

# LECTURES ON WEB GEOMETRY

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## III. Webs by conics on Del Pezzo Surfaces and hyperlogarithmic identities

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**Workshop on Symplectic Varieties and Projective Geometry**

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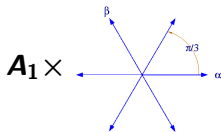
**中国科学院晨兴数学中心**

MORNINGSIDE CENTER OF MATHEMATICS  
CHINESE ACADEMY OF SCIENCES

## Logarithm

$$\text{Log}(x) + \text{Log}(y) - \text{Log}(xy) = 0$$

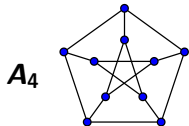
$$dP_6 = \text{Bl}_3(\mathbb{P}^2) \subset \mathbb{P}^6$$



## Dilogarithm

$$R(x) - R(y) \cdots + R\left(\frac{x(1-y)}{y(1-x)}\right) = 0$$

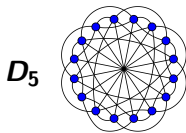
$$dP_5 = \text{Bl}_4(\mathbb{P}^2) \subset \mathbb{P}^5$$



## Weight 3 hyperlog

$$\sum_{i=1}^{10} AH_i^3(U_i(x, y)) = 0$$

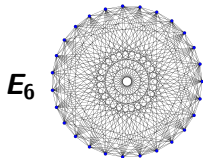
$$dP_4 = Q_1 \cap Q_2 \subset \mathbb{P}^4$$



## Weight 4 hyperlog

$$\sum_{i=1}^{27} AH_i^4(U_i(x, y)) = 0$$

$$dP_3 =$$



## I Introduction

Polylogarithms & their functional identities

**Theorem** :  $\mathbf{HLog}^w = 0$  for  $w = 1, \dots, 6$

## II Proof

Del Pezzo surfaces

Hyperlogarithms

## III An approach à la Gelfand-MacPherson

Gelfand-MacPherson webs on  $G/P$

The case  $r = 5$  :  $\mathbb{S}_5$  (spinor 10-fold)

and  $r = 6$  :  $\mathbb{O}\mathbf{P}^2$  (Cayley plane)

# The logarithm

- $\text{Li}_1(z) = -\text{Log}(1 - z)$  ( $z \in \mathbb{C}$ )
- Integral formula : 
$$\text{Log}(z) = \int \frac{du}{u-0}$$
$$\text{Li}_1(z) = -\int \frac{du}{u-1}$$
- Development in series : 
$$\text{Li}_1(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$
- Monodromy : 
$$\mathcal{M}_0(\text{Log}) = \text{Log} + 2i\pi$$
- Functional identity : 
$$\text{Log}(x) - \text{Log}(y) + \text{Log}\left(\frac{y}{x}\right) = 0$$

# The dilogarithm

- $\text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \quad (|z| < 1)$

- **Integral formula :**  $\text{Li}_2(z) = \mathbf{L}_{01}(z) = -\int^z \log(\mathbf{1} - u) \frac{du}{u-0}$   
 $\mathbf{L}_{10}(z) = \int^z \log(u - \mathbf{0}) \frac{du}{\mathbf{1}-u}$

- **Monodromy :**  $\mathcal{M}_1(\text{Li}_2) = \text{Li}_2 - 2i\pi \mathbf{Log}$

- **Abel's functional identity ( $\mathcal{A}b$ )**

$$\mathbf{R}(\mathbf{x}) - \mathbf{R}(\mathbf{y}) - \mathbf{R}\left(\frac{\mathbf{x}}{\mathbf{y}}\right) - \mathbf{R}\left(\frac{\mathbf{1}-\mathbf{y}}{\mathbf{1}-\mathbf{x}}\right) + \mathbf{R}\left(\frac{\mathbf{x}(\mathbf{1}-\mathbf{y})}{\mathbf{y}(\mathbf{1}-\mathbf{x})}\right) = \mathbf{0}$$

$$\mathbf{R}(\mathbf{x}) = \frac{1}{2} \left( \mathbf{L}_{01}(\mathbf{x}) - \mathbf{L}_{10}(\mathbf{x}) \right)$$

# The $n$ -th polylogarithm $\text{Li}_n$ for $n \geq 1$

- $\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad (|z| < 1)$

- Integral formula :**  $\text{Li}_n(z) = \int^z \text{Li}_{n-1}(u) \frac{du}{u}$

$$\text{Li}_n'(z) = \text{Li}_{n-1}(z)/z$$

- Monodromy :**  $\mathcal{M}_1(\text{Li}_n) = \text{Li}_n - 2i\pi \frac{(\text{Log})^{n-1}}{(n-1)!}$

- Functional identities in one variable :**

$$\text{Li}_n(z^r) = r^{n-1} \sum_{\omega^r=1} \text{Li}_n(\omega z)$$

$$\text{Li}_n(z) + (-1)^n \text{Li}_n(z^{-1}) = -\frac{(2i\pi)^n}{n!} \mathbf{B}_n\left(\frac{\text{Log } z}{2i\pi}\right)$$

# The $n$ -th polylogarithm $\text{Li}_n$ for $n \geq 1$

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- Monodromy :**  $\mathcal{M}_1(\text{Li}_n) = \text{Li}_n - 2i\pi \frac{(\text{Log})^{n-1}}{(n-1)!}$

- Functional identities in several variables (  $\exists?$  ) :**

$$\sum_{i \in I} c_i \text{Li}_n(\mathbf{U}_i) = \text{Elem}_{<n}$$

(  $I$  finite,  $c_i \in \mathbb{Z}$ ,  $\mathbf{U}_i \in \mathbb{Q}(x_1, \dots, x_N)$  )

# The $n$ -th polylogarithm $\text{Li}_n$ for $n \geq 1$

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- **Monodromy :**  $\mathcal{M}_1(\text{Li}_n) = \text{Li}_n - 2i\pi \frac{(\text{Log})^{n-1}}{(n-1)!}$

- **Functional identities in several variables (  $\exists?$  ) :**

$$\sum_{i \in I} c_i \text{Li}_n(\mathbf{U}_i) = \text{Elem}_{<n} \iff \sum_{i \in I} c_i \mathcal{L}_n(\mathbf{U}_i) = \mathbf{0}$$

(  $I$  finite,  $c_i \in \mathbb{Z}$ ,  $\mathbf{U}_i \in \mathbb{Q}(x_1, \dots, x_N)$  )

## Example : $\text{Li}_3$

- $\text{Li}_3(z) = \sum_{k=1}^{\infty} z^k/k^3 = \int^z \text{Li}_2(u) \frac{du}{u}$

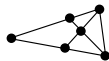
- Spence-Kummer identity ( $\mathcal{SK}$ ) (1809-1840) :**

$$2\mathcal{L}_3(x) + 2\mathcal{L}_3(y) - \mathcal{L}_3\left(\frac{x}{y}\right) + 2\mathcal{L}_3\left(\frac{1-x}{1-y}\right) + 2\mathcal{L}_3\left(\frac{x(1-y)}{y(1-x)}\right) - \mathcal{L}_3(xy) \\ + 2\mathcal{L}_3\left(-\frac{x(1-y)}{(1-x)}\right) + 2\mathcal{L}_3\left(-\frac{(1-y)}{y(1-x)}\right) - \mathcal{L}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right) = 0$$

- $(\mathcal{SK})$  lives on Cayley cubic surface  $\mathfrak{C} \subset \mathbb{P}^3$



blow-up of  $\mathbb{P}^2$  at



$\text{Sing}(\mathfrak{C}) = 4$  nodal points

only 9 pencils of conics  $U_i : \mathfrak{C} \rightarrow \mathbb{P}^1$

## Example : $\text{Li}_4$

- Kummer's functional identity  $\mathcal{K}(4)$  (1840) :

$$\begin{aligned} & \mathcal{L}_4\left(-\frac{x^2y\eta}{\zeta}\right) + \mathcal{L}_4\left(-\frac{y^2x\zeta}{\eta}\right) + \mathcal{L}_4\left(\frac{x^2y}{\eta^2\zeta}\right) + \mathcal{L}_4\left(\frac{y^2x}{\zeta^2\eta}\right) \\ & - 6\mathcal{L}_4(xy) - 6\mathcal{L}_4\left(\frac{xy}{\eta\zeta}\right) - 6\mathcal{L}_4\left(-\frac{xy}{\eta}\right) - 6\mathcal{L}_4\left(-\frac{xy}{\zeta}\right) \\ & - 3\mathcal{L}_4(x\eta) - 3\mathcal{L}_4(y\zeta) - 3\mathcal{L}_4\left(\frac{x}{\eta}\right) - 3\mathcal{L}_4\left(\frac{y}{\zeta}\right) \\ & - 3\mathcal{L}_4\left(-\frac{x\eta}{\zeta}\right) - 3\mathcal{L}_4\left(-\frac{y\zeta}{\eta}\right) - 3\mathcal{L}_4\left(-\frac{x}{\eta\zeta}\right) - 3\mathcal{L}_4\left(-\frac{y}{\eta\zeta}\right) \\ & + 6\mathcal{L}_4(x) + 6\mathcal{L}_4(y) + 6\mathcal{L}_4\left(-\frac{x}{\zeta}\right) + 6\mathcal{L}_4\left(-\frac{y}{\eta}\right) = 0 \end{aligned}$$

$$(\zeta = 1 - x, \eta = 1 - y)$$

- **Functional identities (FI) of polylogarithms  $Li_n$  :**
  - ▶ Hyperbolic geometry
  - ▶ Web geometry (  $n \leq 3$  )
  - ▶ K-theory of number fields (  $n \leq 4$  )
  - ▶ Periods (MZVs)
  - ▶ Particle physics ('*Scattering amplitudes*')
  - ▶ Mathematical physics ('*Y-systèmes*') (  $n = 2$  )
  - ▶ Cluster algebras (  $n \leq 4$  )
  - ▶ Mirror symmetry ('*Scattering diagrams*') (  $n = 2$  )
  
- **Problems :**
  - find some **FI** for  $\mathcal{L}_n$  ( e.g.  $\exists n \geq 8 ?$  )
  - better understand **FI** of polylogarithms

## [ Hain - MacPherson 1990 ] Higher logarithms

*The dilogarithm has properties analogous to those of the logarithm. It has been widely believed, both in the nineteenth century and more recently, that these two functions should be the first two elements of an infinite sequence of higher logarithms which share analogous properties. To date, several sequences of such functions have been proposed, but no function beyond the dilogarithm in any of these sequences is known to possess all the desired properties.*

## [Goncharov-Rudenko 2018] Motivic correlator, cluster algebras...

**Conclusion.** *If  $n > 3$ , the problem of writing explicitly functional equations for  $Li_n$  might not be the “right” problem. It seems that when  $n$  is growing the functional equations become so complicated that one can not write them down on a piece of paper.*

- Considering **hyperlogarithms**, one gives a series of **FI**

$$\mathbf{HLog}^1 \iff \left( \mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}(x/y) = 0 \right)$$

$$\mathbf{HLog}^2 \iff \left( \mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0 \right)$$

⋮

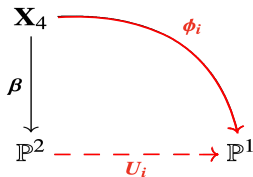
$$\mathbf{HLog}^6 \quad \left( \text{Hyperlogarithmic FI of weight 6} \right)$$

– For  $w = 1, \dots, 6$  :  $\sum_{i=1}^{\kappa} \mathbf{AH}_i^w(\phi_i) = 0 \quad (\mathbf{HLog}^w)$

# A geometric view on Abel's identity

- ( $\mathcal{A}b$ ) 
$$\underbrace{R(x)}_{U_1} - \underbrace{R(y)}_{U_2} - \underbrace{R\left(\frac{x}{y}\right)}_{U_3} - \underbrace{R\left(\frac{1-y}{1-x}\right)}_{U_4} + \underbrace{R\left(\frac{x(1-y)}{y(1-x)}\right)}_{U_5} = 0$$

- **Blow-up**  $\beta : X_4 = \mathbf{Bl}_{p_1, \dots, p_4}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$



$\phi_1, \dots, \phi_5 : X_4 \longrightarrow \mathbb{P}^1$  are  
the five fibrations by conics  
on the del Pezzo surface  $X_4$

- ( $\mathcal{A}b$ )  $\iff \sum_{i=1}^5 \epsilon_i R(\phi_i) = 0 \quad (\text{with } (\epsilon_i)_{i=1}^5 \in \{\pm 1\}^5)$

# Generalisation to del Pezzo surfaces

- $p_1, \dots, p_r \in \mathbb{P}^2$  : points in general position ( $r \in \{3, \dots, 8\}$ )
- **Blow-up**  $\beta_r : X_r = \mathbf{Bl}_{p_1, \dots, p_r}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$  ( $X_r = \mathbf{dP}_{9-r}$ )

**Prop : 1.**  $\exists$  a finite number  $= \kappa_r$   
of conic fibrations on  $X_r$   $\phi_1, \dots, \phi_{\kappa} : X_r \longrightarrow \mathbb{P}^1$

2. For any  $i$  :  $\Sigma_i = \mathbf{SingVal}(\phi_i) \subset \mathbb{P}^1$  has  $r - 1$  elements

**Def<sup>o</sup>** : The complete antisymmetric  
hyperlog. of weight  $r - 2$  :  $\mathbf{AH}_{\Sigma_i}^{r-2} : \widehat{\mathbb{P}^1 \setminus \Sigma_i} \longrightarrow \mathbb{C}$

**Thm [Castravet-P.]**  $\exists (\epsilon_i)_{i=1}^{\kappa} \in \{\pm 1\}^{\kappa}$ ,  $\pm$ -unique, such that

$$\left( \mathbf{HLog}^{r-2} \right) \quad \sum_{i=1}^{\kappa} \epsilon_i \mathbf{AH}_{\Sigma_i}^{r-2}(\phi_i) = 0$$

$$\left(\mathbf{HLog}^{r-2}\right) \quad \sum_{i=1}^{\kappa} \epsilon_i \mathbf{AH}_{\Sigma_i}^{r-2}(\phi_i) = 0$$

- One identity  $\mathbf{HLog}^{r-2}$  for each del Pezzo  $\mathbf{dP}_d = \mathbf{X}_r$  ( $d = 9 - r$ )

**[d = 6]**  $\mathbf{dP}_6$  is unique,  $\mathbf{AH}_{\Sigma_i}^1 = \mathbf{Log}$  for every  $i$

$$\left(\mathbf{HLog}^1\right) : \mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}(x/y) = 0$$

**[d = 5]**  $\mathbf{dP}_5$  is unique :  $\mathbf{AH}_{\Sigma_i}^2 = \frac{1}{2}(\mathbf{L}_{01} - \mathbf{L}_{10}) = \mathbf{R}$  for every  $i$

$$\left(\mathbf{HLog}^2\right) : \sum_{i=1}^5 \epsilon_i \mathbf{R}(\phi_i) = 0 \quad (\mathcal{A}b)$$

$$\left(\mathbf{HLog}^{r-2}\right) \quad \sum_{i=1}^{\kappa} \epsilon_i \mathbf{AH}_{\Sigma_i}^{r-2}(\phi_i) = 0$$

$[d = 4]$   $\mathbf{dP}_4$  moduli  $\infty^2 \rightsquigarrow \infty^2$  identities  $\mathbf{HLog}^3$

$$\begin{aligned} & \mathbf{AH}_1^3(x) + \mathbf{AH}_2^3\left(\frac{1}{y}\right) + \mathbf{AH}_3^3\left(\frac{y}{x}\right) + \dots \\ & \dots + \mathbf{AH}_9^3\left(\frac{y(x-b)}{x(y-a)}\right) + \mathbf{AH}_{10}^3\left(\frac{a(b-x)}{by-ax}\right) = 0 \end{aligned}$$

$[d = 3]$   $\mathbf{dP}_3 =$  cubic surface in  $\mathbb{P}^3 \rightsquigarrow \infty^4$  identities  $\mathbf{HLog}^4$

$$\sum_{i=1}^{27} \mathbf{AH}_i^4(\phi_i) = 0$$

# Del Pezzo surfaces

- $dP_d = X_r = \mathbf{Bl}_{\rho_1, \dots, \rho_r}(\mathbb{P}^2)$        $\mathbf{Pic}(dP_d) = \mathbb{Z} \mathbf{h} \oplus \left( \bigoplus_{i=1}^r \mathbb{Z} \ell_i \right)$


- $\mathbf{Pic}(dP_d) \supset \mathbf{K}^\perp = \langle \rho_1, \dots, \rho_r \rangle$        $\rho_i = \ell_i - \ell_{i+1} \quad i \leq r-1$   
 $\rho_r = 3\mathbf{h} - \sum_{i=1}^3 \ell_i$

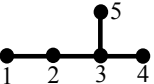
$-(\cdot, \cdot) + \{\rho_i\}_{i=1}^r \rightsquigarrow$  Root system     $E_r \subset R_r = \mathbf{K}^\perp \otimes \mathbb{R}$

- For any root  $\rho$  :       $s_\rho : R_r \longrightarrow R_r$       (orthog. reflection)  
 $d \longmapsto d + (d, \rho)\rho$

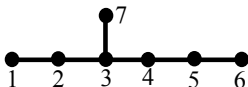
$W_r = W(E_r) = \langle s_{\rho_1}, \dots, s_{\rho_r} \rangle \subset O(R_r)$  : Weyl group of type  $E_r$

# Del Pezzo surfaces

$$E_4 = A_4$$


$$E_5 = D_5$$


$$E_6$$


$$E_7$$


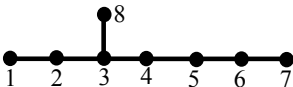
$$E_8$$


Figure – Dynkin diagram  $E_r$  ( $k$  stands for  $\rho_k$  for any  $k = 1, \dots, r$ )

# Lines and conics on $X_r = dP_d$ ( $d = 9 - r$ )

- Lines  $\mathcal{L}_r = \{ \ell \in \mathbf{Pic}(X_r) \mid \ell \cdot K = \ell^2 = -1 \} = W_r \cdot \ell_r$
- Conics  $\mathcal{C}_r = \{ \mathbf{c} \in \mathbf{Pic}(X_r) \mid -K \cdot \mathbf{c} = 2, \mathbf{c}^2 = 0 \} = W_r \cdot \mathbf{c}$   
 $\cup$   
 $\mathbf{c} \rightarrow |\mathbf{c}| \simeq \mathbb{P}^1 \rightsquigarrow$  Conic fibration  $\phi_{\mathbf{c}} : X_r \rightarrow \mathbb{P}^1$

$r$	3	4	5	6	7	8
$E_r$	$A_2 \times A_1$	$A_4$	$D_5$	$E_6$	$E_7$	$E_8$
$W_r = W(E_r)$	$\mathfrak{S}_3 \times \mathfrak{S}_2$	$\mathfrak{S}_5$	$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathfrak{S}_5$	$W(E_6)$	$W(E_7)$	$W(E_8)$
$\omega_r =  W_r $	12	5!	$2^4 \cdot 5!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
$l_r =  \mathcal{L}_r $	6	10	16	27	56	240
$\kappa_r =  \mathcal{K}_r $	3	5	10	27	126	2160

# Reducible conics in $\mathbf{X}_r = \mathbf{dP}_d$

- $L_r = \cup_{\ell \in \mathcal{L}_r} \ell \subset \mathbf{X}_r \rightsquigarrow U_r = \mathbf{X}_r \setminus L_r$

- $\mathcal{C}_r \ni \mathbf{c} \rightsquigarrow$  Conic fibration  $\phi_{\mathbf{c}} : \mathbf{X}_r \rightarrow \mathbb{P}^1$

$$\begin{aligned} \Sigma_{\mathbf{c}} = \mathbf{SingVal}(\phi_{\mathbf{c}}) &= \left\{ \sigma \in \mathbb{P}^1 \mid \phi_{\mathbf{c}}^{-1}(\sigma) \text{ non-irreducible} \right\} \\ &= \left\{ \sigma_{\mathbf{c}}^1, \dots, \sigma_{\mathbf{c}}^{r-2}, \sigma_{\mathbf{c}}^{r-1} = \infty \right\} \subset \mathbb{P}^1 \end{aligned}$$

- For  $\sigma_{\mathbf{c}}^i \in \Sigma_{\mathbf{c}} : \phi_{\mathbf{c}}^{-1}(\sigma_{\mathbf{c}}^i) = L_{\mathbf{c}}^i + \tilde{L}_{\mathbf{c}}^i \quad (L_{\mathbf{c}}^i, \tilde{L}_{\mathbf{c}}^i \in \mathcal{L}_r)$

- $\mathcal{H}_{\mathbf{c}} = \mathbf{H}^0\left(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\text{Log } \Sigma_{\mathbf{c}})\right) = \left\langle \frac{dz}{z - \sigma_{\mathbf{c}}^i} \right\rangle_{i=1}^{r-2} \simeq \mathbb{C}^{r-2}$

|}

- $\mathbf{H}_{\mathbf{c}} = \phi_{\mathbf{c}}^*(\mathcal{H}_{\mathbf{c}}) = \left\langle \frac{d\phi_{\mathbf{c}}}{\phi_{\mathbf{c}} - \sigma_{\mathbf{c}}^i} \right\rangle_{i=1}^{r-2} \subset \mathbf{H}^0\left(\mathbf{X}_r, \Omega_{\mathbf{X}_r}^1(\text{Log } L_r)\right) = \mathbf{H}_{\mathbf{X}_r}$

# Iterated integrals

- Poincaré (1884), Lappo-Danilevski (1928), Chen (1973)

- $\mathbf{Y}$  complex manifold

- $\mathbf{H} = \langle \omega_1, \dots, \omega_m \rangle \subset \mathbf{H}^0(\mathbf{Y}, \Omega_{\mathbf{Y}}^1) + \left[ \begin{array}{l} d\omega_i = 0 \\ \omega_i \wedge \omega_j = 0 \end{array} \right]$

- **Ex :**  $\phi : \mathbf{Y} \rightarrow \mathbf{C}$  and  $\omega_i \in \phi^*(\mathbf{H}^0(\mathbf{C}, \Omega_{\mathbf{C}}^1))$   $i = 1, \dots, m$

- Base point  $y \in \mathbf{Y}$ , path  $\gamma^x = \gamma_y^x : [0, 1] \rightarrow \mathbf{Y}$  from  $y$  to  $x$  :

- $\mathbb{I}_{\omega_i} : x \mapsto \int_{\gamma^x} \omega_i \quad \rightsquigarrow \quad \mathbb{I}_{\omega_i} \in \mathcal{O}_y$

- $\mathbb{I}_{\omega_j \omega_i} : x \mapsto \int_{\gamma^x} \omega_j(u) \cdot \mathbb{I}_{\omega_i}(u) \quad \rightsquigarrow \quad \mathbb{I}_{\omega_j \omega_i} \in \mathcal{O}_y$

- $\mathbb{I}_{\omega_k \omega_j \omega_i} : x \mapsto \int_{\gamma^x} \omega_k(u) \cdot \mathbb{I}_{\omega_j \omega_i}(u) \quad \rightsquigarrow \quad \mathbb{I}_{\omega_k \omega_j \omega_i} \in \mathcal{O}_y$

# Iterated integrals (polylogarithms)

$$\mathbb{H}^w : \mathbf{H}^{\otimes w} \longrightarrow \mathcal{O}_y$$

$$\bullet \quad \underline{\omega} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_w} \longmapsto \mathbb{H}_{\underline{\omega}} : z \mapsto \int_{\gamma^z} \omega_{i_1}(u) \cdot \mathbb{H}_{\omega_{i_2} \cdots \omega_{i_w}}(u)$$

$$\bullet \quad \mathbb{H} : \left( \bigoplus_{w \geq 0} \mathbf{H}^{\otimes w}, \mathbb{H} \right) \longrightarrow \mathcal{O}_y \quad \begin{array}{l} \text{injective morphism} \\ \text{of complex algebras} \end{array}$$

$$\bullet \quad \forall \underline{\omega} : \begin{array}{l} \mathbb{H}_{\underline{\omega}} \in \mathcal{O}_y \cap \tilde{\mathcal{O}}(\mathbf{Y}) \\ \text{unipotent monod.} \end{array} \longrightarrow \text{Symbol } \mathcal{S}(\mathbb{H}_{\underline{\omega}}) = \underline{\omega} \quad \checkmark$$

$$\bullet \quad \text{Ex : } \mathbf{Y} = \mathbb{P}^1 \setminus \Sigma \quad \text{with } \Sigma = \{0, 1, \infty\}$$

$$\mathbf{H} = \left\langle \frac{dz}{z}, \frac{dz}{1-z} \right\rangle = \mathbf{H}^0 \left( \mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\text{Log } \Sigma) \right)$$

$$\mathbf{L}_n = \mathbb{H}^n \left( \left( \frac{dz}{z} \right)^{\otimes (n-1)} \otimes \left( \frac{dz}{1-z} \right) \right) \quad \left( \text{'Polylogarithms'} \right)$$

# Iterated integrals (hyperlogarithms)

- Ex :**  $Y = \mathbb{P}^1 \setminus \Sigma$  avec  $\Sigma = \{ \sigma^1, \dots, \sigma^{r-2}, \sigma^{r-1} = \infty \}$

$$H = \left\langle \frac{dz}{z-\sigma^1}, \dots, \frac{dz}{z-\sigma^{r-2}} \right\rangle = H^0\left(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\text{Log } \Sigma)\right)$$

$$H^n\left(\left(\frac{dz}{z-\sigma^{i_1}}\right) \otimes \dots \otimes \left(\frac{dz}{z-\sigma^{i_n}}\right)\right) \quad \text{“Hyperlogarithm”}$$

- Complete antisymmetric hyperlog of weight  $r-2$  on  $\mathbb{P}^1 \setminus \Sigma$  :**

$$AH_{\Sigma}^{r-2} = H^n\left(\text{Asym}\left(\left(\frac{dz}{z-\sigma^1}\right) \otimes \dots \otimes \left(\frac{dz}{z-\sigma^{r-2}}\right)\right)\right)$$

$$= H^n\left(\frac{1}{(r-2)!} \sum_{\nu \in \mathfrak{S}_{r-2}} (-1)^{\nu} \left(\frac{dz}{z-\sigma^{\nu(1)}}\right) \otimes \dots \otimes \left(\frac{dz}{z-\sigma^{\nu(r-2)}}\right)\right)$$

- Ex :**  $AH_{\{0,1,\infty\}}^2 = \frac{1}{2} H^2\left(\frac{dz}{z} \otimes \frac{dz}{(1-z)} - \frac{dz}{(1-z)} \otimes \frac{dz}{z}\right) = \mathbf{R}$

### III Identity $\mathbf{HLog}^{r-2}$ : proof(s)

$$\left(\mathbf{HLog}^{r-2}\right) : \sum_{\mathbf{c} \in \mathcal{C}} \epsilon_{\mathbf{c}} \mathbf{AH}_{\mathbf{c}}^{r-2}(\phi_{\mathbf{c}}) = 0 \quad \text{with} \quad \mathbf{AH}_{\mathbf{c}}^{r-2} = \mathbf{AH}_{\Sigma_{\mathbf{c}}}^{r-2}$$

- $\phi_{\mathbf{c}} : \mathbf{X}_r \rightarrow \mathbb{P}^1 \supset \Sigma_{\mathbf{c}} = \{ \sigma_{\mathbf{c}}^i \}_{i=1}^{r-1} \quad \mathcal{H}_{\mathbf{c}} = \mathbf{H}^0(\Omega_{\mathbb{P}^1}^1(\text{Log } \Sigma_{\mathbf{c}}))$
- $\phi_{\mathbf{c}}^*(\mathcal{H}_{\mathbf{c}}) = \mathbf{H}_{\mathbf{c}} \subset \mathbf{H}_{\mathbf{X}_r} = \mathbf{H}^0(\Omega_{\mathbf{X}_r}^1(\text{Log } \mathbf{L}_r))$
- $\mathbf{AH}_{\mathbf{c}}^{r-2}(\phi_{\mathbf{c}}) = \mathbf{H} \left( \left( \frac{d\phi_{\mathbf{c}}}{\phi_{\mathbf{c}} - \sigma_{\mathbf{c}}^1} \right) \wedge \dots \wedge \left( \frac{d\phi_{\mathbf{c}}}{\phi_{\mathbf{c}} - \sigma_{\mathbf{c}}^{r-2}} \right) \right) \in \mathbf{H}^{r-2}(\wedge^{r-2} \mathbf{H}_{\mathbf{c}})$   
    ↓  $\mathcal{S}$  (symbole)
- $\Omega_{\mathbf{c}}^{r-2} = \left( \frac{d\phi_{\mathbf{c}}}{\phi_{\mathbf{c}} - \sigma_{\mathbf{c}}^1} \right) \wedge \dots \wedge \left( \frac{d\phi_{\mathbf{c}}}{\phi_{\mathbf{c}} - \sigma_{\mathbf{c}}^{r-2}} \right) \in \wedge^{r-2} \mathbf{H}_{\mathbf{c}} \subset \wedge^{r-2} \mathbf{H}_{\mathbf{X}_r}$

$$\left(\mathbf{HLog}^{r-2}\right) \iff \sum_{\mathbf{c}} \epsilon_{\mathbf{c}} \Omega_{\mathbf{c}}^{r-2} = 0 \quad \text{in} \quad \wedge^{r-2} \mathbf{H}_{\mathbf{X}_r}$$

Proofs of :  $\mathbf{hlog}^{r-2} = \sum_{\mathbf{c}} \epsilon_{\mathbf{c}} \Omega_{\mathbf{c}}^{r-2} = 0$  in  $\wedge^{r-2} \mathbf{H}_{X_r}$

•  $\mathbf{H}_{X_r} = \mathbf{H}^0\left(\Omega_{X_r}^1(\text{Log } L_r)\right) \xrightarrow{\oplus_{\ell} \text{Res}_{\ell}} \mathbb{C}^{\mathcal{L}_r}$  injective

$$\Omega_{\mathbf{c}}^{r-2} \in \wedge^{r-2} \mathbf{H}_{X_r} \hookrightarrow \wedge^{r-2} \mathbb{C}^{\mathcal{L}_r} \curvearrowright \mathbf{W}(E_r)$$

[Pr1] One decomposes  $\mathbf{hlog}^{r-2}$  in a natural basis of  $\wedge^{r-2} \mathbb{C}^{\mathcal{L}_r}$

[Pr2]  $\text{sign}_r \hookrightarrow \bigoplus_{\mathbf{c}} (\mathbf{H}_{\mathbf{c}})^{\wedge(r-2)} \longrightarrow \wedge^{r-2} \mathbb{C}^{\mathcal{L}_r} \quad \langle \text{sign}_r, \wedge^{r-2} \mathbb{C}^{\mathcal{L}_r} \rangle = 0$

$$\mathbf{1} \mapsto (\Omega_{\mathbf{c}}^{r-2})_{\mathbf{c}} \mapsto \sum_{\mathbf{c}} \Omega_{\mathbf{c}}^{r-2} \quad (\text{GAP3})$$

[Pr3] Explicit descrp<sup>o</sup> of lines  $\longrightarrow \mathbb{Z}^{\mathcal{L}_r} \simeq \mathbb{Z}^{|\mathcal{L}_r|}$

+ linear algebra /  $\mathbb{Z}$   $\longrightarrow \sum \epsilon_{\mathbf{c}} \Omega_{\mathbf{c}}^{r-2} = 0$  (Maple)

# Comparison

- $\mathbf{HLog}^2$        $R(x) - R(y) - R\left(\frac{x}{y}\right) - R\left(\frac{1-y}{1-x}\right) + R\left(\frac{x(1-y)}{y(1-x)}\right) = 0$

- $\mathbf{HLog}^3$        $AH_1^3(x) + AH_2^3\left(\frac{1}{y}\right) + AH_3^3\left(\frac{y}{x}\right) + AH_4^3\left(\frac{x-y}{x-1}\right)$   
 $+ AH_5^3\left(\frac{b(a-x)}{ay-bx}\right) + AH_6^3\left(\frac{P(x,y)}{(x-1)(y-b)}\right) + AH_7^3\left(\frac{(x-y)(y-b)}{yP(x,y)}\right)$   
 $+ AH_8^3\left(\frac{xP(x,y)}{(x-y)(x-a)}\right) + AH_9^3\left(\frac{y(x-b)}{x(y-a)}\right) + AH_{10}^3\left(\frac{a(b-x)}{by-ax}\right) = 0$

- $\mathbf{HLog}^2$  et  $\mathbf{HLog}^3$  are very similar... but not quite the same!

- $\mathbf{HLog}^2$  unique      vs      there are  $\infty^2 \mathbf{HLog}_{a,b}^3$

- only  $R$  in  $\mathbf{HLog}^2$       vs       $\exists$  several  $AH_i^3$  in  $\mathbf{HLog}^3$

- $W_{A_4} = \mathbf{Aut}(dP_5)$  acts geometrically      vs      not the case for  $W_{D_5}$

# Gelfand–MacPherson correspondance

- $R(x) - R(y) - R\left(\frac{x}{y}\right) - R\left(\frac{1-y}{1-x}\right) + R\left(\frac{x(1-y)}{y(1-x)}\right) = 0$
- Isomorphism  $\mathbb{C}^2 \setminus \mathbf{A} \simeq \mathcal{M}_{0,5}(\mathbb{C}) : (x, y) \mapsto [0, 1, \infty, x, y]$
- $\left\{ x, y, x/y, \dots, \frac{x(1-y)}{y(1-x)} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Forgetful maps} \\ f_i : \mathcal{M}_{0,5} \rightarrow \mathcal{M}_{0,4} \end{array} \right\}$
- **Gelfand-MacPherson** (over  $\mathbf{k} = \mathbb{R}, \mathbb{C}$ ) :

$$\begin{array}{ccc}
 G_2^o(\mathbf{k}^5) & \xrightarrow{F_i} & G_2^o(\mathbf{k}^5 / \langle e_i \rangle) \\
 \downarrow \bullet / H_4 & & \downarrow \bullet / H_{3,i} \\
 \mathcal{M}_{0,5}(\mathbf{k}) & \xrightarrow{f_i} & \mathcal{M}_{0,4}(\mathbf{k}) \simeq \mathbf{k} \setminus \{0, 1\}
 \end{array}$$

$\xrightarrow{a_i}$  (red curved arrow from  $G_2^o(\mathbf{k}^5)$  to  $G_2^o(\mathbf{k}^5 / \langle e_i \rangle)$ )

The vertical arrows are the quotient maps by the Cartan subgps  $(\mathbf{k}^*)^4 \simeq H_4 \subset GL(\mathbf{k}^5)$   $(\mathbf{k}^*)^3 \simeq H_{3,i} \subset GL(\mathbf{k}^5 / \langle e_i \rangle)$

# Gelfand-MacPherson construction of $(\mathcal{A}b)$

$$\begin{array}{ccc}
 \mathbf{G}_2^o(\mathbb{R}^5) & \xrightarrow{F_i} & \mathbf{G}_2^o(\mathbb{R}^5/\langle e_i \rangle) \\
 \nu=\nu_4 \downarrow & & \downarrow \nu_i=\nu_3 \\
 \mathcal{M}_{0,5}(\mathbb{R}) & \xrightarrow{f_i} & \mathcal{M}_{0,4}(\mathbb{R}) \simeq \mathbb{R} \setminus \{0, 1\}
 \end{array}$$

$\overset{a_i}{\curvearrowright}$

$\mathbf{P}_1$  : 1st Pontryagin class

$$\mathbf{H}^4(\mathbf{G}_2(\mathbb{R}^5)) \ni \mathbf{P}_1 = [\Omega]$$

$$\text{with } \Omega \in \Omega^4(\mathbf{G}_2(\mathbb{R}^5))^{\mathbf{SO}_5(\mathbb{R})}$$

- Fiber  $f$  of the 4-form  $\Omega$   $\rightarrow$ 
  - $\omega_{0,5} = \nu_*(\Omega) \in \Omega^0(\mathcal{M}_{0,5}(\mathbb{R}))$
  - $\omega_{0,4,i} = (\nu_i)_*(\Omega_i) \in \Omega^1(\mathcal{M}_{0,4}(\mathbb{R}))$

- At  $\nu(\xi) : \omega_{0,5} = \int_{\overline{H \cdot \xi}} \Omega + \overline{H \cdot \xi} \simeq \Delta_2^5$  via  $\mathbf{G}_2(\mathbb{R}^5) \xrightarrow{\mu} \Delta_2^5$   
 $\partial[\Delta_2^5] = \sum_{i=1}^5 (-1)^i a_{i*}([\Delta_{2,i}^4])$

- Stokes thm for fiber  $f$  :

$$\underbrace{d\omega_{0,5}}_{=0} = \sum_{i=1}^5 (-1)^i \underbrace{f_i^*(\omega_{0,4})}_{=d\mathbf{R}} \implies 0 = \sum_{i=1}^5 (-1)^i f_i^*(d\mathbf{R}) \quad (\mathcal{A}b)$$

# Cox varieties

- **Question** : How does one obtain  $\mathbf{G}_2(\mathbb{C}^5)$  from  $\mathbf{X}_4 = \mathbf{dP}_5 = \overline{\mathcal{M}}_{0,5}$ ?

**Answer** : It is its **Cox variety**

- $\mathbf{S}$  = projective manifold such that  $\mathbf{Pic}_{\mathbb{Z}}(\mathbf{S}) = \bigoplus_{i=0}^r \mathbb{Z} \ell_i$   
(  $\mathbf{S} = \mathbf{Bl}_{p_1, \dots, p_r}(\mathbb{P}^2)$   $\ell_0 = \mathbf{h} = [H]$  et  $\ell_i = [E_i]$   $i = 1, \dots, r$  )
- **Def**<sup>o</sup> :  $\mathbf{Cox}(\mathbf{S}) = \bigoplus_{n_0, \dots, n_r \in \mathbb{Z}} \mathbf{H}^0(\mathbf{S}, \mathcal{O}_{\mathbf{S}}(n_0 H + n_1 E_1 + \dots + n_r E_r))$
- **Facts** : –  $\mathbf{Cox}(\mathbb{P}^n) = \mathbb{C}_{\text{homog}}[x_0, \dots, x_n]$ 
  - $\mathbf{Cox}(\mathbf{S}) = \mathbb{C}[y_1, \dots, y_m] \iff \mathbf{S}$  toric
  - $\mathbf{Cox}(\mathbf{S})$  of finite type =  $\mathbf{S}$  “Mori Dream Space” (MMP ✓)

||

$$\mathbb{C}[\Gamma_1, \dots, \Gamma_m] / \mathcal{J}_{\mathbf{S}} \longrightarrow \mathbf{A}(\mathbf{S}) = \text{Spec}(\mathbf{Cox}(\mathbf{S})) \subset \mathbb{A}_{\Gamma}^m$$

# Cox varieties of del Pezzo surfaces

- Surface  $\mathbf{S} \supset \ell$  with  $\ell \simeq \mathbb{P}^1$  and  $\ell^2 = -1 \implies \sigma_\ell \in \mathbf{H}^0(\mathcal{O}_{\mathbf{X}_r}(\ell)) \setminus \{0\}$  generator in  $\mathbf{Cox}(\mathbf{S})$
- In  $\mathbf{Bl}_{p_1, \dots, p_9}(\mathbb{P}^2)$  :  $\exists \infty$  of  $(-1)$ -curves  $\implies \mathbf{Cox}(\mathbf{Bl}_{p_1, \dots, p_9}(\mathbb{P}^2))$  not of finite type

**Thm [Batyrev, Popov]** For  $r = 3, \dots, 8$ , one has

$$\mathbf{Cox}(\mathbf{dP}_d = \mathbf{X}_r = \mathbf{Bl}_{p_1, \dots, p_r}(\mathbb{P}^2)) = \mathbb{C}[\sigma_\ell \mid \ell \in \mathcal{L}_r] / \mathcal{J}_{\mathbf{dP}_d}$$

- $\mathbf{T}_{\text{NS}} = \text{Hom}_{\mathbb{Z}}(\mathbf{Pic}_{\mathbb{Z}}(\mathbf{X}_r), \mathbb{C}^*) \circlearrowleft \mathbf{A}(\mathbf{X}_r) \rightsquigarrow \mathbf{X}_r = \mathbf{A}(\mathbf{X}_r) // \mathbf{T}_{\text{NS}}$
- $\mathbf{A}(\mathbf{X}_r) \hookrightarrow \mathbb{C}^{\mathcal{L}_r} + \mathbb{Z}$ -graduation sur  $\mathbf{Cox}(\mathbf{X}_r)$  induite par  $(-K, \cdot)$   
 $\longrightarrow \mathbf{P}(\mathbf{X}_r) = \underbrace{\text{Proj}(\mathbf{Cox}(\mathbf{X}_r))}_{=(\mathbf{A}(\mathbf{X}_r) - \{0\}) / \mathbb{C}^*} \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r}) \circlearrowleft \mathcal{T}_{\text{NS}} = \mathbf{T}_{\text{NS}} / \mathbb{C}^*$

# Cox varieties of del Pezzo surfaces

- $\mathbf{P}(\mathbf{X}_r) = \text{Proj}(\mathbf{Cox}(\mathbf{X}_r)) \subset \mathbb{P}(\mathbb{C}\mathcal{L}_r) \circlearrowleft \begin{matrix} \mathcal{T}_{\text{NS}} = \mathbf{T}_{\text{NS}}/\mathbb{C}^* \\ \mathbf{W}(E_r) \end{matrix} \left. \vphantom{\begin{matrix} \mathcal{T}_{\text{NS}} = \mathbf{T}_{\text{NS}}/\mathbb{C}^* \\ \mathbf{W}(E_r) \end{matrix}} \right\} \rightarrow \mathbf{G}(E_r)$

## Thm [Batyrev, Popov, Derenthal, Serganova-Skorobogatov]

1.  $\mathbb{C}\mathcal{L}_r$  is a **minuscule** representation of  $\mathbf{G}_r = \mathbf{G}(E_r)$
2.  $(\mathcal{T}_{\text{NS}}, \mathbf{H}_r)$ -equivariant embedding  

$$\mathbf{P}(\mathbf{X}_r) \hookrightarrow \mathbf{V}_r = \mathbf{G}_r/\mathbf{P}_r \subset \mathbb{P}(\mathbb{C}\mathcal{L}_r)$$
3. There is an embedding  $f_{\text{SS}} : \mathbf{X}_r \hookrightarrow \mathbf{Y}_r = \mathbf{V}_r//\mathbf{H}_r$  such that

$$\begin{array}{ccc}
 \mathbf{P}(\mathbf{X}_r) \hookrightarrow \mathbf{V}_r \subset \mathbb{P}(\mathbb{C}\mathcal{L}_r) & & \\
 \downarrow & \downarrow & \\
 \mathbf{dP}_{9-r} = \mathbf{X}_r \xrightarrow{f_{\text{SS}}} \mathbf{Y}_r = \mathbf{V}_r//\mathbf{H}_r & & \text{is commutative}
 \end{array}$$

4. There is an isomorphism  $\mathbf{W}(E_r) \simeq \mathbf{Aut}(\mathbf{Y}_r)$

- $r = 4$ ,  $\mathbf{A}_4 : \mathbf{P}(\mathbf{X}_4) = \mathbf{V}_4 = \mathbf{G}_2(\mathbb{C}^5) \hookrightarrow \mathbb{P}(\mathbb{C}^{\mathcal{L}_4}) \simeq \mathbb{P}^9$  (Plücker)

$$\begin{array}{ccccc}
 \mathbf{P}(\mathbf{X}_4) & \xlongequal{\quad} & \mathbf{V}_4 = \mathbf{G}_2(\mathbb{C}^5) & \xrightarrow{\pi_i} & \mathbf{G}_2(\mathbb{C}_i^4) \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbf{dP}_5 = \mathbf{X}_4 & \xrightarrow[\sim]{f_{SS}} & \mathcal{Y}_4 = \overline{\mathcal{M}}_{0,5} & \xrightarrow[\pi_i]{f_i} & \overline{\mathcal{M}}_{0,4} \simeq \mathbb{P}^1
 \end{array}$$

$\longrightarrow \mathcal{W}_{\mathbf{dP}_5} = f_{SS}^*(\mathcal{W}_{\overline{\mathcal{M}}_{0,5}})$

- For  $r \in \{4, \dots, 7\}$  we have

$$\begin{array}{ccc}
 \mathbf{P}(\mathbf{X}_r) & \hookrightarrow & \mathbf{V}_r = \mathbf{G}_r/\mathbf{P}_r \\
 \downarrow & \nearrow F_{SS} & \downarrow \\
 \mathbf{dP}_{9-r} = \mathbf{X}_r & \xrightarrow{f_{SS}} & \mathcal{Y}_r
 \end{array}$$

- Can one obtain  $\mathcal{W}_{\mathbf{dP}_{9-r}}$  and  $\mathbf{HLog}^{r-2}$  from  $\mathbf{V}_r = \mathbf{G}_r/\mathbf{P}_r$  and  $\mathcal{Y}_r$ ?

# Gelfand-MacPherson webs

$\mathbf{G}$  = simple Lie group, Dynkin type  $\mathbf{D}$ , rank  $r$

- $\mathbf{G} \supset \mathbf{P} \supset \mathbf{H}$  :  $\mathbf{P}$  = standard parabolic subgroup (maximal)  
 $\mathbf{H} \simeq (\mathbb{C}^*)^r$  = Cartan subtorus in  $\mathbf{G}$

$\mathbf{V} = \mathbf{G}/\mathbf{P}$  :  $\mathbf{G}$ -homogenous projective variety

$\mathbf{E}_\rho = \text{rep}^\circ$  of  $\mathbf{G}$  ( $\rho : \mathbf{G} \hookrightarrow \mathbf{GL}(\mathbf{E}_\rho)$ ) such that

- $\mathbf{V} \subset \mathbb{P}(\mathbf{E}_\rho)$  :  $\mathbf{V} = \mathbf{G} \cdot [e_\omega]$  with  $e_\omega \in \mathbf{E}_\rho$  of highest weight  $\omega$   
 $\mathbf{P} = \text{Stab}_{\mathbf{G}}([e_\omega]) = \text{stabilizer of } [e_\omega] \in \mathbb{P}(\mathbf{E}_\rho)$

$$\mathfrak{W}_\rho = \{ \text{weights of } \rho \} \subset \mathfrak{h}_{\mathbb{R}}^* \simeq \mathbb{R}^r \quad \left( \mathfrak{h} = \text{Lie}(\mathbf{H}) \right)$$

- Minuscule case :  $\mathbf{E}_\rho = \bigoplus_{w \in \mathfrak{W}} \mathbb{C}e_w$
- **Weight polytope** :  $\Delta = \Delta_{\mathbf{D}, \rho} = \text{Conv}(\mathfrak{W}_\rho) \subset \mathfrak{h}_{\mathbb{R}}^*$

# Gelfand-MacPherson Webs

–  $\mathbf{V} \ni v = \left[ \sum_{w \in \mathfrak{W}} p^w(v) e_w \right] \in \mathbb{P}(\mathbf{E}_\rho)$        $(p^w(v))_w$  generalized Plücker coordinates

- **Moment map** :  $\mu = \mu_{D,\rho} : \mathbf{V} \longrightarrow \mathfrak{h}_{\mathbb{R}}^*$ ,  $v \longmapsto \frac{\sum_w |p^w(v)|^2 w}{\sum_w |p^w(v)|^2}$
- We set  $H_+ = \mathbf{H}(\mathbb{R}_{>0}) \simeq (\mathbb{R}_{>0})^r$

## Thm [Atiyah, Guillemin-Sternberg, Gelfand-Serganova]

1. We have  $\mu(\mathbf{V}) = \Delta$
2. For  $v$  generic (ie.  $v \in \mathbf{V}^\circ = \mu^{-1}(\overset{\circ}{\Delta})$ )
  - $\mu(\overline{H_+ \cdot v}) = \mu(\overline{\mathbf{H} \cdot v}) = \Delta$
  - $\mu$  induces a  $C^\omega$ -isom of varieties with corners  $\overline{H_+ \cdot v} \simeq \Delta$

# Face maps

- $F = \text{a facet of } \Delta \text{ (codim 1)} : \mathbf{V}_F \stackrel{\text{def}}{=} \mu^{-1}(F) \subset \mathbf{V}$

**Prop :** 1.  $\mathbf{V}_F = \mathbf{G}_F / \mathbf{P}_F$  with  $(\mathbf{G}_F, \mathbf{P}_F)$  of type  $(D_F, \omega_F)$   
and  $F \simeq \Delta_{D_F, \omega_F}$

2. We have  $\mathbf{E}_\rho = \mathbf{E}_F \oplus \mathbf{E}^F$  as  $\mathbf{G}_F\text{-rep}^\circ \rightarrow$  Linear project<sup>o</sup>  
 $\Pi_F : \mathbf{E}_\rho \rightarrow \mathbf{E}_F$

$$\mathbf{V}_F = \mathbf{V} \cap \mathbb{P}(\mathbf{E}_F) = \Pi_F(\mathbf{V}) \subset \mathbb{P}(\mathbf{E}_F)$$

3.  $H$ -torsor  $\nu_H : \mathbf{V}^\circ \rightarrow \mathbf{y}^\circ = \mathbf{V}^\circ / H \quad (\mathbf{y}^\circ \subset \mathbf{y} = \mathbf{V}^{ss} // H)$

4.  $\Pi_F : \mathbf{V}^\circ \rightarrow \mathbf{V}_F^\circ$  is  $(H, H_F)$ -equivariant  $(H \twoheadrightarrow H_F)$

# Face maps

Consequence :  $\exists \pi_F : \mathcal{Y}^\circ \rightarrow \mathcal{Y}_F^\circ = \mathbf{V}_F^\circ / \mathbf{H}_F$  such that the diagram

$$\begin{array}{ccc} \mathbf{G}/\mathbf{P} = \mathbf{V} \supset \mathbf{V}^\circ & \xrightarrow{\Pi_F} & \mathbf{V}_F^\circ \subset \mathbf{V}_F = \mathbf{G}_F/\mathbf{P}_F \\ \nu_H \downarrow & & \downarrow \nu_{\mathbf{H}_F} \\ \mathcal{Y}^\circ & \xrightarrow{\pi_F} & \mathcal{Y}_F^\circ \end{array} \quad \text{is commutative}$$

- Definition : **Gelfand-MacPherson webs**

$$\mathcal{W}_{\mathbf{G}/\mathbf{P}}^{\text{GM}} = \mathcal{W} \left( \Pi_F \mid F \text{ facet of } \Delta \right) \leftarrow \mathbf{H}\text{-equivariant}$$

$$\mathcal{W}_{\mathcal{Y}}^{\text{GM}} = \mathcal{W} \left( \pi_F \mid F \text{ facet of } \Delta \right) = \left( \mathcal{W}_{\mathbf{G}/\mathbf{P}}^{\text{GM}} \right) / \mathbf{H}$$

# Gelfand-MacPherson webs : $G_k(\mathbb{C}^N)$

- $G/P = G_k(\mathbb{C}^N) \subset \mathbb{P}(\wedge^k \mathbb{C}^N)$  ( $G = \mathrm{SL}(\mathbb{C}^N)$ , Dynkin type  $A_{N-1}$ )

- **Moment map :**

$$\mu : G_k(\mathbb{C}^N) \longrightarrow \Delta_k^N = \left\{ (t_i)_{i=1}^N \mid \begin{array}{l} 0 \leq t_i \leq 1 \\ \sum_{i=1}^N t_i = k \end{array} \right\} \quad \text{hypersimplex}$$

$$\text{Facets of } \Delta_k^N \quad (\text{ie. codim } 1) = \begin{cases} \Delta_k^N \cap \{t_i = 0\} = \Delta_k^{N-1} & \leftarrow \text{ } G_k(\mathbb{C}^{N-1}) \\ \Delta_k^N \cap \{t_i = 1\} \simeq \Delta_{k-1}^{N-1} & \leftarrow \text{ } G_{k-1}(\mathbb{C}^{N-1}) \end{cases}$$

For every  $i = 1, \dots, N$ , there are two **face maps** :

$$G_{k-1}^\circ(\mathbb{C}^{N-1}_{\{x_i=0\}}) \xleftarrow{\Pi_{\{t_i=1\}}} G_k^\circ(\mathbb{C}^N) \xrightarrow{\Pi_{\{t_i=0\}}} G_k^\circ(\mathbb{C}^N / \langle e_i \rangle)$$

# Gelfand-MacPherson webs : $G_k(\mathbb{C}^N)$

- For every  $i = 1, \dots, N$ , there are two **face maps** :

$$\begin{array}{ccccc}
 G_{k-1}^\circ(\mathbb{C}^{N-1}_{\{x_i=0\}}) & \xleftarrow{\Pi_{\{t_i=0\}}} & G_k^\circ(\mathbb{C}^N) & \xrightarrow{\Pi_{\{t_i=1\}}} & G_k^\circ(\mathbb{C}^N / \langle e_i \rangle) \\
 \downarrow \nu_{i,0} & & \downarrow \nu = \bullet / H_{N-1} & & \downarrow \nu_{i,1} \\
 \text{Conf}_{N-1}^\circ(\mathbb{P}^{k-2}) & \xleftarrow{\pi_{i,0}} & \text{Conf}_N^\circ(\mathbb{P}^{k-1}) & \xrightarrow{\pi_{i,1}} & \text{Conf}_{N-1}^\circ(\mathbb{P}^{k-1}) \\
 [\text{Proj}_{p_i}(p_j)]_{j \neq i} & \xleftarrow{\quad} & [p_1, \dots, p_N] & \xrightarrow{\quad} & [p_1, \dots, \hat{p}_i, \dots, p_N]
 \end{array}$$

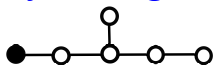
- GM Web** :  $\mathcal{W}_{\text{Conf}_N(\mathbb{P}^{k-1})}^{\text{GM}} = \mathcal{W} \left( \begin{array}{l} N \text{ maps forgetting one point} \\ + N \text{ point-projection maps} \end{array} \right)$

- $k = 2$  :  $\text{Conf}_N^\circ(\mathbb{P}^1) = \mathcal{M}_{0,N}$        $\mathcal{W}_{\mathcal{M}_{0,N}}^{\text{GM}} = \mathcal{W} \left( \mathcal{M}_{0,N} \xrightarrow[\substack{f_i \\ i=1, \dots, N}]{} \mathcal{M}_{0,N-1} \right)$

$+ N = 5$  :  $\overline{\mathcal{M}}_{0,5} \simeq \text{dP}_5 = \mathcal{X}_4$        $\mathcal{W}_{\mathcal{M}_{0,5}}^{\text{GM}} = \mathcal{W}_{\text{dP}_5} \longleftarrow (\mathcal{A}b)$

# Gelfand-MacPherson web : type $E_6$

## Dynkin diagram



## Minuscule representation

$$\mathbf{Herm}_3(\mathbb{O}) = \bigoplus_{i=1}^{27} \mathbb{C}e_i$$

$$\text{highest weight vector } e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- **Minuscule homog. space :** Cayley plane  $E_6/P_1 = \mathbb{O}P^2 = \overline{\text{Im}(v_2)}$

$$v_2 : \mathbb{O}^2 \hookrightarrow \mathbb{P}\mathbf{Herm}_3(\mathbb{O}), (x, y) \mapsto \begin{bmatrix} 1 & x & y \\ \bar{x} & x\bar{x} & y\bar{x} \\ \bar{y} & x\bar{y} & y\bar{y} \end{bmatrix}$$

- **Weight polytope :** Gosset  $2_{21}$  :  $\begin{cases} 27 \text{ orthoplexes (type } (D_5, \omega_1)) \\ 99 \text{ facets} & \begin{cases} 72 \text{ simplices (type } A_5) \end{cases} \end{cases}$

- **Face maps on  $\mathbb{O}P^2$  :**  $F = \text{a } D_5\text{-facet} : \mathbb{O}P^2_F = \mathbb{O}P^2 \cap \mathbb{P}(E_F^{10}) = \mathbb{O}P^1$   
 $\rightsquigarrow \Pi_F \text{ is equivalent to } \Pi_0 : \mathbb{O}^2 \rightarrow \mathbb{O}, (x, y) \mapsto x$

- **GM web :**  $\mathcal{W}_{\mathbb{O}P^2}^{\text{GM}}$  is defined by 27 copies of  $(x, y) \mapsto x$

$$\mathcal{W}_{\mathcal{Y}_{E_6}}^{\text{GM}} \text{ is defined by 27 rational maps } \mathcal{Y}_{E_6} \dashrightarrow \mathbb{P}^3$$

# dP-webs from GM-webs ( $r = 4, 5, 6, 7$ )

- Diagram :

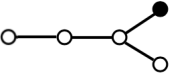
$$\begin{array}{ccccc}
 \mathbf{P}(\mathbf{X}_r) \subset & \longrightarrow & \mathbf{G}_r/\mathbf{P}_r & \xrightarrow{\pi_F} & \mathbb{Q}_F^{2r-4} = \mathbf{G}_F/\mathbf{P}_F \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbf{dP}_{9-r} = \mathbf{X}_r \subset & \xrightarrow{f_{SS}} & \mathbf{y}_r & \xrightarrow{\pi_F} & \mathbf{y}_F \simeq \mathbb{P}^{r-3}
 \end{array}$$

**Thm [P.]**  $\mathcal{W}_{\mathbf{dP}_r} = f_{SS}^*(\mathcal{W}_{\mathbf{y}_r}^{\text{GM}})$

**Question** : Can one get  $\mathbf{HLog}_{\mathbf{X}_r}^{r-2}$  from  $\mathcal{W}_{\mathbf{y}_r}^{\text{GM}}$  ?

**Answer** : Yes we can !

# Example : $\mathcal{W}_{dP_4}$ ( case $r = 5$ )

• Type  $D_5$  :   $\mathbf{G}_5 = \mathrm{Spin}_{10}(\mathbb{C}) \xrightarrow{2:1} \mathrm{SO}_{10}(\mathbb{C})$   
 $\mathbf{G}_5/\mathbf{P}_5 = \mathbb{S}_5 \subset \mathbb{P}(\mathbf{S}_5^+)$

- $\mathbb{S}_5 \simeq \mathbf{OG}_5^+(\mathbb{C}^{10}) = \text{“Spinor 10-fold”}$
- $\mathbf{S}_5^+ \simeq \mathbb{C}^{16} = \text{“half-spin representation”}$

$\mu : \mathbb{S}_5 \rightarrow \Delta_{D_5} =$  **demihypercube** : convexe envelope of the  
 $\frac{1}{2}(\epsilon_1, \dots, \epsilon_5) \in \frac{1}{2}\{\pm 1\}^5$  with  $\epsilon_1 \cdots \epsilon_5 = 1$

Facets :  $\mathbf{F} = \Delta_{D_5, i}^\epsilon = \Delta_{D_5} \cap \{t_i = \frac{\epsilon}{2}\} \simeq \Delta_{D_4, \mathbb{Q}^6}$   $\left( \begin{matrix} i = 1, \dots, 5 \\ \epsilon \in \{\pm 1\} \end{matrix} \right)$

Diagram :

$$\begin{array}{ccccc}
 \mathbf{P}(\mathbf{X}_5) & \hookrightarrow & \mathbb{S}_5 & \xrightarrow{\pi_F} & \mathbb{Q}_F^6 \\
 \downarrow & & \downarrow & & \downarrow \\
 dP_4 = \mathbf{X}_5 & \xrightarrow{f_{SS}} & \mathbf{Y}_5 & \xrightarrow{\pi_F} & \mathbf{Y}_F \simeq \mathbb{P}^2
 \end{array}$$

- Diagram :

$$\begin{array}{ccccc}
 \mathbf{P}(\mathbf{X}_5) \hookrightarrow & \mathbb{S}_5 & \xrightarrow{\pi_F} & \mathbb{Q}_F^6 = \mathbb{S}_4 \\
 \downarrow & \downarrow & & \downarrow \\
 \mathbf{dP}_4 = \mathbf{X}_5 \hookrightarrow & \mathbf{Y}_5 & \xrightarrow{\pi_F} & \mathbf{Y}_F \simeq \mathbb{P}^2
 \end{array}$$

$f_{SS}$  is the map between  $\mathbf{X}_5$  and  $\mathbf{Y}_5$ .

- Gelfand-MacPherson web of  $\mathbb{S}_5$  :**

$$\mathcal{W}_{\mathbf{Y}_5}^{\text{GM}} = \mathcal{W} \left( \begin{array}{l} \pi_i^\varepsilon : \mathbf{Y}_5 \dashrightarrow \mathbb{P}^2 \\ i = 1, \dots, 5, \varepsilon = \pm \end{array} \right) = \text{10-web of codim 2}$$

on  $\mathbf{Y}_5 \simeq_{\text{birat}} \mathbb{C}^5$

- We have  $\mathcal{W}_{\mathbf{dP}_5} = f_{SS}^* \left( \mathcal{W}_{\mathbf{Y}_5}^{\text{GM}} \right)$

- Quadrilateral  $\mathcal{Q} = \overline{\{xyZ = 0\}} \subset \mathbb{P}^2$  (  $Z = 1 + x + y$  )

$$\rightsquigarrow \omega_{\mathcal{Q}} = \mathbf{Asym}_{x,y,Z} \left( \text{Log}(Z) \left( \frac{dx}{x} \wedge \frac{dy}{y} \right) \right)$$

**Thm :** 1. We have  $(\mathbf{HLOG}_{\mathbf{Y}_5}) : \sum_{i,\varepsilon} (\pi_i^\varepsilon)^* (\omega_{\mathcal{Q}}) = 0$

2.  $\mathbf{HLOG}_{\mathbf{Y}_5} \rightsquigarrow \mathbf{HLog}_{\mathbf{dP}_4}^3$  for any  $\mathbf{dP}_4$