Nonlinear Klein-Gordon equation - M2 Exam April 7th of 2025 - 3h

All electronic devices are forbidden

We denote

$$\langle u_1, u_2\rangle = \int_{\mathbb{R}} u_1 u_2, \quad \vec{u} = (u, v), \quad \langle \vec{u}_1, \vec{u}_2\rangle = \int_{\mathbb{R}} u_1 u_2 + \int_{\mathbb{R}} v_1 v_2.$$

For $\vec{u} = (u, v) \in X = H^1 \times L^2$, we set

$$\|\vec{u}\|_X = \sqrt{\|u\|_{H^1}^2 + \|v\|_{L^2}^2}$$

We consider the 1D cubic Klein-Gordon equation

$$\begin{cases} \partial_t U = V \\ \partial_t V = \partial_x^2 U - U + U^3, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \end{cases} \tag{NLKG}$$

We define

$$Q(x) = \frac{\sqrt{2}}{\cosh(x)}$$

and we recall that $U(t, x) = Q(x)$ is a solution of (NLKG). Moreover, we define the linear operator

$$\mathcal{L} = -\partial_x^2 + 1 - 3Q^2.$$

For $u \in H^1$, we use the notation

$$\langle \mathcal{L}u, u\rangle = \int (\partial_x u)^2 + u^2 - 3Q^2 u^2.$$

We recall that $\mathcal{L}$ has two eigenfunctions

$$\mathcal{L}Q' = 0, \quad \mathcal{L}Y = -3Y, \quad Y(x) = \frac{\sqrt{3}}{2} \frac{1}{\cosh^2(x)}, \quad \|Y\|_{L^2} = 1,$$

and that it satisfies the following property: there exist $c_1, c_2 > 0$ such that for any $u \in H^1$,

$$\langle \mathcal{L}u, u\rangle \geq c_1 \|u\|_{H^1}^2 - c_2 \left(\langle u, Y\rangle^2 + \langle u, Q'\rangle^2\right). \tag{$\star$}$$

We also consider the *linearized* Klein-Gordon equation

$$\begin{cases} \partial_t u = v \\ \partial_t v = -\mathcal{L}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \end{cases} \tag{L}$$

Note that equation (L) can be written as

$$\partial_t \vec{u} = A\vec{u}, \quad A = \begin{pmatrix} 0 & 1 \\ -\mathcal{L} & 0 \end{pmatrix}.$$

We also set

$$\vec{Y}^{\pm} = \begin{pmatrix} Y \\ \pm\sqrt{3}Y \end{pmatrix}, \quad \vec{Z}^{\pm} = \begin{pmatrix} \pm\sqrt{3}Y \\ Y \end{pmatrix}.$$

1. Prove that for any $\vec{u}_0 \in X$, there exists a global solution $\vec{u} \in \mathcal{C}(\mathbb{R}, X)$ of (L) with $\vec{u}(0) = \vec{u}_0$.

2. Prove that the solution $\vec{u}(t)$ of (L) constructed above satisfies, for all $t \in \mathbb{R}$,

$$\langle \mathcal{L}u(t), u(t)\rangle + \langle v(t), v(t)\rangle = \langle \mathcal{L}u_0, u_0\rangle + \langle v_0, v_0\rangle.$$

3. Prove that there exist $c_3, c_4 > 0$ such that for any $\vec{u} \in X$,

$$\langle \mathcal{L}u, u\rangle + \langle v, v\rangle \geq c_3\|\vec{u}\|_X^2 - c_4\left(\langle \vec{u}, \vec{Z}^+\rangle^2 + \langle \vec{u}, \vec{Z}^-\rangle^2 + \langle u, Q'\rangle^2\right) \qquad (\star)$$

4. For the initial data $\vec{u}_0 = (aQ', bQ')$, where $a, b \in \mathbb{R}$, what is the corresponding solution $\vec{u}(t)$ of (L)? Are all solutions of (L) bounded in time?

5. Prove that if the initial data $x \mapsto \vec{u}_0(x)$ is even then, for all $t \in \mathbb{R}$, the function $x \mapsto \vec{u}(t, x)$ is even.

From now on, we only consider even functions of the variable x.

6. For the initial data $\vec{u}_0 = b\vec{Y}^+$, where $b \in \mathbb{R}$, what is the corresponding solution $\vec{u}(t)$ of (L)? Same question for $\vec{u}_0 = b\vec{Y}^-$. Are all even solutions of (L) bounded in time?

7. For a solution $\vec{u}$ of (L), give the equation satisfied by $\langle \vec{u}(t), \vec{Z}^+\rangle$. Same question for $\langle \vec{u}(t), \vec{Z}^-\rangle$.

8. Let $\vec{u}_0 \in X$ be such that $\langle \vec{u}_0, \vec{Z}^+\rangle = 0$, and let $\vec{u}(t)$ be the corresponding solution of (L). Prove that $\vec{u}(t)$ is globally bounded in X for $t \geq 0$.

Hint: using $\langle \vec{Y}^-, \vec{Z}^-\rangle \neq 0$ and $\langle \vec{Y}^-, \vec{Z}^+\rangle = 0$, introduce a suitable decomposition of the initial data

$$\vec{u}_0 = b\vec{Y}^- + \vec{u}_0^{\perp}, \quad \langle \vec{u}_0^{\perp}, \vec{Z}^-\rangle = 0.$$

9. For a given function $a \in \mathcal{C}(\mathbb{R})$, determine the solution of the linear non homogeneous model

$$\partial_t \vec{u} = A\vec{u} + a(t)\vec{Y}^-, \quad \vec{u}(0) = 0.$$

10. For any function $\vec{f} \in \mathcal{C}([0,+\infty), X)$, define the space-time norm

$$\|\vec{f}\|_B = \sup_{t \geq 0} \left\{ e^t \, \|\vec{f}(t)\|_X \right\}$$

and the corresponding space (with even functions only)

$$B = \{\vec{f} \in \mathcal{C}([0,+\infty), X) : \|\vec{f}\|_B < +\infty\}.$$

Let $\vec{f} \in B$. Prove that there exists a unique $\vec{u}_0$ satisfying $\langle \vec{u}_0, \vec{Z}^- \rangle = 0$ such that the solution $\vec{u}$ of

$$\partial_t \vec{u} = A\vec{u} + \vec{f}(t), \quad \vec{u}(0) = \vec{u}_0,$$

for $t \geq 0$, satisfies $\|\vec{u}\|_B < +\infty$.

Hint: introduce a suitable decomposition of the source term

$$\vec{f}(t) = a(t)\vec{Y}^- + \vec{f}^\perp(t), \quad \langle \vec{f}^\perp(t), \vec{Z}^- \rangle = 0$$

and choose a function $\vec{u}_0$ depending of $\vec{f}^\perp$.

11. Prove that the map $\mathcal{K} : \vec{f} \in B \mapsto \vec{u} \in B$ constructed in the previous question is a bounded linear map on B.

12. Prove that there exist $C_0 > 0$ and $b_0 > 0$ such that for any $b \in \mathbb{R}$ with $|b| < b_0$, there exists a unique global solution $\vec{u}$ of

$$\begin{cases} \partial_t u = v \\ \partial_t v = -\mathcal{L}u + (Q+u)^3 - Q^3 - 3Q^2 u, \quad (t,x) \in [0,+\infty) \times \mathbb{R}, \end{cases}$$

such that $\langle \vec{u}(0), Z^- \rangle = b$ and $\|\vec{u}\|_B \leq C_0|b|$. We denote this solution by $\vec{u}_b$.

Hint: set $\vec{f} = (0, (Q+u)^3 - Q^3 - 3Q^2 u)$ and reformulate the question as a fixed point problem.

13. For $b \in \mathbb{R}$ such that $|b| < b_0$, define

$$\vec{U}_b = (Q, 0) + \vec{u}_b.$$

Justify that $\vec{U}_b$ satisfies (NLKG). Describe its asymptotic behavior as $t \to +\infty$.

Why does the existence of such a family of solutions $\vec{U}_b$ illustrate the instability of the solution $Q(x)$?