CUSPIDAL ℓ-MODULAR REPRESENTATIONS OF $\mathrm{GL}_n(F)$ DISTINGUISHED BY A GALOIS INVOLUTION, II

by

Robert Kurinczuk, Nadir Matringe & Vincent Sécherre

Abstract. — Let F/F_0 be a quadratic extension of non-Archimedean locally compact fields with residual characteristic $p \neq 2$, and ℓ be a prime number different from p. We classify those ℓ-modular cuspidal irreducible representations of $\mathrm{GL}_n(F)$ which are $\mathrm{GL}_n(F_0)$-distinguished, that is, which carry a non-zero $\mathrm{GL}_n(F_0)$-invariant linear form. In the case when $\ell \neq 2$, an ℓ-modular cuspidal representation of $\mathrm{GL}_n(F)$ is $\mathrm{GL}_n(F_0)$-distinguished if and only if it lifts to a $\mathrm{GL}_n(F_0)$-distinguished cuspidal ℓ-adic representation, whereas when $\ell = 2$, it is $\mathrm{GL}_n(F_0)$-distinguished if and only if it is conjugate-self-dual.

2010 Mathematics Subject Classification: 22E50, 11F70, 20C20
Keywords and Phrases: Cuspidal representations, Distinction, Lifting, Modular representations, Rankin–Selberg gamma factors

1. Introduction

1.1. Let F/F_0 be a quadratic extension of non-Archimedean locally compact fields with residual characteristic $p \neq 2$. Fix a positive integer n and set $G = \mathrm{GL}_n(F)$ and $H = \mathrm{GL}_n(F_0)$. Let R be an algebraically closed field of characteristic $\ell > 0$ different from p. In [**24**], we addressed the following problem.

Problem 1.1. — *Classify the cuspidal, irreducible, smooth R-representations of G which are distinguished by H, that is, which carry a non-zero H-invariant linear form.*

In this article, we give a complete answer to Problem 1.1. The classification can be stated in very simple terms. However, the case where $\ell \neq 2$ and the case where $\ell = 2$ are completely different. Let us explain what happens.

1.2. Let us first briefly recall what happens for complex representations. Let σ be the non-trivial automorphism of F/F_0 and η be the character of $F_0^\times$ with kernel the subgroup of F/F_0-norms of $F_0^\times$. One has the three following facts:

(*Conjugate-self-duality*) Any irreducible complex representation of G distinguished by H is σ-self-dual, that is, its contragredient is isomorphic to its conjugate by σ ([**10**] Proposition 12).

(*Dichotomy and Disjunction*) A σ-self-dual cuspidal complex irreducible representation π of G is distinguished either by H or by η (the latter means that $\mathrm{Hom}_H(\pi, \eta \circ \det)$ is non-zero) but not both ([**18**] Theorem 7, [**1**] Corollary 1.6 if F has characteristic 0, and [**2**] Theorem A.2, [**17**] if F has characteristic p[(1)]).

(*Parametrisation*) Cuspidal complex (irreducible) representations of G distinguished by H are classified in terms of their Langlands parameter ([**12**] Theorem 6.2): they are the σ-self-dual cuspidal representations whose parameter is conjugate-orthogonal in the sense of [**11**] Section 3.

1.3. Let us now consider representations with coefficients in R. For simplicity, we will assume in this introduction that R is an algebraic closure $\overline{\mathbb{F}}_\ell$ of a finite field of cardinality ℓ. First, as in the complex case, any H-distinguished irreducible $\overline{\mathbb{F}}_\ell$-representation of G is σ-self-dual ([**35**] Theorem 4.1). Now let us focus on cuspidal representations.

Recall that over $\overline{\mathbb{F}}_\ell$, the two notions of *cuspidal* and *supercuspidal* representations do not coincide ([**37**]), contrary to the case of complex representations: an irreducible representation of G is cuspidal (respectively, supercuspidal) if it is not isomorphic to a subrepresentation (respectively, a subquotient) of a representation induced from a proper parabolic subgroup of G. Any supercuspidal representation is cuspidal, but the converse does not hold in general.

Let us also fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers and identify the residue field of its ring of integers $\overline{\mathbb{Z}}_\ell$ with $\overline{\mathbb{F}}_\ell$. In this situation, there is a notion of *integral* $\overline{\mathbb{Q}}_\ell$-representation of G and such a representation can be *reduced mod* ℓ, which gives rise to a semi-simple $\overline{\mathbb{F}}_\ell$-representation of finite length of G (see §2.4 for precise definitions). One then defines a $\overline{\mathbb{Q}}_\ell$-*lift* of an irreducible $\overline{\mathbb{F}}_\ell$-representation π of G as an integral $\overline{\mathbb{Q}}_\ell$-representation of G whose reduction mod ℓ is isomorphic to π.

Cuspidal representations of the group G behave well with respect to reduction and lifting: the reduction mod ℓ of an integral cuspidal $\overline{\mathbb{Q}}_\ell$-representation is irreducible and cuspidal, and any cuspidal $\overline{\mathbb{F}}_\ell$-representation has a $\overline{\mathbb{Q}}_\ell$-lift (which is automatically cuspidal) ([**37**] III.1.1, III.5.10). Reduction mod ℓ thus defines a surjective map from isomorphism classes of integral cuspidal $\overline{\mathbb{Q}}_\ell$-representations of G to isomorphism classes of cuspidal $\overline{\mathbb{F}}_\ell$-representations of G.

Moreover, reduction mod ℓ of integral cuspidal $\overline{\mathbb{Q}}_\ell$-representations preserves H-distinction: the reduction mod ℓ of an H-distinguished integral cuspidal $\overline{\mathbb{Q}}_\ell$-representation is H-distinguished ([**23**] Theorem 3.4). (Equivalently, any cuspidal $\overline{\mathbb{F}}_\ell$-representation of G with an H-distinguished $\overline{\mathbb{Q}}_\ell$-lift is H-distinguished.) Conversely, any H-distinguished supercuspidal representation of G has an H-distinguished $\overline{\mathbb{Q}}_\ell$-lift ([**35**] Theorem 10.11, see also [**9**] Theorem 3.4 for a more general statement). There is also a classification of the non-supercuspidal, cuspidal $\overline{\mathbb{F}}_\ell$-representations of G having an H-distinguished $\overline{\mathbb{Q}}_\ell$-lift ([**24**] Propositions 6.1, 6.2). We will come back to it in §1.6.

We show that, surprisingly, assuming that $\ell \neq 2$, this Distinguished Lift Property of supercuspidal representations extends to all cuspidal representations. Our first main theorem is:

Theorem 1.2. — *Assume that $\ell \neq 2$. Then a cuspidal $\overline{\mathbb{F}}_\ell$-representation of G is H-distinguished if and only if it has an H-distinguished $\overline{\mathbb{Q}}_\ell$-lift.*

Together with our classification of the cuspidal $\overline{\mathbb{F}}_\ell$-representations of G having an H-distinguished $\overline{\mathbb{Q}}_\ell$-lift, this thus gives a complete classification of all cuspidal $\overline{\mathbb{F}}_\ell$-representation of G distinguished by H in the case when $\ell \neq 2$.

(1) In characteristic p, there is an issue in the proof of [**2**] Theorem A.1. This is explained in [**17**] p. 303–304 and corrected by [**17**] Theorem 4.7.

We automatically deduce from Theorem 1.2 the following Disjunction Theorem: a cuspidal $\overline{\mathbb{F}}_\ell$-representation of G cannot be both distinguished and η-distinguished (see Corollary 3.19). Observe that Dichotomy holds for supercuspidal $\overline{\mathbb{F}}_\ell$-representations ([**35**] Theorem 10.8) but not for cuspidal ones (see Remark 3.20).

1.4. When $\ell = 2$, the Distinguished Lift Property of H-distinguished cuspidal representations of G does not hold: there are H-distinguished cuspidal representations with no H-distinguished lift (see Remark 3.2). However, the classification of all H-distinguished cuspidal $\overline{\mathbb{F}}_2$-representations of G turns out to be even easier to state.

As has been said in §1.3, any H-distinguished irreducible $\overline{\mathbb{F}}_2$-representation of G is σ-self-dual. Conversely, a supercuspidal $\overline{\mathbb{F}}_2$-representation of G is distinguished by H if and only if it is σ-self-dual ([**35**] Theorem 10.8). (Note that there is no $\overline{\mathbb{F}}_2$-character of $F_0^\times$ with kernel the subgroup of F/F_0-norms.)

We prove that, surprisingly again, this characterisation of distinction for supercuspidal representations extends to all $\overline{\mathbb{F}}_2$-cuspidal representations, in contrast with the case where $\ell \neq 2$. Our second main theorem is:

Theorem 1.3. — *A cuspidal $\overline{\mathbb{F}}_2$-representation of G is distinguished by H if and only if it is σ-self-dual.*

1.5. Let us now explain the mains ideas of our proof of our main theorems, starting with Theorem 1.2. In the complex case, studying the distinction of a quotient of a parabolically induced representation is a notoriously non-trivial problem. In our situation, where representations have coefficients in $\overline{\mathbb{F}}_\ell$, the situation is even worse: a non-supercuspidal, cuspidal representation of G cannot be realized as a quotient of a proper parabolically induced representation, although it is a subquotient of a proper parabolically induced representation. Also, the analytic tools that have been developed in the complex case to study distinction are no longer available in our modular case.

First, thanks to the results of [**24**], we reduce to the case of $\overline{\mathbb{F}}_\ell$-cuspidal representations of level 0. Secondly, our strategy is by contradiction: we assume that there is a cuspidal representation π of level 0 which is distinguished but has no distinguished cuspidal lift. We then compute a Rankin–Selberg γ-factor associated with π in two different ways. The first computation gives -1 and the second one gives 1. Since $-1 \neq 1$ in $\overline{\mathbb{F}}_\ell$ when $\ell \neq 2$, we get the expected contradiction. Note that the idea of using γ-factors in distinction problems goes back to Hakim [**14**] – see also Hakim–Offen [**15**].

These two computations rely on very different ideas. The first one relies on the lifting properties of the Rankin–Selberg γ-factor of [**22**] together with the classification of H-distinguished non-supercuspidal, cuspidal $\overline{\mathbb{F}}_\ell$-representations of G of [**24**]. The second one relies on a ℓ-modular, finite field variant of a theorem of Ok [**33**] on H-distinguished cuspidal complex representations of G. Let us give more details.

1.6. Let π be a cuspidal representation of G. According to [**29**], there are a divisor r of n and a supercuspidal representation ρ of $\mathrm{GL}_{n/r}(F)$ such that π is isomorphic to the unique generic irreducible subquotient of the induced representation $\rho\nu^{(1-r)/2} \times \cdots \times \rho\nu^{(r-1)/2}$, where ν is the absolute value of the determinant and $\times$ denotes normalized parabolic induction. Since our problem is solved for supercuspidal representations, we may and will assume that π is not supercuspidal, that is, $r \geqslant 2$. One then can form the Rankin–Selberg γ-factor $\gamma(X, \pi, \rho^\vee, \psi)$ with respect to any non-trivial smooth $\overline{\mathbb{F}}_\ell$-character ψ of F (see [**22**]). This is a non-zero element of the fraction field

$\overline{\mathbb{F}}_\ell(X)$ which we may evaluate at $q^{-1/2}$, where $q^{1/2}$ is a square root in $\overline{\mathbb{F}}_\ell$ of the cardinality q of the residue field of F. (In the case when F/F_0 is unramified, we always choose for $q^{1/2}$ the cardinality of the residue field of F_0.) A simple use of the lifting properties of this local factor (see Proposition 4.3) gives us

$$\gamma(q^{-1/2}, \pi, \rho^\vee, \psi) = \omega_\pi(-1)^{n/r-1} \cdot (-1)^r \cdot q^{n/2} \tag{1.1}$$

where ω_π is the central character of π. Assuming that π is distinguished but has no distinguished lift, and using our classification of the cuspidal $\overline{\mathbb{F}}_\ell$-representations of G having an H-distinguished lift ([**24**] Propositions 6.1, 6.2), we obtain

$$\gamma(q^{-1/2}, \pi, \rho^\vee, \psi) = -1 \tag{1.2}$$

for any non-trivial ψ (see Proposition 3.15). Let us stress that this classification is crucial in the proof of (1.2). It is stated in [**24**] in terms of type theoretic invariants of cuspidal representations, but it takes a simpler form for representations of level 0 (see §3.6). Assuming that there exists a distinguished cuspidal representation with no distinguished lift, it gives some information on the order of the cardinality of the residue field of F_0 mod ℓ (see §3.9) which allows us to compute $q^{n/2}$.

1.7. We now have to find another way of computing the γ-factor of §1.6. Let $\boldsymbol{k}$ denote the residue field of F. Let V be a cuspidal irreducible representation of $\mathrm{GL}_n(\boldsymbol{k})$ and W be any generic irreducible representation of $\mathrm{GL}_m(\boldsymbol{k})$ for some positive integer $m < n$. Associated with any non-trivial character Ψ of $\boldsymbol{k}$, there is a γ-factor $\gamma(\mathsf{V}, \mathsf{W}, \Psi)$ defined in [**4**]. We show that:

- it is compatible with reduction mod ℓ (see Proposition 5.10),
- it is multiplicative in the second variable (see Proposition 5.13).

Now let $\overline{\pi}$ denote the type of the cuspidal representation π of §1.6. This is the representation of $\mathrm{GL}_n(\boldsymbol{k})$ defined by the action of $\mathrm{GL}_n(\mathcal{O}_F)$ on the vectors of π that are fixed by $1+\mathbf{M}_n(\mathfrak{p}_F)$, where $\mathcal{O}_F$ and $\mathfrak{p}_F$ are the ring of integers of F and its maximal ideal. It is irreducible and cuspidal. Similarly, ρ has a type $\overline{\rho}$, which is a supercuspidal, irreducible representation of $\mathrm{GL}_{n/r}(\boldsymbol{k})$, and $\overline{\pi}$ is isomorphic to the unique generic irreducible subquotient of $\overline{\rho} \times \cdots \times \overline{\rho}$. Moreover, if we let $\boldsymbol{k}_0$ denote the residue field of F_0 and write

$$\overline{H} = \begin{cases} \mathrm{GL}_n(\boldsymbol{k}_0) & \text{if } F/F_0 \text{ is unramified,} \\ \mathrm{GL}_{n/2}(\boldsymbol{k}) \times \mathrm{GL}_{n/2}(\boldsymbol{k}) & \text{if } F/F_0 \text{ is ramified,} \end{cases} \tag{1.3}$$

(note that n is always even in the latter case), it follows from [**24**] that $\overline{\pi}$ is $\overline{H}$-distinguished with no $\overline{H}$-distinguished $\overline{\mathbb{Q}}_\ell$-lift. Assuming that the character ψ of §1.6 has conductor $\mathfrak{p}_F$, we deduce from [**32**] Theorem 3.11 that

$$\gamma(q^{-1/2}, \pi, \rho^\vee, \psi) = \gamma(\overline{\pi}, \overline{\rho}^\vee, \Psi) \tag{1.4}$$

where Ψ is the character of $\boldsymbol{k}$ induced by the restriction of ψ to $\mathcal{O}_F$. We are thus reduced to computing $\gamma(\overline{\pi}, \overline{\rho}^\vee, \Psi)$, which entirely pertains to the theory of representations of finite general linear groups.

1.8. We now change the notation: for any integer $n \geqslant 1$, we set $G_n = \mathrm{GL}_n(\boldsymbol{k})$ and let H_n denote the subgroup defined by (1.3) (we actually use a non-standard Levi subgroup of $\mathrm{GL}_n(\boldsymbol{k})$ in the Levi case, and also need to define H_n in the case when n is odd: see Section 6). Let us fix a non-supercuspidal, cuspidal $\overline{\mathbb{F}}_\ell$-representation π of G_n. It occurs as the unique generic irreducible subquotient of the induced representation $\rho \times \cdots \times \rho$ for some supercuspidal $\overline{\mathbb{F}}_\ell$-representation ρ of

$G_{n/r}$ for some divisor r of n. Let ψ be the character denoted Ψ in §1.7. We prove that

$$\gamma(\pi, \rho^\vee, \psi) = 1 \tag{1.5}$$

when π is H_n-distinguished with no H_n-distinguished lift (see Proposition 3.18). Note that even if this γ-factor does not depend on the choice of ψ, it is convenient to compute it for a specific choice of ψ, namely, a ψ trivial on $\boldsymbol{k}_0$ when $H_n = \mathrm{GL}_n(\boldsymbol{k}_0)$, which we assume in Section 6.

First, associated with π, there is a scalar $c(\pi, \psi) \in \overline{\mathbb{F}}_\ell^\times$, which is a proportionality constant between two explicit non-zero H_n-invariant linear forms on the Whittaker model of π (see Proposition 6.8). Secondly, we prove that, for any H_{n-1}-distinguished representation π' of G_{n-1} of Whittaker type (see Definition 5.1) satisfying a technical condition (6.7) (see Definition 6.12), one has

$$\gamma(\pi, \pi', \psi) = c(\pi, \psi).$$

In particular, this γ-factor does not depend on π'. We call a distinguished representation of Whittaker type satisfying (6.7) *special*. We then show that the class of special representations is large enough: it contains all distinguished cuspidal irreducible representations and is stable under parabolic induction (see Corollary 6.17 and Lemma 6.18). This allow us to prove that

$$\gamma(\pi, \rho^\vee, \psi) = \gamma(\pi, \psi)^{n/r} \tag{1.6}$$

where $\gamma(\pi, \psi)$ is the Godement–Jacquet γ-factor of π. We then use **[25]**, where we have computed $\gamma(\pi, \psi)$ (see Proposition 5.16). The expected result (1.5) follows.

This strategy for computing $\gamma(\pi, \rho^\vee, \psi)$, which relies on the introduction of the constant $c(\pi, \psi)$ is reminiscent of Ok **[33]**. However, Ok's strategy cannot be directly adapted to our modular setting. The main novelty in our approach is to introduce the class $\mathcal{C}(H)$ (see Definition 6.13), which gives a sufficient condition for being special, and to check that it contains enough representations to make Ok's strategy work over finite fields.

1.9. Let us now explain the main ideas of our proof of Theorem 1.3. It is based on the following observation: if π is a non-supercuspidal, cuspidal $\overline{\mathbb{F}}_2$-representation of $\mathrm{GL}_n(\boldsymbol{k})$, then n is even and there is a unique cuspidal $\overline{\mathbb{F}}_2$-representation τ of $\mathrm{GL}_{n/2}(\boldsymbol{k})$ such that π is the unique generic subquotient of the induced representation $\tau \times \tau$. By uniqueness, π is σ-self-dual if and only if τ is σ-self-dual. Since Theorem 1.3 is known to hold for supercuspidal representations, it thus suffices to prove that, if τ is distinguished, then π is distinguished.

For this, we examine the induced representation $\tau \times \tau$. It is indecomposable of length 3. Assuming that τ is distinguished, we construct two independent invariant linear forms on the representation $\tau \times \tau$ vanishing on its socle. If π were not distinguished, one would thus get two independent invariant $\overline{\mathbb{F}}_2$-linear forms on the cosocle of $\tau \times \tau$. This contradicts Proposition 7.7, which says that the dimension of the space of invariant linear forms on this cosocle is at most 1. Theorem 1.3 follows.

1.10. Let us make a couple of comments. It is tempting to conjecture that Theorem 1.2 holds in greater generality. For other involutions of the group $\mathrm{GL}_n(F)$, or more generally for involutions of inner forms of $\mathrm{GL}_n(F)$, there is some hope indeed. It seems plausible as well that a result similar to Theorem 1.3 holds for any involution of any inner form of $\mathrm{GL}_n(F)$. Antonin Casel is investigating these questions in his ongoing PhD thesis at the University of Versailles.

For reductive p-adic groups other than general linear groups, an obstruction to a naive generalisation of Theorem 1.2 is already apparent in the group case: given a connected reductive p-adic group H, diagonally embedded in $G = H \times H$, and an irreducible $\overline{\mathbb{F}}_\ell$-representation π of H, the fact that $\pi\otimes\pi^\vee$ lifts to an irreducible $\overline{\mathbb{Q}}_\ell$-representation of G distinguished by H is equivalent to the fact that π has a $\overline{\mathbb{Q}}_\ell$-lift. But this latter fact does not hold in general (see for instance [**21**] Remark 5.5 for unramified unitary groups in three variables, which have cuspidal $\overline{\mathbb{F}}_\ell$-representations which do not lift to $\overline{\mathbb{Q}}_\ell$).

Acknowledgements

Robert Kurinczuk was supported by the Engineering and Physical Sciences Research Council (EPSRC) grant EP/V001930/1. Nadir Matringe was supported by the Research Start-up Fund of the Shanghai Institute for Mathematics and Interdisciplinary Sciences (SIMIS). Vincent Sécherre was partly supported by the Institut Universitaire de France.

We thank Alberto Mínguez and Shaun Stevens for their interest in this project.

2. Notation

2.1. Given a locally compact, totally disconnected topological group G and an algebraically closed field R, we consider smooth representations of G on R-vector spaces. We abbreviate *smooth R-representation* to *R-representation*, or even *representation* when R is clear from the context.

An *R-character* (or *character*) of G is a group homomorphism from G to $R^\times$ with open kernel.

Given a representation π and a character χ of G, we write $\pi^\vee$ for the contragredient of π and $\pi\chi$ for the representation $g\mapsto\chi(g)\pi(g)$ of G. If π has a central character, we denote it by ω_π.

Given a vector v in the space of π and a linear form ξ in the space of $\pi^\vee$, we write $f_{v,\xi}$ for the matrix coefficient $g\mapsto\xi(\pi(g)v)$ on G.

Given any closed subgroup H of G and any character μ of H, we say that a representation π of G is *μ-distinguished* if the space $\mathrm{Hom}_H(\pi,\mu)$ is non-zero. If μ is the trivial character, we abbreviate *μ-distinguished* to *H-distinguished*, or just *distinguished* when H is clear from the context.

2.2. Let F be a non-archimedean locally compact field of residue characteristic p. We will write $\mathcal{O}$ for its ring of integers, $\mathfrak{p}$ for the maximal ideal of $\mathcal{O}$, $\boldsymbol{k}$ for its residue field and q for the cardinality of $\boldsymbol{k}$.

Given an algebraically closed field R of characteristic different from p and a square root $q^{1/2}$ of q in R, we write $\nu^{1/2}$ for the unramified R-character of $F^\times$ which sends any uniformizer of F to $q^{-1/2}$ and ν for the square of $\nu^{1/2}$, which is unramified and sends any uniformizer of F to q^{-1}.

2.3. Let R be an algebraically closed field of characteristic different from p, and $n\geqslant 1$ be an integer. Given r integers $n_1,\dots,n_r\geqslant 1$ such that $n_1+\dots+n_r=n$ and, for each $i=1,\dots,r$, a representation π_i of $\mathrm{GL}_{n_i}(F)$, we write

$$\pi_1\times\dots\times\pi_r \tag{2.1}$$

for the representation of $\mathrm{GL}_n(F)$ obtained by normalized parabolic induction from $\pi_1\otimes\dots\otimes\pi_r$ along the parabolic subgroup generated by upper triangular matrices and the standard Levi subgroup $\mathrm{GL}_{n_1}(F)\times\dots\times\mathrm{GL}_{n_r}(F)$.

An irreducible representation of $\mathrm{GL}_n(F)$ is said to be *cuspidal* (respectively, *supercuspidal*) if it does not occur as a subrepresentation (respectively, a subquotient) of a representation of the form (2.1) with $r\geqslant 2$. Any supercuspidal representation of $\mathrm{GL}_n(F)$ is cuspidal.

Given a representation π of $\mathrm{GL}_n(F)$ and a character χ of $F^\times$, we will write $\pi\chi$ for $\pi(\chi\circ\det)$.

2.4. Let us fix a prime number $\ell \neq p$ and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field of ℓ-adic numbers. Let $\overline{\mathbb{Z}}_\ell$ denote its ring of integers, and $\overline{\mathbb{F}}_\ell$ denote the residue field of $\overline{\mathbb{Z}}_\ell$.

An irreducible $\overline{\mathbb{Q}}_\ell$-representation π of $\mathrm{GL}_n(F)$ is said to be *integral* if the $\overline{\mathbb{Q}}_\ell$-vector space of π contains a stable $\overline{\mathbb{Z}}_\ell$-lattice. For any such lattice L, the $\overline{\mathbb{F}}_\ell$-representation $L \otimes \overline{\mathbb{F}}_\ell$ of $\mathrm{GL}_n(F)$ has finite length. Its semisimplification is independent of the choice of L: it is called the *reduction modulo* ℓ of π and is denoted by $\mathbf{r}_\ell(\pi)$.

Given an irreducible $\overline{\mathbb{F}}_\ell$-representation π of $\mathrm{GL}_n(F)$, any integral irreducible $\overline{\mathbb{Q}}_\ell$-representation $\widetilde{\pi}$ of $\mathrm{GL}_n(F)$ such that $\mathbf{r}_\ell(\widetilde{\pi}) = \pi$ is called a $\overline{\mathbb{Q}}_\ell$*-lift* of π.

2.5. In the representation theory of the finite general linear group $\mathrm{GL}_n(\boldsymbol{k})$, one can also define parabolic induction, cuspidal and supercuspidal representations, reduction mod ℓ and lift, and we will use the same notation (2.1) for parabolic induction and $\mathbf{r}_\ell$ for reduction mod ℓ as in the non-Archimedean case.

3. The first main result ($\ell \neq 2$)

Let F/F_0 be a quadratic extension of non-Archimedean locally compact fields of residue characteristic $p \neq 2$, σ be its non-trivial automorphism and $\ell \notin \{2, p\}$ be a prime number.

Let G be the group $\mathrm{GL}_n(F)$ for some integer $n \geqslant 1$, equipped with the action of σ componentwise. Let H denote the closed subgroup $G^\sigma = \mathrm{GL}_n(F_0)$ made of the σ-fixed points of G.

3.1. Let R be an algebraically closed field of characteristic different from p. Given any R-representation π of G, we write π^σ for the representation $g \mapsto \pi(\sigma(g))$.

We write $\boldsymbol{k}_0$ for the residue field of F_0 and q_0 for its cardinality, and ν_0 for the unramified character of $F_0^\times$ sending any uniformizer to q_0^{-1}. Given any square root $q_0^{1/2}$ of q_0 in R, we set

$$q^{1/2} = \begin{cases} q_0^{1/2} & \text{if } F/F_0 \text{ is ramified,} \\ q_0 & \text{if } F/F_0 \text{ is unramified,} \end{cases} \tag{3.1}$$

which we use to define the character $\nu^{1/2}$ (§2.2) and normalized parabolic induction (§2.3).

3.2. Here is our first main theorem.

Theorem 3.1. — *A cuspidal irreducible $\overline{\mathbb{F}}_\ell$-representation of G is H-distinguished if and only if it has an H-distinguished cuspidal lift to $\overline{\mathbb{Q}}_\ell$.*

Note that:

– any cuspidal $\overline{\mathbb{F}}_\ell$-representation of G having an H-distinguished cuspidal lift to $\overline{\mathbb{Q}}_\ell$ is H-distinguished ([**23**] Theorem 3.4),

– Theorem 3.1 is known to hold for supercuspidal representations ([**35**] Theorem 10.11).

To prove our Theorem 3.1, it thus suffices to prove that any H-distinguished non-supercuspidal, cuspidal $\overline{\mathbb{F}}_\ell$-representation of G has an H-distinguished cuspidal lift to $\overline{\mathbb{Q}}_\ell$.

Remark 3.2. — If $\ell = 2$, there are examples of $\mathrm{GL}_2(F_0)$-distinguished non-supercuspidal, cuspidal $\overline{\mathbb{F}}_2$-representations of $\mathrm{GL}_2(F)$ with no $\mathrm{GL}_2(F_0)$-distinguished lift ([**24**] Remark 6.4).

3.3. Thanks to [**24**], the proof of Theorem 3.1 can be immediately reduced to the level 0 case.

Proposition 3.3. — *Suppose that Theorem 3.1 holds for cuspidal $\overline{\mathbb{F}}_\ell$-representations of level* 0 *of* $\mathrm{GL}_n(F)$*, for all* $n \geqslant 1$ *and all* F/F_0*. Then it holds for cuspidal representations of any level.*

Proof. — It suffices to prove that any H-distinguished cuspidal $\overline{\mathbb{F}}_\ell$-representation of G has a H-distinguished cuspidal lift to $\overline{\mathbb{Q}}_\ell$. Let π be a H-distinguished cuspidal representation of G. Associated with it in [**24**] §4.11, there are

- a tamely ramified extension T_0 of F_0 such that $T = T_0 \otimes_{F_0} F$ is a quadratic extension of T_0,
- a positive integer m dividing n,
- and a cuspidal representation π_{t} of level 0 of $\mathrm{GL}_m(T)$.

By [**24**] Proposition 4.40, this representation π_{t} is $\mathrm{GL}_m(T_0)$-distinguished. By assumption, since it has level 0, it has a $\mathrm{GL}_m(T_0)$-distinguished cuspidal lift to $\overline{\mathbb{Q}}_\ell$. It then follows from [**24**] Lemma 6.5 that π has an H-distinguished cuspidal lift to $\overline{\mathbb{Q}}_\ell$. □

We are thus reduced to proving the following theorem.

Theorem 3.4. — *Any H-distinguished non-supercuspidal, cuspidal $\overline{\mathbb{F}}_\ell$-representation of level* 0 *of G has a H-distinguished cuspidal lift to $\overline{\mathbb{Q}}_\ell$.*

3.4. Let us fix a square root $q_0^{1/2}$ in $\overline{\mathbb{Q}}_\ell$. By reduction mod the maximal ideal of $\overline{\mathbb{Z}}_\ell$, it defines a square root in $\overline{\mathbb{F}}_\ell$, which we also denote by $q_0^{1/2}$. By applying the rule (3.1), we get a square root $q^{1/2}$ in $\overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{F}}_\ell$.

Let us recall the classification of non-supercuspidal, cuspidal $\overline{\mathbb{F}}_\ell$-representations of G given by [**29**] Théorème 6.14. Let $k \geqslant 1$ be a positive integer and ρ be a supercuspidal $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_k(F)$. According to [**29**] §8.1, for any $r \geqslant 1$, the induced representation

$$\rho\nu^{-(r-1)/2} \times \cdots \times \rho\nu^{(r-1)/2} \tag{3.2}$$

contains a unique generic irreducible subquotient, denoted $\mathrm{Sp}_r(\rho)$. (For the definition of a generic representation, see §4.1.) Let $e(\rho)$ be the smallest integer $i \geqslant 1$ such that $\rho\nu^i$ is isomorphic to ρ.

Proposition 3.5. — *Let π be a non-supercuspidal, cuspidal $\overline{\mathbb{F}}_\ell$-representation of* $\mathrm{GL}_n(F)$*.*

(1) *There are a unique positive integer $r = r(\pi) \geqslant 2$ dividing n and a supercuspidal representation ρ of* $\mathrm{GL}_{n/r}(F)$ *such that π is isomorphic to* $\mathrm{Sp}_r(\rho)$*.*

(2) *There is a unique integer $u \geqslant 0$ such that $r = e(\rho)\ell^u$.*

(3) *Let ρ' be a supercuspidal representation of* $\mathrm{GL}_{n/r}(F)$*. The representation π is isomorphic to* $\mathrm{Sp}_r(\rho')$ *if and only if ρ' is isomorphic to $\rho\nu^i$ for some $i \in \mathbb{Z}$.*

Remark 3.6. — (1) By [**30**] Lemme 3.6, the integer $e(\rho)$ is equal to the order of $q^{t(\rho)}$ mod ℓ, where $t(\rho)$ is the number of unramified characters χ of $\mathrm{GL}_k(F)$ such that $\rho\chi$ is isomorphic to ρ.

(2) The representation π has level 0 if and only if ρ has level 0, in which case the number $t(\rho)$ is equal to k (see [**28**] §3.4).

3.5. Recall from [**24**] the following necessary conditions of distinction. Let e and e_0 denote the orders of q and q_0 mod ℓ, respectively.

Proposition 3.7. — *Let π be an H-distinguished non-supercuspidal, cuspidal representation of level 0 of G. Then there is a supercuspidal representation ρ of $\mathrm{GL}_k(F)$, with $rk = n$, such that π is isomorphic to $\mathrm{Sp}_r(\rho)$ and*

(1) *If r is odd, then:*
   (a) *the representation ρ is $\mathrm{GL}_k(F_0)$-distinguished,*
   (b) *if F/F_0 is unramified, then k, e are odd,*
   (c) *if F/F_0 is ramified, then k is even.*

(2) *If r is even, then:*
   (a) *we have $\rho^{\vee\sigma} = \rho\nu^i$ for some $i \in \{0, 1\}$,*
   (b) *if F/F_0 is unramified or $k \neq 1$, then $\rho\nu^{i/2}$ is $\mathrm{GL}_k(F_0)$-distinguished,*
   (c) *if F/F_0 is unramified, then $i = 0$.*

Proof. — The case where r is odd comes from [**24**] Theorem 5.1. Let us consider the case where r is even. First note that $q^{n/2}$ is congruent to -1 mod ℓ in this case. Indeed, $r = e(\rho)\ell^u$ for some $u \geqslant 0$ and $e(\rho)$ is the order of q^k mod ℓ (see Remark 3.6). Now Property (2.a) follows from [**24**] Proposition 3.8. It follows from (2.a) that $\mu = \rho\nu^{i/2}$ is σ-self-dual. It has a unique non-isomorphic σ-self-dual unramified twist, which is $\mu\nu^{n/2}$. Suppose that F/F_0 is unramified or $k \neq 1$. By [**35**] Proposition 10.12, there exists exactly one distinguished representation among μ and $\mu\nu^{n/2}$. If it is $\mu\nu^{n/2}$, we may replace ρ by $\rho\nu^{n/2}$, which exchanges the roles played by μ and $\mu\nu^{n/2}$ without changing π. This proves (2.b).

Let us prove (2.c). Thanks to (2.b), we may assume that $\rho\nu^{i/2}$ is distinguished. Thus $\omega_\rho(\varpi) = q^{ik/2}$. This implies

$$\omega_\pi(\varpi) = \omega_\rho(\varpi)^r = q^{in/2} = (-1)^i$$

since $q^{n/2} \equiv -1$ mod ℓ. Since π is distinguished, we must have $i = 0$. □

3.6. Recall the necessary and sufficient conditions for a non-supercuspidal, cuspidal representation of level 0 of G to have a H-distinguished cuspidal lift ([**24**] Propositions 6.1, 6.2).

Let $\eta = \eta_{F/F_0}$ be the character of $F_0^\times$ with kernel $\mathrm{N}_{F/F_0}(F^\times)$, the subgroup of F/F_0-norms.

Theorem 3.8. — *A non-supercuspidal, cuspidal representation of level 0 of G has a H-distinguished cuspidal lift if and only if it is of the form $\mathrm{Sp}_r(\rho)$ with ρ a supercuspidal representation of $\mathrm{GL}_k(F)$ with $rk = n$, and one of the following three conditions is satisfied:*

(1) *r is odd, F/F_0 is unramified, e_0 is even and ρ is $\mathrm{GL}_k(F_0)$-distinguished,*
(2) *r is odd, F/F_0 is ramified, k is even, n/e is odd and ρ is $\mathrm{GL}_k(F_0)$-distinguished,*
(3) *r is even, F/F_0 is ramified, $k = 1$ and $\rho|_{F_0^\times} \in \{\nu_0^{-1}, \eta\}$.*

It will be convenient to rephrase Theorem 3.8 as follows.

Theorem 3.9. — *A non-supercuspidal, cuspidal representation of level 0 of G has a H-distinguished cuspidal lift if and only if it is of the form $\mathrm{Sp}_r(\rho)$ with ρ a supercuspidal representation of $\mathrm{GL}_k(F)$ with $rk = n$, and one of the following conditions is satisfied:*

(1) *F/F_0 is unramified, e, k are odd, e_0 is even and ρ is $\mathrm{GL}_k(F_0)$-distinguished,*
(2) *F/F_0 is ramified, e is even and*

(a) *either k is even with the same dyadic valuation as e and ρ is $\mathrm{GL}_k(F_0)$-distinguished,*
(b) *or $k = 1$ and $\rho|_{F_0^\times} \in \{\nu_0^{-1}, \eta\}$.*

Proof. — Remark 3.6 implies that $e(\rho) = e/(e,k)$, thus

$$r = \ell^u e/(e,k), \quad n/e = \ell^u k/(e,k), \quad \text{for some } u \geqslant 0. \tag{3.3}$$

Suppose that F/F_0 is unramified. Then the fact that ρ is $\mathrm{GL}_k(F_0)$-distinguished implies that k is odd (by **[35]** Proposition 9.8). It then follows from (3.3) that r and e have the same parity.

Suppose now that F/F_0 is ramified. Then r is odd if and only if $v_2(e) \leqslant v_2(k)$, and n/e is odd if and only if $v_2(k) \leqslant v_2(e)$, thanks to (3.3), where $v_2(a)$ denotes the dyadic valuation of a positive integer $a \geqslant 1$. □

Remark 3.10. — When the conditions of Theorem 3.9 are satisfied, e_0 is even and e has the same parity as the ramification order of F/F_0.

Remark 3.11. — One can easily extract the following result from Theorem 3.9 and the Dichotomy and Disjunction Property of supercuspidal representations. We won't use it in the present article, but it is worth mentioning it. Let π be a non-supercuspidal, cuspidal representation of level 0 of G, and set $r = r(\pi)$ and $k = n/r$. Then π has a σ-self-dual cuspidal lift if and only if it is σ-self-dual and one of the following conditions is satisfied:
(1) F/F_0 is unramified, e, k are odd and e_0 is even,
(2) F/F_0 is ramified, e is even and either k has the same dyadic valuation as e, or $k = 1$.

One easily obtains a similar result for cuspidal representations of positive level which uses the invariants T/T_0 and m of **[24]** Propositions 6.1, 6.2.

3.7. As has been pointed out to us by A. Mínguez, the conditions of Theorem 3.9 only depend on ρ in the following sense: if ρ is a supercuspidal representation of $\mathrm{GL}_k(F)$ and if we define π_u to be the cuspidal representation $\mathrm{Sp}_{e(\rho)\ell^u}(\rho)$ for any $u \geqslant 0$, then
– either, for all $u \geqslant 0$, the representation π_u has no distinguished cuspidal lift,
– or, for all $u \geqslant 0$, the representation π_u has a distinguished cuspidal lift.

Consequently, in order to prove Theorem 3.1, given a ρ such that π_u is distinguished for some u, it suffices to find a v such that π_v has a distinguished lift. But we won't use this strategy.

Our strategy is by contradiction: we assume that there is a cuspidal representation π of level 0 which is distinguished but has no distinguished lift. We then compute a γ-factor in two different manners, and find two different values (see Propositions 3.15 and 3.16).

3.8. Let us first consider more carefully the case where r is even, F/F_0 is ramified and $k = 1$.

Proposition 3.12. — *Suppose that F/F_0 is ramified and n is even. Let π be a cuspidal representation of level 0 of G such that $r = n$. The following conditions are equivalent:*
(1) *π is H-distinguished.*
(2) *π has an H-distinguished lift.*
(3) *π is isomorphic to $\mathrm{Sp}_n(\rho)$ for some character ρ of $F^\times$ such that $\rho|_{F_0^\times} \in \{\nu_0^{-1}, \eta\}$.*

Proof. — We already know that (3) implies (2) and that (2) implies (1). Let us prove that (1) implies (3). Assume that π is distinguished. By Proposition 3.7(2.a), it is isomorphic to $\mathrm{Sp}_n(\rho)$ for some character ρ of $F^\times$ such that $\rho\nu^{i/2}$ is σ-self-dual for some $i \in \{0, 1\}$. The latter is thus either distinguished or η-distinguished. Since F/F_0 is ramified, η is ramified, thus $\rho\nu^{i/2}$ is distinguished if and only if ρ is unramified. We thus get

$$\rho(\varpi_0) = \begin{cases} q^i & \text{if } \rho \text{ is unramified,} \\ \eta(-1)q^i & \text{if } \rho \text{ is ramified.} \end{cases} \tag{3.4}$$

Since $\omega_\pi(\varpi) = \rho(\varpi)^n = \rho(\varpi_0)^{n/2}$, we thus have

$$\omega_\pi(\varpi) = \begin{cases} q^{in/2} & \text{if } \rho \text{ is unramified,} \\ \eta(-1)^{n/2}q^{in/2} & \text{if } \rho \text{ is ramified.} \end{cases} \tag{3.5}$$

But, since F/F_0 is ramified, the fact that π is distinguished implies (by **[24]** Theorem 4.45, Lemma 6.12) that

$$\omega_\pi(\varpi) = \begin{cases} -1 & \text{if } \rho \text{ is unramified,} \\ \eta(-1)^{n/2} & \text{if } \rho \text{ is ramified.} \end{cases} \tag{3.6}$$

We thus get
- if ρ is unramified, then $i = 1$, thus $\rho|_{F_0^\times} = \nu_0^{-1}$,
- if ρ is ramified, then $i = 0$, thus $\rho|_{F_0^\times} = \eta$.

This proves the proposition. □

3.9. It follows that, if π is a non-supercuspidal, cuspidal representation of level 0 of G which is H-distinguished but has no H-distinguished cuspidal lift, then π is isomorphic to $\mathrm{Sp}_r(\rho)$ and we are in one of the four following cases:

(1) F/F_0 is unramified, e_0 is odd and ρ is $\mathrm{GL}_k(F_0)$-distinguished,
(2) F/F_0 is unramified, e is even and ρ is $\mathrm{GL}_k(F_0)$-distinguished,
(3) F/F_0 is ramified, $k/(k,e)$ is even, $e/(e,k)$ is odd and ρ is $\mathrm{GL}_k(F_0)$-distinguished,
(4) F/F_0 is ramified, k is even and $\rho\nu^{i/2}$ is $\mathrm{GL}_k(F_0)$-distinguished for an $i \in \{0,1\}$.

3.10. Let ψ be a non-trivial character of F. Associated with any generic irreducible representation τ of $\mathrm{GL}_m(F)$ with $m \leqslant n$, there is a Rankin–Selberg γ-factor $\gamma(X,\pi,\tau,\psi) \in \overline{\mathbb{F}}_\ell(X)$ whose definition and main properties are recalled in Section 4.

Lemma 3.13. — *Let π denote a non-supercuspidal, cuspidal $\overline{\mathbb{F}}_\ell$-representation of G isomorphic to $\mathrm{Sp}_r(\rho)$ for some supercuspidal representation ρ of $\mathrm{GL}_k(F)$. Then*

$$\gamma\left(q^{-1/2},\pi,\rho^\vee,\psi\right) = \omega_\pi(-1)^{k-1}\cdot(-1)^r\cdot q^{n/2}$$

where ω_π is the central character of π.

Remark 3.14. — Note that this value is a sign, since n is a multiple of the order e of q mod ℓ.

The proof of this lemma is postponed to §4.7. We deduce from it the following result.

Proposition 3.15. — *Let π be a non-supercuspidal, cuspidal representation of level 0 of G, isomorphic to $\mathrm{Sp}_r(\rho)$ for some supercuspidal representation ρ of $\mathrm{GL}_k(F)$. Suppose that π is H-distinguished but has no H-distinguished lift to $\overline{\mathbb{Q}}_\ell$. Then*

$$\gamma(q^{-1/2},\pi,\rho^\vee,\psi) = -1.$$

Proof. — According to Lemma 3.13, and since π is distinguished, we have

$$\gamma\left(q^{-1/2},\pi,\rho^\vee,\psi\right) = (-1)^r\cdot q^{n/2}.$$

Suppose first that r is even. As in the proof of Proposition 3.7, this implies that $q^{n/2}$ is congruent to -1 mod ℓ. The result follows.

Now suppose that r is odd. We are thus either in Case 1 or in Case 3 of §3.9. If we are in Case 1, then $q^{1/2} = q_0$ since F/F_0 is unramified (see (3.1)) and $e_0 = e$ since e_0 is odd. We thus have $q^{n/2} = q_0^n = 1$ since e divides n. In Case 3, then $q^{n/2} = (q^e)^{k/2(k,e)} = 1$ since $k/(k,e)$ is even. □

In order to prove Theorem 3.4, it thus suffices to prove:

Proposition 3.16. — *Let π be a non-supercuspidal cuspidal representation of level* 0 *of G, isomorphic to $\mathrm{Sp}_r(\rho)$ for some supercuspidal representation ρ of $\mathrm{GL}_k(F)$. Suppose that π is H-distinguished but has no H-distinguished lift to $\overline{\mathbb{Q}}_\ell$. Then*

$$\gamma(q^{-1/2},\pi,\rho^\vee,\psi) = 1.$$

Since $\ell \neq 2$, this will give us the expected contradiction.

We are now going to reduce the proof of Proposition 3.16 to that of a similar proposition for representations of general linear groups over the finite field $\boldsymbol{k}$.

3.11. Let V denote the vector space of π, and consider the subspace W of V made of the vectors invariant by the subgroup $1+\mathbf{M}_n(\mathfrak{p})$. Since π has level 0, this subspace is non-zero. It carries an action of $\mathrm{GL}_n(\mathcal{O})$ which factors into a representation $\overline{\pi}$ of $\mathrm{GL}_n(\boldsymbol{k})$ on W which is irreducible and cuspidal ([**37**] III.3.14 or [**29**] Exemple 5.10, Proposition 5.11). Similarly, one defines a cuspidal, irreducible representation $\overline{\rho}$ of $\mathrm{GL}_k(\boldsymbol{k})$ on the vectors of the space of ρ invariant by $1+\mathbf{M}_k(\mathfrak{p})$.

By [**24**] 4.7, the representation $\overline{\rho}$ is supercuspidal and $\overline{\pi}$ occurs as a subquotient of the induced representation $\overline{\rho}^{\times r} = \overline{\rho}\times\cdots\times\overline{\rho}$ (where $\overline{\rho}$ occurs r times). On the other hand, it follows from [**24**] Theorem 4.45 that $\overline{\pi}$ is distinguished by the subgroup $\overline{H}$ of $\mathrm{GL}_n(\boldsymbol{k})$ defined by

$$\overline{H} = \begin{cases} \mathrm{GL}_n(\boldsymbol{k}_0) & \text{if } F/F_0 \text{ is unramified,}\\ \mathrm{GL}_{n/2}(\boldsymbol{k})\times\mathrm{GL}_{n/2}(\boldsymbol{k}) & \text{if } F/F_0 \text{ is ramified,}\end{cases}$$

and from [**24**] Lemma 6.5 that it has no $\overline{H}$-distinguished lift. Note that n even when F/F_0 is ramified, thanks to Proposition 3.7.

Suppose that ψ has conductor 1, that is, ψ is trivial on $\mathfrak{p}$ but not on $\mathcal{O}$. It defines a non-trivial character of $\boldsymbol{k}$, which we still denote by ψ. We then may associate to the pair $(\overline{\pi},\overline{\rho}^\vee)$ a non-zero scalar $\gamma(\overline{\pi},\overline{\rho}^\vee,\psi)\in R^\times$ which will be defined in Section 5.

We have the following proposition, the proof of which is postponed to §5.5.

Proposition 3.17. — *Suppose that ψ has conductor* 1. *Then*

$$\gamma(q^{-1/2},\pi,\rho^\vee,\psi) = \gamma(\overline{\pi},\overline{\rho}^\vee,\psi).$$

The proof of Proposition 3.16 thus reduces to proving that $\gamma(\overline{\pi},\overline{\rho}^\vee,\psi) = 1$.

3.12. More generally, for any non-supercuspidal, cuspidal representation $\varkappa$ of $\mathrm{GL}_n(\boldsymbol{k})$, there are a unique integer u dividing n and a unique supercuspidal representation ϱ of $\mathrm{GL}_{n/u}(\boldsymbol{k})$ such that $\varkappa$ occurs as a subquotient of the parabolically induced representation $\varrho^{\times u}$ (see **[24]** Proposition 3.9 for instance). We will say that ϱ is the supercuspidal representation *associated with* $\varkappa$.

Changing the notation, the proof of Theorem 3.1 is thus reduced to the proof of the following proposition, which entirely pertains to the theory of representations of $\mathrm{GL}_n(\boldsymbol{k})$.

Proposition 3.18. — *Let π be a non-supercuspidal cuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$, with associated supercuspidal representation ρ. Let H be one of the following subgroups of $\mathrm{GL}_n(\boldsymbol{k})$:*

(1) *either $\boldsymbol{k}$ has a subfield $\boldsymbol{k}_0$ such that $[\boldsymbol{k} : \boldsymbol{k}_0] = 2$ and $H = \mathrm{GL}_n(\boldsymbol{k}_0)$,*

(2) *or $n = 2m$ for some integer $m \geqslant 1$ and H is the standard Levi subgroup $\mathrm{GL}_m(\boldsymbol{k}) \times \mathrm{GL}_m(\boldsymbol{k})$.*

Suppose that π is H-distinguished but has no H-distinguished lift. Then, for any non-trivial character ψ of $\boldsymbol{k}$, we have $\gamma(\pi, \rho^\vee, \psi) = 1$.

We will refer to the first case as the Galois case, and to the second case as the Levi case.

In conclusion, in order to prove our main Theorem 3.1, it remains to prove Lemma 3.13, Proposition 3.17 and Proposition 3.18.

3.13. Before proceeding to the proof of these results, let us end this section by the following Disjunction Theorem, well-known for discrete series representations of G when $R = \mathbb{C}$.

Corollary 3.19. — *A σ-self-dual cuspidal $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_n(F)$ cannot be both distinguished and η-distinguished.*

Proof. — If r is odd, this is **[24]** Corollary 5.2. Assume that r is even and π is both distinguished and η-distinguished. We may and will assume that π has level 0. By Theorem 3.1 together with Theorem 3.8, F/F_0 is ramified, $r = n$ and π is isomorphic to $\mathrm{Sp}_r(\rho)$ for some character ρ of $F^\times$ such that $\rho|_{F_0^\times} \in \{\nu_0^{-1}, \eta\}$.

Let α be a tamely ramified character of $F^\times$ extending η. Since π is η-distinguished, the representation $\pi\alpha = \mathrm{Sp}_r(\rho\alpha)$ is distinguished. It is thus isomorphic to $\mathrm{Sp}_r(\rho')$ for some character ρ' of $F^\times$ such that $\rho'|_{F_0^\times} \in \{\nu_0^{-1}, \eta\}$. Since π is isomorphic to $\mathrm{Sp}_r(\rho'\alpha^{-1})$, it follows from the classification of cuspidal representations that $\rho'\alpha^{-1} = \rho\nu^i$ for some $i \in \mathbb{Z}$.

Since η is ramified and $\nu|_{F_0^\times} = \nu_0^2$, we have a contradiction in any case. □

Remark 3.20. — Note that Dichotomy does not hold: when r is even and either F/F_0 is unramified, or F/F_0 is ramified and $r \neq n$, any σ-self-dual cuspidal representation of level 0 is neither distinguished nor η-distinguished.

4. Rankin–Selberg factors over non-Archimedean fields

In this section, ℓ is any prime number different from p and R is an algebraically closed field of characteristic 0 or ℓ. We fix a non-trivial character

$$\psi : F \to R^\times$$

with conductor $d = d(\psi)$, which is the smallest integer $i \in \mathbb{Z}$ such that $\mathfrak{p}^i \subseteq \mathrm{Ker}(\psi)$. The purpose of this section is to :

– recall the definition and main properties of the Rankin–Selberg local factors of **[22]**,

– prove Lemma 3.13.

4.1. Given an integer $n \geqslant 1$, we write $G = \mathrm{GL}_n(F)$. Let $N = N_n$ be the subgroup of G made of all upper triangular unipotent matrices. The character

$$x \mapsto \psi(x_{1,2} + \cdots + x_{n-1,n})$$

of N will still be denoted by ψ. An irreducible R-representation π of G is *generic* if it embeds in the space $\mathrm{Ind}_N^G(\psi)$ made of all smooth functions $W : G \to R$ such that $W(xg) = \psi(x)W(g)$ for all $x \in N$ and $g \in G$, or equivalently, if the R-vector space $\mathrm{Hom}_N(\pi, \psi)$ is non-zero. When this is the case, the latter has dimension 1, that is, the image of the embedding of π in $\mathrm{Ind}_N^G(\psi)$ is unique. It is called the *Whittaker model* of π with respect to ψ and is denoted $\mathcal{W}(\pi, \psi)$.

4.2. Given integers n and m such that $1 \leqslant m \leqslant n$, let us review the Rankin–Selberg functional equations and local factors for $\mathrm{GL}_n(F) \times \mathrm{GL}_m(F)$ following **[22]**. Let us fix a square root

$$q^{1/2} \in R^\times \tag{4.1}$$

of q. Let π and π' be generic R-representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, respectively. Attached to the pair (π, π') in **[22]** Section 3, there are

(1) an L-*factor* $\mathrm{L}(X, \pi, \pi')$, which is an element of $R(X)$ of the form P^{-1} where P is a polynomial in $R[X]$ such that $P(0) = 1$,

(2) an ε-*factor* $\varepsilon(X, \pi, \pi', \psi)$, which is of the form

$$\varepsilon(X, \pi, \pi', \psi) = eX^{f(\pi,\pi',\psi)} \tag{4.2}$$

for a non-zero scalar $e \in R^\times$ and an integer $f(\pi, \pi', \psi) \in \mathbb{Z}$,

(3) and a γ-*factor* $\gamma(X, \pi, \pi', \psi)$, which is the element of $R(X)$ defined by

$$\gamma(X, \pi, \pi', \psi) = \varepsilon(X, \pi, \pi', \psi) \cdot \mathrm{L}(X, \pi, \pi')^{-1} \cdot \mathrm{L}\left(\frac{1}{qX}, \pi^\vee, \pi'^\vee\right).$$

We give more details, for which we refer to **[22]** Theorem 3.5, Corollary 3.11.

4.3. Assume first that $m = n$. Then the L-factor $\mathrm{L}(X, \pi, \pi')$ is the unique Euler factor generating the fractional ideal of $R[X^{\pm 1}]$ generated by the Laurent series

$$I(X, W, W', \Phi) = \sum_{k \in \mathbb{Z}} X^k \cdot \int_{\mathbf{Y}_{n,k}} W(g)W'(g)\Phi(\eta g)\ \mathrm{d}g,$$

$$W \in \mathcal{W}(\pi, \psi), \quad W' \in \mathcal{W}(\pi', \psi^{-1}), \quad \Phi \in \mathcal{C}_c^\infty(F^n),$$

where $\mathcal{C}_c^\infty(F^n)$ is the space of locally constant, compactly supported functions from F^n to R, the integral is over the set $\mathbf{Y}_{n,k}$ made of all $N_n g \subseteq \mathrm{GL}_n(F)$ such that $\det(g)$ has valuation k and η is the row matrix $(0\ \ldots\ 0\ 1)$. We have the functional equation

$$I\left(\frac{1}{qX}, \widetilde{W}, \widetilde{W}', \widehat{\Phi}\right) = \omega_{\pi'}(-1)^{n-1} \cdot \gamma(X, \pi, \pi', \psi) \cdot I(X, W, W', \Phi) \tag{4.3}$$

where $\widetilde{W} \in \mathcal{W}(\pi^\vee, \psi^{-1})$ is the function given by $g \mapsto W(w_n g^*)$, where w_n is the antidiagonal permutation matrix of maximal length and g^* is the transpose of g^{-1}, and

$$\widehat{\Phi} : y \mapsto \int_{F^n} \Phi(x)\psi(x \cdot y)\ \mathrm{d}x$$

is the Fourier transform of Φ with respect to the unique Haar measure $\mathrm{d}x = \mathrm{d}x(\psi)$ on F^n giving measure $q^{nd/2}$ to the lattice $\mathcal{O}^n$ (and $x \cdot y$ denotes the canonical scalar product of $x, y \in F^n$).

Assume now that $m < n$ and fix an integer $j \in \{0, \dots, n-m-1\}$. The L-factor $\mathrm{L}(X, \pi, \pi')$ is the unique Euler factor generating the fractional ideal of $R[X^{\pm 1}]$ generated by the series

$$(4.4)\ I_j(X, W, W') = \sum_{k \in \mathbb{Z}} (q^{(n-m)/2} X)^k \cdot \int_{\mathbf{M}_{j,m}(F)} \int_{\mathbf{Y}_{m,k}} W \begin{pmatrix} g & 0 & 0 \\ x & 1_j & 0 \\ 0 & 0 & 1_{n-m-j} \end{pmatrix} W'(g)\ \mathrm{d}g\ \mathrm{d}x,$$
$$W \in \mathcal{W}(\pi, \psi), \quad W' \in \mathcal{W}(\pi, \psi^{-1}),$$

where the integral over the vector space $\mathbf{M}_{j,m}(F)$ is with respect to the unique Haar measure giving measure $q^{jmd/2}$ to the lattice $\mathbf{M}_{j,m}(\mathcal{O})$. We have the functional equation

$$I_{n-m-1-j}\left(\frac{1}{qX}, w_{m,n-m} \cdot \widetilde{W}, \widetilde{W}'\right) = \omega_{\pi'}(-1)^{n-1} \cdot \gamma(X, \pi, \pi', \psi) \cdot I_j(X, W, W')$$

where $w_{m,n-m} = \mathrm{diag}(\mathrm{id}_m, w_{n-m})$ in $\mathrm{GL}_n(F)$.

4.4. These Rankin–Selberg local factors enjoy the following properties.

Proposition 4.1. — *Let π and π' be as above.*

(1) *The integer*

$$f(\pi, \pi') = f(\pi, \pi', \psi) + mnd(\psi)$$

does not depend on ψ.

(2) *Given an $a \in F^\times$, let ψ^a denote the character $x \mapsto \psi(ax)$ of F. Then*

$$\varepsilon(X, \pi, \pi', \psi^a) = \omega_\pi(a)^m \cdot \omega_{\pi'}(a)^n \cdot \left(q^{1/2} X\right)^{mn \cdot \mathrm{val}_F(a)} \cdot \varepsilon(X, \pi, \pi', \psi).$$

(3) *If χ is an unramified character of $F^\times$, then*

$$\varepsilon(X, \pi\chi, \pi', \psi) = \varepsilon(X, \pi, \pi'\chi, \psi) = \chi(\varpi)^{f(\pi, \pi', \psi)} \cdot \varepsilon(X, \pi, \pi', \psi)$$

for any uniformizer ϖ of F.

(4) *Let $\mathrm{L}^*(X, \pi, \pi')$ and $\varepsilon^*(X, \pi, \pi', \psi)$ be the local constants obtained by replacing* (4.1) *by the opposite square root $-q^{1/2}$. Then*

$$\begin{aligned} \mathrm{L}^*(X, \pi, \pi') &= \mathrm{L}((-1)^{n-m} X, \pi, \pi'), \\ \varepsilon^*(X, \pi, \pi', \psi) &= (-1)^{(n-m) f(\pi, \pi') + mnd(\psi)} \cdot \varepsilon(X, \pi, \pi', \psi), \end{aligned}$$

and the integer $f(\pi, \pi')$ does not depend on the choice of the square root made in (4.1).

Proof. — The first three properties are well-known when R is the field of complex numbers. For a general R, the first property follows from the second one together with the fact that $d(\psi^a) = d(\psi) - \mathrm{val}_F(a)$, and the third one easily follows from the functional equations. We now prove the second and fourth ones, treating the cases $n = m$ and $m < n$ separately.

Assume first that $n = m$. If one replaces the square root $q^{1/2}$ by its opposite $-q^{1/2}$, the series $I(X, W, W')$ remain unchanged, which gives $\mathrm{L}^*(X, \pi, \pi') = \mathrm{L}(X, \pi, \pi')$, and the Haar measure $\mathrm{d}x$ on F^n is changed to $(-1)^{nd} \cdot \mathrm{d}x$, which implies that

$$\varepsilon^*(X, \pi, \pi', \psi) = (-1)^{nd} \cdot \varepsilon(X, \pi, \pi', \psi)$$

as expected. Given $a \in F^\times$, we define $t = t(a) \in G$ to be the diagonal matrix $\mathrm{diag}(a^{n-1}, \dots, a, 1)$. The functions $W^a : g \mapsto W(tg)$ and $W'^a : g \mapsto W'(tg)$ belong to $\mathcal{W}(\pi, \psi^a)$ and $\mathcal{W}(\pi', \psi^a)$, respectively. We have

$$\begin{aligned} I(X, W^a, W'^a, \Phi) &= \sum_{k\in\mathbb{Z}} X^k \cdot \int_{\mathbf{Y}_{n,k}} W(tg)W'(tg)\Phi(\eta g)\ \mathrm{d}g \\ &= \sum_{k\in\mathbb{Z}} X^k \cdot \int_{\mathbf{Y}_{n,k+\mathrm{val}_F(a)\cdot n(n-1)/2}} W(h)W'(h)\Phi(\eta h)\ \mathrm{d}(t^{-1}h) \\ &= X^{-\mathrm{val}_F(a)\cdot n(n-1)/2} \cdot \mu(a) \cdot I(X, W, W', \Phi) \end{aligned}$$

where μ is the character of $F^\times$ such that $\mathrm{d}(t^{-1}h) = \mu(a) \cdot \mathrm{d}h$ on $N_n\backslash \mathrm{GL}_n(F)$. For all $g \in G$, one has $W(twg^*) = \widetilde{W}(a^{1-n}tg)$. If we denote by $\widehat{\Phi}^a$ the Fourier transform of Φ with respect to the measure $\mathrm{d}x(\psi^a) = |a|^{-n} \cdot \mathrm{d}x(\psi)$, we have $\widehat{\Phi}^a(x) = |a|^{n/2} \cdot \widehat{\Phi}(ax)$ for all $x \in F^n$. This gives

$$\begin{aligned} I\left(\frac{1}{qX}, \widetilde{W^a}, \widetilde{W'^a}, \widehat{\Phi}^a\right) &= \sum_{k\in\mathbb{Z}} q^{-k}X^{-k} \cdot \int_{\mathbf{Y}_{n,k}} \widetilde{W}(a^{1-n}tg)\widetilde{W}'(a^{1-n}tg)\widehat{\Phi}^a(\eta g)\ \mathrm{d}g \\ &= \omega_\pi(a)^{n-1} \cdot \omega_{\pi'}(a)^{n-1} \cdot \sum_{k\in\mathbb{Z}} q^{-k}X^{-k} \cdot \int_{\mathbf{Y}_{n,k}} \widetilde{W}(tg)\widetilde{W}'(tg)\widehat{\Phi}^a(\eta g)\ \mathrm{d}g \\ &= \omega_\pi(a)^{n-1} \cdot \omega_{\pi'}(a)^{n-1} \cdot X^{\mathrm{val}_F(a)\cdot n(n-1)/2} \cdot \mu(a) \cdot I\left(\frac{1}{qX}, \widetilde{W}, \widetilde{W}', \widehat{\Phi}^a\right) \end{aligned}$$

and

$$\begin{aligned} I\left(\frac{1}{qX}, \widetilde{W}, \widetilde{W}', \widehat{\Phi}^a\right) &= |a|^{n/2} \cdot \sum_{k\in\mathbb{Z}} q^{-k}X^{-k} \cdot \int_{\mathbf{Y}_{n,k}} \widetilde{W}(g)\widetilde{W}'(g)\widehat{\Phi}(a\eta g)\ \mathrm{d}g \\ &= |a|^{n/2} \cdot \omega_\pi(a) \cdot \omega_{\pi'}(a) \cdot J_{k,a}(X) \end{aligned}$$

with

$$\begin{aligned} J_{k,a}(X) &= \sum_{k\in\mathbb{Z}} q^{-k}X^{-k} \int_{\mathbf{Y}_{n,k+\mathrm{val}_F(a)\cdot n}} \widetilde{W}(g)\widetilde{W}'(g)\widehat{\Phi}(\eta g)\ \mathrm{d}(a^{-1}g) \\ &= (qX)^{\mathrm{val}_F(a)\cdot n} \cdot |a|^{n(n+1)/2} \cdot I\left(\frac{1}{qX}, \widetilde{W}, \widetilde{W}', \widehat{\Phi}\right), \end{aligned}$$

which implies the expected result.

Assume now that $m < n$, and write $I_j^*(X, W, W')$ for the series (4.4) defined by replacing (4.1) by the opposite square root $-q^{1/2}$. Taking into account that the measure $\mathrm{d}x$ on $\mathbf{M}_{j,m}(F)$ changes, we get

$$I_j^*(X, W, W') = (-1)^{jmd(\psi)} \cdot I_j((-1)^{n-m}X, W, W'),$$

thus $\mathrm{L}^*(X,\pi,\pi') = \mathrm{L}((-1)^{n-m}X,\pi,\pi')$ and

$$\begin{aligned}\varepsilon^*(X,\pi,\pi',\psi) &= (-1)^{jmd(\psi)+(n-m-1-j)md(\psi)} \cdot \varepsilon((-1)^{n-m}X,\pi,\pi',\psi)\\ &= (-1)^{(n-m-1)md(\psi)+(n-m)f(\pi,\pi',\psi)} \cdot \varepsilon(X,\pi,\pi',\psi)\\ &= (-1)^{(n-m)f(\pi,\pi')+(n(n-m)+n-m-1)md(\psi)} \cdot \varepsilon(X,\pi,\pi',\psi)\\ &= (-1)^{(n-m)f(\pi,\pi')+mnd(\psi)} \cdot \varepsilon(X,\pi,\pi',\psi)\end{aligned}$$

where the second identity uses (4.2), the third one uses (1) and the last one uses that the integers mn and $m(n(n-m)+n-m-1)$ have the same parity. The proof of (2) when $m<n$ follows the same pattern as in the case when $m=n$. □

4.5. A fundamental property that we will use is that the γ-factor is multiplicative: if the generic representations π and π' are subquotients of $\rho_1\times\cdots\times\rho_r$ and $\mu_1\times\cdots\times\mu_s$, respectively, where the ρ_i and the μ_j are generic, then

$$\gamma(X,\pi,\pi',\psi) = \prod_{i=1}^{r}\prod_{j=1}^{s}\gamma(X,\rho_i,\mu_j,\psi) \tag{4.5}$$

(see **[22]** Theorem 4.1).

4.6. Any non-trivial $\overline{\mathbb{Q}}_\ell$-character of F takes values in $\overline{\mathbb{Z}}_\ell$. It thus may be reduced mod ℓ and its reduction mod ℓ is a non-trivial $\overline{\mathbb{F}}_\ell$-character of F. Moreover, reduction mod ℓ induces a bijection between non-trivial $\overline{\mathbb{Q}}_\ell$-characters of F and non-trivial $\overline{\mathbb{F}}_\ell$-character of F.

In this paragraph, we fix a non-trivial $\overline{\mathbb{Q}}_\ell$-character ψ of F and denote its reduction mod ℓ by ψ as well. The following lemma is **[22]** Lemma 2.27.

Lemma 4.2. — *Let π be an integral generic $\overline{\mathbb{Q}}_\ell$-representation of G.*

(1) *There is, up to isomorphism, a unique generic $\overline{\mathbb{Q}}_\ell$-representation of G occurring in the reduction mod ℓ of π.*

(2) *It occurs with multiplicity 1 as an irreducible component of the reduction mod ℓ of π.*

We fix a square root of q in $\overline{\mathbb{Q}}_\ell$, which we will use to define the local constants of pairs of generic $\overline{\mathbb{Q}}_\ell$-representations. For local constants of pairs of generic $\overline{\mathbb{F}}_\ell$-representations, we use the reduction mod ℓ of this square root.

Proposition 4.3. — *Let $\widetilde{\pi}$, $\widetilde{\pi}'$ be integral generic $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{GL}_n(F)$, $\mathrm{GL}_m(F)$, respectively, and π, π' be the unique generic subquotients of their reduction mod ℓ, respectively.*

(1) *We have $\mathrm{L}(X,\widetilde{\pi},\widetilde{\pi}')^{-1} \in 1+X\overline{\mathbb{Z}}_\ell[X]$ and $\varepsilon(1,\widetilde{\pi},\widetilde{\pi}',\psi)\in\overline{\mathbb{Z}}_\ell^\times$.*

(2) *The reduction mod ℓ of $\gamma(X,\widetilde{\pi},\widetilde{\pi}',\psi)$ is equal to $\gamma(X,\pi,\pi',\psi)$.*

Proof. — The first part follows from **[22]** Corollary 3.6 (in the statement of which the assumption that $\widetilde{\pi}$ and $\widetilde{\pi}'$ are integral is missing) and **[22]** Lemma 3.12. The second part follows from **[22]** Theorem 3.13. □

4.7. Let us prove Lemma 3.13. Recall that π is a non-supercuspidal, cuspidal $\overline{\mathbb{F}}_\ell$-representation of $G = \mathrm{GL}_n(F)$ isomorphic to $\mathrm{Sp}_r(\rho)$ for some supercuspidal representation ρ of $\mathrm{GL}_k(F)$.

Proof of Lemma 3.13. — Let $\widetilde{\rho}$ be a $\overline{\mathbb{Q}}_\ell$-lift of ρ, whose existence is granted by **[37]** III.5.10. By Proposition 4.3(2), we have

$$\gamma\left(X, \pi, \rho^\vee, \psi\right) = \mathbf{r}_\ell\left(\gamma(X, \mathrm{Sp}_r(\widetilde{\rho}), \widetilde{\rho}^\vee, \psi)\right).$$

By multiplicativity (4.5), we have

$$\gamma\left(X, \mathrm{Sp}_r(\widetilde{\rho}), \widetilde{\rho}^\vee, \psi\right) = \prod_{i=0}^{r-1} \gamma\left(X, \widetilde{\rho}\nu^{i-(r-1)/2}, \widetilde{\rho}^\vee, \psi\right).$$

Write $a = (r-1)/2$ for simplicity. Then

$$\gamma\left(X, \widetilde{\rho}\nu^{i-(r-1)/2}, \widetilde{\rho}^\vee, \psi\right) = \varepsilon\left(X, \widetilde{\rho}\nu^{i-a}, \widetilde{\rho}^\vee, \psi\right) \cdot \frac{\mathrm{L}(q^{-1}X^{-1}, \widetilde{\rho}^\vee\nu^{a-i}, \widetilde{\rho})}{\mathrm{L}(X, \widetilde{\rho}\nu^{i-a}, \widetilde{\rho}^\vee)}.$$

By **[22]** Theorem 4.9, we have

$$\mathrm{L}(X, \widetilde{\rho}\nu^{i-a}, \widetilde{\rho}^\vee) = \frac{1}{1 - q^{k(a-i)}X^k} \quad \text{and} \quad \mathrm{L}(X, \widetilde{\rho}^\vee\nu^{a-i}, \widetilde{\rho}) = \frac{1}{1 - q^{k(i-a)}X^k}.$$

We thus have

$$\gamma(X, \mathrm{Sp}_r(\widetilde{\rho}), \widetilde{\rho}^\vee, \psi) = \varepsilon(X, \widetilde{\rho}, \widetilde{\rho}^\vee, \psi)^r \cdot \prod_{i=0}^{r-1} \frac{1 - q^{k(a-i)}X^k}{1 - q^{k(i-a-1)}X^{-k}}. \tag{4.6}$$

Let us first focus on the product on the right hand side of (4.6). We have

$$\begin{aligned}
\prod_{i=0}^{r-1} \frac{1 - q^{k(a-i)}X^k}{1 - q^{k(i-a-1)}X^{-k}} &= (-X^k)^r \cdot \prod_{i=0}^{r-1} \frac{1 - q^{k(a-i)}X^k}{q^{k(i-a-1)} - X^k} \\
&= (-1)^r \cdot X^n \cdot q^n \cdot \prod_{i=0}^{r-1} \frac{1 - q^{k(a-i)}X^k}{1 - q^{k(a-i+1)}X^k} \\
&= (-1)^r \cdot (qX)^n \cdot \frac{1 - q^{-ka}X^k}{1 - q^{k(a+1)}X^k}
\end{aligned}$$

which we may write under the form

$$\prod_{i=0}^{r-1} \frac{1 - q^{k(a-i)}X^k}{1 - q^{k(i-a-1)}X^{-k}} = (-1)^r \cdot q^{n/2} \cdot (q^{1/2}X)^n \cdot \frac{1 - q^{-ka}X^k}{1 - q^n q^{-ka}X^k}. \tag{4.7}$$

Let us compute the ε-factor occurring on the right hand side of (4.6). By **[7]** Théorème 2 we have $\varepsilon(q^{-1/2}, \widetilde{\rho}, \widetilde{\rho}^\vee, \psi) = \omega_{\widetilde{\rho}}(-1)^{k-1}$, where $\omega_{\widetilde{\rho}}$ is the central character of $\widetilde{\rho}$. Thus

$$\varepsilon(X, \widetilde{\rho}, \widetilde{\rho}^\vee, \psi) = \omega_{\widetilde{\rho}}(-1)^{k-1} \cdot (q^{1/2}X)^f \tag{4.8}$$

for some integer $f = f(\widetilde{\rho}, \widetilde{\rho}^\vee, \psi) \in \mathbb{Z}$. Putting (4.7) and (4.8) together, we get

$$\gamma(X, \mathrm{Sp}_r(\widetilde{\rho}), \widetilde{\rho}^\vee, \psi) = (-1)^r \cdot q^{n/2} \cdot \omega_{\widetilde{\rho}}(-1)^{r(k-1)} \cdot (q^{1/2}X)^{rf} \cdot \frac{1 - q^{-ka}X^k}{1 - q^n q^{-ka}X^k}.$$

Since q^n is congruent to 1 mod ℓ (which follows from the fact that π is cuspidal but non-supercuspidal) and ω_ρ^r is the central character of π, we get

$$\gamma(X,\pi,\rho^\vee,\psi) = (-1)^r \cdot q^{n/2} \cdot \omega_\pi(-1)^{k-1} \cdot (q^{1/2}X)^{rf}.$$

Specializing at $X = q^{-1/2}$, we get the expected result. □

In conclusion, in order to prove Theorem 3.1, it remains to prove Propositions 3.17 and 3.18.

5. Gamma factors over finite fields

In this section, $\boldsymbol{k}$ is a finite field of characteristic p, ℓ is a prime number different from p and R is an algebraically closed field of characteristic 0 or ℓ. We fix a non-trivial character

$$\psi : \boldsymbol{k} \to R^\times.$$

The purpose of this section is to recall the definition and properties of the γ-factors of **[34, 31, 4]** and prove Proposition 3.17.

5.1. Given an integer $n \geqslant 1$, we write $G = G_n$ for the group $\mathrm{GL}_n(\boldsymbol{k})$ and $N = N_n$ for the subgroup of G made of all upper triangular unipotent matrices. The character

$$x \mapsto \psi(x_{1,2} + \cdots + x_{n-1,n})$$

of N will still be denoted by ψ. Let us introduce the following definitions.

Definition 5.1. — (1) A representation π of G is *of Whittaker type* if $\dim_R \mathrm{Hom}_N(\pi,\psi) = 1$.
(2) A representation of G is said to be *generic* if it is both irreducible and of Whittaker type.

Since any two non-trivial characters of N are conjugate under the normalizer of N in G, being of Whittaker type does not depend on the choice of ψ.

Since N is a p-group and the characteristic of R is different from p, it follows that

– a representation of Whittaker type has a unique generic subquotient,
– a representation is of Whittaker type if and only if its contragredient is of Whittaker type.

If π is a representation of Whittaker type and $\xi \in \mathrm{Hom}_N(\pi,\psi)$ is non-zero, the map associating to any vector v in π the matrix coefficient $f_{v,\xi}$ (see Paragraph 2.1) is a morphism from π to $\mathrm{Ind}_N^G(\psi)$ whose image $\mathcal{W}(\pi,\psi)$ does not depend on the choice of ξ. This image is called the *Whittaker model* of π with respect to ψ. If τ is the unique generic subquotient of π, then the unique irreducible subrepresentation of $\mathcal{W}(\pi,\psi)$ is equal to $\mathcal{W}(\tau,\psi)$.

As in the non-Archimedean case (see Paragraph 4.3), we denote by g^* the transpose of the inverse of a matrix $g \in G$ and by w_n the antidiagonal permutation matrix of maximal length of G. If π is of Whittaker type, the representation $\pi^* : g \mapsto \pi(g^*)$ is of Whittaker type and the map

$$W \mapsto \left(\widetilde{W} : g \mapsto W(w_n g^*)\right)$$

is an isomorphism of R-vector spaces from $\mathcal{W}(\pi,\psi)$ to $\mathcal{W}(\pi^*,\psi^{-1})$. Moreover, the unique generic subquotient of π^* is isomorphic to that of $\pi^\vee$.

5.2. We introduce the Bessel function of a representation of Whittaker type.

Proposition 5.2. — *Let π be a representation of Whittaker type of G. There is a unique function $\mathrm{J}_{\pi,\psi} \in \mathcal{W}(\pi,\psi)$ such that*

(1) $\mathrm{J}_{\pi,\psi}(1) = 1$,

(2) $\mathrm{J}_{\pi,\psi}(xgy) = \psi(xy)\mathrm{J}_{\pi,\psi}(g)$ *for all $g \in G$ and $x, y \in N$.*

It is called the Bessel function *of π with respect to ψ.*

Proof. — Since π is of Whittaker type, its Whittaker model is of Whittaker type as well. Thus the space of functions $B \in \mathcal{W}(\pi,\psi)$ such that $B(gx) = \psi(x)B(g)$ has dimension 1. We are thus reduced to proving that $B(1) \neq 0$ for at least one of such B. For this, let V denote the space of π. It decomposes as the direct sum of the line V^{ψ} made of all $v \in V$ such that $\pi(x)(v) = \psi(x)v$ for all $x \in N$ and its unique N-stable complement $V(\psi)$. Fix a non-zero $v \in V^{\psi}$, and a non-zero linear form ξ on V with kernel $V(\psi)$. Then the function $B = f_{v,\xi}$ has the required property. □

We collect a couple of properties of the Bessel function.

Proposition 5.3. — *Let π be a representation of Whittaker type of G and τ be its unique generic subquotient. Then $\mathrm{J}_{\pi,\psi} = \mathrm{J}_{\tau,\psi}$.*

Proof. — We have $\mathrm{J}_{\tau,\psi} \in \mathcal{W}(\tau,\psi) \subseteq \mathcal{W}(\pi,\psi)$. By uniqueness, it follows that $\mathrm{J}_{\tau,\psi} = \mathrm{J}_{\pi,\psi}$. □

Proposition 5.4. — *Let $\widetilde{\pi}$ be a generic $\overline{\mathbb{Q}}_\ell$-representation of G and π be the unique generic subquotient of its reduction mod ℓ. Then $\mathrm{J}_{\widetilde{\pi},\psi}$ takes values in $\overline{\mathbb{Z}}_\ell$, and its reduction mod the maximal ideal of $\overline{\mathbb{Z}}_\ell$ is equal to $\mathrm{J}_{\pi,\psi}$.*

Proof. — The formula given in [**13**] Proposition 4.5 for the Bessel function of $\widetilde{\pi}$ shows that it takes values in $\overline{\mathbb{Z}}_\ell$. Let L be the lattice of $\overline{\mathbb{Z}}_\ell$-valued Whittaker functions in $\mathcal{W}(\widetilde{\pi},\psi)$. Then $L \otimes \overline{\mathbb{F}}_\ell$ is of Whittaker type and its Bessel function is the reduction of $\mathrm{J}_{\widetilde{\pi},\psi} \in L$ mod the maximal ideal of $\overline{\mathbb{Z}}_\ell$. The claimed equality follows from Proposition 5.3 applied to $L \otimes \overline{\mathbb{F}}_\ell$ and π. □

Let P denote the standard mirabolic subgroup of G, that is, the subgroup made of all matrices with last row $(0\ \dots\ 0\ 1)$.

Lemma 5.5. — *Let π be a representation of Whittaker type of G. If $g \in P$ and $\mathrm{J}_{\pi,\psi}(g) \neq 0$, then we have $g \in N$.*

Proof. — The proof for complex representations (see [**13**] Proposition 1.2) applies to R-representations. □

5.3. Let n, m be integers such that $1 \leqslant m < n$. Given representations of Whittaker type π, π' of $\mathrm{GL}_n(\boldsymbol{k})$, $\mathrm{GL}_m(\boldsymbol{k})$, respectively, an integer j such that $0 \leqslant j \leqslant n-m-1$, and Whittaker functions $W \in \mathcal{W}(\pi,\psi)$ and $W' \in \mathcal{W}(\pi',\psi^{-1})$, we define the sum

$$I_j(W,W') = \sum_{g \in N_m \backslash G_m} \sum_{x \in \mathbf{M}_{j,m}(\boldsymbol{k})} W\begin{pmatrix} g & 0 & 0 \\ x & 1_j & 0 \\ 0 & 0 & 1_{n-m-j} \end{pmatrix} W'(g).$$

We introduce the Rankin–Selberg γ-factor of the pair (π,π') following [**34**] Theorems 5.1, 5.4, [**31**] Theorem 2.10 and [**4**] Corollary 3.1.2. We set $w_{m,n-m} = \mathrm{diag}(\mathrm{id}_m, w_{n-m})$ as in §4.3. We also let q denote the cardinality of $\boldsymbol{k}$ and fix a square root $q^{1/2}$ of q in $R^{\times}$.

Proposition 5.6. — *Let π be a cuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$ and π' be a representation of Whittaker type of $\mathrm{GL}_m(\boldsymbol{k})$. Let $\omega_{\pi'}$ denote the central character of the generic subquotient of π'. There exists a unique scalar $\gamma(\pi,\pi',\psi)\in R$ such that*

$$I_{n-m-1-j}\left(w_{m,n-m}\cdot\widetilde{W},\widetilde{W'}\right)=\omega_{\pi'}(-1)^{n-1}\cdot q^{\frac{m(n-m-1-2j)}{2}}\cdot\gamma(\pi,\pi',\psi)\cdot I_j(W,W') \tag{5.1}$$

for all Whittaker functions $W\in\mathcal{W}(\pi,\psi)$, $W'\in\mathcal{W}(\pi',\psi^{-1})$ and all integers $j\in\{0,\dots,n-m-1\}$.

Proof. — The case where $j=0$ is given by [**4**] Corollary 3.1.2 (where the authors have a normalization of the γ-factor different from ours). The functional equation for any j follows from there as in the proof of [**31**] Theorem 2.10. □

Remark 5.7. — Our choice of normalization of the γ-factor $\gamma(\pi,\pi',\psi)$ is dictated by Proposition 5.14 below.

We collect a couple of properties of the γ-factor.

Proposition 5.8. — *Let π be a cuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$ and π' be a representation of Whittaker type of $\mathrm{GL}_m(\boldsymbol{k})$.*

(1) *The scalar $\gamma(\pi,\pi',\psi)$ is non-zero.*

(2) *We have*

$$\gamma(\pi,\pi',\psi)=\omega_{\pi'}(-1)^{n-1}\cdot q^{\frac{m(n-m-1)}{2}}\cdot\sum_{\gamma\in N_m\backslash G_m}\mathrm{J}_{\pi,\psi}\begin{pmatrix}0 & 1_{n-m}\\ g & 0\end{pmatrix}\mathrm{J}_{\pi',\psi^{-1}}(g).$$

(3) *If τ is the unique generic subquotient of π', then $\gamma(\pi,\pi',\psi)=\gamma(\pi,\tau,\psi)$.*

(4) *Given an $a\in\boldsymbol{k}^\times$, and denoting by ψ^a the character $x\mapsto\psi(ax)$ of $\boldsymbol{k}$, one has*

$$\gamma(\pi,\pi',\psi^a)=\omega_\pi(a)^m\cdot\omega_{\pi'}(a)^n\cdot\gamma(\pi,\pi',\psi). \tag{5.2}$$

Proof. — Assertion (2) follows from Lemma 5.5, as in [**31**] Proposition 2.16. Assertion (3) follows from Proposition 5.3 and Assertion (1). Assertion (4) follows from Assertion (2) together with the fact that J_{π,ψ^a} is equal to $g\mapsto\mathrm{J}_{\pi,\psi}(tgt^{-1})$ where t is the diagonal matrix $\mathrm{diag}(a^{n-1},\dots,a,1)$.

Since π^* is cuspidal and π'^* is of Whittaker type, we may apply Proposition 5.6 to the Whittaker functions $w_{m,n-m}\cdot\widetilde{W}\in\mathcal{W}(\pi^*,\psi^{-1})$, $\widetilde{W'}\in\mathcal{W}(\pi'^*,\psi)$ and the integer $n-m-1-j$. Thus

$$I_j\left(W,W'\right)=\omega_{\pi'}(-1)^{n-1}\cdot q^{\frac{-m(n-m-1-2j)}{2}}\cdot\gamma(\pi^\vee,\pi'^*,\psi^{-1})\cdot I_{n-m-1-j}(w_{m,n-m}\cdot\widetilde{W},\widetilde{W'}).$$

Combining this identity with (5.1), we get

$$\gamma(\pi,\pi',\psi)\cdot\gamma(\pi^*,\pi'^*,\psi^{-1})=1$$

which proves that $\gamma(\pi,\pi',\psi)$ is non-zero. □

Remark 5.9. — When m and n are odd, $\gamma(\pi,\pi',\psi)$ depends on the choice of $q^{1/2}$.

Let us recall that the reduction mod ℓ of a cuspidal $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_n(\boldsymbol{k})$ is (irreducible and) cuspidal ([**37**] III.1.1).

Proposition 5.10. — *Let $\widetilde{\pi}$ be a cuspidal $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_n(\boldsymbol{k})$ and $\widetilde{\pi}'$ be a generic $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_m(\boldsymbol{k})$. Let π be the reduction mod ℓ of $\widetilde{\pi}$ and π' be the unique generic subquotient of the reduction mod ℓ of $\widetilde{\pi}'$. Then the reduction mod ℓ of $\gamma(\widetilde{\pi},\widetilde{\pi}',\psi)$ is equal to $\gamma(\pi,\pi',\psi)$.*

Proof. — This follows from Lemma 5.4 and Proposition 5.8. □

5.4. We now state the multiplicativity property of γ-factors that we will need.

Lemma 5.11. — *Let n_1, n_2 be positive integers, and let π_i be a representation of Whittaker type of $\mathrm{GL}_{n_i}(\boldsymbol{k})$, for $i = 1, 2$. Then $\pi_1 \times \pi_2$ is a representation of Whittaker type of $\mathrm{GL}_{n_1+n_2}(\boldsymbol{k})$.*

Proof. — Let U be unipotent radical of the standard parabolic subgroup of $G = \mathrm{GL}_{n_1+n_2}(\boldsymbol{k})$ generated by the standard Levi subgroup $M = \mathrm{GL}_{n_1}(\boldsymbol{k}) \times \mathrm{GL}_{n_2}(\boldsymbol{k})$ and the subgroup N of unipotent upper triangular matrices. Then

$$\begin{aligned} \mathrm{Hom}_N(\pi_1 \times \pi_2, \psi) &\simeq \mathrm{Hom}_G(\pi_1 \times \pi_2, \mathrm{Ind}_N^G(\psi)) \\ &\simeq \mathrm{Hom}_M(\pi_1 \otimes \pi_2, \mathrm{Ind}_N^G(\psi)^U). \end{aligned}$$

The lemma follows from the fact that $\mathrm{Ind}_N^G(\psi)^U \simeq \mathrm{Ind}_{N\cap M}^M(\psi|_{N\cap M})$ by **[5]** Theorem 2.1. □

Lemma 5.12. — *Let π be a cuspidal irreducible representation of $\mathrm{GL}_n(\boldsymbol{k})$ and π' be a representation of Whittaker type of $\mathrm{GL}_m(\boldsymbol{k})$. Let S be an algebraically closed extension of R. Then $\pi \otimes S$ is irreducible and cuspidal, $\pi' \otimes S$ is of Whittaker type and*

$$\gamma(\pi \otimes S, \pi' \otimes S, \psi \otimes S) = \gamma(\pi, \pi', \psi). \tag{5.3}$$

Proof. — The fact that $\pi \otimes S$ is irreducible and cuspidal follows from **[37]** Théorème III.1.1 together with the fact that the base change from R to S of the mirabolic R-representation is isomorphic to the mirabolic S-representation. The fact that $\pi' \otimes S$ is of Whittaker type follows from the fact that, if κ is an irreducible component of the restriction of π' to N, then $\kappa \otimes S$ is irreducible. The identity (5.3) follows from Proposition 5.8(1) together with the fact that, by uniqueness, the Bessel functions of π' and $\pi' \otimes S$ coincide. □

Proposition 5.13. — *Let π be a cuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$, let m_1 and m_2 be positive integers such that $m_1 + m_2 = m$ and let π_i be a representation of Whittaker type of $\mathrm{GL}_{m_i}(\boldsymbol{k})$ for $i = 1, 2$. Then*

$$\gamma(\pi, \pi_1 \times \pi_2, \psi) = \gamma(\pi, \pi_1, \psi) \cdot \gamma(\pi, \pi_2, \psi).$$

Proof. — Suppose first that R is the field $\overline{\mathbb{Q}}_\ell$. Soudry-Zelingher **[36]** defined a factor $\Gamma(\pi, \pi', \psi)$ for any representation π' of Whittaker type of $\mathrm{GL}_m(\boldsymbol{k})$. Comparing our Proposition 5.8 with **[36]** Theorem 3.4(1), we deduce that

$$\Gamma(\pi, \pi', \psi) = \omega_{\pi'}(-1)^n \cdot q^{mn/2} \cdot \gamma(\pi, \pi'^*, \psi). \tag{5.4}$$

(Note that the representation π' is assumed to be irreducible in **[36]** Theorem 3.4. However, since the factor $\Gamma(\pi, \pi', \psi)$ only depends on the generic subquotient τ of π' by definition, the identity of **[36]** Theorem 3.4(1) also makes sense for π' not necessarily irreducible, provided that one replaces $\pi'^\vee$ by π'^* in the right hand side.) Putting (5.4) together with **[36]** Theorem 3.3, and observing that $(\pi_1 \times \pi_2)^*$ and $\pi_1^* \times \pi_2^*$ share the same generic subquotient, we deduce that Proposition 5.13 holds for $R = \overline{\mathbb{Q}}_\ell$.

Suppose now that R is the field $\overline{\mathbb{F}}_\ell$. For $i = 1, 2$, fix a generic $\overline{\mathbb{Q}}_\ell$-representation $\widetilde{\pi}_i$ of $\mathrm{GL}_{m_i}(\boldsymbol{k})$ whose reduction mod ℓ contains the generic subquotient τ_i of π_i and fix a cuspidal $\overline{\mathbb{Q}}_\ell$-lift $\widetilde{\pi}$ of π. We have

$$\gamma(\widetilde{\pi}, \widetilde{\pi}_1 \times \widetilde{\pi}_2, \psi) = \gamma(\widetilde{\pi}, \widetilde{\pi}_1, \psi) \cdot \gamma(\widetilde{\pi}, \widetilde{\pi}_2, \psi) \tag{5.5}$$

and, by Proposition 5.8, the left hand side of (5.5) is equal to $\gamma(\widetilde{\pi},\widetilde{\tau},\psi)$ where $\widetilde{\tau}$ is the generic subquotient of $\widetilde{\pi}_1\times\widetilde{\pi}_2$. Applying Proposition 5.10, we get

$$\gamma(\pi,\tau,\psi)=\gamma(\pi,\tau_1,\psi)\cdot\gamma(\pi,\tau_2,\psi) \tag{5.6}$$

where τ is the generic subquotient of the reduction mod ℓ of $\widetilde{\tau}$, and the right hand side of (5.6) is equal to $\gamma(\pi,\pi_1,\psi)\cdot\gamma(\pi,\pi_2,\psi)$ by Proposition 5.8. The expected result now follows from Proposition 5.8 again, together with the fact that τ is also the generic subquotient of $\pi_1\times\pi_2$.

Now assume that R is an arbitrary algebraically closed field of characteristic ℓ. Fix a field embedding $\iota:\overline{\mathbb{F}}_\ell\to R$. There are a cuspidal $\overline{\mathbb{F}}_\ell$-representation π_ℓ and $\overline{\mathbb{F}}_\ell$-representations $\pi_{1,\ell}$, $\pi_{2,\ell}$ of Whittaker type such that $\pi\simeq\pi_\ell\otimes R$ and $\pi_i\simeq\pi_{i,\ell}\otimes R$ for $i=1,2$, where tensor products are taken with respect to ι. The expected result now follows from Lemma 5.12. □

5.5. Let us prove Proposition 3.17. We will actually prove the following more general statement. Let F be a non-Archimedean locally compact field with residue field $\boldsymbol{k}$. Given a cuspidal representation of level 0 of $\mathrm{GL}_n(F)$, we associate with it a cuspidal representation $\overline{\pi}$ of $\mathrm{GL}_n(\boldsymbol{k})$ as in Paragraph 3.11, called the *type* of π. Let ψ_F be a non-trivial character of F of conductor 1 such that the character of $\boldsymbol{k}$ it induces is equal to ψ.

Proposition 5.14. — *Let π, π' be cuspidal representations of level* 0 *of* $\mathrm{GL}_n(F)$, $\mathrm{GL}_m(F)$, *respectively, with $n>m$. Let $\overline{\pi}$ and $\overline{\pi}'$ be their types. Then*

$$\gamma(X,\pi,\pi',\psi_F)=\gamma(\overline{\pi},\overline{\pi}',\psi).$$

In particular, the γ-factor $\gamma(X,\pi,\pi',\psi_F)$ is constant.

Proof. — In the case when $R=\overline{\mathbb{Q}}_\ell$, it follows from **[32]** Theorem 3.11 that

$$\gamma(X,\pi,\pi',\psi_F)=\omega_{\pi'}(-1)^{n-1}\cdot q^{m(n-m-1)/2}\cdot\overline{\gamma}(\overline{\pi},\overline{\pi}',\psi),$$

where $\overline{\gamma}(\overline{\pi},\overline{\pi}',\psi)$ is the factor associated with the pair $(\overline{\pi},\overline{\pi}')$ by **[31]** Theorem 2.10. Comparing our Proposition 5.8 with **[31]** Proposition 3.6, we deduce that

$$\overline{\gamma}(\overline{\pi},\overline{\pi}',\psi)=\omega_{\pi'}(-1)^{n-1}\cdot q^{-m(n-m-1)/2}\cdot\gamma(\overline{\pi},\overline{\pi}',\psi).$$

Thus Proposition 5.14 holds when $R=\overline{\mathbb{Q}}_\ell$. From there, we deduce Proposition 5.14 when $R=\overline{\mathbb{F}}_\ell$ by using Propositions 4.3 and 5.10, and we pass from $\overline{\mathbb{F}}_\ell$ to any algebraically closed field of characteristic ℓ as in the proof of Proposition 5.13. □

In order to prove Theorem 3.1, it now only remains to prove Proposition 3.18.

5.6. Before proceeding to the proof of Proposition 3.18, let us collect results from **[25]** which we will need. Recall that we have fixed a square root $q^{1/2}$ of q in R. Associated with any cuspidal representation π of $\mathrm{GL}_n(\boldsymbol{k})$, there is its Godement–Jacquet γ-factor $\gamma(\pi,\psi)$ defined in **[20, 34]** for complex representations and in **[25]** for R-representations.

Proposition 5.15. — *For any cuspidal representation π of* $\mathrm{GL}_n(\boldsymbol{k})$, *we have*

$$\gamma(\pi,1,\psi)=\gamma(\pi,\psi)$$

where 1 *in the left hand side is the trivial character of* $\boldsymbol{k}^\times$.

Proof. — The point is that these two γ-factors are defined by two different functional equations, so the fact that they are equal is not immediate. (See **[34]** Theorem 4.1.1 and **[25]** Definition 2.12 for the functional equations defining $\gamma(\pi,\psi)$.) For $R = \overline{\mathbb{Q}}_\ell$, the equality follows from **[34]** Theorem 4.2.1. We deduce it for $R = \overline{\mathbb{F}}_\ell$ by Proposition 5.10 and **[25]** Proposition 6.1, and for arbitrary R as in the proof of Proposition 5.13. □

Now assume that H is one of the subgroups of $\mathrm{GL}_n(\boldsymbol{k})$ of Proposition 3.18, that is:

– either $\boldsymbol{k}$ has a subfield $\boldsymbol{k}_0$ such that $[\boldsymbol{k} : \boldsymbol{k}_0] = 2$ and $H = \mathrm{GL}_n(\boldsymbol{k}_0)$,

– or $n = 2m$ for some integer $m \geqslant 1$ and H is the standard Levi subgroup $\mathrm{GL}_m(\boldsymbol{k}) \times \mathrm{GL}_m(\boldsymbol{k})$.

Given an H-distinguished cuspidal representation π of $\mathrm{GL}_n(\boldsymbol{k})$, the space $\mathrm{Hom}_H(\pi, 1)$ has dimension 1 (**[35]** Remark 4.3 in the Galois case, and **[35]** Corollary 2.16 in the Levi case). In the Levi case, the element

$$s = \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix} \in \mathrm{GL}_n(\boldsymbol{k})$$

normalizes H. It thus acts on this space by a sign, which we denote by $\mathrm{sgn}(\pi) \in \{-1, 1\} \subseteq R^\times$. In **[25]** Corollary 5.2, Theorem 5.3 we proved:

Proposition 5.16. — *Let π be an H-distinguished cuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$. In the Galois case, assume that $q^{1/2}$ is the cardinality of $\boldsymbol{k}_0$ and that ψ is trivial on $\boldsymbol{k}_0$. Then*

$$\gamma(\pi,\psi) = \begin{cases} 1 & \textit{in the Galois case,} \\ \mathrm{sgn}(\pi) & \textit{in the Levi case.} \end{cases}$$

We now proceed to the proof of Proposition 3.18 in the next section.

6. Gamma factors and distinction

For any integer $n \geqslant 1$, let G_n be the group $\mathrm{GL}_n(\boldsymbol{k})$, N_n be the subgroup of its upper triangular unipotent matrices and P_n be its mirabolic subgroup. Let H_n be any of the following subgroups of G_n:

– (Galois case) the subgroup $\mathrm{GL}_n(\boldsymbol{k}_0)$ where $\boldsymbol{k}$ is a quadratic extension of $\boldsymbol{k}_0$,

– (Levi case) the centralizer of the diagonal matrix $\mathrm{diag}(-1, 1, \dots, (-1)^n)$ in G_n. Depending on the parity of n, we thus have

$$H_n = \begin{pmatrix} * & 0 & * & \cdots & 0 & * & 0 \\ 0 & * & 0 & \cdots & * & 0 & * \\ * & 0 & * & \cdots & 0 & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & * & 0 & \cdots & * & 0 & * \\ * & 0 & * & \cdots & 0 & * & 0 \\ 0 & * & 0 & \cdots & * & 0 & * \end{pmatrix} (n \text{ even}) \quad \text{or} \quad H_n = \begin{pmatrix} * & 0 & * & \cdots & * & 0 & * \\ 0 & * & 0 & \cdots & 0 & * & 0 \\ * & 0 & * & \cdots & * & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & 0 & * & \cdots & * & 0 & * \\ 0 & * & 0 & \cdots & 0 & * & 0 \\ * & 0 & * & \cdots & * & 0 & * \end{pmatrix} (n \text{ odd}).$$

Note that, if r is the smallest integer such that $2r \geqslant n$, the subgroup H_n is conjugate to the standard Levi subgroup $\mathrm{GL}_r(\boldsymbol{k}) \times \mathrm{GL}_{n-r}(\boldsymbol{k})$ of G.

6.1. The purpose of this section is to prove the following proposition.

Proposition 6.1. — *Let π be a non-supercuspidal cuspidal representation of G_n, with associated supercuspidal representation ρ. Suppose that π is H_n-distinguished but has no lift distinguished by H_n. Then, for any non-trivial character ψ of $\boldsymbol{k}$, we have $\gamma(\pi, \rho^\vee, \psi) = 1$.*

This proposition is equivalent to Proposition 3.18. This is clear in the Galois case. In the Levi case, the fact that π is H_n-distinguished implies that $n = 2m$ for some $m \geqslant 1$ (by [**35**] Proposition 2.14). Then, as H_n is conjugate to the standard Levi subgroup $G_m \times G_m$ in G_n, distinction by H_n is equivalent to distinction by $G_m \times G_m$.

Remark that, for any $a \in \boldsymbol{k}^\times$, one has $\gamma(\pi, \rho^\vee, \psi^a) = \gamma(\pi, \rho^\vee, \psi)$ from (5.2). It thus suffices to prove Proposition 6.1 for a specific character ψ. *We will thus assume from now on until the end of this section, that, in the Galois case, the character ψ is trivial on $\boldsymbol{k}_0$.* In the Levi case, we make no assumption on ψ.

6.2. In this paragraph, we collect some properties of the groups H_n. First we observe that H_n is the group of fixed points of an involution σ of G_n. In the Galois case, it is the non-trivial automorphism of $\boldsymbol{k}/\boldsymbol{k}_0$ acting componentwise. In the Levi case it is the matrix $\mathrm{diag}(-1, 1, \dots, (-1)^n)$ acting by conjugacy. Thanks to the assumption made on ψ in the previous paragraph, the character ψ of N_n satisfies $\psi \circ \sigma = \psi^{-1}$, which implies that

$$\psi \text{ is trivial on } N_n \cap H_n. \tag{6.1}$$

This explains why we didn't choose for H_n a standard Levi subgroup in the Levi case.

Let $(n_1, \dots, n_r)$ be a family of positive integers whose sum is equal to n, let M be the standard Levi subgroup $G_{n_1} \times \dots \times G_{n_r}$ of G_n, let Q be the standard parabolic subgroup of G_n with Levi subgroup M and Q^- be the opposite parabolic subgroup with respect to M. Finally, let U, U^- be the unipotent radicals of Q, Q^-, respectively. Then

$$U^- M U \cap H_n = (U^- \cap H_n)(M \cap H_n)(U \cap H_n), \tag{6.2}$$

$$M \cap H_n = H_{n_1} \times \dots \times H_{n_r}. \tag{6.3}$$

Recall that g^* denotes the transpose of the inverse of a matrix $g \in G_n$ and w_n is the antidiagonal permutation matrix of maximal length of G_n. Then

$$H_n \text{ is stable by } *, \tag{6.4}$$

$$H_n \text{ is normalized by } w_n. \tag{6.5}$$

This list of properties (6.1) to (6.5) will allow us to treat the Galois and Levi cases uniformly.

We will also need the following properties of H_n-distinguished generic representations of G_n.

Lemma 6.2. — *For any H_n-distinguished cuspidal representation π of G_n, one has*

$$\dim \mathrm{Hom}_{P_n \cap H_n}(\pi, 1) = 1.$$

Proof. — In the Levi case, the result is given by [**35**] Remark 2.15. In the Galois case, is it given by [**3**] Proposition 4.3 for $R = \mathbb{C}$ only. Let us consider the case of a general R in the Galois case. We will write $G = G_n$, $P = P_n$, etc.

First note that restriction of π to P is isomorphic to the induced representation $\mathrm{Ind}_N^P(\psi)$ (see [**37**] III.1.1). It follows from a simple application of Mackey's formula that the dimension of the space $\mathrm{Hom}_{P \cap H}(\pi, R)$ is the number of (H, N)-double cosets $HgN \subseteq G$ such that $\psi|_{N \cap H^g} = 1$.

It follows that, in the case when R is the field of complex numbers, [**3**] Proposition 4.3 tells us that there is only one double coset $HgN \subseteq G$ such that $\psi|_{N\cap H^g} = 1$ (and it is HN).

Since N is a p-group, the character ψ takes values in $\mu_{p^\infty}(R)$, the group of elements of $R^\times$ whose order is a p-power. Fix a group isomorphism

$$\iota : \mu_{p^\infty}(R) \to \mu_{p^\infty}(\mathbb{C})$$

(which exists since R is algebraically closed and has characteristic different from p). Then $\iota\circ\psi$ is well-defined and HN is the unique double coset $HgN \subseteq G$ such that $\iota\circ\psi|_{N\cap H^g} = 1$. Since ι is injective, the same result holds for ψ itself. We thus get the expected result. □

Lemma 6.3. — *Any H_n-distinguished generic representation of G_n is σ-self-dual.*

Proof. — As in the proof of the preceding lemma, we write $G = G_n$, $H = H_n$, etc. In the Galois case, any H-distinguished irreducible representation is σ-self-dual (see [**35**] Remark 4.3). Suppose that we are in the Levi case. Since σ is an inner involution, an irreducible representation of G is σ-self-dual if and only if it is self-dual. Let π be an H-distinguished generic R-representation of G. Arguing as in [**24**] 5.7, we may and will assume that $R = \overline{\mathbb{F}}_\ell$.

By [**24**] Lemma 5.8 (whose proof works for any H-distinguished irreducible representation of G and not only for H-distinguished cuspidal representations), there exists an H-distinguished irreducible $\overline{\mathbb{Q}}_\ell$-representation $\widetilde{\pi}$ of G whose reduction mod ℓ contains π. The main result of [**19**] tells us that this representation $\widetilde{\pi}$ is self-dual.

Let $\mu_1,\dots,\mu_t$ be cuspidal $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{GL}_{n_1}(\boldsymbol{k}),\dots,\mathrm{GL}_{n_t}(\boldsymbol{k})$ respectively, for integers $n_1,\dots,n_t \geqslant 1$ of sum n, such that $\widetilde{\pi}$ occurs in $\mu_1\times\cdots\times\mu_t$. By [**37**] III.1.1, the reduction mod ℓ of μ_i is irreducible and cuspidal. The representation π is thus the unique generic subquotient of the induced representation $\mathbf{r}_\ell(\mu_1)\times\cdots\times\mathbf{r}_\ell(\mu_t)$. It is *a fortiori* the unique generic subquotient of $\mathbf{r}_\ell(\widetilde{\pi})$. Since $\widetilde{\pi}$ is self-dual, π and $\pi^{\vee\sigma}$ are generic subquotients of $\mathbf{r}_\ell(\widetilde{\pi})$. Thus π is self-dual. □

Lemma 6.4. — *Let π be a supercuspidal representation of G_n. If π is σ-self-dual, then*

(1) *In the Galois case, n is odd and π is H_n-distinguished.*

(2) *In the Levi case, either n is even and π is H_n-distinguished, or $n = 1$ and π is a character of order at most 2 of $\boldsymbol{k}^\times$.*

Proof. — In the Galois case, this is [**24**] Lemmas 2.3 and 2.5. In the Levi case, this is [**24**] Lemmas 2.17 and 2.19. □

6.3. In this paragraph, we fix an integer n and abbreviate $G = G_n$, $P = P_n$, etc. Let U be the unipotent radical of P and G' be the image of G_{n-1} in G via the embedding $g \mapsto \mathrm{diag}(g,1)$. We thus have $P = G'U$. We also write $N' = N\cap G'$ and $H' = H\cap G'$. From (6.2), (6.3) one has

$$P\cap H = H'(U\cap H). \tag{6.6}$$

The goal of this paragraph is to associate a non-zero scalar $c(\pi,\psi)$ to any H-distinguished cuspidal representation π of G, which is a proportionality constant between two explicit H-invariant linear forms on the Whittaker model of π (see Proposition 6.8).

Let π be a representation of Whittaker type of G. Associated with it, there are R-linear forms $\Lambda = \Lambda_\pi$ and $\Lambda^* = \Lambda^*_\pi$ on its Whittaker space $\mathcal{W}(\pi,\psi)$ defined for all $W \in \mathcal{W}(\pi,\psi)$ by

$$\begin{aligned}\Lambda(W) &= \sum_{h'\in N'\cap H'\backslash H'} W(h'),\\ \Lambda^*(W) &= \sum_{h'\in N'\cap H'\backslash H'} \widetilde{W}(h') = \sum_{h'\in N'\cap H'\backslash H'} W(w_n h'^*).\end{aligned}$$

The first sum over $N' \cap H'\backslash H'$ is well-defined thanks to (6.1). For the second sum, observe that the conjugate of $(N' \cap H')^*$ by w_n is contained in $w_n(N \cap H)^* w_n^{-1} = N \cap w_n H^* w_n^{-1}$, which is equal to $N \cap H$ thanks to (6.4) and (6.5).

Lemma 6.5. — *The linear form Λ is $P \cap H$-invariant and non-zero.*

Proof. — It follows from (6.1) and (6.6) that, for all $W \in \mathcal{W}(\pi,\psi)$, we have

$$\sum_{g\in P\cap H} W(g) = \sum_{u\in U\cap H}\sum_{h'\in H'} W(uh') = |U \cap H| \cdot |N' \cap H'| \cdot \Lambda(W)$$

and the left hand side is $P \cap H$-invariant as a linear form on $\mathcal{W}(\pi,\psi)$. By Lemma 5.5, we have

$$\Lambda(\mathrm{J}_{\pi,\psi}) = \mathrm{J}_{\pi,\psi}(1) = 1$$

hence Λ is non-zero. □

Lemma 6.6. — *The linear form Λ^* is $(P \cap H)^*$-invariant and non-zero.*

Proof. — First, it follows from (6.6) that

$$(P \cap H)^* = (H'(U \cap H))^* = H'^*(U \cap H)^*$$

and Λ^* is clearly invariant by H'^*. It thus remains to prove that it is invariant by $(U \cap H)^*$. For any $x \in (U \cap H)^*$, we have

$$\Lambda^*(x \cdot W) = \sum_{h'\in N'\cap H'\backslash H'} W(w_n h'^* x).$$

For a given $h' \in H'$, write $u = w_n h'^* x h'^{*-1} w_n^{-1}$. Since H' normalizes $U \cap H$, we have

$$u \in w_n(U \cap H)^* w_n^{-1} \subseteq w_n(N \cap H)^* w_n^{-1} = N \cap w_n H^* w_n^{-1} = N \cap H.$$

It thus follows from (6.1) that $W(w_n h'^* x) = W(u w_n h'^*) = \psi(u) W(w_n h'^*) = W(w_n h'^*)$. The expected result follows.

Now let us consider the Bessel function $\mathrm{J}_{\widetilde{\pi},\psi^{-1}} \in \mathcal{W}(\widetilde{\pi},\psi^{-1})$ and its image $\widetilde{\mathrm{J}}_{\widetilde{\pi},\psi^{-1}} \in \mathcal{W}(\pi,\psi)$ by the map $W \mapsto \widetilde{W}$ defined in Paragraph 5.1. By Lemma 5.5, we have

$$\Lambda^*(\widetilde{\mathrm{J}}_{\widetilde{\pi},\psi^{-1}}) = \sum_{h'\in N'\cap H'\backslash H'} \mathrm{J}_{\widetilde{\pi},\psi^{-1}}(h') = \mathrm{J}_{\widetilde{\pi},\psi^{-1}}(1) = 1$$

hence Λ^* is non-zero. □

Lemma 6.7. — *Let π be an H-distinguished generic representation of G. Suppose that*

$$\dim \mathrm{Hom}_{P\cap H}(\pi, 1) = \dim \mathrm{Hom}_{P\cap H}(\pi^\vee, 1) = 1.$$

Then there exists a unique non-zero scalar $c(\pi,\psi) \in R^\times$ such that $\Lambda^* = c(\pi,\psi) \cdot \Lambda$.

Proof. — We will identify π with its Whittaker model $\mathcal{W}(\pi,\psi)$. By assumption, the containment

$$\mathrm{Hom}_H(\pi,1) \subseteq \mathrm{Hom}_{P\cap H}(\pi,1)$$

is an equality. The linear form Λ, which is $P \cap H$-invariant and non-zero by Lemma 6.5, is thus H-invariant.

On the other hand, the fact that the spaces $\mathrm{Hom}_{P\cap H}(\pi^*,1)$ and $\mathrm{Hom}_{(P\cap H)^*}(\pi,1)$ are isomorphic together with the fact that $\pi^* \simeq \pi^\vee$ imply that

$$\dim \mathrm{Hom}_{(P\cap H)^*}(\pi,1) = \dim \mathrm{Hom}_{P\cap H}(\pi^\vee,1) = 1.$$

Since $(P \cap H)^* = P^* \cap H^* \subseteq H^* = H$, where the latter equality follows from (6.4), we have

$$\mathrm{Hom}_H(\pi,1) = \mathrm{Hom}_{H^*}(\pi,1) \subseteq \mathrm{Hom}_{(P\cap H)^*}(\pi,1)$$

and the containment is an equality. The form Λ^*, which is $(P \cap H)^*$-invariant and non-zero by Lemma 6.6, is thus H-invariant. Since the space $\mathrm{Hom}_H(\pi,1)$ has dimension 1 by assumption, it follows that there exists a unique non-zero scalar $c(\pi,\psi) \in R^\times$ such that $\Lambda^* = c(\pi,\psi)\cdot\Lambda$. □

Proposition 6.8. — *For any H-distinguished cuspidal representation π of G, there is a unique non-zero scalar $c(\pi,\psi) \in R^\times$ such that $\Lambda^* = c(\pi,\psi)\cdot\Lambda$.*

Proof. — As $\pi^\vee$ is isomorphic to π^* and $H^* = H$, the spaces $\mathrm{Hom}_H(\pi^\vee,1)$ and $\mathrm{Hom}_H(\pi,1)$ have the same dimension. The representation $\pi^\vee$ is thus both cuspidal and H-distinguished. The proposition now follows from Lemmas 6.7 and 6.2. □

6.4. In preparation of the next paragraph, we prove the following lemma. As in the previous paragraph, we fix an integer n and abbreviate $G = G_n$, $H = H_n$, etc.

Let RG denote the space of R-valued functions on G equipped with the action of G by right translations. Let I_ψ be the idempotent endomorphism of RG defined by

$$I_\psi(f)(g) = \frac{1}{|N|}\sum_{u\in N}\psi^{-1}(u)f(ug), \quad f \in RG,\ g \in G.$$

Its image is equal to the induced representation $\mathrm{Ind}_N^G(\psi)$. We denote by $\mathrm{Ind}_H^G(1)$ the representation of G induced by the trivial character of H, whose space is made of all functions of RG which are invariant by left translations by H.

Lemma 6.9. — *Given any $W \in \mathrm{Ind}_N^G(\psi^{-1})$, the following assertions are equivalent.*

(1) *For all $g \in G$, one has*

$$\sum_{h\in H} W(hg) = 0.$$

(2) *For all functions $\phi \in I_\psi(\mathrm{Ind}_H^G(1))$, one has*

$$\sum_{g\in G}\phi(g)W(g) = 0.$$

Proof. — Given any $W \in \mathrm{Ind}_N^G(\psi^{-1})$ and $f \in \mathrm{Ind}_H^G(1)$, we have

$$\begin{aligned}\sum_{g\in G} I_\psi(f)(g)W(g) &= \sum_{g\in G}\frac{1}{|N|}\sum_{u\in N}\psi^{-1}(u)f(ug)W(g)\\ &= \frac{1}{|N|}\sum_{g\in G}\sum_{u\in N} f(ug)W(ug)\\ &= \sum_{g\in G} f(g)W(g)\\ &= \sum_{x\in H\backslash G} f(x)\Big(\sum_{h\in H} W(hx)\Big).\end{aligned}$$

The lemma follows. □

6.5. The following proposition will be crucial to our proof of Proposition 6.1. In this proposition, we introduce the technical assumption (6.7) which we will discuss in §6.6 and §6.7.

Proposition 6.10. — *Let π be an H_n-distinguished cuspidal representation of G_n and let π' be a representation of Whittaker type of G_{n-1} such that*

$$\mathcal{W}(\pi',\psi^{-1}) \cap I_{\psi^{-1}}\left(\mathrm{Ind}_{H_{n-1}}^{G_{n-1}}(1)\right) \neq \{0\}. \tag{6.7}$$

Then $\gamma(\pi,\pi',\psi) = c(\pi,\psi)$.

Proof. — By the functional equation, for all $W \in \mathcal{W}(\pi,\psi)$ and all $W' \in \mathcal{W}(\pi',\psi^{-1})$, we have

$$\sum_{g\in N_{n-1}\backslash G_{n-1}} W\begin{pmatrix}0&1\\g&0\end{pmatrix}W'(g) = \gamma(\pi,\pi',\psi)\cdot\sum_{g\in N_{n-1}\backslash G_{n-1}} W\begin{pmatrix}g&0\\0&1\end{pmatrix}W'(g).$$

Let Δ denote the function

$$\Delta(g) = W\begin{pmatrix}0&1\\g&0\end{pmatrix} - c(\pi,\psi)\cdot W\begin{pmatrix}g&0\\0&1\end{pmatrix}$$

for all $g \in G_{n-1}$. It belongs to the space $\mathrm{Ind}_{N_{n-1}}^{G_{n-1}}(\psi)$. By Proposition 6.8, we have

$$\sum_{h\in H_{n-1}} \Delta(h) = |N'\cap H'|(\Lambda^*(W) - c(\pi,\psi)\Lambda(W)) = 0.$$

By Lemma 6.9 applied to G_{n-1}, H_{n-1} and ψ^{-1}, we thus have

$$\sum_{g\in N_{n-1}\backslash G_{n-1}}\left(W\begin{pmatrix}0&1\\g&0\end{pmatrix} - c(\pi,\psi)W\begin{pmatrix}g&0\\0&1\end{pmatrix}\right)\phi(g) = 0$$

for all $\phi \in I_{\psi^{-1}}\left(\mathrm{Ind}_{H_{n-1}}^{G_{n-1}}(1)\right)$. Taking a non-zero W' in the left hand side of (6.7), we have

$$\sum_{g\in N_{n-1}\backslash G_{n-1}} W\begin{pmatrix}0&1\\g&0\end{pmatrix}W'(g) = c(\pi,\psi)\sum_{g\in N_{n-1}\backslash G_{n-1}} W\begin{pmatrix}g&0\\0&1\end{pmatrix}W'(g).$$

Choosing a $g_0 \in G_{n-1}$ such that $W'(g_0) \neq 0$, and setting $W = g_0^{-1} \cdot \mathrm{J}_{\pi,\psi}$, we get

$$\sum_{g \in N_{n-1} \backslash G_{n-1}} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W'(g) = \sum_{g \in N_{n-1} \backslash G_{n-1}} \mathrm{J}_{\pi,\psi} \begin{pmatrix} gg_0^{-1} & 0 \\ 0 & 1 \end{pmatrix} W'(g) = W'(g_0) \neq 0.$$

As $\gamma(\pi, \pi', \psi)$ is the unique scalar which satisfies this equation, we get $\gamma(\pi, \pi', \psi) = c(\pi, \psi)$. □

Corollary 6.11. — *Let π be an H_n-distinguished cuspidal representation of G_n. Let π' and π'' be H_{n-1}-distinguished representations of Whittaker type of G_{n-1}. Suppose that $\mathcal{W}(\pi', \psi^{-1})$ and $\mathcal{W}(\pi'', \psi^{-1})$ have a non-zero intersection with $I_{\psi^{-1}}(\mathrm{Ind}_{H_{n-1}}^{G_{n-1}}(1))$. Then*

$$\gamma(\pi, \pi', \psi) = \gamma(\pi, \pi'', \psi).$$

Proof. — This follows from the equalities $\gamma(\pi, \pi', \psi) = c(\pi, \psi) = \gamma(\pi, \pi'', \psi)$ given by Proposition 6.10 applied to π' and π''. □

6.6. As in §6.3, the integer n is fixed and we abbreviate $G = G_n$, $H = H_n$, etc. Let us introduce the following definition.

Definition 6.12. — A representation π of Whittaker type of G is said to be H-*special*, or *special* with respect to H, if the intersection $\mathcal{W}(\pi, \psi) \cap I_\psi(\mathrm{Ind}_H^G(1))$ is non-zero.

We need to produce enough H-special representations of Whittaker type in order to prove Proposition 6.1. For this, we introduce the following criterion.

Definition 6.13. — An H-distinguished representation π of Whittaker type of G is said to be *of class* $\mathcal{C}(H)$ if there is an H-invariant linear form on π which is non-zero on the 1-dimensional space $\mathrm{Hom}_N(\psi, \pi)$ of vectors v of π such that $\pi(u)v = \psi(u)v$ for all $u \in N$.

This gives us a sufficient condition for being H-special.

Lemma 6.14. — *Any representation of G of class $\mathcal{C}(H)$ is H-special.*

Proof. — Let ξ be an H-invariant linear form on π. Given any vector $v \in \pi$, the matrix coefficient $f_{v,\xi}$ is in $\mathrm{Ind}_H^G(1)$ and the function $I_\psi(f_{v,\xi})$ is equal to $f_{v,\xi'}$ where

$$\xi' = \frac{1}{|N|} \sum_{u \in N} \psi(u) \xi \circ \pi(u^{-1}) \in \mathrm{Hom}_N(\pi, \psi).$$

If ξ' is non-zero, the image of $v \mapsto f_{v,\xi'}$ is $\mathcal{W}(\pi, \psi)$, thus $\mathcal{W}(\pi, \psi)$ is contained in $I_\psi(\mathrm{Ind}_H^G(1))$.

Now let $j \in \mathrm{Hom}_N(\psi, \pi)$ be non-zero and suppose that $\xi(j)$ is non-zero. One has

$$\xi'(j) = \frac{1}{|N|} \sum_{u \in N} \psi(u) \xi(\pi(u^{-1})j) = \frac{1}{|N|} \sum_{u \in N} \xi(j) = \xi(j)$$

which is non-zero. □

Remark 6.15. — In the case when $R = \mathbb{C}$, it follows from [**3**] Theorem 1.1 that, in the Galois case, any H-distinguished generic representation of G is of class $\mathcal{C}(H)$.

Recall that P is the mirabolic subgroup of G.

Lemma 6.16. — *Let π be an H-distinguished generic representation of G. Suppose that the space $\mathrm{Hom}_{P \cap H}(\pi, 1)$ has dimension 1. Then π is of class $\mathcal{C}(H)$.*

Proof. — We identify π with its Whittaker model $\mathcal{W}(\pi,\psi)$. By assumption, the containment

$$\mathrm{Hom}_H(\pi,1) \subseteq \mathrm{Hom}_{P\cap H}(\pi,1)$$

is an equality. Let ξ be the linear form on $\mathcal{W}(\pi,\psi)$ defined by

$$\xi(W) = \sum_{h\in P\cap H} W(h).$$

It is $P\cap H$-invariant, thus H-invariant. Evaluating it on the Bessel function $\mathrm{J}_{\pi,\psi}$, it follows from Lemma 5.5 that

$$\xi(\mathrm{J}_{\pi,\psi}) = \sum_{h\in N\cap H} \mathrm{J}_{\pi,\psi}(h) = |N\cap H| \neq 0.$$

Hence π is of class $\mathcal{C}(H)$. □

It follows immediately from Lemma 6.2 that:

Corollary 6.17. — *Any H-distinguished cuspidal representation of G is of class $\mathcal{C}(H)$.*

6.7. We now prove a preservation property of $\mathcal{C}(H)$ under parabolic induction. Let n_1 and n_2 be positive integers such that $n = n_1 + n_2$. Let M be the standard Levi subgroup $G_{n_1}\times G_{n_2}$ of G_n and Q be the standard parabolic subgroup of G generated by M and N. Let U be the unipotent radical of Q and U^- be the unipotent radical of the parabolic subgroup Q^- opposite to Q with respect to M. We will write ψ_i for the character of N_{n_i} induced by ψ.

Lemma 6.18. — *For $i = 1,2$, let π_i be an H_{n_i}-distinguished representation of Whittaker type of class $\mathcal{C}(H_{n_i})$ of G_{n_i}. The induced representation $\pi_1\times\pi_2$ is of class $\mathcal{C}(H_n)$.*

Proof. — Let π be the representation of G_n parabolically induced from $\pi_M = \pi_1\otimes\pi_2$ along the parabolic subgroup Q^-. By **[16]** Theorem 1.1, it is isomorphic to $\pi_1\times\pi_2$. It thus suffices to prove that π is of class $\mathcal{C}(H)$. First, a simple application of Mackey's formula gives us

$$\mathrm{Hom}_N(\psi,\pi) \simeq \bigoplus_{g\in N\backslash G/Q^-} \mathrm{Hom}_{N^g\cap Q^-}(\psi^g,\pi_{Q^-})$$

where π_{Q^-} is the inflation of π_M to Q^-. By Lemma 5.11, we know that $\pi_1\times\pi_2$ (thus π) is a representation of Whittaker type. There is thus a unique double coset NgQ^- contributing to this decomposition. For $g\in NQ^-$, we have $N\cap Q^- = N\cap M = N_{n_1}\times N_{n_2}$, thus

$$\mathrm{Hom}_{N\cap Q^-}(\psi,\pi_{Q^-}) \simeq \mathrm{Hom}_{N_{n_1}}(\psi_1,\pi_1)\otimes \mathrm{Hom}_{N_{n_2}}(\psi_2,\pi_2)$$

which has dimension 1. Thus NQ^- is the only contributing double coset and one has an isomorphism of R-vector spaces

$$\mathrm{Hom}_N(\psi,\pi) \simeq \mathrm{Hom}_{N_{n_1}}(\psi_1,\pi_1)\otimes \mathrm{Hom}_{N_{n_2}}(\psi_2,\pi_2)$$

given by $f\mapsto f(1)$.

For $i=1,2$, fix a non-zero vector $j_i\in \mathrm{Hom}_{N_{n_i}}(\psi_i,\pi_i)$, and let $j\in\mathrm{Hom}_N(\psi,\pi)$ be the unique element of π such that $j(1) = j_1\otimes j_2$. Thus j is supported on $Q^-N = U^-MU$ and

$$j(u^-mu) = \psi(u)\pi_M(m)(j_1\otimes j_2), \quad u^-\in U^-,\ m\in M,\ u\in U,$$

and more generally one has $j(gx) = \psi(x)j(g)$ for all $g\in G$ and all $x\in N$.

For $i = 1, 2$, fix $\xi_i \in \mathrm{Hom}_{H_{n_i}}(\pi_i, 1)$ such that $\xi_i(j_i) \neq 0$. Define an H_n-invariant linear form ξ on the space of π by

$$\xi(f) = \sum_{h \in Q^- \cap H_n \backslash H_n} \langle f(h), \xi_1 \otimes \xi_2 \rangle$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket. It is well-defined thanks to (6.2) and (6.3). Let us now compute $\xi(j)$. Thanks to (6.2), (6.3) and (6.1), we have

$$\begin{aligned} \xi(j) &= \sum_{h \in Q^- \cap H_n \backslash H_n} \langle j(h), \xi_1 \otimes \xi_2 \rangle \\ &= \sum_{u \in U \cap H_n} \langle j(u), \xi_1 \otimes \xi_2 \rangle \\ &= |U \cap H_n| \cdot \xi_1(j_1) \cdot \xi_2(j_2) \end{aligned}$$

which is non-zero. This finishes the proof. □

Corollary 6.19. — *Let $n_1, \dots, n_r$ be positive integers of sum n. For $i = 1, \dots, r$, let π_i be some H_{n_i}-distinguished cuspidal representation of G_{n_i}. Then $\pi_1 \times \cdots \times \pi_r$ is of class $\mathcal{C}(H_n)$.*

Proof. — This follows from Corollary 6.17 and Lemma 6.18. □

6.8. We now prove Proposition 6.1, which will end the proof of our main Theorem 3.1. We will actually prove the more general following result.

Proposition 6.20. — *Let π be an H_n-distinguished cuspidal representation of G_n and let μ be an H_m-distinguished supercuspidal representation of G_m for some $m < n$. Then*

$$\gamma(\pi, \mu, \psi) = \begin{cases} \mathrm{sgn}(\pi) & \text{in the Levi case with } m = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Let us first explain how to deduce Proposition 6.1 from Proposition 6.20. Let π be a non-supercuspidal cuspidal representation of G_n, and ρ be the supercuspidal representation of G_k associated with it. Suppose that π is H_n-distinguished but has no H_n-distinguished lift. By Lemma 6.3, it is σ-self-dual. By uniqueness of its supercuspidal support, ρ is σ-self-dual as well. By Lemma 6.4, ρ is distinguished, unless we are in the Levi case and ρ is the character of order 2 of $\boldsymbol{k}^\times$. But the Levi case with $k = 1$ is excluded by **[24]** Lemma 6.10(2.b), which says that π has a distinguished $\overline{\mathbb{Q}}_\ell$-lift in that case. To deduce Proposition 6.1, it thus remains to apply Proposition 6.20 to the distinguished representation $\mu = \rho^\vee$.

Let us prove Proposition 6.20 in the exceptional case where H_n is a Levi subgroup and $m = 1$. In this case, μ is the trivial character of $\boldsymbol{k}^\times$. We thus have $\gamma(\pi, \mu, \psi) = \mathrm{sgn}(\pi)$ thanks to Propositions 5.15 and 5.16. It now remains to prove Proposition 6.20:

– either in the Galois case (in which case ψ is trivial on $\boldsymbol{k}_0$),
– or in the Levi case with $m \neq 1$ (in which case m is even by Lemma 6.4).

Let π be an H_n-distinguished cuspidal representation of G_n. Denote by 1^a, for any integer $a \geqslant 1$, the induced representation $1 \times \cdots \times 1$ where the trivial character 1 of $\boldsymbol{k}^\times$ occurs a times. It is of Whittaker type and satisfies $\mathcal{C}(H_a)$ by Corollary 6.19. By Corollary 6.17, the representation μ satisfies $\mathcal{C}(H_m)$. By Corollary 6.19 again, the representation $\tau = \mu \times 1^{n-1-m}$ satisfies $\mathcal{C}(H_{n-1})$.

By Lemma 6.14, the representations 1^{n-1} and τ are both H_{n-1}-special. It follows from Propositions 5.13 and 6.10 that

$$\begin{aligned}\gamma(\pi,\mu,\psi) &= \gamma(\pi,\mu\times 1^{n-1-m},\psi)\cdot\gamma(\pi,1^{n-1-m},\psi)^{-1}\\ &= \gamma(\pi,1^{n-1},\psi)\cdot\gamma(\pi,1,\psi)^{-n+1+m}\\ &= \gamma(\pi,1,\psi)^{m}\end{aligned}$$

which is equal to $\gamma(\pi,\psi)^m$ by Proposition 5.15. Proposition 6.20 follows automatically from Proposition 5.16, together with the fact that m is even in the Levi case. Theorem 3.1 is proven.

Corollary 6.21. — *For any H_n-distinguished cuspidal representation π of G_n, one has*

$$c(\pi,\psi)=\begin{cases}1 & \textit{in the Galois case,}\\ \mathrm{sgn}(\pi)^{n-1} & \textit{in the Levi case.}\end{cases}$$

Proof. — By Proposition 6.10, we have $c(\pi,\psi)=\gamma(\pi,1^{n-1},\psi)=\gamma(\pi,1,\psi)^{n-1}$. The result follows from Propositions 5.15 and 5.16. □

Remark 6.22. — Let π be a non-supercuspidal cuspidal representation of G_n, with associated supercuspidal representation ρ of degree k. Suppose that π is H_n-distinguished. Then

$$\gamma(\pi,\rho^\vee,\psi)=\begin{cases}-1 & \text{in the Levi case with } k=1,\\ \ \ 1 & \textit{otherwise.}\end{cases}$$

The only case which is not covered by Proposition 6.20 is the Levi case where ρ is the character of order 2 of $\boldsymbol{k}^\times$. In this case, we have $\gamma(\pi,\rho^\vee,\psi)=\gamma(\pi',1,\psi)$ where $\pi'=\pi\rho$ is the unique cuspidal subquotient of 1^n. We then have $\gamma(\pi',1,\psi)=\mathrm{sgn}(\pi')$ thanks to Propositions 5.15, 5.16, and the result now follows from **[24]** Lemma 6.12.

6.9. We give the following corollaries to Theorem 3.1.

Corollary 6.23. — *Let $\boldsymbol{k}/\boldsymbol{k}_0$ be a quadratic extension of finite fields of characteristic p and π be a non-supercuspidal, cuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$. The following assertions are equivalent.*

(1) *π is $\mathrm{GL}_n(\boldsymbol{k}_0)$-distinguished.*
(2) *π has a $\mathrm{GL}_n(\boldsymbol{k}_0)$-distinguished cuspidal lift to $\overline{\mathbb{Q}}_\ell$.*
(3) *π has a σ-self-dual cuspidal lift to $\overline{\mathbb{Q}}_\ell$.*
(4) *π is σ-self-dual, n is odd and e_0 is even.*

Corollary 6.24. — *Let $\boldsymbol{k}$ be a finite field of characteristic p and π be a non-supercuspidal, cuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$ with $n=2m$. The following assertions are equivalent.*

(1) *π is $\mathrm{GL}_m(\boldsymbol{k})\times\mathrm{GL}_m(\boldsymbol{k})$-distinguished.*
(2) *π has a $\mathrm{GL}_m(\boldsymbol{k})\times\mathrm{GL}_m(\boldsymbol{k})$-distinguished cuspidal lift to $\overline{\mathbb{Q}}_\ell$.*
(3) *π has a self-dual cuspidal lift to $\overline{\mathbb{Q}}_\ell$.*
(4) *π is self-dual and either r, n/e are odd, or $r=n$.*

Proof. — We will prove the two corollaries at the same time. In both cases, we have the chain of implications (4) ⇒ (3) ⇒ (2) ⇒ (1). It thus remains to prove that (1) implies (4) in both cases.

Fix a quadratic extension F/F_0 of p-adic fields such that $\boldsymbol{k}$, $\boldsymbol{k}_0$ are the residue fields of F, F_0 respectively, where we write $\boldsymbol{k}_0=\boldsymbol{k}$ in the Levi case. Fix a uniformizer ϖ of F such that $\varpi^{e_{F/F_0}}$ is a uniformizer of F_0, where e_{F/F_0} denotes the ramification order of F/F_0. Thus $\sigma(\varpi)\in\{-\varpi,\varpi\}$ in any case. Let $\boldsymbol{J}^0$ denote

– the maximal compact open subgroup $\mathrm{GL}_n(\mathcal{O})$ if F/F_0 is unramified,
– the conjugate of $\mathrm{GL}_n(\mathcal{O})$ by

$$\mathrm{diag}(\varpi, \dots, \varpi, 1, \dots, 1) \in \mathrm{GL}_n(F)$$

where ϖ occurs m times, if F/F_0 is ramified.

Let $\boldsymbol{J}^1$ be the normal maximal pro-p-subgroup of $\boldsymbol{J}^0$ and $\boldsymbol{J}$ be its normalizer in $\mathrm{GL}_n(F)$. The natural group isomorphism $\boldsymbol{J}^0/\boldsymbol{J}^1 \simeq \mathrm{GL}_n(\boldsymbol{k})$ transports the action of $\sigma \in \mathrm{Gal}(F/F_0)$ on $\boldsymbol{J}^0/\boldsymbol{J}^1$ to
– the action of $\sigma \in \mathrm{Gal}(\boldsymbol{k}/\boldsymbol{k}_0)$ if F/F_0 is unramified,
– the adjoint action of

$$\begin{pmatrix} -\mathrm{id}_m & 0 \\ 0 & \mathrm{id}_m \end{pmatrix} \in \mathrm{GL}_n(\boldsymbol{k})$$

on $\mathrm{GL}_n(\boldsymbol{k})$ if F/F_0 is ramified.

Let us consider π as a representation of $\boldsymbol{J}^0$ by inflation, and extend it to a representation $\boldsymbol{\lambda}$ of $\boldsymbol{J} = F^\times \boldsymbol{J}^0$ by demanding that the central character of $\boldsymbol{\lambda}$ at a uniformizer ϖ of F is equal to 1 if F/F_0 is unramified, and to $\mathrm{sgn}(\pi)$ if F/F_0 is ramified. In any case, $(\boldsymbol{J}, \boldsymbol{\lambda})$ is a generic σ-self-dual level 0 type in the sense of [**24**] Definition 4.31. Inducing $\boldsymbol{\lambda}$ to $\mathrm{GL}_n(F)$, we get a cuspidal (irreducible) representation π_F of level 0 of $\mathrm{GL}_n(F)$. By [**24**] Theorem 4.45, the fact that π is distinguished implies that π_F is $\mathrm{GL}_n(F_0)$-distinguished. It follows from Theorem 3.1 that π_F has a $\mathrm{GL}_n(F_0)$-distinguished cuspidal $\overline{\mathbb{Q}}_\ell$-lift, and from [**24**] Lemma 6.5 that π has an H-distinguished cuspidal $\overline{\mathbb{Q}}_\ell$-lift. □

7. The second main result ($\ell = 2$)

We now assume that the field R has characteristic $\ell = 2$.

7.1. Here is our second main theorem.

Theorem 7.1. — *Suppose that $\ell = 2$. A cuspidal R-representation of $\mathrm{GL}_n(F)$ is $\mathrm{GL}_n(F_0)$-distinguished if and only if it is σ-self-dual.*

Note that:
– any H-distinguished cuspidal representation of G is σ-self-dual ([**35**] Theorem 4.1),
– Theorem 7.1 holds for supercuspidal representations ([**35**] Theorem 10.8).

To prove Theorem 7.1, it thus suffices to prove that any σ-self-dual non-supercuspidal, cuspidal representation of G is H-distinguished. As in the proof of Proposition 3.3, it follows from [**24**] Proposition 4.40 that the proof of Theorem 7.1 reduces to the level 0 case. As $\ell = 2$, the central character of any σ-self-dual irreducible representation of G is trivial on $F_0^\times$. Applying [**24**] Theorem 4.45, we are thus reduced to proving the following result.

Theorem 7.2. — *Let σ be one of the two following involutions of $\mathrm{GL}_n(\boldsymbol{k})$:*
– either $\boldsymbol{k}$ is a quadratic extension of $\boldsymbol{k}_0$ and σ is the non-trivial automorphism of $\boldsymbol{k}/\boldsymbol{k}_0$,
– or $n = 2m$ for some integer $m \geqslant 1$ and σ is the inner automorphism of conjugacy by the diagonal element $\delta_n = \mathrm{diag}(1, -1, 1, \dots, -1)$,
and let H be the subgroup of σ-fixed elements of $\mathrm{GL}_n(\boldsymbol{k})$. Then any σ-self-dual cuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$ is H-distinguished.

Remark 7.3. — When $\ell \neq 2$, there are σ-self-dual cuspidal $\overline{\mathbb{F}}_\ell$-representations of $\mathrm{GL}_2(F)$ which are not distinguished (see Remark 3.20).

7.2. Let $G_n = \mathrm{GL}_n(\boldsymbol{k})$ for some $n \geqslant 1$, let σ be one of the involutions of G_n of Theorem 7.2 and let H_n be the subgroup of σ-fixed points of G_n. The following lemma follows from **[29]** 2.4.

Lemma 7.4. — *Let π be a cuspidal representation of G_n.*

(1) *There are a unique divisor r of n and a supercuspidal representation ρ of $G_{n/r}$, unique up to isomorphism, such that π occurs as a subquotient of the induced representation $\rho^{\times r}$.*

(2) *One has $r = 2^a$ for some $a \geqslant 0$.*

(3) *π is the unique cuspidal subquotient of $\rho^{\times r}$ and it occurs with multiplicity* 1.

In this situation, we will write $\pi = \mathrm{sp}_a(\rho)$.

Let π be a cuspidal representation of G_n.

Lemma 7.5. — *The induced representation $\pi \times \pi$ is indecomposable of length* 3. *It has a unique cuspidal subquotient, which occurs with multiplicity* 1, *and a non-cuspidal irreducible subquotient occurring with multiplicity* 2.

Proof. — First, $\pi \times \pi$ contains a unique generic subquotient, occurring with multiplicity 1. Writing π under the form $\mathrm{sp}_a(\rho)$, it also contains the cuspidal (thus generic) representation $\mathrm{sp}_{a+1}(\rho)$. This proves the second assertion. Its unique non-zero proper Jacquet module has length 2 and is made of $\pi \otimes \pi$ with multiplicity 2. It thus has at most two non-cuspidal irreducible subquotients. Since its cuspidal subquotient can't appear as a subrepresentation nor a quotient, this length has to be 3 and $\pi \times \pi$ is indecomposable. By **[29]** Théorème 5.3, the number of its non-isomorphic irreducible subquotients is equal to the number of partitions of 2, which is 2. Thus the non-cuspidal irreducible subquotients are isomorphic. □

Let us denote by $\mathrm{sp}_1(\pi)$ the cuspidal subquotient of $\pi \times \pi$ and by τ its unique irreducible quotient, which is isomorphic to its unique irreducible subrepresentation. In this paragraph, we reduce the proof of Theorem 7.2 to that of the the following proposition.

Proposition 7.6. — *Let π be an H_n-distinguished cuspidal representation of G_n for some $n \geqslant 1$. Then the cuspidal representation $\mathrm{sp}_1(\pi)$ of G_{2n} is H_{2n}-distinguished.*

Let us explain how this proposition implies Theorem 7.2. Let π be a σ-self-dual cuspidal representation of G_n for some $n \geqslant 1$. It is of the form $\mathrm{sp}_a(\rho)$ for an integer $a \geqslant 0$ and a supercuspidal representation ρ. By uniqueness, ρ is σ-self-dual, thus it is distinguished by **[35]** Theorem 10.8. On the other hand, we denote by π_i the cuspidal representation $\mathrm{sp}_i(\rho)$. One has $\pi_{i+1} = \mathrm{sp}_1(\pi_i)$ for all $i \geqslant 0$. By induction, thanks to Proposition 7.6 and the supercuspidal case, π_i is distinguished for all $i \geqslant 0$. In particular, $\pi_a = \pi$ is distinguished.

It thus remains to prove Proposition 7.6. We will prove it by contradiction, assuming that the cuspidal representation $\mathrm{sp}_1(\pi)$ is not H_{2n}-distinguished.

7.3. Let π be an H_n-distinguished cuspidal representation of G_n for some $n \geqslant 1$, and let τ be the unique irreducible quotient of $\pi \times \pi$. We first prove the following multiplicity 1 result.

Proposition 7.7. — *The space $\mathrm{Hom}_{H_{2n}}(\tau, 1)$ has dimension at most* 1.

Proof. — In the Galois case, this is [**35**] Remark 4.3. We will thus assume from now on that we are in the Levi case. For simplicity, we write $H = H_{2n}$, $P = P_{2n}$ and $G = G_{2n}$. Since $\mathrm{Hom}_H(\tau, 1)$ is contained in $\mathrm{Hom}_{P\cap H}(\tau, 1)$, it suffices to prove

$$\dim \mathrm{Hom}_{P\cap H}(\tau, 1) \leqslant 1.$$

Our argument is inspired from [**3**] Corollary 4.4. We will use the theory of derivatives for R-representations of general linear groups over $\boldsymbol{k}$, for which we refer to [**37**] III.1. (Unlike [**37**], we will use the usual notation Φ^+, Φ^-, Ψ^+, Ψ^-, the definition of which can be found in [**3**] Section 4 for instance.) It will be convenient to define $G_0 = H_0$ to be the trivial group. We will need the following property (see [**3**] Proposition 4.3 in the Galois case for $R = \mathbb{C}$).

Lemma 7.8. — (1) *For any $i \geqslant 2$ and any representation κ of P_{i-1}, one has an isomorphism of R-vector spaces*

$$\mathrm{Hom}_{P_i\cap H_i}(\Phi^+\kappa, 1) \simeq \mathrm{Hom}_{P_{i-1}\cap H_{i-1}}(\kappa, 1). \tag{7.1}$$

(2) *For any $i \geqslant 1$ and any representation μ of G_{i-1}, one has an equality*

$$\mathrm{Hom}_{P_i\cap H_i}(\Psi^+\mu, 1) = \mathrm{Hom}_{G_{i-1}\cap H_{i-1}}(\mu, 1). \tag{7.2}$$

Proof. — Let us embed G_{i-1} in G_i via $g \mapsto \mathrm{diag}(g, 1)$. Then (7.2) immediately follows from the fact that $P_i \cap H_i = H_{i-1}(U_i \cap H_i)$ (this is (6.6)), and (7.1) is given by [**35**] Lemma 2.10. □

First, it follows from [**37**] III.1.3 that

$$\dim \mathrm{Hom}_{P\cap H}(\tau, 1) \leqslant \sum_{i=1}^{2m} \dim \mathrm{Hom}_{P\cap H}((\Phi^+)^{i-1}\Psi^+\tau^{(i)}, 1). \tag{7.3}$$

We are going to prove that $\tau^{(i)}$ is zero, unless $i = n$ in which case $\tau^{(n)} = \pi$. It will follow that

$$\dim \mathrm{Hom}_{P\cap H}(\tau, 1) \leqslant \dim \mathrm{Hom}_{P\cap H}((\Phi^+)^{i-1}\Psi^+\pi, 1) = \dim \mathrm{Hom}_{H_n}(\pi, 1) = 1$$

where the first equality follows from Lemma 7.8 and the second one from Lemma 6.2.

Recall that the semi-simplification of $\pi\times\pi$ is equal to $\mathrm{sp}_1(\pi)+2\tau$. Since the derivative functors are exact, the semi-simplification of $(\pi \times \pi)^{(i)}$ is equal to $\mathrm{sp}_1(\pi)^{(i)} + 2\tau^{(i)}$ for all $i \geqslant 0$. The Leibniz rule ([**37**] III.1.10) together with the fact that π is cuspidal imply that $(\pi\times\pi)^{(i)}$ is zero, unless $i \in \{n, 2n\}$, and that $(\pi \times \pi)^{(n)}$ has length 2 with subquotient π occurring twice. For $i = 2n$, the derivative $\tau^{(2n)}$ is zero since τ is not generic. For $i = n$, $\mathrm{sp}_1(\pi)^{(n)}$ is zero since $\mathrm{sp}_1(\pi)$ is cuspidal, thus $\tau^{(n)} = \pi$. □

7.4. We are now going to construct two independent non-zero H_{2n}-invariant linear forms on the induced representation $\pi \times \pi$ vanishing on the socle of $\pi \times \pi$. Assuming that the cuspidal representation $\mathrm{sp}_1(\pi)$ is not H_{2n}-distinguished, they will thus vanish on the maximal proper subrepresentation of $\pi\times\pi$, thus inducing two independent non-zero H_{2n}-invariant linear forms on τ which will contradict Proposition 7.7. For simplicity, we will write $H = H_{2n}$, $P = P_{2n}$ and $G = G_{2n}$.

Let Q be the standard parabolic subgroup of G of size (n, n) together with its standard Levi subgroup M and unipotent radical U. The endomorphism algebra $\mathrm{End}_G(\pi \times \pi)$ contains an element T defined by

$$Tf(g) = \sum_{u\in U} Af(sug)$$

for $f \in \pi \times \pi$ and $g \in G$, where A is the isomorphism $v \otimes w \mapsto w \otimes v$ on the tensor square of the space of π. Denoting by s the element

$$s = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \in G,$$

this isomorphism is the unique intertwiner between the representation $\pi \otimes \pi$ of M and its conjugate by s such that A^2 is the identity. The element T satisfies $T^2 = 1$, or equivalently $(T+1)^2 = 0$, and the image of $T+1$ is the socle of $\pi \times \pi$. A linear form Λ on the space of $\pi \times \pi$ thus vanishes on the socle of $\pi \times \pi$ if and only if $\Lambda \circ T = \Lambda$.

Note that Q, U and M are σ-stable and that $s \in H$. We will need the following lemma.

Lemma 7.9. — (1) *(Galois case) For $i = 0, \ldots, n$, define*

$$\gamma_i = \begin{pmatrix} 1_i & & & \\ & 0 & 1_{n-i} & \\ & 1_{n-i} & 0 & \\ & & & 1_i \end{pmatrix}.$$

For any choice of $x_i \in G$ such that $\sigma(x_i)x_i^{-1} = \gamma_i$, the set $\{x_0, \ldots, x_n\}$ is a set of representatives of (Q, H)-double cosets of G.

(2) *(Levi case) For $i, j \in \{0, \ldots, n\}$ such that $0 \leqslant i + j \leqslant n$, define*

$$\gamma_{i,j} = \delta_{2n} \begin{pmatrix} 1_i & & & & & \\ & -1_j & & & & \\ & & 0 & 1_{n-i-j} & & \\ & & 1_{n-i-j} & 0 & & \\ & & & & 1_j & \\ & & & & & -1_i \end{pmatrix}.$$

For any choice of $x_{i,j} \in G$ such that $\sigma(x_{i,j})x_{i,j}^{-1} = \gamma_{i,j}$, the set $\{x_{i,j}, 0 \leqslant i + j \leqslant n\}$ is a set of representatives of (Q, H)-double cosets of G.

Proof. — Let V be the $\boldsymbol{k}$-vector space $\boldsymbol{k}^{2n}$ equipped with its canonical basis $(e_1, \ldots, e_{2n})$ and $W_\circ$ be the subspace of V spanned by $(e_1, \ldots, e_n)$. The map $g \mapsto g^{-1}W_\circ$ induces a bijection between the (Q, H)-double cosets of G and the H-orbits of subspaces of dimension n of V.

Let us start with the Galois case. The space V is equipped with the action of σ componentwise. By **[26]** Section 3, these H-orbits are in bijection with $\{0, \ldots, n\}$ through $W \mapsto \dim(W \cap \sigma(W))$. For any $g \in G$, one has

$$\dim(g^{-1}W_\circ \cap \sigma(g^{-1}W_\circ)) = \dim(\sigma(g)g^{-1}W_\circ \cap W_\circ).$$

It thus follows that $x_i^{-1}W_\circ$ corresponds to i for all $i \in \{0, \ldots, n\}$, and the result follows. See **[26]** Proposition 3.7.

Let us now consider the Levi case. The space V is equipped with the natural action of $\delta = \delta_{2n}$. By **[27]** Section 3, the H-orbits of subspaces of dimension n of V are in bijection with the set of pairs of integers $(i, j) \in \{0, \ldots, n\}^2$ such that $i + j \leqslant n$ through

$$W \mapsto (\dim \operatorname{Ker}(\delta - 1 \mid W \cap \delta W), \dim \operatorname{Ker}(\delta + 1 \mid W \cap \delta W)).$$

For any $g \in G$ and any sign $\varepsilon \in \{-1, 1\}$, one has

$$\dim \operatorname{Ker}(\delta - \varepsilon \cdot 1 \mid g^{-1}W_\circ \cap \delta g^{-1}W_\circ) = \dim \operatorname{Ker}(g\delta g^{-1} - \varepsilon \cdot 1 \mid g\delta g^{-1}W_\circ \cap W_\circ).$$

It then follows that $x_{i,j}^{-1}W_\circ$ corresponds to the pair (i,j) for all $i,j \in \{0,\dots,n\}$ such that $i+j \leqslant n$. The result follows. See [**27**] Proposition 3.2. □

7.5. Let us define a first H-invariant linear form on $\pi \times \pi$. Since π is H_n-distinguished, we fix a non-zero linear form $\lambda \in \mathrm{Hom}_{H_n}(\pi, 1)$ and write $\Lambda_\pi^0 = \lambda \otimes \lambda \in \mathrm{Hom}_{H_n \times H_n}(\pi \otimes \pi, 1)$. It defines an H-invariant linear form Λ^0 on $\pi \times \pi$ given by

$$\Lambda^0(f) = \sum_{h \in Q \cap H \backslash H} \langle f(h), \Lambda_\pi^0 \rangle. \tag{7.4}$$

This linear form is non-zero. Indeed, if f is supported on Q and $f(1) = v \otimes v$ where v is a vector in the space of π such that $\lambda(v) \neq 0$, one has $\Lambda^0(f) = \langle f(1), \Lambda_\pi^0 \rangle = \lambda(v)^2 \neq 0$. Let us prove that Λ^0 vanishes on the socle of $\pi \times \pi$.

Proposition 7.10. — *One has $\Lambda^0 \circ T = \Lambda^0$.*

Proof. — Since Λ^0 is H-invariant, it suffices to prove that $\Lambda^0(Tf) = \Lambda^0(f)$ for any $f \in \pi \times \pi$ supported on Qg, where g ranges over a set of representatives of (Q,H)-double cosets of G.

First, an $h \in Q \cap H \backslash H$ contributes to the sum in (7.4) if and only if $h \in H \cap Qg$, which is non-empty if and only if $g \in QH$, in which case we may assume that $g = 1$, thus $\Lambda^0(f) = \langle f(1), \Lambda_\pi^0 \rangle$.

Since $\Lambda_\pi^0 \circ A = \Lambda_\pi^0$, one has

$$\Lambda^0(Tf) = \sum_{h \in Q \cap H \backslash H} \sum_{u \in U} \langle f(suh), \Lambda_\pi^0 \rangle.$$

We are now going to determine the $h \in Q \cap H \backslash H$ and $u \in U$ such that $suh \in Qg$. Set $\gamma = \sigma(g)g^{-1}$. By Lemma 7.9, we may assume that $Q\gamma Q$ is equal to $Q\gamma_i Q$ for some parameter i (in the Galois case) or $Q\gamma_{i,j}Q$ for some i,j (in the Levi case). If $suh \in Qg$, then $s\sigma(u)u^{-1}s \in Q\gamma Q \cap U^-$ with

$$Q\gamma Q = Q \begin{pmatrix} 1_k & & & \\ & 0 & 1_{n-k} & \\ & 1_{n-k} & 0 & \\ & & & 1_k \end{pmatrix} Q \tag{7.5}$$

for some $k \in \{0,\dots,n\}$. (More precisely, one has $k = i$ in the Galois case if $\gamma = \gamma_i$, and $k = i+j$ in the Levi case if $\gamma = \gamma_{i,j}$.) This double coset has a non-empty intersection with U^- if and only if $k = n$, in which case this intersection is reduced to $\{1\}$. It follows that $s\sigma(u)u^{-1}s = 1$, whence $u \in U \cap H$. Since $s \in H$, the intersection $(U \cap H)sQg \cap H$ is non-empty if and only if $g \in QH$, in which case we may assume that $g = 1$. One then has

$$h \in (U \cap H)sQ \cap H = (U \cap H)s(Q \cap H) = (Q \cap H)s(U \cap H).$$

Setting $h = sv$ with $v \in U \cap H$, one has $susv \in Q$ if and only if $sus \in Q \cap U^- = \{1\}$. Thus $u = 1$ and

$$\Lambda^0(Tf) = \sum_{v \in U \cap H} \langle f(v), \Lambda_\pi^0 \rangle = |U \cap H| \cdot \langle f(1), \Lambda_\pi^0 \rangle = \Lambda^0(f)$$

since $|U \cap H| \equiv 1 \bmod 2$. □

7.6. Let us now define a second H-invariant linear form on $\pi\times\pi$. In order to treat the Galois and Levi cases uniformly, we set $\delta = 1$ in the Galois case and $\delta = \delta_{2n}$ in the Levi case. We also set

$$\theta = \begin{cases} \sigma & \text{in the Galois case,} \\ \mathrm{id} & \text{in the Levi case.} \end{cases}$$

Fix an $x \in G$ such that $\sigma(x)x^{-1} = \delta s$. Note that δs is γ_0 in the Galois case and $\gamma_{0,0}$ in the Levi case, and that

$${}^{x}H \cap Q = {}^{x}H \cap M = \{(g, \theta(g)) \in M \mid g \in G_n\}.$$

Fix an isomorphism between π and $\pi^{\vee\theta}$, or equivalently a non-zero linear form $\Lambda^1_\pi : \pi \otimes \pi \to R$ such that $\Lambda^1_\pi(\theta(g)v, gw) = \Lambda^1_\pi(v, w)$. It defines an H-invariant linear form Λ^1 on $\pi \times \pi$ given by

$$\Lambda^1(f) = \sum_{h \in M^x \cap H \backslash H} \langle f(xh), \Lambda^1_\pi \rangle. \tag{7.6}$$

It is non-zero. Indeed, if f is supported on Qx and $f(x) \notin \mathrm{Ker}(\Lambda^1_\pi)$, then $\Lambda^1(f) = \langle f(x), \Lambda^1_\pi\rangle \neq 0$. Let us prove that Λ^1 vanishes on the socle of $\pi \times \pi$.

Proposition 7.11. — *One has $\Lambda^1 \circ T = \Lambda^1$.*

Proof. — Since Λ^1 is H-invariant, it suffices to prove that $\Lambda^1(Tf) = \Lambda^1(f)$ for any $f \in \pi\times\pi$ supported on Qg where g ranges over a set of representatives of (Q, H)-double cosets of G.

We set $\gamma = \sigma(g)g^{-1}$. By Lemma 7.9, we may assume that $Q\gamma Q$ is equal to $Q\gamma_i Q$ for some i (in the Galois case) or $Q\gamma_{i,j}Q$ for some i, j (in the Levi case). In any case, we can write it as in (7.5) for some k.

First, an $h \in M^x \cap H\backslash H$ contributes to the sum in (7.6) if and only if $xh \in xH \cap Qg$. If this intersection is non-empty, then, by applying the map $y \mapsto \sigma(y)y^{-1}$, we get $\delta s \in Q\gamma Q$, which can happen if and only if $k = 0$, thus $g \in QxH$, in which case we may assume that $g = x$. One then has $h \in H \cap Q^x = H \cap M^x$ and $\Lambda^1(f) = \langle f(x), \Lambda^1_\pi\rangle$.

By uniqueness of the isomorphism $\pi \simeq \pi^{\vee\theta}$ up to a non-zero scalar, the linear form Λ^1_π that we have fixed is unique up to a non-zero scalar. Thus $\Lambda^1_\pi \circ A = \mu\cdot\Lambda^1_\pi$ for some $\mu \in R^\times$. Since A^2 is the identity and R has characteristic 2, it follows that $\mu = 1$, that is, $\Lambda^1_\pi \circ A = \Lambda^1_\pi$. One has

$$\Lambda^1(Tf) = \sum_{h \in M^x \cap H\backslash H} \sum_{u \in U} \langle f(suxh), \Lambda^1_\pi\rangle.$$

We are now going to determine the $h \in M^x \cap H\backslash H$ and $u \in U$ such that $suxh \in Qg$. If $suxh \in Qg$, then $s\sigma(u)\delta s u^{-1}s \in Q\gamma Q$. Let us write

$$s\sigma(u)\delta s u^{-1} s = s\begin{pmatrix} 1 & \sigma(t) \\ & 1 \end{pmatrix} \delta s \begin{pmatrix} 1 & -t \\ & 1\end{pmatrix} s = \delta\begin{pmatrix} -t & 1 \\ 1-\theta(t)t & \theta(t)\end{pmatrix} \quad \text{with} \quad u = \begin{pmatrix} 1 & t \\ & 1\end{pmatrix}$$

for some $t \in \mathbf{M}_n(\boldsymbol{k})$. Such an element is in $Q\gamma Q$ if and only if $k = 0$, in which case we may assume that $g = x$ and $\gamma = \delta s$. Then QsQ is made of those $a \in \mathbf{M}_2(\mathbf{M}_n(\boldsymbol{k}))$ such that $a_{2,1}$ is invertible. Let $t \in \mathbf{M}_n(\boldsymbol{k})$ and suppose that $1-\theta(t)t \in G_n$. Then $suxh \in Qx$ implies that $xhx^{-1} \in QsQ \cap {}^{x}H$. Since ${}^{x}H$ is the subgroup of elements of G fixed by the involution $g \mapsto s\theta(g)s$, an easy calculation shows that it is made of the matrices of G of the form

$$\begin{pmatrix} a & b \\ \theta(b) & \theta(a)\end{pmatrix}, \quad a, b \in \mathbf{M}_n(\boldsymbol{k}) \tag{7.7}$$

thus $QsQ \cap {}^xH$ corresponds to those matrices with b invertible. Since we consider h up to multiplication by $M^x \cap H$ on the left, and since

$$M \cap {}^xH = \begin{pmatrix} a & \\ & \theta(a) \end{pmatrix}, \quad a \in G_n,$$

we may assume that xhx^{-1} is of the form (7.7) with $b = 1_n$. Thus

$$suxhx^{-1} = \begin{pmatrix} 1 & -\theta(t) \\ t+a & 1+t\theta(a) \end{pmatrix}$$

belongs to Q if and only if $a = -t$. We thus get

$$\begin{aligned} \Lambda^1(Tf) &= \sum_{t\in\Omega} \langle f\left(\begin{pmatrix} 1 & -\theta(t) \\ 0 & 1-t\theta(t) \end{pmatrix} x\right), \Lambda^1_\pi\rangle \\ &= \sum_{t\in\Omega} \langle \begin{pmatrix} 1 & \\ & 1-t\theta(t) \end{pmatrix} \cdot f(x), \Lambda^1_\pi\rangle \end{aligned}$$

where Ω denotes the set of $t \in \mathbf{M}_n(\boldsymbol{k})$ such that $1-\theta(t)t \in G_n$. Note that $0 \in \Omega$ and that t, $-t$ have the same image by $t \mapsto 1-t\theta(t)$, thus Ω is stable by $t \mapsto -t$. Fix a set Ω^+ of representatives of the cosets $\{t, -t\}$ in $\Omega - \{0\}$. We then have

$$\Lambda^1(Tf) = \langle f(x), \Lambda^1_\pi\rangle + 2\sum_{t\in\Omega^+} \langle \begin{pmatrix} 1 & \\ & 1-t\theta(t) \end{pmatrix} \cdot f(x), \Lambda^1_\pi\rangle = \langle f(x), \Lambda^1_\pi\rangle.$$

This finishes the proof. □

7.7. In conclusion, we have defined two independent non-zero H-invariant linear forms Λ^0, Λ^1 on $\pi\times\pi$ vanishing on its socle. If $\mathrm{sp}_1(\pi)$ were not H-distinguished, these linear forms would induce two independent non-zero H-invariant linear forms on the irreducible quotient τ of $\pi\times\pi$, thus contradicting Proposition 7.7. This proves Proposition 7.6 and finishes the proof of our second main Theorem 7.1, as explained in §7.2.

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Robert Kurinczuk, School of Mathematics and Statistics, University of Sheffield, Sheffield, S3 7RH, United Kingdom • *E-mail :* robkurinczuk@gmail.com

Nadir Matringe, Shanghai Institute for Mathematics and Interdisciplinary Sciences, Block A, International Innovation Plaza, N°657 Songhu Road, Yangpu District, Shanghai, China
E-mail : nadirmatringe@outlook.fr

Vincent Sécherre, Laboratoire de Mathématiques de Versailles, UVSQ, CNRS, Université Paris-Saclay, 78035, Versailles, France • *E-mail :* vincent.secherre@uvsq.fr