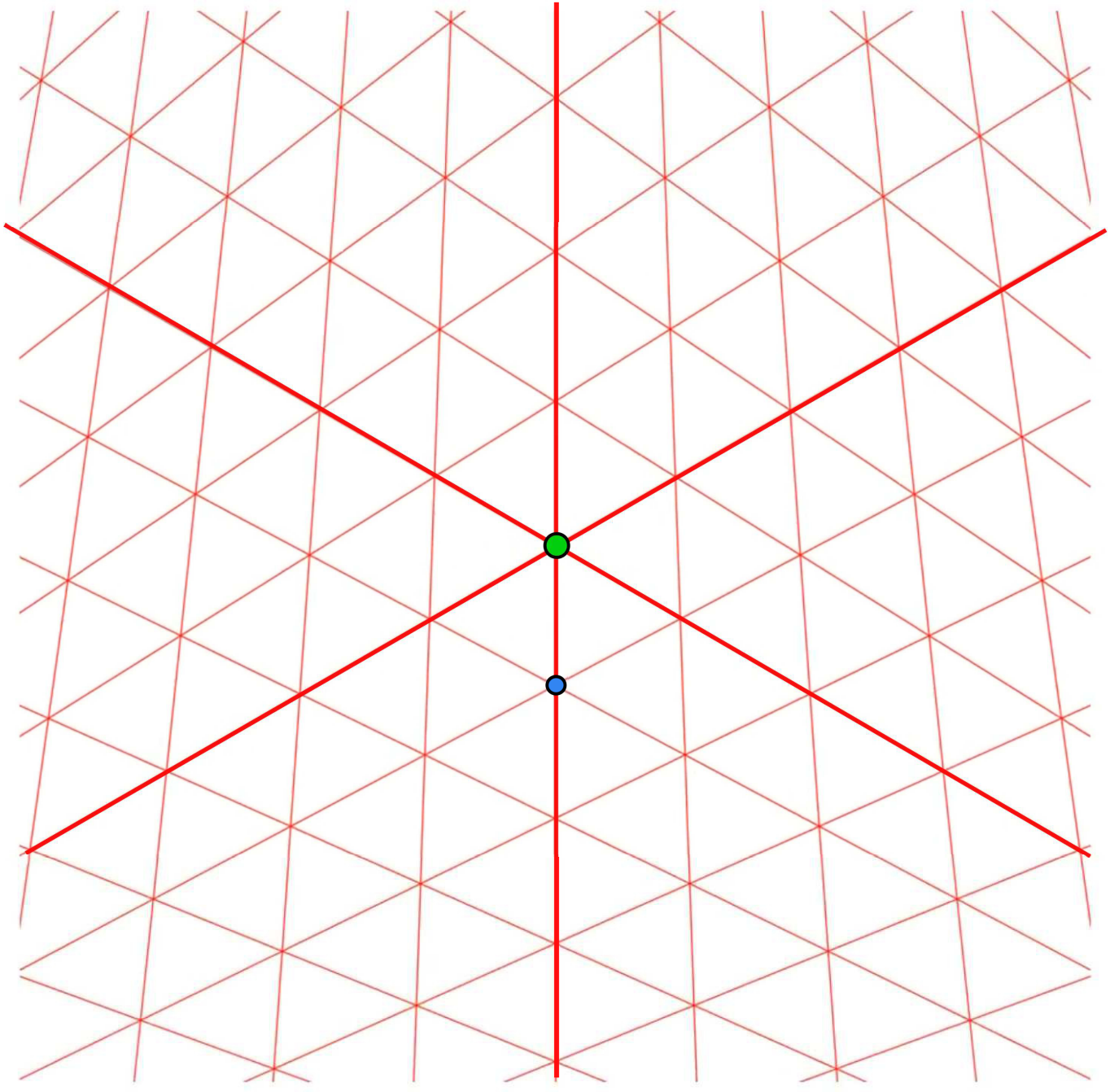
A planar algebraic 3-web

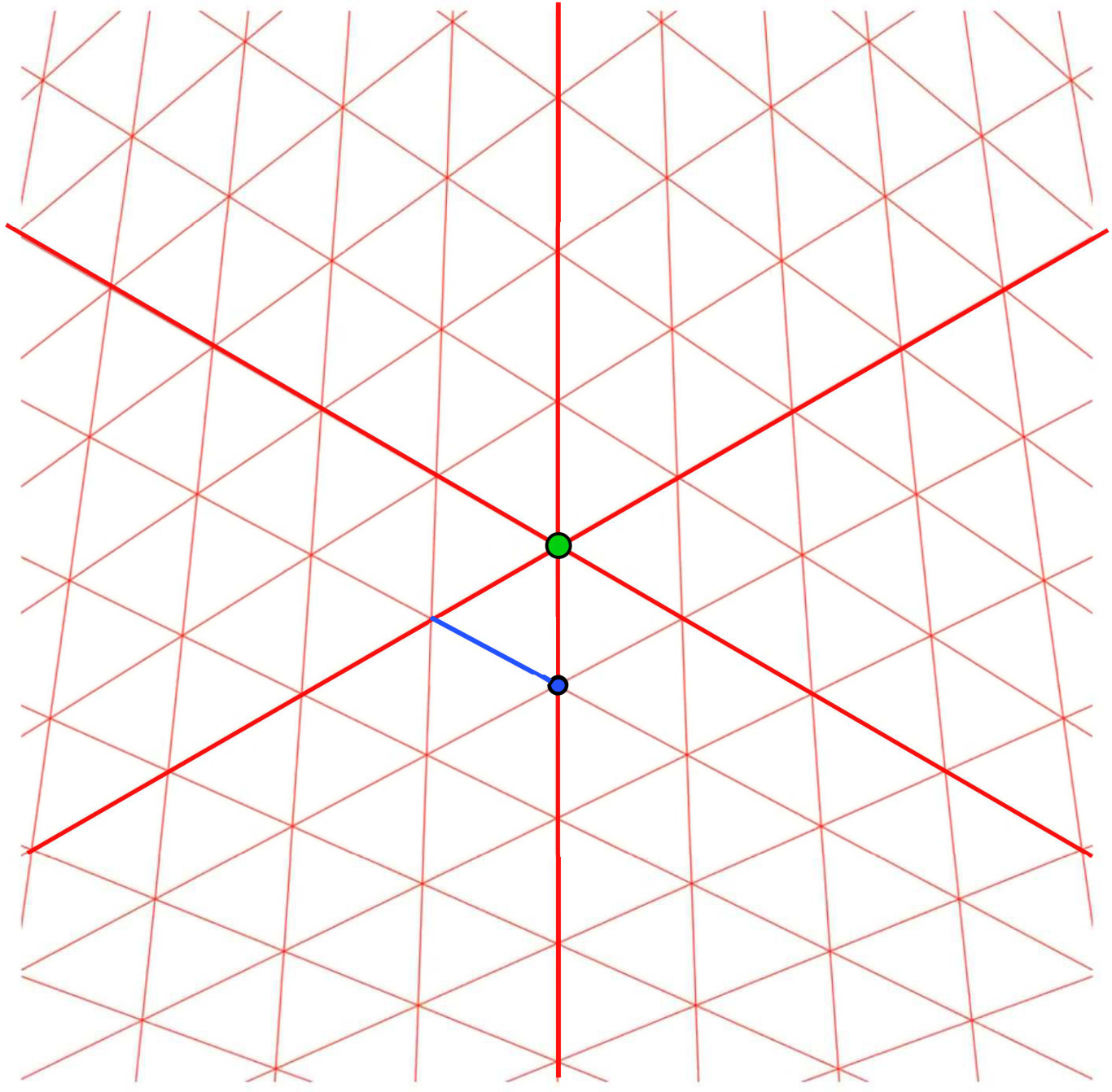


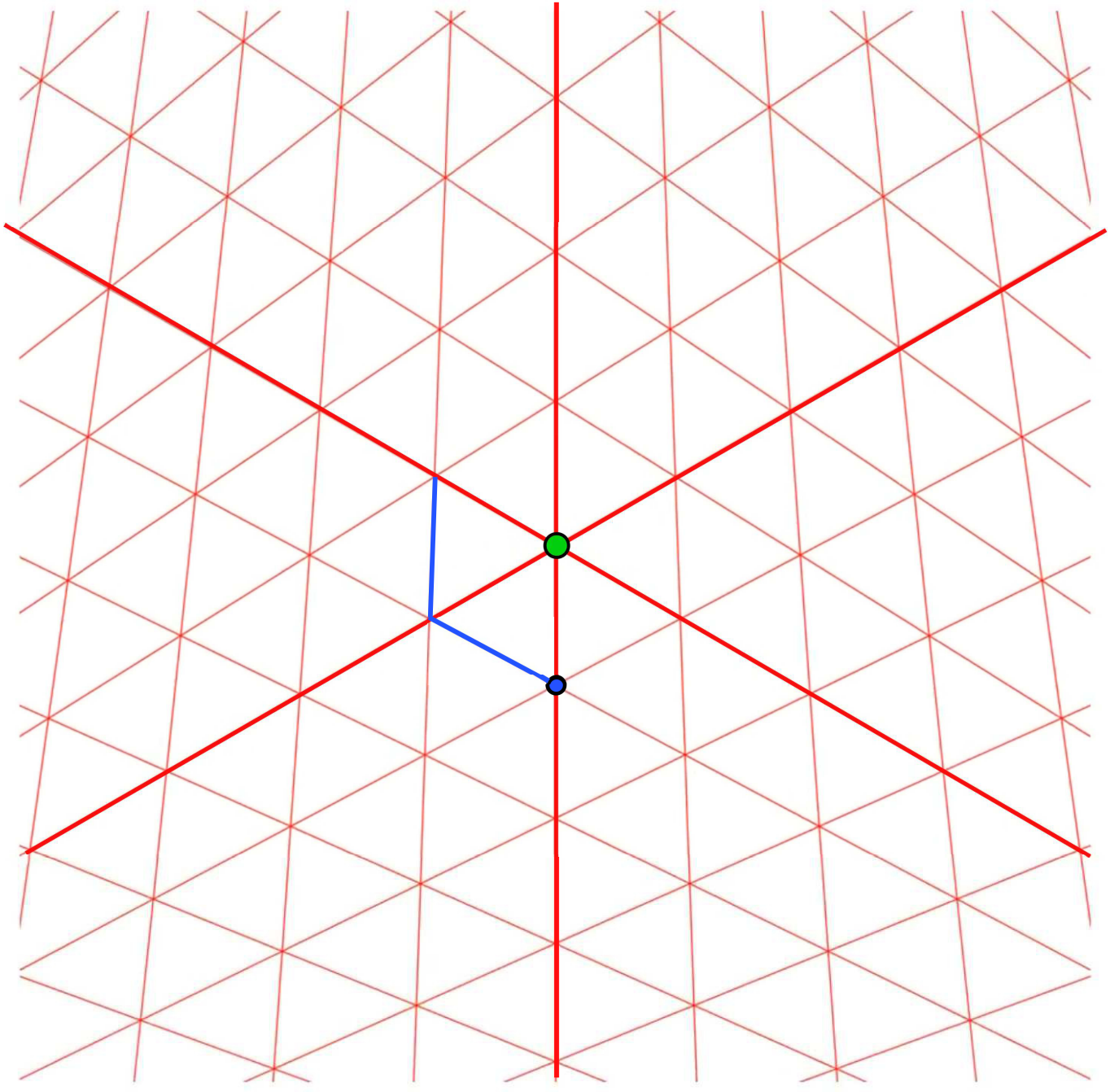


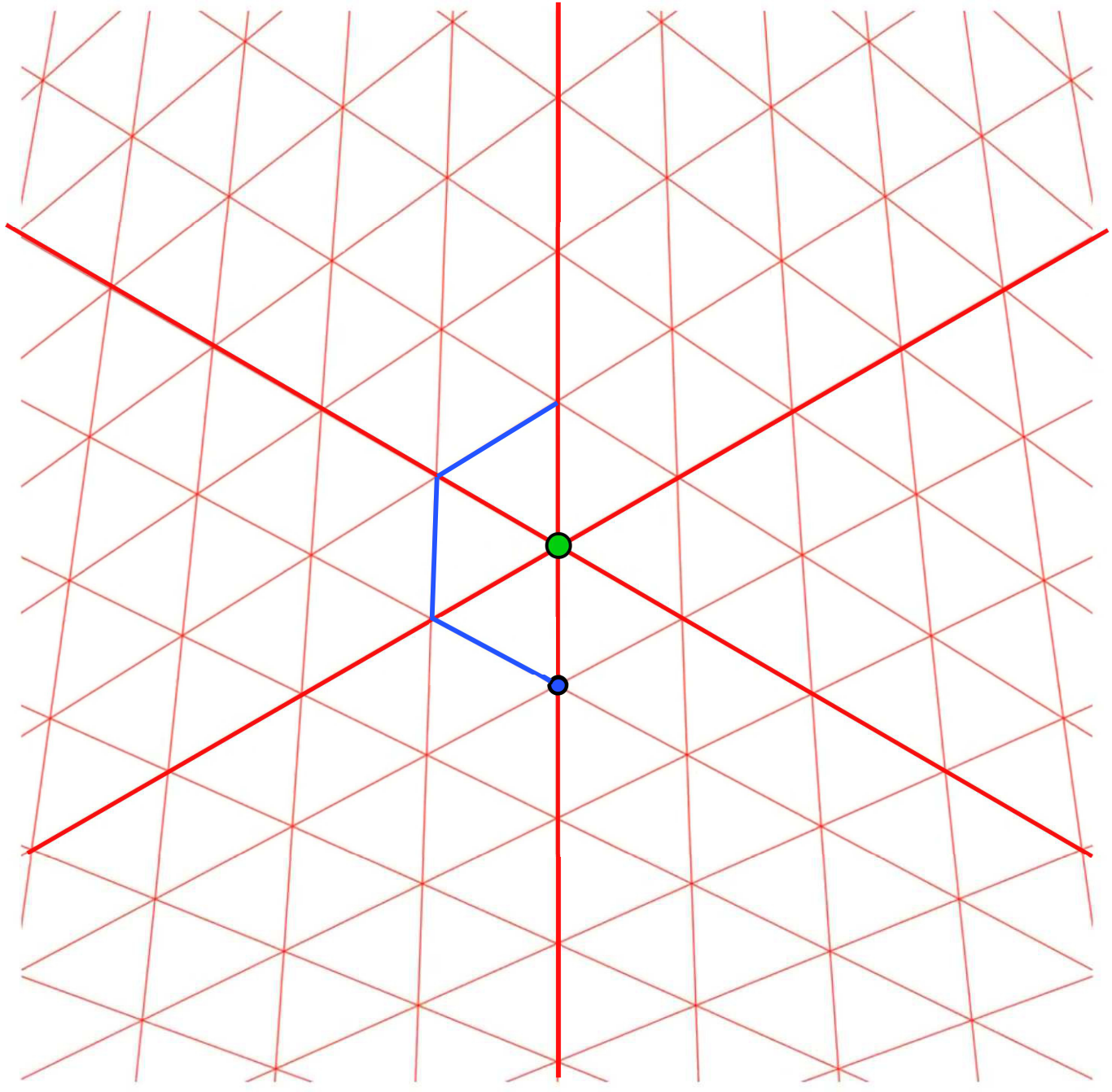


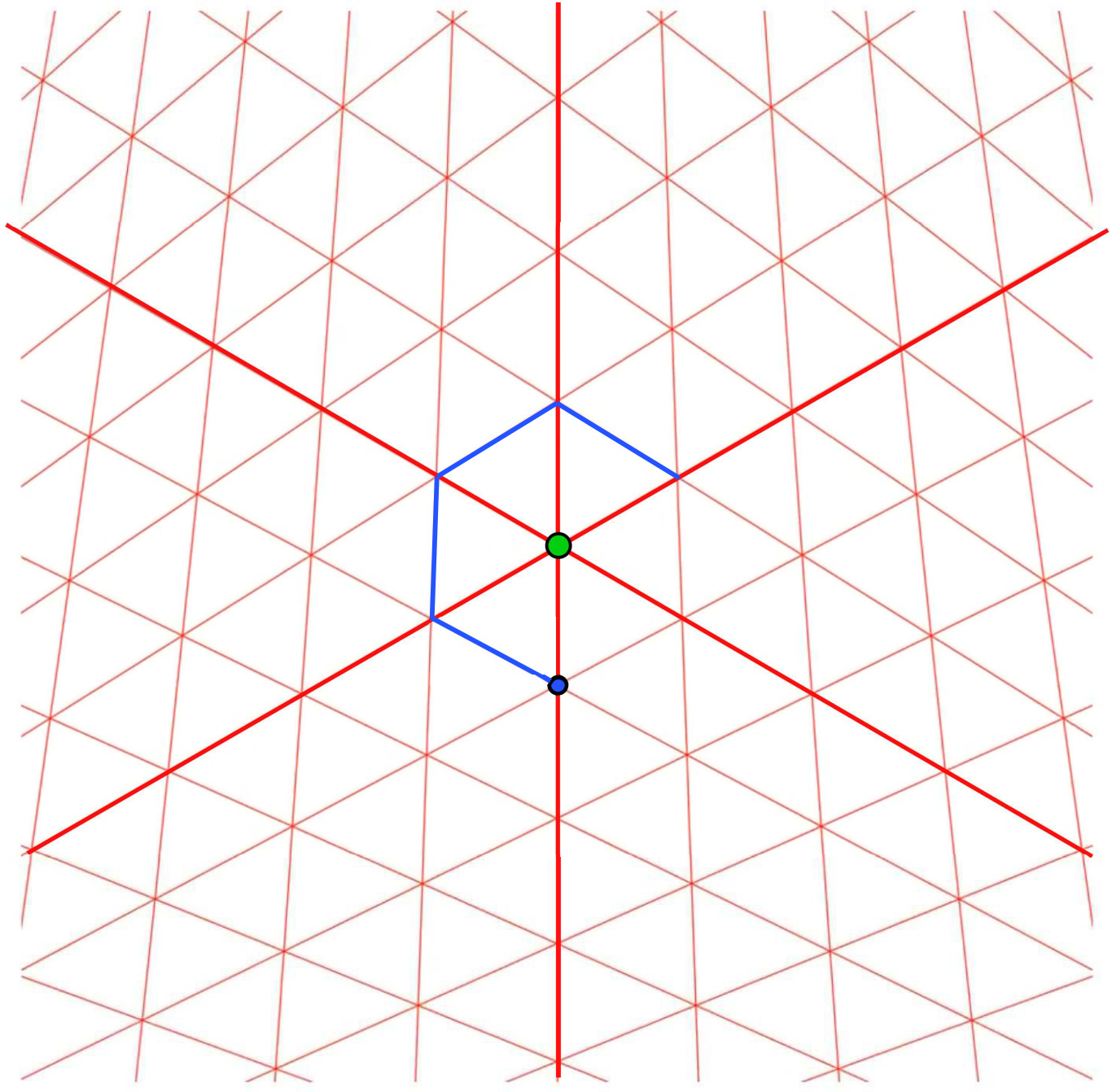


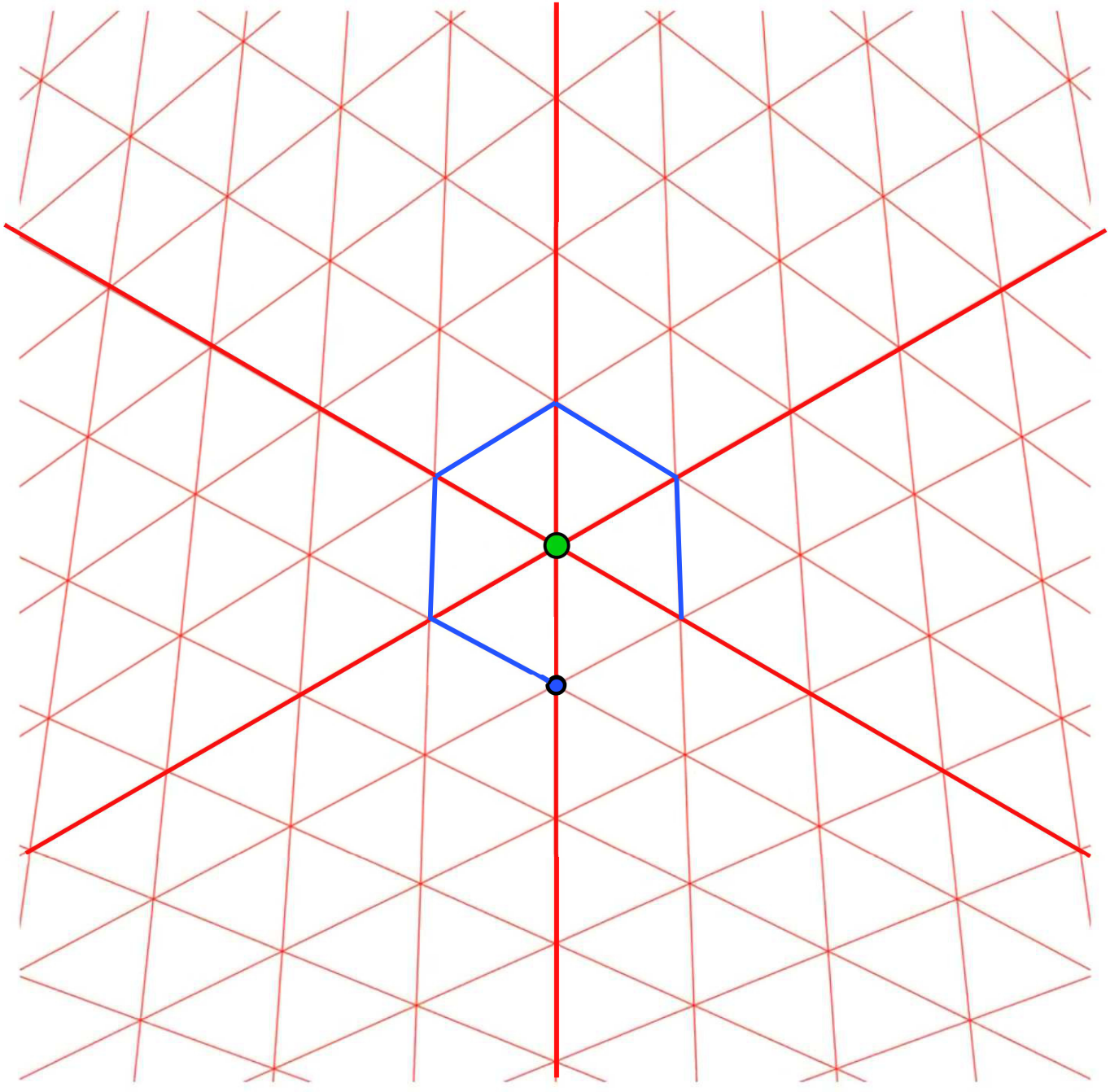


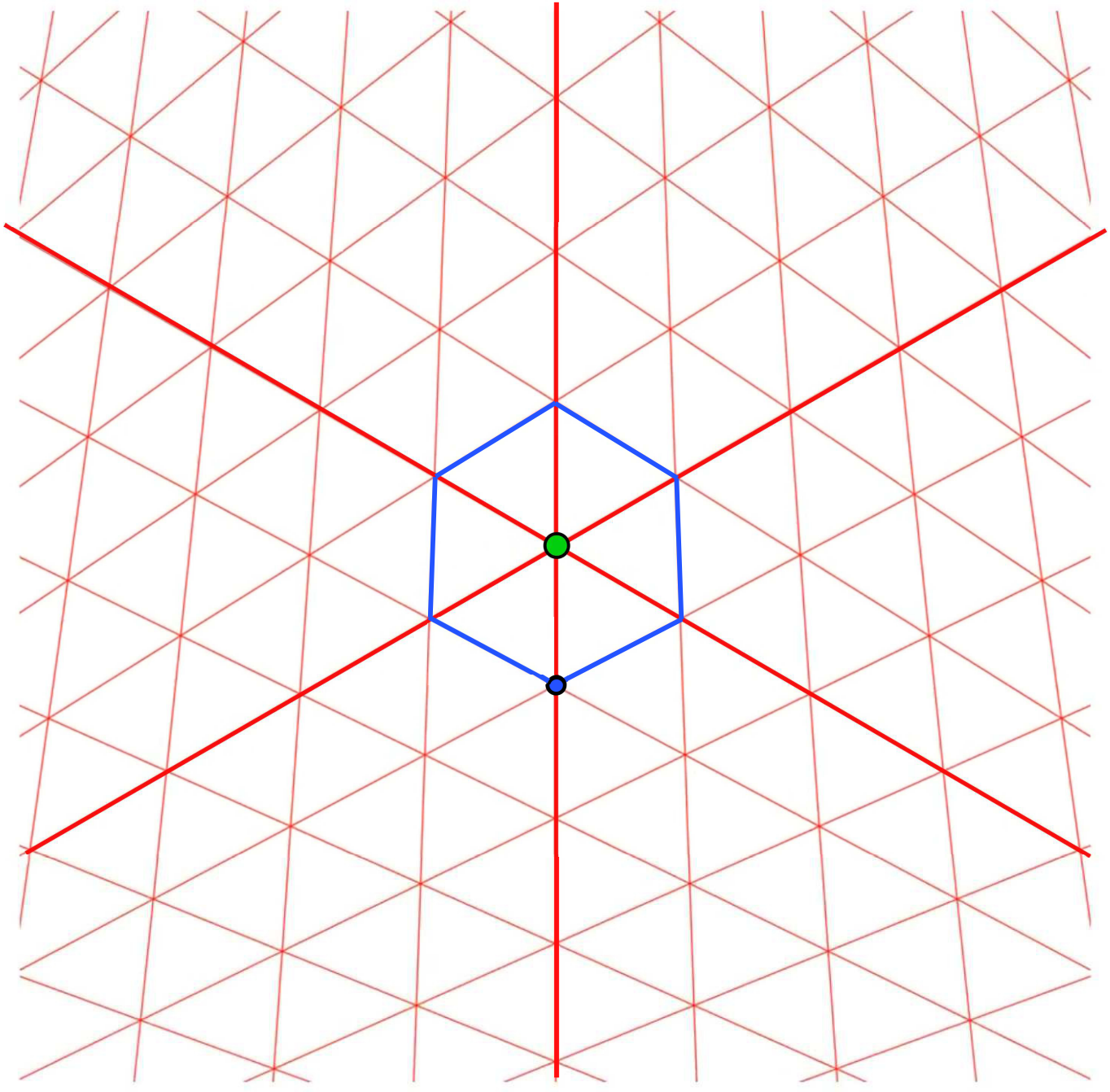


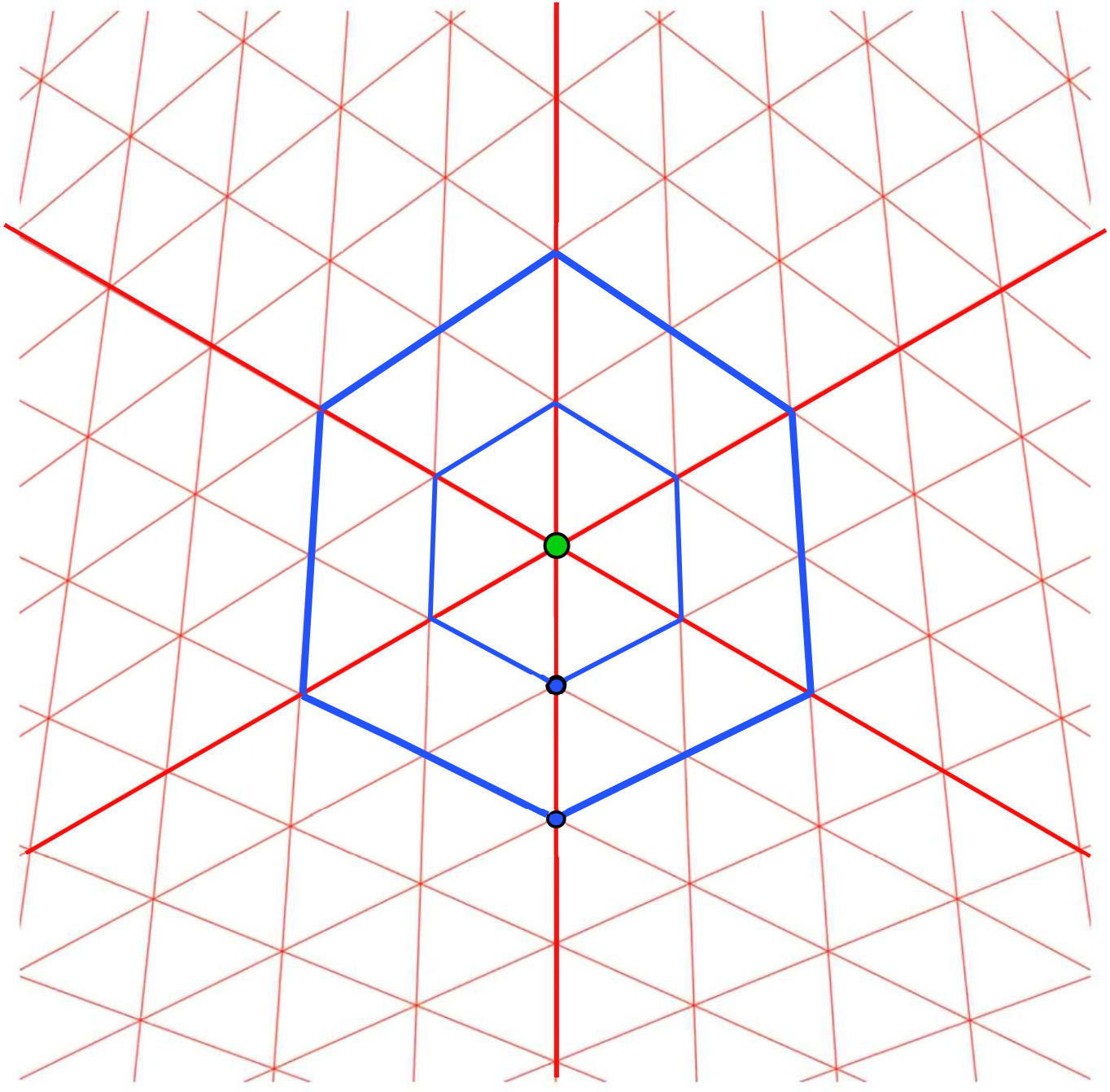


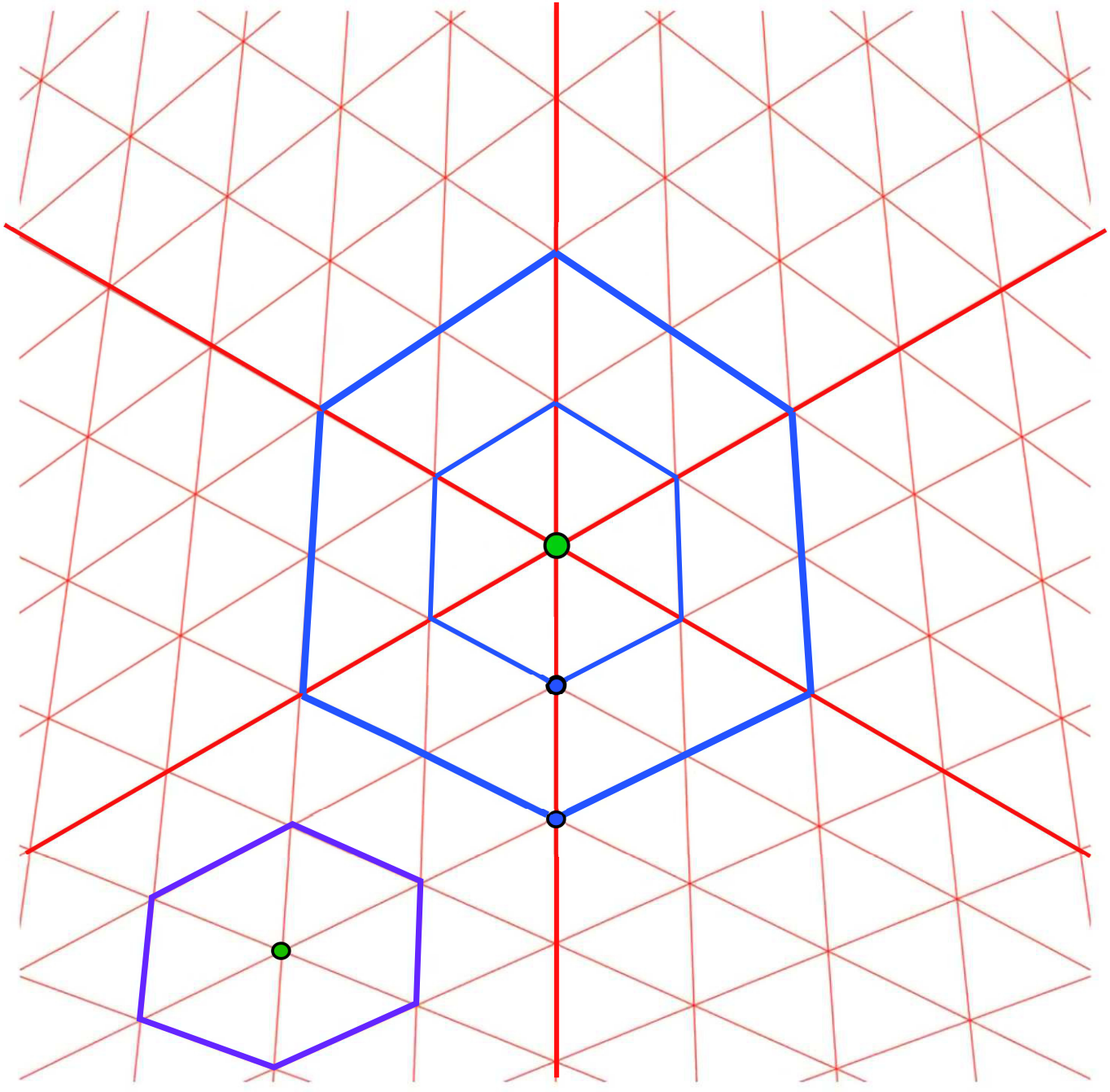


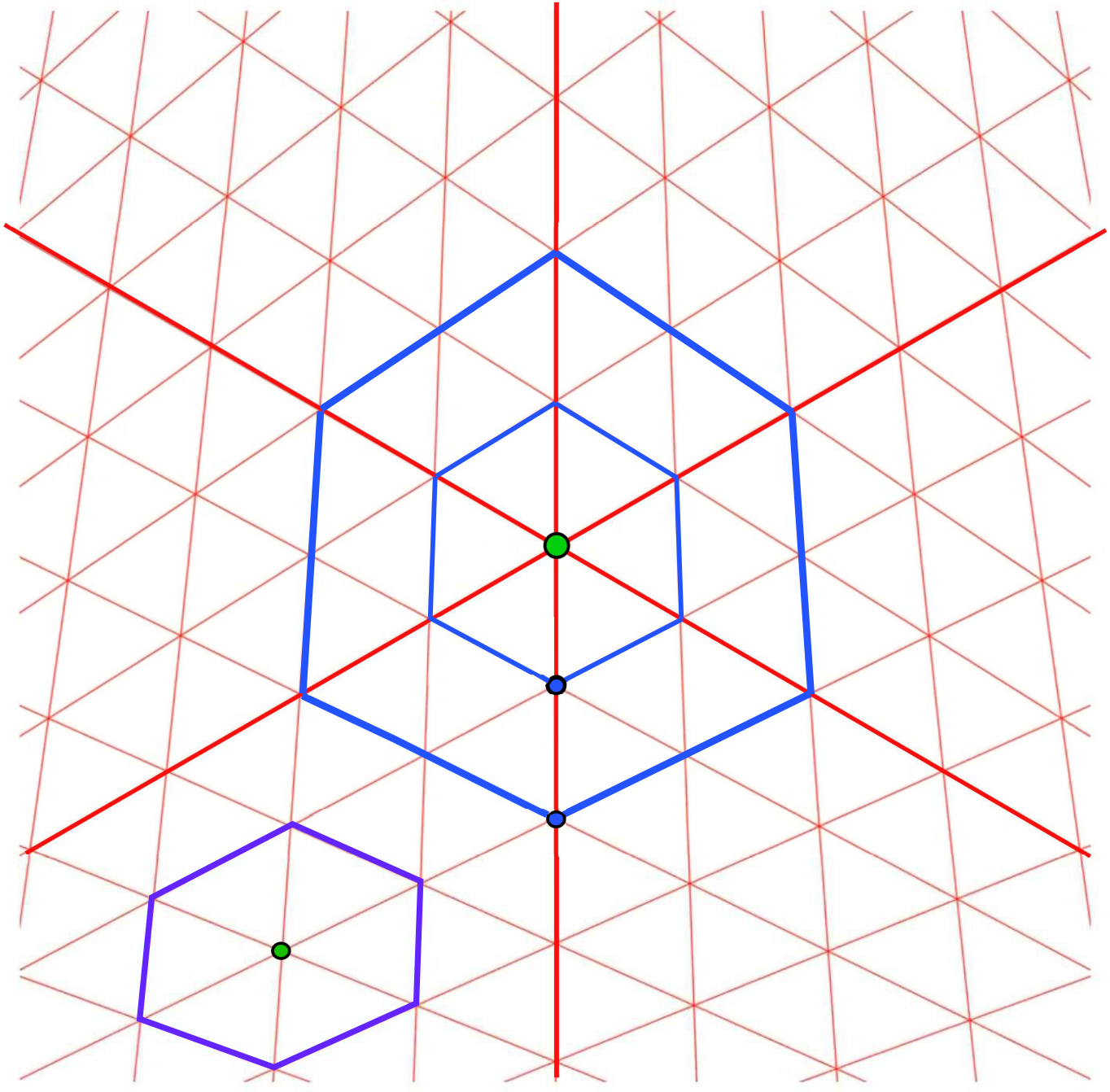


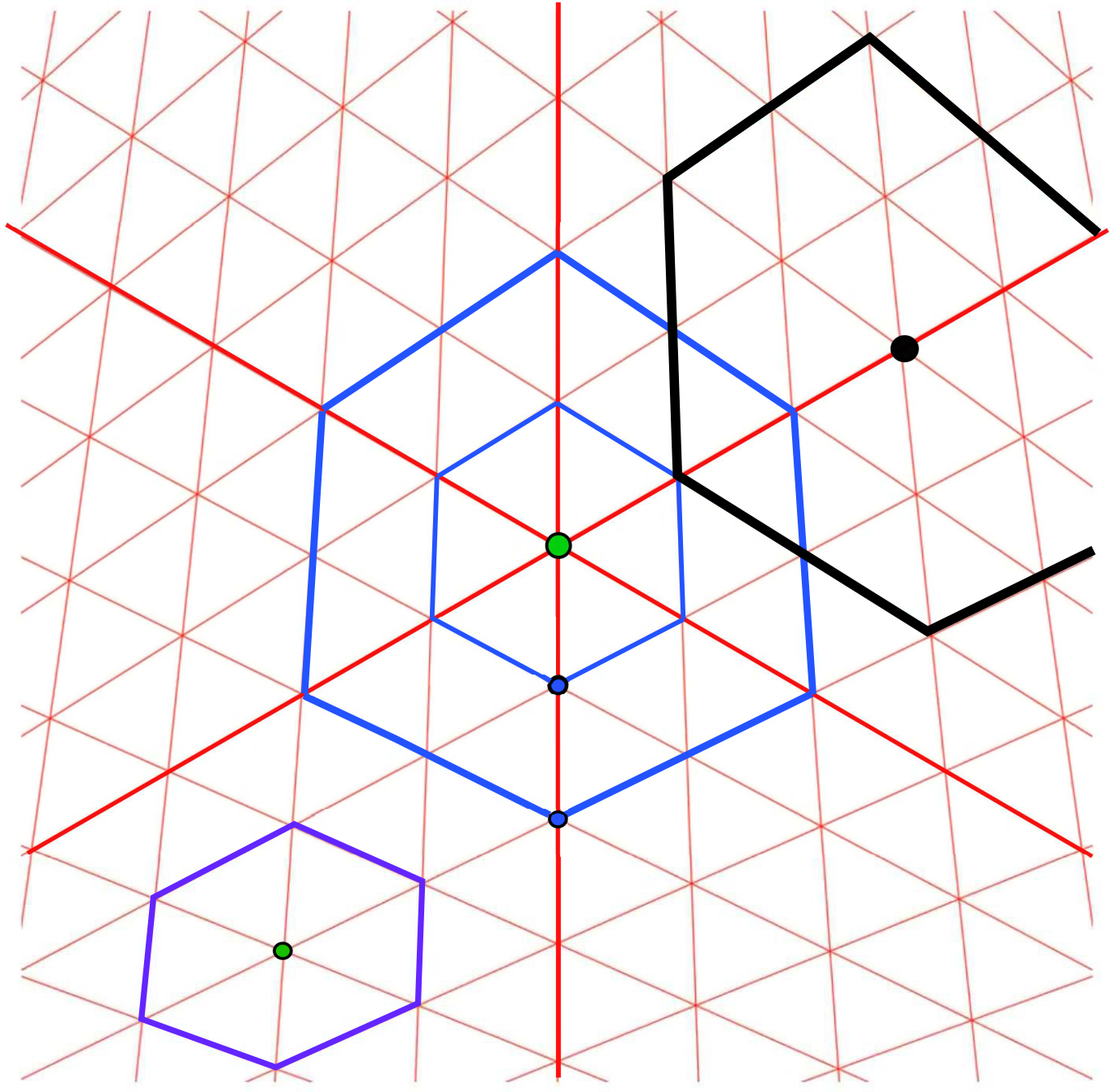




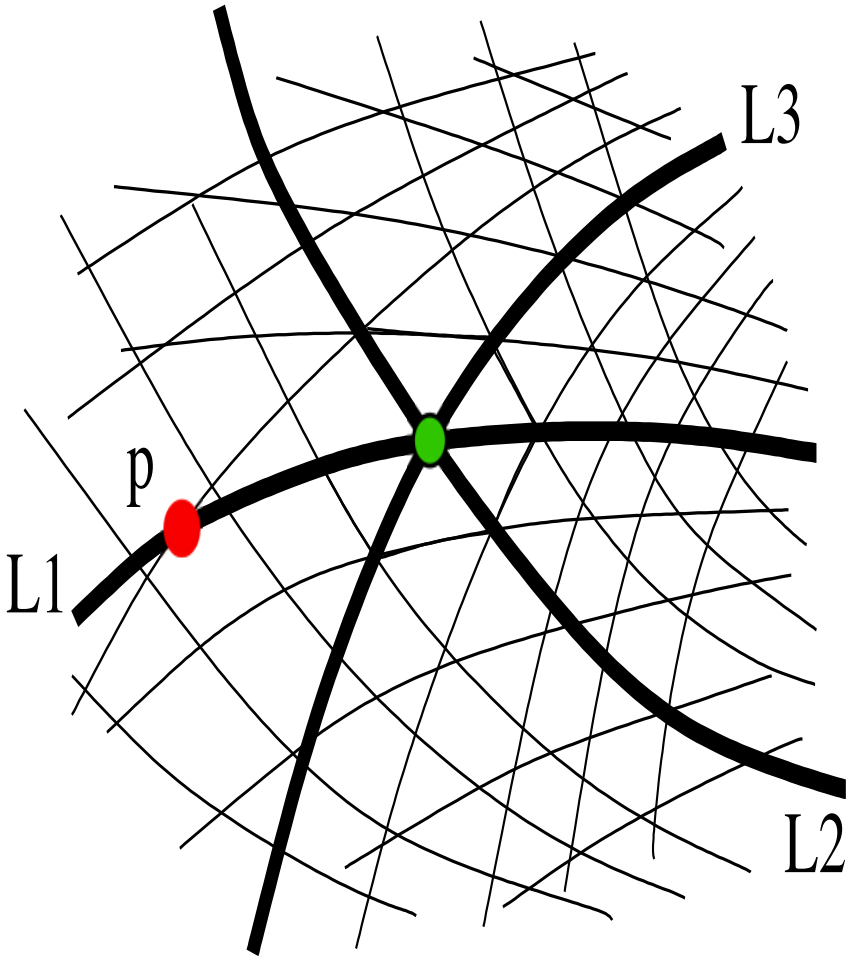








Hexagonality of a planar 3-web



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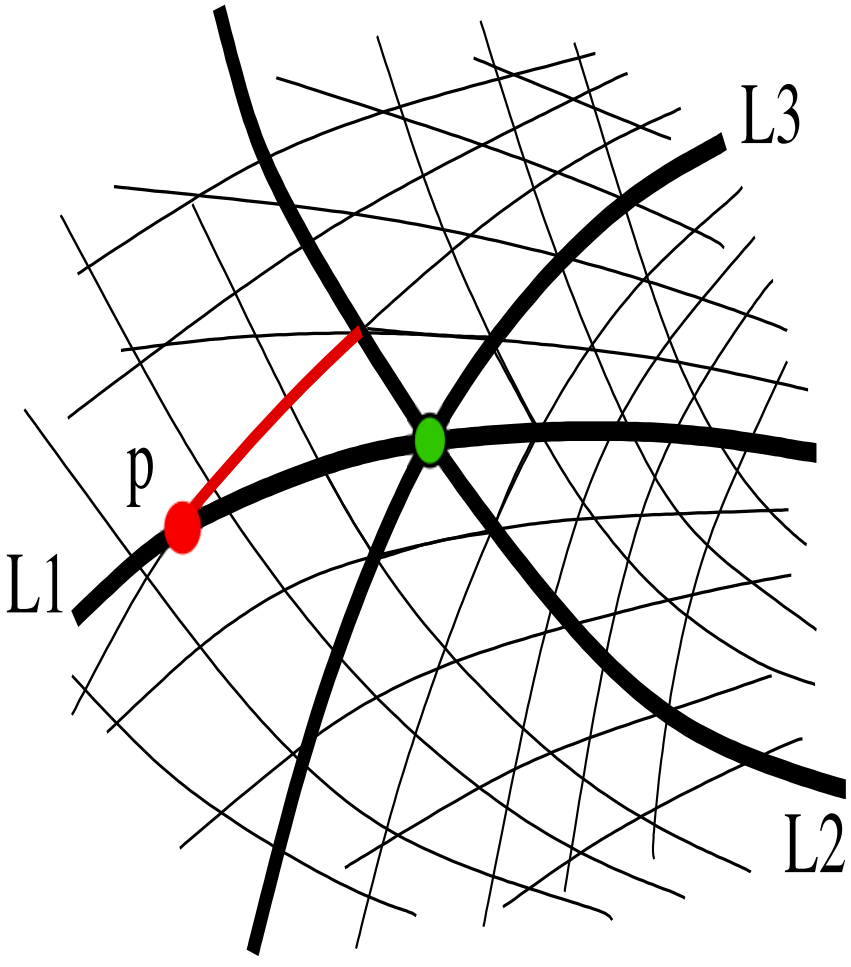

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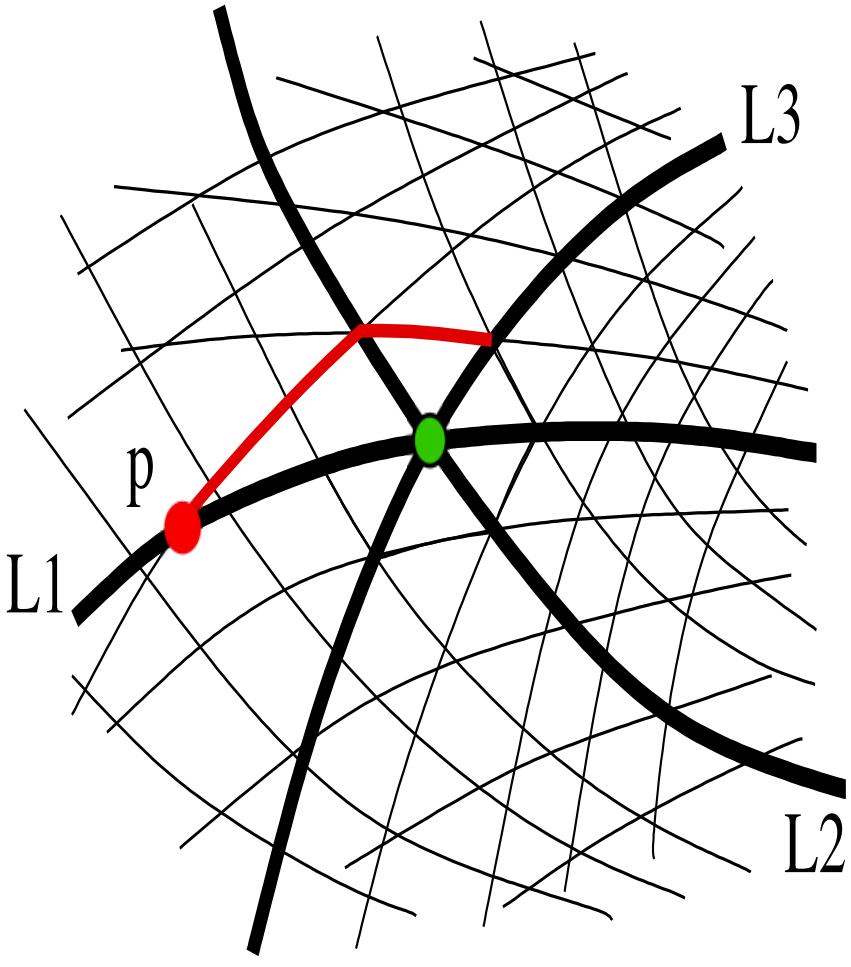

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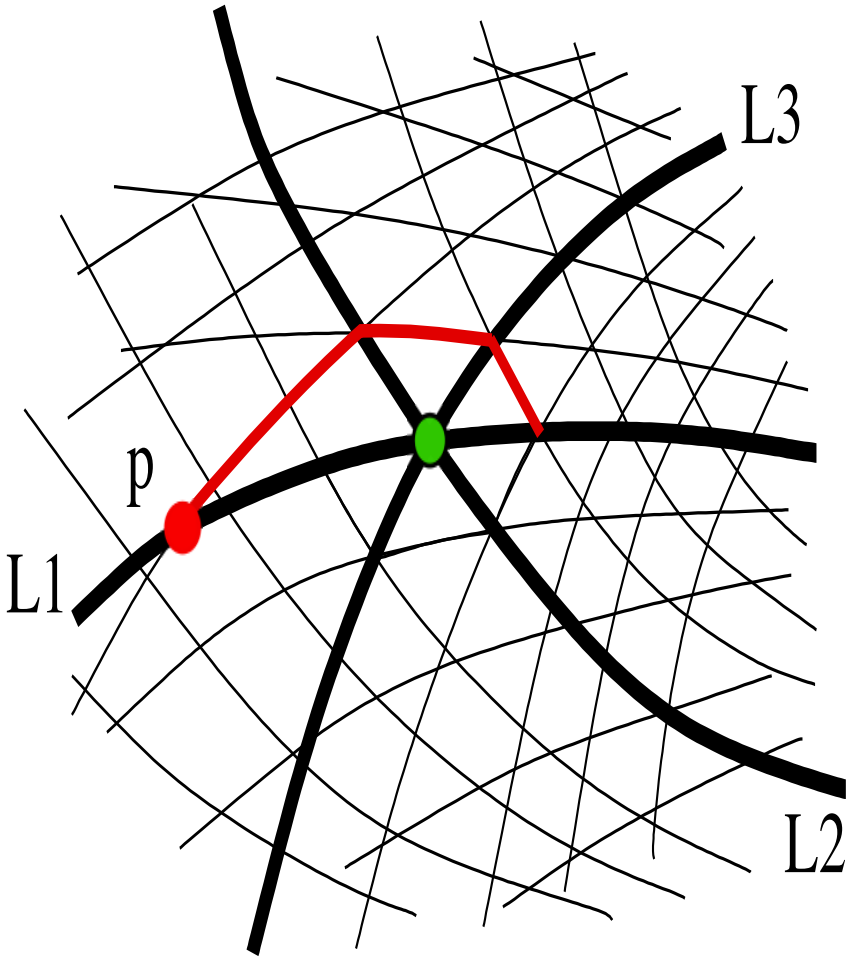

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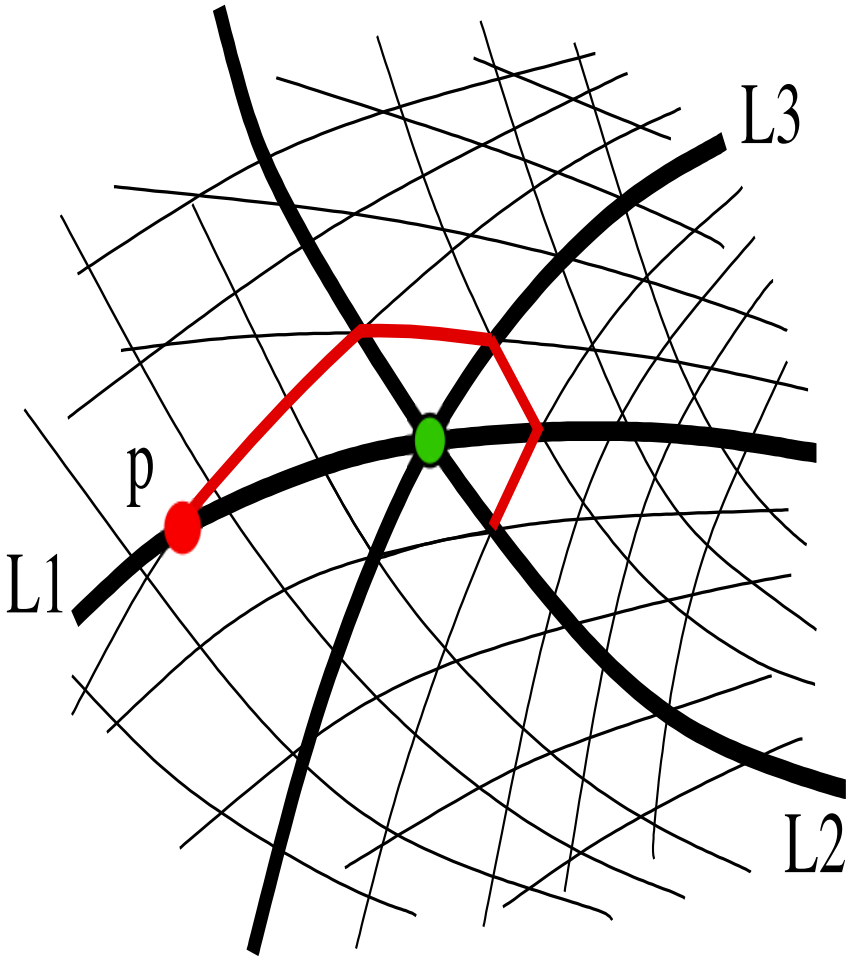

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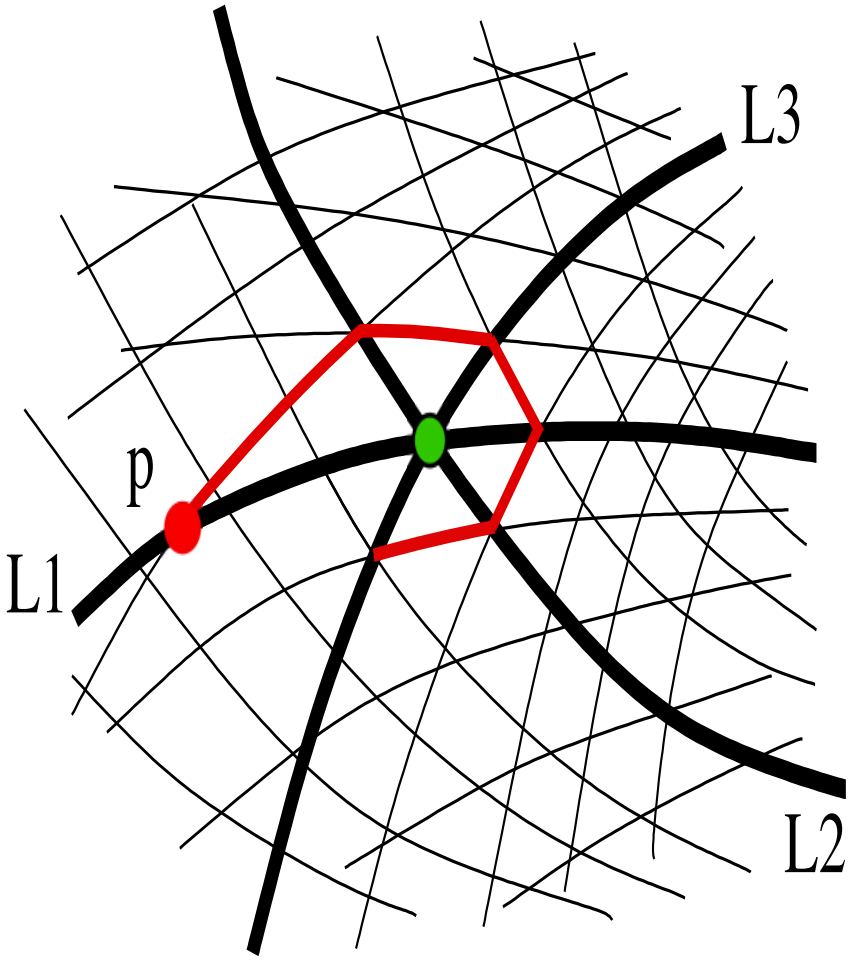

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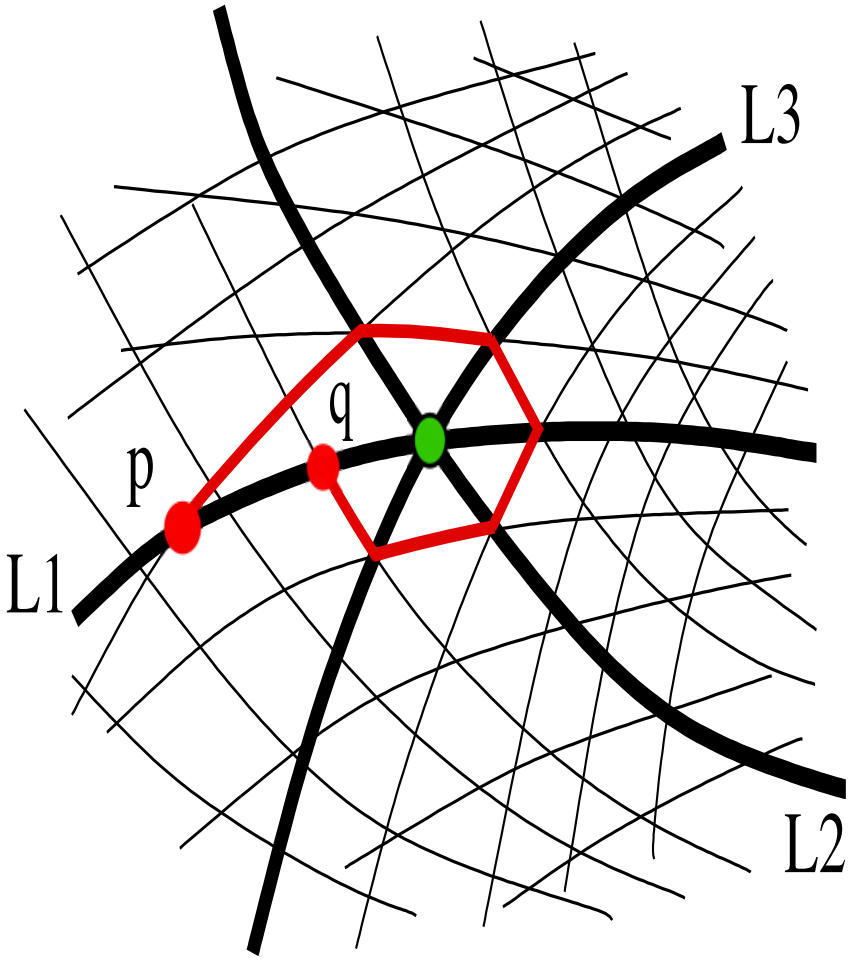

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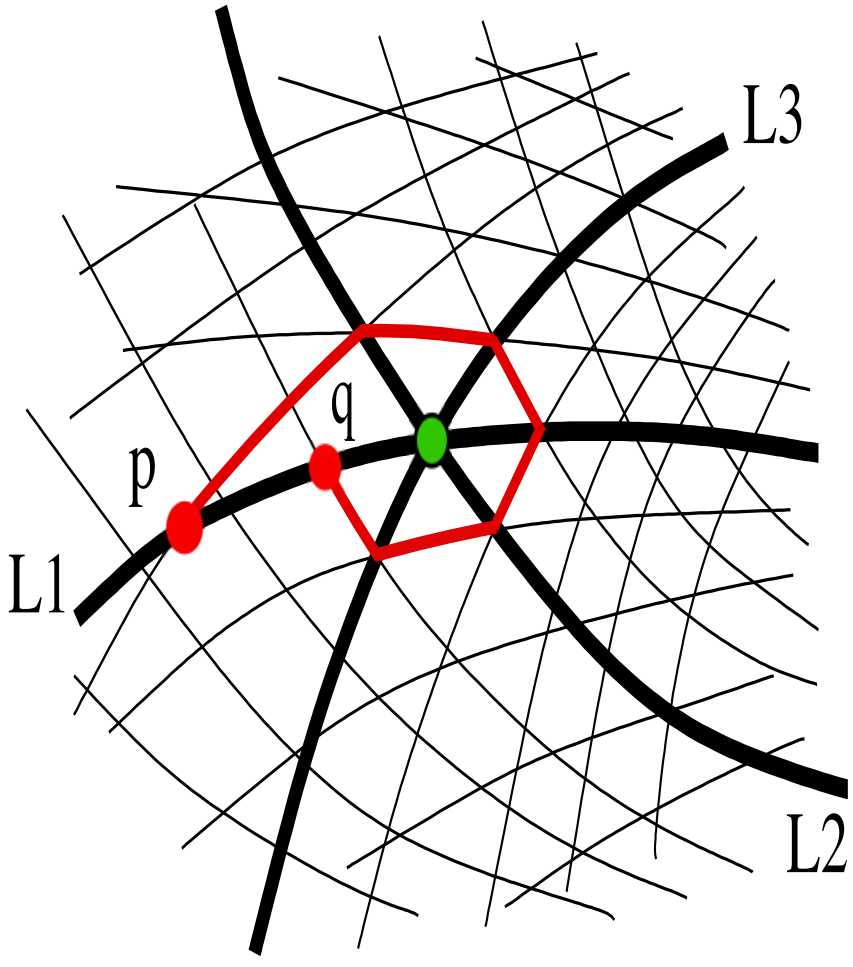

Definition : $\mathcal{W}_3$ is **hexagonal** if all 'hexagons' are closed

The Queen Elizabeth II Great Court roof at the British Museum

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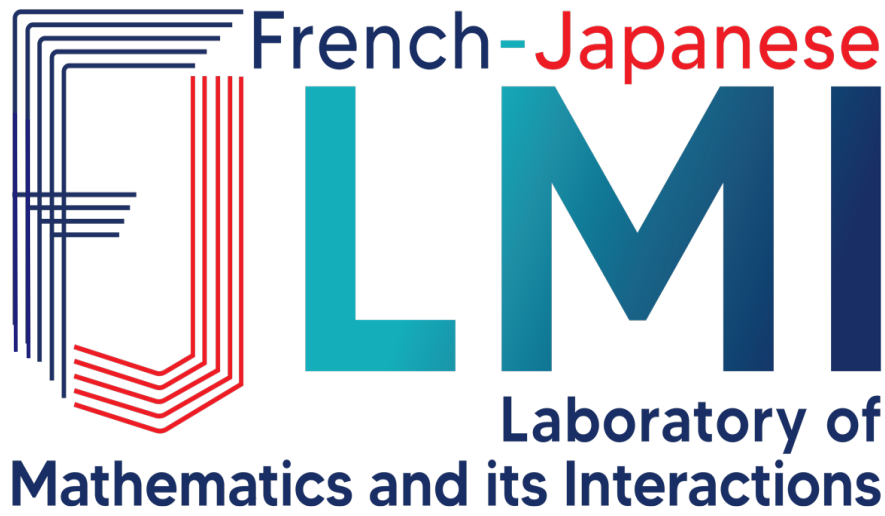
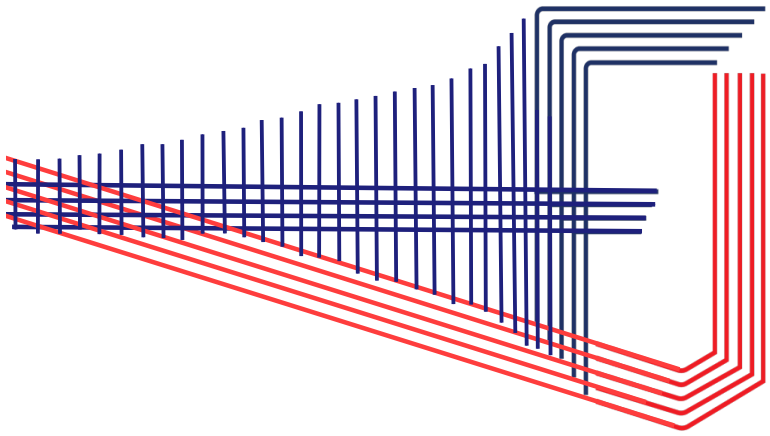

French-Japanese
LMI
Laboratory of
Mathematics and its Interactions



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A classical theorem on webs

- Let $\mathcal{W}_3 = \mathcal{W}(U_1, U_2, U_3)$ be a 3-web on $D \subset \mathbb{C}^2$ defined by some first integrals $U_i : D \to \mathbb{C}$

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Theorem. The following properties are equivalent :

1. $\mathcal{W}_3$ is **hexagonal**
2. $\mathcal{W}_3$ is **parallelizable**
3. $\mathcal{W}_3$ is **flat**
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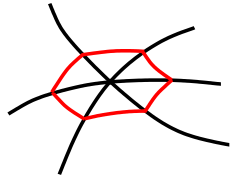
1. $\mathcal{W}_3$ is **hexagonal** 
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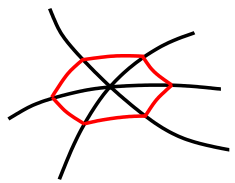
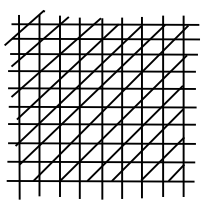
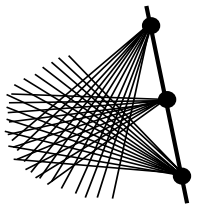
1. $\mathcal{W}_3$ is **hexagonal** 

2. $\mathcal{W}_3$ is **parallelizable** $\simeq \mathcal{W}(x, y, x-y) =$  $=$ 

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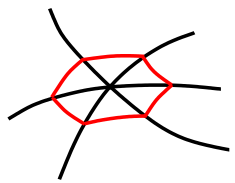
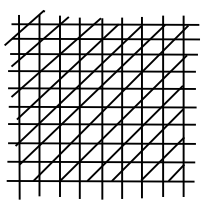
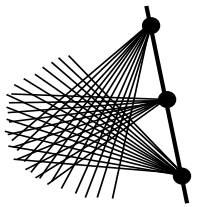
1. $\mathcal{W}_3$ is **hexagonal** 

2. $\mathcal{W}_3$ is **parallelizable** $\simeq \mathcal{W}(x, y, x-y) =$  $=$ 

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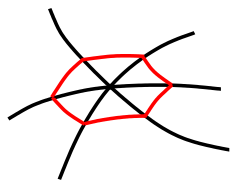
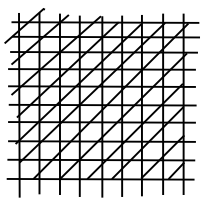
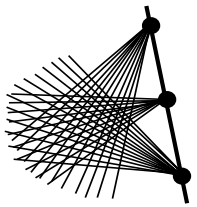
1. $\mathcal{W}_3$ is **hexagonal** 

2. $\mathcal{W}_3$ is **parallelizable** $\simeq \mathcal{W}(x, y, x-y) =$  $=$ 

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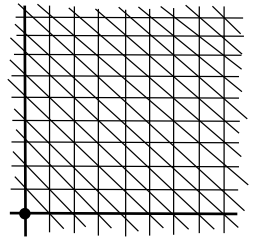
$$\exists\, (F_i)_{i=1}^{3} \quad \text{such that} \quad F_1(U_1) + F_2(U_2) + F_3(U_3) \equiv 0$$

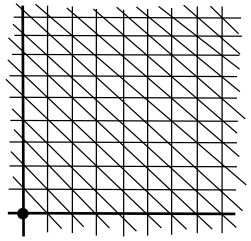
$$\mathcal{W}\big(\, x\,,\, y\,,\, x+y\,\big)$$



'Identity equation'

$$(x)+(y)-(x+y)=0$$

$\mathcal{W}\left(x\,,\, y\,,\, x+y \right)$



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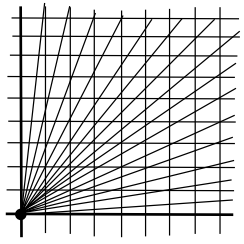

Cauchy's equation

$$\mathbf{Log}(x) - \mathbf{Log}(y) + \mathbf{Log}\,(y/x) = 0$$

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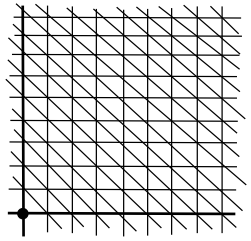

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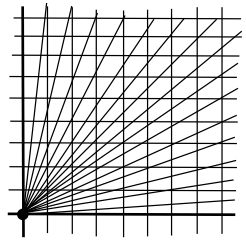

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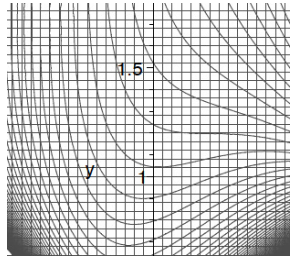

Euler's equation

$$\int^{x} \frac{ds}{\sqrt{g(s)}} + \int^{y} \frac{ds}{\sqrt{g(s)}} - \int^{Q} \frac{ds}{\sqrt{g(s)}} = 0$$

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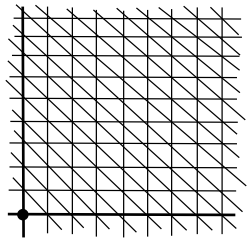

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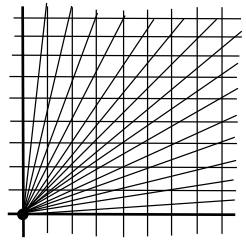

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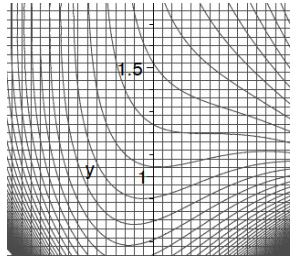

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Euler's Addition Theorem (1753). We have

$$\int_0^{x} \frac{ds}{\sqrt{g(s)}} + \int_0^{y} \frac{ds}{\sqrt{g(s)}} - \int_0^{Q} \frac{ds}{\sqrt{g(s)}} = 0$$

for $g(x) = (1-x^2)(1-k^2x^2)$ and $Q = Q(x,y) = \frac{x\sqrt{g(y)}+y\sqrt{g(x)}}{\sqrt{1-k^2x^2y^2}}$

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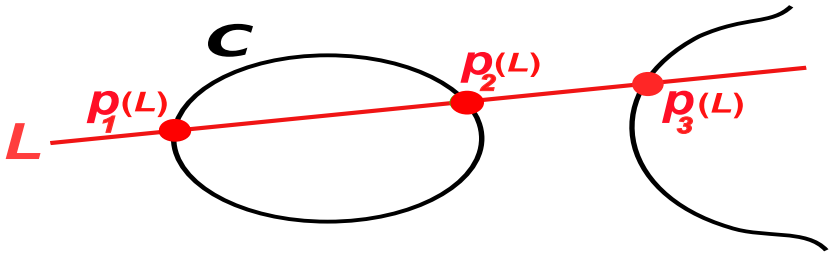

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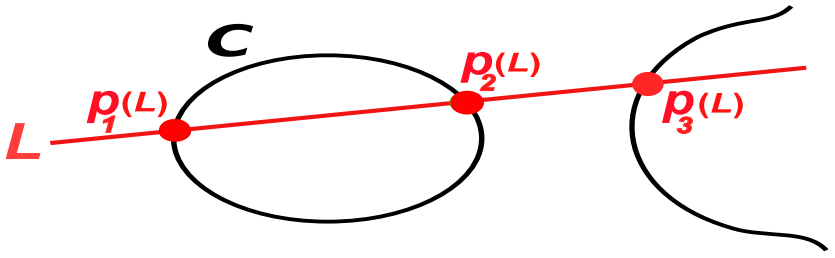
- 

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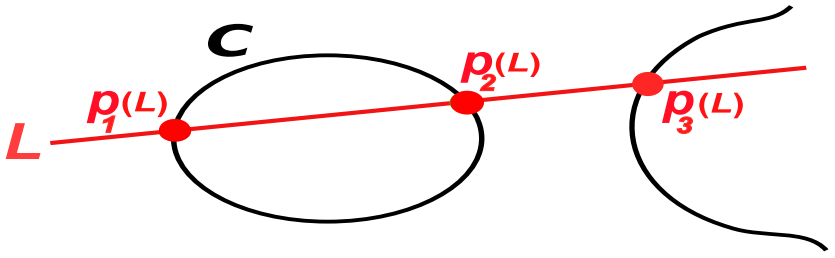

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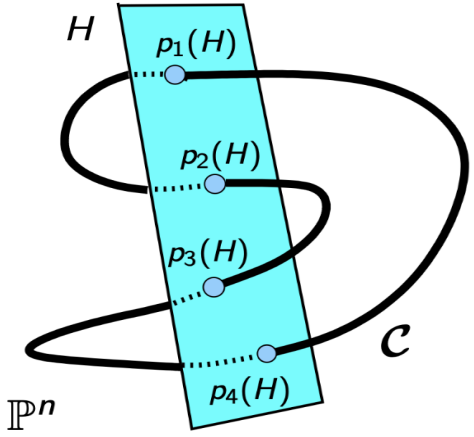
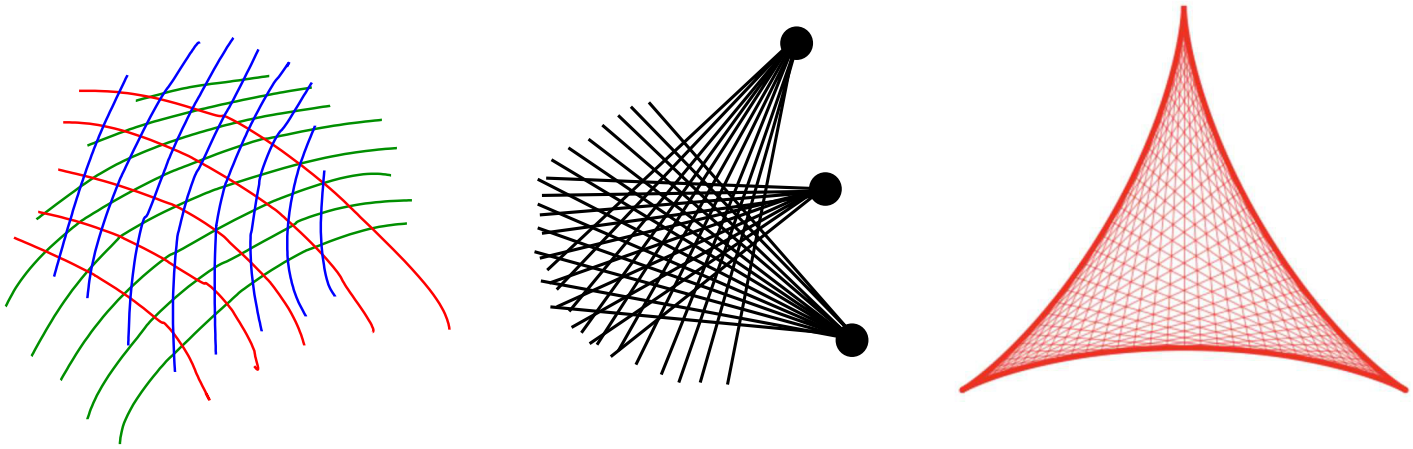


Abel's Addition Theorem

$\forall\, \omega \in \mathbf{H}^0(\omega^1_C) \quad : \quad \sum_{i=1}^{d} \int^{p_i} \omega = 0$

Webs and Functional Equations

- The study of webs $\mathcal{W}(U_1, \ldots, U_d)$ is closely related...



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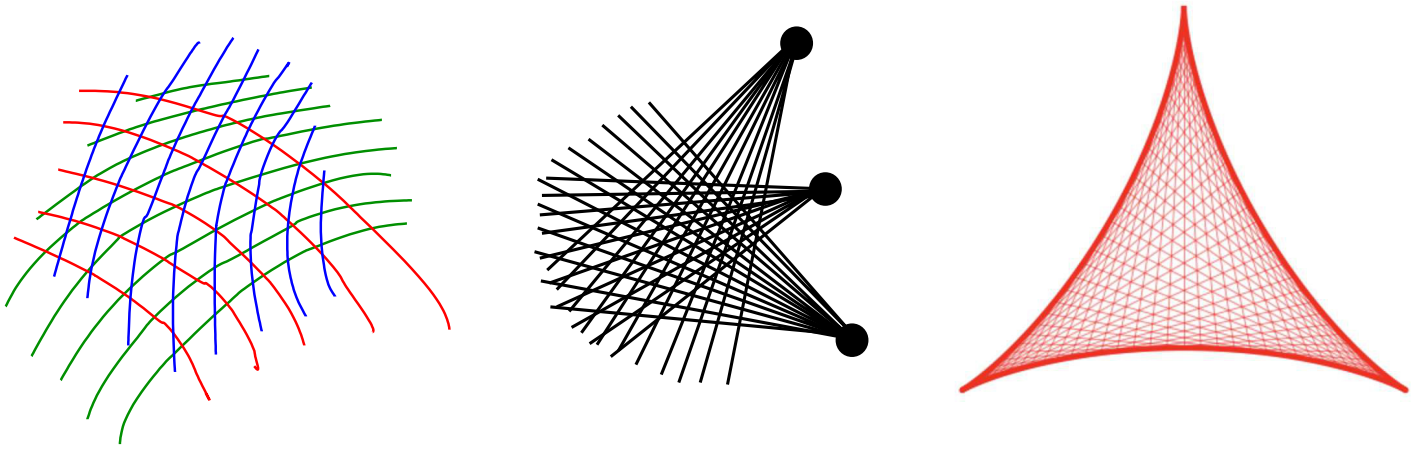

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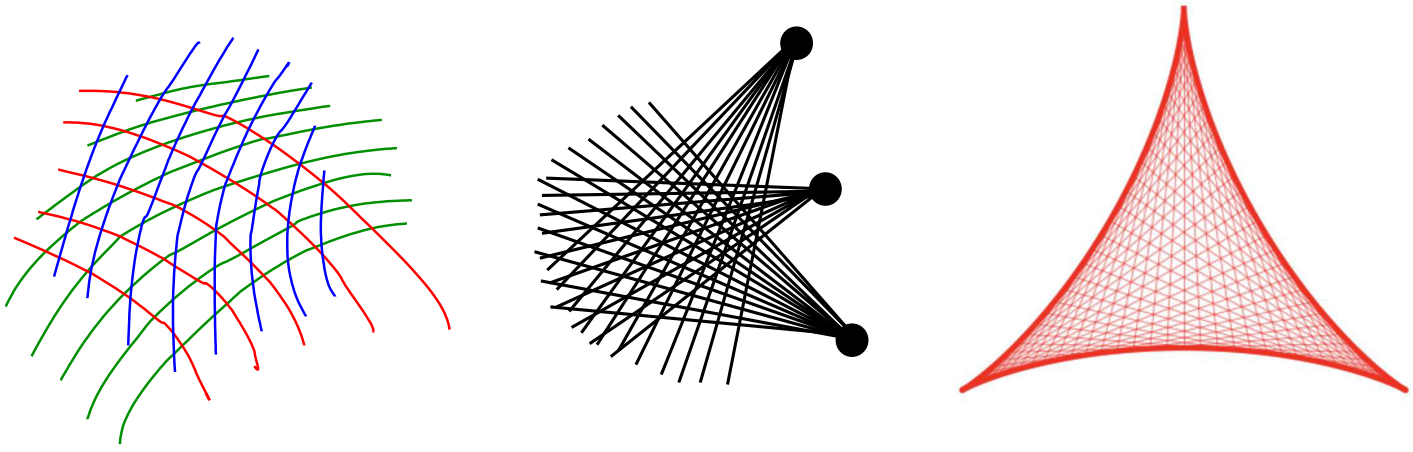

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Webs and Functional Equations

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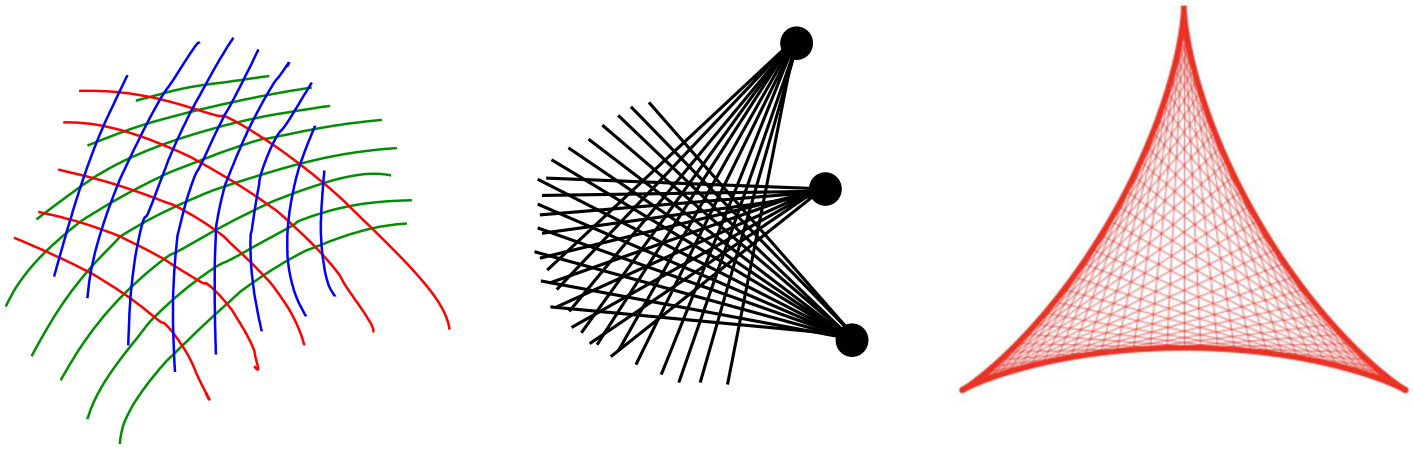

... to that of functional identities $F_1(U_1) + \cdots + F_d(U_d) = 0$

- $(x) + (y) - (x + y) = 0$ $\qquad$ $2(x)^2 + 2(y)^2 - (x + y)^2 - (x - y)^2 = 0$
- $\int^{p_1} \omega + \cdots + \int^{p_d} \omega = \mathbf{0}$

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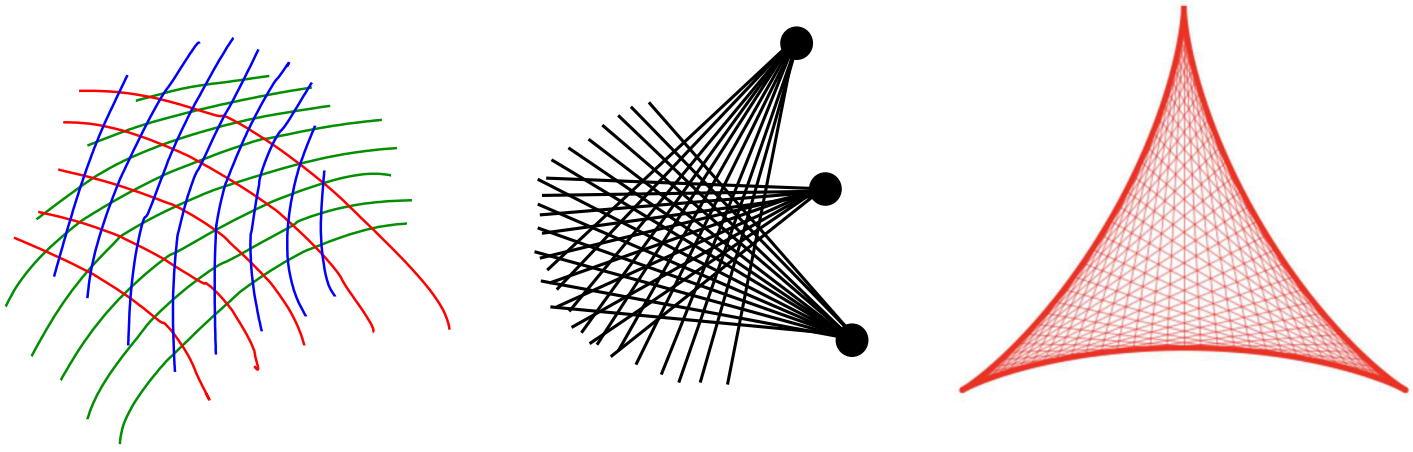

... to that of functional identities $F_1(U_1) + \cdots + F_d(U_d) = 0$

- $(x) + (y) - (x + y) = 0$ $\qquad 2(x)^2 + 2(y)^2 - (x + y)^2 - (x - y)^2 = 0$
- $\int^{p_1} \omega + \cdots + \int^{p_d} \omega = \mathbf{0}$
- $\mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}\left(\frac{x}{y}\right) = 0$

$$\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$$

Thank you very much for your attention

ご清聴ありがとうございました。

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