# LECTURES ON WEB GEOMETRY

## II. Webs on (some) projective varieties

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# A planar algebraic 3-web

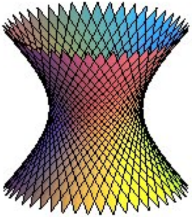
# Surfaces carring families of lines

- **[Wren 1669]** *Generatio corporis cylindroidis hyperbolici ...*

*the one-sheet hyperboloid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

*carries 2 rulings by lines*



**Kobe port tower**

**Canton tower**

**Corp° street bridge**

# The quadric surface

- Let $\boldsymbol{Q} \subset \mathbb{P}^3$ be a smooth surface with deg $\boldsymbol{Q} = 2$

$\longrightarrow$ $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sim} \boldsymbol{Q} \subset \mathbb{P}^3$ embedded by $\left| \mathcal{O}(1,1) \right|$

$\boldsymbol{Q}$ carries two rulings by lines $=$ a linear 2-web

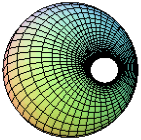
**Thm** Let $\boldsymbol{\Sigma} \subset \mathbb{P}^3$ be a smooth analytic surface st $\langle\, \boldsymbol{\Sigma} \,\rangle = \mathbb{P}^3$.

If $\boldsymbol{\Sigma}$ carries two rulings by (segments of) lines, then

there exists a smooth quadric surface $\boldsymbol{Q}$ such that $\boldsymbol{\Sigma} \subset \boldsymbol{Q}$.

# Surfaces carrying families of 'simple' curves

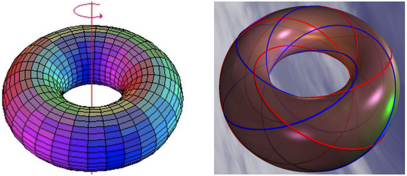
- **[Dupin 1802]** *Discovery of **cyclids** :* *surfaces $\mathcal{S} \subset \mathbb{E}^3$ whose lines of curvature are circles*



- **[Liouville 1847]** : *A Dupin cyclid can be obtained from the inversion of a torus $\mathbf{T} \subset \mathbb{E}^3$ wrt a sphere*

- **[Villarceau 1848]**

  *A torus $\mathbf{T} \subset \mathbb{E}^3$ carries four families of circles contained in it*



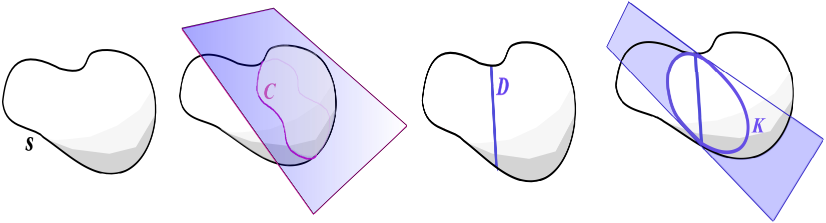
- **[Mannheim 1860]** *Through a generic point of a cyclid $\mathcal{S} = \boldsymbol{i}(\mathbf{T}) \subset \mathbb{E}^3$ pass at least four circles contained in $\mathcal{S}$*

# The cubic surface and its 27-web in conics

- $\boldsymbol{S} = \{\, \boldsymbol{F} = 0 \,\} \subset \mathbb{P}^3$ with $\boldsymbol{F} \in \mathbb{C}[x_0, \ldots, x_3]$ homogeneous, $\mathbf{d}^\circ \boldsymbol{F} = 3$

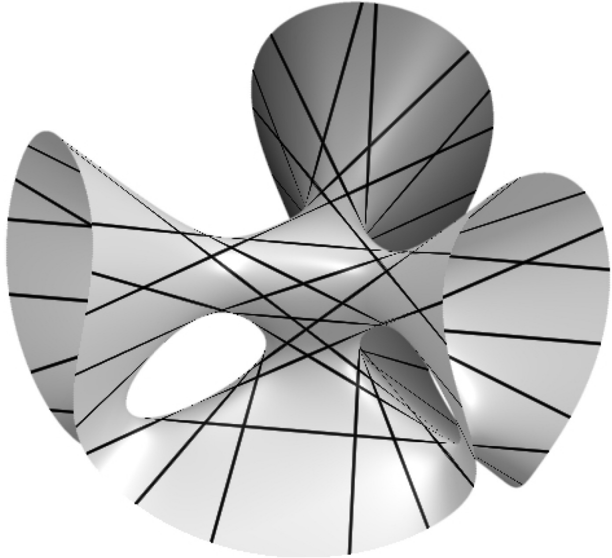
**Theorem [Cayley-Salmon 1849]**

*A smooth cubic surface $\boldsymbol{S} \subset \mathbb{P}^3$ contains exactly 27 lines*



- **Fact** : $\forall$ line $D \subset \boldsymbol{S}$ defines a pencil of conics contained in $\boldsymbol{S}$

- **Cor.** *A smooth cubic surface is covered by 27 families of conics*



# Surfaces carrying families of 'simple' curves

**Coro.** A smooth cubic $\boldsymbol{S} \subset \mathbb{P}^3$ carries a 27-web in conics $\boldsymbol{\mathcal{W}_S}$

**Thm [Bol].** $\boldsymbol{\mathcal{W}}_d =$ a planar hexagonal $d$-web : $\boldsymbol{\mathcal{H}ex_{\mathcal{W}}} \equiv 1$.

Either $\boldsymbol{\mathcal{W}}_d$ is linearizable $\Longrightarrow$ it is equivalent to $d$ pencils of lines

Otherwise : $d = 5$ and $\boldsymbol{\mathcal{W}}_5 \simeq \boldsymbol{\mathcal{B}}$.

**Theorem [Burau 1936].** Let $\boldsymbol{S}$ be a smooth cubic in $\mathbb{P}^3$.

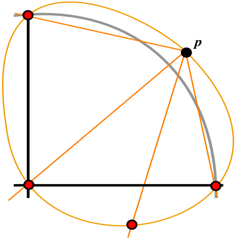
**1.** The 3-subweb $\boldsymbol{\mathcal{W}}_{\ell,\ell',\ell''}$ of $\boldsymbol{\mathcal{W}_S}$ is hexagonal $\Longleftrightarrow$ the lines $\ell, \ell', \ell''$ are pairwise skew

**2.** For a planar 27-web $\boldsymbol{\mathcal{W}}_{27}$ :

$$\boldsymbol{\mathcal{H}ex}_{\boldsymbol{\mathcal{W}}_{27}} \simeq \boldsymbol{\mathcal{H}ex}_{\boldsymbol{\mathcal{W}_S}} \quad \Longrightarrow \quad \boldsymbol{\mathcal{W}} \simeq \boldsymbol{\mathcal{W}_S}$$

# Bol's web and del Pezzo quintic surface

- $(\mathcal{A}\mathbf{b})$ $$\underset{\underset{U_1}{\|}}{\mathsf{R}(x)} - \underset{\underset{U_2}{\|}}{\mathsf{R}(y)} - \underset{\underset{U_3}{\|}}{\mathsf{R}\left(\frac{x}{y}\right)} - \underset{\underset{U_4}{\|}}{\mathsf{R}\left(\frac{1-y}{1-x}\right)} + \underset{\underset{U_5}{\|}}{\mathsf{R}\left(\frac{x(1-y)}{y(1-x)}\right)} = 0$$



$$\mathcal{B} = \mathcal{W}(U_1, U_2, \ldots, U_5)$$

**Base points of the $U_i$'s :**

$p_1 = [1, 0, 0]$ $p_2 = [0, 1, 0]$

$p_3 = [0, 0, 1]$ $p_4 = [1, 1, 1]$

- **Blow-up** $\beta : \boldsymbol{X}_4 = \mathbf{Bl}_{p_1,\ldots,p_4}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$
- The **quintic del Pezzo surface** is $\mathbf{dP}_5 = \phi_{|-K|}(\boldsymbol{X}_4) \subset \mathbb{P}^5$

# Bol's web and del Pezzo quintic surface

- $\mathbf{Bl}_{p_1,\ldots,p_4}(\mathbb{P}^2) = \boldsymbol{X}_4 \xrightarrow{\;|-K|\;} \phi_{|-K|}(\boldsymbol{X}_4) = \mathbf{dP}_5 \subset \mathbb{P}^5$

$$\begin{tikzcd}
\boldsymbol{X_4} \arrow[d, "\beta"'] \arrow[dr, bend left, "\psi_i"] & \\
\mathbb{P}^2 \arrow[r, dashed, "U_i"'] & \mathbb{P}^1
\end{tikzcd}$$

$\psi_1, \ldots, \psi_5 : \boldsymbol{X}_4 = \mathbf{dP}_5 \longrightarrow \mathbb{P}^1$ **are the five fibrations by conics on the quintic del Pezzo surface**

- $\mathbf{dP}_5 \subset \mathbb{P}^5$ is covered by 5 pencils of conics

  $\boldsymbol{S} \subset \mathbb{P}^3$ cubic surface is covered by 27 pencils of conics

⟶ What about the other surfaces of degree $d$ in $\mathbb{P}^d$ ?

⟶ What about the other del Pezzo surfaces ?

# Del Pezzo surfaces : lines and conics

- Let $\boldsymbol{S}$ be a smooth del Pezzo surface : $-K_{\boldsymbol{S}}$ ample
  - $d = \deg \boldsymbol{S} \overset{def}{=} \big( - K_{\boldsymbol{S}}\big)^2 \in \{1, \ldots, 9\}$
  - $\boldsymbol{S} = \boldsymbol{X}_r = \mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2) \qquad \left\{ \begin{array}{l} p_1, \ldots, p_r \in \mathbb{P}^2 \\ \boldsymbol{r} = 9 - d \text{ points} \\ \text{in general position} \end{array} \right.$
  - $\mathbf{Pic}(\boldsymbol{X}_r) = \mathbb{Z}\boldsymbol{h} \oplus \Big( \oplus_{i=1}^{r} \mathbb{Z}\boldsymbol{e}_i \Big) \qquad -K_{\boldsymbol{X}_r} = 3\boldsymbol{h} - \sum_i \boldsymbol{e}_i$
  - $R_r = (-K_{\boldsymbol{X}_r})^{\perp} \otimes \mathbb{R}$ root system of type $E_r \rightsquigarrow$ Weyl group $\boldsymbol{W}_r$
  - Lines $\boldsymbol{\mathcal{L}}_r = \Big\{ \ell \in \mathbf{Pic}(\boldsymbol{X}_r) \,\Big|\, \ell \cdot \boldsymbol{K} = \ell^2 = -1 \Big\} = \boldsymbol{W}_r \cdot \boldsymbol{e}_i$
  - Conics $\boldsymbol{\mathcal{C}}_r = \Big\{ \mathfrak{c} \in \mathbf{Pic}(\boldsymbol{X}_r) \,\Big|\, \mathfrak{c} \cdot \boldsymbol{K} = -2 \,,\, \mathfrak{c}^2 = 0 \Big\} = \boldsymbol{W}_r \cdot \mathfrak{c}_0$

| $\boldsymbol{r}$ | **3** | **4** | **5** | **6** | **7** | **8** |
|---|---|---|---|---|---|---|
| $\lvert\boldsymbol{\mathcal{L}}_{\boldsymbol{r}}\rvert$ | 6 | 10 | 16 | 27 | 56 | 240 |
| $\lvert\boldsymbol{\mathcal{C}}_{\boldsymbol{r}}\rvert$ | 3 | 5 | 10 | 27 | 126 | 2160 |

$\mathbf{dP}_5 \qquad \boldsymbol{S} \subset \mathbb{P}^3$

# Del Pezzo webs

**Def$^{\mathbf{o}}$.** Given $\mathbf{dP}_d$ $(d = 1, \ldots, 6)$, its **del Pezzo web** is

$$\mathcal{W}_{\mathbf{dP}_d} = \mathcal{W}\Big( \varphi_{\mathfrak{c}} : \mathbf{dP}_d \to |\,\mathfrak{c}\,| \simeq \mathbb{P}^1 \ \Big| \ \mathfrak{c} \in \mathcal{C}_r \Big)$$

- $\mathbf{dP}_6 = \mathbf{Bl}_3(\mathbb{P}^2) \rightsquigarrow \mathcal{W}_{\mathbf{dP}_6} \simeq \mathcal{W}(\, x \,,\, y \,,\, x/y)$

  $$\rightsquigarrow \ \mathbf{Log}(x) + \mathbf{Log}(y) - \mathbf{Log}\,(x/y) = 0$$

- $\mathbf{dP}_5 = \overline{\mathcal{M}}_{0,5} \quad \rightsquigarrow \mathcal{W}_{\mathbf{dP}_5} \simeq \mathcal{W}(\, f_1 \,, \ldots ,\, f_5)$

  $$\rightsquigarrow \ \mathbf{R}(f_1) - \mathbf{R}(f_2) + \cdots + \mathbf{R}(f_5) = 0 \qquad \longleftarrow \quad \mathcal{A}b$$

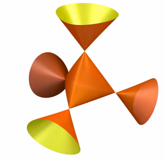
**Question.** Does a web $\mathcal{W}_{\mathbf{dP}_d}$ carry interesting ARs

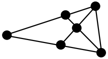
$$\sum_{\mathfrak{c} \in \mathcal{C}_r} F_{\mathfrak{c}}(\varphi_{\mathfrak{c}}) = 0 \ ? \qquad \Big( \text{ see } \textbf{Lecture III} \Big)$$

# Looking for some generalizations of del Pezzo webs

- **Singular del Pezzo surfaces** carries many interesting webs of conics

**Ex :** Cayley nodal cubic surface $\mathfrak{C} = \overline{\{1/x + 1/y + 1/z = 1\}} \subset \mathbb{P}^3$



blow-up of $\mathbb{P}^2$ at 

Sing($\mathfrak{C}$)= 4 nodal points

only 9 pencils of conics $\boldsymbol{U}_i : \mathfrak{C} \to \mathbb{P}^1$

**Fact :** Del Pezzo webs are models of webs on any (sufficiently generic) analytic surface $\boldsymbol{\Sigma} \subset \mathbb{P}^d$ $(d = 3, 4, 5, 6)$ :

**Ex :** [Moutard 1865, Darboux 1880]

$$\boldsymbol{\Sigma} \subset \mathbb{P}^3 \quad \rightsquigarrow \quad \text{27-web } \mathcal{W}_{\boldsymbol{\Sigma}} \text{ on it}$$

# Curvilinear webs on Fano threefolds

- Del Pezzo surfaces are the 2-dimensional examples of a interesting class of projective varieties, that of **Fano varieties** (ie. $-K$ ample)

**Thm.** A Fano manifold $\boldsymbol{X}$ is

**1. uniruled** : $\mathrm{RatCurves}(\boldsymbol{X}, x) \neq \emptyset$ for $x \in \boldsymbol{X}$ general

**2. rationally connected** : $\mathrm{RatCurves}(\boldsymbol{X}, x, y) \neq \emptyset$ for $x, y$ $\mathrm{g}^{\mathrm{al}}$

- Fano 3-folds have been classified : $\rho =$ Picard number

  $-\ \boldsymbol{\rho = 1}$ : $\exists$ 17 families **[ Fano-Iskovskikh ]**

  ( $-\ \boldsymbol{\rho > 1}$ : $\exists$ 88 families **[ Mori-Mukai ]** )

# Curvilinear webs on Fano threefolds

- Given a Fano 3-fold $\boldsymbol{X}$ with $\rho(\boldsymbol{X}) = 1$ :
  - **Index** $\boldsymbol{i = i(X)}$ $= \mathbf{max}\left\{ k > 0 \,\middle|\, \begin{array}{c} -K_{\boldsymbol{X}} = kH \\ \text{for } H \text{ ample} \end{array} \right\} \in \{ 1, \ldots, 4 \}$
  - **Degree** $\boldsymbol{d = d(X)}$ $= H^3$

**Thm.** $\boldsymbol{i} = 4 \Rightarrow \boldsymbol{X} \simeq \mathbb{P}^3$ [Kobayashi] , $\boldsymbol{i} = 3 \Rightarrow \boldsymbol{X} \simeq \boldsymbol{Q}^3$ [Ochiai]

- Through a point $x$ in $\mathbb{P}^3$ (resp. in $\boldsymbol{Q}^3$) pass $\infty^2$ (resp. $\infty^1$) lines

  $\longrightarrow$ Both $\mathbb{P}^3$ and $\boldsymbol{Q}^3$ are excluded

# Curvilinear webs on Fano threefolds

- Let $\boldsymbol{X}$ be a Fano 3-fold with $\rho = 1$ and index $\boldsymbol{i} = \boldsymbol{i}(\boldsymbol{X}) \in \{1, 2\}$.

- Set $\boldsymbol{H} = (-1/\boldsymbol{i})K_{\boldsymbol{X}}$. A **line** (resp. a **conic**) is an irreducible rational curve $C \subset \boldsymbol{X}$ such that $C \cdot H = 1$ (resp. $C \cdot H = 2$)

$$\boldsymbol{F}(\boldsymbol{X}) = \{\text{ line } \ell \subset \boldsymbol{X}\, \} \qquad \boldsymbol{C}(\boldsymbol{X}) = \{\text{ conic } C \subset \boldsymbol{X}\, \}$$

**Fact.** – $\boldsymbol{i} = 2$ ($\boldsymbol{X}$ = **del Pezzo** 3fold) $\Longrightarrow$ $\boldsymbol{X}$ is covered by $\infty^2$ lines
– $\boldsymbol{i} = 1$ ($\boldsymbol{X}$ = **prime Fano** 3fold) $\Longrightarrow$ $\boldsymbol{X}$ is covered by $\infty^2$ conics

- **Del Pezzo** : $\boldsymbol{F}(\boldsymbol{X})$ is a surface $\Leftrightarrow$ $\ell(\boldsymbol{X}) = \#\left\{\begin{array}{c}\text{lines through}\\ x \in \boldsymbol{X} \text{ general}\end{array}\right\} \in \mathbb{N}^*$

  $\exists$ five types of del Pezzo 3folds, one for each $\boldsymbol{d} = H^3 \in \{1, \ldots, 5\}$

- **Prime Fano** : $\boldsymbol{C}(\boldsymbol{X})$ surface $\Leftrightarrow$ $\boldsymbol{c}(\boldsymbol{X}) = \#\left\{\begin{array}{c}\text{conics through}\\ x \in \boldsymbol{X} \text{ general}\end{array}\right\} \in \mathbb{N}^*$

  $\exists$ ten types of prime Fano 3folds, one for each $\boldsymbol{g} \in \{2, \ldots 10, \cancel{11}, 12\}$

# Webs by lines on del Pezzo threefolds

| $\boldsymbol{X}$ | $\boldsymbol{d}$ | $\boldsymbol{\ell(X)}$ | $\boldsymbol{F(X)}$ |
|---|---|---|---|
| Veronese double cone | 1 | 60 | non-reduced<br>$\boldsymbol{F}_{\text{red}}$ smooth |
| Quartic double solid | 2 | 12 | $\boldsymbol{F}$ |
| Cubic hypersurface in $\mathbb{P}^4$ | 3 | 6 | $\boldsymbol{F}$ |
| $\boldsymbol{Q}_1 \cap \boldsymbol{Q}_2 \subset \mathbb{P}^5$ | 4 | 4 | $\boldsymbol{A}$ abelian |
| $G_2(\mathbb{C}^5) \cap H_1 \cap H_2 \cap H_3 \subset \mathbb{P}^6$ | 5 | 3 | $\mathbb{P}^2$ |

**Def$^{\text{o}}$ :** The **del Pezzo web $\mathcal{LW}_X$** is the $\boldsymbol{\ell(X)}$-web of lines on $\boldsymbol{X}$

**Question :** What about the $k$-ARs of $\mathcal{LW}_X$ for $k = 1$ or $k = 2$ ?

# Webs by lines on del Pezzo threefolds

- The del Pezzo web $\boldsymbol{\mathcal{LW}_X}$ is defined by the algebraic correspondance

$$\boldsymbol{F}^{2} \xleftarrow{\;\varphi\;} \boldsymbol{\ell X} = \Big\{ (\ell, x) \;\Big|\; x \in \ell \Big\}^{3} \xrightarrow[(\boldsymbol{\ell(X)}:1)]{\;\boldsymbol{\tau}\;} \boldsymbol{X}^{3}$$

**Propº :** For $\omega \in \mathbf{H}^0(\Omega^k_{\boldsymbol{F}})$ $(k = 1, 2)$ : $\mathbf{Tr}(\omega) \overset{\text{def}}{=} \boldsymbol{\tau}_* \boldsymbol{\varphi}^*(\omega) = 0$

- If $\boldsymbol{\mathcal{LW}_X} \overset{\text{loc}}{=} \mathcal{W}\big( L_i \big)$ with $L_i : (\boldsymbol{X}, x_0) \to \boldsymbol{F}$ $\qquad (i = 1, \ldots, \ell(\boldsymbol{X}))$

$$0 = \mathbf{Tr}(\omega) \overset{\text{loc}}{=} \textstyle\sum_{i=1}^{\ell} L_i^*(\omega) \quad \Longrightarrow \quad \big(L_i^*(\omega)\big)_{i=1}^{\ell} \in \boldsymbol{AR}^k\big(\boldsymbol{\mathcal{LW}_X}\big)$$

**Cor :** There are linear injective maps $\mathbf{H}^0(\Omega^k_{\boldsymbol{F}}) \hookrightarrow \boldsymbol{AR}^k\big(\boldsymbol{\mathcal{LW}_X}\big)$

$$\omega \longmapsto \big(L_i^*(\omega)\big)_{i=1}^{\ell}$$

# Webs by lines on del Pezzo threefolds

| $\boldsymbol{X}$ | $\boldsymbol{d}$ | $\boldsymbol{\ell(X)}$ | $\boldsymbol{F(X)}$ | $\boldsymbol{q(F)}$ | $\boldsymbol{p_g(F)}$ |
|---|---|---|---|---|---|
| Veronese double cone | 1 | 60 | non-reduced<br>$\boldsymbol{F}_{\mathrm{red}}$ smooth | 42 | ? |
| Quartic double solid | 2 | 12 | $\boldsymbol{F}$ | 10 | 101 |
| Cubic in $\mathbb{P}^4$ | 3 | 6 | $\boldsymbol{F}$ | 5 | 10 |
| $\boldsymbol{Q}_1 \cap \boldsymbol{Q}_2 \subset \mathbb{P}^5$ | 4 | 4 | $\boldsymbol{A}$ abelian | 2 | 1 |
| $G_2(\mathbb{C}^5) \cap L \subset \mathbb{P}^6$ | 5 | 3 | $\mathbb{P}^2$ | 0 | 0 |

- For any del Pezzo 3-fold $\boldsymbol{X}$ with $\rho = 1$ distinct from $\mathbb{P}^3$ and $\boldsymbol{Q}^3$ :

  $\mathcal{LW}_{\boldsymbol{X}}$ is a $\boldsymbol{\ell(X)}$-web of lines on $\boldsymbol{X}$ with $\mathbf{rk}^1(\mathcal{LW}_{\boldsymbol{X}}) \geq \boldsymbol{q(F)}$

  and $\mathbf{rk}^2(\mathcal{LW}_{\boldsymbol{X}}) \geq \boldsymbol{p_g(F)}$

# Curvilinear webs on analytic 3folds in $\mathbf{P}^d$ $(d = 4, 5)$

**[Sisam 1911]** Let $M^3 \subset \mathbb{P}^4$ be a sufficiently generic analytic 3-fold.

$\rightsquigarrow$ curvilinear 6-web $\mathcal{W}_M$ on $M^3 \subset \mathbb{P}^4$

**[Lalan 1923, Vangeldere 1964]** Let $M^3 \subset \mathbb{P}^5$ be a sufficiently generic analytic 3-fold.

$\rightsquigarrow$ curvilinear 4-web $\mathcal{W}_M$ on $M^3 \subset \mathbb{P}^5$

**Question.** What $\mathcal{W}_M$ says about $M$ ?

Ex : given $M, \widetilde{M} \subset \mathbb{P}^d$, what can be say if $\mathcal{W}_M \simeq \mathcal{W}_{\widetilde{M}}$ ?

**Theorem [Sisam 1930]** Let $M \subset \mathbb{P}^4$ be an analytic hypersurface with $\mathcal{W}_M =$ **linear** 6-web. Then there is a cubic $X \subset \mathbb{P}^4$ such that

$$M \subset X \quad \left( \text{ and } \mathcal{W}_M = \mathcal{LW}_X\big|_M \right)$$

# Varieties with 0-dimensional VMRT

- Let $\boldsymbol{X}$ be a smooth Fano $n$-fold
- $\mathcal{K} \subset \mathrm{RatCurves}(\boldsymbol{X})$ an irreducible covering family of rational curves of minimal $(-K_{\boldsymbol{X}})$-degree
- For $x \in \boldsymbol{X}$ general : $\mathcal{K}_x = \Big\{ [C] \in \mathcal{K} \,|\, x \in C \Big\}$ with associated family of tangent directions :

$$\mathscr{C}_x = \mathbf{Im}\left( \begin{array}{c} \tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}\mathrm{T}_x\boldsymbol{X} \\ C \mapsto \mathrm{T}_x C \end{array} \right) \subset \mathbb{P}\mathrm{T}_x\boldsymbol{X} \simeq \mathbb{P}^{n-1}$$

- $\mathscr{C}_x$ = **variety of minimal rat. tangents (VMRT)** at $x$

  $\longrightarrow$ **Cone structure** $\mathscr{C}_{\boldsymbol{X}} = \{\mathscr{C}_x\}_{x \in \boldsymbol{X}_0} \subset \mathbb{P}\mathrm{T}\boldsymbol{X}_0$

**Question :** To what extent is $\boldsymbol{X}$ locally determined by $\mathscr{C}_{\boldsymbol{X}}$ ?

# Varieties with 0-dimensional VMRT

**Question :** For $\boldsymbol{X}$, $\widetilde{\boldsymbol{X}}$ as above, let $\varphi : (\boldsymbol{X}, x) \simeq (\widetilde{\boldsymbol{X}}, \tilde{x})$ be a germ of biholomorphism such that $[d\varphi] : (\mathbb{P}\mathrm{T}\boldsymbol{X}, x) \simeq (\mathbb{P}\mathrm{T}\widetilde{\boldsymbol{X}}, \tilde{x})$ induces an isomorphism of the cone structures $(\mathscr{C}_{\boldsymbol{X}}, x) \simeq (\mathscr{C}_{\widetilde{\boldsymbol{X}}}, \tilde{x})$.

Then **1.** is $\varphi$ the restriction of a global isom $\Phi : \boldsymbol{X} \simeq \widetilde{\boldsymbol{X}}$ ?

**2.** is $\varphi$ induced by an algebraic correspondence $\boldsymbol{\Gamma} \subset \boldsymbol{X} \times \widetilde{\boldsymbol{X}}$ ?

- Let $\boldsymbol{X}$, $\widetilde{\boldsymbol{X}}$ be smooth Fano varieties of Picard rank 1

**Thm [Hwang-Mok]** **dim** $\mathscr{C}_x > 0 \implies \boldsymbol{X} \simeq \widetilde{\boldsymbol{X}}$ ( **1.** holds )

- **Case dim $\mathscr{C}_x = 0$ :** $\boldsymbol{\delta} = \#\mathscr{C}_x \in \mathbb{N}_{>0}$ for $x \in \boldsymbol{X}$ general

$\rightsquigarrow$ $\boldsymbol{\delta}$-web $\boldsymbol{\mathcal{W}}_{\boldsymbol{X}}^{\mathrm{VMRC}}$ by RCs of minimal d° (eg. $\boldsymbol{\mathcal{W}}_{\mathbf{dP}_d}$, $\boldsymbol{\mathcal{L}\mathcal{W}}_{\boldsymbol{X}}$)

**Question :** Does $\boldsymbol{\mathcal{W}}_{\boldsymbol{X}}^{\mathrm{VMRC}}$ characterise $\boldsymbol{X}$ locally ?

# Varieties with VMRC-webs

**Def^o :** A web $\mathcal{W} \overset{\text{loc}}{=} \mathcal{W}(\mathcal{F}_1, \ldots, \mathcal{F}_d)$ is

- **(P) pairwise nonintegrable** if $\forall\ i,\ \exists\ j$ such that $\mathrm{T}\mathcal{F}_i \oplus \mathrm{T}\mathcal{F}_j$ is nonintegrable
- **(B) bracket-generating** if $\sum_{i=1}^{d} \mathrm{T}\mathcal{F}_i$ is

**Thm [Hwang]** Let $\mathcal{W}$, $\widetilde{\mathcal{W}}$ be webs of curves on two projective varieties $\boldsymbol{X}$, $\widetilde{\boldsymbol{X}}$ such that

1. both $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ satisfy **(P)** and **(B)**
2. $\exists$ a germ $\varphi : (\boldsymbol{X}, x) \simeq (\widetilde{\boldsymbol{X}}, \tilde{x})$ of biholom s.t. $\varphi_*(\mathcal{W}) = \widetilde{\mathcal{W}}$

Then $\mathrm{Gr}(\varphi) \subset \boldsymbol{\Gamma}$ for a generically finite algebraic correspondence

$$\boldsymbol{\Gamma} \subset \boldsymbol{X} \times \widetilde{\boldsymbol{X}}$$

# Varieties with webs by lines

**Thm [Hwang]** Let $\boldsymbol{X}$, $\widetilde{\boldsymbol{X}}$ be two proj. manifolds with $\rho = 1$ and $\boldsymbol{F}_1(\boldsymbol{X}, x)$ finite for $x$ general in $\boldsymbol{X}$ (idem $\widetilde{\boldsymbol{X}}$).

Let $\varphi : (\boldsymbol{X}, x) \simeq (\widetilde{\boldsymbol{X}}, \tilde{x})$ be a local biholomorphism inducing an equivalence between two (global) webs by lines on $\boldsymbol{X}$ and $\widetilde{\boldsymbol{X}}$.

Then $\varphi$ is the germ of a global isomorphism $\Phi : \boldsymbol{X} \simeq \widetilde{\boldsymbol{X}}$.

**Cor.** For $\boldsymbol{X}$, $\widetilde{\boldsymbol{X}}$ = del Pezzo 3-manifolds of Picard rank 1 :

$$\mathcal{LW}_{\boldsymbol{X}} \simeq \mathcal{LW}_{\widetilde{\boldsymbol{X}}} \text{ locally} \quad \Longrightarrow \quad \boldsymbol{X} \simeq \widetilde{\boldsymbol{X}} \text{ globally}$$

**Problem :** Find local analytic charcterizations of webs $\mathcal{LW}_{\boldsymbol{X}}$

# Webs on some more general projective varieties

**Prop.** A general hypersurface $\boldsymbol{X} \subset \mathbb{P}^n$ of degree $n-1$ carries a $(n-1)!$-web by lines

**Prop.** A general hypersurface $\boldsymbol{X} \subset \mathbb{P}^n$ of degree $n$ carries a $\delta_n$-web by conics with $\delta_n = (2n)!/2^{n+1} - (n!)^2/2$

**Prop. [Reid]** Through a general point of the smooth intersection

$$\boldsymbol{X}^{2n+1} = \boldsymbol{Q}_1 \cap \boldsymbol{Q}_2 \subset \mathbb{P}^{2n+3}$$

of two hyperquadrics pass $2^{n+1}$ $n$-planes included in $\boldsymbol{X}$.

In other terms : $\boldsymbol{X}$ carries a $2^{n+1}$-web by $n$-planes.

- Many projective varieties carries webs by “simple” subvarieties...

# Webs on moduli spaces of totally reducible cycles

- Webs on moduli spaces of cycles with many irreducible components may be relevant in the study of webs carrying interesting ARs

- **[Blaschke-Walberer]** Let $\boldsymbol{X} \subset \mathbb{P}^4$ be a smooth cubic 3fold.

  Consider the moduli space of triangles in $\boldsymbol{X}$ :

$$\boldsymbol{\Delta_X} = \Big\{ \ \boldsymbol{\Pi} \in G_2(\mathbb{P}^4) \ \Big| \ \boldsymbol{\Pi} \cap \boldsymbol{X} =$$

$$\Big\}$$

# Blaschke-Walberer webs

- A generically $(12,3)$ algebraic correspondance :

$$\boldsymbol{X} \xleftarrow[(1:12)]{} \boldsymbol{p\Delta} = \Big\{ (x,\triangle) \;\Big|\; \overset{x\bullet}{\blacktriangle} \Big\} \xrightarrow[(3:1)]{} \boldsymbol{\Delta}_{\boldsymbol{X}}^{3}$$

- Another algebraic correspondance (with $\boldsymbol{F} = \boldsymbol{F}_1(\boldsymbol{X})$)

$$\boldsymbol{F}^{2} \xleftarrow{\varphi} \ell\boldsymbol{\Delta} = \Big\{ (\ell,\triangle) \;\Big|\; {}^{\ell}\!\blacktriangle \Big\}^{3} \xrightarrow[(3:1)]{\tau} \boldsymbol{\Delta}_{\boldsymbol{X}}^{3}$$

**Blaschke-Walberer web** $\boldsymbol{\Delta}\mathcal{W}_{\boldsymbol{X}} =$ curvilinear 3-web on $\boldsymbol{\Delta}_{\boldsymbol{X}}$ induced by $\ell\boldsymbol{\Delta}$

**Prop.** **1.** $\boldsymbol{\Delta}\mathcal{W}_{\boldsymbol{X}}$ is **skew** : if $\boldsymbol{\Delta}\mathcal{W}_{\boldsymbol{X}} \overset{\textbf{loc}}{=} \{\mathcal{F}_i\}_{i=1}^{3}$ then $T_{\mathcal{F}_i} \oplus T_{\mathcal{F}_j}$ is non-integrable for $i \neq j$

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**2.** If $\boldsymbol{\Delta}\mathcal{W}_{\boldsymbol{X}} \overset{\textbf{loc}}{=} \mathcal{W}(\ell_1,\ell_2,\ell_3)$ with $\ell_i : (\boldsymbol{\mathcal{T}}_{\boldsymbol{X}}, \triangle_0) \to \boldsymbol{F}$ then

$$\forall\, \omega \in \mathbf{H}^0(\Omega^1_{\boldsymbol{F}}) \; : \; \mathbf{Tr}(\omega) = \boldsymbol{\tau}_*\boldsymbol{\varphi}^*(\omega) \overset{\textbf{loc}}{=} \textstyle\sum_i \ell_i^*(\omega) = 0$$

# Blaschke-Walberer algebraization theorem

**Prop. [Kähler]** For a skew curvilinear 3-web $\mathcal{W}$ in dimension 3 :

$$\mathbf{rk}^1(\mathcal{W}) \leq 5$$

- We have
  1. $\mathbf{H}^0(\Omega^1_{\boldsymbol{F}}) \hookrightarrow \boldsymbol{AR}^1(\boldsymbol{\Delta\mathcal{W}_X})$, $\omega \longmapsto \big(\ell_i^*(\omega)\big)_{i=1}^3$
  2. $q(\boldsymbol{F}) = \mathbf{h}^0(\Omega^1_{\boldsymbol{F}}) = 5$ and $\mathbf{H}^0(\Omega^1_{\boldsymbol{F}}) \simeq \boldsymbol{AR}^1(\boldsymbol{\Delta\mathcal{W}_X})$

$\Longrightarrow$ $\mathbf{rk}^1(\boldsymbol{\Delta\mathcal{W}_X}) = 5$ i.e. $\boldsymbol{\Delta\mathcal{W}_X}$ has maximal 1-rank

**Theorem [BW]** A skew curvilinear 3-web $\mathcal{W}$ in dimension 3 of maximal 1-rank (=5) is algebraizable :

$$\exists \text{ a hypercubic } \boldsymbol{X} \subset \mathbb{P}^4 \text{ such that } \mathcal{W} \simeq \boldsymbol{\Delta\mathcal{W}_X}$$

# Blaschke-Walberer algebraization theorem

**Theorem [BW]** A skew curvilinear 3-web $\mathcal{W}$ in dimension 3 of maximal 1-rank is algebraizable :

$$\exists \text{ a hypercubic } \boldsymbol{X} \subset \mathbb{P}^4 \text{ such that } \mathcal{W} \simeq \boldsymbol{\Delta\mathcal{W}_X}$$

**Rks : 1.** The proof gives a way to reconstruct $\boldsymbol{X}$ from $\boldsymbol{AR}^1(\mathcal{W})$

**2.** This applies to many singular cubic hypersurfaces. E.g. :

Segre cubic primal (10 nodes), the chordal cubic (sing $= v_4(\mathbb{P}^1)$)

**Prop°.** Given a smooth cubic $\boldsymbol{X} \subset \mathbb{P}^4$, its 6-web by lines $\mathcal{L}\mathcal{W}_{\boldsymbol{X}}$

**i.** is linear(izable) **ii.** $\mathbf{rk}^1(\mathcal{W}_{\boldsymbol{X}}) \geq 5$ **iii.** $\mathbf{rk}^2(\mathcal{W}_{\boldsymbol{X}}) = 10$ (maximal 2-rank)

**Quest° :** For a curvilinear 6-web $\mathcal{W}$ in three variables :

$\big[$ **ii.** and **iii.** (+ **i.** ?) $\Longrightarrow$ $\mathcal{W} \simeq \mathcal{L}\mathcal{W}_{\boldsymbol{X}}$ for a cubic $\boldsymbol{X} \subset \mathbb{P}^4$ $\big]$ **?**

# End

# Gelfand-MacPherson webs : $\boldsymbol{G_k}\big(\mathbb{C}^{\boldsymbol{N}}\big)$

- $\boldsymbol{G}/_{\boldsymbol{P}} = \boldsymbol{G}_k(\mathbb{C}^N) \subset \mathbb{P}(\wedge^k \mathbb{C}^N)$ $\Big(\boldsymbol{G} = \mathrm{SL}(\mathbb{C}^N)$, Dynkin type $\boldsymbol{A_{N-1}}\Big)$

- **Moment map :**

$$\boldsymbol{\mu} : \boldsymbol{G}_k(\mathbb{C}^N) \longrightarrow \boldsymbol{\Delta}_k^N = \left\{ (t_i)_{i=1}^N \;\middle|\; \begin{array}{l} 0 \leq t_i \leq 1 \\ \sum_{i=1}^N t_i = k \end{array} \right\}$$ **hypersimplex**

Facets of $\boldsymbol{\Delta}_k^N$ (ie. codim 1)

$$= \begin{cases} \boldsymbol{\Delta}_k^N \cap \{t_i = 0\} = \boldsymbol{\Delta}_k^{N-1} & \leftsquigarrow \quad \boldsymbol{G_k}\big(\mathbb{C}^{\boldsymbol{N-1}}\big) \\ \boldsymbol{\Delta}_k^N \cap \{t_i = 1\} \simeq \boldsymbol{\Delta}_{k-1}^{N-1} & \leftsquigarrow \quad \boldsymbol{G_{k-1}}\big(\mathbb{C}^{\boldsymbol{N-1}}\big) \end{cases}$$

For every $i = 1, \ldots, N$, there are two **face maps** :

$$\boldsymbol{G}^\circ_{\boldsymbol{k-1}}\Big(\mathbb{C}^{\boldsymbol{N-1}}_{\{\boldsymbol{x_i=0}\}}\Big) \xleftarrow{\Pi_{\{t_i=1\}}} \boldsymbol{G}^\circ_{\boldsymbol{k}}\big(\mathbb{C}^{\boldsymbol{N}}\big) \xrightarrow{\Pi_{\{t_i=0\}}} \boldsymbol{G}^\circ_{\boldsymbol{k}}\Big(\mathbb{C}^{\boldsymbol{N}}/_{\langle \boldsymbol{e_i}\rangle}\Big)$$

# Gelfand-MacPherson webs : $G_k\big(\mathbb{C}^N\big)$

- For every $i = 1, \dots, N$, there are two **face maps** :

$$\begin{tikzcd}
\boldsymbol{G}^{\circ}_{\boldsymbol{k-1}}\Big(\mathbb{C}^{\boldsymbol{N-1}}_{\{\boldsymbol{x_i=0}\}}\Big) \arrow[d, "\nu_{i,0}"] & \boldsymbol{G}^{\circ}_{\boldsymbol{k}}\big(\mathbb{C}^{\boldsymbol{N}}\big) \arrow[l, "\Pi_{\{t_i=0\}}"'] \arrow[r, "\Pi_{\{t_i=1\}}"] \arrow[d, "\nu=\bullet/\boldsymbol{H}_{N-1}"] & \boldsymbol{G}^{\circ}_{\boldsymbol{k}}\Big(\mathbb{C}^{\boldsymbol{N}}/_{\langle \boldsymbol{e_i}\rangle}\Big) \arrow[d, "\nu_{i,1}"] \\
\mathbf{Conf}^{\circ}_{N-1}\big(\mathbb{P}^{k-2}\big) & \mathbf{Conf}^{\circ}_{N}\big(\mathbb{P}^{k-1}\big) \arrow[l, "\pi_{i,0}"'] \arrow[r, "\pi_{i,1}"] & \mathbf{Conf}^{\circ}_{N-1}\big(\mathbb{P}^{k-1}\big) \\
\Big[\,\mathbf{Proj}_{p_i}(p_j)\Big]_{j\neq i} & \big[\,p_1, \dots, p_N\,\big] \arrow[l, mapsto] \arrow[r, mapsto] & \big[\,p_1, \dots, \widehat{p_i}, \dots, p_N\,\big]
\end{tikzcd}$$

- **GM Web :** $\mathcal{W}^{\mathrm{GM}}_{\mathbf{Conf}_N(\mathbb{P}^{k-1})} = \mathcal{W}\left(\begin{array}{c} N \text{ maps forgetting one point} \\ +\ N \text{ point-projection maps}\end{array}\right)$

- $\boldsymbol{k = 2}$ **:** $\mathbf{Conf}^{\circ}_N\big(\mathbb{P}^1\big) = \mathcal{M}_{0,N}$ $\qquad \mathcal{W}^{\mathrm{GM}}_{\mathcal{M}_{0,N}} = \mathcal{W}\Big(\mathcal{M}_{0,N} \xrightarrow[i\,=\,1,\dots,N]{f_i} \mathcal{M}_{0,N-1}\Big)$

$\boldsymbol{+\, N = 5}$ **:** $\overline{\mathcal{M}}_{0,5} \simeq \mathbf{dP}_5 = \boldsymbol{X}_4$ $\qquad \mathcal{W}^{\mathrm{GM}}_{\mathcal{M}_{0,5}} = \mathcal{W}_{\mathbf{dP}_5} \qquad \longleftarrow \ (\,\mathcal{A}\mathbf{b}\,)$

# Gelfand-MacPherson webs : Dynkin type $D$

- $G \supset P \supset H$ : $\quad G$ = simple Lie group, Dynkin type $D$, rank $r$
  $P$ = standard parabolic subgroup (maximal)
  $H \simeq (\mathbb{C}^*)^r$ = Cartan subtorus in $G$

$V = G/P$ : $G$-homogenous projective variety

- $V \subset \mathbb{P}(E_\rho)$ : $\quad E_\rho$ = rep° of $G$ $\big( \rho : G \hookrightarrow \mathbf{GL}(E_\rho) \big)$ such that
  $V = G \cdot [e_\omega]$ with $e_\omega \in E_\rho$ of highest weight $\omega$
  $P = \mathrm{Stab}_G([e_\omega])$ = stabilizer of $[e_\omega] \in \mathbb{P}(E_\rho)$

$\mathfrak{W}_\rho = \Big\{ \text{weights of } \rho \Big\} \subset \mathfrak{h}^*_{\mathbb{R}} \simeq \mathbb{R}^r \qquad \Big( \mathfrak{h} = \mathrm{Lie}(H) \Big)$

- Minuscule case : $E_\rho = \bigoplus_{w \in \mathfrak{W}} \mathbb{C} e_w$

- **Weight polytope :** $\quad \boldsymbol{\Delta = \Delta_{D,\rho}} = \mathrm{Conv}(\mathfrak{W}_\rho) \subset \mathfrak{h}^*_{\mathbb{R}}$

# Face maps

- Let $F$ be a **facet** (= face of codim 1) of $\boldsymbol{\Delta} = \boldsymbol{\Delta}_{\boldsymbol{D},\omega}$ :

$$\boldsymbol{E} = \boldsymbol{E}_F \oplus \boldsymbol{E}^F \quad \text{with} \quad \boldsymbol{E}_F = \oplus_{w \in F} \mathbb{C} e_w \ \text{ and } \ \boldsymbol{E}^F = \oplus_{w \notin F} \mathbb{C} e_w$$

**Facts.** – $F \simeq \boldsymbol{\Delta}_{\boldsymbol{D}_F, \omega_F}$, $(\boldsymbol{D}_F, \omega_F)$ minuscule, $\boldsymbol{D}_F \subset \boldsymbol{D}$, $|\boldsymbol{D}_F| = r - 1$

– $\boldsymbol{E}_F$ is the minuscule rep° of $(\boldsymbol{G}_F, \boldsymbol{P}_F)$ of type $(\boldsymbol{D}_F, \omega_F)$

– One has $\boldsymbol{V}_F \overset{def}{=} \boldsymbol{V} \cap \mathbb{P}(\boldsymbol{E}_F) = \boldsymbol{G}_F/_{\boldsymbol{P}_F} \subset \mathbb{P}(\boldsymbol{E}_F)$

– If $\Pi_F : \boldsymbol{E} \longrightarrow \boldsymbol{E}_F$ is the linear projection from $\boldsymbol{E}^F$, one has

$$\Pi_F(\boldsymbol{V}) = \boldsymbol{V}_F \subset \mathbb{P}(\boldsymbol{E}_F)$$

– $\exists$ $\boldsymbol{V}^\circ$, $\boldsymbol{V}_F^\circ$ Zariski-open subsets such that $\Pi_F : \boldsymbol{V}^\circ \to \boldsymbol{V}_F^\circ$ is a fibration (in weighted proj. spaces) + is $(\boldsymbol{H}, \boldsymbol{H}_F)$-equivariant

# Face maps

- Let $F$ be a **facet** (= a face of codim 1) of $\boldsymbol{\Delta}$ : $\boldsymbol{E} = \boldsymbol{E}_F \oplus \boldsymbol{E}^F$

**Facts.** **1.** $\nu : \boldsymbol{V}^\circ \to \mathcal{Y}^\circ = \boldsymbol{V}^\circ/\boldsymbol{H}$ torsor for $\boldsymbol{H} \subset \boldsymbol{G}$

$\nu_F : \boldsymbol{V}_F^\circ \to \mathcal{Y}_F^\circ = \boldsymbol{V}^\circ/\boldsymbol{H}_F$ torsor for $\boldsymbol{H}_F = \boldsymbol{H}/_{\mathbb{C}_F^*} \subset \boldsymbol{G}_F$

**2.** $\exists$ $\pi_F : \mathcal{Y}^\circ \to \mathcal{Y}_F^\circ$ s.t. the following diagram commutes

$$\begin{tikzcd}
\mathbb{P}(\boldsymbol{E}) \supset \boldsymbol{G}/_{\boldsymbol{P}} \supset & \boldsymbol{V}^\circ \arrow[r, "\Pi_F"] \arrow[d, "\nu"'] & \boldsymbol{V}_F^\circ \arrow[d, "\nu_F"] & \subset \boldsymbol{G}_F/_{\boldsymbol{P}_F} \subset \mathbb{P}(\boldsymbol{E}_F) \\
\boldsymbol{V}^{ss}/\!\!/_{\boldsymbol{H}} = \mathcal{Y} \supset & \mathcal{Y}^\circ \arrow[r, "\pi_F"'] & \mathcal{Y}_F^\circ & \subset \mathcal{Y}_{\boldsymbol{F}} = \boldsymbol{V}_F^{ss}/\!\!/_{\boldsymbol{H}_F}
\end{tikzcd}$$

**Definition** : **Gelfand-MacPherson webs**

$\mathcal{W}^{\mathbf{GM}}_{\boldsymbol{G}/\boldsymbol{P}} = \mathcal{W}\Big(\Pi_F \mid F \text{ facet of } \boldsymbol{\Delta}\Big) \leftarrow$ web on $\boldsymbol{G}/_{\boldsymbol{P}}$, $\boldsymbol{H}$-equivariant

$\mathcal{W}^{\mathbf{GM}}_{\mathcal{Y}} = \mathcal{W}\Big(\pi_F \mid F \text{ facet of } \boldsymbol{\Delta}\Big) = \Big(\mathcal{W}^{\mathbf{GM}}_{\boldsymbol{G}/\boldsymbol{P}}\Big)/\boldsymbol{H} \leftarrow$ $\boldsymbol{W}_{\boldsymbol{D}}$-invariant web on $\mathcal{Y}$

# Gelfand-MacPherson webs : type $A_N$

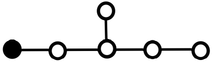
$$\begin{tikzcd}
\boldsymbol{G}^{\circ}_{\boldsymbol{k-1}}\left(\mathbb{C}^{\boldsymbol{N-1}}_{\boldsymbol{\{x_i=0\}}}\right) \arrow[d, "\nu_i"] & & \boldsymbol{G}^{\circ}_{\boldsymbol{k}}\left(\mathbb{C}^{\boldsymbol{N}}\right) \arrow[ll, "\Pi_{\{t_i=0\}}"'] \arrow[rr, "\Pi_{\{t_i=1\}}"] \arrow[d, "\nu"] & & \boldsymbol{G}^{\circ}_{\boldsymbol{k}}\left(\mathbb{C}^{\boldsymbol{N}}/_{\langle \boldsymbol{e_i} \rangle}\right) \arrow[d, "\nu_i"] \\
\boldsymbol{\mathcal{Y}}^{\boldsymbol{N-1}}_{\boldsymbol{k-1}} \arrow[d, equal] & & \boldsymbol{\mathcal{Y}}^{\boldsymbol{N}}_{\boldsymbol{k}} \arrow[ll, "\mathrm{pr}_i"'] \arrow[rr, "f_i"] \arrow[d, equal] & & \boldsymbol{\mathcal{Y}} \arrow[d, equal] \\
\mathbf{Conf}^{\circ}_{N-1}\left(\mathbb{P}^{k-2}\right) & & \mathbf{Conf}^{\circ}_{N}\left(\mathbb{P}^{k-1}\right) \arrow[ll, "\mathrm{pr}_i"'] \arrow[rr, "f_i"] & & \mathbf{Conf}^{\circ}_{N-1}\left(\mathbb{P}^{k-1}\right)
\end{tikzcd}$$

- **GM web :** $$\mathcal{W}^{\mathrm{GM}}_{\mathbf{Conf}_N(\mathbb{P}^k)} = \mathcal{W}\Big(f_1 \, , \ldots , \, f_N \, , \, \mathrm{pr}_1 \, , \ldots , \, \mathrm{pr}_N\Big)$$

- **$k = 2$ :** $$\mathcal{W}^{\mathrm{GM}}_{\mathcal{M}_{0,N}} = \mathcal{W}\big(f_1 \, , \ldots , \, f_N\big)$$ $N$-web by RCs on $\mathcal{M}_{0,N}$

# Gelfand-MacPherson web : type $E_6$

**Dynkin diagram**



**Minuscule representation**

$$\mathbf{Herm}_3(\mathbb{O}) = \oplus_{i=1}^{27} \mathbb{C} e_i$$

highest weight vector $e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- **Minuscule homog. space :** Cayley plane $\boldsymbol{E_6}/\boldsymbol{P_1} = \mathbb{O}\mathbf{P}^2 = \overline{\mathbf{Im}(v_2)}$

$$v_2 : \mathbb{O}^2 \hookrightarrow \mathbb{P}\mathbf{Herm}_3(\mathbb{O}),\ (x,y) \longmapsto \begin{bmatrix} 1 & x & y \\ \overline{x} & x\overline{x} & y\overline{x} \\ \overline{y} & x\overline{y} & y\overline{y} \end{bmatrix}$$

- **Weight polytope :** Gosset $\mathbf{2_{21}}$ 99 facets : $\left\{ \begin{array}{l} 27 \text{ orthoplexes (type } (D_5, \omega_1)) \\ 72 \text{ simplices (type } A_5) \end{array} \right.$

- **Face maps on $\mathbb{O}\mathbf{P}^2$ :** $F$ = a $D_5$-facet : $\mathbb{O}\mathbf{P}^2_F = \mathbb{O}\mathbf{P}^2 \cap \mathbb{P}(E_F^{10}) = \mathbb{O}\mathbf{P}^1$

$\rightsquigarrow$ $\Pi_F$ is equivalent to $\Pi_0 : \mathbb{O}^2 \to \mathbb{O},\ (x,y) \mapsto x$

- **GM web :** $\mathcal{W}^{\mathrm{GM}}_{\mathbb{O}\mathbf{P}^2}$ is defined by 27 copies of $(x,y) \mapsto x$

$\mathcal{W}^{\mathrm{GM}}_{\boldsymbol{\mathcal{Y}}_{E_6}}$ is defined by 27 rational maps $\boldsymbol{\mathcal{Y}}_{E_6} \dashrightarrow \mathbb{P}^3$