CORRIGENDUM TO: LOCAL TRANSFER FOR QUASI-SPLIT CLASSICAL GROUPS AND CONGRUENCES MOD ℓ

by

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Abstract. — Proposition B.1 of [**1**] is false. We prove a weaker statement which is sufficient for our purpose.

1.

Proposition B.1 of [**1**] is false: given a p-adic field F with $p \neq 2$ and an integer $n \geqslant 2$, the split even special orthogonal group $\mathrm{SO}_{2n}(F)$ has no cuspidal representation of level 0 whose transfer to $\mathrm{GL}_{2n}(F)$ is cuspidal. The error lies in the proof of [**1**] Lemma B.2.

We prove that [**1**] Proposition B.1 holds for the split odd special orthogonal group $\mathrm{SO}_{2n-1}(F)$ and the *unramified* non-split quasi-split even special orthogonal group. We then show that this is enough for proving the main theorem of [**1**].

1.1.

Let p be a prime number different from 2, let F be a p-adic field and let W_F be the absolute Weil group of F.

Let ϕ be an irreducible smooth representation of W_F of dimension $2n$ for some integer $n \geqslant 1$. Suppose that ϕ is self-dual. It is thus

– either symplectic, that is, its image is contained in a conjugate of $\mathrm{Sp}_{2n}(\mathbb{C})$ in $\mathrm{GL}_{2n}(\mathbb{C})$,
– or orthogonal, that is, its image is contained in a conjugate of $\mathrm{O}_{2n}(\mathbb{C})$ in $\mathrm{GL}_{2n}(\mathbb{C})$.

If it is symplectic, it factors through a local Langlands parameter φ for $\mathrm{SO}_{2n+1}(F)$. The packet $\Pi_\varphi(\mathrm{SO}_{2n+1}(F))$ thus contains a cuspidal representation whose transfer to $\mathrm{GL}_{2n}(F)$ is the cuspidal representation with parameter ϕ.

If it is orthogonal, it factors through a Langlands parameter φ for a quasi-split special orthogonal group $\mathrm{SO}^{\alpha}_{2n}(F)$ for some $\alpha \in F^\times/F^{\times 2}$ (see [**1**] 5.1). More precisely (see [**1**] 5.3), the determinant of ϕ corresponds through the local class field theory to the character

$$x \mapsto (\alpha, x)_F \tag{1.1}$$

of $F^\times$, where $(\cdot,\cdot)_F$ is the Hilbert symbol over F. The packet $\Pi_\varphi(\mathrm{SO}^\alpha_{2n}(F))$ associated with the $\mathrm{O}_{2n}(\mathbb{C})$-conjugacy class of φ thus contains a cuspidal representation whose transfer to $\mathrm{GL}_{2n}(F)$ is the cuspidal representation with parameter ϕ.

1.2.

We prove the following result.

Proposition 1.1. — *Suppose that either $G = \mathrm{SO}_{2n+1}(F)$, or $G = \mathrm{SO}^\alpha_{2n}(F)$ for an $\alpha \in F^\times$ such that $F(\sqrt{\alpha})$ is quadratic and unramified over F. Then there is a cuspidal representation of level 0 of G whose transfer to $\mathrm{GL}_{2n}(F)$ is cuspidal.*

Thanks to Paragraph 1.1, it suffices to prove that there exist

– a symplectic self-dual irreducible representation ϕ of W_F of dimension $2n$ of level 0,

– an orthogonal self-dual irreducible representation ϕ of W_F of dimension $2n$ of level 0 whose determinant is unramified and has order 2.

1.3.

Let L be the unramified extension of degree $2n$ of F in $\overline{\mathbb{Q}}_p$, and let $K \subseteq L$ be the unramified extension of degree n of F. Thus L has degre 2 over K. Let

$$\xi : L^\times \to \mathbb{C}^\times$$

be a tamely ramified character such that all conjugates ξ^α, $\alpha \in \mathrm{Gal}(L/F)$, are pairwise distinct. Let η denote the unramified character of $L^\times$ of order 2. Thus

$$\sigma = \mathrm{Ind}_{L/F}(\xi\eta)$$

(where $\mathrm{Ind}_{L/F}$ denotes induction from W_L to W_F) is an irreducible $2n$-dimensional representation of level 0 of W_F. Through local class field theory, the determinant of σ corresponds to the restriction of ξ to $F^\times$. (See for instance [**2**] Theorem 2).

Likewise, $\tau = \mathrm{Ind}_{L/K}(\xi\eta)$ is an irreducible 2-dimensional representation of W_K whose determinant corresponds to the restriction of ξ to $K^\times$. One has $\sigma = \mathrm{Ind}_{K/F}(\tau)$.

Let $\gamma \in \mathrm{Gal}(L/K) \subseteq \mathrm{Gal}(L/F)$ denote the element of order 2. One thus has $L^\gamma = K$. Suppose that σ is self-dual. This is equivalent to $\xi^\gamma = \xi^{-1}$. Indeed, the fact that the representation σ is self-dual implies that $\xi^{-1} = \xi^\alpha$ for some $\alpha \in \mathrm{Gal}(L/F)$. Applying α twice gives $\xi^{\alpha^2} = \xi$, which implies that $\alpha^2 = \mathrm{id}_L$, thus $\alpha = \gamma$ thanks to the regularity assumption on ξ. Note that the restriction of ξ to $K^\times$ is unramified since ξ is trivial on $\mathrm{N}_{L/K}(L^\times)$.

Note that τ is self-dual, with same parity as σ. Indeed, if $\langle\cdot,\cdot\rangle_\tau$ is a τ-invariant ε-symmetric non-degenerate bilinear form on the space of τ, for some sign $\varepsilon \in \{-1, 1\}$, then

$$\langle f, g\rangle_\sigma = \sum_{w \in \mathrm{W}_K\backslash\mathrm{W}_F} \langle f(w), g(w)\rangle_\tau$$

is a σ-invariant ε-symmetric non-degenerate bilinear form on the space of $\sigma = \mathrm{Ind}_{K/F}(\tau)$, where w ranges over a set of representatives of $\mathrm{W}_K\backslash\mathrm{W}_F$ in W_F.

Suppose that ξ is trivial on $K^\times$. Then the representation τ has determinant 1, that is, it takes values in $\mathrm{SL}_2(\mathbb{C}) = \mathrm{Sp}_2(\mathbb{C})$. It is thus symplectic. It follows that σ is symplectic.

Now suppose that ξ is non-trivial on $K^\times$. The representation τ is orthogonal, thus σ is orthogonal. Its determinant is the restriction of ξ to $F^\times$, which is unramified non-trivial (since the restriction of ξ to $K^\times$ is unramified non-trivial and K is unramified over F). It has order 2 since σ is self-dual.

In order to prove Proposition 1.1, it thus remains to prove the existence of a tamely ramified character $\xi : L^\times \to \mathbb{C}^\times$ such that

(1) all conjugates ξ^α, $\alpha \in \mathrm{Gal}(L/F)$, are pairwise distinct,

(2) the restriction of ξ to $K^\times$ is a given character of $K^\times$ trivial on $\mathrm{N}_{L/K}(L^\times)$.

A tamely ramified character ξ of $L^\times$ is entirely determined by

– the character $\chi : \boldsymbol{k}_L^\times \to \mathbb{C}^\times$, where $\boldsymbol{k}_L$ denotes the residue field of L, whose inflation to $\mathcal{O}_L^\times$ is the restriction of the character ξ to $\mathcal{O}_L^\times$,

– the non-zero scalar $z = \xi(\varpi_F) \in \mathbb{C}^\times$, where ϖ_F is a fixed uniformizer of F.

Then the two conditions (1) and (2) are equivalent to the following two conditions:

(1′) all conjugates $\chi, \chi^q, \dots, \chi^{q^{2n-1}}$ are pairwise distinct, where q is the cardinality of the residue field of F,

(2′) one has $\chi^{-1} = \chi^{q^n}$ and the scalar z takes a given value in $\{-1, 1\}$.

The existence of characters ξ satisfying the conditions (1) and (2) thus follows for instance from (the proof of) [**3**] Lemma 2.17.

1.4.

Let us now adapt the proof of [**1**] Lemma 9.1 in the case where G is a quasi-split special orthogonal group over F. Let Q be a non-degenerate quadratic form over F such that $G = \mathrm{SO}(Q)$. Let k, w and q be as in [**1**] Theorem 2.8. Thus

– k is a totally real number field of even degree,

– w is a finite place of k such that $k_w = F$,

– q is a non-degenerate quadratic form over k such that $q \otimes F$ and Q are equivalent, and the group $\mathrm{SO}(q \otimes k_v)$ is compact for all real places v and quasi-split for all finite places v.

We may even assume that the discriminant of q (in the sense of [**1**] 2.1) is equal to any given element $\delta \in k^\times/k^{\times 2}$ such that δ_w is equal to the discriminant of Q, and $\delta_v > 0$ for all real places v of k (see [**1**] Propositions 2.2, 2.4). We thus may assume that there is a finite place $u \neq w$, not dividing 2ℓ, such that the extension of k_u generated by a square root of $(-1)^n\delta_u$ is unramified and of degree 2.

Let $\mathbf{G}$ be the k-group $\mathrm{SO}(q)$. Let ℓ be a prime number different from p.

Lemma 1.2. — *There is a finite place u of k different from w, not dividing ℓ, such that there is a unitary cuspidal irreducible complex representation ρ of $\mathbf{G}(k_u)$ with the following properties:*

(1) *ρ is compactly induced from some compact mod centre, open subgroup of $\mathbf{G}(k_u)$,*

(2) *the local transfer of ρ to $\mathrm{GL}_{2n}(k_u)$ is cuspidal.*

Proof. — Recall that, if u does not divide 2, any cuspidal representation of $\mathbf{G}(k_u)$ is compactly induced from some compact mod centre, open subgroup of $\mathbf{G}(k_u)$.

If $\dim(q)$ is odd, it suffices to choose any finite place $u \neq w$ not dividing 2ℓ, then apply Proposition 1.1.

If $\dim(q) = 2n$ for some $n \geqslant 1$, it suffices to choose any finite place $u \neq w$ not dividing 2ℓ such that the extension of k_u generated by a square root of $(-1)^n \delta_u$ is quadratic and unramified, that is, such that $\mathrm{SO}(q \otimes k_u)$ is non-split and unramified, then apply Proposition 1.1. □

The main theorem of [**1**] now follows, since its proof (see [**1**] 9.1) relies on Lemma 9.1, Proposition 6.3, Theorems 4.4, 5.5, 5.6, 8.2 only.

References

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