DISCRETE SERIES REPRESENTATIONS OF QUATERNIONIC $\mathrm{GL}_n(D)$ WITH SYMPLECTIC PERIODS

by

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Abstract. — For a non-Archimedean locally compact field F of odd residue characteristic and characteristic 0, we prove a conjecture of D. Prasad predicting that, for an integer $n \geqslant 1$ and a non-split quaternionic F-algebra D, a discrete series representation of $\mathrm{GL}_n(D)$ has a symplectic period if and only if it is cuspidal and its Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ is non-cuspidal.

2010 Mathematics Subject Classification: 22E50, 11F70
Keywords and Phrases: Cuspidal representations, Discrete series, Jacquet–Langlands correspondence, Quaternionic forms, Symplectic periods

1. Introduction

1.1. Let F be a non-Archimedean locally compact field, and D be a non-split quaternion algebra of centre F. Fix an integer $n \geqslant 1$, and set $G = \mathrm{GL}_n(D)$. This is an inner form of $\mathrm{GL}_{2n}(F)$, which can be equipped with an involution σ whose fixed point subgroup H is equal to $\mathrm{Sp}_n(D)$, the non-quasi-split inner form of the symplectic group $\mathrm{Sp}_{2n}(F)$. In the framework of the local relative Langlands program, one is interested in the classification of the irreducible (smooth, complex) representations of G which are distinguished by H, that is, which admit non-zero H-invariant linear forms. The split version where the pair (G, H) is replaced by $(G', H') = (\mathrm{GL}_{2n}(F), \mathrm{Sp}_{2n}(F))$ has been thoroughly investigated by Jacquet–Rallis [**32**], Heumos-Rallis [**28**] and Offen [**42, 43**], both locally and globally. The pair (G', H') is a vanishing pair in the sense that there is no cuspidal representation of G' distinguished by H', both locally and globally. The striking difference with the pair (G, H) under consideration in this work is that the latter does not share this property, as observed by Verma in [**64**]. More precisely, for discrete series representations of G, Dipendra Prasad proposed the following conjecture (see also [**64**] Conjecture 7.1) stated in terms of the local Jacquet–Langlands correspondence, a bijection between the discrete series of G and $\mathrm{GL}_{2n}(F)$. For the definition of the notation St_2, see §4.1 below.

Conjecture 1.1. — (1) *There is a discrete series representation of* $\mathrm{GL}_n(D)$ *distinguished by* $\mathrm{Sp}_n(D)$ *if and only if* n *is odd.*

(2) *Suppose that the integer n is odd. The discrete series representations of* $\mathrm{GL}_n(D)$ *which are distinguished by* $\mathrm{Sp}_n(D)$ *are exactly the cuspidal representations of* $\mathrm{GL}_n(D)$ *whose Jacquet–Langlands transfer to* $\mathrm{GL}_{2n}(F)$ *is of the form* $\mathrm{St}_2(\tau)$ *for some cuspidal representation* τ *of* $\mathrm{GL}_n(F)$.

Note that, if one replaces the groups G and H by their split forms $\mathrm{GL}_{2n}(F)$ and $\mathrm{Sp}_{2n}(F)$, then [**46**] Theorem 1 implies that there is no generic (in particular, no discrete series) representation of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$, whatever the parity of n.

1.2. In this article, we prove the following results. Let p be the residue characteristic of F.

Theorem 1.2. — *Suppose that F has odd residue characteristic and that the Jacquet–Langlands transfer of any cuspidal representation of* $\mathrm{GL}_n(D)$ *distinguished by* $\mathrm{Sp}_n(D)$ *is non-cuspidal. Then any cuspidal representation of* $\mathrm{GL}_n(D)$ *whose Jacquet–Langlands transfer is non-cuspidal is distinguished by* $\mathrm{Sp}_n(D)$.

It then follows from well-known properties of the Jacquet–Langlands correspondence (see §4.1 and §4.2) that

– there is a cuspidal representation of $\mathrm{GL}_n(D)$ distinguished by $\mathrm{Sp}_n(D)$ if and only if n is odd,

– if n is odd, a cuspidal representation π of $\mathrm{GL}_n(D)$ is distinguished by $\mathrm{Sp}_n(D)$ if and only if its Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ is a discrete series representation of the form $\mathrm{St}_2(\tau)$, where τ is a cuspidal representation of $\mathrm{GL}_n(F)$ uniquely determined by π up to isomorphism.

In the case when F has characteristic 0, Verma ([**64**] Theorem 1.2) proved, by using a globalisation argument, that any cuspidal representation of G which is distinguished by H has a non-cuspidal Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ (see also Theorem 5.2 below). It follows that, when F has characteristic 0 and $p \neq 2$, any cuspidal representation of $\mathrm{GL}_n(D)$ whose Jacquet–Langlands transfer is non-cuspidal is distinguished by $\mathrm{Sp}_n(D)$.

Theorem 1.3. — *Suppose that F has characteristic* 0*. Then any discrete series representation of* $\mathrm{GL}_n(D)$ *distinguished by* $\mathrm{Sp}_n(D)$ *is cuspidal.*

Putting Theorems 1.2 and 1.3 together with Verma's result, we obtain the following corollary.

Corollary 1.4. — *Suppose that F is a non-Archimedean locally compact field of characteristic* 0 *and odd residue characteristic. Then Prasad's Conjecture 1.1 holds.*

The proofs of Theorems 1.2 and 1.3 use quite different tools and methods. Let us first explain how we prove Theorem 1.2.

1.3. The strategy of the proof of Theorem 1.2 is, given a cuspidal representation π of G whose Jacquet–Langlands transfer is non-cuspidal, to produce a pair $(\boldsymbol{J}, \boldsymbol{\lambda})$ made of a compact mod centre, open subgroup $\boldsymbol{J}$ of G and an irreducible representation $\boldsymbol{\lambda}$ of $\boldsymbol{J}$ such that:

– $\boldsymbol{\lambda}$ is distinguished by $\boldsymbol{J} \cap H$,

– the compact induction of $\boldsymbol{\lambda}$ to G is isomorphic to π.

By a simple application of Mackey's formula, this will imply that π is distinguished by H. The construction of a suitable pair $(\boldsymbol{J},\boldsymbol{\lambda})$ is based on Bushnell–Kutzko's theory of types, as we explain below.

1.4. Start with a cuspidal irreducible representation π of G. By [**19, 55**], it is compactly induced from a Bushnell–Kutzko type: this is a pair $(\boldsymbol{J},\boldsymbol{\lambda})$ with the following properties:

– the group $\boldsymbol{J}$ is open and compact mod centre, it has a unique maximal compact subgroup $\boldsymbol{J}^0$ and a unique maximal normal pro-p-subgroup $\boldsymbol{J}^1$,

– the representation $\boldsymbol{\lambda}$ of $\boldsymbol{J}$ is irreducible and factors (non-canonically) as $\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$, where $\boldsymbol{\kappa}$ is a representation of $\boldsymbol{J}$ whose restriction to $\boldsymbol{J}^1$ is irreducible and $\boldsymbol{\rho}$ is an irreducible representation of $\boldsymbol{J}$ whose restriction to $\boldsymbol{J}^1$ is trivial,

– the quotient $\boldsymbol{J}^0/\boldsymbol{J}^1$ is isomorphic to $\mathrm{GL}_m(\boldsymbol{l})$ for some integer m dividing n and some finite extension $\boldsymbol{l}$ of the residue field of F, and the restriction of $\boldsymbol{\rho}$ to $\boldsymbol{J}^0$ is the inflation of a cuspidal representation ϱ of $\mathrm{GL}_m(\boldsymbol{l})$.

Our first task is to prove that, if the Jacquet–Langlands transfer of π to $\mathrm{GL}_{2n}(F)$ is non-cuspidal, then, among all possible Bushnell–Kutzko types $(\boldsymbol{J},\boldsymbol{\lambda})$ whose compact induction to G is isomorphic to π (they form a single G-conjugacy class), there is one such that $\boldsymbol{J}$ is stable by σ and $\boldsymbol{\kappa}$ can be chosen to be distinguished by $\boldsymbol{J}\cap H$.

1.5. Assuming this has been done, our argument is as follows:

(1) The fact that $\boldsymbol{\kappa}$ is distinguished by $\boldsymbol{J}\cap H$ together with the decomposition

$$\mathrm{Hom}_{\boldsymbol{J}\cap H}(\boldsymbol{\kappa}\otimes\boldsymbol{\rho},\mathbb{C})\simeq\mathrm{Hom}_{\boldsymbol{J}\cap H}(\boldsymbol{\kappa},\mathbb{C})\otimes\mathrm{Hom}_{\boldsymbol{J}\cap H}(\boldsymbol{\rho},\mathbb{C})$$

implies that $\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$ is distinguished by $\boldsymbol{J}\cap H$ if and only if $\boldsymbol{\rho}$ is distinguished by $\boldsymbol{J}\cap H$.

(2) The representation $\boldsymbol{\rho}$ is distinguished by $\boldsymbol{J}\cap H$ if and only if the cuspidal representation ϱ of $\mathrm{GL}_m(\boldsymbol{l})$ is distinguished by a unitary group, or equivalently, ϱ is invariant by the non-trivial automorphism of $\boldsymbol{l}/\boldsymbol{l}_0$, where $\boldsymbol{l}_0$ is a subfield of $\boldsymbol{l}$ over which $\boldsymbol{l}$ is quadratic.

(3) The fact that the Jacquet–Langlands transfer of π is non-cuspidal implies that ϱ is invariant by $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$.

Note that (2) is reminiscent of [**64**] Section 5. See Section 9 below for more details.

1.6. It remains to prove that $\boldsymbol{J}$ and $\boldsymbol{\kappa}$ can be chosen as in §1.4. The construction of $\boldsymbol{\kappa}$ relies on the notion of simple character, which is the core of Bushnell–Kutzko's type theory. The cuspidal representation π of §1.4 contains a simple character, and the set of simple characters contained in π form a single G-conjugacy class. We first prove that, if the Jacquet–Langlands transfer of π to $\mathrm{GL}_{2n}(F)$ is non-cuspidal, then, among all simple characters contained in π, there is a simple character θ such that $\theta\circ\sigma=\theta^{-1}$. Zou [**69**] proved a similar result for cuspidal representations of $\mathrm{GL}_n(F)$ with respect to an orthogonal involution, and we explain how to transfer it to G in an appropriate manner. (Note that, if the Jacquet–Langlands transfer of π is cuspidal, such a θ may not exist.)

Next, fix a simple character θ as above, and let $\boldsymbol{J}$ denote its normalizer in G, which is stable by σ. A standard construction (see for instance [**54, 33**]) provides us with:

– a representation $\boldsymbol{\kappa}$ of $\boldsymbol{J}$ such that the contragredient of $\boldsymbol{\kappa} \circ \sigma$ is isomorphic to $\boldsymbol{\kappa}$,
– a quadratic character χ of $\boldsymbol{J} \cap H$ such that the vector space $\mathrm{Hom}_{\boldsymbol{J}\cap H}(\boldsymbol{\kappa}, \chi)$ is non-zero.

To prove that the character χ is trivial, we show that, if χ were non-trivial, one could construct an H-distinguished cuspidal representation of G with cuspidal transfer to $\mathrm{GL}_{2n}(F)$, thus contradicting the assumption of Theorem 1.2.

Together with the argument of §1.5, this finishes the proof of Theorem 1.2.

1.7. Now let us go back to §1.2, assuming that n is odd. Associated with any cuspidal representation π of $\mathrm{GL}_n(D)$ whose Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ is non-cuspidal, there exists a unique cuspidal representation τ of $\mathrm{GL}_n(F)$ such that the transfer of π is equal to $\mathrm{St}_2(\tau)$. This defines a map

$$\pi \mapsto \tau$$

from cuspidal representations of $\mathrm{GL}_n(D)$ whose Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ is non-cuspidal to cuspidal representations of $\mathrm{GL}_n(F)$, and this map is a bijection (see Remark 5.1). As suggested by Prasad, the inverse of this map can be thought of as a 'non-abelian' base change, denoted $\mathbf{b}_{D/F}$, from cuspidal representations of $\mathrm{GL}_n(F)$ to those of $\mathrm{GL}_n(D)$. For instance, if $n = 1$, the map $\mathbf{b}_{D/F}$ is just $\chi \mapsto \chi \circ \mathrm{Nrd}$, where Nrd is the reduced norm from $D^\times$ to $F^\times$ and χ ranges over the set of all characters of $F^\times$. When F has characteristic 0 and odd residue characteristic, it follows from Corollary 1.4 that the image of the map $\mathbf{b}_{D/F}$ is made of those cuspidal representations of $\mathrm{GL}_n(D)$ which are distinguished by $\mathrm{Sp}_n(D)$.

A type theoretic, explicit description of $\mathbf{b}_{D/F}$ can be extracted from [**16, 56, 21**], at least up to inertia, that is: given a cuspidal representation τ of $\mathrm{GL}_n(F)$, described as the compact induction of a Bushnell–Kutzko type, one has an explicit description of the type of the inertial class of the cuspidal representation $\mathbf{b}_{D/F}(\tau)$ in terms of the type of τ.

The case of cuspidal representations of depth 0 has been considered in [**64**] Section 5. The explicit description of $\mathbf{b}_{D/F}$ provided by [**64**] Proposition 5.1, Remark 5.2 is somewhat incomplete (see Remarks 10.1 and 10.4 below). In Section 10, thanks to [**58, 59, 17**], we provide a full description of $\mathbf{b}_{D/F}$ for cuspidal representations of depth 0.

1.8. We now explain how we prove Theorem 1.3. The proof is based on an idea from the first author's previous work [**37**]: we study the functional equation of local intertwining periods in order to reduce the study of distinction of discrete series representations to the cuspidal case. We note that this idea has been successfully applied in [**61**] as well. However in the case at hand, the argument here is different from, and in fact more involved than, the one used in [**37**] and [**61**], since the result to prove is of different flavour, and we cannot avoid using intertwining periods attached to orbits which are neither closed, nor open, hence using the results of [**38**].

Let π be an irreducible discrete series representation of G. Associated with it (via the classification of the discrete series of G which we recall in §4.1), there are a divisor m of n, a cuspidal irreducible representation ρ of $\mathrm{GL}_{n/m}(D)$ and an integer $r \in \{1, 2\}$, such that, if we set

$$I(s, \rho) = \rho\nu^{sr(m-1)} \times \rho\nu^{sr(m-3)} \times \cdots \times \rho\nu^{sr(1-m)} \tag{1.1}$$

for any $s \in \mathbb{C}$, where ν denotes the character "normalized absolute value of the reduced norm" of $\mathrm{GL}_{n/m}(D)$ (see Section 3 for the notation), then π is the unique irreducible quotient of $I(-1, \rho)$. Note that π is cuspidal if and only if $m = 1$. We set $t = n/m$.

1.9. The proof of Theorem 1.3 is by contradiction, assuming that π is distinguished by H and $m \geqslant 2$. First, by using Offen's geometric lemma [**44**] and Verma [**64**] Theorem 1.2, we prove that the distinction of π implies that ρ is distinguished by $\mathrm{Sp}_t(D)$ and $r = 2$ (Proposition 13.1 and Corollary 13.2).

Let M be the standard Levi subgroup $\mathrm{GL}_t(D) \times \cdots \times \mathrm{GL}_t(D)$ and P be the standard parabolic subgroup generated by M and upper triangular matrices of G. Fix an $\eta \in G$ such that $P\eta H$ is open in G and $M \cap \eta H \eta^{-1}$ is equal to $\mathrm{Sp}_t(D) \times \cdots \times \mathrm{Sp}_t(D)$. Since ρ is distinguished, there is a non-zero $M \cap \eta H \eta^{-1}$-invariant linear form μ on the inducing representation of (1.1). For any flat section $\varphi_s \in I(s, \rho)$ (see §11.2 for a definition), the integral

$$J(s, \varphi_s, \mu) = \int_{(\eta^{-1}P\eta \cap H)\backslash H} \mu(\varphi_s(\eta h))\, \mathrm{d}h$$

converges for $\mathrm{Re}(s)$ in a certain right half plane, has meromorphic continuation to $\mathbb{C}$ and defines an H-invariant linear form $J(s, \cdot, \mu)$ on $I(s, \rho)$, called the *open intertwining period*.

Let $M(s, w)$ denote the standard intertwining operator from $I(s, \rho)$ to $I(-s, \rho)$ associated to the longest element w of the symmetric group $\mathfrak{S}_m$. As $I(s, \rho)$ is irreducible for generic s and the space $\mathrm{Hom}_H(I(s, \rho), \mathbb{C})$ has dimension at most 1 for such s by Verma [**64**] Theorem 1.1, there is a meromorphic function $\alpha(s, \rho)$ satisfying

$$J(-s, M(s, w)\varphi_s, \mu) = \alpha(s, \rho) J(s, \varphi_s, \mu)$$

for any flat section $\varphi_s \in I(s, \rho)$. The next assertion is the key to the proof of Theorem 1.3.

***Proposition 1.5* (Corollary 15.3).** — *The meromorphic function $\alpha(s, \rho)$ is holomorphic and non-zero at $s = 1$.*

Let $M^* \in \mathrm{Hom}_G(I(-1, \rho), I(1, \rho))$ be a non-zero intertwining operator. Since $m \geqslant 2$, its image is isomorphic to π and $M^*M(1, w)$ is zero. Since $\mathrm{Hom}_H(I(-1, \rho), \mathbb{C})$ has dimension 1 by Proposition 13.1, there is a $\Lambda \in \mathrm{Hom}_H(\pi, \mathbb{C})$ such that $J(-1, \varphi, \mu) = \Lambda M^* \varphi$ for all $\varphi \in I(-1, \rho)$. Then we see that

$$\alpha(1, \rho) J(1, \varphi, \mu) = J(-1, M(1, w)\varphi, \mu) = \Lambda M^* M(1, w)\varphi = 0$$

for $\varphi \in I(1, \rho)$. Combined with Proposition 1.5, this implies that $J(1, \varphi, \mu) = 0$ for all $\varphi \in I(1, \rho)$. This contradicts Proposition 13.4, which asserts that $J(1, \cdot, \mu)$ is a non-zero linear form.

1.10. For the proof of Proposition 1.5, we compute $\alpha(s, \rho)$ via a global method. First we globalise ρ. There are

– a totally imaginary number field k, with ring of adèles $\mathbb{A}$, such that there is a unique place u above p and k_u, the completion of k at u, is equal to F,

– a quaternion algebra B over k such that $B_u = B \otimes k_u$ is equal to D,

– a cuspidal automorphic representation Π of $\mathrm{GL}_t(B \otimes_k \mathbb{A})$ with a non-zero $\mathrm{Sp}_t(B \otimes_k \mathbb{A})$-period and whose local component at u is isomorphic to ρ.

Given $s \in \mathbb{C}$, let φ be a non-zero automorphic form on $\mathrm{GL}_n(B \otimes_k \mathbb{A})$ in the parabolically induced representation $\Pi \times \cdots \times \Pi$ which decomposes into a product of local factors φ_v. Similarly to the local setting, we can define the global intertwining period by the meromorphic continuation of

$$J(s,\varphi) = \int_{(\eta^{-1}\mathrm{P}\eta\cap\mathrm{H})(\mathbb{A})\backslash\mathrm{H}(\mathbb{A})} \left(\int_{(\mathrm{M}\cap\eta\mathrm{H}\eta^{-1})(k)\backslash(\mathrm{M}\cap\eta\mathrm{H}\eta^{-1})(\mathbb{A})} \varphi(m\eta h)\, \mathrm{d}m \right) e^{\langle s\rho_P, H_P(\eta h)\rangle}\, \mathrm{d}h.$$

(For unexplained notations, see the later sections.) It has the product decomposition

$$J(s,\varphi) = \prod_v J_v(s,\phi_v,\mu_v)$$

and, for each v, we have the functional equation

$$J_v(-s, M_v(s,w)\varphi_v, \mu_v) = \alpha_v(s) J_v(s,\varphi_v,\mu_v)$$

where $\alpha_v(s)$ is a meromorphic function such that $\alpha_u(s) = \alpha(s,\rho)$. In §14.5, we show that $\alpha_v(s)$ can be written in terms of Rankin–Selberg γ-factors for all v at which B_v is split. We also obtain a formula for $J_v(s,\phi_v,\mu_v)$ at almost all places. Using the functional equation of global intertwining periods of Section 14, we obtain an equality of the form

$$\prod_{v\in S} \alpha_v(s) = \prod_{v\in S} \prod_{1\leqslant i<j\leqslant m} \gamma(2(j-i)s+2, \Sigma_v, \Sigma_v^\vee, \psi_v)\gamma(2(j-i)s, \Sigma_v, \Sigma_v^\vee, \psi_v)^{-1} \tag{1.2}$$

where S is a finite set of finite places of k such that B_v splits for all $v \notin S$ and Σ is a certain cuspidal automorphic representation of $\mathrm{GL}_t(\mathbb{A})$ associated with Π via the global Jacquet–Langlands correspondence. We deduce from (1.2) and the fact that u is the only place of k above p that

$$\alpha(s,\rho) = c \cdot \prod_{1\leqslant i<j\leqslant m} \gamma(2(j-i)s+2, \Sigma_u, \Sigma_u^\vee, \psi_u)\gamma(2(j-i)s, \Sigma_u, \Sigma_u^\vee, \psi_u)^{-1}$$

for some constant $c \in \mathbb{C}^\times$. Hence we see that $\alpha(1,\rho) \neq 0$.

1.11. Let us comment on the assumption on the residue characteristic in Theorem 1.2. The only places where we use this assumption are: Proposition 7.1, Lemma 8.2 and Proposition 9.2. Proposition 7.1 is the main difficulty: this proposition might not hold when $p = 2$.

1.12. Let us comment on the assumption on the characteristic of F in Theorem 1.3. Our proof uses the theory of local and global intertwining periods, which are only available in characteristic 0 so far, although at least locally when $p \neq 2$ this restriction is probably removable just by checking the original sources. We also use this assumption when we apply Theorem 5.2 in the proof of Corollary 13.2 (see also Remark 5.3).

Acknowledgements

We thank D. Prasad for giving his talk at the conference *Representations of p-adic Groups and the Langlands Correspondence, in honour of Colin Bushnell*, September 2024, which brought our attention to his Conjecture 1.1, and in particular for suggesting that we restrict to $\mathrm{SL}_n(D)$ in the proof of Theorem 5.2.

Nadir Matringe thanks Kyoto University for its hospitality, as parts of this work were elaborated while he was invited there. Vincent Sécherre was supported by the Institut Universitaire de France. Shaun Stevens was supported by the Engineering and Physical Sciences Research Council (grant EP/V061739/1). Miyu Suzuki was supported by JSPS KAKENHI (JP22K13891 and JP23K20785).

2. Quaternion algebras and symplectic groups

2.1. Let F be a field. Given any finite-dimensional central division F-algebra $\mathit{\Delta}$ and any integer $n \geqslant 1$, we write $\mathbf{M}_n(\mathit{\Delta})$ for the central simple F-algebra made of all $n \times n$ matrices with entries in $\mathit{\Delta}$ and $\mathrm{GL}_n(\mathit{\Delta})$ for the group of its invertible elements. The latter is the group of F-rational points of a connected reductive algebraic group defined over F.

2.2. Fix an integer $n \geqslant 1$, and set $A = \mathbf{M}_n(\mathit{\Delta})$ and $G = \mathrm{GL}_n(\mathit{\Delta})$. We write $\mathrm{Nrd}_{A/F}$ and $\mathrm{trd}_{A/F}$ for the reduced norm and trace of A over F, respectively.

Let $(n_1, n_2, \dots, n_r)$ be a composition of n, that is, a family of positive integers whose sum is equal to n. Associated with it, there are the standard Levi subgroup

$$M = \mathrm{GL}_{n_1}(\mathit{\Delta}) \times \cdots \times \mathrm{GL}_{n_r}(\mathit{\Delta})$$

considered as a subgroup of block diagonal matrices of G, and the standard parabolic subgroup P of G generated by M and all upper triangular matrices.

Denoting by N the unipotent radical of P, we have the standard Levi decomposition $P = MN$.

2.3. Let D be a quaternion algebra over F, that is, a central simple F-algebra of dimension 4. The algebra D is either isomorphic to $\mathbf{M}_2(F)$ – in which case we say that it is split – or a division algebra. In both cases, it is equipped with the canonical anti-involution

$$x \mapsto \overline{x} = \mathrm{trd}_{D/F}(x) - x. \tag{2.1}$$

One has the identity $x\overline{x} = \overline{x}x = \mathrm{Nrd}_{D/F}(x)$ for any $x \in D$. Note that an element of D is invertible if and only if its reduced norm is non-zero.

Given an $a \in A = \mathbf{M}_n(D)$, for an $n \geqslant 1$, we write ${}^{\mathsf{t}}a$ for the transpose of a with respect to the antidiagonal and $\overline{a}$ for the matrix obtained by applying (2.1) to each entry of a. We define an anti-involution

$$a \mapsto a^* = {}^{\mathsf{t}}\overline{a} \tag{2.2}$$

on the F-algebra A. The group $G = \mathrm{GL}_n(D)$, made of invertible elements of A, is then equipped with the involution $\sigma : x \mapsto (x^*)^{-1}$. The subgroup G^σ made of all elements of G that are fixed by σ is denoted by $\mathrm{Sp}_n(D)$.

When D is split, any isomorphism from D to $\mathbf{M}_2(F)$ transports the canonical anti-involution of D to that of $\mathbf{M}_2(F)$, and induces an algebra isomorphism $A \simeq \mathbf{M}_{2n}(F)$ transporting (2.2) to the anti-involution $x \mapsto \Omega \cdot {}^{\mathsf{T}}x \cdot \Omega^{-1}$ where

$$\Omega = \Omega_{2n} = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} \in \mathrm{GL}_{2n}(F) \tag{2.3}$$

and ${}^{\mathsf{T}}x$ is the transpose of $x \in \mathbf{M}_{2n}(F)$ with respect to the antidiagonal. It thus induces a group isomorphism $G \simeq \mathrm{GL}_{2n}(F)$ which sends the subgroup $\mathrm{Sp}_n(D)$ to the symplectic group $\mathrm{Sp}_{2n}(F)$, the latter being defined as the subgroup of $\mathrm{GL}_{2n}(F)$ made of those matrices $x \in \mathrm{GL}_{2n}(F)$ such that ${}^{\mathsf{T}}x\Omega x = \Omega$.

When D is non-split, the group $\mathrm{GL}_n(D)$ is an inner form of $\mathrm{GL}_{2n}(F)$ and $\mathrm{Sp}_n(D)$ is an inner form of $\mathrm{Sp}_{2n}(F)$.

Remark 2.1. — The reader may be more familiar with the transpose with respect to the diagonal. Let $J = J_n$ denote the antidiagonal matrix of $\mathrm{GL}_n(F) \subseteq G$ with antidiagonal entries all equal to 1. Then the transpose of a matrix $a \in \mathbf{M}_n(D)$ with respect to the diagonal is the conjugate of ${}^{\mathsf{t}}a$ by J.

3. Preliminaries on groups and representations

3.1. Let G be a locally compact, totally disconnected topological group. By *representation* of a closed subgroup H of G, we mean a smooth, complex representation of H. By *character* of H, we mean a group homomorphism from H to $\mathbb{C}^\times$ with open kernel. If π is a representation of H, we denote by $\pi^\vee$ its contragredient. Given a character χ of H, we denote by $\pi\chi$ the representation $h \mapsto \chi(h)\pi(h)$ of H.

If σ is a continous involution of G, we denote by π^σ the representation $\pi \circ \sigma$ of $\sigma(H)$. Given a closed subgroup K of H, the representation π is said to be *distinguished by* K if its underlying vector space V carries a non-zero linear form Λ such that $\Lambda(\pi(x)v) = \Lambda(v)$ for all $x \in K$, $v \in V$.

We also denote by δ_H the modulus character of H.

3.2. Given a non-Archimedean locally compact field F, we will denote by $\mathcal{O}_F$ its ring of integers, by $\mathfrak{p}_F$ the maximal ideal of $\mathcal{O}_F$, by $\boldsymbol{k}_F$ its residue field and by $|\cdot|_F$ the absolute value on F sending any uniformizer to the inverse of the cardinality of $\boldsymbol{k}_F$.

Similarly, given a finite-dimensional central division F-algebra Δ, we denote by $\mathcal{O}_\Delta$ its ring of integers, by $\mathfrak{p}_\Delta$ the maximal ideal of $\mathcal{O}_\Delta$ and by $\boldsymbol{k}_\Delta$ its residue field.

3.3. Let F be a non-Archimedean locally compact field, Δ be a finite-dimensional central division F-algebra and n be a positive integer. The group $G = \mathrm{GL}_n(\Delta)$ is locally compact and totally disconnected.

Let $(n_1, \dots, n_r)$ be a composition of n and $P = MN$ be the standard parabolic subgroup of G associated with it (§2.2). Given a representation σ of M, we denote by $\mathrm{Ind}_P^G(\sigma)$ the representation of G obtained from σ by (normalized) parabolic induction along P.

For $i = 1, \dots, r$, let π_i be a representation of $\mathrm{GL}_{n_i}(\varDelta)$. We write

$$\pi_1 \times \cdots \times \pi_r$$

for the parabolically induced representation $\mathrm{Ind}_P^G(\pi_1 \otimes \cdots \otimes \pi_r)$ of G.

3.4. Suppose that $\varDelta = F$ and let U denote the subgroup of upper triangular unipotent matrices of $G = \mathrm{GL}_n(F)$. Fix a non-trivial additive character $\psi : F \to \mathbb{C}^\times$. It gives rise to a character

$$u \mapsto \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n})$$

of U, which we still denote by ψ.

Given any irreducible representation π of G, the dimension of the vector space $\mathrm{Hom}_U(\pi, \psi)$ is at most 1 (see [**23**]). We say that π is *generic* if this space is non-zero.

4. The Jacquet–Langlands correspondence

4.1. Let F be a non-Archimedean locally compact field, $\varDelta$ be a finite-dimensional central division F-algebra of reduced degree denoted d and n be a positive integer. Let us recall the classification of the discrete series of the groups $\mathrm{GL}_n(\varDelta)$, $n \geqslant 1$ ([**68, 63, 7**]).

Given any cuspidal representation ρ of $\mathrm{GL}_n(\varDelta)$ for some $n \geqslant 1$, there is a unique positive integer $r = r(\rho)$ such that, for any integer $m \geqslant 2$, the parabolically induced representation

$$\rho\nu^{r(1-m)/2} \times \rho\nu^{r(3-m)/2} \times \cdots \times \rho\nu^{r(m-1)/2} \tag{4.1}$$

is reducible, where ν denotes the character of $\mathrm{GL}_n(\varDelta)$ defined as the composition of the normalized absolute value of F with the reduced norm. For $m \geqslant 1$, the representation (4.1) has a unique irreducible quotient, which we denote by $\mathrm{St}_m(\rho)$. This quotient is a discrete series representation of $\mathrm{GL}_{nm}(\varDelta)$, which is unitary if and only if ρ is unitary.

The integer r associated with ρ (it is denoted $s(\rho)$ in [**54**] §3.5) has the following properties:

– it divides the reduced degree d of $\varDelta$ ([**54**] Remark 3.15(1)),

– it is prime to n ([**54**] Remark 3.15(2)).

In particular, when $\varDelta$ is isomorphic to F, one has $r = 1$ for all cuspidal representations ρ.

Conversely, if π is a discrete series representation of $\mathrm{GL}_n(\varDelta)$ for some $n \geqslant 1$, there are a unique integer m dividing n and a cuspidal representation ρ of $\mathrm{GL}_{n/m}(\varDelta)$, uniquely determined up to isomorphism, such that π is isomorphic to $\mathrm{St}_m(\rho)$.

Remark 4.1. — Note that (4.1) also has a unique irreducible subrepresentation, which we will denote by $\mathrm{Sp}_m(\rho)$.

4.2. The (local) Jacquet–Langlands correspondence ([**50, 20, 4**]) is a bijection between the discrete series of $\mathrm{GL}_n(\varDelta)$ and that of $\mathrm{GL}_{nd}(F)$ characterised by a character relation on elliptic regular conjugacy classes. If π is a discrete series representation of $\mathrm{GL}_n(\varDelta)$ for some $n \geqslant 1$, its Jacquet–Langlands transfer will be denoted by ${}^{\mathrm{JL}}\pi$.

Let ρ be a cuspidal representation of $\mathrm{GL}_n(\varDelta)$ for some $n \geqslant 1$, and set $r = r(\rho)$. Its Jacquet–Langlands transfer ${}^{\mathrm{JL}}\rho$ is isomorphic to $\mathrm{St}_r(\tau)$ for a cuspidal representation τ of $\mathrm{GL}_{nd/r}(F)$ and, for all $m \geqslant 1$, the Jacquet–Langlands transfer of $\mathrm{St}_m(\rho)$ is $\mathrm{St}_{mr}(\tau)$ (see for instance [**39**] Proposition 12.2).

Conversely, given a positive integer $n \geqslant 1$, a divisor k of nd and a cuspidal representation τ of $\mathrm{GL}_{nd/k}(F)$, the discrete series representation π of $\mathrm{GL}_n(\Delta)$ whose Jacquet–Langlands transfer is $\mathrm{St}_k(\tau)$ is of the form $\mathrm{St}_m(\rho)$ for some cuspidal representation ρ of $\mathrm{GL}_{nd/m}(\Delta)$, where m is the greatest common divisor of k and n ([**54**] Remark 3.15(3)).

4.3. Let k be a number field and B be a finite-dimensional central division k-algebra of reduced degree d. Let $\mathbb{A} = \mathbb{A}_k$ be the ring of adèles of k. For each place v of k, let k_v be the completion of k at v and set $B_v = B \otimes k_v$.

We recall the classification of the discrete series automorphic representations of $\mathrm{GL}_n(B \otimes_k \mathbb{A})$ for all $n \geqslant 1$ (see [**41**] and [**6**] Proposition 5.7, Remark 5.6 and [**8**] Proposition 18.2).

Given an integer $n \geqslant 1$, we denote by ν the automorphic character of $\mathrm{GL}_n(B \otimes_k \mathbb{A})$ obtained by composing the reduced norm $\mathrm{GL}_n(B \otimes_k \mathbb{A}) \to \mathbb{A}^\times$ with the idelic norm $\mathbb{A}^\times \to \mathbb{C}^\times$. Thus, for each place v of k, the local component of ν at v, denoted by ν_v, is the character "normalized absolute value of the reduced norm" of $\mathrm{GL}_n(B_v)$.

Given any cuspidal automorphic representation Σ of $\mathrm{GL}_n(B \otimes_k \mathbb{A})$ for some $n \geqslant 1$, there is a positive integer $r = r(\Sigma)$ such that, for any $m \geqslant 1$, the parabolically induced representation

$$\Sigma\nu^{r(1-m)/2} \times \Sigma\nu^{r(3-m)/2} \times \cdots \times \Sigma\nu^{r(m-1)/2} \tag{4.2}$$

has a unique constituent which is a discrete series automorphic representation of $\mathrm{GL}_{nm}(B \otimes_k \mathbb{A})$. This constituent is denoted by $\mathrm{MW}_m(\Sigma)$. Note that, if $B \simeq k$, one has $r = 1$ for all cuspidal automorphic representations Σ.

Conversely, if Π is a discrete series automorphic representation of $\mathrm{GL}_n(B \otimes_k \mathbb{A})$ for some integer $n \geqslant 1$, there are a unique integer m dividing n and a unique cuspidal automorphic representation Σ of $\mathrm{GL}_{n/m}(B \otimes \mathbb{A})$ such that Π is isomorphic to $\mathrm{MW}_m(\Sigma)$.

4.4. The (global) Jacquet–Langlands correspondence is an injection $\Pi \mapsto {}^{\mathrm{JL}}\Pi$ from the automorphic discrete series of $\mathrm{GL}_n(B \otimes_k \mathbb{A})$ to that of $\mathrm{GL}_{nd}(\mathbb{A})$ characterised by the fact that, for any discrete series automorphic representation Π of $\mathrm{GL}_n(B \otimes_k \mathbb{A})$ and any place v of k such that B_v is split, the local components of ${}^{\mathrm{JL}}\Pi$ and Π at v are isomorphic once $\mathrm{GL}_n(B_v)$ and $\mathrm{GL}_{nd}(k_v)$ are identified ([**6**] Theorem 5.1 and [**8**] Theorem 1.4).

At finite places of k where B does not split, we will only need the following result.

Lemma 4.2. — *Let Π be a discrete series automorphic representation of $\mathrm{GL}_n(B \otimes_k \mathbb{A})$. Let v be a finite place such that B_v is not split and let π denote the local component of Π at v. Suppose that the Jacquet–Langlands transfer of π to $\mathrm{GL}_{nd}(k_v)$ is cuspidal. Then ${}^{\mathrm{JL}}\Pi$ is cuspidal and its local component at v is ${}^{\mathrm{JL}}\pi$.*

Proof. — Let π' denote the local component of ${}^{\mathrm{JL}}\Pi$ at v. Let $\mathbf{LJ}$ denote the Langlands–Jacquet morphism (defined in [**6**] §2.7) from the Grothendieck group of the category of representations of $\mathrm{GL}_{nd}(k_v)$ of finite length to that of $\mathrm{GL}_n(B_v)$. By [**6**] Theorem 5.1(a), there is a sign $\epsilon \in \{-1, 1\}$ such that $\mathbf{LJ}(\pi') = \epsilon \cdot \pi$.

On the other hand, given a unitary representation κ' of $\mathrm{GL}_{nd}(k_v)$ such that $\mathbf{LJ}(\kappa')$ is non-zero, it follows from the classification of the unitary dual of $\mathrm{GL}_{nd}(k_v)$ in [**62**] and the description of the image of unitary representations by $\mathbf{LJ}$ in [**6**] Section 3 that, if κ is the unique unitary representation of $\mathrm{GL}_n(B_v)$ such that $\mathbf{LJ}(\kappa') \in \{-\kappa, \kappa\}$ and if we denote by $\rho_1 + \cdots + \rho_r$ the cuspidal support of κ (which means that κ is an irreducible component of $\rho_1 \times \cdots \times \rho_r$), then

$$\mathrm{cusp}(\kappa') = \mathrm{cusp}({}^{\mathrm{JL}}\rho_1) + \cdots + \mathrm{cusp}({}^{\mathrm{JL}}\rho_r).$$

Applying this to the unitary representation π', we obtain $\pi' = {}^{\mathrm{JL}}\pi$. □

5. A necessary condition of distinction for cuspidal representations

In this section, F is a non-Archimedean locally compact field and D is a non-split quaternion F-algebra. Fix a positive integer $n \geqslant 1$ and write $G = \mathrm{GL}_n(D)$. It is equipped with the involution σ defined in §2.3. In §5.3 only, the field F will be assumed to have characteristic 0.

5.1. Let π be a cuspidal representation of G. Associated with it in §4.1, there is a positive integer $r = r(\pi)$ which divides the reduced degree of D and is prime to n.

As the reduced degree of D is equal to 2, we immediately deduce that, if π has a non-cuspidal Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$, then $r = 2$ and n is odd. Its Jacquet–Langlands transfer ${}^{\mathrm{JL}}\pi$ thus has the form $\mathrm{St}_2(\tau)$ for some cuspidal representation of $\mathrm{GL}_n(F)$.

Conversely, if n is odd, and if τ is any cuspidal representation of $\mathrm{GL}_n(F)$, it follows from §4.2 that the unique discrete series representation of G whose Jacquet–Langlands transfer is $\mathrm{St}_2(\tau)$ is cuspidal.

5.2. The following lemma will be used in the proof of Theorem 5.2, and later in Section 15.

Lemma 5.1. — *Let π be a unitary cuspidal representation of G distinguished by G^σ. Let*

- *k be a global field together with a finite place u dividing p such that k_u is isomorphic to F,*
- *B be a (non-split) quaternion k-algebra such that $B_u = B \otimes_k k_u$ is non-split.*

Then there exists a cuspidal automorphic representation Π of $\mathrm{GL}_n(B \otimes_k \mathbb{A})$ such that

(1) *Π has a non-zero $\mathrm{Sp}_n(B \otimes_k \mathbb{A})$-period, that is, there is a $\varphi \in \Pi$ such that*

$$\int_{\mathrm{Sp}_n(B)\backslash\mathrm{Sp}_n(B\otimes\mathbb{A})} \varphi(h)\ \mathrm{d}h \neq 0,$$

(2) *the local component of Π at u is isomorphic to π.*

Proof. — Let π be a unitary cuspidal irreducible representation of G. Assume that π is distinguished by G^σ. Let Z denote the centre of G, which is isomorphic to $F^\times$, and let $G' = \mathrm{SL}_n(D)$ be the kernel of the reduced norm from $\mathrm{GL}_n(D)$ to $F^\times$. By [**27**] Theorem 4.2, the restriction of π to the normal, cocompact, closed subgroup $G_1 = ZG'$ is semisimple of finite length. Let π_1 be an irreducible summand of this restriction. The centre Z acts on it through ω, the central character of π. The restriction of π_1 to G', denoted π', is thus irreducible.

Let k be a global field together with a finite place u dividing p such that k_u is isomorphic to F. Thus k is a finite extension of $\mathbb{Q}$ when F has characteristic 0, and the field of rational functions over a smooth irreducible projective curve defined over a finite field of characteristic p if F has characteristic p.

Let B be a quaternion algebra over k such that $B \otimes_k k_u$ is non-split (it is thus isomorphic to D). Let G be the k-group $\mathrm{GL}_n(B)$ and G′ be the k-group $\mathrm{SL}_n(B)$. The latter is an inner form of SL_{2n} over k which contains $\mathrm{H} = \mathrm{Sp}_n(B)$. The connected component of the centre of G′ is trivial and H is a closed algebraic k-subgroup of G′ with no non-trivial character. Let V denote the k-vector space made of all matrices $a \in \mathbf{M}_n(B)$ such that $a^* = a$ and consider the algebraic representation of G′ on V defined by $(g, a) \mapsto \sigma(g)ag^{-1}$. This representation is semisimple (it is even irreducible) and the G′-stabilizer of the identity matrix (on V) is H.

We now apply either [**49**] Theorem 4.1 (if F has characteristic 0) or [**22**] Theorem 1.3 (if F has characteristic p): there exists a cuspidal automorphic representation Π' of $\mathrm{G}'(\mathbb{A})$ with a non-zero $\mathrm{H}(\mathbb{A})$-period and such that the local component of Π' at u is isomorphic to π'. (Here $\mathbb{A}$ denotes the ring of adèles of k.)

By [**34**] Theorem 5.2.2, the representation Π' occurs as a subrepresentation in the restriction to $\mathrm{G}'(\mathbb{A})$ of a cuspidal automorphic representation Π of $\mathrm{G}(\mathbb{A})$. Since G′ contains H, the representation Π has a non-zero $\mathrm{H}(\mathbb{A})$-period. It follows that:

(1) for any finite place v, the local component Π_v of Π at v is distinguished by $\mathrm{H}(k_v)$,
(2) the restriction to $\mathrm{G}'(k_u)$ of the local component Π_u of Π at u contains π'.

More precisely, let us prove that Π_u is isomorphic to $\pi \otimes (\chi \circ \mathrm{Nrd}_{A/F})$ for some character χ of the group $F^\times$. Arguing as at the beginning of the proof of the theorem, the restriction of Π_u to G_1 is semisimple of finite length and contains an irreducible summand π_2 whose restriction to G' is isomorphic to π'. If μ denotes the central character of Π_u, we thus have $\pi_2(zx) = \mu(z)\pi'(x)$ for all $z \in Z$ and $x \in G'$. The representation Π_u is unitary as a local component of the unitary representation Π. Its central character μ is thus unitary. Similarly, we have $\pi_1(zx) = \omega(z)\pi'(x)$ for all $z \in Z$ and $x \in G'$. By twisting π by a unitary character of G, we may assume that $\omega = \mu$. By [**34**] Proposition 2.2.2, we get the expected result.

Let us fix a unitary character Θ of $\mathbb{A}^\times/k^\times$ whose local component at u is χ. By twisting Π by Θ^{-1} composed with the reduced norm from $\mathrm{G}(\mathbb{A})$ to $\mathbb{A}^\times$, we obtain a cuspidal automorphic representation of $\mathrm{G}(\mathbb{A})$ having the required properties. □

5.3. In this paragraph, F is a non-Archimedean locally compact field of characteristic 0.

Theorem 5.2. — *Suppose that F has characteristic 0, and let π be a cuspidal representation of G. If π is distinguished by G^σ, then ${}^{\mathrm{JL}}\pi$ is non-cuspidal.*

Proof. — This follows from [**64**] Theorem 1.2 but we give a more detailed proof, based on Lemmas 4.2 and 5.1. Our argument is inspired by [**64**] Section 6, in particular the proof of Theorem 6.3.

Let π be a cuspidal irreducible representation of G distinguished by G^σ. Assume that ${}^{\mathrm{JL}}\pi$ is cuspidal.

Since any unramified character of G is trivial on G^σ, and since the Jacquet–Langlands correspondence is compatible with torsion by unramified characters, we may and will assume, by twisting by an appropriate unramified character, that π is unitary.

Let k be a number field together with a finite place u dividing p such that k_u is isomorphic to F. Let B be a quaternion division algebra over k such that B_u is non-split (it is thus isomorphic to D) and B_v is split for all Archimedean places v. By Lemma 5.1, there exists a cuspidal automorphic representation Π of $\mathrm{GL}_n(B \otimes_k \mathbb{A})$ such that

(1) Π has a non-zero $\mathrm{Sp}_n(B \otimes_k \mathbb{A})$-period,

(2) the local component of Π at u is isomorphic to π.

The Jacquet–Langlands transfer ${}^{\mathrm{JL}}\Pi$ of such a Π is a discrete series automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ with the following properties:

(3) by Lemma 4.2, its local component at u is cuspidal, isomorphic to ${}^{\mathrm{JL}}\pi$, thus ${}^{\mathrm{JL}}\Pi$ is cuspidal,

(4) for any finite place v such that B_v is split, the local components of ${}^{\mathrm{JL}}\Pi$ and Π at v are isomorphic.

Since ${}^{\mathrm{JL}}\Pi$ is cuspidal, it is generic. Therefore:

(5) for any finite place v, the local component of ${}^{\mathrm{JL}}\Pi$ at v is generic.

On the other hand, it follows from (1) that

(6) for any finite place v, the local component of Π at v is distinguished by $\mathrm{Sp}_n(B_v)$.

It follows from (4), (5) and (6) that, if v is a finite place of k such that B_v is split, Π_v is an irreducible representation of $\mathrm{GL}_{2n}(k_v)$ which is generic and distinguished by $\mathrm{Sp}_n(B_v) \simeq \mathrm{Sp}_{2n}(k_v)$. This contradicts [**46**] Theorem 1, which says that no generic irreducible representation of $\mathrm{GL}_{2n}(k_v)$ is distinguished by $\mathrm{Sp}_{2n}(k_v)$. □

Remark 5.3. — We assumed that F has characteristic 0 in Theorem 5.2 because a global Jacquet–Langlands correspondence for discrete series automorphic representations of $\mathrm{GL}_n(B \otimes_k \mathbb{A})$ is not known to exist in characteristic p, except when $n = 1$ (see [**9**]).

Sections 6 to 9 are devoted to the proof of Theorem 1.2: assuming that F is any non-Archimedean locally compact field with odd residue characteristic, and that the Jacquet–Langlands transfer of any cuspidal representation of G distinguished by G^σ is non-cuspidal (which holds for instance when F has characteristic 0 by Theorem 5.2), we prove that any cuspidal representation of G whose Jacquet–Langlands transfer is non-cuspidal is distinguished by G^σ.

6. Type theoretic material

In this section, we introduce the type theoretic material which we will need in Sections 7–10. Let Δ be *any finite-dimensional central division F-algebra* (this extra generality will be useful in §8.3.) Let A be the central simple F-algebra $\mathbf{M}_n(\Delta)$ of $n \times n$ matrices with coefficients in Δ for some integer $n \geqslant 1$, and $G = A^\times = \mathrm{GL}_n(\Delta)$. Let us fix a character

$$\psi : F \to \mathbb{C}^\times \tag{6.1}$$

trivial on $\mathfrak{p}_F$ but not on $\mathcal{O}_F$. For the definitions and main results stated in this section, we refer the reader to [**19, 18**] (see also [**54, 33**]).

6.1. A *simple stratum* in A is a pair $[\mathfrak{a},\beta]$ made of a hereditary $\mathcal{O}_F$-order $\mathfrak{a}$ of A and an element $\beta \in A$ such that the F-algebra $E = F[\beta]$ is a field, and the multiplicative group $E^\times$ normalizes $\mathfrak{a}$ (plus an extra technical condition on β which it is not necessary to recall here). The centralizer B of E in A is a central simple E-algebra, and $\mathfrak{b} = \mathfrak{a} \cap B$ is a hereditary $\mathcal{O}_E$-order in B.

Associated to a simple stratum $[\mathfrak{a},\beta]$, there are a pro-p-subgroup $H^1(\mathfrak{a},\beta)$ of G and a non-empty finite set $\mathcal{C}(\mathfrak{a},\beta)$ of characters of $H^1(\mathfrak{a},\beta)$ called *simple characters*, depending on ψ.

Remark 6.1. — This includes the case where $\beta = 0$. The simple stratum $[\mathfrak{a},0]$ is then said to be *null*. One has $H^1(\mathfrak{a},0) = 1 + \mathfrak{p}_\mathfrak{a}$ (where $\mathfrak{p}_\mathfrak{a}$ is the Jacobson radical of $\mathfrak{a}$) and the set $\mathcal{C}(\mathfrak{a},0)$ is reduced to the trivial character of $1 + \mathfrak{p}_\mathfrak{a}$.

When the order $\mathfrak{b}$ is maximal in B, the simple stratum $[\mathfrak{a},\beta]$ is said to be *maximal*, and the simple characters in $\mathcal{C}(\mathfrak{a},\beta)$ are said to be *maximal*. If this is the case, and if $[\mathfrak{a}',\beta']$ is another simple stratum in A such that $\mathcal{C}(\mathfrak{a},\beta) \cap \mathcal{C}(\mathfrak{a}',\beta')$ is non-empty, then

$$\mathcal{C}(\mathfrak{a}',\beta') = \mathcal{C}(\mathfrak{a},\beta), \quad \mathfrak{a}' = \mathfrak{a}, \quad [F[\beta'] : F] = [F[\beta] : F], \tag{6.2}$$

and the simple stratum $[\mathfrak{a}',\beta']$ is maximal ([**54**] Proposition 3.6).

6.2. Let Δ' be a finite dimensional central division F-algebra and $[\mathfrak{a}',\beta']$ be a simple stratum in $\mathbf{M}_{n'}(\Delta')$ for some $n' \geqslant 1$. Assume that there is a morphism of F-algebras $\varphi : F[\beta] \to F[\beta']$ such that $\varphi(\beta) = \beta'$. Then there is a natural bijection from $\mathcal{C}(\mathfrak{a},\beta)$ to $\mathcal{C}(\mathfrak{a}',\beta')$ called transfer.

6.3. Let $\mathcal{C}$ denote the union of the sets $\mathcal{C}(\mathfrak{a}',\beta')$ for all maximal simple strata $[\mathfrak{a}',\beta']$ of $\mathbf{M}_{n'}(\Delta')$, for all $n' \geqslant 1$ and all finite dimensional central division F-algebras Δ'. Any two maximal simple characters $\theta_1,\theta_2 \in \mathcal{C}$ are said to be *endo-equivalent* if they are transfers of each other, that is, if there exist

– maximal simple strata $[\mathfrak{a}_1,\beta_1]$ and $[\mathfrak{a}_2,\beta_2]$,
– a morphism of F-algebras $\varphi : F[\beta_1] \to F[\beta_2]$ such that $\varphi(\beta_1) = \beta_2$,

such that $\theta_i \in \mathcal{C}(\mathfrak{a}_i,\beta_i)$ for $i = 1, 2$, and θ_2 is the image of θ_1 by the transfer map from $\mathcal{C}(\mathfrak{a}_1,\beta_1)$ to $\mathcal{C}(\mathfrak{a}_2,\beta_2)$. This defines an equivalence relation on $\mathcal{C}$, called *endo-equivalence*. An equivalence class for this equivalence relation is called an *endoclass*.

The *degree* of an endoclass $\boldsymbol{\Theta}$ is the degree of $F[\beta']$ over F, for any choice of $[\mathfrak{a}',\beta']$ such that $\mathcal{C}(\mathfrak{a}',\beta') \cap \boldsymbol{\Theta}$ is non-empty. (It is well defined thanks to (6.2).)

6.4. Let $\mathcal{C}(G)$ be the union of the sets $\mathcal{C}(\mathfrak{a},\beta)$ for all maximal simple strata $[\mathfrak{a},\beta]$ of A. Any two maximal simple characters $\theta_1,\theta_2 \in \mathcal{C}(G)$ are endo-equivalent if and only if they are G-conjugate.

Given a cuspidal representation π of G, there exists a maximal simple character $\theta \in \mathcal{C}(G)$ contained in π, and any two maximal simple characters contained in π are G-conjugate. The maximal simple characters contained in π thus all belong to the same endoclass $\boldsymbol{\Theta}$, called the endoclass of π. Conversely, any maximal simple character $\theta \in \mathcal{C}(G)$ of endoclass $\boldsymbol{\Theta}$ is contained in π ([**54**] Corollaire 3.23).

6.5. Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a simple character with respect to a maximal simple stratum $[\mathfrak{a}, \beta]$ in A as in §6.1. There are a divisor m of n and a finite dimensional central division E-algebra $\mathit{\Gamma}$ such that B is isomorphic to $\mathbf{M}_m(\mathit{\Gamma})$. Let $\boldsymbol{J}_\theta$ be the normalizer of θ in G. Then

(1) the group $\boldsymbol{J}_\theta$ has a unique maximal compact subgroup $\boldsymbol{J}^0 = \boldsymbol{J}^0_\theta$ and a unique maximal normal pro-p-subgroup $\boldsymbol{J}^1 = \boldsymbol{J}^1_\theta$,

(2) the group $\boldsymbol{J}_\theta \cap B^\times$ is the normalizer of $\mathfrak{b}$ in $B^\times$ and $\boldsymbol{J}^0 \cap B^\times = \mathfrak{b}^\times$, $\boldsymbol{J}^1 \cap B^\times = 1 + \mathfrak{p}_\mathfrak{b}$,

(3) one has $\boldsymbol{J}_\theta = (\boldsymbol{J}_\theta \cap B^\times)\boldsymbol{J}^0$ and $\boldsymbol{J}^0 = (\boldsymbol{J}^0 \cap B^\times)\boldsymbol{J}^1$.

Since $\mathfrak{b}$ is a maximal order in B, it follows from (2) and (3) that there is a group isomorphism

$$\boldsymbol{J}^0/\boldsymbol{J}^1 \simeq \mathrm{GL}_m(\boldsymbol{l}) \tag{6.3}$$

where $\boldsymbol{l}$ is the residue field of $\mathit{\Gamma}$, and an element $\varpi \in B^\times$ normalizing $\mathfrak{b}$ such that $\boldsymbol{J}_\theta$ is generated by $\boldsymbol{J}^0$ and ϖ.

There is an irreducible representation $\eta = \eta_\theta$ of $\boldsymbol{J}^1$ whose restriction to $H^1(\mathfrak{a}, \beta)$ contains θ. It is unique up to isomorphism, and it is called the *Heisenberg representation* associated with θ. It extends to the group $\boldsymbol{J}_\theta$ (thus its normalizer in G is equal to $\boldsymbol{J}_\theta$).

If $\boldsymbol{\kappa}$ is a representation of $\boldsymbol{J}_\theta$ extending η, any other extension of η to $\boldsymbol{J}_\theta$ has the form $\boldsymbol{\kappa\xi}$ for a unique character $\boldsymbol{\xi}$ of $\boldsymbol{J}_\theta$ trivial on $\boldsymbol{J}^1$. More generally, the map

$$\boldsymbol{\tau} \mapsto \boldsymbol{\kappa} \otimes \boldsymbol{\tau} \tag{6.4}$$

is a bijection between isomorphism classes of irreducible representations of $\boldsymbol{J}_\theta$ trivial on $\boldsymbol{J}^1$ and isomorphism classes of irreducible representations of $\boldsymbol{J}_\theta$ whose restriction to $\boldsymbol{J}^1$ contains η.

7. σ-self-dual simple characters

From this section until Section 9, the residue characteristic p of F is assumed to be odd.

We go back to the group $G = \mathrm{GL}_n(D)$ of §3.2 and fix a cuspidal representation π of G with non-cuspidal transfer to $\mathrm{GL}_{2n}(F)$. In particular, as explained in §5.1, the integer n is odd. The main result of this section is the following proposition. Recall that $*$ has been defined in §2.3.

Proposition 7.1. — *There are a maximal simple stratum $[\mathfrak{a}, \beta]$ in A and a maximal simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ contained in π such that*

(1) *the group $H^1(\mathfrak{a}, \beta)$ is stable by σ and $\theta \circ \sigma = \theta^{-1}$,*

(2) *the order $\mathfrak{a}$ is stable by $*$ and β is invariant by $*$.*

7.1. Let $\boldsymbol{\Theta}$ denote the endoclass of π. Since π contains any maximal simple character of $\mathcal{C}(G)$ of endoclass $\boldsymbol{\Theta}$, it suffices to prove the existence of a maximal simple stratum $[\mathfrak{a}, \beta]$ in A and a character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ of endoclass $\boldsymbol{\Theta}$ satisfying Conditions (1) and (2) of Proposition 7.1.

Since the Jacquet–Langlands transfer of π is non-cuspidal, it follows from §4.2 that this transfer is of the form $\mathrm{St}_2(\tau)$ for a cuspidal irreducible representation τ of $\mathrm{GL}_n(F)$. By Dotto [**21**], the representations π and τ have the same endoclass. It follows that the degree of $\boldsymbol{\Theta}$ divides n.

7.2. Let d denote the degree of $\boldsymbol{\Theta}$. Thanks to §7.1, it is an odd integer dividing n.

Let σ_0 be the involution $x \mapsto {}^{\mathsf{T}}x^{-1}$ on $\mathrm{GL}_d(F)$ where T denotes transposition with respect to the antidiagonal. The fixed points of $\mathrm{GL}_d(F)$ by σ_0 is a split orthogonal group.

By [**69**] Theorem 4.1, there are a maximal simple stratum $[\mathfrak{a}_0, \beta]$ in $\mathbf{M}_d(F)$ and a maximal simple character $\theta_0 \in \mathcal{C}(\mathfrak{a}_0, \beta)$ of endoclass $\boldsymbol{\Theta}$ such that

- the group $H^1(\mathfrak{a}_0, \beta)$ is stable by σ_0 and $\theta_0 \circ \sigma_0 = \theta_0^{-1}$,
- the order $\mathfrak{a}_0$ is stable by T and β is invariant by T.

Write E for the sub-F-algebra $F[\beta] \subseteq \mathbf{M}_d(F)$. It is made of T-invariant matrices. Its centralizer in $\mathbf{M}_d(F)$ is equal to E itself. The intersection $\mathfrak{a}_0 \cap E$ is $\mathcal{O}_E$, the ring of integers of E.

7.3. Let us write $n = md$. We identify $\mathbf{M}_n(F)$ with $\mathbf{M}_m(\mathbf{M}_d(F))$ and E with its diagonal image in $\mathbf{M}_n(F)$. The centralizer of E in $\mathbf{M}_n(F)$ is thus $\mathbf{M}_m(E)$.

Now consider $\mathbf{M}_n(F)$ as a sub-F-algebra of A. The centralizer B of E in A is equal to $\mathbf{M}_m(C)$ where C is an E-algebra isomorphic to $E \otimes_F D$. Since the degree d of E over F is odd, C is a non-split quaternion E-algebra. Denote by $*_B$ the anti-involution on B analogous to (2.2).

Proposition 7.2. — *The restriction of $*$ to B is equal to $*_B$.*

Proof. — It suffices to treat the case where $m = 1$. We will thus assume that $m = 1$, in which case we have $B = C$. We thus have to prove that

$$c^* = \mathrm{trd}_{C/E}(c) - c \tag{7.1}$$

for all $c \in C \subseteq \mathbf{M}_d(D)$. Let us identify $\mathbf{M}_d(D)$ with $\mathbf{M}_d(F) \otimes_F D$. Then $(a \otimes x)^* = {}^{\mathsf{T}}a \otimes \overline{x}$ for all $a \in \mathbf{M}_n(F)$ and $x \in D$, and C identifies with $E \otimes_F D$. Thus (7.1) is equivalent to

$$e \otimes \overline{x} = \mathrm{trd}_{C/E}(e \otimes x) - e \otimes x \tag{7.2}$$

for all $e \in E$ and $x \in D$. Thanks to (2.1), we are thus reduced to proving that

$$\mathrm{trd}_{C/E}(x) = \mathrm{trd}_{D/F}(x) \tag{7.3}$$

for all $x \in D$, where the F-algebra D is embedded in C via $x \mapsto 1 \otimes_F x$.

Let L be a quadratic unramified extension of F. Since the degree of E over F is odd, $E \otimes_F L$ is a field, denoted EL. The reduced trace is invariant by extension of scalars (see [**13**] §17.3, Proposition 4). Thus $\mathrm{trd}_{D/F}(x)$ is the trace of x in $D \otimes_F EL \simeq \mathbf{M}_2(EL)$. (By the Skolem-Noether theorem, the computation of this trace does not depend on the choice of the isomorphism.) Similarly, $\mathrm{trd}_{C/E}(x)$ is the trace of x in $C \otimes_E EL \simeq \mathbf{M}_2(EL)$. The proposition is proven. □

Let $\mathfrak{b}$ denote the standard maximal order $\mathbf{M}_m(\mathcal{O}_C)$ in B. Then $\mathfrak{b}^\times$ is a maximal open compact subgroup of $B^\times$ which is stable by σ. Let $\mathfrak{a}$ denote the unique $\mathcal{O}_F$-order in A normalized by $E^\times$ such that $\mathfrak{a} \cap B = \mathfrak{b}$ (see [**52**] Lemme 1.6). We thus obtain a maximal simple stratum $[\mathfrak{a}, \beta]$ in A where $\mathfrak{a}$ is stable by $*$ and $\beta^* = \beta$, and E is made of $*$-invariant matrices.

Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be the transfer of θ_0. We are going to prove that the group $H^1(\mathfrak{a}, \beta)$ is stable by σ and $\theta \circ \sigma = \theta^{-1}$, which will finish the proof of Proposition 7.1. For this, set $\theta^* = \theta^{-1} \circ \sigma$. This is a character of $\sigma(H^1(\mathfrak{a}, \beta))$. We thus have to prove that $\theta^* = \theta$.

Let ϑ_0 be any character of $\mathcal{C}(\mathfrak{a}_0,\beta)$ and ϑ be its transfer to $\mathcal{C}(\mathfrak{a},\beta)$. Let us define the characters ${}^{\mathsf{T}}\vartheta_0=\vartheta_0^{-1}\circ\sigma_0$ and $\vartheta^*=\vartheta^{-1}\circ\sigma$. By Lemma 16.1 (which we will prove in a separate section since its proof requires techniques which are not used anywhere else in the paper), we have

$$\vartheta^*\in\mathcal{C}(\mathfrak{a}^*,\beta^*),\quad {}^{\mathsf{T}}\vartheta_0\in\mathcal{C}({}^{\mathsf{T}}\mathfrak{a}_0,{}^{\mathsf{T}}\beta).$$

On the one hand, by [**60**] Proposition 6.3, the transfer of ${}^{\mathsf{T}}\vartheta_0\in\mathcal{C}({}^{\mathsf{T}}\mathfrak{a}_0,{}^{\mathsf{T}}\beta)$ to $\mathcal{C}(\mathfrak{a}^*,\beta^*)$ is equal to ϑ^*. On the other hand, we have $\mathcal{C}(\mathfrak{a}^*,\beta^*)=\mathcal{C}(\mathfrak{a},\beta)$ since $\mathfrak{a}$ is stable by $*$ and $\beta^*=\beta$, and likewise $\mathcal{C}({}^{\mathsf{T}}\mathfrak{a}_0,{}^{\mathsf{T}}\beta)=\mathcal{C}(\mathfrak{a}_0,\beta)$. Now choose $\vartheta_0=\theta_0$. Since ${}^{\mathsf{T}}\theta_0=\theta_0$, we deduce that $\theta^*=\theta$.

8. σ-self-dual extensions of Heisenberg representations

In this section, the residue characteristic p of F is odd. We focus on the maximal simple stratum $[\mathfrak{a},\beta]$ and the maximal simple character $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ constructed in Section 7, forgetting temporarily about the cuspidal representation π. We thus have $\mathfrak{a}^*=\mathfrak{a}$, $\beta^*=\beta$ and $\theta^{-1}\circ\sigma=\theta$. Recall that the centralizer B of E in A is equal to $\mathbf{M}_m(C)$ where $m[E:F]=n$ and C is a quaternion E-algebra isomorphic to $E\otimes_F D$, and $\mathfrak{b}$ is the standard maximal order $\mathbf{M}_m(\mathcal{O}_C)$ in B. Let $\boldsymbol{\Theta}$ denote the endoclass of θ.

8.1. Let $\boldsymbol{J}_\theta$ be the normalizer of θ in G. According to §6.5, it has a unique maximal compact subgroup $\boldsymbol{J}^0=\boldsymbol{J}^0_\theta$ and a unique maximal normal pro-p-subgroup $\boldsymbol{J}^1=\boldsymbol{J}^1_\theta$. One has the identity $\boldsymbol{J}_\theta=C^\times\boldsymbol{J}^0$, where $C^\times$ is diagonally embedded in $\mathrm{GL}_m(C)=B^\times\subseteq G$, and a group isomorphism

$$\boldsymbol{J}^0/\boldsymbol{J}^1\simeq\mathrm{GL}_m(\boldsymbol{l}) \tag{8.1}$$

where $\boldsymbol{l}$ is the residue field of C, coming from the identities $\boldsymbol{J}^0=(\boldsymbol{J}^0\cap B^\times)\boldsymbol{J}^1$, $\boldsymbol{J}^0\cap B^\times=\mathfrak{b}^\times$ and $\boldsymbol{J}^1\cap B^\times=1+\mathfrak{p}_\mathfrak{b}$.

8.2. Let η denote the Heisenberg representation associated with θ and $\boldsymbol{\kappa}$ be a representation of $\boldsymbol{J}_\theta$ extending η. Let ϱ be a cuspidal irreducible representation of $\boldsymbol{J}^0/\boldsymbol{J}^1$. Its inflation to $\boldsymbol{J}^0$ will still be denoted by ϱ. The normalizer $\boldsymbol{J}$ of ϱ in $\boldsymbol{J}_\theta$ satisfies

$$E^\times\boldsymbol{J}^0\subseteq\boldsymbol{J}\subseteq\boldsymbol{J}_\theta.$$

Since $E^\times\boldsymbol{J}^0$ has index 2 in $\boldsymbol{J}_\theta=C^\times\boldsymbol{J}^0$, there are only two possible values for $\boldsymbol{J}$, namely $E^\times\boldsymbol{J}^0$ and $\boldsymbol{J}_\theta$. More precisely, $\boldsymbol{J}_\theta$ is generated by $\boldsymbol{J}^0$ and a uniformizer ϖ of C, and the action of ϖ on $\boldsymbol{J}^0$ by conjugacy identifies through (8.1) with the action on $\mathrm{GL}_m(\boldsymbol{l})$ of the generator of $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$, where $\boldsymbol{l}_0$ is the residue field of E. It follows that $\boldsymbol{J}=\boldsymbol{J}_\theta$ if and only if ϱ is $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$-stable. For the following three assertions, see for instance [**54**] 3.5.

Let $\boldsymbol{\rho}$ be a representation of $\boldsymbol{J}$ extending ϱ. Then the representation of G compactly induced from $\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$ is irreducible and cuspidal, of endoclass $\boldsymbol{\Theta}$.

Conversely, any cuspidal representation of G of endoclass $\boldsymbol{\Theta}$ is obtained this way, for a suitable choice of ϱ and of an extension $\boldsymbol{\rho}$ to $\boldsymbol{J}$.

Two pairs $(\boldsymbol{J},\boldsymbol{\kappa}\otimes\boldsymbol{\rho})$ and $(\boldsymbol{J}',\boldsymbol{\kappa}\otimes\boldsymbol{\rho}')$ constructed as above give rise to the same cuspidal representation of G if and only if they are $\boldsymbol{J}_\theta$-conjugate, that is, if and only if $\boldsymbol{J}'=\boldsymbol{J}$ and $\boldsymbol{\rho}'$ is $\boldsymbol{J}_\theta$-conjugate to $\boldsymbol{\rho}$.

The following theorem will be crucial in our proof of Theorem 1.2.

Theorem 8.1. — *The Jacquet–Langlands transfer to* $\mathrm{GL}_{2n}(F)$ *of the cuspidal representation of* G *compactly induced from* $(\boldsymbol{J}, \boldsymbol{\kappa} \otimes \boldsymbol{\rho})$ *is cuspidal if and only if* $\boldsymbol{J} = E^\times \boldsymbol{J}^0$.

Proof. — The Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ of the cuspidal representation compactly induced from $(\boldsymbol{J}, \boldsymbol{\kappa} \otimes \boldsymbol{\rho})$ is cuspidal if and only if the integer r associated to it (in §4.2) is 1. By [**54**] Remarque 3.15(1) (which is based on [**16**]), this integer is equal to the order of the stabilizer of ϱ in $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$, that is, to the index of $E^\times \boldsymbol{J}^0$ in $\boldsymbol{J}$. □

8.3. Let us prove that there exists a representation $\boldsymbol{\kappa}$ of $\boldsymbol{J}_\theta$ extending η such that $\boldsymbol{\kappa}^{\sigma\vee}$ is isomorphic to $\boldsymbol{\kappa}$. As in [**54**] Lemme 3.28, we prove it in a more general context (see Section 6).

Lemma 8.2. — *Let* Δ *be a finite dimensional central division* F*-algebra, let* τ *be a continuous automorphism of* $\mathrm{GL}_r(\Delta)$ *for some integer* $r \geqslant 1$*, let* ϑ *be a maximal simple character of* $\mathrm{GL}_r(\Delta)$ *such that* $\vartheta \circ \tau = \vartheta^{-1}$*, let* $\boldsymbol{J}_\vartheta$ *be its* $\mathrm{GL}_r(\Delta)$*-normalizer and* η *be its Heisenberg representation.*

(1) *The representation* $\eta^{\tau\vee}$ *is isomorphic to* η.

(2) *For any representation* $\boldsymbol{\kappa}$ *of* $\boldsymbol{J}_\vartheta$ *extending* η*, there exists a unique character* $\boldsymbol{\xi}$ *of* $\boldsymbol{J}_\vartheta$ *trivial on* $\boldsymbol{J}^1$ *such that* $\boldsymbol{\kappa}^{\tau\vee}$ *is isomorphic to* $\boldsymbol{\kappa\xi}$.

(3) *Assume that the order of* τ *is finite and prime to* p*. There exists a representation* $\boldsymbol{\kappa}$ *of* $\boldsymbol{J}_\vartheta$ *extending* η *such that* $\boldsymbol{\kappa}^{\tau\vee}$ *is isomorphic to* $\boldsymbol{\kappa}$.

Proof. — The first two assertions are given by [**54**] Lemme 3.28. For the third one, note that

$$\mathrm{val}_F \circ \mathrm{Nrd} \circ \tau = \epsilon(\tau) \cdot \mathrm{val}_F \circ \mathrm{Nrd}$$

where val_F is any valuation on F, Nrd is the reduced norm on $\mathbf{M}_r(\Delta)$ and $\epsilon(\tau)$ is a sign uniquely determined by τ. Indeed, the left hand side is a morphism from $\mathrm{GL}_r(\Delta)$ to $\mathbb{Z}$. As τ is continuous, it stabilizes the kernel of $\mathrm{val}_F \circ \mathrm{Nrd}$, which is generated by compact subgroups. The left hand side thus factors through $\mathrm{val}_F \circ \mathrm{Nrd}$, and the surjective morphisms from $\mathbb{Z}$ to $\mathbb{Z}$ are the identity and $x \mapsto -x$.

If $\epsilon(\tau) = 1$, the result is given by [**54**] Lemme 3.28. (Note that, in this case, the assumption on the order of τ is unnecessary.) We thus assume that $\epsilon(\tau) = -1$. Let $\boldsymbol{\kappa}$ be such that $\det(\boldsymbol{\kappa})$ has p-power order on $\boldsymbol{J}_\vartheta$ (whose existence is granted by [**54**] Lemme 3.12). The representation $\boldsymbol{\kappa}^{\tau\vee}$ is then isomorphic to $\boldsymbol{\kappa\xi}$ for some character $\boldsymbol{\xi}$ of $\boldsymbol{J}_\vartheta$ trivial on $\boldsymbol{J}^1$. As in the proof of [**54**] Lemma 3.28, since p is odd, this $\boldsymbol{\xi}$ is trivial on $\boldsymbol{J}^0$ and it has p-power order.

The group $\boldsymbol{J}_\vartheta$ is generated by $\boldsymbol{J}^0$ and an element ϖ whose reduced norm has non-zero valuation (see §6.5). Since $\epsilon(\tau) = -1$ and $\boldsymbol{J}_\vartheta$ is stable by τ, we have $\tau(\varpi) \in \varpi^{-1}\boldsymbol{J}^0$. And since $\boldsymbol{\xi}$ is trivial on $\boldsymbol{J}^0$, we deduce that $\boldsymbol{\xi} \circ \tau = \boldsymbol{\xi}^{-1}$.

Now write a for the order of τ, which we assume to be prime to p. Then the identity $\boldsymbol{\kappa}^{\tau\vee} \simeq \boldsymbol{\kappa\xi}$ applied $2a$ times shows that $\boldsymbol{\kappa\xi}^{2a} \simeq \boldsymbol{\kappa}$ so that $\boldsymbol{\xi}^{2a} = 1$. But since $\boldsymbol{\xi}$ has p-power order, and $2a$ is prime to p, we deduce that $\boldsymbol{\xi}$ is trivial. □

Remark 8.3. — In the case when $\epsilon(\tau) = -1$ and the order of τ is finite and prime to p, we even proved that any $\boldsymbol{\kappa}$ such that $\det(\boldsymbol{\kappa})$ has p-power order satisfies $\boldsymbol{\kappa}^{\tau\vee} \simeq \boldsymbol{\kappa}$. We also have

$$\boldsymbol{J}_\vartheta \cap \mathrm{GL}_r(\varDelta)^\tau = \boldsymbol{J}^0 \cap \mathrm{GL}_r(\varDelta)^\tau. \tag{8.2}$$

Indeed, if $x \in \boldsymbol{J}_\vartheta$ is τ-invariant, its valuation has to be equal to its opposite: it is thus 0.

8.4. Now let us go back to the situation of §8.2 with the group $G = \mathrm{GL}_n(D)$ equipped with the involution σ. Note that $\epsilon(\sigma) = -1$ (in the notation of the proof of Lemma 8.2) and the order of σ is prime to p, so Lemma 8.2 and Remark 8.3 apply. We will need the following lemma, which is [**54**] Lemme 3.30.

Lemma 8.4. — *Let $\boldsymbol{\kappa}$ be a representation of $\boldsymbol{J}_\theta$ extending η such that $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}$.*

(1) *There is a unique character χ of $\boldsymbol{J}_\theta \cap G^\sigma = \boldsymbol{J}^0 \cap G^\sigma$ trivial on $\boldsymbol{J}^1 \cap G^\sigma$ such that*

$$\mathrm{Hom}_{\boldsymbol{J}_\theta \cap G^\sigma}(\boldsymbol{\kappa}, \chi) \neq \{0\}$$

and this χ is quadratic (that is, $\chi^2 = 1$).

(2) *Let $\boldsymbol{\rho}$ be an irreducible representation of $\boldsymbol{J}_\theta$ trivial on $\boldsymbol{J}^1$. The canonical linear map:*

$$\mathrm{Hom}_{\boldsymbol{J}^1 \cap G^\sigma}(\eta, \mathbb{C}) \otimes \mathrm{Hom}_{\boldsymbol{J}_\theta \cap G^\sigma}(\boldsymbol{\rho}, \chi) \to \mathrm{Hom}_{\boldsymbol{J}_\theta \cap G^\sigma}(\boldsymbol{\kappa} \otimes \boldsymbol{\rho}, \mathbb{C})$$

is an isomorphism.

9. Proof of Theorem 1.2

Recall that F has odd residue characteristic, and assume that the Jacquet–Langlands transfer of any cuspidal representation of G distinguished by G^σ is non-cuspidal (which is known to be true when F has characteristic 0, thanks to Verma [**64**] Theorem 1.2, and also Theorem 5.2.)

Let π be a cuspidal irreducible representation of G with non-cuspidal transfer to $\mathrm{GL}_{2n}(F)$, as in Section 7. By Proposition 7.1, there are a maximal simple stratum $[\mathfrak{a}, \beta]$ in A and a maximal simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ such that $\mathfrak{a}^* = \mathfrak{a}$, $\beta^* = \beta$ and $\theta^{-1} \circ \sigma = \theta$. We use the notation of Section 8. In particular, we have groups $\boldsymbol{J}_\theta$, $\boldsymbol{J}^0$ and $\boldsymbol{J}^1$.

9.1. Identify $\boldsymbol{J}^0/\boldsymbol{J}^1$ with $\mathrm{GL}_m(\boldsymbol{l})$ thanks to the group isomorphism (8.1). Through this identification, and thanks to Proposition 7.2, the involution σ on $\boldsymbol{J}^0/\boldsymbol{J}^1$ identifies with the unitary involution

$$x \mapsto {}^{\mathsf{T}}\overline{x}^{-1} \tag{9.1}$$

on $\mathrm{GL}_m(\boldsymbol{l})$, where T denotes transposition with respect to the antidiagonal and $x \mapsto \overline{x}$ is the action of the non-trivial element of $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$ componentwise. It follows that $(\boldsymbol{J}^0 \cap G^\sigma)/(\boldsymbol{J}^1 \cap G^\sigma)$ identifies with the unitary group $\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)$. The following lemma will be useful. Note that m is odd since it divides n, which is odd by §5.1.

Lemma 9.1. — *Let ϱ be a cuspidal irreducible representation of $\mathrm{GL}_m(\boldsymbol{l})$. The following assertions are equivalent.*

(1) *The representation ϱ is $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$-invariant.*

(2) *The representation* ϱ *is distinguished by* $\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)$.

Moreover, there exist $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$*-invariant cuspidal irreducible representations of* $\mathrm{GL}_m(\boldsymbol{l})$.

Proof. — For the equivalence between (1) and (2), see for instance [**48**] Theorem 2 or [**25**] Theorem 2.4. For the last assertion, see for instance [**53**] Lemma 2.3. □

Let $\boldsymbol{\kappa}$ be a representation of $\boldsymbol{J}_\theta$ extending η such that $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}$ (whose existence is given by Lemma 8.2) and χ be the quadratic character of the group $\boldsymbol{J}_\theta \cap G^\sigma$ (which is equal to $\boldsymbol{J}^0 \cap G^\sigma$ by (8.2)) given by Lemma 8.4.

Proposition 9.2. — *The character* χ *is trivial.*

Proof. — Assume this is not the case. Then χ, considered as a character of $\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)$, is trivial on unipotent elements because these elements have p-power order and $p \neq 2$. Thus χ is trivial on the subgroup generated by all transvections. By [**26**] Theorem 11.15, this subgroup is $\mathrm{SU}_m(\boldsymbol{l}/\boldsymbol{l}_0)$. Thus $\chi = \alpha \circ \det$ for some quadratic character α of $\boldsymbol{l}^1$, where det is the determinant on $\mathrm{GL}_m(\boldsymbol{l})$ and $\boldsymbol{l}^1$ is the subgroup of $\boldsymbol{l}^\times$ made of elements of $\boldsymbol{l}/\boldsymbol{l}_0$-norm 1. Let β extend α to $\boldsymbol{l}^\times$, and let $\varkappa$ be the character of $\boldsymbol{J}^0$ inflated from $\beta \circ \det$. It extends χ.

Since m is odd, there is a cuspidal irreducible representation ϱ of $\mathrm{GL}_m(\boldsymbol{l})$ which is invariant by $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$ (equivalently, which is distinguished by $\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)$), thanks to Lemma 9.1.

Let ϱ' be the cuspidal representation $\varrho\varkappa$. Let us prove that it is not $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$-invariant. Let γ denote the generator of $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$. If ϱ' were $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$-invariant, $\varrho\varkappa^\gamma$ would be isomorphic to $\varrho\varkappa$. Comparing the central characters, one would get $(\beta^\gamma\beta^{-1})^m = 1$, that is, $\alpha(x^\gamma x^{-1})^m = 1$ for all $x \in \boldsymbol{l}^\times$, or equivalently $\alpha^m = 1$. But α is quadratic and m is odd: contradiction.

Let $\boldsymbol{\rho}'$ be a representation of $\boldsymbol{J}' = E^\times \boldsymbol{J}^0$ whose restriction to $\boldsymbol{J}^0$ is the inflation of ϱ'. Since ϱ' is not $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$-invariant, the normalizer of $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}'$ in $\boldsymbol{J}_\theta$ is $\boldsymbol{J}'$ (which has index 2 in $\boldsymbol{J}_\theta$).

On the one hand, the representation π' compactly induced by $(\boldsymbol{J}', \boldsymbol{\kappa} \otimes \boldsymbol{\rho}')$ is irreducible and cuspidal, and its Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ is cuspidal by Theorem 8.1.

On the other hand, the map

$$\mathrm{Hom}_{\boldsymbol{J}^1 \cap G^\sigma}(\eta, \mathbb{C}) \otimes \mathrm{Hom}_{\boldsymbol{J}' \cap G^\sigma}(\boldsymbol{\rho}', \chi) \to \mathrm{Hom}_{\boldsymbol{J}' \cap G^\sigma}(\boldsymbol{\kappa} \otimes \boldsymbol{\rho}', \mathbb{C})$$

is an isomorphism (by Lemma 8.4) and the space $\mathrm{Hom}_{\boldsymbol{J}' \cap G^\sigma}(\boldsymbol{\rho}', \chi)$ is non-zero by construction. This implies that $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}'$ is $\boldsymbol{J}' \cap G^\sigma$-distinguished. Thus π' is distinguished, which contradicts the assumption of Theorem 1.2. Thus χ is trivial. □

9.2. According to §8.2, our cuspidal representation π of G contains a representation of the form $(\boldsymbol{J}, \boldsymbol{\kappa} \otimes \boldsymbol{\rho})$, where

– the group $\boldsymbol{J}$ satisfies $E^\times \boldsymbol{J}^0 \subseteq \boldsymbol{J} \subseteq \boldsymbol{J}_\theta$,

– the representation $\boldsymbol{\kappa}$ is the restriction to $\boldsymbol{J}$ of a representation of $\boldsymbol{J}_\theta$ extending η,

– the representation $\boldsymbol{\rho}$ of $\boldsymbol{J}$ is trivial on $\boldsymbol{J}^1$ and its restriction to $\boldsymbol{J}^0$ is the inflation of a cuspidal representation ϱ of $\boldsymbol{J}^0/\boldsymbol{J}^1 \simeq \mathrm{GL}_m(\boldsymbol{l})$ whose normalizer in $\boldsymbol{J}_\theta$ is equal to $\boldsymbol{J}$.

Thanks to Lemma 8.2 and Proposition 9.2, we may and will assume that $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}$ and $\boldsymbol{\kappa}$ is distinguished by $\boldsymbol{J}_\theta \cap G^\sigma$.

Thanks to Theorem 8.1, the fact that the Jacquet–Langlands transfer of π is non-cuspidal implies that $\boldsymbol{J} = \boldsymbol{J}_\theta$. By §8.2, this implies that ϱ is $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$-invariant. It follows from Lemma 9.1 that ϱ is distinguished by $\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)$, thus $\boldsymbol{\rho}$ is distinguished by $\boldsymbol{J}^0 \cap G^\sigma = \boldsymbol{J}_\theta \cap G^\sigma$. By Lemma 8.4, the representation $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$ is distinguished by $\boldsymbol{J}_\theta \cap G^\sigma$. It follows from Mackey's formula

$$\mathrm{Hom}_{G^\sigma}(\pi, \mathbb{C}) \simeq \prod_g \mathrm{Hom}_{\boldsymbol{J}_\theta \cap g G^\sigma g^{-1}}(\boldsymbol{\kappa} \otimes \boldsymbol{\rho}, \mathbb{C})$$

(where g ranges over a set of representatives of $(\boldsymbol{J}_\theta, G^\sigma)$-double cosets of G) that π is distinguished by G^σ. This finishes the proof of Theorem 1.2.

10. The depth 0 case

In this section, we discuss in more detail the case of representations of depth 0. We assume throughout the section that n is odd. We no longer assume that F has odd residue characteristic.

10.1. Let π be a cuspidal irreducible representation of G with non-cuspidal transfer to $\mathrm{GL}_{2n}(F)$ as in Section 9. Assume moreover that π has depth 0. In that case, we are in the situation described by Remark 6.1. In this situation, we have $m = n$ and $\boldsymbol{l}$ is the residue field of D (thus $\boldsymbol{l}_0$ is that of F). We have $\boldsymbol{J} = D^\times \mathrm{GL}_n(\mathcal{O}_D)$ and $\boldsymbol{J}^0 = \mathrm{GL}_n(\mathcal{O}_D)$, and one can choose for $\boldsymbol{\kappa}$ the trivial character of $\boldsymbol{J}$. The representation π is compactly induced from an irreducible representation $\boldsymbol{\rho}$ of $\boldsymbol{J}$ whose restriction to $\boldsymbol{J}^0$ is the inflation of a $\mathrm{Gal}(\boldsymbol{k}_D/\boldsymbol{k}_F)$-invariant, cuspidal representation ϱ of $\mathrm{GL}_n(\boldsymbol{k}_D)$.

Remark 10.1. — In [**64**] Proposition 5.1, the inducing subgroup should be $D^\times \mathrm{GL}_n(\mathcal{O}_D)$ and not $F^\times \mathrm{GL}_n(\mathcal{O}_D)$. Inducing from the latter subgroup gives a representation which is not irreducible. The same comment applies to [**64**] Remark 5.2(1). See also Remark 10.4 below.

10.2. Let us now consider the map $\mathbf{b}_{D/F}$ defined in §1.7. This is a bijection from cuspidal representations of $\mathrm{GL}_n(F)$ to those cuspidal representations of $\mathrm{GL}_n(D)$ which are distinguished by the subgroup $\mathrm{Sp}_n(D)$. In this paragraph, given a cuspidal representation τ of depth 0 of $\mathrm{GL}_n(F)$, we describe explicitly the cuspidal representation $\pi = \mathbf{b}_{D/F}(\tau)$, that is, the unique cuspidal representation of G whose Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ is $\mathrm{St}_2(\tau)$.

On the one hand, it follows from [**58**] Proposition 3.2 that the representation π has depth 0. It can thus be described as in §10.1, that is, it is compactly induced from a representation $\boldsymbol{\rho}$ of the group $\boldsymbol{J} = D^\times \mathrm{GL}_n(\mathcal{O}_D)$ whose restriction to $\boldsymbol{J}^0$ is the inflation of ϱ.

On the other hand, the representation τ can be described in a similar way: it is compactly induced from a representation of $F^\times \mathrm{GL}_n(\mathcal{O}_F)$ whose restriction to $\mathrm{GL}_n(\mathcal{O}_F)$ is the inflation of a cuspidal representation ϱ_0 of $\mathrm{GL}_n(\boldsymbol{k}_F)$. (See for instance [**15**] 1.2.)

[**58**] Theorem 4.1 provides a simple and natural relation between ϱ and ϱ_0: the representation ϱ is the base change (that is, the Shintani lift) of ϱ_0. (This relation was pointed out in [**64**] Remark 5.2(1) without reference to [**58**].)

The knowledge of ϱ does not quite determine the representation π. In order to completely determine it, fix a uniformizer ϖ_F of F and a uniformizer $\varpi = \varpi_D$ of D such that $\varpi^2 = \varpi_F$. As the group $\boldsymbol{J}$ is generated by $\boldsymbol{J}^0$ and ϖ, it remains to compute the operator $A = \boldsymbol{\rho}(\varpi)$, which intertwines ϱ with ϱ^γ, where γ is the non-trivial element of $\mathrm{Gal}(\boldsymbol{k}_D/\boldsymbol{k}_F)$, that is, one has

$$A \circ \varrho(x) = \varrho(x^\gamma) \circ A, \quad x \in \mathrm{GL}_n(\boldsymbol{k}_D).$$

The space of intertwining operators between ϱ and ϱ^γ has dimension 1. To go further, we have to identify this space with $\mathbb{C}$ in a natural way.

Fix a non-trivial character ψ_0 of $\boldsymbol{k}_F$, and let ψ be the character of $\boldsymbol{k}_D$ obtained by composing ψ_0 with the trace of $\boldsymbol{k}_D/\boldsymbol{k}_F$. Let U denote the subgroup of $\mathrm{GL}_n(\boldsymbol{k}_D)$ made of all unipotent upper triangular matrices, and consider ψ as the character $u \mapsto \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n})$ of U. It is well-known that, if V is the underlying vector space of ϱ, then

$$\varrho^\psi = \{v \in V \mid \varrho(u)(v) = \psi(u)v,\ u \in U\}$$

has dimension 1. Since the character ψ is $\mathrm{Gal}(\boldsymbol{k}_D/\boldsymbol{k}_F)$-invariant, this 1-dimensional space is stable by A. There is thus a non-zero scalar $\alpha \in \mathbb{C}^\times$ such that $A(v) = \alpha v$ for all $v \in \varrho^\psi$, and A is uniquely determined by α. Let ω_0 denote the central character of τ. (Note that the representation τ is entirely determined by ϱ_0 and ω_0.)

Proposition 10.2. — *One has* $\alpha = \omega_0(-\varpi_F)$.

We now have completely determined π from the knowledge of τ. The proof of this proposition, based on [**59**] and [**17**], will be done in the next paragraph.

Remark 10.3. — The proposition thus implies that the result does not depend on the choice of a $\varpi \in D$ such that $\varpi^2 = \varpi_F$. Replacing ϖ by $-\varpi$ should thus lead to the same result, that is, one should have $\boldsymbol{\rho}(-\varpi) = \boldsymbol{\rho}(\varpi)$, or equivalently, the central character of π should be trivial at -1. This is the case indeed, since π is distinguished by $\mathrm{Sp}_n(D)$, which contains -1.

Remark 10.4. — This paragraph corrects the description made in [**64**] Remark 5.2(1), which is incorrect due to the error pointed out in Remark 10.1. Note that [**64**] Remark 5.2(2) is correct: it follows from [**58**] Theorem 4.1 or [**16**] Theorem 6.1.

10.3. We now proceed to the proof of Proposition 10.2, which is essentially an exercise of translation into the language of [**59**] and [**17**]. Fix a separable closure $\overline{F}$ of F.

A *tame admissible pair* is a pair $(K/F, \xi)$ made of an unramified finite extension K of F contained in $\overline{F}$ together with a tamely ramified character $\xi : K^\times \to \mathbb{C}^\times$ all of whose $\mathrm{Gal}(K/F)$-conjugate ξ^γ, $\gamma \in \mathrm{Gal}(K/F)$, are pairwise distinct. The *degree* of such a pair is the degree of K over F.

Given any integer $m \geqslant 1$ and any inner form H of $\mathrm{GL}_m(F)$, Silberger–Zink [**59**] have defined a bijection $\mathit{\Pi}^H$ between:

(1) the set of Galois conjugacy classes of tame admissible pairs of degree dividing m,
(2) the set of isomorphism classes of discrete series representations of depth 0 of H.

They have also described (in [**59**] Theorem 3) the behavior of this parametrization of the discrete series of inner forms of $\mathrm{GL}_m(F)$ with respect to the Jacquet–Langlands correspondence: if H is isomorphic to $\mathrm{GL}_r(\mathit{\Delta})$ for some divisor r of m and some central division F-algebra $\mathit{\Delta}$ of reduced degree m/r, and if $(K/F, \xi)$ is a tame admissible pair of degree f dividing m, then the Jacquet–Langlands transfer of $\Pi^H(K/F, \xi)$ to $\mathrm{GL}_m(F)$ is equal to

$$\Pi^{\mathrm{GL}_m(F)}\left(K/F, \xi\mu_K^{m-r+(f,r)-f}\right) \tag{10.1}$$

where μ_K is the unique unramified character of $K^\times$ of order 2 and (a, b) denotes the greatest common divisor of two integers $a, b \geqslant 1$. (Silberger–Zink state their result by using the multiplicative group of a central division F-algebra of reduced degree m as an inner form of reference, but it is more convenient for us to use $\mathrm{GL}_m(F)$ as the inner form of reference.)

Let us start with our cuspidal representation τ of depth 0 of $\mathrm{GL}_n(F)$. Let $(K/F, \xi)$ be a tame admissible pair associated with it by the bijection $\Pi^{\mathrm{GL}_n(F)}$. It follows from [**17**] 5.1 that this pair has degree n, and that the central character ω_0 of τ is equal to the restriction of ξ to $F^\times$.

Now form the discrete series representation $\mathrm{St}_2(\tau)$ of $\mathrm{GL}_{2n}(F)$. By [**17**] 5.2, the tame admissible pair associated with it by the bijection $\Pi^{\mathrm{GL}_{2n}(F)}$ is of the form $(K/F, \xi')$ where ξ' coincides with ξ on the units of $\mathcal{O}_K$. By [**17**] 6.4, one has

$$\xi'(\varpi_F) = -\omega_0(\varpi_F) = -\xi(\varpi_F).$$

One thus has $\xi' = \xi\mu_K$. We now claim that the representation π is parametrized, through the bijection Π^G, by the tame admissible pair $(K/F, \xi)$. Indeed, by (10.1), and since n is odd and μ_K is quadratic, we have

$$^{\mathrm{JL}}\Pi^G(K/F, \xi) = \Pi^{\mathrm{GL}_{2n}(F)}(K/F, \xi\mu_K^{2n-n+(n,n)-n}) = \Pi^{\mathrm{GL}_{2n}(F)}(K/F, \xi\mu_K)$$

which is equal to $\mathrm{St}_2(\tau)$. It now follows from [**59**] (8), p. 196, that the scalar α by which A acts on the line ϱ^ψ is equal to $\xi(-\varpi_F) = \omega_0(-\varpi_F)$ as expected.

11. More preliminaries

The remainder of the article is devoted to the proof of Theorem 1.3. In this section, we give more preliminaries, in addition to those of Section 3.

11.1. Fix an integer $n \geqslant 1$ and set $G = \mathrm{GL}_n(\mathbb{C})$. By *representation* of G we mean a smooth admissible Fréchet representation of moderate growth.

We denote by $|\cdot|_{\mathbb{C}}$ the normalized absolute value of $\mathbb{C}$, that is, the square of the usual modulus of $\mathbb{C}$.

Let $P = MN$ be a standard parabolic subgroup of G together with its standard Levi decomposition. As in the non-Archimedean case (§3.3), we denote by $\mathrm{Ind}_P^G(\sigma)$ the representation of G obtained from a representation σ of M by (normalized) parabolic induction along P.

Fix a non-trivial unitary additive character ψ of $\mathbb{C}$. As in the non-Archimedean case (§3.4), it defines a character (still denoted by ψ) of the subgroup U of upper triangular unipotent matrices of G. An irreducible representation π of G is said to be generic if $\mathrm{Hom}_U(\pi, \psi)$ is non-zero.

Let ν denote the character "normalized absolute value of the determinant" of G.

11.2. Let F be either $\mathbb{C}$ or a non-Archimedean locally compact field and let Δ be a finite-dimensional central division F-algebra. (If $F = \mathbb{C}$, we thus have $\Delta = \mathbb{C}$.) Fix an integer $n \geqslant 1$ and set $G = \mathrm{GL}_n(\Delta)$. It is a locally compact group.

Let K be the compact subgroup $\mathrm{GL}_n(\mathcal{O}_\Delta)$ if F is non-Archimedean, and the compact unitary group $\mathrm{U}_n(\mathbb{C}/\mathbb{R})$ if $F = \mathbb{C}$. In either case, K is a maximal compact subgroup of G.

Let $(n_1, \ldots, n_r)$ be a composition of n, and let $P = MN$ be the standard parabolic subgroup of G associated with it. One has the Iwasawa decomposition $G = PK$. For $i = 1, \ldots, r$, let π_i be a representation of $\mathrm{GL}_{n_i}(\Delta)$. Given any $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$, restriction of functions from G to K induces an isomorphism

$$\pi_1\nu^{s_1} \times \cdots \times \pi_r\nu^{s_r} = \mathrm{Ind}_P^G(\pi_1\nu^{s_1} \otimes \cdots \otimes \pi_r\nu^{s_r}) \to \mathrm{Ind}_{K\cap P}^K(\pi_1 \otimes \cdots \otimes \pi_r) \tag{11.1}$$

of representations of K, which we denote by r_s. A family of functions

$$\varphi_s \in \pi_1\nu^{s_1} \times \cdots \times \pi_r\nu^{s_r}, \quad s \in \mathbb{C}^r,$$

is called a *flat section* if $r_s(\varphi_s)$ is independent of s, that is, if there exists a function f in the right hand side of (11.1) such that $\varphi_s = r_s^{-1}(f)$ for all $s \in \mathbb{C}^r$. The reader may refer to [**65**] IV.1 for more detail.

Let $\mathsf{X}(M)$ be the free $\mathbb{Z}$-module of algebraic characters of M, set $\mathfrak{a}_P = \mathsf{X}(M) \otimes_{\mathbb{Z}} \mathbb{R}$ and let $\mathfrak{a}_P^*$ be its dual. The *Harish-Chandra map* $H_P : G \to \mathfrak{a}_P^*$ is defined by

$$e^{\langle \chi, H_P(muk)\rangle} = |\chi(m)|_F$$

for $m \in M$, $u \in N$, $k \in K$ and $\chi \in \mathsf{X}(M)$.

11.3. Now assume that $\Delta = F$, and let π, π' be generic irreducible representations of $\mathrm{GL}_n(F)$. Associated with them in [**31**] if F is non-Archimedean, and [**30**] if $F = \mathbb{C}$, there are the Rankin–Selberg local factors

$$L(s, \pi, \pi'), \quad \varepsilon(s, \pi, \pi', \psi), \quad \gamma(s, \pi, \pi', \psi),$$

related by the identity

$$\gamma(s, \pi, \pi', \psi) = \varepsilon(s, \pi, \pi', \psi)\frac{L(1-s, \pi^\vee, \pi'^\vee)}{L(s, \pi, \pi')}.$$

Note that, if F is non-Archimedean and if π, π' are cuspidal, the local L-factor $L(s, \pi, \pi')$ is nowhere vanishing, and it has a pole at $s_0 \in \mathbb{C}$ if and only if π' is isomorphic to $\pi^\vee\nu^{-s_0}$ (see [**31**] Proposition 8.1).

11.4. Let k be a totally imaginary number field. Fix a non-trivial additive character of $\mathbb{A} = \mathbb{A}_k$ which is trivial on k. For any place v of k, let ψ_v denote the local component of ψ at v. It is a non-trivial unitary additive character of k_v.

Let Π, Π' be cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$. Given an Archimedean place v, their local components Π_v, Π'_v are Harish-Chandra modules, having Casselman–Wallach completions denoted $\overline{\Pi}_v^\infty$, $\overline{\Pi}_v'^\infty$, respectively. These completions are generic irreducible representations of $\mathrm{GL}_n(k_v)$ (see [**66**] for details) and we define

$$L(s, \Pi_v, \Pi'_v) = L(s, \overline{\Pi}_v^\infty, \overline{\Pi}_v'^\infty)$$

and similarly for ε-factors and γ-factors.

Proposition 11.1 **([29] Theorem 5.3).** — *Let S be a finite set of places of k containing all Archimedean places and all places at which at least one of Π, Π' or ψ is ramified. The product*

$$L^S(s,\Pi,\Pi') = \prod_{v\notin S} L(s,\Pi_v,\Pi'_v)$$

converges absolutely for $\mathrm{Re}(s)$ *sufficiently large and extends to a meromorphic function on $\mathbb{C}$ satisfying the functional equation*

$$L^S(s,\Pi,\Pi') = \prod_{v\in S} \gamma(s,\Pi_v,\Pi'_v,\psi_v)\cdot L^S(1-s,\Pi^\vee,\Pi'^\vee).$$

12. Geometry of the symmetric space

In this section, F is a field of characteristic different from 2, and A is either $\mathsf{M}_2(F)$ or a non-split quaternion F-algebra. For any integer $n \geqslant 1$, we write $G_n = \mathrm{GL}_n(A)$ and $H_n = \mathrm{Sp}_n(A)$.

12.1. Fix an integer $n \geqslant 1$ and write $G = G_n$ and $H = H_n$ for simplicity. The symmetric space corresponding to the symmetric pair (G,H) is

$$X = \{x \in G \mid x\sigma(x) = 1\}.$$

It is endowed with a transitive action of G defined by $g\cdot x = gx\sigma(g)^{-1}$ for $g \in G$, $x \in X$. Since H is the stabilizer of the identity matrix in X, we can identify X with G/H via $g \mapsto g\sigma(g)^{-1}$.

12.2. Fix integers $m,t \geqslant 1$ such that $n = mt$ and let $P = MN$ be the standard parabolic subgroup of G associated with the composition $(t,\dots,t)$. The subgroups P, M and N are σ-stable.

Let M_0 be the group of diagonal matrices in G. We write W_M for the Weyl group of M, that is, the quotient of the M-normalizer of M_0 by M_0. It is isomorphic to $\mathfrak{S}_t \times \cdots \times \mathfrak{S}_t$, where $\mathfrak{S}_t$ is the symmetric group of order $t!$. We also write W for the Weyl group of G.

Any (W_M,W_M)-double coset in W contains a unique element w with minimal length in both W_Mw and wW_M. This defines a set of representatives of (W_M,W_M)-double cosets in W, which we denote by ${}_MW_M$.

Let $W(M)$ be the set of elements of ${}_MW_M$ normalizing M. It naturally identifies with $\mathfrak{S}_m$.

12.3. We describe the P-orbits of X, or equivalently the (P,H)-double cosets of G. For more details, we refer the reader to [**40**] §3 in the case when A is split, and [**57**] §3 in the case when A is non-split (see also [**42, 44**]). Given an $x \in X$, there is a unique $w \in {}_MW_M$ such that the P-orbit $P\cdot x$ is contained in PwP, since P and M are σ-stable. Hence we obtain a map

$$\iota_M : P\backslash X \to {}_MW_M. \tag{12.1}$$

A P-orbit $P\cdot x$ is said to be M-*admissible* if $w = \iota_M(P\cdot x)$ normalizes M, that is, if $w \in W(M)$.

Let $P\cdot x$ be an M-admissible P-orbit of X and consider $w = \iota_M(P\cdot x) \in W(M)$ as an element of $\mathfrak{S}_m$. Let w_m be the element of maximal length in $\mathfrak{S}_m$. The permutation $\tau = ww_m \in \mathfrak{S}_m$ satisfies $\tau^2 = 1$. The map

$$P\cdot x \mapsto \tau \tag{12.2}$$

is a bijection from the set of M-admissible P-orbits of X to the set $\{\tau \in \mathfrak{S}_m \mid \tau^2 = 1\}$.

To describe the inverse of the bijection (12.2), let $w \mapsto [w]_t$ denote the bijection from $\mathfrak{S}_m$ to the group of permutation matrices of $\mathrm{GL}_m(F)$ composed with the natural embedding of $\mathrm{GL}_m(F)$ in $\mathrm{GL}_m(\mathbf{M}_t(A)) = G$. Given a permutation $\tau \in \mathfrak{S}_m$ such that $\tau^2 = 1$, the element $x = [\tau w_m]_t$ is in X and the image of its P-orbit by (12.2) is equal to τ. For such an x, the stabilizer M_x of x in $M = G_t \times \cdots \times G_t$ is equal to

$$M_x = \left\{(g_1,\dots,g_m) \in G_t \times \cdots \times G_t \mid g_{\tau(i)} = \sigma(g_i) \text{ for all } i = 1,\dots,m\right\}. \tag{12.3}$$

In particular, one has $g_i \in H_t$ for all i fixed by τ. Thus M_x is the group of F-rational points of a connected reductive algebraic group defined over F.

Example 12.1. — (1) In particular, the choice $\tau = 1 \in \mathfrak{S}_m$ gives the representative

$$x = [w_m]_t = \begin{pmatrix} & & 1_t \\ & \iddots & \\ 1_t & & \end{pmatrix} \in X \tag{12.4}$$

where 1_t is the identity matrix in $\mathbf{M}_t(A)$. The M-admissible orbit $P\cdot x$ is the open P-orbit of X and we have $M_x = H_t \times \cdots \times H_t$ where H_t occurs m times.

(2) Let us give an explicit representative $\eta \in G$ such that $\eta\sigma(\eta)^{-1} = x$. First, fix non-zero elements $\alpha, \beta \in A^\times$ such that $\mathrm{Nrd}_{A/F}(\alpha) = -1/2$ and $\mathrm{Nrd}_{A/F}(\beta) = 1/2$, which is possible since the reduced norm $\mathrm{Nrd}_{A/F}$ is surjective. If m is even, write $m = 2k$ and set

$$\eta = \begin{pmatrix} \alpha\cdot 1_{kt} & -\alpha\cdot[w_k]_t \\ \beta\cdot[w_k]_t & \beta\cdot 1_{kt} \end{pmatrix}.$$

If m is odd, set $m = 2k+1$ and

$$\eta = \begin{pmatrix} \alpha\cdot 1_{kt} & & -\alpha\cdot[w_k]_t \\ & 1_{kt} & \\ \beta\cdot[w_k]_t & & \beta\cdot 1_{kt}. \end{pmatrix}.$$

Such an η satisfies $\eta\sigma(\eta)^{-1} = x$ and $P\eta H$ is the open (P,H)-double coset of G.

12.4. In this paragraph, F is either $\mathbb{C}$ or a non-Archimedean local field of characteristic 0. Let us recall some known properties of symplectic periods which we will use later. The next theorem is called the *multiplicity one property* of symplectic periods.

Theorem 12.2. — *For any irreducible representation π of G, the vector space $\mathrm{Hom}_H(\pi,\mathbb{C})$ has dimension at most* 1.

Proof. — In the non-Archimedean case, this is proven by Heumos and Rallis [**28**] Theorem 2.4.2 when $A = \mathbf{M}_2(F)$ and by Verma [**64**] Theorem 3.8 when A is non-split. Their argument is based on Prasad's idea [**47**] and the key is to find a certain anti-automorphism of G acting trivially on H-bi-invariant distributions on G. Here we are in the simplest case where the H-double cosets are invariant under the anti-involution at hand. Due to [**1**], these arguments also apply to the Archimedean case where $A = \mathbf{M}_2(\mathbb{C})$. □

12.5. In this paragraph, F is still either $\mathbb{C}$ or a non-Archimedean locally compact field of characteristic 0, and we assume moreover that $A = \mathbf{M}_2(F)$. We will identify G with $\mathrm{GL}_{2n}(F)$.

Given any integer $k \in \{0, \dots, n\}$, consider the subgroup ${}_kH$ of G defined by

$${}_kH = \left\{ \begin{pmatrix} u & a \\ 0 & h \end{pmatrix} \;\middle|\; u \in U_{2r},\ h \in \mathrm{Sp}_{2k}(F),\ a \in \mathbf{M}_{2r,2k}(F) \right\}$$

where $r = n - k$ and U_{2r} is the group of upper triangular unipotent matrices in $\mathrm{GL}_{2r}(F)$. Note that ${}_nH = H = \mathrm{Sp}_{2n}(F)$. Define a character ${}_k\Psi$ of ${}_kH$ by

$${}_k\Psi \begin{pmatrix} u & a \\ 0 & h \end{pmatrix} = \psi(u_{1,2} + u_{2,3} + \cdots + u_{2r-1,2r}).$$

We say that an irreducible representation π of G has a *Klyachko model* if there exists an integer $k \in \{0, \dots, n\}$ such that $\mathrm{Hom}_{{}_kH}(\pi, {}_k\Psi)$ is non-zero.

The following theorem is a consequence of results of Offen–Sayag ([**46**] Theorem 1) and Aizenbud–Offen–Sayag ([**2**] Theorem 1.1).

Theorem 12.3. — *Let π be an irreducible representation of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$. Then the space $\mathrm{Hom}_{{}_kH}(\pi, {}_k\Psi)$ is zero for all $k \in \{0, \dots, n-1\}$.*

12.6. In this paragraph, the assumptions of §12.5 on F and A still hold. For later use, we prepare the following lemma. Let $(n_1, \dots, n_r)$ be a composition of n and, for $i = 1, \dots, r$, let δ_i be a discrete series representation of $\mathrm{GL}_{n_i}(F)$. Suppose that $\pi = \delta_1 \times \cdots \times \delta_r$ is a unitary generic irreducible representation of $\mathrm{GL}_n(F)$. Then

$$\pi\nu^{1/2} \times \pi\nu^{-1/2}$$

has a unique irreducible quotient, denoted $\mathrm{Sp}_2(\pi)$. (Recall that ν denotes the character "normalized absolute value of the determinant".) Note that the representation $\mathrm{Sp}_2(\pi)$ is isomorphic to $\mathrm{Sp}_2(\delta_1) \times \cdots \times \mathrm{Sp}_2(\delta_r)$. We refer the reader to [**5**] §4.1 (and [**63**]) when F is non-Archimedean, and when $F = \mathbb{C}$ the argument there applies as well.

Lemma 12.4. — *In the above situation, suppose that π is a unitary generic principal series representation of $\mathrm{GL}_n(F)$ and $\pi\nu^{1/2} \times \pi\nu^{-1/2}$ is distinguished by H. Any linear form in*

$$\mathrm{Hom}_H(\pi\nu^{1/2} \times \pi\nu^{-1/2}, \mathbb{C})$$

factors through the quotient map from $\pi\nu^{1/2} \times \pi\nu^{-1/2}$ to $\mathrm{Sp}_2(\pi)$.

Proof. — It suffices to prove that the kernel κ of this map is not distinguished by H. By assumption on π, we have $n_1 = \cdots = n_r = 1$, that is, the representations δ_i are characters of $F^\times$, and $\pi = \delta_1 \times \cdots \times \delta_n$ is unitary and generic. Up to semi-simplification, we have

$$\begin{aligned} \pi\nu^{1/2} \times \pi\nu^{-1/2} &= \left(\delta_1\nu^{1/2} \times \delta_1\nu^{-1/2}\right) \times \cdots \times \left(\delta_n\nu^{1/2} \times \delta_n\nu^{-1/2}\right), \\ \delta_i\nu^{1/2} \times \delta_i\nu^{-1/2} &= \mathrm{St}_2(\delta_i) + \mathrm{Sp}_2(\delta_i), \end{aligned}$$

for all $i = 1, \ldots, r$. The semi-simplification of κ is thus equal to

$$\bigoplus_{I \subsetneq \{1,\ldots,n\}} \mathrm{S}_1^I \times \cdots \times \mathrm{S}_n^I, \quad \mathrm{S}_i^I = \begin{cases} \mathrm{Sp}_2(\delta_i) & \text{if } i \in I, \\ \mathrm{St}_2(\delta_i) & \text{if } i \notin I, \end{cases}$$

where the sum ranges over the proper subsets I of $\{1, \ldots, n\}$. By [**45**] Theorem 3.7 when $F = \mathbb{C}$ and [**24**] Theorem A when F is non-Archimedean, each irreducible summand of κ has a Klyachko model which is not the symplectic model. By local disjointness (Theorem 12.3), we see that κ is not distinguished by H. □

13. Proof of Theorem 1.3

In all this section, F is a non-Archimedean locally compact field of characteristic 0 and D is a quaternion division F-algebra. Recall that $G = G_n = \mathrm{GL}_n(D)$ and $H = H_n = \mathrm{Sp}_n(D)$.

We deduce Theorem 1.3 from the key Proposition 13.5 which will be proven in Section 15.

13.1. Let π be a discrete series representation of G. According to §4.1, this representation is of the form $\mathrm{St}_m(\rho)$ for a unique divisor m of n and a cuspidal representation ρ of $\mathrm{GL}_{n/m}(D)$, uniquely determined up to isomorphism. Let $r = r(\rho)$ denote the positive integer associated with ρ. It is equal to either 1 or 2. We set $t = n/m$.

Let $P = MU$ denote the standard parabolic subgroup of G corresponding to the composition $(t, \ldots, t)$ of n. Given a complex number $s \in \mathbb{C}$, define the parabolically induced representation

$$I(s, \rho) = \rho\nu^{sr(m-1)/2} \times \rho\nu^{sr(m-3)/2} \times \cdots \times \rho\nu^{sr(1-m)/2}.$$

In particular, the representation π is the unique irreducible quotient of $I(-1, \rho)$. It is also isomorphic to the unique irreducible subrepresentation of $I(1, \rho)$.

Proposition 13.1. — *Let $s \in \mathbb{C}$.*

(1) *If ρ is distinguished by H_t, then $I(s, \rho)$ is distinguished by H.*

(2) *Conversely, suppose that $I(s, \rho)$ is distinguished by H and the real part of s does not belong to the set $\{(kr)^{-1} \mid k = 1, 2, \ldots, m-1\}$. Then*

(a) *The representation ρ is distinguished by H_t.*

(b) *The dimension of $\mathrm{Hom}_H(I(s, \rho), \mathbb{C})$ is equal to 1.*

(c) *Any non-zero linear form in $\mathrm{Hom}_H(I(s, \rho), \mathbb{C})$ does not vanish on the subspace made of functions of $I(s, \rho)$ supported in the open (P, H)-double coset of G.*

Proof. — First assume that $I(s,\rho)$ is distinguished by H. It follows from [**44**] Corollary 5.2 that, with the notation and definitions of Section 12, there is an $x \in X$ such that $P\cdot x$ is M-admissible and the space

$$\mathrm{Hom}_{M_x}\left(\rho\nu^{sr(m-1)/2}\otimes\rho\nu^{sr(m-3)/2}\otimes\cdots\otimes\rho\nu^{sr(1-m)/2},\delta_{P_x}\delta_P^{-1/2}\right) \tag{13.1}$$

is non-zero, where M_x and P_x are the stabilizers of x in M and P, and δ_P and δ_{P_x} are the modulus characters of P and P_x, respectively. We are going to prove that $P\cdot x$ is the open orbit of the symmetric space X.

Thanks to the description of M_x given in (12.3) for a suitable choice of representative x, and according to [**40**] §3.3.5 and [**57**] §3.4, we have

$$\delta_{P_x}\delta_P^{-1/2}(g) = \prod_{i<\tau(i)}\nu(g_i)$$

for all $g = (g_1, g_2, \dots, g_m) \in M_x \subseteq M = G_t \times \cdots \times G_t$, where the product is over the integers $i \in \{1,\dots,m\}$ such that $i < \tau(i)$. Associated with $P\cdot x$, there is a $\tau \in \mathfrak{S}_m$ such that $\tau^2 = 1$. Let us prove that τ is the identity. It will follow from Example 12.1 that $P\cdot x$ is the open orbit.

Suppose to the contrary that $i < \tau(i)$ for some i. Then the space

$$\mathrm{Hom}_{G_t}\left(\rho\nu^{sr(m+1-2i)/2}\otimes\rho\nu^{sr(m+1-2\tau(i))/2},\nu\otimes 1\right)$$

is non-zero, where G_t is identified with the subgroup $\{(g,\sigma(g)) \mid g \in G_t\}$ of $G_t \times G_t$. Looking at the central character, and thanks to the fact that $\sigma(z) = z^{-1}$ for all z in the centre of G_t, we obtain

$$\frac{1}{2}\cdot\mathrm{Re}(s)\cdot r(m+1-2i) - \left(\frac{1}{2}\cdot\mathrm{Re}(s)\cdot r(m+1-2\tau(i))\right) = 1$$

or equivalently $\mathrm{Re}(s)\cdot r(\tau(i)-i) = 1$, where $\mathrm{Re}(s)$ is the real part of s. Since r is an integer and thanks to the assumption on $\mathrm{Re}(s)$, this contradicts the assumption $i < \tau(i)$.

The open P-orbit of X is thus the unique P-orbit such that (13.1) holds. Example 12.1 shows that we thus may choose $x = [w_m]_t$. For such a choice of representative of the open P-orbit of X, the subgroup M_x is equal to $H_t \times \cdots \times H_t$. It thus follows from [**44**] Section 4 that

$$\begin{aligned}
\mathrm{Hom}_H\left((I(s,\rho),\mathbb{C}\right) &\simeq \mathrm{Hom}_H\left(\mathrm{Ind}_P^{P\eta H}(s,\rho),\mathbb{C}\right)\\
&\simeq \mathrm{Hom}_{M_x}\left(\rho\nu^{sr(m-1)/2}\otimes\cdots\otimes\rho\nu^{sr(1-m)/2},\mathbb{C}\right)\\
&\simeq \mathrm{Hom}_{H_t}\left(\nu^{sr(m-1)/2}\rho,\mathbb{C}\right)\otimes\cdots\otimes\mathrm{Hom}_{H_t}\left(\nu^{sr(1-m)/2}\rho,\mathbb{C}\right)\\
&\simeq \mathrm{Hom}_{H_t}(\rho,\mathbb{C})\otimes\cdots\otimes\mathrm{Hom}_{H_t}(\rho,\mathbb{C})
\end{aligned}$$

where the first isomorphism is induced by the restriction map from G to the open double coset $P\eta H$ (where $\eta \in G$ satisfies $\eta\sigma(\eta)^{-1} = x$) and the last one follows from the fact that the character ν is trivial on H_t. Hence assertion (2) follows from the previous series of isomorphisms, together with the fact that the dimension of $\mathrm{Hom}_{H_t}(\rho,\mathbb{C})$ is at most 1 by Theorem 12.2.

Now assume that ρ is distinguished by H_t, and let $x = [w_m]_t$ be the representative of the open P-orbit given in Example 12.1, as above. Since $\rho\nu^{sr(m-1)/2}\otimes\cdots\otimes\rho\nu^{sr(1-m)/2}$ is distinguished by M_x, and since $\sigma(P) = P$ and xPx^{-1} is the parabolic subgroup of G opposite to P with respect to M, it follows from [**44**] Proposition 7.2 that $I(s,\rho)$ is distinguished by H. □

Corollary 13.2. — *Let $\pi = \mathrm{St}_m(\rho)$ as above. If π is H-distinguished, then $r = 2$.*

Proof. — If π is distinguished by H, then so is $I(-1,\rho)$, and it follows from Proposition 13.1(2.a) that ρ is distinguished by H_t. Theorem 5.2 thus implies that ${}^{\mathrm{JL}}\rho$ is not cuspidal, that is, $r \neq 1$. Hence we obtain $r = 2$. □

13.2. Assume now that the discrete series representation π is distinguished by H. In particular, the induced representation $I(-1,\rho)$ is distinguished by H, thus ρ is distinguished by H_t thanks to Proposition 13.1(2.a). Our goal is to prove that $m = 1$, that is, π is cuspidal. Suppose to the contrary that $m \geqslant 2$.

We fix a non-zero linear form $\Lambda_\rho \in \mathrm{Hom}_{H_t}(\rho,\mathbb{C})$. Since the character ν is trivial on H_t, we can regard it as an H_t-invariant linear form on $\rho\nu^s$ for any $s \in \mathbb{C}$. Let x and η correspond to the open orbit, as in Example 12.1. In particular, M_x is equal to $H_t \times \cdots \times H_t$. Set

$$\mu = \Lambda_\rho \otimes \cdots \otimes \Lambda_\rho \in \mathrm{Hom}_{M_x}(\rho \otimes \cdots \otimes \rho, \mathbb{C}).$$

Given a flat section $\varphi_s \in I(s,\rho)$ (see §11.2), we consider the integral

$$J(s,\varphi_s,\mu) = \int_{(\eta^{-1}P\eta\cap H)\backslash H} \mu(\varphi_s(\eta h))\,\mathrm{d}h. \tag{13.2}$$

The following theorem follows from [**12**] Théorème 2.8, Théorème 2.16 (see [**12**] Remarque 2.17 and [**35**] Théorème 4). At this point, and more generally in this paragraph, the choice of invariant measures is not important, and we postpone the discussion of this matter to §14.2 below.

Theorem 13.3. — (1) *There is an $x_0 \in \mathbb{R}$ such that, for any flat section $\varphi_s \in I(s,\rho)$, the integral* (13.2) *converges when* $\mathrm{Re}(s) > x_0$.

(2) *There exists a non-zero Laurent polynomial $P(X) \in \mathbb{C}[X,X^{-1}]$ such that, for any flat section $\varphi_s \in I(s,\rho)$, the function*

$$P(q^{-s}) \cdot J(s,\varphi_s,\mu)$$

extends to a function in $\mathbb{C}[q^{-s},q^s]$.

This defines a meromorphic family of H-invariant linear forms $J(s,\cdot,\mu)$ on $I(s,\rho)$, called *open intertwining periods.*

Proposition 13.4. — *The open intertwining period $J(s,\cdot,\mu)$ on $I(s,\rho)$ is holomorphic and non-zero at $s = 1$ and $s = -1$.*

Proof. — Note that $r = 2$ by Corollary 13.2. Let $e \in \{-1,1\}$. As ρ is distinguished by H_t, Proposition 13.1(1) says that $\mathrm{Hom}_H(I(e,\rho),\mathbb{C})$ has dimension 1 and Proposition 13.1(2.c) says that any non-zero linear form in this 1-dimensional space has a non-zero restriction to the subspace of functions in $I(e,\rho)$ supported in the open double coset $P\eta H$. The assertion now follows from the same argument as [**37**] Proposition 10.4. □

Let $w = w_m$ be the longest element in $W(M) \simeq \mathfrak{S}_m$. Let $M(s,w) : I(s,\rho) \to I(-s,\rho)$ denote the standard intertwining operator given by the convergent integral

$$M(s,w)\varphi_s(g) = \int_N \varphi_s(wug)\ \mathrm{d}u$$

for a flat section $\varphi_s \in I(s,\rho)$ when $\mathrm{Re}(s)$ is sufficiently large. It has a meromorphic continuation to the whole complex plane. Note that $M(s,w)$ is holomorphic and non-zero at $s = 1$, as follows for example from [**41**] I.1(4) together with the aforementioned results on the location of poles of local Rankin–Selberg L-factors of pairs of cuspidal representations.

We thus get two meromorphic families $J(s,\cdot,\mu)$ and $J(-s, M(s,w)\ \cdot,\mu)$ of H-invariant linear forms on $I(s,\rho)$. It follows from Theorem 12.2 together with generic irreducibility of $I(s,\rho)$ that the space $\mathrm{Hom}_H(I(s,\rho),\mathbb{C})$ has dimension at most 1 for $s \in \mathbb{C}$ generic. There is thus a meromorphic function $\alpha(s,\rho)$ such that

$$J(-s, M(s,w)\varphi_s,\mu) = \alpha(s,\rho)J(s,\varphi_s,\mu) \tag{13.3}$$

for any flat section $\varphi_s \in I(s,\rho)$. By Theorem 13.3(2), the function $\alpha(s,\rho)$ is in $\mathbb{C}(q^{-s})$. We will prove in Section 15 the following property of $\alpha(s,\rho)$.

***Proposition 13.5* (Proposition 15.3)**. — *The meromorphic function $\alpha(s,\rho)$ is holomorphic and non-zero at $s = 1$.*

Let us show how Theorem 1.3 immediately follows from Proposition 13.5.

13.3. In this paragraph, we prove Theorem 1.3 assuming Proposition 13.5.

Write π as $\mathrm{St}_m(\rho)$, where m,t are positive integers such that $n = mt$ and ρ is a cuspidal representation of G_t. Since π is distinguished by H, the space $\mathrm{Hom}_H(I(-1,\rho),\mathbb{C})$ has dimension 1. As $r = r(\rho) = 2$ by Corollary 13.2, it follows from Proposition 13.1 that ρ is distinguished by H_t, thus $\mathrm{Hom}_H(I(1,\rho),\mathbb{C})$ has dimension 1.

Recall that we assume $m \geqslant 2$. Let $M^*(-1)$ be a non-zero element in $\mathrm{Hom}_G(I(-1,\rho), I(1,\rho))$. Note that such a morphism is unique up to scalar, its image is isomorphic to π and its kernel is the image of $M(1,w)$. In particular, $M^*(-1)M(1,w)$ vanishes on $I(1,\rho)$.

We fix a non-zero H_t-invariant linear form Λ_ρ on ρ, and associate to it the meromorphic family of open intertwining periods $J(s,\cdot,\mu)$. As $J(-1,\cdot,\mu)$ is a non-zero element of the 1-dimensional space $\mathrm{Hom}_H(I(-1,\rho),\mathbb{C})$ by Proposition 13.4, there is a non-zero $\Lambda_\pi \in \mathrm{Hom}_H(\pi,\mathbb{C})$ such that

$$J(-1,\varphi,\mu) = \Lambda_\pi(M^*(-1)\varphi)$$

for all $\varphi \in I(-1,\rho)$. By (13.3) applied at $s = 1$, we deduce that

$$\alpha(1,\rho)J(1,\varphi,\mu) = J(-1, M(1,w)\varphi,\mu) = \Lambda_\pi(M^*(-1)M(1,w)\varphi)$$

for all $\varphi \in I(1,\rho)$. Since $M^*(-1)M(1,w)\varphi = 0$ for all $\varphi \in I(1,\rho)$ and $\alpha(1,\rho) \neq 0$ from Proposition 13.5, it follows that $J(1,\varphi,\mu) = 0$ for all $\varphi \in I(1,\rho)$. This contradicts Proposition 13.4.

14. Global theory and computation at split places

We study the local intertwining period and the meromorphic function $\alpha(s,\rho)$ via a global argument.

14.1. Let k be a totally imaginary number field and let $\mathbb{A} = \mathbb{A}_k$ denote its ring of adèles. Let B be a non-split quaternion k-algebra. Given any integer $n \geqslant 1$, the groups $\mathrm{GL}_n(B)$ and $\mathrm{Sp}_n(B)$ are the groups of k-rational points of reductive algebraic k-groups which we denote by G_n and H_n.

Given an Archimedean place v of k, any choice of isomorphism $k_v \simeq \mathbb{C}$ of topological fields induces a group isomorphism $\mathrm{G}_n(k_v) \simeq \mathrm{GL}_{2n}(\mathbb{C})$. Note that there are only two such isomorphisms from k_v to $\mathbb{C}$, which are conjugate to each other.

Given a non-Archimedean place v where B splits, any choice of isomorphism $B_v \simeq \mathbf{M}_2(k_v)$ induces a group isomorphism $\mathrm{G}_n(k_v) \simeq \mathrm{GL}_{2n}(k_v)$. By the Skolem–Noether theorem, any two isomorphisms from B_v to $\mathbf{M}_2(k_v)$ are $\mathrm{GL}_2(k_v)$-conjugate to each other. It follows that any two isomorphisms $\mathrm{G}_n(k_v) \simeq \mathrm{GL}_{2n}(k_v)$ obtained as above are $\mathrm{GL}_{2n}(k_v)$-conjugate to each other.

Given any irreducible automorphic representation Π of $\mathrm{G}_n(\mathbb{A})$, there is a decomposition of Π into local components Π_v for each place v of k. If v is non-Archimedean, Π_v is an irreducible representation of $\mathrm{G}_n(k_v)$, well defined up to isomorphism. If, in addition, the k-algebra B splits at v, then Π_v defines an irreducible representation of $\mathrm{GL}_{2n}(k_v)$, well defined up to isomorphism. If v is Archimedean, Π_v defines an irreducible Harish-Chandra module of $\mathrm{GL}_{2n}(\mathbb{C})$, well defined up to conjugacy.

14.2. We did not fix choices of Haar measures before since it did not matter much. In what follows, it will be helpful to do so, in order to normalize spherical vectors and invariant linear forms, especially at unramified places of automorphic representations.

Fix an integer $n \geqslant 1$ and write $\mathrm{G} = \mathrm{G}_n$ and $\mathrm{H} = \mathrm{H}_n$ for simplicity. For any place v of k, we will write $G_v = \mathrm{G}(k_v)$ and $H_v = \mathrm{H}(k_v)$, and similarly for any group defined over k.

At any place v where B_v is split, we fix once and for all an isomorphism $B_v \simeq \mathbf{M}_2(k_v)$ of k_v-algebras. It induces a group isomorphism $G_v \simeq \mathrm{GL}_{2n}(k_v)$. We will identify these groups through this isomorphism, without any further discussion, whenever convenient.

Fix integers $m, t \geqslant 1$ such that $mt = n$ and let $\mathrm{P} = \mathrm{MN}$ be the standard parabolic subgroup of G associated with the composition $(t, \dots, t)$. We will write $P_v = \mathrm{P}(k_v)$, etc.

As in §11.2, we set $K_v = \mathrm{G}(\mathcal{O}_v)$ when v is finite, whereas we set $K_v = \mathrm{U}_{2n}(\mathbb{C}/\mathbb{R})$ when it is Archimedean. This fixes a good maximal compact subgroup K of $\mathrm{G}(\mathbb{A})$, defined as the product of the K_v for all v.

Given any closed subgroup X_v of G_v, we define its Haar measure to give volume 1 to the intersection $X_v \cap K_v$. This also normalizes the right invariant measures on $(X_v \cap K_v)\backslash X_v$, and it also normalizes similar global right invariant measures.

14.3. Let $\mathfrak{z}_{\mathrm{M}}$ be the centre of the universal enveloping algebra of the Lie algebra of $\mathrm{M}(k \otimes_{\mathbb{Q}} \mathbb{R})$. A complex function on $\mathrm{M}(k)\backslash\mathrm{M}(\mathbb{A})$ is called an automorphic form for $\mathrm{M}(\mathbb{A})$ if it is smooth, of moderate growth, right $K \cap \mathrm{M}(\mathbb{A})$-finite and $\mathfrak{z}_{\mathrm{M}}$-finite. We refer for example to [**11**, Section 2.7] for a detailed discussion of automorphic forms in the smooth setting.

An automorphic form $f : \mathrm{M}(k)\backslash\mathrm{M}(\mathbb{A}) \to \mathbb{C}$ is called a cusp form if, for any proper parabolic subgroup Q of M with unipotent radical V, we have

$$\int_{\mathrm{V}(k)\backslash\mathrm{V}(\mathbb{A})} f(vm)\ \mathrm{d}v = 0, \quad m \in \mathrm{M}(\mathbb{A}).$$

Let $\mathcal{A}_{\mathrm{P}}(\mathrm{G})$ be the space of right K-finite functions $\varphi : \mathrm{N}(\mathbb{A})\mathrm{M}(k)\backslash\mathrm{G}(\mathbb{A}) \to \mathbb{C}$ such that, for each $\kappa \in K$, the function $g \mapsto \varphi(g\kappa)$ on $\mathrm{M}(\mathbb{A})$ is an automorphic form.

Given a cuspidal automorphic representation Π of $\mathrm{G}_t(\mathbb{A})$, let $\mathcal{A}^{\Pi}_{\mathrm{P}}(\mathrm{G})$ denote the subspace of all $\varphi \in \mathcal{A}_{\mathrm{P}}(\mathrm{G})$ such that for any $\kappa \in K$, the function $m \mapsto \varphi(m\kappa)$ is in the space of the cuspidal automorphic representation $\Pi \otimes \Pi \otimes \cdots \otimes \Pi$ of $\mathrm{M}(\mathbb{A})$.

We have the standard intertwining operator $M(s,w) : \mathcal{A}_{\mathrm{P}}(\mathrm{G}) \to \mathcal{A}_{\mathrm{P}}(\mathrm{G})$, given by the absolutely convergent integral

$$M(s,w)\varphi(g) = e^{\langle s\rho_P, H_P(g)\rangle} \int_{\mathrm{N}(\mathbb{A})} \varphi(w^{-1}ug) e^{\langle s\rho_P, H_P(w^{-1}ug)\rangle}\ \mathrm{d}u$$

where $\rho_P = ((m-1)/2, (m-3)/2, \cdots, (1-m)/2) \in \mathfrak{a}_P^*$ and s is a complex number with sufficiently large real part. It has a meromorphic continuation to $\mathbb{C}$.

14.4. Let $x = [w_m]_t$ be the representative of the open $\mathrm{P}(k)$-orbit of $\mathrm{G}(k)/\mathrm{H}(k)$ given by (12.4) and $\eta \in \mathrm{G}(k)$ be some representative of the open $(\mathrm{P}(k), \mathrm{H}(k))$-double coset of $\mathrm{G}(k)$ such that x is equal to $\eta\sigma(\eta)^{-1}$ (see Example 12.1). Set $\mathrm{M}_x = \mathrm{M} \cap \eta\mathrm{H}\eta^{-1} = \mathrm{H}_t \times \cdots \times \mathrm{H}_t$.

We define the global open intertwining period by

$$J(s,\varphi) = \int_{(\eta^{-1}\mathrm{P}(\mathbb{A})\eta\cap\mathrm{H}(\mathbb{A}))\backslash\mathrm{H}(\mathbb{A})} \left(\int_{\mathrm{M}_x(k)\backslash\mathrm{M}_x(\mathbb{A})} \varphi(m\eta h)\ \mathrm{d}m \right) e^{\langle s\rho_P, H_P(\eta h)\rangle}\ \mathrm{d}h \tag{14.1}$$

for $s \in \mathbb{C}$ and $\varphi \in \mathcal{A}^{\Pi}_{\mathrm{P}}(\mathrm{G})$. Before adressing its convergence and meromorphic continuation, we already observe that the inner integral is convergent and factorizable. Namely, from [**3**] Proposition 1, the non-zero linear form on $\Pi \otimes \cdots \otimes \Pi$ given by the period integral

$$p_{M_x} : \phi \mapsto \int_{\mathrm{M}_x(k)\backslash\mathrm{M}_x(\mathbb{A})} \phi(m)\ \mathrm{d}m$$

converges absolutely and provides, for each place v, an $\mathrm{M}_x(k_v)$-invariant linear form μ_v on the local component $\Pi_v \otimes \cdots \otimes \Pi_v$. Thanks to Theorem 12.2, there is for each place v a non-zero linear form $\mu_v \in \mathrm{Hom}_{\mathrm{M}_{x,v}}(\Pi_v, \mathbb{C})$ such that if ϕ decomposes as $\phi = \bigotimes_v \phi_v$, then

$$p_{M_x}(\phi) = \prod_v \mu_v(\phi_v). \tag{14.2}$$

Note that for this to make sense, we have fixed a choice of $\mathrm{M}_x(\mathcal{O}_v)$-spherical vectors ϕ_v at all finite places v of k such that B_v splits and Π_v is unramified, and normalized the linear forms μ_v at these places such that $\mu_v(\phi_v) = 1$. We then naturally normalize the $\mathrm{GL}_{2n}(\mathcal{O}_v)$-spherical function $\varphi_v \in \Pi_v \times \cdots \times \Pi_v$ by requiring that its value at the identity element $1_{2n,v}$ of $\mathrm{GL}_{2n}(k_v)$ is ϕ_v.

Proposition 14.1. — *Let Π be a cuspidal automorphic representation of $\mathrm{G}_t(\mathbb{A})$, and let $\varphi \in \mathcal{A}^{\Pi}_{\mathrm{P}}(\mathrm{G})$.*

(1) *The integral* (14.1) *is absolutely convergent when* $\mathrm{Re}(s)$ *is sufficiently large, and has a meromorphic continuation to* $\mathbb{C}$.

(2) *We have the global functional equation*

$$J(s,\varphi) = J(-s, M(s,w)\varphi). \tag{14.3}$$

Proof. — First we adress convergence and meromorphic continuation. We may assume that φ is decomposable into $\prod_v \varphi_v$, and at almost all places v of k, the function φ_v is the $\mathrm{GL}_{2n}(\mathcal{O}_v)$-spherical function normalized as above. According to (14.2), the inner integral in (14.1) can be factorized as the well-defined product

$$\prod_v \mu_v(\varphi_{v,s}(\eta h_v)).$$

Now we observe that for a place v at which B splits, the local factor $\mu_v(\varphi_{v,s}(\eta h_v))$ identifies with that for intertwining periods on the representation of $\mathrm{GL}_{2n}(\mathbb{A})$ induced from ${}^{\mathrm{JL}}\Pi \otimes \cdots \otimes {}^{\mathrm{JL}}\Pi$ with respect to $\mathrm{Sp}_{2n}(\mathbb{A})$. It thus follows from [**67**] Proposition 3.1, that if we fix S_0 a finite set of places of k, outside of which B is split, then there is a $r_0 \in \mathbb{R}$ independent of φ such that, for $\mathrm{Re}(s) > r_0$, the quantity

$$\prod_{v \notin S_0} \mu_v(\varphi_{v,s}(\eta h_v))$$

is integrable on the restricted product of the $(\eta^{-1}\mathrm{P}(k_v)\eta \cap \mathrm{H}(k_v))\backslash\mathrm{H}(k_v)$ for $v \notin S_0$ with respect to $(\eta^{-1}\mathrm{P}(k_v)\eta \cap \mathrm{H}(k_v) \cap K_v)\backslash\mathrm{H}(k_v) \cap K_v$. But, up to taking r_0 larger, the finite product

$$\prod_{v \in S_0} \mu_v(\varphi_{v,s}(\eta h_v))$$

is integrable on the product of the $(\eta^{-1}\mathrm{P}(k_v)\eta \cap \mathrm{H}(k_v))\backslash\mathrm{H}(k_v)$ for $v \in S_0$ thanks to [**12**] and [**14**], and the convergence statement is proved. The meromorphy then follows from [**12**], [**14**] and [**67**] Theorem 3.5. This concludes the proof of Assertion (1).

The functional equation (14.3) follows from the same argument as that for intertwining periods of $\mathrm{GL}_{2n}(\mathbb{A})$ with respect to $\mathrm{Sp}_{2n}(\mathbb{A})$ proved by Offen in [**42**] Theorem 7.7, and we observe that this also gives another proof of the meromorphy of the global intertwining period. □

Lemma 14.2. — *Let* Π *be a cuspidal automorphic representation of* $\mathrm{G}_t(\mathbb{A})$ *having a non-zero* $\mathrm{H}_t(\mathbb{A})$*-period. There exists a cuspidal automorphic representation* Σ *of* $\mathrm{GL}_t(\mathbb{A})$ *such that* ${}^{\mathrm{JL}}\Pi$ *is equal to* $\mathrm{MW}_2(\Sigma)$.

Proof. — According to §4.3, there are a positive integer t dividing 2 and a cuspidal representation Σ of $\mathrm{GL}_{2a/t}(\mathbb{A})$ such that ${}^{\mathrm{JL}}\Pi = \mathrm{MW}_t(\Sigma)$. From [**64**] Theorem 1.3, we know that ${}^{\mathrm{JL}}\Pi$ is distinguished by $\mathrm{Sp}_{2n}(\mathbb{A})$, hence [**3**] Theorem implies that ${}^{\mathrm{JL}}\Pi$ is not cuspidal. Thus $t = 2$. □

14.5. From now on and until the end of Section 14, we assume that Π is a cuspidal automorphic representation of $\mathrm{G}_t(\mathbb{A})$ having a non-zero $\mathrm{H}_t(\mathbb{A})$-period.

Fix a finite set S of places of k containing the set S_∞ of Archimedean places and such that, for all $v \notin S$, one has

(1) the k_v-algebra B_v is split,

(2) the character ψ_v has conductor $\mathcal{O}_v$ if v is finite,
(3) the local component Π_v is unramified.

Fix a $\varphi \in \Pi \times \cdots \times \Pi \subseteq \mathcal{A}_{\mathrm{P}}^{\Pi}(G)$ and assume that it decomposes as a tensor product

$$\varphi = \bigotimes_v \varphi_v. \tag{14.4}$$

We further assume that, for any place $v \notin S$, the vector φ_v is $\mathrm{GL}_{2n}(\mathcal{O}_v)$-spherical in $\Pi_v \times \cdots \times \Pi_v$ and normalized by requiring that its value at $1_{2n,v}$ is ϕ_v, as in the discussion before Proposition 14.1 above.

By its very definition, and up to potentially modifying one of the linear forms μ_v at one place by a non-zero scalar, the global intertwining period factorises into the product of local intertwining periods as

$$J(s,\varphi) = \prod_v J_v(s, \varphi_{v,s}, \mu_v)$$

where $\varphi_{v,s}$ is the flat section in $\Pi_v\nu^{s(m-1)} \times \cdots \times \Pi_v\nu^{s(1-m)}$ such that $\varphi_{v,0} = \varphi_v$. (As usual, ν denotes the character "normalized absolute value of the reduced norm".)

We set

$$J_S(s,\varphi) = \prod_{v\in S} J_v(s, \varphi_{v,s}, \mu_v), \quad J^S(s,\varphi) = \prod_{v\notin S} J_v(s, \varphi_{v,s}, \mu_v).$$

As in (13.3), for each place v, there exists a meromorphic function $\alpha_v(s)$ satisfying

$$J_v(-s, M_v(s,w)\varphi_{v,s}, \mu_v) = \alpha_v(s) J_v(s, \varphi_{v,s}, \mu_v) \tag{14.5}$$

where $w = w_m$ is the longest element of $W(M_v) \simeq \mathfrak{S}_m$ and $M_v(s,w)$ is the standard intertwining operator from $\Pi_v\nu^{s(m-1)} \times \cdots \times \Pi_v\nu^{s(1-m)}$ to $\Pi_v\nu^{s(1-m)} \times \cdots \times \Pi_v\nu^{s(m-1)}$. From the global functional equation Proposition 14.1(2), we obtain

$$\begin{aligned} \prod_{v\in S} \alpha_v(s) &= \frac{J_S(-s, M(s,w)\varphi)}{J_S(s,\varphi)} \\ &= \frac{J(-s, M(s,w)\varphi) J^S(s,\varphi)}{J(s,\varphi) J^S(-s, M(s,w)\varphi)} \\ &= \frac{J^S(s,\varphi)}{J^S(-s, M(s,w)\varphi)}. \end{aligned} \tag{14.6}$$

Hence, the functional equations of $J_S(s,\varphi)$ and $J^S(s,\varphi)$ are related by inversion.

14.6. Recall that, according to Lemma 14.2, there is a cuspidal automorphic representation Σ of $\mathrm{GL}_t(\mathbb{A})$ such that ${}^{\mathrm{JL}}\Pi = \mathrm{MW}_2(\Sigma)$. Fix a place $v \notin S - S_\infty$, and write $\sigma = \Sigma_v$. By our choice of S made at the beginning of §14.5, the representation σ is a generic irreducible principal series representation of $\mathrm{GL}_t(k_v)$, which is moreover unramified when v is finite.

Since B_v is split, the local components of Π and ${}^{\mathrm{JL}}\Pi$ at v are isomorphic. We thus have

$$\Pi_v \simeq \mathrm{MW}_2(\Sigma)_v \simeq \mathrm{Sp}_2(\sigma) \tag{14.7}$$

where the notation $\mathrm{Sp}_2(\sigma)$ has been defined earlier in §12.6 and the last isomorphism comes from [**41**] I.11. We fix such an isomorphism, and identify Π_v and $\mathrm{Sp}_2(\sigma)$ with no further notice, when convenient.

Let $\varOmega_t$ denote the diagonal matrix of $\mathrm{GL}_t(k_v)$ with diagonal entries $(1,-1,1,\dots,(-1)^{t-1})$. It has already been defined by (2.3) when t is even, and we have

$$\varOmega_{2t} = \begin{pmatrix} \varOmega_t & \\ & (-1)^t\varOmega_t \end{pmatrix}.$$

Given any representation π of $\mathrm{GL}_t(k_v)$, let $\pi^\diamond$ be the representation $g \mapsto \pi(g^\diamond)$ of $\mathrm{GL}_t(k_v)$, where $g^\diamond = \varOmega_t \cdot {}^{\mathsf{t}}g^{-1} \cdot \varOmega_t$ for any $g \in \mathrm{GL}_t(k_v)$. If π is irreducible, then $\pi^\diamond$ is isomorphic to $\pi^\vee$ (thanks to [**10**] Theorem 7.3 if v is non-Archimedean and [**1**] Theorem 8.2.1 if v is Archimedean). We observe that the intersection of $\mathrm{Sp}_{2t}(k_v)$ with the standard Levi subgroup $\mathrm{GL}_t(k_v)\times\mathrm{GL}_t(k_v)$ of $\mathrm{GL}_{2t}(k_v)$ is equal to the group $C = \{\mathrm{diag}(g,g^\diamond) \mid g \in \mathrm{GL}_t(k_v)\}$.

The representation $\sigma\nu^{1/2}\times\sigma\nu^{-1/2}$ affords a closed intertwining period, given by a compact integration which we now describe. We first need to describe the inducing linear form. For this, we fix an isomorphism between σ and the parabolically induced representation $\mathrm{Ind}_B^{\mathrm{GL}_t(k_v)}(\chi)$ for some character χ of $(k_v^\times)^t$, where B is the upper triangular Borel subgroup of $\mathrm{GL}_t(k_v)$. We identify these two representations with no further notice, when convenient.

By irreducibility of σ, there exists a unique up to scalar isomorphism

$$M : \mathrm{Ind}_B^{\mathrm{GL}_t(k_v)}(\chi) \simeq \mathrm{Ind}_B^{\mathrm{GL}_t(k_v)}(\chi^{w_t}) \tag{14.8}$$

where we recall that w_t is the longest element of $\mathfrak{S}_t$. We then set, for f_1 and f_2 in σ:

$$\gamma(f_1\otimes f_2) = \int_{B\backslash\mathrm{GL}_t(k_v)} f_1(g)Mf_2(g^\diamond)dg.$$

The map γ is a non-zero C-invariant linear form on the space of $\sigma\otimes\sigma$. Note that, for any complex numbers $a,b\in\mathbb{C}$, the representations $\sigma\otimes\sigma$ and $\sigma\nu^a\otimes\sigma\nu^b$ have the same underlying space. Then, for $f\in\sigma\nu^{1/2}\times\sigma\nu^{-1/2}$, one can consider the well-defined integral

$$\widetilde{\varLambda}(f) = \int_{(R\cap\mathrm{H}_t(k_v))\backslash\mathrm{H}_t(k_v)} \gamma(f(h))dh$$

where R is the standard parabolic subgroup of $\mathrm{G}_t(k_v)\simeq\mathrm{GL}_{2t}(k_v)$ associated with the composition (t,t). The map $\widetilde{\varLambda}$ is a non-zero $\mathrm{H}_t(k_v)$-invariant linear form on $\sigma\nu^{1/2}\times\sigma\nu^{-1/2}$, which is the closed intertwining period we were referring to above. We now fix a quotient map

$$\mathsf{p} : \sigma\nu^{1/2}\times\sigma\nu^{-1/2} \to \mathrm{Sp}_2(\sigma) = \Pi_v.$$

It follows from Lemma 12.4 that there is a linear form $\varLambda$ on Π_v such that $\widetilde{\varLambda} = \varLambda\circ\mathsf{p}$. For $s\in\mathbb{C}$, we set

$$\begin{aligned} \widetilde{\tau}_s &= (\sigma\nu^{1/2}\times\sigma\nu^{-1/2})\nu^{s(m-1)}\otimes\cdots\otimes(\sigma\nu^{1/2}\times\sigma\nu^{-1/2})\nu^{s(1-m)}, \\ \tau_s &= \Pi_v\nu^{s(m-1)}\otimes\cdots\otimes\Pi_v\nu^{s(1-m)}. \end{aligned}$$

The map p induces a quotient map from $\widetilde{\tau}_s$ to τ_s for all s, inducing the surjection

$$\mathsf{q}_s : \mathrm{Ind}_{P_v}^{G_v}(\widetilde{\tau}_s) \to \mathrm{Ind}_{P_v}^{G_v}(\tau_s) = \Pi_v\nu^{s(m-1)}\times\cdots\times\Pi_v\nu^{s(1-m)}$$

where we recall that $P_v = M_v N_v$ is the standard parabolic subgroup of G_v corresponding to the composition $(2t, \ldots, 2t)$. Note that if $\widetilde{\varphi}_s$ is a flat section of $\mathrm{Ind}_{P_v}^{G_v}(\widetilde{\tau}_s)$, then $\varphi_s = \mathsf{q}_s(\widetilde{\varphi}_s)$ is a flat section of $\mathrm{Ind}_{P_v}^{G_v}(\tau_s)$. Finally, write $\widetilde{\tau} = \widetilde{\tau}_0$ and $\tau = \tau_0$ and set

$$\widetilde{\mu} = \widetilde{\Lambda} \otimes \cdots \otimes \widetilde{\Lambda} \in \mathrm{Hom}_{\mathrm{M}_x(k_v)}(\widetilde{\tau}, \mathbb{C}), \quad \mu = \Lambda \otimes \cdots \otimes \Lambda \in \mathrm{Hom}_{\mathrm{M}_x(k_v)}(\tau, \mathbb{C}).$$

We observe that, up to modifying the isomorphism M of (14.8) above by a non-zero scalar, the linear form μ agrees with μ_v in (14.2). We now consider the open intertwining periods

$$J_{P_v,\widetilde{\tau}}(s, \widetilde{\varphi}_s, \widetilde{\mu}) = \int_{(\eta^{-1} P_v \eta \cap H_v)\backslash H_v} \widetilde{\mu}(\widetilde{\varphi}_s(\eta h))\, \mathrm{d}h,$$

and

$$J_{P_v,\tau}(s, \varphi_s, \mu) = \int_{(\eta^{-1} P_v \eta \cap H_v)\backslash H_v} \mu(\varphi_s(\eta h))\, \mathrm{d}h.$$

The convergence and meromorphic continuation of these integral are proved in [**12**] and [**14**], and we refer to [**38**] for further generalization and properties. By definition, it is immediate that

$$J_{P_v,\widetilde{\tau}}(s, \widetilde{\varphi}_s, \widetilde{\mu}) = J_{P_v,\tau}(s, \varphi_s, \mu) \tag{14.9}$$

whenever the flat sections $\widetilde{\varphi}_s$ and φ_s are related by $\varphi_s = \mathsf{q}_s(\widetilde{\varphi}_s)$.

Recall that $w = w_m$ is the longest element of $W(M_v) \simeq \mathfrak{S}_m$, that is, we have $w(i) = m+1-i$ for $i = 1, 2, \ldots, m$. Let $w(\widetilde{\tau}_s)$ be the representation of M_v defined as

$$\begin{aligned} w(\widetilde{\tau}_s) &= (\sigma\nu^{1/2} \times \sigma\nu^{-1/2})\nu^{s(1-m)} \otimes \cdots \otimes (\sigma\nu^{1/2} \times \sigma\nu^{-1/2})\nu^{s(m-1)} \\ &= (\sigma\nu^{1/2} \times \sigma\nu^{-1/2})\nu^{-s(m-1)} \otimes \cdots \otimes (\sigma\nu^{1/2} \times \sigma\nu^{-1/2})\nu^{-s(1-m)} \\ &= \widetilde{\tau}_{-s}. \end{aligned}$$

Let $M_{P_v,\widetilde{\tau}}(s, w)$ denote the standard intertwining operator

$$\mathrm{Ind}_{P_v}^{G_v}(\widetilde{\tau}_s) \to \mathrm{Ind}_{P_v}^{G_v}(w(\widetilde{\tau}_s)) = \mathrm{Ind}_{P_v}^{G_v}(\widetilde{\tau}_{-s}).$$

In this notation, (14.5) becomes

$$J_{P_v,\widetilde{\tau}}(-s, M_{P_v,\widetilde{\tau}}(s, w)\widetilde{\varphi}_s, \widetilde{\mu}) = \alpha_v(s) J_{P_v,\widetilde{\tau}}(s, \widetilde{\varphi}_s, \widetilde{\mu}). \tag{14.10}$$

Similarly, let $\mathrm{Q} = \mathrm{LV}$ be the standard parabolic subgroup of GL_{2n} associated with $(t, \ldots, t)$. We observe that the Q_v-orbit of x is L_v-admissible, although it is not open anymore. Let λ denote the linear form

$$\lambda = \gamma \otimes \cdots \otimes \gamma \in \mathrm{Hom}_{\mathrm{L}_x(k_v)}((\sigma \otimes \sigma) \otimes \cdots \otimes (\sigma \otimes \sigma), \mathbb{C})$$

where L_x denotes the stabilizer of x in L. Note that $\mathrm{L}_x(k_v)$ is equal to the subgroup $C \times \cdots \times C$ (where C occurs m times). For $s \in \mathbb{C}$ and $i = 1, \ldots, m$, write $\sigma_{i,s} = \sigma\nu^{s(m-2i+1)}$ and

$$\omega_s = \sigma_{1,s}\, \nu^{1/2} \otimes \sigma_{1,s}\, \nu^{-1/2} \otimes \cdots \otimes \sigma_{m,s}\nu^{1/2} \otimes \sigma_{m,s}\nu^{-1/2}$$

and $\omega = \omega_0$. Given a flat section

$$f_s \in \sigma_{1,s}\, \nu^{1/2} \times \sigma_{1,s}\, \nu^{-1/2} \times \cdots \times \sigma_{m,s}\nu^{1/2} \times \sigma_{m,s}\nu^{-1/2} = \mathrm{Ind}_{Q_v}^{G_v}(\omega_s)$$

we consider the local intertwining period, in the sense of [**38**], given by the meromorphic continuation of the integral

$$J_{Q_v,\omega}(s,f_s,\lambda)=\int_{(\eta^{-1}Q_v\eta\cap H_v)\backslash H_v}\lambda(f_s(\eta h))e^{\langle -\rho',H_{Q_v}(\eta h)\rangle}\,\mathrm{d}h,$$

where $\rho'=(1/2,-1/2,\dots,1/2,-1/2)\in\mathfrak{a}^*_{Q_v}$. We observe that this is not an open intertwining period anymore, however it is well defined, since the modulus assumption of [**38**], namely $\delta_{\mathrm{Q}_x(k_v)}=\delta_{Q_v}^{1/2}$ on $\mathrm{L}_x(k_v)$, is satisfied. Suppose that f_s and $\widetilde{\varphi}_s$ correspond to each other under the natural isomorphism between the induced representations $\mathrm{Ind}_{Q_v}^{G_v}(\omega_s)$ and $\mathrm{Ind}_{P_v}^{G_v}(\widetilde{\tau}_s)$. Then we have

$$J_{Q_v,\omega}(s,f_s,\lambda)=J_{P_v,\widetilde{\tau}}(s,\widetilde{\varphi}_s,\widetilde{\mu}) \tag{14.11}$$

by [**36**] Proposition 3.7, or rather its proof, which requires only the unimodularity of the vertices involved.

Let $\widetilde{w}$ denote the element of $W(L_v)\simeq\mathfrak{S}_{2m}$ given by

$$\widetilde{w}(k)=\begin{cases}2(m+1-i) & \text{if } k=2i \text{ is even,}\\ 2(m+1-i)-1 & \text{if } k=2i-1 \text{ is odd}\end{cases}$$

for $k=1,2,\dots,2m$. Let $\widetilde{w}(\omega_s)$ be the representation of L defined as

$$\begin{aligned}\widetilde{w}(\omega_s)&=\sigma_{m,s}\nu^{1/2}\otimes\sigma_{m,s}\nu^{-1/2}\otimes\cdots\otimes\sigma_{1,s}\,\nu^{1/2}\otimes\sigma_{1,s}\,\nu^{-1/2}\\&=\sigma_{1,-s}\nu^{1/2}\otimes\sigma_{1,-s}\nu^{-1/2}\otimes\cdots\otimes\sigma_{m,-s}\nu^{1/2}\otimes\sigma_{m,-s}\nu^{-1/2}\\&=\omega_{-s}.\end{aligned}$$

Let $M_{Q_v,\omega}(s,\widetilde{w})$ denote the standard intertwining operator

$$\mathrm{Ind}_{Q_v}^{G_v}(\omega_s)\to\mathrm{Ind}_{Q_v}^{G_v}(\widetilde{w}(\omega_s))=\mathrm{Ind}_{Q_v}^{G_v}(\omega_{-s}).$$

Then the functional equation (14.10) can be rewritten as

$$J_{Q_v,\omega}(-s,M_{Q_v,\omega}(s,\widetilde{w})f_s,\lambda)=\alpha_v(s)J_{Q_v,\omega}(s,f_s,\lambda) \tag{14.12}$$

for any flat section f_s as above.

14.7. Let w' and w'' be the elements of $W(L)\simeq\mathfrak{S}_{2m}$ defined by

$$w'(i)=\begin{cases}2m+1-i/2 & \text{if } i \text{ is even,}\\ (i+1)/2 & \text{if } i \text{ is odd,}\end{cases}\qquad w''(i)=\begin{cases}m+i/2 & \text{if } i \text{ is even,}\\ m+1-(i+1)/2 & \text{if } i \text{ is odd,}\end{cases}$$

for $i=1,2,\dots,2m$, so that we have $w''\widetilde{w}=w'$. We define a linear form λ' on the representation

$$w'(\omega_s)=(\sigma_{1,s}\,\nu^{1/2}\otimes\cdots\otimes\sigma_{m,s}\nu^{1/2})\otimes(\sigma_{m,s}\nu^{-1/2}\otimes\cdots\otimes\sigma_{1,s}\,\nu^{-1/2}).$$

of L by

$$\lambda'(x_1\otimes\dots x_m\otimes y_m\otimes\cdots\otimes y_1)=\prod_{i=1}^{m}\lambda_i(x_i\otimes y_i)$$

for $x_i\in\sigma_{i,s}\nu^{1/2}$ and $y_i\in\sigma_{i,s}\nu^{-1/2}$. Given a flat section

$$f'_s\in\mathrm{Ind}_{Q_v}^{G_v}(w'(\omega_s))=(\sigma_{1,s}\,\nu^{1/2}\times\cdots\times\sigma_{m,s}\nu^{1/2})\times(\sigma_{m,s}\nu^{-1/2}\times\cdots\times\sigma_{1,s}\,\nu^{-1/2})$$

we have the closed intertwining period

$$J'_{Q_v,\omega}(s, f'_s, \lambda') = \int_{(Q_v \cap H_v)\backslash H_v} \lambda'(f'_s(h))dh.$$

Let $M_{Q_v,\omega}(s, w')$ denote the standard intertwining operator from $\mathrm{Ind}_{Q_v}^{G_v}(\omega_s)$ to $\mathrm{Ind}_{Q_v}^{G_v}(w'(\omega_s))$. As before, suppose that the flat sections f_s and φ_s correspond to each other under the natural isomorphism $\mathrm{Ind}_{Q_v}^{G_v}(\omega_s) \simeq \mathrm{Ind}_{P_v}^{G_v}(\widetilde{\tau}_s)$. Then we have

$$\begin{aligned} J_{P_v,\widetilde{\tau}}(s, \widetilde{\varphi}_s, \widetilde{\mu}) &= J_{Q_v,\omega}(s, f_s, \lambda) \\ &= J'_{Q_v,\omega}(s, M_{Q_v,\omega}(s, w')f_s, \lambda'). \end{aligned} \tag{14.13}$$

The first equality is (14.11) and the second equality is verified by repeated use of [**38**] Proposition 5.1 along an appropriate reduced expression of $w' \in \mathfrak{S}_{2m}$. Consider flat sections f_s^-, $\widetilde{\varphi}_s^-$ in $\mathrm{Ind}_{Q_v}^{G_v}(\omega_{-s})$ and $\mathrm{Ind}_{P_v}^{G_v}(\tau_{-s})$ respectively, corresponding to each other under the natural isomorphism $\mathrm{Ind}_{Q_v}^{G_v}(\omega_{-s}) \simeq \mathrm{Ind}_{P_v}^{G_v}(\widetilde{\tau}_{-s})$. Let $M_{Q_v,\widetilde{w}(\omega)}(s, w'')$ be the standard intertwining operator

$$\mathrm{Ind}_{Q_v}^{G_v}(\omega_{-s}) = \mathrm{Ind}_{Q_v}^{G_v}(\widetilde{w}(\omega_s)) \to \mathrm{Ind}_{Q_v}^{G_v}(w''\widetilde{w}(\omega_s)) = \mathrm{Ind}_{Q_v}^{G_v}(w'(\omega_s)).$$

From Equation (14.11), and applying [**38**] Proposition 5.1 repeatedly along an appropriate reduced expression of $w'' \in \mathfrak{S}_{2m}$, we obtain

$$\begin{aligned} J_{P_v,\widetilde{\tau}}(-s, \widetilde{\varphi}_s^-, \widetilde{\mu}) &= J_{Q_v,\omega}(-s, f_s^-, \lambda) \\ &= J'_{Q_v,\omega}(s, M_{Q_v,\widetilde{w}(\omega)}(s, w'')f_s^-, \lambda'). \end{aligned} \tag{14.14}$$

In the rest of this section, we write

$$L_\sigma(s) = L(s, \sigma, \sigma^\vee), \quad \varepsilon_\sigma(s) = \varepsilon(s, \sigma, \sigma^\vee, \psi_v), \quad \gamma_\sigma(s) = \gamma(s, \sigma, \sigma^\vee, \psi_v).$$

Following [**41**] I.1, let us introduce the normalized intertwining operators $N_\omega(s, w')$, $N_{\widetilde{w}(\omega)}(s, w'')$ and $N_\omega(s, \widetilde{w})$ defined as

$$\begin{aligned} N_\omega(s, w') &= r_\omega(s, w')^{-1} M_{Q_v,\omega}(s, w'), \\ N_{\widetilde{w}(\omega)}(s, w'') &= r_{\widetilde{w}(\omega)}(s, w'')^{-1} M_{Q_v,\widetilde{w}(\omega)}(s, w''), \\ N_\omega(s, \widetilde{w}) &= r_\omega(s, \widetilde{w})^{-1} M_{Q_v,\omega}(s, \widetilde{w}), \end{aligned}$$

where $r_\omega(s, w')$, $r_{\widetilde{w}(\omega)}(s, w'')$ and $r_\omega(s, \widetilde{w})$ are meromorphic functions given by

$$\begin{aligned} r_\omega(s, w') &= \prod_{1 \leqslant i < j \leqslant m} \frac{L_\sigma(2(j-i)s - 1)}{\varepsilon_\sigma(2(j-i)s - 1)\varepsilon_\sigma(2(j-i)s)L_\sigma(2(j-i)s + 1)}, \\ r_{\widetilde{w}(\omega)}(s, w'') &= \prod_{1 \leqslant i < j \leqslant m} \frac{L_\sigma(2(i-j)s - 1)}{\varepsilon_\sigma(2(i-j)s - 1)\varepsilon_\sigma(2(i-j)s)L_\sigma(2(j-i)s + 1)}, \end{aligned}$$

and

$$\begin{aligned} r_\omega(s, \widetilde{w}) = \prod_{1 \leqslant i < j \leqslant m} &\varepsilon_\sigma(2(j-i)s)^{-1}\varepsilon_\sigma(2(j-i)s+1)^{-1}\varepsilon_\sigma(2(j-i)s-1)^{-1}\varepsilon_\sigma(2(j-i)s)^{-1} \\ &\times \prod_{1 \leqslant i < j \leqslant m} \frac{L_\sigma(2(j-i)s - 1)L_\sigma(2(j-i)s)}{L_\sigma(2(j-i)s + 2)L_\sigma(2(j-i)s + 1)}. \end{aligned}$$

From [**41**] I.1 we have

$$N_{\widetilde{w}(\omega)}(s, w'') \circ N_\omega(s, \widetilde{w}) = N_\omega(s, w').$$

This is equivalent to

$$M_{Q_v,\widetilde{w}(\omega)}(s, w'') \circ M_{Q_v,\omega}(s, \widetilde{w}) = \kappa_v(s) M_{Q_v,\omega}(s, w'), \tag{14.15}$$

where we set $\kappa_v(s) = r_{\widetilde{w}(\omega)}(s, w'') \cdot r_\omega(s, \widetilde{w}) \cdot r_\omega(s, w')^{-1}$, that is

$$\kappa_v(s) = \prod_{1\leqslant i<j\leqslant m} \varepsilon_\sigma(2(j-i)s)^{-1} \varepsilon_\sigma(2(j-i)s+1)^{-1} \varepsilon_\sigma(2(i-j)s-1)^{-1} \varepsilon_\sigma(2(i-j)s)^{-1} \times \prod_{1\leqslant i<j\leqslant m} \frac{L_\sigma(2(j-i)s) L_\sigma(2(i-j)s-1)}{L_\sigma(2(j-i)s+2) L_\sigma(2(i-j)s+1)}.$$

From (14.15), (14.14) and (14.13), we obtain

$$\begin{aligned} J_{Q_v,\omega}(-s, M_{Q_v,\omega}(s, \widetilde{w}) f_s, \lambda) &= J'_{Q_v,\omega}(s, M_{Q_v,\widetilde{w}(\omega)}(s, w'') \circ M_{Q_v,\omega}(s, \widetilde{w}) f_s, \lambda') \\ &= \kappa_v(s) J'_{Q_v,\omega}(s, M_{Q_v,\omega}(s, w') f_s, \lambda') \\ &= \kappa_v(s) J_{Q_v,\omega}(s, f_s, \lambda). \end{aligned}$$

Comparing with the functional equation (14.12), we deduce that $\alpha_v(s) = \kappa_v(s)$. We state this result as a proposition.

Proposition 14.3. — *Let*

$$M_{Q_v,\omega}(s, \widetilde{w}) : \mathrm{Ind}_{Q_v}^{G_v}(\omega_s) \to \mathrm{Ind}_{Q_v}^{G_v}(\omega_{-s})$$

denote the standard intertwining operator. For any flat section $f_s \in \mathrm{Ind}_{Q_v}^{G_v}(\omega_s)$, we have the functional equation of local intertwining period

$$J_{Q_v,\omega}(-s, M_{Q_v,\omega}(s, \widetilde{w}) f_s, \lambda) = \alpha_v(s) J_{Q_v,\omega}(s, f_s, \lambda)$$

with

$$\alpha_v(s) = \prod_{1\leqslant i<j\leqslant m} \varepsilon_\sigma(2(j-i)s)^{-1} \varepsilon_\sigma(2(j-i)s+1)^{-1} \varepsilon_\sigma(2(i-j)s-1)^{-1} \varepsilon_\sigma(2(i-j)s)^{-1} \times \prod_{1\leqslant i<j\leqslant m} \frac{L_\sigma(2(j-i)s) L_\sigma(2(i-j)s-1)}{L_\sigma(2(j-i)s+2) L_\sigma(2(i-j)s+1)}. \tag{14.16}$$

If σ and ψ_v are moreover unramified, which happens if v is finite, then $\varepsilon_\sigma(s) = 1$ and (14.16) simplifies to

$$\alpha_v(s) = \prod_{1\leqslant i<j\leqslant m} \frac{\gamma_\sigma(2(j-i)s+2)}{\gamma_\sigma(2(j-i)s)}.$$

This observation also follows from the unramified computation of intertwining periods below, together with the Gindikin–Karpelevich formula.

Proposition 14.4. — *Suppose that v is finite, that σ is unramified, and that $\varphi \in \Pi_v \times \cdots \times \Pi_v$ is the spherical vector such that $\mu(\varphi(1_{2n,v})) = 1$. Then the following equality holds good:*

$$J_{P_v,\mu}(s,\varphi_s,\mu) = \prod_{1\leqslant i<j\leqslant m} \frac{L_\sigma(2(j-i)s-1)}{L_\sigma(2(j-i)s+1)}.$$

Proof. — Putting Equations (14.9) and (14.13) together, we obtain

$$J_{P_v,\tau}(s,\varphi_s,\mu) = J'_{Q_v,\omega}(s, M_{Q_v,\omega}(s,w')f_s,\lambda'),$$

where φ_s and f_s are any flat sections related as in the above discussion. If φ is $\mathrm{GL}_{2n}(\mathcal{O}_v)$-spherical and normalized as in the statement, which is as in the discussion before Theorem 14.1, then $\widetilde{\varphi}$ can be chosen to be $\mathrm{GL}_{2n}(\mathcal{O}_v)$-spherical, and $\widetilde{\mu}(\widetilde{\varphi}(1_{2n,v})) = \mu(\varphi(1_{2n,v})) = 1$. In turn this implies that the function f is $\mathrm{GL}_{2n}(\mathcal{O}_v)$-spherical, and that one has $\lambda(f(1_{2n,v})) = 1$. The computation for $J'_{Q_v,\omega}(s, M_{Q_v,\omega}(s,w')f_s,\lambda')$ now follows from the Gindikin–Karpelevich formula in the form recalled in [**37**] Lemma 9.1. □

15. Conclusion

We now deduce Proposition 13.5 from the computations in Section 14. Thanks to §13.3, this will end the proof of Theorem 1.3.

Recall that F is a non-Archimedean locally compact field of characteristic 0, and D is a non-split quaternion F-algebra. We also have a discrete series representation π of $G = \mathrm{GL}_n(D)$ distinguished by $H = \mathrm{Sp}_n(D)$. It is of the form $\mathrm{St}_m(\rho)$ for some divisor m of n and some cuspidal representation ρ of $\mathrm{GL}_t(D)$ distinguished by $\mathrm{Sp}_t(D)$, with $t = n/m$.

For any unramified character χ of G, the representation $\pi\chi$ is a discrete series representation of G distinguished by H, which is cuspidal if and only if π is cuspidal. Without loss of generality, we thus may (and will) assume that π, or equivalently ρ, is unitary.

Let us fix

– a totally imaginary number field k with a finite place u so that $k_u = F$ and u is the only place of k above p,

– a quaternion division algebra B over k which is non-split at u.

By Lemma 5.1, there exists an irreducible cuspidal automorphic representation Π of $\mathrm{GL}_t(B\otimes_k\mathbb{A})$ such that $\Pi_u \simeq \rho$ and Π is $\mathrm{Sp}_t(B\otimes_k\mathbb{A})$-distinguished, that is, there is an $f\in\Pi$ such that

$$\int_{\mathrm{Sp}_t(B)\backslash\mathrm{Sp}_t(B\otimes_k\mathbb{A})} f(h)\,\mathrm{d}h \neq 0.$$

By Lemma 14.2, there is a cuspidal automorphic representation Σ of $\mathrm{GL}_t(\mathbb{A})$ such that ${}^{\mathrm{JL}}\Pi$ is equal to $\mathrm{MW}_2(\Sigma)$. Let S be the finite set of places of k consisting of u, the Archimedean places of k and all finite places v such that Σ_v is ramified. Let S_∞ be the set of Archimedean places of k. The next proposition is a consequence of Propositions 11.1, 14.3, 14.4 and [**37**] Lemma 9.1.

Proposition 15.1. — *Let $\varphi \in \Pi \times \cdots \times \Pi$ be the automorphic form fixed in* (14.4) *where, for any place $v \notin S$, the vector φ_v is the unique* $\mathrm{GL}_{2n}(\mathcal{O}_v)$*-spherical function in $\Pi_v \times \cdots \times \Pi_v$ given by Proposition 14.4. We have*

$$J^S(s,\varphi) = \prod_{1\leqslant i<j\leqslant m} \frac{L^S(2(j-i)s-1,\Sigma,\Sigma^\vee)}{L^S(2(j-i)s+1,\Sigma,\Sigma^\vee)}$$

and hence

$$\frac{J^S(-s,M(s,w)\varphi)}{J^S(s,\varphi)} = \prod_{v\in S}\prod_{1\leqslant i<j\leqslant m} \frac{\gamma(2(j-i)s,\Sigma_v,\Sigma_v^\vee,\psi_v)}{\gamma(2(j-i)s+2,\Sigma_v,\Sigma_v^\vee,\psi_v)}.$$

For f and g two functions of a complex variable s, write $f(s) \sim g(s)$ if there exists a $c \in \mathbb{C}^\times$ such that $g(s) = cf(s)$.

Theorem 15.2. — *With the above notation, we have*

$$\alpha(s,\rho) \sim \prod_{1\leqslant i<j\leqslant m} \frac{\gamma(2(j-i)s+2,\Sigma_u,\Sigma_u^\vee,\psi_u)}{\gamma(2(j-i)s,\Sigma_u,\Sigma_u^\vee,\psi_u)}.$$

Proof. — Recall from (14.6) that we have

$$\frac{J^S(s,\varphi)}{J^S(-s,M(s,w)\varphi)} = \prod_{s\in S}\alpha_v(s).$$

By Proposition 15.1, we obtain

$$\prod_{v\in S}\alpha_v(s) = \prod_{v\in S}\prod_{1\leqslant i<j\leqslant m} \frac{\gamma(2(j-i)s+2,\Sigma_v,\Sigma_v^\vee,\psi_v)}{\gamma(2(j-i)s,\Sigma_v,\Sigma_v^\vee,\psi_v)}.$$

Since Archimedean root numbers are constant (see [**30**] 16 Appendix), it follows from Proposition 14.3 for Archimedean places that we can simplify the above identity to

$$\prod_{v\in S\setminus S_\infty}\alpha_v(s) \sim \prod_{v\in S\setminus S_\infty}\prod_{1\leqslant i<j\leqslant m} \frac{\gamma(2(j-i)s+2,\Sigma_v,\Sigma_v^\vee,\psi_v)}{\gamma(2(j-i)s,\Sigma_v,\Sigma_v^\vee,\psi_v)}.$$

Since u is the only place above p and $\alpha(s,\rho) = \alpha_u(s)$, the assertion now follows from [**37**] Lemma 9.3. □

Now we obtain the next proposition as promised in §13.2.

Proposition 15.3. — *The meromorphic function $\alpha(s,\rho)$ is holomorphic and non-zero at $s = 1$.*

Proof. — Note that Σ_u is cuspidal and $\mathrm{JL}(\rho) = \mathrm{St}_2(\Sigma_u)$. From the properties of L factors recalled in Section 11.3 and central character considerations, $\gamma(s,\Sigma_u,\Sigma_u^\vee,\psi_u)$ and its inverse can only vanish for s of real part equal to 0 or 1. It then follows from Theorem 15.2 that each quotient of γ-factors in the formula for $\alpha(s,\rho)$ is holomorphic and non-vanishing at $s = 1$. □

16. Appendix

In this section, A is as in Section 5. Let $[\mathfrak{a},\beta]$ be a simple stratum in A. Let ψ^A denote the character $x \mapsto \psi(\mathrm{trd}_{A/F}(x))$ of A, where $\mathrm{trd}_{A/F}$ is the reduced trace of A over F.

We prove Lemma 16.1 (which has been used in Section 7), whose proof was postponed to this last section since it requires techniques which are not used anywhere else in the paper.

Lemma 16.1. — *Let $\theta \in \mathcal{C}(\mathfrak{a},\beta)$ be a simple character. Then*

(1) *$[\mathfrak{a}^*,\beta^*]$ is a simple stratum realizing β, and*
(2) *θ^* is a simple character in $\mathcal{C}(\mathfrak{a}^*,\beta^*)$.*

Proof. — If $[\mathfrak{a},\beta]$ is the null stratum, there is nothing to prove. We will thus assume that $[\mathfrak{a},\beta]$ has positive depth.

The map $\iota : x \mapsto x^*$ is an F-linear involution of A such that $\iota(xy) = \iota(y)\iota(x)$. Restricting to the commutative F-algebra $E = F[\beta]$, it is thus an embedding of F-algebras from E to A. This proves (1). Note that, if B is the centralizer of E in A, then the centralizer of E^* in A is B^*.

Lemma 16.2. — *One has $\mathrm{Nrd}_{B/E}(x^*) = \mathrm{Nrd}_{B^*/E^*}(x)^*$ for all $x \in B^*$.*

Proof. — The proof is similar to [**54**] Lemme 5.15. □

We will prove (2) by induction on the integer $q = -k_0(\mathfrak{a},\beta)$. (See [**51**] §2.1 for the definition of $k_0(\mathfrak{a},\beta)$.) Define $r = \lfloor q/2 \rfloor + 1$. First note that $k_0(\mathfrak{a}^*,\beta^*) = k_0(\mathfrak{a},\beta)$ and θ^* is normalized by $\mathcal{K}(\mathfrak{a}^*) \cap B^{*\times}$ as θ is normalized by $\mathcal{K}(\mathfrak{a}) \cap B^\times$. (Here, $\mathcal{K}(\mathfrak{a})$ denotes the normalizer in G of the order $\mathfrak{a}$.) For any integer $i \geqslant 1$, let us write $U^i(\mathfrak{a}) = 1 + \mathfrak{p}_{\mathfrak{a}}^i$.

Assume first that β is minimal over F (see [**51**] §2.3.3). In this case, we have

– $H^1(\mathfrak{a},\beta) = U^1(\mathfrak{b})U^r(\mathfrak{a})$,
– the restriction of θ to $U^r(\mathfrak{a})$ is the character $\psi^A_\beta : 1 + x \mapsto \psi^A(\beta x)$,
– the restriction of θ to $U^1(\mathfrak{b})$ is equal to $\xi \circ \mathrm{Nrd}_{B/E}$ for some character ξ of $1 + \mathfrak{p}_E$.

The character θ^* is defined on the group $\sigma(H^1(\mathfrak{a},\beta)) = U^1(\mathfrak{b}^*)U^r(\mathfrak{a}^*) = H^1(\mathfrak{a}^*,\beta^*)$. Its restriction to $U^r(\mathfrak{a}^*)$ is the character

$$1 + y \mapsto \psi^A(\beta y^*) = \psi^A(\beta^* y) = \psi^A_{\beta^*}(1 + y)$$

since ψ^A is invariant by $*$. By Lemma 16.2, its restriction to $U^1(\mathfrak{b}^*)$ is $\xi^* \circ \mathrm{Nrd}_{B^*/E^*}$ where ξ^* is the character $x \mapsto \xi(x^*)$ of $1 + \mathfrak{p}_{E^*}$. It follows from [**51**] Proposition 3.47 that θ^* is a simple character in $\mathcal{C}(\mathfrak{a}^*,\beta^*)$.

Now assume that β is not minimal over F, and that γ is an approximation of β with respect to $\mathfrak{a}$ (see [**51**] §2.1). We have

– $H^1(\mathfrak{a},\beta) = U^1(\mathfrak{b})H^r(\mathfrak{a},\gamma)$,
– the restriction of θ to $H^r(\mathfrak{a},\gamma)$ is equal to $\psi^A_{\beta-\gamma}\theta'$ for some simple character $\theta' \in \mathcal{C}(\mathfrak{a},\gamma)$,
– the restriction of θ to $U^1(\mathfrak{b})$ is equal to $\xi \circ \mathrm{Nrd}_{B/E}$ for some character ξ of $1 + \mathfrak{p}_E$.

The character θ^* is defined on the group

$$\sigma(H^1(\mathfrak{a},\beta)) = U^1(\mathfrak{b}^*)\sigma(H^1(\mathfrak{a},\gamma)) = U^1(\mathfrak{b}^*)H^1(\mathfrak{a}^*,\gamma^*) = H^1(\mathfrak{a}^*,\beta^*)$$

since γ^* is an approximation of β^* with respect to $\mathfrak{a}^*$. By induction, its restriction to $H^1(\mathfrak{a}^*, \gamma^*)$ is the character $\psi^A_{\beta^*-\gamma^*}\theta'^*$ where $\theta'^* \in \mathcal{C}(\mathfrak{a}^*, \gamma^*)$ is the transfer of θ'. Its restriction to $U^1(\mathfrak{b}^*)$ is the character $\xi^* \circ \mathrm{Nrd}_{B^*/E^*}$. It follows from [**51**] Proposition 3.47 that $\theta^* \in \mathcal{C}(\mathfrak{a}^*, \beta^*)$. □

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