From Cauchy and Abel to Hyperlogarithmic Functional Identities on Del Pezzo Surfaces

Luc PIRIO (FJ-LMI)

Seminar on Geometric Complex Analysis (Tōdai)

May 11, 2026

Logarithm

$$\mathbf{Log}(x) - \mathbf{Log}(y) + \mathbf{Log}\left(y/x\right) = 0$$

Logarithm

$$\mathbf{Log}(x) - \mathbf{Log}(y) + \mathbf{Log}\left(y/x\right) = 0$$

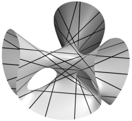
⋮

Hyperlogarithm

$$\sum_{i=1}^{2160} \mathbf{HLog}^6\Big(U_i(x,y)\Big) = 0$$

Logarithm

$$\mathbf{Log}(x) - \mathbf{Log}(y) + \mathbf{Log}\,(y/x) = 0$$

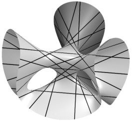


Hyperlogarithm

$$\sum_{i=1}^{2160} \mathbf{HLog}^{6}\Big(U_i(x,y)\Big) = 0$$

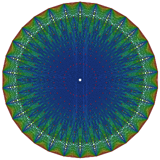
Logarithm

$$\mathbf{Log}(x) - \mathbf{Log}(y) + \mathbf{Log}\,(y/x) = 0$$



Hyperlogarithm

$$\sum_{i=1}^{2160} \mathbf{HLog}^6\Big(U_i(x,y)\Big) = 0$$



Plan

I Introduction

Polylogarithms & their functional identities

Theorem : $\mathbf{HLog}^w = 0$ for $w = 1, \ldots, 6$

II Ingredients and proof

Del Pezzo surfaces, Hyperlogarithms

Proof of $\mathbf{HLog}^w = 0$

Comparison

III An approach à la Gelfand-MacPherson

Face maps on $\boldsymbol{G/P}$

Example : the case $r = 6$ ($\boldsymbol{G/P} = \mathbb{O}\mathbf{P}^{\mathbf{2}} =$ Cayley plane)

The logarithm

- $\mathsf{Li}_1(z) = -\mathsf{Log}(1-z)$ $\qquad (z \in \mathbb{C})$
- **Development in series :** $\mathsf{Li}_1(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k}$
- **Integral formula :** $\mathsf{Log}(z) \;=\; \int^{z} \frac{\mathrm{d}u}{u-0}$

  $$\mathsf{Li}_1(z) \;=\; -\int^{z} \frac{\mathrm{d}u}{u-1}$$
- **Monodromy :** $\mathcal{M}_0\,(\,\mathsf{Log}\,) \;=\; \mathsf{Log} + 2i\pi$
- **Cauchy's functional identity :**

$$\mathsf{Log}(\, x \,) - \mathsf{Log}(\, y \,) + \mathsf{Log}\left(\frac{y}{x} \right) = 0$$

The dilogarithm

- $\mathrm{Li}_2(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ $\qquad \big(\; |z| < 1 \;\big)$
- **Integral formula :**

$$\begin{aligned} \mathrm{Li}_2(z) \;=\; \mathbf{L}_{01}(z) &\;=\; -\int^z \mathrm{Log}(1-u)\,\frac{du}{u-0} \\ \mathbf{L}_{10}(z) &\;=\; \int^z \mathrm{Log}(u-0)\,\frac{du}{1-u} \end{aligned}$$

- **Monodromy :** $\mathcal{M}_1\,(\,\mathrm{Li}_2\,) \;=\; \mathrm{Li}_2 - 2i\pi\,\mathrm{Log}$
- **Abel's functional equation** $(\mathcal{A}b)$ **[Spence 1809]**

$$\mathrm{Li}_2(x) - \mathrm{Li}_2(y) - \mathrm{Li}_2\Big(\frac{x}{y}\Big) - \mathrm{Li}_2\Big(\frac{1-y}{1-x}\Big) + \mathrm{Li}_2\left(\frac{x(1-y)}{y(1-x)}\right) = \mathrm{Log}(y)\,\mathrm{Log}\left(\frac{1-y}{1-x}\right) - \frac{\pi^2}{6}$$

The dilogarithm

- $\mathbf{L}\mathrm{i}_2(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ $\qquad\qquad \big(\, |z| < 1 \,\big)$
- **Integral formula :** $\quad \mathbf{L}\mathrm{i}_2(z) \;=\; \mathbf{L}_{01}(z) \;=\; -\int^{z} \mathbf{L}\mathrm{og}(1-u)\,\frac{du}{u-0}$

$$\mathbf{L}_{10}(z) \;=\; \int^{z} \mathbf{L}\mathrm{og}(u-0)\,\frac{du}{1-u}$$

- **Monodromy :** $\quad \mathcal{M}_1\,(\,\mathbf{L}\mathrm{i}_2\,) \;=\; \mathbf{L}\mathrm{i}_2 - 2i\pi\,\mathbf{L}\mathrm{og}$
- **Abel's functional identity** $(\mathcal{A}b)$ $\qquad$ **[Spence 1809]**

$$\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\Big(\frac{x}{y}\Big) - \mathbf{R}\Big(\frac{1-y}{1-x}\Big) + \mathbf{R}\Big(\frac{x(1-y)}{y(1-x)}\Big) = \mathbf{0}$$

$$\mathbf{R}(x) = \frac{1}{2}\Big(\mathbf{L}_{01}(x) - \mathbf{L}_{10}(x)\Big)$$

The n-th polylogarithm $\mathbf{Li}_n$ for $n \geq 1$

- $\mathbf{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad\qquad \big(|z| < 1 \big)$

- **Integral formula :**

$$\mathbf{Li}_n(z) = \int^z \mathbf{Li}_{n-1}(u) \frac{du}{u}$$

$$\mathbf{Li}_n{}'(z) = \mathbf{Li}_{n-1}(z)/z$$

- **Monodromy :** $\mathcal{M}_1\,(\,\mathbf{Li}_n\,) = \mathbf{Li}_n - 2i\pi \frac{(\mathbf{Log})^{n-1}}{(n-1)!}$

- **Functional identities in one variable :**

$$\mathbf{Li}_n\Big(z^m\Big) = r^{m-1} \sum_{\omega^m=1} \mathbf{Li}_n(\omega\, z)$$

$$\mathbf{Li}_n(z) + (-1)^n \mathbf{Li}_n\Big(z^{-1}\Big) = -\frac{(2i\pi)^n}{n!}\, \mathbf{B}_n\Big(\frac{\mathbf{Log}\, z}{2i\pi}\Big)$$

The n-th polylogarithm $\mathbf{Li}_n$ for $n \geq 1$

- $\mathbf{Li}_n(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad\qquad\qquad\qquad\qquad\qquad \big(\ |z| < 1\ \big)$

- **Integral formula :** $$\mathbf{Li}_n(z) \;=\; \int^z \mathbf{Li}_{n-1}(u)\frac{du}{u}$$ $$\mathbf{Li}_n{}'(z) \;=\; \mathbf{Li}_{n-1}(z)/z$$

- **Monodromy :** $$\mathcal{M}_1\,(\,\mathbf{Li}_n\,) \;=\; \mathbf{Li}_n - 2i\pi\,\frac{(\mathbf{Log})^{n-1}}{(n-1)!}$$

- **Functional identities in several variables ($\exists$?) :**

$$\sum_{i\in I} c_i\,\mathbf{Li}_n(\boldsymbol{U_i}) = \mathbf{Elem}_{<n}$$

$$\Big(\ I\ \text{ finite}\,,\ c_i \in \mathbb{Z}\,,\ \boldsymbol{U_i} \in \mathbb{Q}(x_1,\ldots,x_N)\ \Big)$$

The n-th polylogarithm $\mathbf{Li}_n$ for $n \geq 1$

- $\mathbf{Li}_n(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad\qquad \Big(\ |z|< 1\ \Big)$

- **Integral formula :** $\qquad \mathbf{Li}_n(z) \;=\; \int^{z} \mathbf{Li}_{n-1}(u)\frac{du}{u}$

  $$\mathbf{Li}_n{}'(z) \;=\; \mathbf{Li}_{n-1}(z)/z$$

- **Monodromy :** $\qquad \boldsymbol{\mathcal{M}}_1\,(\,\mathbf{Li}_n\,) \;=\; \mathbf{Li}_n - 2i\pi\,\frac{(\mathbf{L}\mathrm{og})^{n-1}}{(n-1)!}$

- **Functional identities in several variables ($\exists$?) :**

$$\sum_{i\in I} c_i\,\mathbf{Li}_n\big(\boldsymbol{U_i}\big) = \mathbf{Elem}_{<n} \qquad \Longleftrightarrow \qquad \sum_{i\in I} c_i\,\boldsymbol{\mathcal{L}}_n\big(\boldsymbol{U_i}\big) = \mathbf{0}$$

$\Big(\ I\ \text{finite}\,,\ c_i \in \mathbb{Z}\,,\ \boldsymbol{U_i} \in \mathbb{Q}(x_1,\ldots,x_N)\ \Big)$

Example : $\mathbf{L}\mathrm{i}_3$

- $\mathbf{L}\mathrm{i}_3(z) \;=\; \sum_{k=1}^{\infty} z^k/k^3 \;=\; \int^z \mathbf{L}\mathrm{i}_2(u)\frac{du}{u}$
- **Spence-Kummer identity $(\mathcal{SK})$ (1809-1840) :**

$$
\begin{aligned}
&2\,\mathbf{L}\mathrm{i}_3(\,\boldsymbol{x}\,) + 2\,\mathbf{L}\mathrm{i}_3(\,\boldsymbol{y}\,) - \mathbf{L}\mathrm{i}_3\Big(\frac{\boldsymbol{x}}{\boldsymbol{y}}\Big) + 2\,\mathbf{L}\mathrm{i}_3\Big(\frac{\mathbf{1}-\boldsymbol{x}}{\mathbf{1}-\boldsymbol{y}}\Big) + 2\,\mathbf{L}\mathrm{i}_3\Big(\frac{\boldsymbol{x}(\mathbf{1}-\boldsymbol{y})}{\boldsymbol{y}(\mathbf{1}-\boldsymbol{x})}\Big) - \mathbf{L}\mathrm{i}_3\big(\,\boldsymbol{xy}\,\big)\\
&\qquad + 2\,\mathbf{L}\mathrm{i}_3\Big(-\frac{\boldsymbol{x}(\mathbf{1}-\boldsymbol{y})}{(\mathbf{1}-\boldsymbol{x})}\Big) + 2\,\mathbf{L}\mathrm{i}_3\Big(-\frac{(\mathbf{1}-\boldsymbol{y})}{\boldsymbol{y}(\mathbf{1}-\boldsymbol{x})}\Big) - \mathbf{L}\mathrm{i}_3\Big(\frac{\boldsymbol{x}(\mathbf{1}-\boldsymbol{y})^2}{\boldsymbol{y}(\mathbf{1}-\boldsymbol{x})^2}\Big)\\
&\qquad = 2\,\mathbf{L}\mathrm{i}_3(1) - \mathbf{L}\mathrm{og}\,(y)^2\mathbf{L}\mathrm{og}\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3}\mathbf{L}\mathrm{og}(y) + \frac{1}{3}\mathbf{L}\mathrm{og}\,(y)^3
\end{aligned}
$$

Example : Li_3

- $\mathrm{Li}_3(z) \;=\; \sum_{k=1}^{\infty} z^k/k^3 \;=\; \int^z \mathrm{Li}_2(u)\frac{du}{u}$

- **Spence-Kummer identity $(\mathcal{SK})$ (1809-1840) :**

$$
\begin{aligned}
&2\,\mathrm{Li}_3(x) + 2\,\mathrm{Li}_3(y) - \mathrm{Li}_3\left(\frac{x}{y}\right) + 2\,\mathrm{Li}_3\left(\frac{1-x}{1-y}\right) + 2\,\mathrm{Li}_3\left(\frac{x(1-y)}{y(1-x)}\right) - \mathrm{Li}_3(xy)\\
&\quad + 2\,\mathrm{Li}_3\left(-\frac{x(1-y)}{(1-x)}\right) + 2\,\mathrm{Li}_3\left(-\frac{(1-y)}{y(1-x)}\right) - \mathrm{Li}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right)\\
&\quad = 2\,\mathrm{Li}_3(1) - \mathrm{Log}\,(y)^2\mathrm{Log}\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3}\mathrm{Log}(y) + \frac{1}{3}\mathrm{Log}\,(y)^3
\end{aligned}
$$

$$
\begin{aligned}
&2\,\mathcal{L}_3(x) + 2\,\mathcal{L}_3(y) - \mathcal{L}_3\left(\frac{x}{y}\right) + 2\,\mathcal{L}_3\left(\frac{1-x}{1-y}\right) + 2\,\mathcal{L}_3\left(\frac{x(1-y)}{y(1-x)}\right) - \mathcal{L}_3(xy)\\
&\quad + 2\,\mathcal{L}_3\left(-\frac{x(1-y)}{(1-x)}\right) + 2\,\mathcal{L}_3\left(-\frac{(1-y)}{y(1-x)}\right) - \mathcal{L}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right) = 0
\end{aligned}
$$

$$\mathcal{L}_3(z) = \mathrm{Li}_3(z) - \mathrm{Li}_2(z)\,\mathrm{Log}|z| + \tfrac{1}{3}\,\mathrm{Li}_1(z)\big(\mathrm{Log}|z|\big)^2$$

Example : Li_4

- **Kummer's functional identity $\mathcal{K}(4)$ (1840) :**

$$\begin{aligned}
&\mathcal{L}_4\left(-\frac{x^2y\eta}{\zeta}\right)+\mathcal{L}_4\left(-\frac{y^2x\zeta}{\eta}\right)+\mathcal{L}_4\left(\frac{x^2y}{\eta^2\zeta}\right)+\mathcal{L}_4\left(\frac{y^2x}{\zeta^2\eta}\right)\\
&-6\,\mathcal{L}_4\left(xy\right)-6\,\mathcal{L}_4\left(\frac{xy}{\eta\zeta}\right)-6\,\mathcal{L}_4\left(-\frac{xy}{\eta}\right)-6\,\mathcal{L}_4\left(-\frac{xy}{\zeta}\right)\\
&-3\,\mathcal{L}_4\left(x\eta\right)-3\,\mathcal{L}_4\left(y\zeta\right)-3\,\mathcal{L}_4\left(\frac{x}{\eta}\right)-3\,\mathcal{L}_4\left(\frac{y}{\zeta}\right)\\
&-3\,\mathcal{L}_4\left(-\frac{x\eta}{\zeta}\right)-3\,\mathcal{L}_4\left(-\frac{y\zeta}{\eta}\right)-3\,\mathcal{L}_4\left(-\frac{x}{\eta\zeta}\right)-3\,\mathcal{L}_4\left(-\frac{y}{\eta\zeta}\right)\\
&+6\,\mathcal{L}_4\left(x\right)+6\,\mathcal{L}_4(y)+6\,\mathcal{L}_4\left(-\frac{x}{\zeta}\right)+6\,\mathcal{L}_4\left(-\frac{y}{\eta}\right)=\mathbf{0}
\end{aligned}$$

$$\Big(\ \zeta=1-x,\ \eta=1-y\ \Big)$$

- **Functional identities (FI) of polylogarithms Li_n :**
  - Hyperbolic geometry
  - Web geometry ($n \leq 3$)
  - K-theory of number fields ($n \leq 4$)
  - Periods (MZVs)
  - Particle physics ('*Scattering amplitudes*')
  - Mathematical physics ('*Y-systems*') ($n = 2$)
  - Cluster algebras ($n \leq 4$)
  - Mirror symmetry ('*Scattering diagrams*') ($n = 2$)

'Scattering amplitudes' and functional identities

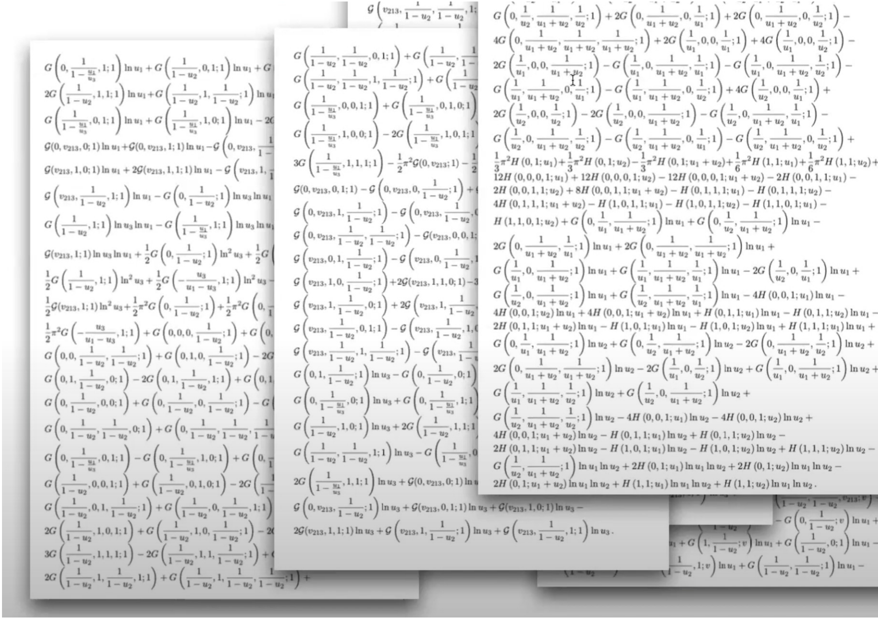
- *'Scattering amplitudes'* $\boldsymbol{A} = \int_{\Delta} \boldsymbol{\Psi}$ (important in HEPP)

$$\boldsymbol{A} = \boldsymbol{A}' + \mathcal{R}$$

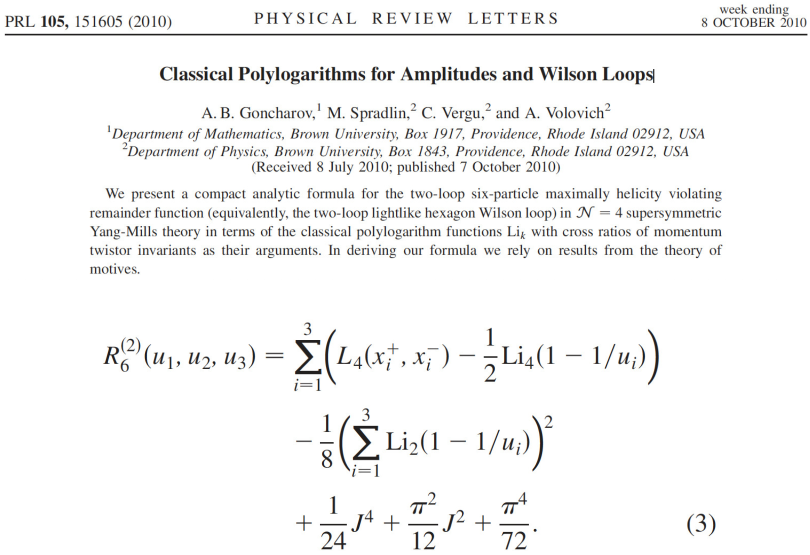
$$\boldsymbol{A} = \boldsymbol{A}' + \sum_{i \in I} \mathbf{F}_i(x_i)$$

- **[dDDS]** *'The 2-loop hexagon Wilson loop in* $\mathcal{N} = 4$ SYM*'* (2010)

$\mathcal{R}^{(2)}_{6,WL}$ = 'remainder' : a 17 pages formula !



$$\begin{aligned}
&G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2},\frac{1}{u_2};1\right)+2G\left(0,\frac{1}{u_1+u_2},0,\frac{1}{u_1};1\right)+2G\left(0,\frac{1}{u_1+u_2},0,\frac{1}{u_2};1\right)-\\
&4G\left(0,\frac{1}{u_1+u_2},\frac{1}{u_1+u_2},\frac{1}{u_1+u_2};1\right)+2G\left(\frac{1}{u_1},0,0,\frac{1}{u_1};1\right)+4G\left(\frac{1}{u_1},0,0,\frac{1}{u_2};1\right)-\\
&2G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2},\frac{1}{u_1};1\right)-G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2},\frac{1}{u_2};1\right)-\\
&G\left(\frac{1}{u_1},\frac{1}{u_1+u_2},0,\frac{1}{u_1};1\right)-G\left(\frac{1}{u_1},\frac{1}{u_1+u_2},0,\frac{1}{u_2};1\right)+4G\left(\frac{1}{u_2},0,0,\frac{1}{u_1};1\right)+\\
&2G\left(\frac{1}{u_2},0,0,\frac{1}{u_2};1\right)-2G\left(\frac{1}{u_2},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2},\frac{1}{u_1};1\right)-\\
&G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2},\frac{1}{u_2};1\right)-G\left(\frac{1}{u_2},\frac{1}{u_1+u_2},0,\frac{1}{u_1};1\right)-G\left(\frac{1}{u_2},\frac{1}{u_1+u_2},0,\frac{1}{u_2};1\right)+\\
&\frac{1}{3}\pi^2H\left(0,1;u_1\right)+\frac{1}{3}\pi^2H\left(0,1;u_2\right)-\frac{1}{3}\pi^2H\left(0,1;u_1+u_2\right)+\frac{1}{6}\pi^2H\left(1,1;u_1\right)+\frac{1}{6}\pi^2H\left(1,1;u_2\right)+\\
&12H\left(0,0,0,1;u_1\right)+12H\left(0,0,0,1;u_2\right)-12H\left(0,0,0,1;u_1+u_2\right)-2H\left(0,0,1,1;u_1\right)-\\
&2H\left(0,0,1,1;u_2\right)+8H\left(0,0,1,1;u_1+u_2\right)-H\left(0,1,1,1;u_1\right)-H\left(0,1,1,1;u_2\right)-\\
&4H\left(0,1,1,1;u_1+u_2\right)-H\left(1,0,1,1;u_1\right)-H\left(1,0,1,1;u_2\right)-H\left(1,1,0,1;u_1\right)-\\
&H\left(1,1,0,1;u_2\right)+G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)\ln u_1+G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)\ln u_1-\\
&2G\left(0,\frac{1}{u_1+u_2},\frac{1}{u_1};1\right)\ln u_1+2G\left(0,\frac{1}{u_1+u_2},\frac{1}{u_1+u_2};1\right)\ln u_1+\\
&G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)\ln u_1+G\left(\frac{1}{u_1},\frac{1}{u_1+u_2},\frac{1}{u_1};1\right)\ln u_1-2G\left(\frac{1}{u_2},0,\frac{1}{u_1};1\right)\ln u_1+\\
&G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)\ln u_1+G\left(\frac{1}{u_2},\frac{1}{u_1+u_2},\frac{1}{u_1};1\right)\ln u_1-4H\left(0,0,1;u_1\right)\ln u_1-\\
&4H\left(0,0,1;u_2\right)\ln u_1+4H\left(0,0,1;u_1+u_2\right)\ln u_1+H\left(0,1,1;u_1\right)\ln u_1-H\left(0,1,1;u_2\right)\ln u_1-\\
&2H\left(0,1,1;u_1+u_2\right)\ln u_1-H\left(1,0,1;u_1\right)\ln u_1-H\left(1,0,1;u_2\right)\ln u_1+H\left(1,1,1;u_1\right)\ln u_1+\\
&G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)\ln u_2+G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)\ln u_2-2G\left(0,\frac{1}{u_1+u_2},\frac{1}{u_2};1\right)\ln u_2+\\
&2G\left(0,\frac{1}{u_1+u_2},\frac{1}{u_1+u_2};1\right)\ln u_2-2G\left(\frac{1}{u_1},0,\frac{1}{u_2};1\right)\ln u_2+G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)\ln u_2+\\
&G\left(\frac{1}{u_1},\frac{1}{u_1+u_2},\frac{1}{u_2};1\right)\ln u_2+G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)\ln u_2+\\
&G\left(\frac{1}{u_2},\frac{1}{u_1+u_2},\frac{1}{u_2};1\right)\ln u_2-4H\left(0,0,1;u_1\right)\ln u_2-4H\left(0,0,1;u_2\right)\ln u_2+\\
&4H\left(0,0,1;u_1+u_2\right)\ln u_2-H\left(0,1,1;u_1\right)\ln u_2+H\left(0,1,1;u_2\right)\ln u_2-\\
&2H\left(0,1,1;u_1+u_2\right)\ln u_2-H\left(1,0,1;u_1\right)\ln u_2-H\left(1,0,1;u_2\right)\ln u_2+H\left(1,1,1;u_2\right)\ln u_2-\\
&G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)\ln u_1\ln u_2+2H\left(0,1;u_1\right)\ln u_1\ln u_2+2H\left(0,1;u_2\right)\ln u_1\ln u_2-\\
&2H\left(0,1;u_1+u_2\right)\ln u_1\ln u_2+H\left(1,1;u_1\right)\ln u_1\ln u_2+H\left(1,1;u_2\right)\ln u_1\ln u_2\,.
\end{aligned}$$



PRL **105,** 151605 (2010) PHYSICAL REVIEW LETTERS week ending 8 OCTOBER 2010

Classical Polylogarithms for Amplitudes and Wilson Loops

A. B. Goncharov,[1] M. Spradlin,[2] C. Vergu,[2] and A. Volovich[2]
[1]*Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA*
[2]*Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA*

(Received 8 July 2010; published 7 October 2010)

We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

$$R_6^{(2)}(u_1, u_2, u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathrm{Li}_4(1 - 1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1 - 1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}. \qquad (3)$$

‘Scattering amplitudes’ and functional identities

- *‘Scattering amplitudes’* $\boldsymbol{A} = \int_\Delta \boldsymbol{\Psi}$ (important in HEPP)

$$\boldsymbol{A} = \boldsymbol{A}' + \mathcal{R}$$

$$\boldsymbol{A} = \boldsymbol{A}' + \sum_{i\in I} \mathbf{F}_i(x_i)$$

- **[dDDS]** *‘The 2-loop hexagon Wilson loop in $\mathcal{N}=4$ SYM’* (2010)

  $\mathcal{R}^{(2)}_{6,WL}$ = ‘remainder’ : a 17 pages formula !

- **[GSVV]** $\mathcal{R}^{(2)}_{6,WL} = \sum_{i=1}^3 \Big(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathbf{L}\mathrm{i}_4(v_i)\Big) - \frac{1}{8}\Big(\sum_{i=1}^3 \mathbf{L}\mathrm{i}_2(v_i)\Big)^2 + \cdots$

- Relevance of simplifying $\sum_{i\in I} \mathbf{F}_i(x_i)$ for $\mathbf{F}_i$ = polylogarithms
  $\mathbf{F}_i$ = hyperlogarithms
  $\mathbf{F}_i$ = elliptic polylogs

- Justifies the study of functional identities $\sum_{j\in J} \mathbf{F}_j(x_j) = \mathrm{cst}$

'Scattering amplitudes' and functional identities

- *'Scattering amplitudes'* $\boldsymbol{A} = \int_\Delta \boldsymbol{\Psi}$ (important in HEPP)

$$\boldsymbol{A} = \boldsymbol{A}' + \mathcal{R}$$

$$\boldsymbol{A} = \boldsymbol{A}' + \sum_{i\in I} \mathbf{F}_i(\,x_i\,)$$

- **[dDDS]** *'The 2-loop hexagon Wilson loop in* $\mathcal{N}=4$ SYM*'* (2010)

$$\mathcal{R}^{(2)}_{6,WL} = \text{'remainder' : a 17 pages formula !}$$

- **[GSVV]** $\mathcal{R}^{(2)}_{6,WL} = \sum_{i=1}^3 \Big(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathbf{L}\mathrm{i}_4(v_i)\Big) - \frac{1}{8}\Big(\sum_{i=1}^3 \mathbf{L}\mathrm{i}_2(v_i)\Big)^2 + \cdots$

- Relevance of simplifying $\sum_{i\in I} \mathbf{F}_i(\,x_i\,)$ for $\mathbf{F}_i$ = polylogarithms
  $\mathbf{F}_i$ = hyperlogarithms
  $\mathbf{F}_i$ = elliptic polylogs

- Justifies the study of functional identities $\sum_{j\in J} \mathbf{F}_j(\,x_j\,) = \mathrm{cst}$

[Hain - MacPherson 1990] Higher logarithms

> *The dilogarithm has properties analogous to those of the logarithm. It has been widely believed, both in the nineteenth century and more recently, that these two functions should be the first two elements of an infinite sequence of higher logarithms which share analogous properties. To date, several sequences of such functions have been proposed, but no function beyond the dilogarithm in any of these sequences is known to possess all the desired properties.*

(Similar questions were raised in **[Griffiths 2002]**, **[Gangl 2013]**, etc.)

[Goncharov-Rudenko 2018] Motivic correlator, cluster algebras...

> **Conclusion.** *If* $\boldsymbol{n > 3}$, ***the problem of writing explicitly functional equations for*** $\mathbf{Li}_{\boldsymbol{n}}$ ***might not be the "right" problem****. It seems that when n is growing the functional equations become so complicated that one cannot write them down on a piece of paper.*

- **Main problems regarding polylogarithms :**
  - Find explicit **FI** for $\mathcal{L}_n$ $\quad \big(e.g.\ \exists\ n \geq 8\ ?\ \big)$
  - $\exists$ a sequence $(\boldsymbol{FI}_n)_{n\geq 1}$ of **FI** for the polylogarithms ?
  - Better understanding of polylogarithmic **FI**

- **In this talk, we give a series of hyperlogarithmic FIs :**
  - $\mathbf{HLog}^1 \Longleftrightarrow \Big(\mathbf{Log}(\boldsymbol{x}) - \mathbf{Log}(\boldsymbol{y}) - \mathbf{Log}(\boldsymbol{x}/\boldsymbol{y}) = 0 \Big)$

    $\mathbf{HLog}^2 \Longleftrightarrow \Big(\mathbf{R}(\boldsymbol{x}) - \mathbf{R}(\boldsymbol{y}) - \mathbf{R}\Big(\frac{\boldsymbol{x}}{\boldsymbol{y}}\Big) - \mathbf{R}\Big(\frac{1-\boldsymbol{y}}{1-\boldsymbol{x}}\Big) + \mathbf{R}\Big(\frac{\boldsymbol{x}(1-\boldsymbol{y})}{\boldsymbol{y}(1-\boldsymbol{x})}\Big) = 0 \Big)$

    $\vdots$

    $\mathbf{HLog}^6 \qquad \Big($ weight 6 hyperlogarithmic **FI** $\Big)$
  - we have $\big(\mathbf{HLog}^w\big) : \sum_{i=1}^{\kappa} \boldsymbol{AH}_i^w\big(\boldsymbol{\phi_i}\big) = 0 \quad (w = 1, \ldots, 6)$

A geometric view on Abel's identity

- $(\mathcal{A}\mathbf{b})$ $$\mathsf{R}\Big(\underset{\underset{U_1}{\parallel}}{x}\Big) - \mathsf{R}\Big(\underset{\underset{U_2}{\parallel}}{y}\Big) - \mathsf{R}\Big(\underset{\underset{U_3}{\parallel}}{\frac{x}{y}}\Big) - \mathsf{R}\Big(\underset{\underset{U_4}{\parallel}}{\frac{1-y}{1-x}}\Big) + \mathsf{R}\Big(\underset{\underset{U_5}{\parallel}}{\frac{x(1-y)}{y(1-x)}}\Big) = 0$$

- **Blow-up** $\beta : X_4 = \mathrm{Bl}_{p_1,\ldots,p_4}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$

$$\begin{tikzcd}
\mathbf{X}_4 \arrow[d, "\beta"'] \arrow[rr, bend left, "\phi_i"] & & \\
\mathbb{P}^2 \arrow[rr, dashed, "U_i"'] & & \mathbb{P}^1
\end{tikzcd}$$

$\phi_1, \ldots, \phi_5 : X_4 \longrightarrow \mathbb{P}^1$ **are the five fibrations by conics on the del Pezzo surface** X_4

- $(\mathcal{A}\mathbf{b}) \Longleftrightarrow \sum_{i=1}^{5} \epsilon_i \,\mathsf{R}(\phi_i) = 0$ (with $(\epsilon_i)_{i=1}^{5} \in \{\pm 1\}^5$)

Generalisation to del Pezzo surfaces

- $p_1, \ldots, p_r \in \mathbb{P}^2$: points in general position $\big(r \in \{3, \ldots, 8\} \big)$
- **Blow-up** $\beta_r : X_r = \mathrm{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$ $\big(X_r = \mathrm{dP}_{9-r} \big)$

Prop : 1. $\exists$ *a finite number* $= \kappa_r$ *of conic fibrations on* X_r $\quad \phi_1, \ldots, \phi_\kappa : X_r \longrightarrow \mathbb{P}^1$

2. *For any* i : $\Sigma_i = \mathbf{SingVal}(\phi_i) \subset \mathbb{P}^1$ *has* $r-1$ *elements*

Def$^\circ$: *The* **complete antisymmetric hyperlog. of weight $r-2$** : $AH^{r-2}_{\Sigma_i} : \widetilde{\mathbb{P}^1 \setminus \Sigma_i} \longrightarrow \mathbb{C}$

Thm [Castravet-P.] $\exists\ (\epsilon_i)_{i=1}^{\kappa} \in \{\pm 1\}^{\kappa}$, ±-unique, such that

$$\big(\mathbf{HLog}^{r-2}\big) \qquad \sum_{i=1}^{\kappa} \epsilon_i \, AH^{r-2}_{\Sigma_i}(\phi_i) = 0$$

$$\left(\mathbf{HLog}^{r-2}\right) \qquad \sum_{i=1}^{\kappa} \epsilon_i \, \boldsymbol{AH}^{r-2}_{\boldsymbol{\Sigma_i}}(\phi_i) = 0$$

- One identity $\mathbf{HLog}^{r-2}$ for each del Pezzo $\mathbf{dP}_d = \boldsymbol{X}_r$ $(d = 9 - r)$

$[\, d = 6 \,]$ $\mathbf{dP}_6$ is unique, $\boldsymbol{AH}^1_{\boldsymbol{\Sigma_i}} = \mathbf{Log}$ for every i

$$\left(\mathbf{HLog}^1\right) \; : \; \mathbf{Log}(\, \boldsymbol{x} \,) - \mathbf{Log}(\, \boldsymbol{y} \,) - \mathbf{Log}(\, \boldsymbol{x}/\boldsymbol{y} \,) = \mathbf{0}$$

$[\, d = 5 \,]$ $\mathbf{dP}_5$ is unique : $\boldsymbol{AH}^2_{\boldsymbol{\Sigma_i}} = \frac{1}{2}\Big(\mathbf{L}_{01} - \mathbf{L}_{10}\Big) = \mathbf{R}$ for every i

$$\left(\mathbf{HLog}^2\right) \; : \quad \sum_{\boldsymbol{i=1}}^{\mathbf{5}} \boldsymbol{\epsilon_i} \, \mathbf{R}(\, \phi_i \,) = \mathbf{0} \qquad (\boldsymbol{\mathcal{A}}\mathbf{b})$$

$$\boxed{\left(\mathbf{HLog}^{r-2}\right) \qquad \sum_{i=1}^{\kappa} \epsilon_i \, AH^{r-2}_{\Sigma_i}(\phi_i) = 0}$$

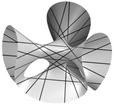
$[\, d = 4 \,]$ $\mathbf{dP}_4$ moduli ∞^2 $\rightsquigarrow$ ∞^2 identities $\mathbf{HLog}^3$

$$AH^3_1(x) + AH^3_2\left(\frac{1}{y}\right) + AH^3_3\left(\frac{y}{x}\right) + \cdots$$

$$\cdots + AH^3_9\left(\frac{y(x-b)}{x(y-a)}\right) + AH^3_{10}\left(\frac{a\,(b-x)}{by-ax}\right) = 0$$

$[\, d = 3 \,]$ $\mathbf{dP}_3$ = cubic surface in $\mathbb{P}^3$ $\rightsquigarrow$ ∞^4 identities $\mathbf{HLog}^4$

$$\sum_{i=1}^{27} AH^4_i(\phi_i) = 0$$



II Del Pezzo surfaces

- $\boldsymbol{S}$ del Pezzo $\Leftrightarrow$ $-\boldsymbol{K_S}$ ample $\longrightarrow$ $\boldsymbol{d} = \left(-\boldsymbol{K_S}\right)^2 \in \{1, \dots, 9\}$
- $\mathbf{dP}_d = \mathbf{X}_r = \mathbf{Bl}_{p_1,\dots,p_r}(\mathbb{P}^2)$ $\quad$ $\mathbf{Pic}(\mathbf{dP}_d) = \mathbb{Z}\,\boldsymbol{h} \oplus \left(\bigoplus_{i=1}^r \mathbb{Z}\,\boldsymbol{\ell}_i\right)$

  $-\boldsymbol{K}_{\mathbf{dP}_d} = 3\boldsymbol{h} - \sum_{i=1}^r \boldsymbol{\ell}_i$ ample $\rightsquigarrow$ $\varphi_{|-\boldsymbol{K}|} : \mathbf{dP}_d \hookrightarrow \mathbb{P}^d$ embedding

- $\mathbf{Pic}(\mathbf{dP}_d) \supset \boldsymbol{K}^\perp = \langle\, \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_r \,\rangle$ $\quad$ $\boldsymbol{\rho}_i = \boldsymbol{\ell}_i - \boldsymbol{\ell}_{i+1} \quad i \leq r-1$

  $\boldsymbol{\rho}_r = \boldsymbol{h} - \sum_{i=1}^3 \boldsymbol{\ell}_i$

  $-(\cdot,\cdot) + \{\,\boldsymbol{\rho}_i\,\}_{i=1}^r$ $\rightsquigarrow$ Root system $E_r \subset \mathbf{R}_r = \boldsymbol{K}^\perp \otimes \mathbb{R}$

- For any root $\boldsymbol{\rho}$: $\quad s_{\boldsymbol{\rho}} : \mathbf{R}_r \longrightarrow \mathbf{R}_r$ $\quad$ (orthog. reflection)

  $$d \longmapsto d + (d, \boldsymbol{\rho})\boldsymbol{\rho}$$

  $\boldsymbol{W}_r = \boldsymbol{W}(E_r) = \langle\, s_{\rho_1}, \dots, s_{\rho_r} \,\rangle \subset \mathrm{O}(\mathbf{R}_r)$: Weyl group of type E_r

Del Pezzo surfaces

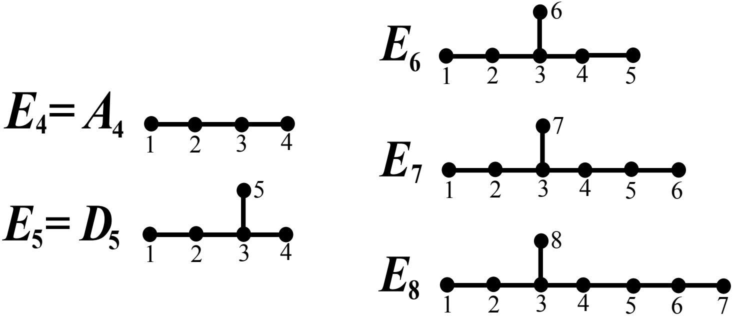


Figure – Dynkin diagram E_r $\big(k$ stands for ρ_k for any $k = 1, \ldots, r\big)$

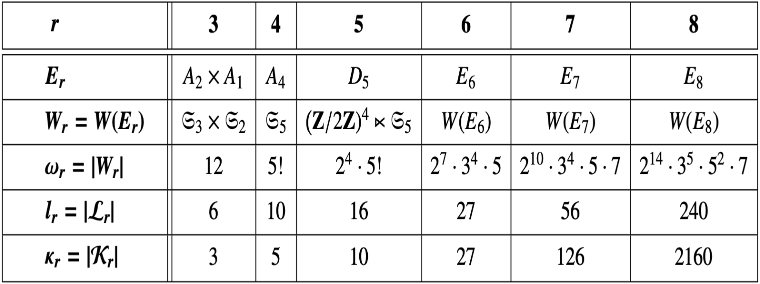
Lines and conics on $X_r = \mathrm{dP}_d$ $\quad (d = 9 - r)$

- Lines $\quad \mathcal{L}_r = \left\{ \ell \in \mathbf{Pic}(X_r) \;\middle|\; \ell \cdot K = \ell^2 = -1 \right\} = W_r \cdot \ell_r$

- Conics $\quad \mathcal{C}_r = \left\{ \mathfrak{c} \in \mathbf{Pic}(X_r) \;\middle|\; -K \cdot \mathfrak{c} = 2 \,,\; \mathfrak{c}^2 = 0 \right\} = W_r \cdot \mathfrak{c}$

  $\rotatebox{90}{$\in$}$

  $\mathfrak{c} \quad \longrightarrow \; |\,\mathfrak{c}\,| \simeq \mathbb{P}^1 \rightsquigarrow$ Conic fibration $\phi_{\mathfrak{c}} : X_r \to \mathbb{P}^1$



r	3	4	5	6	7	8
E_r	$A_2 \times A_1$	A_4	D_5	E_6	E_7	E_8
$W_r = W(E_r)$	$\mathfrak{S}_3 \times \mathfrak{S}_2$	$\mathfrak{S}_5$	$(\mathbf{Z}/2\mathbf{Z})^4 \ltimes \mathfrak{S}_5$	$W(E_6)$	$W(E_7)$	$W(E_8)$
$\omega_r = \lvert W_r \rvert$	12	5!	$2^4 \cdot 5!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
$l_r = \lvert \mathcal{L}_r \rvert$	6	10	16	27	56	240
$\kappa_r = \lvert \mathcal{K}_r \rvert$	3	5	10	27	126	2160

Reducible conics in $X_r = \mathrm{dP}_d$

- $\boldsymbol{L}_r = \cup_{\ell \in \boldsymbol{\mathcal{L}}_r} \ell \subset \boldsymbol{X}_r \quad \rightsquigarrow \quad \boldsymbol{U}_r = \boldsymbol{X}_r \setminus \boldsymbol{L}_r$
- $\mathcal{C}_r \ni \mathfrak{c} \quad \rightsquigarrow$ Conic fibration $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1$

$$\begin{aligned}\Sigma_{\mathfrak{c}} = \mathbf{SingVal}(\phi_{\mathfrak{c}}) &= \left\{ \sigma \in \mathbb{P}^1 \;\middle|\; \phi_{\mathfrak{c}}^{-1}(\sigma) \text{ non-irreducible } \right\} \\ &= \left\{ \sigma_{\mathfrak{c}}^1 \,,\, \ldots \,,\, \sigma_{\mathfrak{c}}^{r-2} \,,\, \sigma_{\mathfrak{c}}^{r-1} = \infty \right\} \subset \mathbb{P}^1\end{aligned}$$

- For $\sigma_{\mathfrak{c}}^i \in \Sigma_{\mathfrak{c}} \; : \; \phi_{\mathfrak{c}}^{-1}(\sigma_{\mathfrak{c}}^i) = \ell_{\mathfrak{c}}^i + \widetilde{\ell}_{\mathfrak{c}}^i \qquad \left(\ell_{\mathfrak{c}}^i \,,\, \widetilde{\ell}_{\mathfrak{c}}^i \in \boldsymbol{\mathcal{L}}_r \right)$
- $\mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\mathbb{P}^1 \,,\, \Omega^1_{\mathbb{P}^1}(\operatorname{Log} \Sigma_{\mathfrak{c}})\Big) = \left\langle \frac{dz}{z - \sigma_{\mathfrak{c}}^i} \right\rangle_{i=1}^{r-2} \simeq \mathbb{C}^{r-2}$

$$\wr|$$

- $\mathbf{H}_{\mathfrak{c}} = \phi_{\mathfrak{c}}^*\Big(\mathcal{H}_{\mathfrak{c}}\Big) = \left\langle \frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}} - \sigma_{\mathfrak{c}}^i} \right\rangle_{i=1}^{r-2} \subset \mathbf{H}^0\Big(\boldsymbol{X}_r \,,\, \Omega^1_{\boldsymbol{X}_r}(\operatorname{Log} \boldsymbol{L}_r)\Big) = \mathbf{H}_{\boldsymbol{X}_r}$

Iterated integrals

- **Poincaré (1884), Lappo-Danilevski (1928), Chen (1973)**

- $\boldsymbol{Y}$ complex manifold

- $\mathbf{H} = \Big\langle \omega_1 \,, \ldots , \omega_m \Big\rangle \subset \mathbf{H}^0\Big(\boldsymbol{Y}, \Omega^1_{\boldsymbol{Y}} \Big) \; + \left[\begin{array}{c} d\omega_i = 0 \\ \omega_i \wedge \omega_j = 0 \end{array} \right]$

- **Ex :** $\phi : \boldsymbol{Y} \to \boldsymbol{C}$ and $\omega_i \in \phi^*\Big(\mathbf{H}^0\big(\boldsymbol{C}, \Omega^1_{\boldsymbol{C}} \big)\Big)$ $\quad i = 1, \ldots, m$

- Base point $y \in \boldsymbol{Y}$, path $\boldsymbol{\gamma}^x = \gamma^x_y : [0,1] \to \boldsymbol{Y}$ from y to x :
  - $\mathbb{I}_{\omega_i} : x \longmapsto \int_{\gamma^x} \omega_i \quad \rightsquigarrow \quad \mathbb{I}_{\omega_i} \in \mathcal{O}_y$
  - $\mathbb{I}_{\omega_j\omega_i} : x \longmapsto \int_{\gamma^x} \omega_j(u) \cdot \mathbb{I}_{\omega_i}(u) \quad \rightsquigarrow \quad \mathbb{I}_{\omega_j\omega_i} \in \mathcal{O}_y$
  - $\mathbb{I}_{\omega_k\omega_j\omega_i} : x \longmapsto \int_{\gamma^x} \omega_k(u) \cdot \mathbb{I}_{\omega_j\omega_i}(u) \quad \rightsquigarrow \quad \mathbb{I}_{\omega_k\omega_j\omega_i} \in \mathcal{O}_y$

Iterated integrals (polylogarithms)

- $$\mathbf{II}^w \;:\; \mathbf{H}^{\otimes w} \longrightarrow \mathcal{O}_y$$

$$\underline{\omega} = \omega_{i_1} \otimes \left(\otimes_{s=2}^{w} \omega_{i_s} \right) \longmapsto \left[\; \mathbb{I}_{\underline{\omega}} \;:\; z \mapsto \int_y^z \omega_{i_1} \, \mathbb{I}_{\omega_{i_2}\cdots\omega_{i_w}} \; \right]$$

- $\mathbf{II} : \left(\bigoplus_{w\geq 0} \mathbf{H}^{\otimes w} \, , \, ш \right) \longrightarrow \mathcal{O}_y$ injective morphism of complex algebras

- $\forall\, \underline{\omega}$: $\mathbb{I}_{\underline{\omega}} \in \mathcal{O}_y \cap \widetilde{\mathcal{O}}(\boldsymbol{Y})$ has unipotent monodromy $\longrightarrow$ Symbol $\boldsymbol{\mathcal{S}}(\mathbb{I}_{\underline{\omega}}) = \underline{\omega}$

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \boldsymbol{\Sigma}$ with $\boldsymbol{\Sigma} = \{0, 1, \infty\}$

$$\mathbf{H} = \left\langle \frac{dz}{z} \, , \, \frac{dz}{1-z} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 \, , \, \Omega^1_{\mathbb{P}^1}\big(\mathrm{Log}\, \boldsymbol{\Sigma} \big) \Big)$$

$$\mathbf{Li}_n = \mathbf{II}^n\Big(\left(\tfrac{dz}{z}\right)^{\otimes(n-1)} \otimes \left(\tfrac{dz}{1-z}\right) \Big)$$ (**'Polylogarithms'**)

Iterated integrals (hyperlogarithms)

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \boldsymbol{\Sigma}$ with $\boldsymbol{\Sigma} = \left\{ \sigma^1 \, , \, \ldots \, , \, \sigma^{r-2} \, , \, \sigma^{r-1} = \infty \right\}$

$$\mathbf{H} = \left\langle \tfrac{dz}{z-\sigma^1}, \ldots, \tfrac{dz}{z-\sigma^{r-2}} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 \, , \, \Omega^1_{\mathbb{P}^1}\big(\mathrm{Log}\, \boldsymbol{\Sigma} \big) \Big)$$

$$\mathbf{II}^n\Big(\big(\tfrac{dz}{z-\sigma^{i_1}}\big) \otimes \cdots \otimes \big(\tfrac{dz}{z-\sigma^{i_n}}\big) \Big) \qquad \text{'Hyperlogarithm'}$$

- **Complete antisymmetric hyperlog of weight $r-2$ on $\mathbf{P}^1 \setminus \boldsymbol{\Sigma}$:**

$$\boldsymbol{AH}^{r-2}_{\boldsymbol{\Sigma}} = \mathbf{II}^n\left(\mathbf{Asym}\left(\left(\frac{dz}{z-\sigma^{\mathbf{1}}}\right) \otimes \cdots \otimes \left(\frac{dz}{z-\sigma^{\boldsymbol{r-2}}}\right) \right) \right)$$

$$= \mathbf{II}^n\left(\tfrac{1}{(r-2)!} \textstyle\sum_{\nu \in \mathfrak{S}_{r-2}} (-1)^{\nu} \left(\tfrac{dz}{z-\sigma^{\nu(1)}} \right) \otimes \cdots \otimes \left(\tfrac{dz}{z-\sigma^{\nu(r-2)}} \right) \right)$$

- **Ex :** $\boldsymbol{AH}^2_{\{0,1,\infty\}} = \frac{1}{2}\, \mathbf{II}^2\left(\frac{dz}{z} \otimes \frac{dz}{(1-z)} - \frac{dz}{(1-z)} \otimes \frac{dz}{z} \right) = \mathbf{R}$

Identity $\mathbf{HLog}^{r-2}$: proof(s)

$$\boxed{\Big(\mathbf{HLog}^{r-2}\Big) \ : \quad \sum\nolimits_{\mathfrak{c}\in\mathcal{C}} \epsilon_{\mathfrak{c}}\, \boldsymbol{AH}^{r-2}_{\mathfrak{c}}(\phi_{\mathfrak{c}}) = 0 \quad \text{with} \quad \boldsymbol{AH}^{r-2}_{\mathfrak{c}} = \boldsymbol{AH}^{r-2}_{\Sigma_{\mathfrak{c}}}}$$

- $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1 \supset \Sigma_{\mathfrak{c}} = \big\{\, \sigma^i_{\mathfrak{c}} \,\big\}_{i=1}^{r-1} \qquad \mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\Omega^1_{\mathbb{P}^1}\big(\,\mathrm{Log}\,\Sigma_{\mathfrak{c}}\,\big)\Big)$
- $\phi^*_{\mathfrak{c}}\big(\,\mathcal{H}_{\mathfrak{c}}\,\big) = \mathbf{H}_{\mathfrak{c}} \subset \mathbf{H}_{\boldsymbol{X}_r} = \mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\,\boldsymbol{L}_r\,\big)\Big)$
- $\boldsymbol{AH}^{r-2}_{\mathfrak{c}}(\phi_{\mathfrak{c}}) = \mathbf{II}\left(\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma^1_{\mathfrak{c}}}\right) \wedge \cdots \wedge \left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma^{r-2}_{\mathfrak{c}}}\right)\right) \in \mathbf{II}^{r-2}\Big(\,\wedge^{r-2}\,\mathbf{H}_{\mathfrak{c}}\Big)$

$$\Big\downarrow \quad \mathcal{S} \ \text{(symbol)}$$

- $\boldsymbol{\Omega}^{r-2}_{\mathfrak{c}} = \left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma^1_{\mathfrak{c}}}\right) \wedge \cdots \wedge \left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma^{r-2}_{\mathfrak{c}}}\right) \quad \in \wedge^{r-2}\mathbf{H}_{\mathfrak{c}} \subset \wedge^{r-2}\mathbf{H}_{X_r}$

$$\boxed{\Big(\mathbf{HLog}^{r-2}\Big) \qquad \Longleftrightarrow \qquad \sum\nolimits_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}^{r-2}_{\mathfrak{c}} = 0 \quad \text{in} \quad \wedge^{r-2}\mathbf{H}_{X_r}}$$

Identity HLog^{r-2} : proof(s)

$$\boxed{\textbf{Proofs of : } \; \mathbf{hlog}^{r-2} = \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}} \, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \textbf{in } \wedge^{r-2} \mathbf{H}_{X_r}}$$

$$\mathbf{H}_{\boldsymbol{X}_r} = \mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\, \boldsymbol{L}_r\,\big)\Big) \xrightarrow{\;\oplus_\ell \mathbf{Res}_\ell\;} \mathbb{C}^{\mathcal{L}_r} \qquad \text{injective}$$

$$\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} \in \wedge^{r-2} \mathbf{H}_{X_r} \lhook\joinrel\longrightarrow \wedge^{r-2}\, \mathbb{C}^{\mathcal{L}_r} \; \curvearrowleft \; \boldsymbol{W}(E_r)$$

[Pr1] One decomposes $\mathbf{hlog}^{r-2}$ in a natural basis of $\wedge^{r-2}\, \mathbb{C}^{\mathcal{L}_r}$

[Pr2] $\mathbf{sign}_r \hookrightarrow \oplus_{\mathfrak{c}} (\mathbf{H}_{\mathfrak{c}})^{\wedge(r-2)} \longrightarrow \wedge^{r-2}\, \mathbb{C}^{\mathcal{L}_r} \qquad \Big\langle\, \mathbf{sign}_r \,,\, \wedge^{r-2}\, \mathbb{C}^{\mathcal{L}_r} \Big\rangle = 0$

$\mathbf{1} \longmapsto \big(\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2}\big)_{\mathfrak{c}} \longmapsto \sum_{\mathfrak{c}} \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2}$ (GAP3)

[Pr3] Explicit descrp° of lines $\longrightarrow$ $\mathbb{Z}^{\mathcal{L}_r} \simeq \mathbb{Z}^{|\mathcal{L}_r|}$

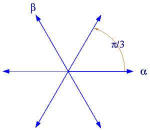
$+$ linear algebra / $\mathbb{Z}$ $\longrightarrow$ $\sum \epsilon_{\mathfrak{c}} \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0$ (Maple)

Logarithm

$\mathbf{Log}(x) + \mathbf{Log}(y) - \mathbf{Log}(xy) = 0$

$$\mathrm{dP}_6 = \mathrm{Bl}_3(\mathbb{P}^2) \subset \mathbb{P}^6$$

$A_1 \times$

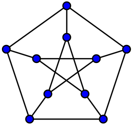


Dilogarithm

$\mathrm{R}(x) - \mathrm{R}(y) \cdots + \mathrm{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$

$$\mathrm{dP}_5 = \mathrm{Bl}_4(\mathbb{P}^2) \subset \mathbb{P}^5$$

A_4

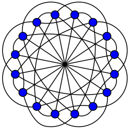


Weight 3 hyperlog

$\sum_{i=1}^{10} AH_i^3(U_i(x,y)) = 0$

$$\mathrm{dP}_4 = Q_1 \cap Q_2 \subset \mathbb{P}^4$$

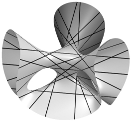
D_5



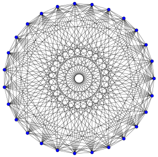
Weight 4 hyperlog

$\sum_{i=1}^{27} AH_i^4(U_i(x,y)) = 0$

$\mathrm{dP}_3 =$



E_6



Comparison

- $\boxed{\mathbf{HLog}^2}$ $\quad \mathbf{R}(x)-\mathbf{R}(y)-\mathbf{R}\left(\frac{x}{y}\right)-\mathbf{R}\left(\frac{1-y}{1-x}\right)+\mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right)=0$

- $\boxed{\mathbf{HLog}^3}$ $\quad AH^3_1(x)+AH^3_2\left(\frac{1}{y}\right)+AH^3_3\left(\frac{y}{x}\right)+AH^3_4\left(\frac{x-y}{x-1}\right)$

$$+AH^3_5\left(\frac{b(a-x)}{ay-bx}\right)+AH^3_6\left(\frac{P(x,y)}{(x-1)(y-b)}\right)+AH^3_7\left(\frac{(x-y)(y-b)}{y\,P(x,y)}\right)$$

$$+AH^3_8\left(\frac{x\,P(x,y)}{(x-y)(x-a)}\right)+AH^3_9\left(\frac{y(x-b)}{x(y-a)}\right)+AH^3_{10}\left(\frac{a\,(b-x)}{by-ax}\right)=0$$

- $\mathbf{HLog}^2$ and $\mathbf{HLog}^3$ look very similar... but are not quite the same !
  - $\mathbf{HLog}^2$ unique ***vs*** there are ∞^2 $\mathbf{HLog}^3_{a,b}$
  - only $\mathbf{R}$ in $\mathbf{HLog}^2$ ***vs*** $\exists$ several AH^3_i in $\mathbf{HLog}^3$
  - $W_{A_4}=\mathbf{Aut}(\mathrm{dP}_5)$ acts geometrically ***vs*** not the case for $W(E_5)=W_{D_5}$

III Gelfand–MacPherson correspondence

- $\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0 \quad \text{on } \mathbf{dP}_5$
- Up to the natural isomorphism $\mathbf{dP}_5 \simeq \overline{\mathcal{M}}_{0,5}$:

$$\left\{ x\,,\ \ y\ ,\ x/y\ ,\ \ldots,\ \frac{x(1-y)}{y(1-x)} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Forgetful maps} \\ f_i : \mathcal{M}_{0,5} \to \mathcal{M}_{0,4} \end{array} \right\}$$

- **Gelfand-MacPherson** (over $\mathbf{k} = \mathbb{R}, \mathbb{C}$) **:**

$$\begin{tikzcd}
G_2^o(\mathbf{k}^5) \arrow[r, "F_i"] \arrow[d, "\bullet/_{H_4}"] & G_2^o\left(\mathbf{k}^5/_{\langle e_i\rangle}\right) \arrow[l, bend right, "a_i"'] \arrow[d, "\bullet/_{H_{3,i}}"] \\
\mathcal{M}_{0,5}(\mathbf{k}) \arrow[r, "f_i"] & \mathcal{M}_{0,4}(\mathbf{k}) \ \simeq \mathbf{k} \setminus \{0,1\}
\end{tikzcd}$$

⟵ The vertical arrows
are the quotient maps
by the Cartan subgps
$(\mathbf{k}^*)^4 \simeq H_4 \subset \mathrm{GL}(\mathbf{k}^5)$
$(\mathbf{k}^*)^3 \simeq H_{3,i} \subset \mathrm{GL}\left(\mathbf{k}^5/_{\langle e_i\rangle}\right)$

Gelfand-MacPherson construction of $(\mathcal{A}b)$

$$\begin{tikzcd}
G_2^o(\mathbb{R}^5) \arrow[r, "F_i"] \arrow[d, "\nu=\nu_4"'] & G_2^o\left(\mathbb{R}^5/_{\langle e_i\rangle}\right) \arrow[l, bend right, "a_i"'] \arrow[d, "\nu_i=\nu_3"] \\
\mathcal{M}_{0,5} \arrow[r, "f_i"] & \mathcal{M}_{0,4} \simeq \mathbb{R}\setminus\{0,1\}
\end{tikzcd}$$

$\mathbf{P}_1$: 1st Pontryagin class

$\mathbf{H}^4\Big(\boldsymbol{G}_2(\mathbb{R}^5)\Big) \ni \mathbf{P}_1 = [\,\boldsymbol{\Omega}\,]$

with $\boldsymbol{\Omega} \in \Omega^4\Big(\boldsymbol{G}_2(\mathbb{R}^5)\Big)^{\mathbf{SO}_5(\mathbb{R})}$

- Fiber $\int$ of the 4-form $\boldsymbol{\Omega}$ $\longrightarrow$
  - $\omega_{0,5} = \nu_*(\boldsymbol{\Omega}) \in \Omega^0(\mathcal{M}_{0,5})$
  - $\omega_{0,4,i} = (\nu_i)_*(\boldsymbol{\Omega}_i) \in \Omega^1(\mathcal{M}_{0,4})$
- At $\nu(\xi)$: $\omega_{0,5} = \int_{\overline{H\cdot\xi}} \boldsymbol{\Omega}$ $\quad + \quad$ $\overline{\boldsymbol{H}\cdot\xi} \simeq \boldsymbol{\Delta}_2^5$ via $\boldsymbol{G}_2(\mathbb{R}^5) \xrightarrow{\mu} \boldsymbol{\Delta}_2^5$
- Stokes thm for fiber integ° : $\quad d\omega_{0,5} = \sum_{i=1}^5 (-1)^i \boldsymbol{f_i}^*(\omega_{0,4})$

$+ \quad \omega_{0,5} = 0 \qquad \omega_{0,4} = \mathrm{d}\mathbf{R} \quad \Longrightarrow \quad 0 = \sum_{i=1}^5 (-1)^i\, \boldsymbol{f_i}^*(\mathrm{d}\mathbf{R}) \quad (\mathcal{A}\mathbf{b})$

Cox varieties

- **Question :** How can one obtain $\boldsymbol{G}_2(\mathbb{C}^5)$ from $\boldsymbol{X}_4 = \mathbf{dP}_5 = \overline{\boldsymbol{\mathcal{M}}}_{0,5}$?

  Answer : $\boldsymbol{G}_2(\mathbb{C}^5)$ is the projective **Cox variety** of $\mathbf{dP}_5$

- $\boldsymbol{S}$ = projective manifold such that $\mathbf{Pic}_{\mathbb{Z}}(\boldsymbol{S}) = \oplus_{i=0}^{r} \mathbb{Z}\,\boldsymbol{\ell}_i$

  $\Big[\ \boldsymbol{S} = \mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2) \quad \boldsymbol{\ell}_0 = \boldsymbol{h} = [\,H\,] \quad \text{et} \quad \boldsymbol{\ell}_i = [\,E_i\,] \quad i = 1,\ldots,r\ \Big]$

- **Def**$^\circ$: $\mathbf{Cox}(\boldsymbol{S}) = \bigoplus_{n_0,\ldots,n_r\in\mathbb{Z}} \mathbf{H}^0\Big(\boldsymbol{S}, \mathcal{O}_{\boldsymbol{S}}(n_0H + n_1E_1 + \cdots + n_rE_r)\Big)$

- **Facts** : – $\mathbf{Cox}(\boldsymbol{S}) = \mathbb{C}[y_1,\ldots,y_m] \iff \boldsymbol{S}$ toric

  – $\mathbf{Cox}(\boldsymbol{S})$ of finite type $=$ $\boldsymbol{S}$ "Mori Dream Space" (MMP ✓)

$$\begin{tikzcd} \mathbf{Cox}(\boldsymbol{S}) \arrow[d, phantom, "\parallel"] & \\ \mathbb{C}[\Gamma_1,\ldots,\Gamma_m]/\boldsymbol{\mathcal{J}}_{\boldsymbol{S}} \arrow[r] & \mathbf{A}(\boldsymbol{S}) = \mathrm{Spec}\Big(\mathbf{Cox}(\boldsymbol{S})\Big) \subset \mathbb{A}^m_{\Gamma} \end{tikzcd}$$

« Affine Cox variety » of $\boldsymbol{S}$

Cox varieties of del Pezzo surfaces

- Surface $\boldsymbol{S} \supset \ell$ with $\ell \simeq \mathbb{P}^1$ and $\ell^2 = -1$ $\Longrightarrow$ $\sigma_\ell \in \mathbf{H}^0(\mathcal{O}_{\boldsymbol{X}_r}(\ell)) \setminus \{0\}$ generator in $\mathbf{Cox}(\boldsymbol{S})$

Thm [Batyrev, Popov] For $r = 3, \ldots, 8$, one has

$$\mathbf{Cox}\Big(\mathbf{dP}_d = \boldsymbol{X_r} = \mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2)\Big) = \mathbb{C}\Big[\,\sigma_\ell \,\big|\, \ell \in \boldsymbol{\mathcal{L}}_r\,\Big] / \boldsymbol{\mathcal{J}}_{\mathbf{dP_d}}$$

- $\mathbf{T}_{\mathsf{NS}} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{Pic}_{\mathbb{Z}}(\boldsymbol{X}_r), \mathbb{C}^*) \circlearrowleft \mathbf{A}(\boldsymbol{X}_r) \rightsquigarrow \quad \boldsymbol{X}_r = \mathbf{A}(\boldsymbol{X}_r) /\!/ \, \mathbf{T}_{\mathsf{NS}}$

- $\mathbf{A}(\boldsymbol{X}_r) \hookrightarrow \mathbb{C}^{\mathcal{L}_r}$ + $\mathbb{Z}$-grading on $\mathbf{Cox}(\boldsymbol{X_r})$ induced by $(-K, \cdot)$

- **Def°** : The **Projective Cox variety** of $\boldsymbol{X}_r$ is

$$\mathbf{P}(\boldsymbol{X}_r) = \mathrm{Proj}\Big(\mathbf{Cox}(\boldsymbol{X_r})\Big) \subset \mathbb{P}\Big(\mathbb{C}^{\mathcal{L}_r}\Big) \circlearrowleft \mathcal{T}_{\mathsf{NS}} = \mathbf{T}_{\mathsf{NS}}/_{\mathbb{C}^*}$$

Cox varieties of del Pezzo surfaces

- $\mathbf{P}(\boldsymbol{X}_r) = \mathrm{Proj}\Big(\mathbf{Cox}(\boldsymbol{X_r})\Big) \subset \mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}) \circlearrowright \left.\begin{matrix} \boldsymbol{\mathcal{T}}_{\mathrm{NS}} = \mathbf{T}_{\mathrm{NS}}/\mathbb{C}^* \\ \boldsymbol{W}(E_r) \end{matrix}\right\} \rightsquigarrow \boldsymbol{G}(E_r)$

Thm **[Batyrev, Popov, Derenthal, Serganova-Skorobogatov]**

1. $\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ is a **minuscule** representation of $\boldsymbol{G}_r = \boldsymbol{G}(E_r)$
2. There is a $\boldsymbol{\mathcal{T}}_{\mathrm{NS}} = \boldsymbol{H}_r$-equivariant embedding
$$\mathbf{P}(\boldsymbol{X}_r) \hookrightarrow \boldsymbol{G}_r/\boldsymbol{P}_r \subset \mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{L}}_r})$$
3. There is an embedding $f_{SS} : \boldsymbol{X}_r \hookrightarrow \boldsymbol{\mathcal{Y}}_r = \big(\boldsymbol{G}_r/\boldsymbol{P}_r\big)/\!\!/\boldsymbol{H}_r$ st
$$\begin{tikzcd} \mathbf{P}(\boldsymbol{X}_r) \arrow[r, hook] \arrow[d] & \boldsymbol{G}_r/\boldsymbol{P}_r \subset \mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}) \arrow[d] \\ \mathbf{dP}_{9-r} = \boldsymbol{X}_r \arrow[r, hook, "f_{SS}"] & \boldsymbol{\mathcal{Y}}_r \end{tikzcd}$$
is commutative
4. There is an isomorphism $\boldsymbol{W}(E_r) \simeq \mathbf{Aut}(\boldsymbol{\mathcal{Y}}_r)$

- $r=4$, $\boldsymbol{A_4}$: $\mathbf{P}(\boldsymbol{X}_4) = \boldsymbol{G}_4/\boldsymbol{P}_4 = \boldsymbol{G}_2(\mathbb{C}^5) \hookrightarrow \mathbb{P}(\mathbb{C}^{\mathcal{L}_4}) \simeq \mathbb{P}^9$ (Plücker)

$$\begin{tikzcd}
\mathbf{P}(\boldsymbol{X}_4) \arrow[r, equal] \arrow[d] & \boldsymbol{G}_2(\mathbb{C}^5) \arrow[r, dashed, "F_i"] \arrow[d] & \boldsymbol{G}_2(\mathbb{C}^4_i) \arrow[d] \\
\mathbf{dP}_5 = \boldsymbol{X}_4 \arrow[r, hook, "f_{SS}", "\sim"'] & \boldsymbol{\mathcal{Y}}_4 = \overline{\boldsymbol{\mathcal{M}}}_{0,5} \arrow[r, dashed, "f_i"] & \overline{\boldsymbol{\mathcal{M}}}_{0,4} \simeq \mathbb{P}^1
\end{tikzcd}$$

$\longrightarrow$ *'Everything comes from the right square diagrams'*

- For $r = 4, \ldots, 7$ we have

$$\begin{tikzcd}
\mathbf{P}(\boldsymbol{X}_r) \arrow[r, hook] \arrow[d] & \boldsymbol{G}_r/\boldsymbol{P}_r \arrow[d] \\
\mathbf{dP}_d = \boldsymbol{X}_r \arrow[r, hook, "f_{SS}"] & \boldsymbol{\mathcal{Y}}_r
\end{tikzcd}$$

- **Question :** Can the pencils of conics on $\mathbf{dP}_d$ and the identity $\mathbf{HLog}^{r-2}$ be obtained from $\boldsymbol{G}_r/\boldsymbol{P}_r$ and $\boldsymbol{\mathcal{Y}}_r$?

Face maps

- $\mathbb{C}^{\mathcal{L}_r}$ is a minuscule representation of G

  $G/P \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}}) =$ minuscule G-homogenous projective variety

  H Cartan subgroup of G with Lie algebra $\mathfrak{h} \simeq \mathbb{C}^r$

- **Fact** : The set of weights identifies with $\mathcal{L}_r$

- **Def°** : **Weight polytope :** $\boldsymbol{\Delta} = \mathrm{Conv}(\mathcal{L}_r) \subset \mathfrak{h}^*_{\mathbb{R}} \simeq \mathbb{R}^r$

- Given a facet $F \subset \Delta$ $\rightsquigarrow$ Linear projection $\Pi_F : \mathbb{C}^{\mathcal{L}_r} \to \mathbb{C}^F$

Prop. For any facet $F \subset \Delta$, the proj° Π_F induces a rational map

$$\Pi_F : G/P \dashrightarrow \mathbb{P}(\mathbb{C}^F) \cap G/P = G_F/P_F$$

with G_F/P_F minuscule of rank $r-1$

Face maps

- **Def**$^\circ$: $\Pi_{\boldsymbol{F}} : \boldsymbol{G}/\boldsymbol{P} \dashrightarrow \boldsymbol{G_F}/\boldsymbol{P_F}$ is the **face map** associated to $\boldsymbol{F}$

- There exists a quotient map $\pi_{\boldsymbol{F}}$ such that the diagram

$$\begin{tikzcd}
\boldsymbol{G}/_{\boldsymbol{P}} \arrow[r, "\Pi_F"] \arrow[d, "\nu_{\boldsymbol{H}}"'] & \boldsymbol{G}_F/_{\boldsymbol{P}_F} \arrow[d, "\nu_{\boldsymbol{H}_F}"] \\
\boldsymbol{\mathcal{Y}} \arrow[r, "\pi_F"] & \boldsymbol{\mathcal{Y}}_F
\end{tikzcd}$$

is commutative

- For each $\boldsymbol{F}$:

$$\begin{tikzcd}
\mathbf{P}(\boldsymbol{X}_r) \arrow[r, hook] \arrow[d] & \boldsymbol{G_r}/_{\boldsymbol{P_r}} \arrow[r, dashed, "\Pi_F"] \arrow[d] & \mathbb{Q}_F^{2r-4} = \boldsymbol{G}_F/\boldsymbol{P}_F \arrow[d] \\
\mathbf{dP}_{9-r} = \boldsymbol{X}_r \arrow[r, hook, "f_{SS}"] & \boldsymbol{\mathcal{Y}}_r \arrow[r, dashed, "\pi_F"] & \boldsymbol{\mathcal{Y}}_F \simeq \mathbb{P}^{r-3}
\end{tikzcd}$$

Prop. The $\pi_F \circ \boldsymbol{f_{SS}}$'s give the pencils of conics on $\boldsymbol{X}_r$

Differential identities $\mathbf{HLOG}_{\mathcal{Y}_r}$

$$\begin{tikzcd}
 & \mathbf{P}(X_r) \arrow[r, hook] \arrow[d] & G_r/P_r \arrow[r, dashed, "\Pi_F"] \arrow[d] & \mathbb{Q}_F^{2r-4} = G_F/P_F \arrow[d] \\
\mathbf{dP}_{9-r} = & X_r \arrow[r, hook, "f_{SS}"] & \mathcal{Y}_r \arrow[r, dashed, "\pi_F"] & \mathcal{Y}_F \simeq \mathbb{P}^{r-3}
\end{tikzcd}$$

- Setting $\mathrm{U} = 1 + \sum_i u_i$, we define a multivalued $(r-3)$-form on $\mathbb{P}^{r-3}$

$$\Omega_r = \mathbf{Asym}_{u_1,\ldots,u_{r-3},\mathrm{U}}\left(\mathrm{Log}(\mathrm{U})\Big(\frac{du_1}{u_1}\wedge\cdots\wedge\frac{du_{r-3}}{u_{r-3}}\Big)\right)$$

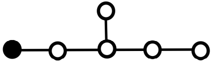
Thm. **1.** The $\pi_F \circ f_{SS}$'s give the pencils of conics on X_r

2. We have $\mathbf{HLOG}_{\mathcal{Y}_r}$: $\sum_F \pi_F^*\Big(\Omega_r\Big) = 0$

3. $\mathbf{HLOG}_{\mathcal{Y}_r} \rightsquigarrow \mathbf{HLog}^{r-2}_{\mathbf{dP}_{9-r}}$ for any $\mathbf{dP}_{9-r}$

Case $r = 6$: type E_6

Dynkin diagram



Minuscule representation

$$\mathbf{Herm}_3(\mathbb{O}) = \oplus_{i=1}^{27} \mathbb{C} e_i$$

highest weight vector $e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- **Minuscule homog. space :** Cayley plane $E_6 / P_1 = \mathbb{O}\mathbf{P}^2 = \overline{\mathbf{Im}(v_2)}$

$$v_2 : \mathbb{O}^2 \hookrightarrow \mathbb{P}\mathbf{Herm}_3(\mathbb{O}),\ (x, y) \longmapsto \begin{bmatrix} 1 & x & y \\ \overline{x} & x\overline{x} & yx \\ \overline{y} & \overline{yx} & y\overline{y} \end{bmatrix}$$

- **Weight polytope :** Gosset $\mathbf{2_{21}}$ 99 facets : $\begin{cases} \text{27 orthoplexes (type } (D_5, \omega_1)) \\ \text{72 simplices (type } A_5) \end{cases}$

- **Face maps on $\mathbb{O}\mathbf{P}^2$:** F = a D_5-facet : $\mathbb{O}\mathbf{P}^2_F = \mathbb{O}\mathbf{P}^2 \cap \mathbb{P}(E_F^{10}) = \mathbb{O}\mathbf{P}^1$

  $\rightsquigarrow$ Π_F is equivalent to $\mathbb{O}^2 \to \mathbb{O},\ (x, y) \mapsto x$

- We can recover the 27 pencils of conics on a cubic surface $\mathbf{dP}_3$ and the identity $\mathbf{HLog}^4_{\mathbf{dP}_3}$ from the 27 faces maps $\mathbb{O}\mathbf{P}^2 \dashrightarrow \mathbb{O}\mathbf{P}^1 \simeq \mathbb{Q}^8$

Thank you very much for your attention

ご清聴ありがとうございました。