LECTURES ON WEB GEOMETRY

I. Introduction and some basics

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Workshop on Symplectic Varieties and Projective Geometry

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中国科学院晨兴数学中心
MORNINGSIDE CENTER OF MATHEMATICS
CHINESE ACADEMY OF SCIENCES

General overview of the lectures

Lecture 1 **Introduction and some Basics**

- Webs, Equivalence, Hexagonality
- Some basic examples
- Abelian relations & Polylogarithms

Lecture 2 **Examples of Webs in Algebraic Geometry**

- Webs on surfaces, on Fano manifolds, on moduli spaces on homogeneous spaces G/P, etc

Lecture 3 **Webs by conics on Del Pezzo Surfaces**

- Geometry of Del Pezzo Surfaces
- Hyperlogarithms
- Gelfand-MacPherson Webs

A Foreword on the Development of Web Geometry

- **Web geometry** : Developed by Blaschke's School in Hamburg
- **1927-1938 :** Blaschke and others published more than 60 papers
  « Topologie Fragen der Differentialgeometrie »
- **1932 :** Blaschke was invited to give lectures at Peking University
- **Chern** was a master student at Tsinghua Univ. at that time, he
  attended & was influenced by Blascke's lectures on webs
- **1934 :** Chern went to Hamburg to work under Blaschke's guidance
- **1936 :** Chern defended his thesis (2 papers on webs)
- **1977-80 :** Chern and Griffiths worked together on the
  problem of the algebraization of maximal-rank webs

First definitions : foliations

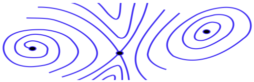
- **General setting :** analytic or algebraic / $\mathbb{C}$ (∃ others : C^∞, etc)

- ***M*** complex manifold of dimension $\mu \geq 2$

- A regular **foliation $\mathcal{F}$** of codimension r on $\boldsymbol{M}$ is given by a integrable distribution $\mathrm{T}\mathcal{F} \subset \mathrm{T}\boldsymbol{M}$ of corank r.

  A **first integral** for $\mathcal{F}$ on $D \subset \boldsymbol{M}$ is a map $U : D \to \mathbb{C}^r$ such that $\mathrm{T}\mathcal{F}|_D = \mathbf{ker}(\mathrm{d}U)$.

- In general, foliations may have singularities...



- ... but in web geometry, we usually stay away from the singularities of the foliations we are working with.

First definitions : webs

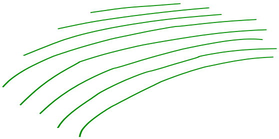
- **Geneneral position (GP)** for subspaces $V_1, \ldots, V_d \subset \mathbb{C}^{\mu}$:

$$\forall\ I \subset \{1, \ldots, d\} : \ \mathbf{codim}\big(\cap_{i \in I} V_i\big) = \max\Big(\mu\,,\ \textstyle\sum_{i \in I} \mathbf{codim}(V_i)\Big)$$

Definition : A (TD) *d*-web is a d-tuple $\mathcal{W} = (\mathcal{F}_1, \ldots, \mathcal{F}_d)$ of foliations $\mathcal{F}_i$ whose tangent distributions $\mathrm{T}\mathcal{F}_i$ are in **GP**

(wGP) : $\forall\ i \neq j$, $\mathrm{T}_m\mathcal{F}_i$ and $\mathrm{T}_m\mathcal{F}_j$ are in **GP** in $\mathrm{T}_m\boldsymbol{M}$

or **(GP) :** the tangent spaces $\mathrm{T}_m\mathcal{F}_1, \ldots, \mathrm{T}_m\mathcal{F}_d$ of the foliations of the web $\mathcal{W}$ at m are in **GP** in $\mathrm{T}_m\boldsymbol{M}$



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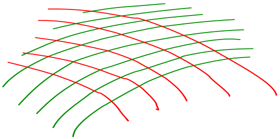

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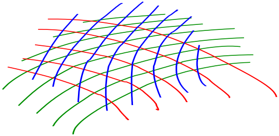
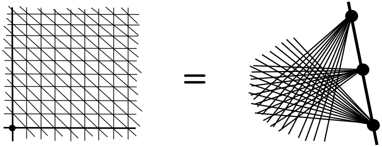


Definition : a **d-web** is obtained by gluing totally decomposable webs

Some examples of webs : Lie groups

Notation : $\mathcal{W}(U_1, \ldots, U_d)$ = d-web whose foliations are those admitting the U_i's as first integrals

- **Planar parallel 3-web :** $\mathcal{W}(x, y, x+y)$



- **Let G be a Lie group.** Then G^2 comes with three natural maps :
  - $\pi_i : G \times G \to G$: the projection onto the i-th factor ($i = 1, 2$)
  - $\Pi : G \times G \to G : (x, y) \mapsto xy$ the group multiplication

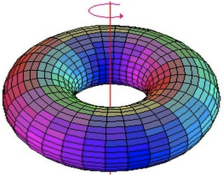
$$\rightsquigarrow \mathcal{W}_G = \mathcal{W}(\pi_1 \, , \, \pi_2 \, , \, \Pi \,) \text{ is 3-web on } G^2$$

- A classical theme in web geometry is the investigation of the interplay

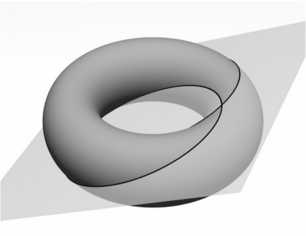
Differential properties of $\mathcal{W}_G$ $\longleftrightarrow$ Algebraic properties of $\boldsymbol{G}$

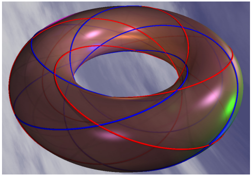
Some examples of webs : on surfaces in R^3

- There are two families of circles on a torus $\mathbb{T}^2$ in $\mathbb{R}^3$:



- But there are also two **Villarceau's circles** through each point





- Hence $\mathbb{T}^2$ carries a 4-web in circles

Some examples of webs : in algebraic geometry

- **Algebraic surface $S \subset \mathbf{P}^3$** : if deg $S = 2$ then $S = Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$
  $\rightsquigarrow$ Q carries a 2-web by lines

  Cubic surface $S \subset \mathbf{P}^3$: $\exists$ 27 lines contained in S 
  $\rightsquigarrow$ 27-web by conics on S

- **Cubic threefold $X \subset \mathbf{P}^4$** : through x general pass 6 lines $\subset X$
  $\rightsquigarrow$ 6-web by lines on X

- **Moduli spaces I :** $\exists$ n forgetful morphisms $\varphi_i : \mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1}$
  $\rightsquigarrow$ n-web $\mathcal{W}_{\mathcal{M}_{0,n}}$ by rational curves on $\mathcal{M}_{0,n}$

- **Moduli spaces II :** Let Y^r = moduli space of marked del Pezzo surfaces of degree $9 - r$. Blowing-down a -1 curve gives a map $Y^r \to Y^{r-1}$. There is a finite number ℓ_r of lines on each $\mathbf{dP}_{9-r}$
  $\rightsquigarrow$ ℓ_r-web by surfaces on Y^r

Web geometry

Singular set $\Sigma(\mathcal{W})$ $= \Big(\cup_i \mathrm{Sing}(\mathcal{F}_i)\Big) \cup \{$ where **GP** is not satisfied $\}$

- Given a web $\mathcal{W}$, we always work on **$\mathrm{Reg}(\mathcal{W})$** $= \boldsymbol{M} \setminus \Sigma(\mathcal{W})$

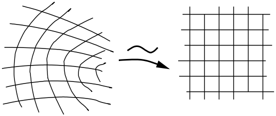
  Classically, one even works locally, at a generic point $m \in \mathrm{Reg}(\mathcal{W})$

Definition. Two webs $\mathcal{W} = (\mathcal{F}_i)_{i=1}^d$ and $\widetilde{\mathcal{W}} = (\widetilde{\mathcal{F}}_i)_{i=1}^d$ on two manifolds $\boldsymbol{M}$ and $\widetilde{\boldsymbol{M}}$ are **equivalent** if there exists a germ of biholomorphism $\varphi : (\boldsymbol{M}, m) \simeq (\widetilde{\boldsymbol{M}}, \tilde{m})$ (with $m \in \mathrm{Reg}(\mathcal{W})$, ...) such that (possibly up to reindexing the foliations) :

$$\varphi_*(\mathcal{W}) = \big(\varphi_*(\mathcal{F}_i)\big)_{i=1}^d = \widetilde{\mathcal{W}}$$

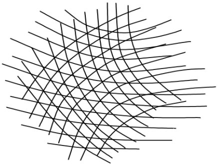
Web geometry is the study of webs up to equivalence

Example :



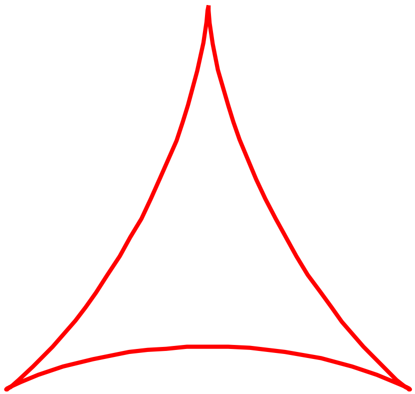
A planar 2-web
is locally trivial

Web Geometry

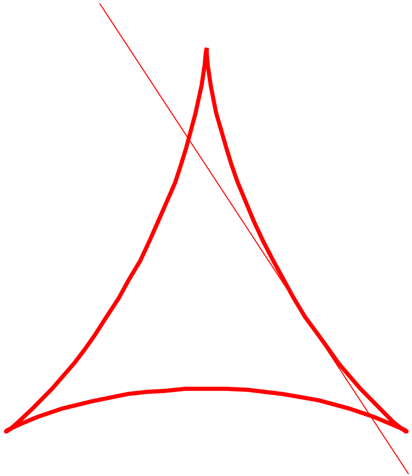


A planar algebraic curve : the deltoid

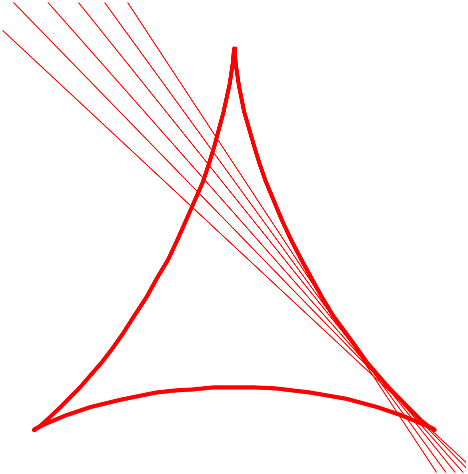
A planar algebraic 3-web



A planar algebraic 3-web



A planar algebraic 3-web



A planar algebraic 3-web

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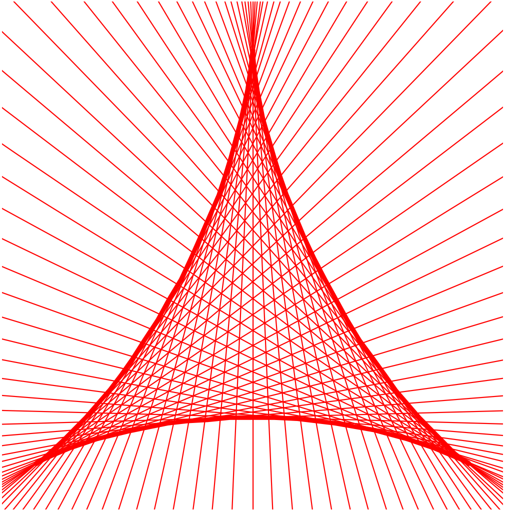
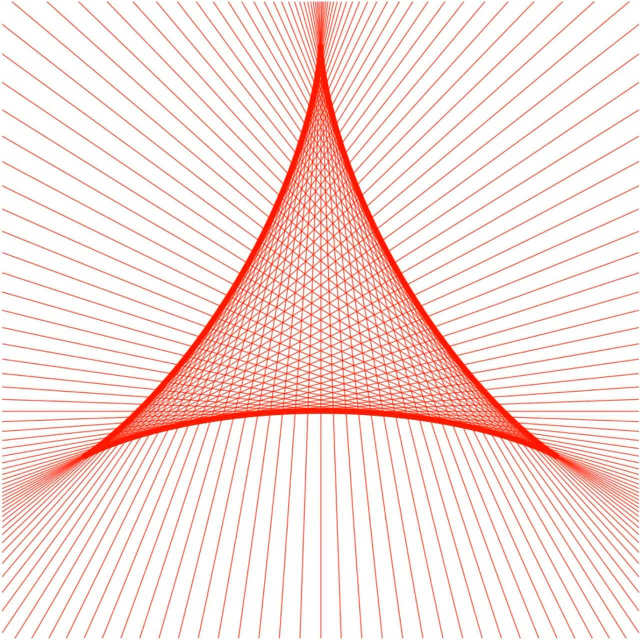
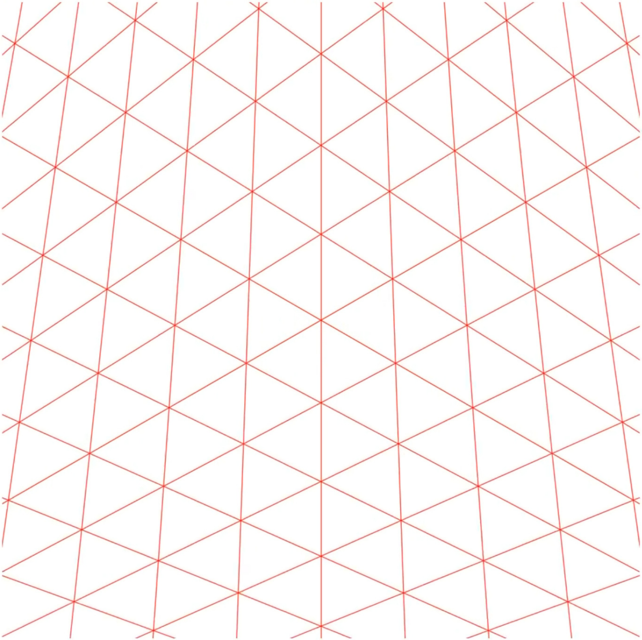
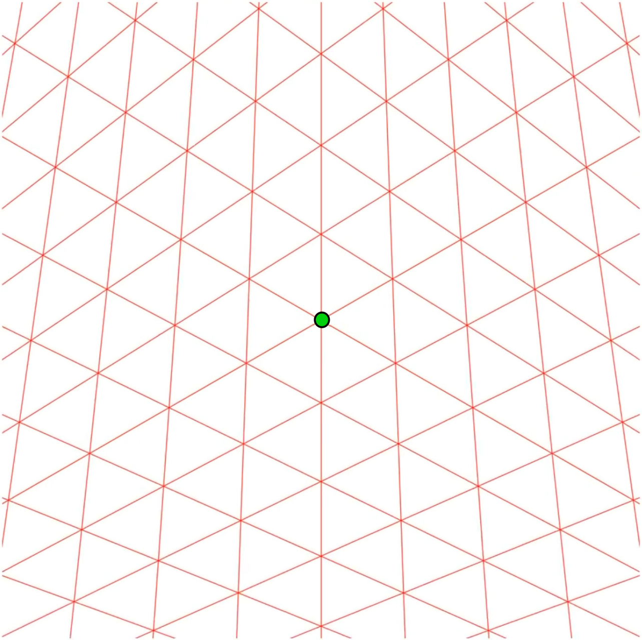
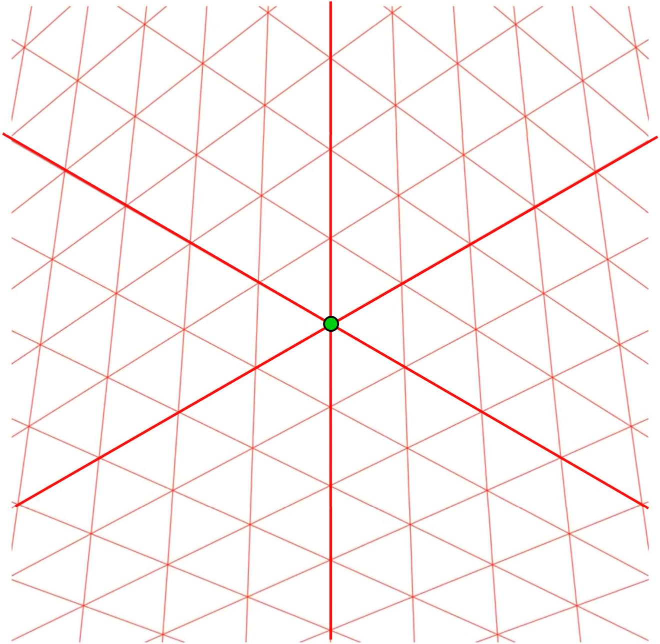


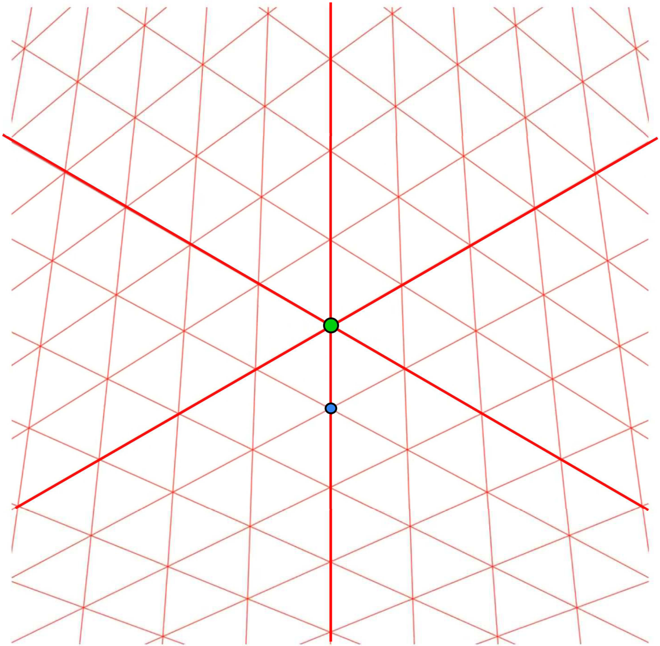
The algebraic 3-web $\mathcal{W}_C$ associated to a plane cubic $C \subset \mathbb{P}^2$

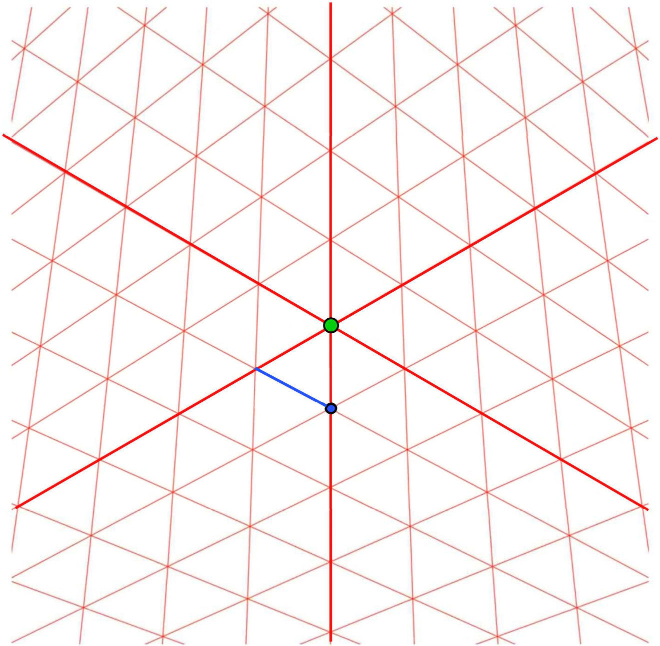


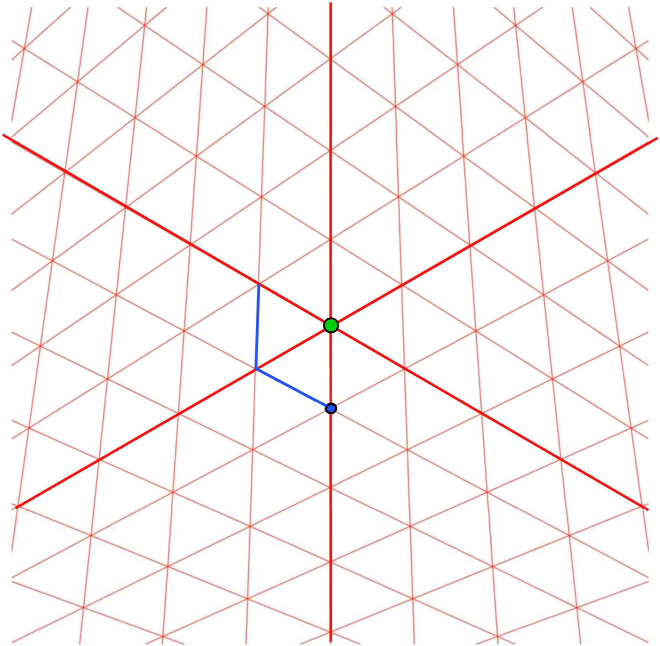


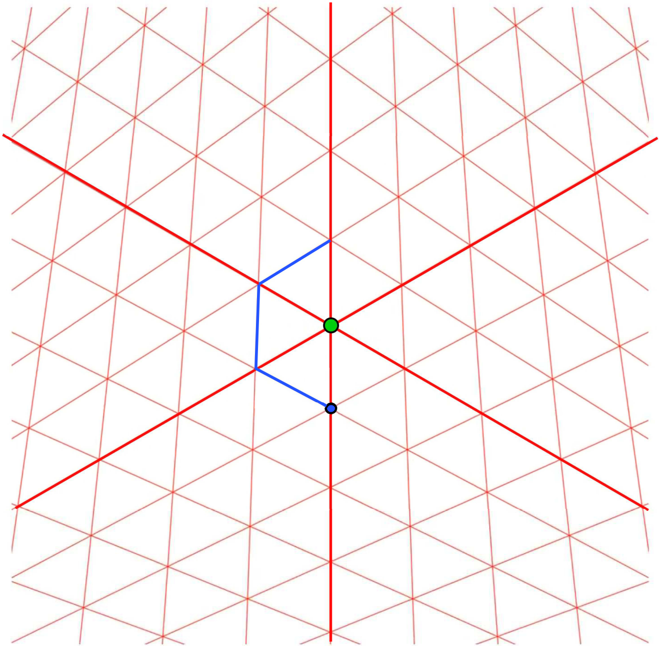


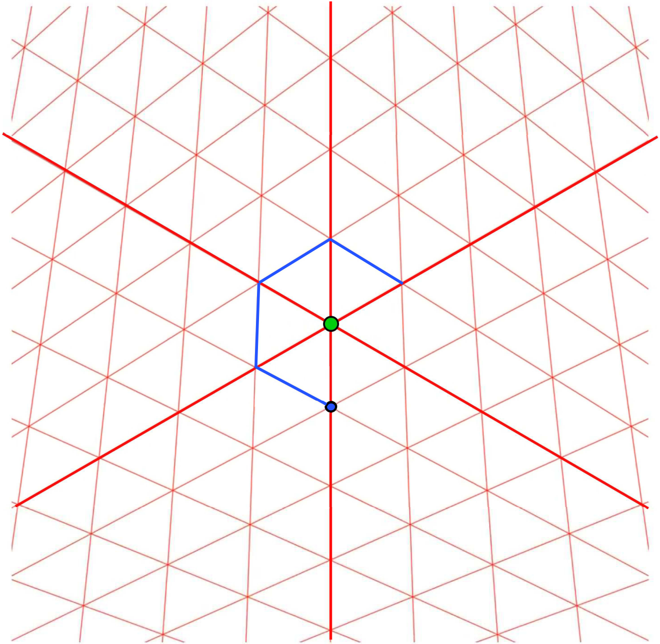


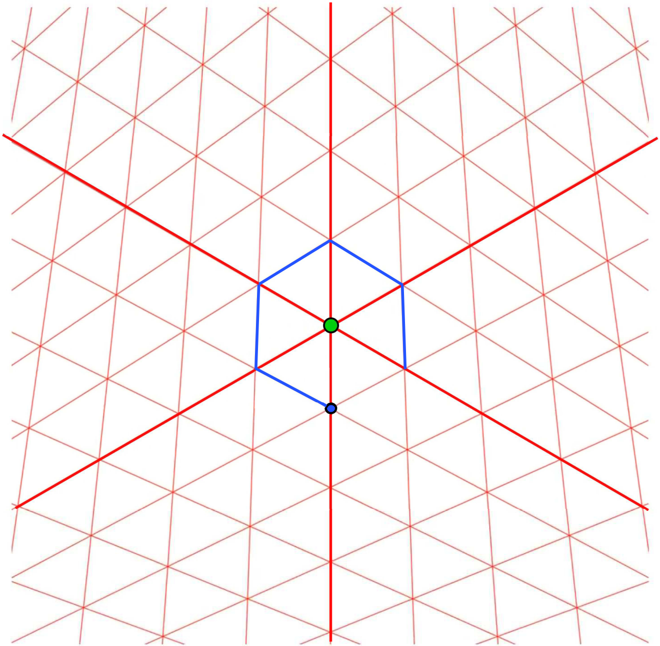


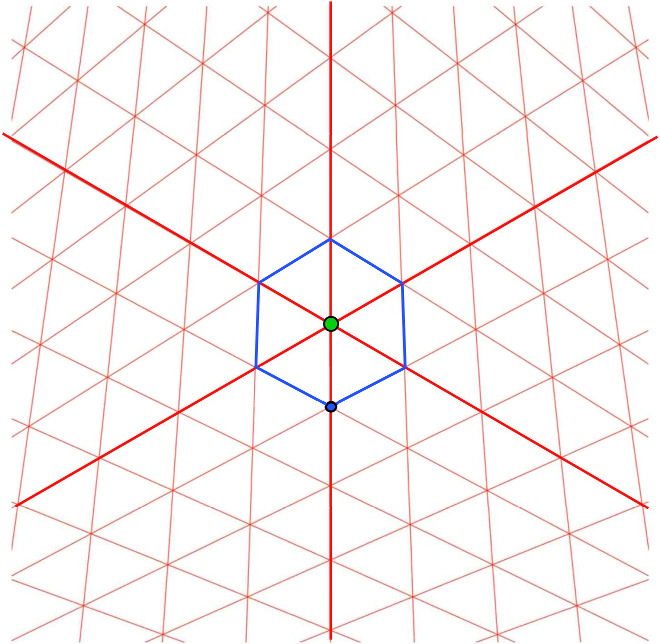


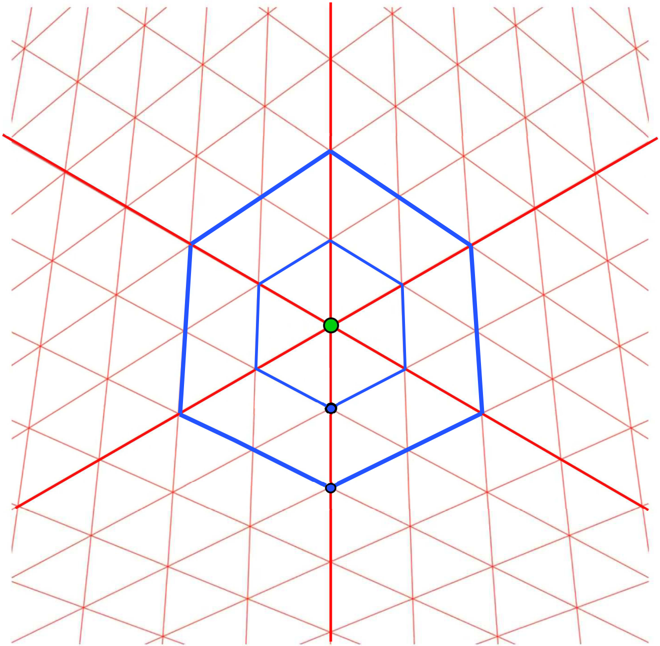


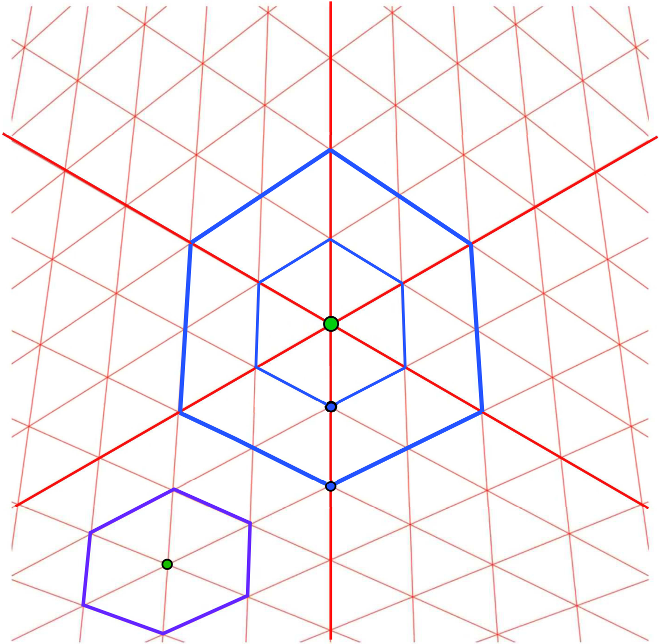


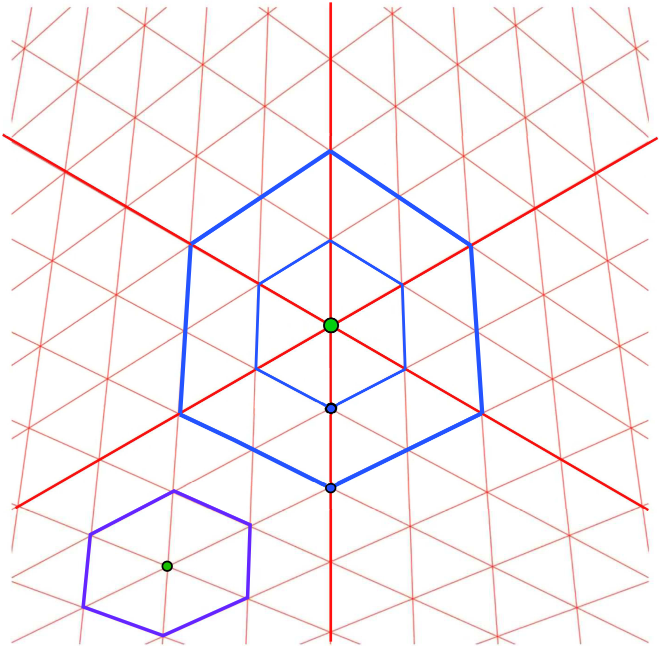


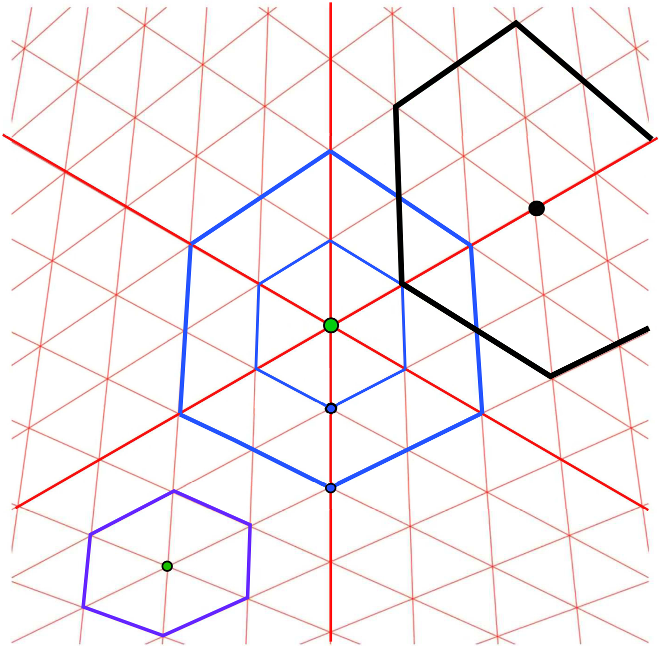




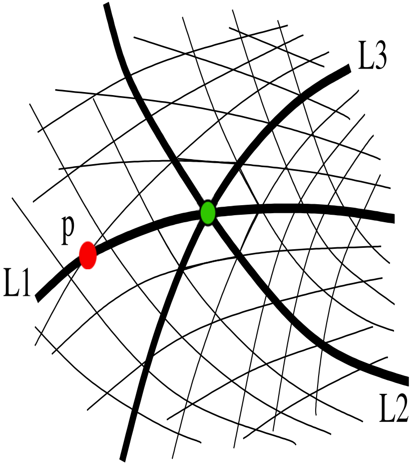


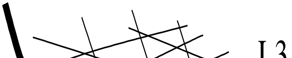






Hexagonality of a planar 3-web





The Queen Elizabeth II Great Court roof at the British Museum

A classical theorem

- Let $\mathcal{W}_3$ be a 3-web on an open domain $D \subset \mathbb{C}^2$
- Let $U_1, U_2, U_3 : D \to \mathbb{C}$ be first integrals : $\mathcal{W}_3 = \mathcal{W}(U_1, U_2, U_3)$

Theorem. The following properties are equivalent :

1. $\mathcal{W}_3$ is **hexagonal**

2. $\mathcal{W}_3$ is **parallelizable**

3. $\mathcal{W}_3$ is **flat**

4. $\mathcal{W}_3$ carries a non-trivial **abelian relation**

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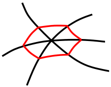

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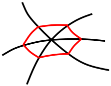
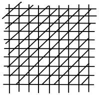
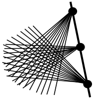
1. $\mathcal{W}_3$ is **hexagonal** 

2. $\mathcal{W}_3$ is **parallelizable** $\simeq \mathcal{W}(x, y, x - y) =$  $=$ 

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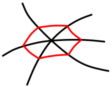
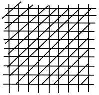
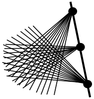
1. $\mathcal{W}_3$ is **hexagonal** 

2. $\mathcal{W}_3$ is **parallelizable** $\simeq \mathcal{W}(x, y, x-y) =$  $=$ 

3. $\mathcal{W}_3$ is **flat** i.e. its curvature vanishes : $\boldsymbol{K}_{\mathcal{W}_3} = 0$ in $\Omega^2(D)$

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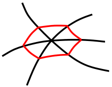
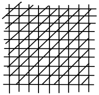
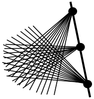
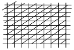
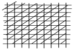
1. $\mathcal{W}_3$ is **hexagonal** 

2. $\mathcal{W}_3$ is **parallelizable** $\simeq \mathcal{W}(x, y, x-y) =$  $=$ 

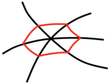
3. $\mathcal{W}_3$ is **flat**, ie. its curvature vanishes : $\boldsymbol{K}_{\mathcal{W}_3} = 0$ in $\Omega^2(D)$

4. $\mathcal{W}_3$ carries a non-trivial **abelian relation** : $\exists\, (\boldsymbol{F}_i)_{i=1}^3 \in \mathcal{O}(D)^3$ such that $\boldsymbol{F}_1(U_1) + \boldsymbol{F}_2(U_2) + \boldsymbol{F}_3(U_3) \equiv 0$

- Hexagonality is stable up to local topological equivalence :

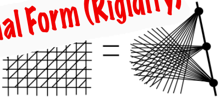
$\mathcal{W}_3 \simeq$  topologically $\Longleftrightarrow$ $\mathcal{W}_3 \simeq$  analytically

Theorem. The following properties are equivalent :

1. $\mathcal{W}_3$ is **hexagonal** 

Topological condition

Normal Form (Rigidity)

2. $\mathcal{W}_3$ is **parallelizable** $\simeq \mathcal{W}(x, y, x + \ldots)$ 

3. $\mathcal{W}_3$ is **flat**, i[illegible]nishes : $K_{\mathcal{W}_3} = 0$ in $\Omega^2(D)$

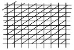
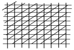
Differential condition

4. $\mathcal{W}_3$ carri[illegible]trivial **abelian relation** : $\exists\, (F_i)_{i=1}^3 \in \mathcal{O}(D)^3$ [illegible] that $F_1(U_1) + F_2(U_2) + F_3(U_3) \equiv 0$

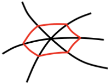
Functional condition

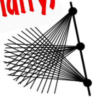
- Hence « **Topologische Fragen der Differentialgeometrie** »

- Hexagonality is stable up to local topological equivalence :

$\mathcal{W}_3 \simeq$  topologically $\Longleftrightarrow$ $\mathcal{W}_3 \simeq$  analytically

Theorem. The following properties are equivalent :

1. $\mathcal{W}_3$ is **hexagonal**  Topological condition

2. $\mathcal{W}_3$ is **parallelizable** $\simeq \mathcal{W}(x, y, x$ - ... Normal Form (Rigidity)  = 

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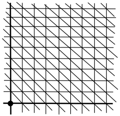
4. $\mathcal{W}_3$ carri... Functional condition ...trivial **abelian relation** : $\exists\,(F_i)_{i=1}^3 \in \mathcal{O}(D)^3$ ...that $F_1(U_1) + F_2(U_2) + F_3(U_3) \equiv 0$

- Hence **« Topological Questions in Differential Geometry »**

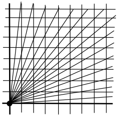
$\mathcal{W}(x, y, x+y)$



'Identity equation'

$(x) + (y) - (x+y) = 0$

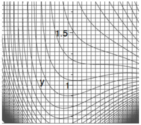
$\mathcal{W}(x, y, y/x)$



Cauchy's equation

$\mathbf{Log}(x) - \mathbf{Log}(y) + \mathbf{Log}(y/x) = 0$

$\mathcal{W}(x, y, Q)$



Euler's equation

$\int^{x} \frac{ds}{\sqrt{g(s)}} + \int^{y} \frac{ds}{\sqrt{g(s)}} - \int^{Q} \frac{ds}{\sqrt{g(s)}} = 0$

Euler's Addition Theorem (1753). We have

$$\int_0^{x} \frac{ds}{\sqrt{g(s)}} + \int_0^{y} \frac{ds}{\sqrt{g(s)}} - \int_0^{Q} \frac{ds}{\sqrt{g(s)}} = 0$$

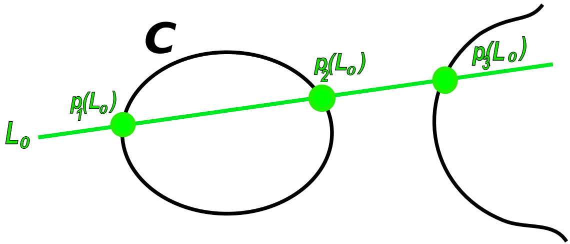
for $g(x) = (1-x^2)(1-k^2x^2)$ and $Q = Q(x,y) = \frac{x\sqrt{g(y)}+y\sqrt{g(x)}}{\sqrt{1-k^2x^2y^2}}$

Euler's Addition Theorem, geometrically

- To $\boldsymbol{y^2 = (1-x^2)(1-k^2x^2)}$ is associated a cubic curve $\mathcal{C} \subset \mathbb{P}^2$

  To $\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{dx}{y}$ is associated $\omega \in \mathbf{H}^0(\Omega^1_{\mathcal{C}}) \simeq \mathbb{C}$

- Let $L_0 \in \check{\mathbb{P}}^2$ be a line such that $L_0 \cdot \mathcal{C} = \sum_{i=1}^3 \boldsymbol{p_i}(L_0)$



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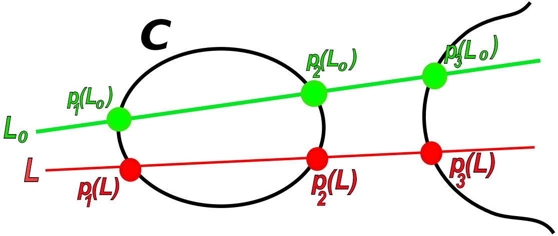
- $\exists$ germs $p_i : (\check{\mathbb{P}}^2, L_0) \to \mathcal{C}$ s.t. for $L \sim L_0$, then $L \cdot \mathcal{C} = \sum_{i=1}^3 p_i(L)$



Euler's Addition Theorem. For $\omega \in \mathbf{H}^0(\Omega^1_{\mathcal{C}})$, we have

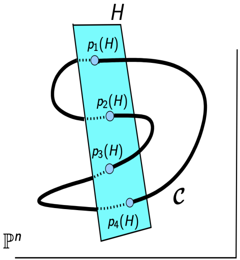
$$\sum_{i=1}^3 p_i{}^*(\omega) = 0 \iff \int^{p_1} \omega + \int^{p_2} \omega + \int^{p_3} \omega = \mathbf{cst}$$

Abel's Addition Theorem : curves

- Let $\mathcal{C} \subset \mathbb{P}^n$ be a reduced algebraic curve of degree $d \geq 3$

- Given $H_0 \in \check{\mathbb{P}}^n$ s.t. $H_0 \cdot \mathcal{C}_{reg} = \sum_{i=1}^d p_i(H_0)$ (transverse intersection)

  $\exists$ germs $\boldsymbol{p_i} : (\check{\mathbb{P}}^n, H_0) \to \mathcal{C}$ s.t. for $H \sim H_0$: $H \cdot \mathcal{C} = \sum_{i=1}^d \boldsymbol{p_i}(H)$



Abel's Addition Theorem

$\forall\ \omega \in \mathbf{H}^0(\omega^1_{\mathcal{C}})$, we have

$\sum_{i=1}^d \boldsymbol{p_i}^*(\omega) = 0$ on $\check{\mathbb{P}}^n$

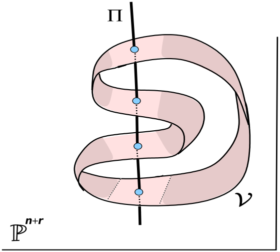
- $\mathcal{C} \rightsquigarrow \mathcal{W}_{\mathcal{C}} \overset{locally}{=} \mathcal{W}(p_1, \ldots, p_d)$: d-web by hyperplanes on $\check{\mathbb{P}}^n$

- $\forall\ \omega \in \mathbf{H}^0(\omega^1_{\mathcal{C}})$: $\left(\boldsymbol{p_i}^*(\omega)\right)_{i=1}^d$ is an **abelian relation** for $\mathcal{W}_{\mathcal{C}}$

Abel's Addition Theorem : projective varieties

- Let $\mathcal{V}^r \subset \mathbb{P}^{n+r-1}$ be a reduced proj. variety, $\deg \mathcal{V} = d$, $\dim \mathcal{V} = r$
- Let $\Pi_0 \in G_{n-1}(\mathbb{P}^{n+r-1}) = G_n(\mathbb{C}^{n+r})$ s.t. $\Pi_0 \cdot \mathcal{V}_{reg} = \sum_{i=1}^d p_i(\Pi_0)$

  $\exists$ germs $\boldsymbol{p_i} : \Big(G_{n-1}(\mathbb{P}^{n+r-1}), \Pi_0\Big) \to \mathcal{V}$ s.t. $\Pi \cdot \mathcal{V} = \sum_{i=1}^d \boldsymbol{p_i}(\Pi)$



Abel's Addition Theorem

For $k \leq r$, $\forall\ \eta \in \mathbf{H}^0(\omega^k_{\mathcal{V}})$:

$$\sum_{i=1}^d \boldsymbol{p_i}^*(\eta) = 0 \text{ on } G_n(\mathbb{C}^{n+r})$$

- $\mathcal{V} \rightsquigarrow \mathcal{W}_{\mathcal{V}} \overset{loc}{=} \mathcal{W}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_d})$: d-web on $G_n(\mathbb{C}^{n+r})$ by Schubert subvarieties
- $\forall\ \eta \in \mathbf{H}^0(\omega^k_{\mathcal{V}})$: $\big(\boldsymbol{p_i}^*(\eta)\big)_{i=1}^d$ is a ***k*-abelian relation** for $\mathcal{W}_{\mathcal{V}}$

Abelian relations

- Let $\mathcal{W}_d = d$-web on $D \subset \mathbb{C}^N$ defined by submersions $U_i : D \to \mathbb{C}^r$

Definitions. – A ***k*-abelian relation (*k*-AR)** for $\mathcal{W}_d$ is a d-tuple $(\eta_i)_{i=1}^d \in \prod_i \Omega^k_{\mathbb{C}^r}$ such that in $\Omega^k(D)$, one has

$$\sum_{i=1}^{d} U_i^*(\eta_i) = 0$$

– $AR^k(\mathcal{W}_d)$ = $\mathbb{C}$-vector space of k-**AR**s of $\mathcal{W}_d$

– $\mathrm{rk}^k(\mathcal{W}_d) = \dim AR^k(\mathcal{W}_d)$ is the ***k*-rank** of $\mathcal{W}_d$

Abel's Thm For $V^r \subset \mathbb{P}^{n+r}$ algebraic and $k \leq r$

the map $\mathbf{H}^0(\omega_V^k) \longrightarrow AR^k(\mathcal{W}_V)$ is **linear and injective**

$$\eta \longmapsto \left(p_i^*(\eta)\right)_{i=1}^d$$

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Abel's Thm + its converse. For $V^r \subset \mathbb{P}^{n+r}$ algebraic and $k \leq r$ the map

$$\begin{aligned} \mathbf{H}^0(\omega^k_V) &\longrightarrow AR^k(\mathcal{W}_V) \\ \eta &\longmapsto \big(p_i^*(\eta)\big)_{i=1}^d \end{aligned}$$

is an **isomorphism**

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the map $\mathbf{H}^0(\omega_V^k) \longrightarrow AR^k(\mathcal{W}_V)$ is an **isomorphism**

$\eta \longmapsto \big(p_i^*(\eta)\big)_{i=1}^d \quad \Longrightarrow \quad \mathbf{h}^{k,0}(V) = \mathbf{rk}^k(\mathcal{W}_V)$

Algebraic & Algebraizable Webs

Def$^{\text{o}}$. A d-web $\mathcal{W}_d$ is **algebraizable** if it is equivalent to an **algebraic web** = a web $\mathcal{W}_{\boldsymbol{V}}$ for $\boldsymbol{V}^r \subset \mathbb{P}^{n+r-1}$ alg. subvariety

Rk. If $\mathcal{W}_d$ of codimension r on $D \subset \mathbb{C}^N$ is equivalent to $\mathcal{W}_{\boldsymbol{V}}$ which is a web on $G_n(\mathbb{C}^{n+r}) \Rightarrow N = \dim G_{n-1}(\mathbb{P}^{n+r-1}) = nr$

Chern in [*Web Geometry*, Bull. AMS (1982)] :

> *In order to keep algebraic geometry in sight, we will consider [...] only webs of codimension r in $\mathbb{R}^N$, with $N = rn$. Even so the subject is a wide generalization of the geometry of projective algebraic varieties. Just as intrinsic algebraic varieties are generalized to Kähler manifolds and complex manifolds, such a generalization to web geometry seems justifiable.*

Rk. An algebraic web $\mathcal{W}_{\boldsymbol{V}}$ carries many ARs : $\mathbf{rk}^k(\mathcal{W}_{\boldsymbol{V}}) = \mathbf{h}^0(\Omega^k_{\boldsymbol{V}})$

Problem : Study webs with ‘many’ abelian relations

Abelian relations : some examples

- $\mathbf{Log}(x) - \mathbf{Log}(y) + \mathbf{Log}\left(\frac{y}{x}\right) = 0 \rightsquigarrow \left(\mathbf{Log}, -\mathbf{Log}, \mathbf{Log}\right) \in \boldsymbol{AR}\left(\mathcal{W}\left(x, y, \frac{y}{x}\right)\right)$

- **Abel's identity of the dilogarithm [Spence 1807,...]**

Rogers' dilogarithm : $\mathbf{R}(x) = \frac{1}{2}\left(\mathbf{Li}_2(x) - \mathbf{Li}_2(1-x)\right)$ with $\mathbf{Li}_2(x) = \sum_{k\geq 1} \frac{x^k}{k^2}$

$$(\mathcal{A}b) \qquad \mathbf{R}\left(x\right) - \mathbf{R}\left(y\right) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = -\frac{\pi^2}{12}$$

$$\rightsquigarrow \boldsymbol{Ab} = \left(\mathbf{R}, -\mathbf{R}, \ldots, \mathbf{R}\right) \in \boldsymbol{AR}\left(\mathcal{W}\left(x, y, \frac{y}{x}, \frac{1-y}{1-x}, \frac{x(1-y)}{y(1-x)}\right)\right)$$

Bol's web : $\mathcal{B} = \mathcal{W}\left(x, y, \frac{y}{x}, \frac{1-y}{1-x}, \frac{x(1-y)}{y(1-x)}\right) \simeq \mathcal{W}_{\mathcal{M}_{0,5}}$

Abelian Relations : bounds on the ranks

- $\mathcal{W} = \mathcal{W}(\mathsf{U}_1, \ldots, \mathsf{U}_d)$ with 1st integrals $\mathsf{U}_i : \boldsymbol{M} = (\mathbb{C}^{nr}, 0) \to \mathbb{C}^r$

- **Notations.** – Foliation $\mathcal{F}_i = \mathcal{F}_{\mathsf{U}_i}$ with $\mathsf{T}\mathcal{F}_i = \ker(\mathsf{d}\mathsf{U}_i)$

– $\mathsf{U}_i = (u_i^1, \ldots, u_i^r)$ for $S \subset \{1, \ldots, r\}$ with $|S| = k$:

$$\mathbf{d}^{\boldsymbol{S}}\mathbf{U}_{\boldsymbol{i}} = \mathsf{d}u_i^{s_1} \wedge \cdots \wedge \mathsf{d}u_i^{s_k} \in \Omega^k(\boldsymbol{M}) \qquad \mathbf{d}^{\boldsymbol{r}}\mathbf{U}_{\boldsymbol{i}} = \wedge_{s=1}^r \mathsf{d}u_i^s \in \Omega^r(\boldsymbol{M})$$

- **(GP)** : For all $i_1, \ldots, i_n \in \{1, \ldots, d\}$ pairwise distinct, we have

$$\bigoplus_{m=1}^{n} \mathsf{T}\mathcal{F}_{i_m} = \mathsf{T}\boldsymbol{M} \iff \mathsf{d}^r\mathsf{U}_{i_1} \wedge \cdots \wedge \mathsf{d}^r\mathsf{U}_{i_n} \neq 0$$

- $\boldsymbol{AR}^k(\mathcal{W}) = \left\{ (\eta_i)_{i=1}^d \in \left(\Omega^k(\mathbb{C}^r)\right)^d \mid \sum_{i=1}^d \mathsf{U_i}^*(\eta_i) = 0 \text{ in } \Omega^k_{\boldsymbol{M}} \right\}$

- A k- AR corresponds to a tuple $\left(\boldsymbol{F}_i^K(\mathsf{U}_i)\right)_{i,K}$ of holomorphic funct°

such that $$\sum_{i=1}^{d} \Big(\sum_{|K|=k} \boldsymbol{F}_i^K(\mathsf{U}_i) \cdot \mathsf{d}^K\mathsf{U}_i \Big) = 0 \qquad \text{in } \Omega^k_{\boldsymbol{M}}$$

Abelian Relations : valuations

- $\mathfrak{M} = (z_1, \ldots, z_r)$ maximal ideal of $\mathcal{O}(\mathbb{C}^r, 0) = \mathbb{C}\{z_1, \ldots, z_r\}$

Def. – For $\eta \in \Omega^r(\mathbb{C}^r, 0)$, its **valuation** (at $0 \in \mathbb{C}^r$) is

$$\mathbf{v}(\eta) = \mathbf{v}_{\mathfrak{M}}(\eta) = \sup \Big\{ \ell \geq 0 \,\big|\, \eta \in \mathfrak{M}^\ell \cdot \Omega^k(\mathbb{C}^r, 0) \Big\} \in \overline{\mathbb{N}}$$

– The **valuat°** of $\underline{\boldsymbol{\eta}} = (\eta_i)_{i=1}^d \in \boldsymbol{AR}^k(\boldsymbol{\mathcal{W}})$ is $\mathbf{v}(\underline{\boldsymbol{\eta}}) = \min \{\mathbf{v}(\eta_i)\}_{i=1}^d$

– Decreasing filtrat° $F^\nu \boldsymbol{AR}^k = \Big\{ \underline{\boldsymbol{\eta}} \in \boldsymbol{AR}^k \text{ with } \mathbf{v}(\underline{\eta}) \geq \nu \Big\}$ $(\nu \geq 0)$

Fact. – We have : $\mathbf{rk}^k(\boldsymbol{\mathcal{W}}) = \dim \boldsymbol{AR}^k \leq \sum_\nu \dim \mathrm{Gr}^\nu \boldsymbol{AR}^k$

Def. The **linearisation of $\boldsymbol{\mathcal{W}}$** (at the origin of $V = \mathbb{C}^{nr}$) is the constant web defined by the linear maps $\ell_i = \mathrm{dU}_i(0) = (\ell_i^s)_{s=1}^r$

Fact. $\mathrm{Gr}^\nu \boldsymbol{AR}^k \hookrightarrow \{ \sum_{i,|K|=k} P_i^K \mathrm{d}^K \ell_i = 0 \,|\, P_i^K \in \mathbb{C}^\nu_{\mathrm{hom}}[\ell_i^1, \ldots, \ell_i^r] \}$

Abelian Relations : bounds on the ranks ($k = r$)

- $\ell_i = (\ell_i^s)_{s=1}^r \in \mathbf{L}(V, \mathbb{C}^r)$ with $\wedge^r \ell_i \in \wedge^r V^\vee$ satisfying **(GP)**

Proposition. For $\nu \geq 0$, with $P_i \in \mathbb{C}^\nu_{\mathrm{hom}}[\ell_i]$ for all i, we have :

$$\dim \left\{ (P_i)_{i=1}^d \;\middle|\; \sum_i P_i \wedge^r \ell_i = 0 \;\text{ in }\; \wedge^r V^\vee \right\} \leq \pi^r_\nu(d,n,r)$$

with $\pi^r_\nu(d,n,r) = \binom{r-1+\nu}{r-1} \max\left(d - (r+\nu)(n-1) - 1, 0\right)$

Cor. 1. For a d-web $\mathcal{W}$ of codim r on $\mathbb{C}^{nr}$ satisfying **GP** :

$$\mathbf{rk}^r(\mathcal{W}) \leq \boldsymbol{\pi^r(d,n,r)} := \sum_{\nu \geq 0} \pi^r_\nu(d,n,r)$$

[Bol, Chern, Chern-Griffiths]

2. For $\boldsymbol{V}^r \subset \mathbb{P}^{n+r-1}$ alg. irred. non-degenerate with $\deg(\boldsymbol{V}) = d$:

$$p_g(\boldsymbol{V}) = \mathbf{h}^0(K_{\widetilde{\boldsymbol{V}}}) \leq \pi^r(d,n,r)$$

[Harris]

Castelnovo varieties and maximal rank webs

Castelnuovo (1937)-Harris (1981) bound on the genus.

1. For $\boldsymbol{V}^r \subset \mathbb{P}^{n+r-1}$ alg. irred. non-degenerate with $\deg(\boldsymbol{V}) = d$

$$p_g(\boldsymbol{V}) = \mathbf{h}^0(K_{\widetilde{\boldsymbol{V}}}) \leq \pi^r(d,n,r)$$

2. For d such that $\pi^r(d,n,r) > 0$, $\exists$ $\boldsymbol{V}$ with $p_g(\boldsymbol{V}) = \pi^r(d,n,r)$

$\boldsymbol{V}$ is a **'Castelnuovo variety'** and can be described as follows :

– $\big|I_{\boldsymbol{V}}(2)\big|$ cut out $\boldsymbol{X}^{r+1} \subset \mathbb{P}^{n+r-1}$ with $\deg(\boldsymbol{X}) = n-1$ minimal

– the class of $\boldsymbol{V}$ in $\mathrm{Pic}(\boldsymbol{X})$ is determined

Consequence. For $\boldsymbol{V}$ Castelnuovo, $\mathcal{W}_{\boldsymbol{V}}$ is of **maximal** (r-)**rank**, that is $\mathbf{rk}^k(\mathcal{W}_{\boldsymbol{V}}) = \pi^r(d,n,r)$ is maximal. Such webs exist.

Question. To what extent does a d-web $\mathcal{W}_d$ of maximal r-rank look like a 'Castelnuovo web' $\mathcal{W}_{\boldsymbol{V}}$ (with $\boldsymbol{V}^r \subset \mathbb{P}^{n+r-1}$ Castelnuovo) ?

Webs of maximal rank : algebraization

- One assumes d large enough (in particular for that $\pi^r(d,n,r) > 0$)

Thm. [Hénaut ($n=2$), Trépreau ($r=1$), P.-Trépreau ($r \geq 2$)]

In '*most cases*', a d-web $\mathcal{W}_d$ of codim r on $(\mathbb{C}^{nr}, 0)$ with maximal r-rank is algebraizable in the classical sense :

$$\text{one has } \mathcal{W}_d \simeq \mathcal{W}_{\boldsymbol{V}} \text{ for a Castelnuovo variety } \boldsymbol{V}$$

Thm. [P.] In some '*special cases*' ($n=2$, $r \geq 2$, $d = 2r+5$), one can construct maximal r-rank webs which are not of Castelnuovo type but which are algebraic in a generalized sense (Jordan type)

- **Notable exception :** Planar webs !

  $\exists$ webs which are **exceptional** $=$ nonalgebraizable but of max. rank

Bol's web

Prop. – For $\mathcal{W}_d$ planar : $\mathbf{rk}(\mathcal{W}_d) \leq \frac{1}{2}(d-1)(d-2)$ **[Bol]**

– Let $C \subset \mathbb{P}^2$ be a degree d reduced algebraic curve :

$$\mathbf{h}^0(\omega^1_C) = \frac{1}{2}(d-1)(d-2) \quad \Longrightarrow \quad \mathcal{W}_C \text{ has maximal rank}$$

Thm [Lie-Poincaré] A planar 4-web $\mathcal{W}_4$ of maximal rank 3 is algebraizable : $\exists$ a quartic curve $C \subset \mathbb{P}^2$ such that $\mathcal{W}_4 \simeq \mathcal{W}_C$

- **Bol's 5-web :** $\mathcal{B} = \mathcal{W}\left(x, y, \frac{y}{x}, \frac{1-y}{1-x}, \frac{x(1-y)}{y(1-x)}\right) \simeq \mathcal{W}_{\mathcal{M}_{0,5}}$

$(\mathcal{A}b)$ $\quad \mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = -\frac{\pi^2}{12} \quad + \quad \mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}\left(\frac{x}{y}\right) = 0$

......

- $AR(\mathcal{B}) = \Big\langle \text{Logarithmic ARs} \Big\rangle^5 \oplus \langle \mathbf{Ab} \rangle^1 \longleftarrow \quad \dim 6 = \frac{(5-1)(5-2)}{2}$

Exceptional webs & Chern's problem

- Bol's web $\mathcal{B}$ $\begin{cases} \text{has maximal rank 6} \\ \text{is not linearizable} \end{cases}$ $\Longrightarrow$ $\mathcal{B}$ is **exceptional**

- **$(\mathcal{A}b)$** $$\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = -\frac{\pi^2}{12}$$

Chern [**My Mathematical Education** (1992)] :

> *I think the determination of such webs which are exotic is a problem of great interest and importance.*
>
> ⋮
>
> *Due to my background I like algebraic manipulation* [...]*. Local differential geometry calls for such work. But good local theorems are difficult to come by. The problem on maximum rank webs discussed above is clearly an important problem, and will receive my attention.*

Chern's Probem : Determine the webs of maximal ranks

感谢大家的聆听。

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80 (1978) 13–110

Abel's Theorem and Webs

By S. S. CHERN*) and PHILLIP GRIFFITHS**)

Dedicated to Professors
G. Bol, E. Kaehler, and E. Sperner

Table of Contents

Chern [**The mathematical works of W. Blaschke** (1973)]

It was Professor Blaschke whose influence on me cannot be overstated. In 1932 he visited Peking as part of his world tour. I was a young college student in his audience. I was immediately impressed by his fresh ideas and his insistence of mathematics to be a lively and intelligible subject. This contact with him was instrumental in making me to decide to come to Hamburg as a student. I started my study in Hamburg in November 1934 and received my doctor's degree in February 1936.

Abzählungen für Gewebe.

Von SHIING-SHEN CHERN in Hamburg.

In einer früheren Arbeit[1]) hat BLASCHKE einige Sätze bewiesen, die sich auf Abzählungen für Kurvengewebe der Ebene und Flächengewebe des Raumes beziehen. In der vorliegenden Arbeit wollen wir diese Sätze auf höhere Dimensionen verallgemeinern. Es wird also der „Höchstrang" für alle Hyperflächengewebe eines N-dimensionalen Raumes bestimmt.

§ 1. Der allgemeine Fall.

Es sei R_N ein N-dimensionaler Euklidischer Raum mit den Koordinaten $x_1, x_2, \cdots, x_N$. Wir sprechen von einem „n-Gewebe von Hyperflächen" in einem zusammenhängenden Gebiet G des Raumes R_N, wenn n Hyperflächenscharen

(1) $$t_i(x_1, x_2, \cdots, x_N) = \text{konst.} \qquad (i = 1, 2, \cdots, n)$$

sich dort so darstellen lassen, daß für alle ungleichen $i_1, i_2, \cdots, i_N$

(2) $$\frac{\partial(t_{i_1}, t_{i_2}, \cdots, t_{i_N})}{\partial(x_1, x_2, \cdots, x_N)} \neq 0$$

in G gilt. Um den trivialen Fall zu vermeiden, nehmen wir $n > N$ an. Wenn es nun m und nicht mehr linear unabhängige Identitäten in $x_1, x_2, \cdots, x_N$ von der Gestalt

(3) $$\sum_{i=1}^{n} f_i^{(k)}(t_i) = 0 \qquad (k = 1, 2, \cdots, m)$$

gibt (dabei bedeutet k einen Index), so sagen wir: das Hyperflächengewebe hat den „Rang" m. Linearkombination ist dabei mit festen Koeffizienten gemeint, und die vorkommenden Funktionen sollen regulär und analytisch in G sein.

Wir denken uns die Identitäten (3) abgeleitet und finden zwischen den „Pfaffschen Formen" dt_i die linearen Abhängigkeiten

(4) $$\sum_{i=1}^{n} \frac{df_i^{(k)}}{dt_i}\, dt_i = 0 \qquad (k = 1, 2, \cdots, m).$$

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[1]) W. BLASCHKE, T_{40}, Hamb. Abhandl. 9 (1933), S. 299–312.

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