

A STORY OF WEBS: THE WEBS BY CONICS ON DEL PEZZO QUARTIC SURFACES AND GELFAND-MACPHERSON'S WEB OF THE SPINOR TENFOLD

LUC PIRIO[†]

ABSTRACT. In a previous paper, we studied the web by conics $\mathcal{W}_{\text{dP}_4}$ on a del Pezzo quartic surface dP_4 and proved that it enjoys suitable versions of most of the remarkable properties satisfied by Bol's web $\mathcal{B}$. In particular, Bol's web can be seen as the toric quotient of the Gelfand-MacPherson web naturally defined on the A_4 -grassmannian variety $G_2(\mathbb{C}^5)$ and we have shown that $\mathcal{W}_{\text{dP}_4}$ can be obtained in a similar way from the web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ which is the quotient by the Cartan torus of $\text{Spin}_{10}(\mathbb{C})$, of the Gelfand-MacPherson 10-web naturally defined on the tenfold spinor variety $\mathbb{S}_5$, a peculiar projective homogenous variety of type D_5 . In the present paper, by means of direct and explicit computations, we show that many of the remarkable similarities between $\mathcal{B}$ and $\mathcal{W}_{\text{dP}_4}$ actually can be extended to, or from an opposite perspective, can be seen as coming from some similarities between Bol's web and $\mathcal{W}_{\mathcal{Y}_5}^{GM}$. The latter web can be seen as a natural uniquely defined rank 5 generalization of Bol's web. In particular, it carries a peculiar 2-abelian relation, denoted by $\mathbf{HLOG}_{\mathcal{Y}_5}$, which appears as a natural generalization of Abel's five terms relation of the dilogarithm and from which one can recover the weight 3 hyperlogarithmic functional identity of any quartic del Pezzo surface.

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[†] Laboratoire de Mathématiques de Versailles, Univ. Paris-Saclay & CNRS (UMR 8100), 78000 Versailles, France.

This paper may be viewed as a continuation of [CP] and [Pi5], to which the reader is referred for a more detailed exposition of the background and motivations underlying the questions addressed here. Nevertheless, the Introduction below should suffice to provide most readers with a general overview of the main themes investigated in this work.

Here are a few words about the setting(s) and some general notations we will work with: except in some cases which will be explicitly indicated and for which the setting is real algebraic or real analytic, we work over the field $\mathbf{C}$ of complex numbers, in an algebraic or analytic framework. We denote indifferently ‘ln’ or ‘Log’ the usual complex logarithm (defined as the primitive vanishing at 1 of the logarithmic differential du/u on the punctured complex plane $\mathbf{C}^*$). Given any positive integer n , we set $[[n]]$ for the set of positive integers less than or equal to n : one has $[[n]] = \{1, 2, \dots, n\}$. We will denote by x, y the affine coordinates associated to the affine embedding $\mathbf{C}^2 \hookrightarrow \mathbf{P}^2$, $(x, y) \mapsto [x : y : 1]$.

1. Introduction

‘Cauchy’s identity’ of the logarithm

$$(C) \quad \text{Log}(x) - \text{Log}(y) - \text{Log}\left(\frac{x}{y}\right) = 0$$

admits a ‘weight 2’ generalization, the so-called ‘Abel’s identity’

$$(Ab) \quad R(x) - R(y) - R\left(\frac{x}{y}\right) - R\left(\frac{1-y}{1-x}\right) + R\left(\frac{x(1-y)}{y(1-x)}\right) = 0,$$

satisfied for any $x, y \in \mathbf{R}$ such that $0 < x < y < 1$, by Rogers’ *dilogarithm* R , which is the function defined by $R(x) = \text{Li}_2(x) + \text{Log}(x)\text{Log}(1-x)/2 - \pi^2/6$ for $x \in]0, 1[$.¹

In [HM] (see also the last paragraph of [Gr, §4.1]), it is written (p. 393) that it has been widely believed that the logarithm and the dilogarithm, together with the two functional identities above that they satisfy, are the first two elements of an infinite sequence of higher logarithms which share analogous properties: in particular each satisfies a peculiar functional identity, which even may be fundamental in a certain sense. In the as yet unpublished preprint [GR], the authors wrote that ‘writing explicitly functional equations for the classical n -th polylogarithm might not be the “right” problem’, because ‘it seems that when n is growing, the functional equations become so complicated that one can not write them down on a piece of paper’.

In [CP], it has been shown that to work with hyperlogarithms instead of just polylogarithms, a simple and natural geometric construction allows to get a uniform series of functional identities, up to weight 6, which are very concise and whose first two elements precisely are the classical identities (C) and (Ab) of the logarithm and the dilogarithm. Given a del Pezzo surface dP_d of degree $d \in \{1, \dots, 6\}$ with canonical class K , its set of conic classes $\mathcal{K} = \{\mathfrak{c} \in \text{Pic}_{\mathbf{Z}}(dP_d) \mid (-K, \mathfrak{c}) = 2 \text{ and } \mathfrak{c}^2 = 0\}$ is known to be finite and for each $\mathfrak{c}$ in it, if $\phi_{\mathfrak{c}} : dP_d \rightarrow |\mathfrak{c}| \simeq \mathbf{P}^1$ stands for the associated fibration in conics, then the set $\Sigma_{\mathfrak{c}} \subset \mathbf{P}^1$ of its singular values has exactly $8-d$ elements. To $\Sigma_{\mathfrak{c}}$, one can associate the ‘complete weight $w = 7-d$ antisymmetric hyperlogarithm $\mathbf{AH}_{\mathfrak{c}}^w$ ’, which is a multivalued function on $\mathbf{P}^1$, ramified at the points of $\Sigma_{\mathfrak{c}}$ and which is canonically defined up to sign. The main result of [CP] is the following:

¹Here Li_2 stands for the classical bilogarithm: one has $\text{Li}_2(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\text{Log}(1-u)}{u} du$ for any x such that $|x| < 1$. Soustracting $\pi^2/6$ in the given definition of Rogers dilogarithm is in order that the RHS of Abel’s identity be zero.

Theorem. *Given a point $x \in \text{dP}_d$ not lying on a line and given a determination of the hyperlogarithms $\mathbf{AH}_\epsilon^w$ at $\phi_\epsilon(x)$ for any $\epsilon \in \mathcal{K}$, there exists a $\mathcal{K}$ -tuple $(\epsilon_\epsilon)_{\epsilon \in \mathcal{K}} \in \{\pm 1\}^{\mathcal{K}}$, unique up to a global sign,² such that the following identity is satisfied in the (complex analytic) vicinity of x :*

$$(\mathbf{Hlog}_{\text{dP}_d}^w) \quad \sum_{\epsilon \in \mathcal{K}} \epsilon_\epsilon \mathbf{AH}_\epsilon^w(\phi_\epsilon) = 0.$$

For $d = 6$ and $d = 5$, the ramification sets Σ_ϵ have cardinalities 2 and 3, respectively, and can therefore be assumed to coincide with $\{0, \infty\}$ and $\{0, 1, \infty\}$, respectively. With this normalization, $\mathbf{AH}_\epsilon^1$ and $\mathbf{AH}_\epsilon^2$ are just the usual logarithm and Rogers dilogarithm $\mathbf{R}$ respectively. The del Pezzo surfaces dP_6 and dP_5 are unique (there is no modulus), the corresponding hyperlogarithmic functional identities as well and it is easy to check that they respectively coincide with Cauchy's and Abel's identity: one has

$$(\mathbf{Hlog}_{\text{dP}_6}^1) \simeq (\mathcal{C}) \quad \text{and} \quad (\mathbf{Hlog}_{\text{dP}_5}^2) \simeq (\mathcal{Ab})$$

(where the symbol $\simeq$ here means a coincidence between two identities up to a (possibly local) change of coordinates).

Motivated by the question of whether the ‘del Pezzo identities $\mathbf{Hlog}_{\text{dP}_d}^w$ ’ for $w = 1, \dots, 6$, genuinely can be considered as the most natural higher weights generalizations of Abel's relation or not, we carried out a thorough comparison of $\mathbf{Hlog}_{\text{dP}_5}^2$ and $\mathbf{Hlog}_{\text{dP}_4}^3$ in [Pi5]. We did that by taking a web geometer perspective. For $X = \text{dP}_d$ with $d = 4, 5$, let $\mathbf{W}$ be the corresponding Weyl group acting on $\mathbf{Pic}(X)$ and let us denote by $\mathcal{W}_{\text{dP}_d}$ the web formed by the pencils of conics on X .

The main outcome of [Pi5] is that virtually all the remarkable properties of various kinds satisfied by the pair $(\mathcal{W}_{\text{dP}_5}, \mathbf{Hlog}_{\text{dP}_5}^2 \simeq \mathcal{Ab})$ admit natural analogues for $(\mathcal{W}_{\text{dP}_4}, \mathbf{Hlog}_{\text{dP}_4}^3)$, which also hold true. Below is a list of some of the remarkable properties shared by both webs $\mathcal{W}_{\text{dP}_5}$ and $\mathcal{W}_{\text{dP}_4}$ share (see §1.1 and §1.2 of [Pi5] for further details):

- geometric definition as the webs formed by the pencils of conics on a del Pezzo surface;
- all their abelian relations are hyperlogarithmic; the Weyl group $\mathbf{W}$ acts on this space and when viewed as an abelian relation, the identity $(\mathbf{Hlog}_{\text{dP}_d}^w)$ transforms according to the signature under the action of $\mathbf{W}$ and spans all the subspaces of antisymmetric hyperlogarithmic ARS by residues/monodromy; dues and monodromy;
- maximality of the rank together with non linearizability (they are exceptional webs);
- combinatorial characterization by means of the hexagonal subwebs;
- modularity (ie. definition by means of a web naturally defined on a moduli space of configurations of points in a projective space);
- cluster character (ie. definition by means of some cluster variables of a cluster algebra);
- geometric construction à la Gelfand-MacPherson.

The fact that the two pairs $(\mathcal{W}_{\text{dP}_5}, \mathcal{Ab})$ and $(\mathcal{W}_{\text{dP}_4}, \mathbf{Hlog}_{\text{dP}_4}^3)$ share so many similarities is quite striking, and may suggest that ‘the hyperlogarithmic identity $\mathbf{Hlog}_{\text{dP}_4}^3$ is, in some sense, the most natural weight-3 generalization of Abel's identity $\mathcal{Ab} \simeq \mathbf{Hlog}_{\text{dP}_5}^2$ ’.

²Actually, one can chose the determinations of the $\mathbf{AH}_\epsilon^w$'s in such a way that one has $\epsilon_\epsilon = 1$ for every $\epsilon \in \mathcal{K}$, see [CP] (in particular formula (9) in Theorem 3.1 therein).

However, such a strong claim should be approached with caution. First—and quite evidently—the rigorous meaning of ‘most natural generalization’ is far from being clear or universally accepted. Second, what makes Abel’s identity for the dilogarithm particularly compelling is its relevance across many seemingly unrelated areas of mathematics. By contrast, to date, we are not aware of significant applications of $\mathbf{Hlog}_{\mathbf{dP}_4}^3$ beyond the theory of functional identities itself and the realm of web geometry.

But there are more substantial considerations that temper the naive assumption of viewing $\mathbf{Hlog}_{\mathbf{dP}_4}^3$ as ‘the’ most natural weight-3 analogue of Abel’s identity:

- the (smooth) del Pezzo quintic surface has no moduli hence the dilogarithmic identity $\mathbf{Hlog}_{\mathbf{dP}_5}^2$ is genuinely unique. This is not the case for del Pezzo surfaces of degree $d = 4$ since the moduli space of these surfaces is of dimension 2. Hence there is a 2-dimensional family of functional identities $\mathbf{Hlog}_{\mathbf{dP}_4}^3$, and no unique/well-defined weight 3 hyperlogarithmic identity. The same phenomenon occurs for the hyperlogarithmic identities of higher weights: for each $d = 1, 2, \dots, 5$, the $\mathbf{Hlog}_{\mathbf{dP}_d}^{7-d}$ ’s form an irreducible complex analytic family of dimension $2(5 - d)$ of functional relations and in each family, no identity appears as being more canonical or particular than the others;
- if $\mathbf{Hlog}_{\mathbf{dP}_5}^2$ and $\mathbf{Hlog}_{\mathbf{dP}_4}^3$ share an important number of nice features, there is a formal difference between these identities which may make the latter identity appear as less fundamental than the former. Indeed, if Abel’s identity $\mathcal{A}\mathbf{b} \simeq \mathbf{Hlog}_{\mathbf{dP}_5}^2$ involves only one function, namely Rogers dilogarithm R , this is not the case for $\mathbf{Hlog}_{\mathbf{dP}_4}^3$ since for a generic quartic surface $\mathbf{dP}_4$, the ten weight 3 hyperlogarithms $\mathbf{AH}_{\mathbf{c}}^3$ for $\mathbf{c} \in \mathcal{K}$ do not coincide, even up to sign and to precomposition by a projective automorphism;³
- the Weyl group $W_{\mathbf{dP}_d}$ of a del Pezzo surface $\mathbf{dP}_d$ acts on the space of ARs of $\mathcal{W}_{\mathbf{dP}_d}$ but there is a major difference between the case $d = 5$ and $d \in \{1, \dots, 4\}$. The natural group embedding $\text{Aut}(\mathbf{dP}_d) \hookrightarrow W_{\mathbf{dP}_d}$ is an isomorphism only for $d = 5$ hence the Weyl group action on $\mathbf{AR}(\mathcal{W}_{\mathbf{dP}_d})$ is not geometric (that is is not induced by automorphisms of the considered del Pezzo surface) for any $d \in \{1, \dots, 4\}$;
- if the identities $\mathbf{Hlog}_{\mathbf{dP}_4}^3$ were truly the ‘right’ weight 3 generalizations of Abel’s identity, it would be natural to expect the same for the identities $\mathbf{Hlog}_{\mathbf{dP}_d}^w$ ’s in higher weights $w = 8 - d$. It turns out that it does not seem to be the case since some of the most striking remarkable properties shared by the webs $\mathcal{W}_{\mathbf{dP}_5}$ and $\mathcal{W}_{\mathbf{dP}_4}$ are no longer satisfied by the webs $\mathcal{W}_{\mathbf{dP}_d}$ for $d \leq 3$. For instance, by a direct computation, we have verified that the curvature⁴ of a 27-web $\mathcal{W}_{\mathbf{dP}_3}$ is non zero, which implies that this web is not of maximal rank, contrarily to the webs $\mathcal{W}_{\mathbf{dP}_d}$ for $d = 4, 5$.

The above considerations naturally led us to seek a generalization of the relation $\mathcal{A}\mathbf{b} \simeq \mathbf{Hlog}_{\mathbf{dP}_5}^2$ that satisfies the following desirable properties:

³As it follows easily from the explicit form for $\mathbf{Hlog}_{\mathbf{dP}_4}^3$ given in [CP, §4], the quotient $\{\mathbf{AH}_{\mathbf{c}}^3 \mid \mathbf{c} \in \mathcal{K}\} / \sim$ is of cardinal 5 if $\sim$ stands for the following equivalence relation: given two (germs of) functions F, G , one has $F \sim G$ if and only if $\pm F = G \circ \gamma$ for some projective automorphism $\gamma \in \text{Aut}(\mathbf{P}^1) = \mathbf{PGL}_2(\mathbf{C})$.

⁴By definition, this is the sum of the Blaschke-Dubourdiu’s curvatures of all of the 3-subwebs of $\mathcal{W}_{\mathbf{dP}_3}$ (see [CL]).

- It is defined on a space equipped with a natural action of the Weyl group of type D_5 , and this action operates via automorphisms. Moreover, the looked for generalization behaves coherently and compatibly with respect to this group action.
- It consists of a single, well-defined identity, without any module entering into the picture.
- This identity involves a unique element whose sum of pullbacks under a specific family of maps vanishes identically.
- Moreover, every hyperlogarithmic functional identity $\mathbf{Hlog}_{\mathrm{dP}_4}^3$ can be recovered from this single identity, and in a natural, canonical way.

In this paper, we provide a fully explicit formulation of such an identity and investigate several of its noteworthy properties. In particular, we show that it is unique (up to multiplication by a nonzero scalar) and that it satisfies all four of the criteria listed above.

An additional remarkable feature of $\mathcal{W}_{\mathrm{dP}_5}$ is that it is a ‘cluster web’. We further demonstrate that a certain lift of the generalization of $\mathcal{W}_{\mathrm{dP}_5}$ discussed here also possesses a cluster structure—albeit with respect to a generalized notion of cluster variety.

In the following sections, we present our main results in greater detail.

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In their interesting paper [GM], Gelfand and MacPherson describe a geometric setting which allows them to give a cohomological-analytic construction of Abel’s identity. On the purely geometric side, they show that the del Pezzo quintic surface $\mathrm{dP}_5 \simeq \overline{\mathcal{M}}_{0,5}$ together with the five fibrations in conics, which correspond to the five forgetful morphisms $\overline{\mathcal{M}}_{0,5} \rightarrow \overline{\mathcal{M}}_{0,4} \simeq \mathbf{P}^1$, are equivariant quotients, under the action of the Cartan torus H_{A_4} of $\mathrm{GL}_5(\mathbf{C})$, of the grassmannian $G_2(\mathbf{C}^5)$ and of the five natural rational maps $G_2(\mathbf{C}^5) \dashrightarrow G_2(\mathbf{C}^5/\langle e_i \rangle) \simeq G_2(\mathbf{C}^4)$, induced for each $i \in \{1, \dots, 5\}$ by the linear projection $\mathbf{C}^5 \rightarrow \mathbf{C}^5/\langle e_i \rangle$ onto the quotient of $\mathbf{C}^5$ by the line spanned by the i -th element of the canonical basis $(e_k)_{k=1}^5$ of $\mathbf{C}^5$.

In view of generalizing Gelfand and MacPherson approach to the webs $\mathcal{W}_{\mathrm{dP}_d}$ for any $d \in \{1, 2, \dots, 4\}$, we remarked in [Pi5] that there is an intrinsic way to recover $G_2(\mathbf{C}^5)$ from $X_4 = \mathrm{dP}_5 \simeq \overline{\mathcal{M}}_{0,5}$: the former variety is the (projective) Cox variety of the latter.

Let dP_d be a fixed smooth del Pezzo surface of degree $d \in \{2, \dots, 5\}$. We will also denote it by X_r to emphasize that it can be obtained as the blow-up of $\mathbf{P}^2$ at $r = 9 - d$ points in general position. We denote by $\mathcal{L}$ the set of lines contained in X_r and by E_r the Dynkin type of the considered del Pezzo surface. The associated Weyl group acting transitively by permutations on $\mathcal{L}$ will be denoted by W_r .

Then we have the following (see [Pi5, §4.5.2] for more details and references):

- there is a projective Cox variety $\mathbf{P}(X_r)$ which is acted upon by a rank r torus $\mathcal{T}_{\mathrm{NS}} = \mathrm{T}_{\mathrm{NS}}/\mathbf{C}^* \simeq (\mathbf{C}^*)^r$ which is the quotient of the Neron-Severi torus $\mathrm{T}_{\mathrm{NS}} = \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}(X_r), \mathbf{C}^*)$ by a one-parameter subgroup associated to the (anti)canonical class of X_r ;
- the Cox ring of X_r is generated by the lines contained in it.⁵ Consequently, the projective Cox variety $\mathbf{P}(X_r)$ embeds into the projective space $\mathbf{P}(\mathbf{C}^{\mathcal{L}})$;

⁵More rigorously, the Cox ring of X_r is generated by any set $\{\sigma_\ell\}_{\ell \in \mathcal{L}}$ where for any line $\ell \subset X_r$, σ_ℓ is a non zero element of $\mathbf{H}^0(X_r, \mathcal{O}_{X_r}(\ell)) \simeq \mathbf{C}$.

- from some results by several authors (Popov, Batyrev-Popov, Derenthal, Serganova-Skorobogatov) it follows that $\mathbf{C}^{\mathcal{L}}$ naturally is a minuscule representation of a simple complex Lie group G_r of type E_r . The weights of this representation identify with the lines included in X_r and the Weyl group W_r acts transitively on their set;
- viewed as contained in $\mathbf{P}(\mathbf{C}^{\mathcal{L}})$, the Cox variety $\mathbf{P}(X_r)$ actually is a subvariety of the associated minuscule homogeneous space $\mathcal{G}_r = G_r/P_r \subset \mathbf{P}(\mathbf{C}^{\mathcal{L}})$, where P_r is a suitable maximal parabolic subgroup of P_r ;
- moreover, up to a natural isomorphism $\mathcal{T}_{\text{NS}} \simeq H_r$ between the torus acting on $\mathbf{P}(X_r)$ and the Cartan torus H_r of G_r , the map $\mathbf{P}(X_r) \hookrightarrow \mathcal{G}_r \subset \mathbf{P}(\mathbf{C}^{\mathcal{L}})$ turns out to be a torus-equivariant embedding;
- let $\mathcal{Y}_r$ be the GIT quotient $\mathcal{G}_r//H_r$ (??). Then:
 - the linear action of the Weyl group W_r on $\mathbf{C}^{\mathcal{L}}$ gives rise to an isomorphism $W_r \simeq \text{Aut}(\mathcal{Y}_r)$ (see [Sk, Theorem 2.2]);
 - there is a natural embedding $f_{\text{SS}} : X_r \hookrightarrow \mathcal{Y}_r$ inducing an isomorphism of the Picard lattices $f^* : \text{Pic}_{\mathbf{Z}}(\mathcal{Y}_r) \simeq \text{Pic}_{\mathbf{Z}}(X_r)$ and making commutative the following diagram:

$$(1) \quad \begin{array}{ccc} \mathbf{P}(X_r) & \hookrightarrow & \mathcal{G}_r \subset \mathbf{P}(\mathbf{C}^{\mathcal{L}}) \\ \downarrow & & \downarrow \\ X_r & \xrightarrow{f_{\text{SS}}} & \mathcal{Y}_r. \end{array}$$

The interest of the material above is that it allows to construct the web $\mathcal{W}_{\text{dP}_d}$, for an arbitrary del Pezzo surface, from a unique web on $\mathcal{Y}_r$ obtained as the quotient of a H_r -equivariant web naturally defined on $\mathcal{G}_r$ and induced by linear projection on $\mathbf{C}^{\mathcal{L}}$. Indeed, denoting by $\mathfrak{h}_{\mathbf{R}}$ the Lie algebra of the real part of H_r and by $H_r^+ \simeq (\mathbf{R}_{>0})^r$ the ‘positive part’ of the latter, we have the following :

- there is a (real analytic) moment map $\mu : \mathcal{G}_r \rightarrow \mathfrak{h}_{\mathbf{R}}^*$ whose image is the associated ‘minuscule weight polytope’ $\Delta_r = \Delta_{G_r, P_r}$, that is the convex envelope of the weights of the minuscule representation $\mathbf{C}^{\mathcal{L}}$;
- for ζ generic, that is in a certain dense Zariski open subset $\mathcal{G}_r^\circ$, the moment map induces an isomorphism between the positive orbit $H_r^+ \cdot \zeta$ and the interior $\mathring{\Delta}_r$ of the weight polytope, which extends to an isomorphism of real analytic manifolds with corners $\mu : \overline{H_r^+} \cdot \zeta \simeq \Delta_r$;
- for each facet (that is a face of codimension 1) F of Δ_r , let $\mathcal{L}_F$ be the set of lines belonging to F (in other terms $\mathcal{L}_F$ is the set of vertices of F) and let $\Pi_F : \mathbf{C}^{\mathcal{L}} \rightarrow \mathbf{C}^{\mathcal{L}_F}$ be the linear projection associated to the inclusion $\mathcal{L}_F \subset \mathcal{L}$ (i.e. $\text{Ker}(\Pi_F) = \mathbf{C}^{\mathcal{L} \setminus \mathcal{L}_F}$);
- there exists a quotient G_F of the subgroup of G letting invariant the decomposition in direct sum $\mathbf{C}^{\mathcal{L}} = \mathbf{C}^{\mathcal{L}_F} \oplus \mathbf{C}^{\mathcal{L} \setminus \mathcal{L}_F}$ and a homogeneous projective space $\mathcal{G}_F = G_F/P_F \subset \mathbf{P}(\mathbf{C}^{\mathcal{L}_F})$ (for a certain parabolic subgroup P_F constructed from P) such that (i). one has $\mathcal{G}_F = \mathcal{G}_r \cap \mathbf{P}(\mathbf{C}^{\mathcal{L}_F})$; and (ii). the restriction to $\mathcal{G}_r$ of the (projectivization of the) linear projection Π_F gives rise to a dominant rational map $\mathcal{G}_r \dashrightarrow \mathcal{G}_F$, again denoted by Π_F ;
- let H_F be the Cartan torus of G_F . There is a natural epimorphism of tori $H \rightarrow H_F$ with respect to which $\Pi_F : \mathcal{G}_r \dashrightarrow \mathcal{G}_F$ is equivariant. It follows that there exists a dominant

rational map π_F from $\mathcal{Y}_r$ onto the GIT quotient $\mathcal{Y}_F = \mathcal{G}_F // \mathbf{H}_F$ such that the following diagram commutes:

$$(2) \quad \begin{array}{ccc} \mathcal{G}_r & \xrightarrow{\Pi_F} & \mathcal{G}_F \subset \mathbf{P}(\mathbf{C}^{\mathcal{L}_F}) \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xrightarrow{\pi_F} & \mathcal{Y}_F; \end{array}$$

- for any conic class $\mathbf{c} \in \mathcal{K}$ on X_r , there exists a uniquely determined facet $F_{\mathbf{c}}$ of Δ_r such that gluing the two diagrams (24) and (2) gives the following commutative one

$$\begin{array}{ccccc} \mathbf{P}(X_r) & \hookrightarrow & \mathcal{G}_r & \xrightarrow{\Pi_{F_{\mathbf{c}}}} & \mathcal{G}_{F_{\mathbf{c}}} \\ \downarrow & & \downarrow & & \downarrow \\ X_r & \xrightarrow{f_{SS}} & \mathcal{Y}_r & \xrightarrow{\pi_{F_{\mathbf{c}}}} & \mathcal{Y}_{F_{\mathbf{c}}}. \end{array}$$

which is such that the composition of the rational maps of the bottom line coincides with the conic fibration $\phi_{\mathbf{c}}$ associated to $\mathbf{c}$: as rational maps on X_r , one has $\phi_{\mathbf{c}} = \pi_{F_{\mathbf{c}}} \circ f_{SS}$.

At this point, one can define the ‘Gelfand-MacPherson’s webs’ $\mathcal{W}_{\mathcal{G}_r}^{GM}$ and $\mathcal{W}_{\mathcal{Y}_r}^{GM}$, which are respectively the (generalized) web on $\mathcal{G}_r$ and $\mathcal{Y}_r$, induced by the rational maps $\Pi_{F_{\mathbf{c}}}$ and $\pi_{F_{\mathbf{c}}}$ for $\mathbf{c}$ ranging in the set $\mathcal{K}$ of all conic classes of X_r : one has

$$\mathcal{W}_{\mathcal{G}_r}^{GM} = \mathcal{W}(\Pi_{F_{\mathbf{c}}} \mid \mathbf{c} \in \mathcal{K}) \quad \text{and} \quad \mathcal{W}_{\mathcal{Y}_r}^{GM} = \mathcal{W}(\pi_{F_{\mathbf{c}}} \mid \mathbf{c} \in \mathcal{K}).$$

From the last two statements in the list above, we deduce that $\mathcal{W}_{\mathcal{Y}_r}^{GM}$ can be seen as the quotient of the web $\mathcal{W}_{\mathcal{G}_r}^{GM}$ which is $\mathbf{H}_r$ -equivariant, and also that del Pezzo’s web $\mathcal{W}_{\text{dP}_d}$ is the pull-back of the web $\mathcal{W}_{\mathcal{Y}_r}^{GM}$ under Skorobogatov-Serganova’s embedding $f_{SS} : \text{dP}_d = X_r \rightarrow \mathcal{Y}_r$: one has

$$(3) \quad \mathcal{W}_{\text{dP}_d} = f_{SS}^*(\mathcal{W}_{\mathcal{Y}_r}^{GM}).^6$$

In this paper, we focus on the webs by conics of del Pezzo quartic surfaces and their relations to the Gelfand-MacPherson web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$. More precisely, since the $\mathcal{W}_{\text{dP}_4}$ ’s can all be obtained from Gelfand-MacPherson web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, it is not unreasonable to ask whether this latter web cannot be seen as a more natural generalization of $\mathcal{W}_{\text{dP}_5} \simeq \mathcal{W}_{\mathcal{Y}_4}^{GM}$ than the $\mathcal{W}_{\text{dP}_4}$ ’s.

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The web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ is defined on the variety $\mathcal{Y}_5$, which is rational and of dimension 5, by means of 10 rational first integrals $\psi_i^\epsilon : \mathcal{Y}_5 \dashrightarrow \mathbf{P}^2$ with $i = 1, \dots, 5$ and $\epsilon = \pm$ (here we use notations similar to those of [Pi5]). The main theme of the present paper is the study of the k -abelian relations of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ for $k = 0, 1, 2$ and how these are related to the abelian relations of a given $\mathcal{W}_{\text{dP}_4}$. Recall that a k -abelian relation (ab. k -AR) for $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, is a 10-tuple of k -forms $(\eta_i^\epsilon)_{i,\epsilon}$ such that $\sum_{i,\epsilon} (\psi_i^\epsilon)^*(\eta_i^\epsilon) = 0$ (possibly just locally) on $\mathcal{Y}_5$. They form a vector space which we denote by $\mathbf{AR}^k(\mathcal{W}_{\mathcal{Y}_5}^{GM})$, and whose dimension $\text{rk}^k(\mathcal{W}_{\mathcal{Y}_5}^{GM})$ is the ‘ k -rank’ of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$. Although $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ is a

⁶This is proved by direct computations for $d \in \{2, \dots, 4\}$ (Maple worksheets of these computations are available under request). For more details in the case when $d = 4$ (which is equivalent to $r = 5$), see [Pi5, Prop. 4.16].

‘web’ only in a generalized sense, a similar approach to the one described in [Pi2, §1.3.4] can be applied to it and one can define the ‘virtual k -rank’ $\rho^k(\mathcal{W})$ of any subweb $\mathcal{W}$ of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, this for $k \in \{0, 1, 2\}$. An interesting fact is that all the virtual ranks of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ are finite (see Proposition 3.3 further) and in this paper we will describe the spaces of k -ARs of the web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ for $k = 0, 1, 2$, see from §3.2 to §3.4 (the two tables on page 8 provide a concise summary of the results obtained in these three subsections).

The most interesting case is that of 2-ARs of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ and how they give rise to 1-ARs for any del Pezzo web $\mathcal{W}_{\text{dP}_4}$. In order to state our main result about the 2-ARs, introducing some terminology will be useful. For ψ standing for one of the first integrals ψ_i^ϵ of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, we denote by

- $\mathbf{C}(\psi)$ the subalgebra of $\mathbf{C}(\mathcal{Y}_5)$ formed by compositions $f \circ \psi$ with $f \in \mathbf{C}(\mathbf{P}^2)$;
- $\text{LogC}(\psi)$ the family of multivalued functions on $\mathcal{Y}_5$ of the form $\text{Log}(\phi)$ with $\phi \in \mathbf{C}(\psi)$;
- $d\text{LogC}(\psi)$ the space of ψ -logarithmic differential 1-forms, that is of rational 1-forms on $\mathcal{Y}_5$ of the form $d\text{Log}(\phi) = d\phi/\phi$ with $\phi \in \mathbf{C}(\psi)$.

The following theorem gathers some of the most interesting results obtained in this paper:

- Theorem 1.1.** 1. One has $\rho^2(\mathcal{W}_{\mathcal{Y}_5}^{GM}) = 11$ and $\rho^2(\mathcal{W}) \leq 1$ for every 5-subweb $\mathcal{W}$ of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$.
2. Among all the 5-subwebs of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, exactly 16 have virtual 2-rank equal to 1. These are the subwebs $\mathcal{W}^\epsilon = \mathcal{W}(\psi_1^{\epsilon_1}, \dots, \psi_5^{\epsilon_5})$ for the sixteen 5-tuples $\underline{\epsilon} = (\epsilon_i)_{i=1}^5 \in \{\pm 1\}^5$ such that $p(\underline{\epsilon}) = \#\{i \mid \epsilon_i = 1\}$ is odd.⁷ Each such subweb $\mathcal{W}^\epsilon$ actually has maximal rank 1, with $\text{AR}^2(\mathcal{W}^\epsilon)$ spanned by a 2-AR LogAR^ϵ which is complete, irreducible and logarithmic, in the sense that the $\psi_i^{\epsilon_i}$ -th component of LogAR^ϵ belongs to $d\text{LogC}(\psi_i^{\epsilon_i}) \wedge d\text{LogC}(\psi_i^{\epsilon_i})$ for every $i = 1, \dots, 5$.
3. The LogAR^ϵ ’s for all odd 5-tuples $\underline{\epsilon}$ ’s span a vector space denoted by $\text{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM})$ and called the space of ‘combinatorial ARs’ of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$. Moreover, one has
- $$\text{rk}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM}) = \dim \text{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM}) = \rho^2(\mathcal{W}_{\mathcal{Y}_5}^{GM}) - 1 = 10.$$
4. There exists a 2-AR of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, denoted by $\mathbf{HLOG}_{\mathcal{Y}_5}$, which is complete, irreducible and whose components are ‘dil logarithmic’ in the sense that for any first integral ψ_i^ϵ of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, the ψ_i^ϵ -th component of $\mathbf{HLOG}_{\mathcal{Y}_5}$ belongs to $\text{LogC}(\psi_i^\epsilon) d\text{LogC}(\psi_i^\epsilon) \wedge d\text{LogC}(\psi_i^\epsilon)$. Moreover, $\mathbf{HLOG}_{\mathcal{Y}_5}$ is unique up to multiplication by a non-zero scalar and one has
- $$(4) \quad \text{AR}^2(\mathcal{W}_{\mathcal{Y}_5}^{GM}) = \text{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM}) \oplus \langle \mathbf{HLOG}_{\mathcal{Y}_5} \rangle,$$
- which implies $\text{rk}^2(\mathcal{W}_{\mathcal{Y}_5}^{GM}) = \rho^2(\mathcal{W}_{\mathcal{Y}_5}^{GM}) = 11$: the web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ has maximal 2-rank.⁸
5. One can associate a divisor $\mathcal{D}_w$ of $\mathcal{Y}_5$ to each weight w of the minuscule half-spin representation S_5^+ . Then the 2-ARs of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ are regular on the complement in $\mathcal{Y}_5$ of the union of all the $\mathcal{D}_w$ ’s which coincides with $\mathcal{Y}_5^\times$. Moreover, the sixteen ARs LogAR^ϵ ’s are exactly the logarithmic ARs obtained by considering the residues of $\mathbf{HLOG}_{\mathcal{Y}_5}$ along the

⁷This has to be compared with the description of Bol’s subwebs of $\mathcal{W}_{\text{dP}_4}$ given in [Pi5, §4.3].

⁸It would be more rigorous to state this as ‘the 2-rank of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ is AMP’, using the terminology introduced in [Pi2, §1.3.5].

$\mathcal{D}_w$'s: for any odd 5-tuple $\underline{\epsilon}$, there exists a uniquely defined weight $w(\underline{\epsilon})$ such that, up to multiplication by a non-zero constant, one has

$$\text{Res}_{\mathcal{D}_{w(\underline{\epsilon})}}(\mathbf{HLOG}_{\mathcal{Y}_5}) = \mathbf{LogAR}^{\underline{\epsilon}}.$$

6. The action of the Weyl group W_{D_5} by automorphisms on $\mathcal{Y}_5$ (cf. [Sk, Theorem 2.2]) gives rise to a linear W_{D_5} -action on $\mathbf{AR}^2(\mathcal{W}_{\mathcal{Y}_5}^{GM})$ which lets invariant the decomposition in direct sum (4). This decomposition actually is the one into W_{D_5} -irreducibles: the irrep associated to the 1-dimensional component $\langle \mathbf{HLOG}_{\mathcal{Y}_5} \rangle$ is the signature representation whereas $\mathbf{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM})$ is the W_{D_5} -irreducible module $V_{[11,111]}^{10}$.⁹
7. For any del Pezzo surface dP_4 , using Serganova-Skorobogatov embedding $f_{SS} : \text{dP}_4 \hookrightarrow \mathcal{Y}_5$ (cf. (24) above), the weight 3 hyperlogarithmic abelian relation $\mathbf{HLog}_{\text{dP}_4}^3$ of $\mathcal{W}_{\text{dP}_4}$ (resp. the sixteen elements of $\mathbf{HLogAR}_{\text{asym}}^2$ equivalent to $\mathbf{HLog}^2 \simeq \mathcal{A}\mathbf{b}$) can be obtained in a natural way from $\mathbf{HLOG}_{\mathcal{Y}_5}$ (resp. from the sixteen logarithmic 2-abelian relations $\mathbf{LogAR}^{\underline{\epsilon}} \in \mathbf{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM})$) by means of residues.

An interesting feature of $\mathcal{W}_{\text{dP}_5}$ is that it is a cluster web. More precisely, it admits a birational model which admits as first integrals the $\mathcal{X}$ -cluster variables of type A_2 . In [Pi5], we established that each del Pezzo web $\mathcal{W}_{\text{dP}_4}$ can also be defined by $\mathcal{X}$ -cluster variables (of type D_4). It is natural to wonder whether Gelfand-MacPherson's web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ is cluster as well. Unfortunately we do not have a complete answer for this web yet, but we have one for its lift $\mathcal{W}_{\mathbb{S}_5}^{GM}$. In [Du], Ducat gave the construction of a generalized cluster structure on $\mathbb{S}_5$ which is not a classical cluster one, but enjoys the nice property of being of finite type. We prove the following result:

Proposition 1.2. *Gelfand-MacPherson's web $\mathcal{W}_{\mathbb{S}_5}^{GM}$ is cluster with respect to Ducat's generalized cluster structure on $\mathbb{S}_5$.*

The similarities between the statements above to the corresponding ones for Bol's web $\mathcal{B} \simeq \mathcal{W}_{\text{dP}_5}$ in [Pi5, §1.1] are even more striking than those in [Pi5, §1.2] about $\mathcal{W}_{\text{dP}_4}$. For this reason, and also because the theorem above can be generalized to all the Gelfand-MacPherson webs $\mathcal{W}_{\mathcal{Y}_r}^{GM}$ for $r = 4, \dots, 8$ (see §7 further), we believe that these latter webs are those which must really be considered as the most direct/fundamental generalizations of $\mathcal{B} \simeq \mathcal{W}_{\text{dP}_5} \simeq \mathcal{W}_{\mathcal{Y}_4}^{GM}$, and not really the del Pezzo's webs $\mathcal{W}_{\text{dP}_{9-r}}$ which actually are 2-dimensional slices of the corresponding Gelfand-MacPherson webs. All these considerations make it natural to ask the following

Question 1.3. *Can the 2-abelian relation $\mathbf{HLOG}_{\mathcal{Y}_5}$ of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ be obtained following the geometric approach of Gelfand-MacPherson, that is by integrating an invariant differential form Ω_P on a real form $\mathbb{S}_5$ of the spinor tenfold $\mathbb{S}_5$ which represents a certain characteristic class $P \in \mathbf{H}^*(\mathbb{S}_5, \mathbf{R})$ along the orbits of the action on $\mathbb{S}_5$ of the Cartan torus of a split real form of the group $\text{Spin}(\mathbf{C}^{10})$?*

This question is the subject of ongoing research by the author at the time of writing.

The rest of the article is organised as follows: Section 2 consists in some preliminaries. After recalling some elements of web geometry in §2.1, we succinctly review the notion of Gelfand-MacPherson web in §2.2. Then we discuss the geometry around the spinor tenfold in §2.3. The

⁹Seeing it as a decomposition in W_{D_5} -irreducibles, (4), must be compared with some results given in §5: in some sense which could be made precise (cf. Proposition 5.2), $\mathbf{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM})$ corresponds to $\mathbf{HLogAR}_{\text{asym}}^2$ and $\mathbf{HLog}^2$ to $\mathbf{HLog}^3$.

third section is devoted to the study of $\mathcal{W}_{\mathbf{y}_5}^{GM}$ especially from the point of view of its ranks and abelian relations. First, in §3.1, working with some adapted rational coordinates previously introduced, we give a list of explicit rational first integrals U_i (see (31)) for a birational model of $\mathcal{W}_{\mathbf{y}_5}^{GM}$, denoted by $\mathcal{W}_{Y_5}^{GM}$. After having computed the virtual and ordinary ranks of $\mathcal{W}_{Y_5}^{GM}$, we start to study the abelian relations of this web in the following subsections. The most important of the ARs is the 2-AR with logarithmic coefficient $\mathbf{HLOG}_{Y_5}$ which corresponds to the differential identity denoted the same which is given in Proposition 3.4. Then the space of abelian relations $\mathbf{AR}^k(\mathcal{W}_{Y_5}^{GM})$ for $k = 2, 1, 0$ are successively studied in the subsections §3.2, §3.3 and §3.4 respectively. In particular, we describe the structures of these spaces relatively to the action of the Weyl group W_{D_5} and indicate how these spaces (or rather some subspaces of them) are related with respect to the total derivatives or to taking residues of ARs. Many of the results concerning the ARs of $\mathcal{W}_{Y_5}^{GM}$ are brought together succinctly in the tables page 1.

2. Preliminaries: elements of web geometry and birational geometry around $\mathbb{S}_5$

We start by recalling/introducing general notions of web geometry in a generalized setup (namely, for ‘generalized webs’). Then we quickly review the notion of ‘Gelfand-MacPherson web’ introduced in [Pi5] in §2.2 before discussing in §2.3 several properties of the most important space considered in this paper, namely the spinor tenfold $\mathbb{S}_5$.

2.1. Elements of web geometry. We introduce here basic notions of web geometry. The webs we are considering here are quite general, and in particular more general than the webs encountered in the classical literature on the subject. For a more detailed overview of web geometry as we need it here, see [Pi2, §1].

Let M be an irreducible analytic (hence possibly singular) variety. In this text, a ‘ d -web’ on M (for a positive integer d) is a finite collection $\mathcal{W} = (\mathcal{F}_1, \dots, \mathcal{F}_d)$ of d pairwise distinct foliations on M , all of the same codimension.¹⁰ We will also assume that for $m \in M$ generic, then the tangent spaces of the foliations of $\mathcal{W}$ at m span the whole tangent space of M at this point, ie. $T_m M = \langle T_m \mathcal{F}_i \mid i \in \llbracket d \rrbracket \rangle$, and that for i, j distinct, $T_m \mathcal{F}_i$ and $T_m \mathcal{F}_j$ intersect transversally in $T_m M$. When these two conditions are fulfilled, m is said to be a ‘smooth or a regular point’ of $\mathcal{W}$.

Another d -web $\mathcal{W}' = (\mathcal{F}'_i)_{i \in \llbracket d \rrbracket}$ defined on another manifold M' is ‘equivalent’ to $\mathcal{W}$ if there exist $m \in M$ and $m' \in M'$, regular points for $\mathcal{W}$ and $\mathcal{W}'$ respectively, as well as a local biholomorphism $\varphi : (M, m) \rightarrow (M', m')$ such that $\varphi^*(\mathcal{W}') = (\varphi^*(\mathcal{F}'_i))_{i \in \llbracket d \rrbracket}$ coincides with the germ of $\mathcal{W}$ at m , possibly up to reindexing the foliations $\mathcal{F}_i$ of this web. One writes $\mathcal{W} \simeq \mathcal{W}'$ when this is occurring.

For simplicity, assume that M is a connected open subset in $\mathbb{C}^n$ and that each $\mathcal{F}_i$ is defined by a global holomorphic submersion $U_i \rightarrow \mathbb{C}^c$, where $c \in \{1, \dots, n-1\}$ stands for the codimension of the web. For any $i \in \llbracket d \rrbracket$, any $k \leq c$ and any subset $I = \{i_1, \dots, i_k\}$ with $1 \leq i_1 < i_2 < \dots < i_k \leq c$, one sets $\Omega_{i,I}^k = dU_{i,i_1} \wedge \dots \wedge dU_{i,i_k}$. Then one defines $\Omega_{\mathcal{F}_i}^k$ as the locally-free sheaf of $\mathcal{O}_M$ -modules freely spanned by the $\Omega_{i,I}^k$ ’s, for all $I \subset \{1, \dots, c\}$ of cardinality k . Given an open subset $O \subset M$ and $k \in \{0, \dots, c\}$, one defines a k -abelian relation (ab. k -ARs) of $\mathcal{W}$ on O as a d -tuple $(\eta_i)_{i=1}^d \in \prod_{i=1}^d \Omega_{\mathcal{F}_i}^k$ such that $\sum_{i=1}^d \eta_i = 0$ in $\Omega^k(O)$. The abelian relations form a vector subspace of $\Omega^k(O)^{\oplus d}$ and letting O vary among the open subsets of M , one defines a local system which we will denote by $\mathbf{AR}^k(\mathcal{W})$ or just $\mathbf{AR}^k$ whenever there is no ambiguity about the web under consideration. When working on a fixed domain D , we will allow ourselves to use the same notation $\mathbf{AR}^k(\mathcal{W})$ for the vector space of k -ARs on D , which, rigorously, should be denoted by $\mathbf{AR}^k(\mathcal{W})(D)$.

By definition, the k -rank $r^k(\mathcal{W})$ is the rank of $\mathbf{AR}^k(\mathcal{W})$. It is an element of $\overline{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$ which is invariantly attached to $\mathcal{W}$: two equivalent webs have the same k -rank, for any k less than or equal to the codimension of the webs. Given a point $m \in M$ smooth for $\mathcal{W}$, let $T_m \mathcal{W}$ be the constant web on $T_m M \simeq \mathbb{C}^n$ with first integrals the linear maps $dU_i(m) : T_m M \rightarrow T_{U_i(m)} \mathbb{C}^c \simeq \mathbb{C}^c$. For $k \in \{0, \dots, c\}$ and any $\sigma \geq 0$ (with $\sigma > 0$ if $k = 0$), let $\mathbf{AR}_\sigma^k(T_m \mathcal{W})$ be the space of k -ARs $(\eta_i)_{i=1}^d$ of $T_m \mathcal{W}$ with each η_i being a k -differential form on $T_m M$ with coefficients in $\text{Sym}^\sigma(T_m^* M)$ (that is, polynomials of degree σ on $T_m M$). Then one defines the ‘ σ -th virtual k -rank of $\mathcal{W}$ at m ’, denoted by $\rho_{m,\sigma}^k(\mathcal{W})$, as the dimension of $\mathbf{AR}_\sigma^k(T_m \mathcal{W})$. One verifies easily that the dimension of

¹⁰Classically, one requires that the tangent spaces of the leaves of a web satisfy a certain general position assumption (everywhere or only at the generic point of M), but it is not the case here.

the space of germs of k -ARs at m is less than or equal to the ‘total virtual k -rank at m ’ $\rho_m^k(\mathcal{W}) := \sum_{\sigma} \rho_{m,\sigma}^k(\mathcal{W})$. We thus define the ‘ σ -th (resp. the total) virtual k -rank’ $\rho_{\sigma}^k(\mathcal{W})$ resp. $\rho^k(\mathcal{W})$ of the considered web as the value $\rho_{m,\sigma}^k(\mathcal{W})$ (resp. $\rho_m^k(\mathcal{W})$) for m generic.

The web $\mathcal{W}$ is said to have ‘as maximal as possible’ (ab. AMP) k -rank when $r^k(\mathcal{W}) = \rho^k(\mathcal{W})$, with this integer being positive and finite. In this case, we will also say that $\mathcal{W}$ is ‘ k -AMP’. Being (k) -AMP has to be view as a strong feature of a web, a wide generalization of the classical notion in web geometry of being of ‘maximal rank’ (for more perspective on this notion, see [Pi2, §1.3]).

For any nonnegative k such that $k \leq c$, the total derivative gives rise to a linear map

$$(5) \quad d^k : \mathbf{AR}^k(\mathcal{W}) \longrightarrow \mathbf{AR}^{k+1}(\mathcal{W}).$$

$$(\eta_i)_{i=1}^d \longmapsto (d\eta_i)_{i=1}^d$$

Accordingly to the usual terminology, abelian relations in the kernel of d^k will be said to be ‘closed’, and those in its image, ‘exact ARs’.

Of course, one always have an inclusion $d^k(\mathbf{AR}^k(\mathcal{W})) \subset \text{Ker}(d^{k+1})$ in $\mathbf{AR}^{k+1}(\mathcal{W})$ but, even if one is considering the ARs locally, on a simply connected domain or even at the level of germs, in general it is not true that this inclusion is an equality. Indeed, let $\chi = (\chi_i)_{i=1}^d$ be a germ of $(k+1)$ -AR for $\mathcal{W}$, at a smooth point $m \in M$ of this web. This means that $\chi_i \in \Omega^{k+1}(M, m)$ is $\mathcal{F}_i$ basic for each $i \in \llbracket d \rrbracket$ and that $\sum_{i=1}^d \chi_i = 0$. Let us assume moreover that χ is closed, i.e. one has $d\chi_i = 0$ for any i . Hence any χ_i is locally exact thus one may first think that it is possible to integrate χ and construct a (germ of) k -AR $\eta = (\eta_i)_{i=1}^d$ such that $d^k(\eta) = \chi$.

But this is not always possible. Indeed, the η_i ’s, in addition to being primitives of the corresponding χ_i ’s, must satisfy two additional conditions: (1) each η_i must be $\mathcal{F}_i$ -basic, and (2) $\sum_{i=1}^d \eta_i = 0$ in $\Omega^k(M, m)$. These two conditions cannot always be simultaneously fulfilled. In particular, the 2-abelian relation $\mathbf{HLOG}_{\mathbf{y}_5}$ considered in this paper, although closed, is not exact as an abelian relation (see the decompositions in direct sums (44) and (50) in Coro/Theorem(?) 3.10 and Theorem 3.14 respectively).¹¹

2.2. Gelfand-MacPherson webs. We review very quickly some material introduced in [Pi5, §4.5.1] to which we refer for further details.

Let G be a simple complex Lie group of Dynkin type D , and let $H \subset G$ be a Cartan torus, with associated set of simple roots $\Phi \subset \mathfrak{h}_{\mathbf{R}}^*$, where $\mathfrak{h}_{\mathbf{R}}$ stands for the real part of the Lie algebra of H . Let P be a standard parabolic subgroup, assumed to be maximal (to simplify), and let ω_P be the vertex of D corresponding to it. If $V = V_P$ stands for the G -representation of highest weight ω_P , there exists a ‘highest-weight vector’ $v_P \in V$ wch is of weight ω_P and such that P coincides with the stabilizer of the line $[v_P] \in \mathbf{P}(V)$ and the orbit $X = G \cdot [v_P]$ is closed in $\mathbf{P}(V)$. It follows that X is a homogeneous algebraic submanifold of $\mathbf{P}(V)$ which naturally identifies with G/P . Let $\mathfrak{B}_P$ be the set of weights of the considered representation, namely the set of weights $w \in \mathfrak{h}_{\mathbf{R}}^*$ for which the corresponding weight subspace $V_w \subset V$ is non-trivial. The ‘weight polytope’ $\Delta = \Delta_{D, \omega_P}$ of the representation V is the convex envelope of $\mathfrak{B}_P$ in $\mathfrak{h}_{\mathbf{R}}^*$. The Weyl group $W_D = N_G(H)/H$ acts on $\mathfrak{h}_{\mathbf{R}}^*$ and the set of vertices of Δ can be proved to coincide with the Weyl orbit $W \cdot \omega_P$ of the highest weight.

¹¹Another place to look is the last line of Table 1.

To a facet F of Δ (that is, a face of codimension 1), one can associate a type (D_F, ω_F) which is a marked Dynkin diagram obtained from (D, ω_P) by removing an extremal vertex. Then V can be seen as a representation for the simple complex Lie group of type D_F , noted by G_F , and there exists a decomposition of G_F -subrepresentations $V = V^F \oplus V_F$ where V_F is the sub-representation with highest weight ω_F . To the facet F is associated a one-parameter subgroup $\mathcal{H}_F$ of H and denoting by H_F the Cartan torus of G_F , there is a surjective map of Cartan tori $H \rightarrow H_F$ with kernel $\mathcal{H}_F$ which induces an identification $H/\mathcal{H}_F \xrightarrow{\sim} H_F$. Then if P_F stands for the maximal standard parabolic subgroup of G_F associated to ω_F , then the two following facts occur:

- (i). setting $X_F = G_F/P_F \subset \mathbf{P}V_F$, one has $X_F = X \cap \mathbf{P}V_F$;
- (ii). the linear projection from $\mathbf{P}V^F$ onto $\mathbf{P}V_F$ gives rise to a surjective rational map $\Pi_F : X \dashrightarrow X_F$. Moreover, this map is equivariant with respect to the epimorphism of tori $H \rightarrow H_F$.

Now we restrict the discussion to some particularly nice cases¹² which contain those of the minuscule homogeneous spaces hence the specific case we will be interested in. In [Sk] (see also [SS]), the author(s) consider a Zariski-open subset $X^{sf} \subset X$ on which H acts nicely, and such that the quotient $\mathcal{Y} = X^{sf}/H$ is a rational quasi-projective variety. From the point (ii). above, it follows that Π_F descends to a rational map $\pi_F : \mathcal{Y} \dashrightarrow X_F/H_F$ where the target space has to be understood as a quotient with respect to the action of H_F on X_F viewed as a rational action. A facet is said to be ‘ $\mathcal{W}$ -relevant’ whenever X_F/H_F has positive dimension. We define the ‘Gelfand-MacPherson webs’ $\mathcal{W}_X^{GM}$ and $\mathcal{W}_{\mathcal{Y}}^{GM}$ as the webs respectively defined by the face maps Π_F and π_F , for all ‘ $\mathcal{W}$ -relevant’ facets F of the weight polytope $\Delta_{G,P}$. From the equivariance property stated in (ii). above, it follows that $\mathcal{W}_X^{GM}$ is H -equivariant and that its quotient by H is precisely $\mathcal{W}_{\mathcal{Y}}^{GM}$.

The most basic example to have in mind is the one when $X = G_2(\mathbf{C}^5)$, which corresponds to the case of type (A_4, ω_2) .¹³ The moment/weight polytope is the hypersimplex $\Delta_{2,5} = \{(t_i)_{i=1}^5 \in [0, 1]^5 \mid \sum_{i=1}^5 t_i = 2\}$. This polytope has 10 facets which are obtained by intersecting it with the affine hyperplanes cut out by $t_i = \tau$ for $i = 1, \dots, 5$ and $\tau \in \{0, 1\}$. All the facets when $\tau = 1$ are 3-simplices, and the associated type is (A_3, ω_3) , hence these facets are not ‘ $\mathcal{W}$ -relevant’. Any facet $F_i = \Delta_{2,5} \cap \{t_i = 0\}$ is a hypersimplex of type (A_3, ω_2) , with $X_{F_i} \simeq G_2(\mathbf{C}^4)$. Denoting by $(e_i)_{i=1}^5$ the canonical basis of $\mathbf{C}^5$, the face map Π_{F_i} associated to F_i identifies with the rational map $G_2(\mathbf{C}^5) \dashrightarrow G_2(\mathbf{C}^5/\langle e_i \rangle) \simeq G_2(\mathbf{C}^4)$ induced by the linear quotient map $\mathbf{C}^5 \rightarrow \mathbf{C}^5/\langle e_i \rangle$. If one denotes by H and H_{F_i} the Cartan tori of the linear groups acting on X and X_{F_i} , then the corresponding GIT quotients respectively are $X//H = G_2(\mathbf{C}^5)//H \simeq \overline{\mathcal{M}}_{0,5}$ and $X_{F_i}//H_{F_i} \simeq \overline{\mathcal{M}}_{0,4} \simeq \mathbf{P}^1$. And the H -equivariant quotient of Π_{F_i} is the i -th forgetful map $\pi_{F_i} : \overline{\mathcal{M}}_{0,5} \dashrightarrow \overline{\mathcal{M}}_{0,4} = \mathbf{P}^1$ (which actually is a morphism). It follows that in this case, Gelfand-MacPherson web on $\mathcal{Y} = \overline{\mathcal{M}}_{0,5}$ is the 5-web with the five forgetful maps as first integrals, hence is a model of the dilogarithmic Bol’s web $\mathcal{B}$: one has

$$\mathcal{W}_{\overline{\mathcal{M}}_{0,5}}^{GM} = \mathcal{W}(\pi_{F_i} : \overline{\mathcal{M}}_{0,5} \longrightarrow \overline{\mathcal{M}}_{0,4} \simeq \mathbf{P}^1)_{i=1}^5 \simeq \mathcal{B}.$$

The main object of study in this paper is Gelfand-MacPherson’s web of the quotient, by the rank 5 Cartan torus, of the spinor 10-fold $\mathbb{S}_5$ which is a homogenous projective variety of type D_5 .

2.3. Birational geometry around the spinor 10-fold $\mathbb{S}_5$. We recall some facts we will need further in the paper. For details, see our previous article [Pi5] and the references therein.

¹²See [Sk], and in particular the list of cases to be excluded given in Proposition 2.1 therein.

¹³The best reference on this case is the great paper [GM] by Gelfand and MacPherson.

2.3.1. Groups of Lie type D_5 . Let $q \in \text{Sym}^2(V^\vee)$ be the non-degenerate complex quadratic form on $V = \mathbb{C}^{10}$ given by

$$q(x) = \sum_{i=1}^5 x_i x_{i+5}$$

in the standard coordinate system $(x_i)_{i=1}^{10}$ on V . The associated orthogonal group $O(V, q) = O_{10}(\mathbb{C})$ is a simple complex Lie group of type D_5 . Its universal covering is the so-called ‘*spin group*’ $\text{Spin}_{10}(\mathbb{C})$ and the canonical covering $\text{Spin}_{10}(\mathbb{C}) \rightarrow O_{10}(\mathbb{C})$ is known to be 2-to-1. The corresponding Dynkin diagram of type D_5 we are going to work with is labeled as follows:

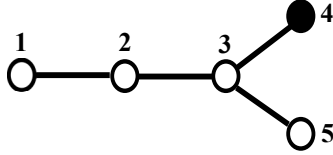


FIGURE 1. The marked Dynkin diagram of type (D_5, ω_4) . (The marking of the fourth vertex corresponds to the choice of the standard maximal parabolic subgroup used to construct the spinor tenfold, see below).

Coxeter matrix associated to the Dynkin D diagram of type D_5 : it is the symmetric 5×5 matrix $(m_{ij})_{i,j=1}^5$ where for all i, j such that $1 \leq i \leq j \leq 5$, one has $m_{ii} = 1$ and when $i < j$, one has $m_{ij} = 2$ if there is no arrow between the i -th and j -th vertices of the diagram, and $m_{ij} = 3$ otherwise. The associated ‘*Weyl group of type D_5* ’ is the group with generators s_i for $i = 1, \dots, 5$, with relations $(s_i s_j)^{m_{ij}}$ for any $i, j \in \llbracket 5 \rrbracket$. In the case under scrutiny, the generators s_i ’s of W_{D_5} are involutions, which pairwise commute, i.e. $(s_i s_j)^2 = 1$ except if the i -th and j -th vertices of D_5 are linked by one of its arrows, in which case one has $(s_i s_j)^3 = 1$. To summarise, setting $J = \{(j, j+1) \mid j = 1, \dots, 4\} \cup \{(3, 5)\}$, then for all distinct $i, j \in \llbracket 5 \rrbracket$, one has

$$(6) \quad \begin{aligned} & - s_i^2 = 1; \\ & - (s_i s_j)^2 = 1 \text{ (that is } s_i \text{ and } s_j \text{ commute) if } (i, j) \notin J; \\ & - (s_i s_j)^3 = 1 \text{ if the pair } (i, j) \text{ belongs to } J. \end{aligned}$$

The explicit form of these relations will be used further in §2.3.6.

2.3.2. The orthogonal grassmannian $OG_5(\mathbb{C}^{10})$, the spinor 10-fold $\mathbb{S}_5$ and the half-spin representations. By definition, the ‘*orthogonal grassmannian $OG_5(V)$* ’ is the subvariety of the standard grassmannian $G_5(V)$ whose points are the 5-dimensional subspaces $\Pi \subset V$ which are ‘totally isotropic’, that is such that $q|_{\Pi} \equiv 0$. It is known that $OG_5(V)$ is the disjoint union of two indistinguishable isomorphic irreducible components which will be denoted by $OG_5^\epsilon(V)$ with $\epsilon = \pm$.

The component $OG_5^+(V)$, which will be the privileged one for us, is homogeneous under the natural action of the ‘*spin group*’ $\text{Spin}_{10}(\mathbb{C})$. In fact, if P_4 stands for the standard maximal parabolic subgroup of the spin group associated to the fourth vertex of the Dynkin diagram of Figure 1, then one defines the ‘*spinor 10-fold*’ as the homogeneous variety $\mathbb{S}_5 = \text{Spin}_{10}(\mathbb{C})/P_4$ which can be proved to be isomorphic to $OG_5^+(V)$. Indeed, the parabolic subgroup P_4 being maximal, $\mathbb{S}_5$ has Picard number 1 with its Picard group spanned by an ample class denoted by H . Moreover, $\mathbb{S}_5$ is Fano with $-K_{\mathbb{S}_5} = h_{D_5} H$ where h_{D_5} stands for the Coxeter number in type D_5 (namely

$h_{D_5} = 8$). Actually, $-K_{\mathbb{S}_5}$ is very ample and the anticanonical linear system gives rise to an equivariant embedding $\varphi : \mathbb{S}_5 \hookrightarrow \mathbf{P}(S^+)$ where $S^+ \simeq \mathbf{H}^0(\mathbb{S}_5, -K_{\mathbb{S}_5})^\vee$ is a 16-dimensional $\mathrm{Spin}_{10}(\mathbf{C})$ -representation, one of the so-called *half-spin representations* $S^\pm$.¹⁴ Then it can be proved that the second Veronese embedding of $\mathbb{S}_5 \subset \mathbf{P}(S^+) \simeq \mathbf{P}^{15}$ coincides with the image of $OG_5^+(\mathbf{C}^5)$ in $\mathbf{P}(\wedge^5 V)$ obtained by taking the image of its inclusion into the ordinary grassmannian of 5-planes in V post-composed with the standard Plücker embedding $G_5(V) \hookrightarrow \mathbf{P}(\wedge^5 V)$.

The half-spin representations are known to be minuscule representations: for $\epsilon = \pm$, the Weyl group W_{D_5} acts transitively on the set of weights $\mathbb{W}^\epsilon$ which is of cardinality 16. Each weight space is a complex line and S^ϵ admits a basis $(v_\varpi)_{\varpi \in \mathbb{W}^\epsilon}$ with v_ϖ being a weight vector of weight ϖ for any $\varpi \in \mathbb{W}^\epsilon$. In other terms, there is a decomposition as a direct sum in which each weight subspace is of dimension 1:

$$(7) \quad S^\epsilon = \bigoplus_{\varpi \in \mathbb{W}^\epsilon} \mathbf{C} v_\varpi.$$

The set of weights in $\mathbb{W}^\epsilon$ can be described quite explicitly as the set of 5-tuples $\underline{\varepsilon}/2$ for all $\underline{\varepsilon} = (\varepsilon_i)_{i=1}^5 \in \{\pm 1\}^5$ whose parity $p(\underline{\varepsilon}) = \varepsilon_1 \cdots \varepsilon_5 \in \{\pm 1\}$ coincides with ϵ . More explicitly, denoting by $(e_1, \dots, e_5)$ the standard basis of $\mathbf{R}^5 \simeq \mathfrak{h}_{D_5}^{\mathbf{R}}$ and setting $e_K = \sum_{k \in K} e_k$ for any $K \subset \llbracket 5 \rrbracket$, the elements of $\mathbb{W}^+$ (resp. of $\mathbb{W}^-$) are the vectors

$$(8) \quad w_L = \frac{1}{2}(e_{\llbracket 5 \rrbracket \setminus L} - e_L)$$

for all subsets $L \subset \llbracket 5 \rrbracket$ of even (resp. of odd) cardinal.

2.3.3. Wick's embedding. As it is well known, a generic 5-plane in V can be represented by a matrix $M_A = [\mathbf{Id}_5, A] \in \mathrm{Mat}_{5 \times 10}(\mathbf{C})$ where $\mathbf{Id}_5$ stands for the identity 5×5 matrix and A a square matrix of the same size but arbitrary otherwise, the associated 5-plane to M_A , denoted by $\zeta_A = \langle [\mathbf{Id}_5, A] \rangle$, being the one spanned by the vectors whose coordinates (in the standard coordinate system) are the given by the five lines of M . Then one verifies easily that the 5-plane ζ_M is q -isotropic if and only if the matrix A is antisymmetric. Moreover, it can be proved that for $A, B \in \mathrm{Asym}_5(\mathbf{C})$ the two following facts are satisfied : (1) ζ_A and ζ_B belong to the same components of $OG_5(V)$, that we choose to be $OG_5^+(V)$; (2) $\zeta_A = \zeta_B$ if and only if $A = B$. We then have that the affine map

$$\widetilde{W} : \mathrm{Asym}_5(\mathbf{C}) \longrightarrow OG_5^+(V)$$

is an embedding which induces a birational equivalence

$$(9) \quad \mathrm{Asym}_5(\mathbf{C}) \simeq OG_5^+(V) \simeq \mathbb{S}_5.$$

One can deduce from it a birational parametrization of the image of $\mathbb{S}_5$ in $\mathbf{P}(S^+) \simeq \mathbf{P}^{15}$, known as the Wick's parametrization, which is easily described using the following notations:

- for $X \in \mathrm{Asym}_5(\mathbf{C})$ and $i \in \llbracket 5 \rrbracket$, we denote by X_i the symmetric 4×4 matrix obtained by deleting the i -th line and the i -th column from X ;
- Pf stands for the pfaffian of an antisymmetric matrix;
- for any $i, j \in \llbracket 5 \rrbracket$, x_{ij} denotes the rational function on $\mathrm{Asym}_5(\mathbf{C})$ associating the (i, j) -th coefficient of an antisymmetric 5×5 matrix;

¹⁴The two half-spin representations S^+ and S^- are isomorphic, but not in a canonical way.

- let $\llbracket 5 \rrbracket_{<}^2$ be the ordered set of pairs (i, j) such that $1 \leq i < j \leq 5$, with the lexicographic order. Then for $k = 1, \dots, 10$, one denotes by X_k the coordinate x_{ij} if (i, j) is the k -th element of $\llbracket 5 \rrbracket_{<}^2$: one has $X_1 = x_{12}, X_2 = x_{13}, \dots, X_9 = x_{35}$ and $X_{10} = x_{45}$.

The projectivization

$$(10) \quad W = [\widehat{W}] : \text{Asym}_5(\mathbf{C}) \longrightarrow \mathbf{P}^{15}$$

of the affine map

$$(11) \quad \begin{aligned} \widehat{W} : \text{Asym}_5(\mathbf{C}) &\longrightarrow \mathbf{C}^{16} \\ X = (x_{ij})_{i,j=1}^5 &\longmapsto (1, X_1, \dots, X_{10}, \text{Pf}(X_{\hat{1}}), \dots, \text{Pf}(X_{\hat{5}})) \end{aligned}$$

is an affine parametrization of $\mathbb{S}_5 \subset \mathbf{P}^{15}$, known as *Wick's parametrization* of the spinor variety. It enjoys the nice property of having its components (as given in (11)) corresponding to the direct sum decomposition of S^+ into weight subspaces: for each component C of $\widehat{W}$, there is a well-defined weight $w(C) \in \mathbb{B}^+$ such that C is the composition of $\widehat{W}$ with the linear projection onto the weight subspace $\mathbf{C}_{V_{w(C)}}$ (with respect to the decomposition in direct sum (7)). The weights $w(C)$ are given by the following formulas where we use the notation (8): one has

$$w(1) = w_0, \quad w(x_{ij}) = w_{\{i,j\}} \quad \text{and} \quad w(\text{Pf}(X_{\hat{k}})) = w_{\llbracket 5 \rrbracket \setminus \{k\}}$$

for all $(i, j) \in \llbracket 5 \rrbracket_{<}^2$ and all $k = 1, \dots, 5$.

2.3.4. The action of the Cartan torus on $\mathbb{S}_5$ and a birational model of the associated torus quotient. We now recall some results of Serganova and Skorobogatov about the action of the Cartan torus H_{D_5} of $\text{Spin}_{10}(\mathbf{C})$ on $\mathbb{S}_5$.

To simplify the notation, we write H instead of H_5 below. Following Serganova and Skorobogatov, we denote by $\mathbb{S}_5^{sf}$ the subset of points $x \in \mathbb{S}_5$ which are stable under the action of H with stabilizer $\text{Stab}_H(x) = Z() = \{\pm 1\}$. It is open in $\mathbb{S}_5$, and of codimension ≥ 2 . One can prove that it contains the points of $\mathbb{S}_5$ whose at most one of the 16 Wick coordinates vanish. In particular, if H_w stands for the coordinate hyperplane in $\mathbf{P}(S^+)$ given by the vanishing of the w -coordinate, $\mathbb{S}_5^{sf}$ contains the generic point of the coordinate hyperplane section $\mathbb{S}_5 \cap H_w$.

From a direct application of GIT, one obtains that the following points hold:

- the quotient $\mathcal{Y}_5 = \mathbb{S}_5^{sf}/H$ is a 5-dimensional quasi-projective variety which is a Zariski-open subset of the GIT quotient $\overline{\mathcal{Y}}_5 = \mathbb{S}_5//H$. Moreover, the canonical map $\nu : \mathbb{S}_5^{sf} \rightarrow \mathcal{Y}_5$ is a geometric quotient: the preimages of ν are the H -orbits in $\mathbb{S}_5^{sf}$;
- for any weight $w \in \mathbb{B}^+$,

$$(12) \quad D_w = \nu(\mathbb{S}_5^{sf} \cap H_w)$$

is an irreducible divisor in $\mathcal{Y}_5$, that we will call ‘the weight divisor of weight w ’;

- one has $\nu(\mathbb{S}_5^*) = \mathcal{Y}_5^*$, where we have set

$$\mathbb{S}_5^* = \mathbb{S}_5 \setminus \left(\bigcup_{w \in \mathbb{B}^+} H_w \right) \quad \text{and} \quad \mathcal{Y}_5^* = \mathcal{Y}_5 \setminus \left(\bigcup_{w \in \mathbb{B}^+} D_w \right).$$

Moreover, Serganova and Skorobogatov proved very nice results about the quasi-projective variety $\mathcal{Y}_5$:

- let $\widehat{T}$ be the character lattice of the diagonal subtorus T of $\mathrm{GL}(S^+)$ generated by the 1-parameter subgroup of scalar matrices $\mathbf{C}^* \mathrm{Id}_{S^+}$ and the image of H by the natural embedding $\mathrm{Spin}_{10}(\mathbf{C}) \subset \mathrm{GL}(S^+)$. Then there is a natural isomorphism of lattices $\widehat{T} \simeq \mathbf{Pic}_{\mathbf{Z}}(\mathcal{Y}_5)$ (see the very end of [SS]);
- the set $\{D_w\}_{w \in \mathbb{W}^+}$ is the unique minimal set of $\mathbf{Z}_{>0}$ -generators of the the semi-group of effective divisor classes in $\mathbf{Pic}_{\mathbf{Z}}(\mathcal{Y}_5)$ (see [Sk, Theorem 1.6]);
- it follows from [Sk, Theorem 2.2.2] that there is a canonical isomorphism of group

$$(13) \quad W_{D_5} \simeq \mathrm{Aut}(\mathcal{Y}_5).$$

The variety $\mathcal{Y}_5$ is rational. We recall below the construction of the birational equivalence $\mathbf{C}^5 \simeq \mathcal{Y}_5$ considered in [Pi5]. We will work with it further to get in explicit form the birational realization of the Weyl group $W_{D_5} = \mathrm{Aut}(\mathcal{Y}_5)$ induced by this birational identification.

The quotient of $\mathbb{S}_5$ by H_{D_5} can be birationally identified with that of $OG_5^+(\mathbf{C}^{10})$ by the Cartan torus H'_{D_5} of $\mathrm{SO}_5(\mathbf{C}^{10})$, hence to the one of $\mathcal{H} = (\mathbf{C}^*)^5$ on $\mathrm{Asym}_5(\mathbf{C})$, where the action of the latter torus is given by

$$A \cdot h = \left(h_i h_j A_{ij} \right)_{i,j=1}^5 \quad {}^{15}$$

for any $h = (h_i)_{i=1}^5 \in \mathcal{H} = (\mathbf{C}^*)^5$ and any $A = (A_{ij})_{i,j=1}^5 \in \mathrm{Asym}_5(\mathbf{C})$. Identifying $\mathbf{C}(\mathrm{Asym}_5(\mathbf{C}))$ with $\mathbf{C}(x_{i,j} \mid 1 \leq i < j \leq 10)$ (where $x_{i,j}$ stands for the map associating the (i, j) -th coefficient), one can prove the

Proposition 2.1. *1. The algebra of rational functions on $\mathrm{Asym}_5(\mathbf{C})$ invariant by the action of $\mathcal{H}$ is free and generated by the components of the rational map $\mathcal{P}_5 : \mathrm{Asym}_5(\mathbf{C}) \dashrightarrow \mathbf{C}^5$ given by*

$$\begin{bmatrix} 0 & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ -x_{1,2} & 0 & x_{2,3} & x_{2,4} & x_{2,5} \\ -x_{1,3} & -x_{2,3} & 0 & x_{3,4} & x_{3,5} \\ -x_{1,4} & -x_{2,4} & -x_{3,4} & 0 & x_{4,5} \\ -x_{1,5} & -x_{2,5} & -x_{3,5} & -x_{4,5} & 0 \end{bmatrix} \longmapsto \left(\frac{x_{4,5} x_{1,3}}{x_{1,5} x_{3,4}}, \frac{x_{1,5} x_{2,4}}{x_{1,2} x_{4,5}}, \frac{x_{1,2} x_{3,5}}{x_{1,5} x_{2,3}}, \frac{x_{2,3} x_{1,4}}{x_{3,4} x_{1,2}}, \frac{x_{3,4} x_{2,5}}{x_{2,3} x_{4,5}} \right).$$

Consequently, $\mathcal{P}_5$ is a birational model of the quotient map $\nu : \mathbb{S}_5 \dashrightarrow \mathcal{Y}_5$.

2. Moreover, the map $\mathcal{P}_5$ admits the following map as a rational section:

$$Y : y = (y_{13}, y_{14}, y_{24}, y_{25}, y_{35}) \longmapsto Y(y) = \begin{bmatrix} 0 & 1 & y_{13} & y_{14} & 1 \\ -1 & 0 & 1 & y_{24} & y_{25} \\ -y_{13} & -1 & 0 & 1 & y_{35} \\ -y_{14} & -y_{24} & -1 & 0 & 1 \\ -1 & -y_{25} & -y_{35} & -1 & 0 \end{bmatrix},$$

i.e. setting $\mathbf{C}(y) = \mathbf{C}(y_{13}, y_{14}, y_{24}, y_{25}, y_{35})$, one has $\mathcal{P}_5 \circ Y = \mathrm{Id}_{\mathbf{C}(y)}$ (equality as rational maps).

Considering this proposition, let Y_5 be the affine space $\mathbf{C}^5$ with the rational coordinates $y_1, \dots, y_5$ related to the y_{ij} 's appearing in the definition of Y above, via the relations $y_1 = y_{13}$, $y_2 = y_{14}$,

¹⁵From a matricial point of view, it is more natural to see the action of $(\mathbf{C}^*)^5$ on $\mathrm{Asym}_5(\mathbf{C})$ as a right-action, but this is just a matter of notation.

$y_3 = y_{24}, y = y_{25}, y = y_{35}$. From the relation $\mathcal{P}_5 \circ Y = \text{Id}_{\mathbf{C}(y)}$, one deduces that the map Θ defined by requiring that the following diagram of rational maps commutes

$$(14) \quad \begin{array}{ccc} \text{Asym}_5(\mathbf{C}) & \xrightarrow{W} & \mathbb{S}_5 \subset \mathbf{P}(S^+) \\ \downarrow Y \quad \downarrow \mathcal{P}_5 & & \downarrow \nu \\ Y_5 & \xrightarrow{\Theta} & \mathcal{Y}_5, \end{array}$$

is birational. We thus have a birational identification

$$(15) \quad \Theta : \mathbf{C}^5 = Y_5 \xrightarrow{\sim} \mathcal{Y}_5$$

and it is the one we will always work with in what follows.

The weight divisors D_w in $\mathcal{Y}_5$ will be important for our purpose hence it is interesting to investigate how they appear (or do not appear) on the birational model $\mathcal{Y}_5$. Remark first that Y is polynomial hence so is the composition $W \circ Y : Y_5 \rightarrow \mathbb{S}_5 \subset \mathbf{P}(S^+) \simeq \mathbf{P}^{15}$. Viewed the definition of the weight divisors in $\mathcal{Y}_5$, we get that their pull-backs under Θ are cut out by the components of $\widehat{W} \circ Y$. Since some of the entries of $Y(y)$ are equal to 1, we see that some of the weight divisors in $\mathcal{Y}_5$ have no pull-back as divisors in Y_5 under Θ . Clearly, there are 10 divisors in $\mathcal{Y}_5$ corresponding to some D_w 's, and these are the five coordinate hyperplanes $\{y_i = 0\}$ with $i = 1, \dots, 5$, and the other five are given by the vanishing of the equations obtained by taking the pfaffians of the 4×4 principal antisymmetric submatrices $Y_1, \dots, Y_5$ of $Y(y)$. These subpfaffians, denoted by $P_i = \text{Pf}(Y_i)$ for $i = 1, \dots, 5$, are given by:

$$(16) \quad \begin{aligned} P_1 &= 1 + y_{25} - y_{24}y_{35}, & P_2 &= 1 + y_{13} - y_{35}y_{14}, & P_3 &= 1 + y_{24} - y_{14}y_{25} \\ P_4 &= 1 + y_{35} - y_{13}y_{25}, & P_5 &= 1 + y_{14} - y_{13}y_{24}. \end{aligned}$$

Then setting

$$(17) \quad \zeta_1 = y_{13}, \quad \zeta_2 = y_{14}, \quad \zeta_3 = y_{24}, \quad \zeta_4 = y_{25}, \quad \zeta_5 = y_{35} \quad \text{and} \quad \zeta_{i+5} = P_i \quad \text{for } i \in \llbracket 5 \rrbracket,$$

we define $Z_k = \{\zeta_k = 0\} \subset Y_5$ for $k = 1, \dots, 10$ and one sets

$$(18) \quad Z = \cup_{k=1}^{10} Z_k = \{\zeta_1 \cdots \zeta_{10} = 0\} \quad \text{and} \quad Y_5^* = \mathbf{C}^5 \setminus Z.$$

One verifies easily the

Proposition 2.2. *The map Θ is defined at every point of Y_5^* and sends it into $\mathcal{Y}_5^*$. Actually, one has $\Theta(Y_5^*) = \mathcal{Y}_5^*$ and the restriction $\Theta|_{Y_5^*}$ induces an isomorphism from Y_5^* onto $\mathcal{Y}_5^*$.*

2.3.5. Del Pezzo quartic surfaces and Serganova-Skorobogatov embeddings. The geometry of a del Pezzo quartic surface (and in particular the geometry of the lines contained in it) is related in a nice way to that of the spinor tenfold $\mathbb{S}_5$ and of its torus quotient $\mathcal{Y}_5$. This is important for our purpose and because it also allows a convenient way to work with the weights of the minuscule representation S^+ , we recall it now.

Let dP_4 be a fixed smooth del Pezzo surface. In finitely many ways (none of which more canonical than the others), it is isomorphic to the total space $X = \mathbf{Bl}_P(\mathbf{P}^2)$ of the projective plane along a subset $P = \{p_1, \dots, p_5\}$ of five points p_i in general position in $\mathbf{P}^2$. There are 16 lines in X and each can be identified with its class in $\text{Pic}_{\mathbb{Z}}(X)$. If e_i stands for the class of the exceptional divisor over p_i in the blow-up $\beta = \beta_P : X = \mathbf{Bl}_P(\mathbf{P}^2) \rightarrow \mathbf{P}^2$ along P (for any $i \in \llbracket 5 \rrbracket$), and if h

denotes the class of $\beta^{-1}(\ell)$ for any line $\ell \subset \mathbf{P}^2 \setminus P$, then the Picard lattice $\mathbf{Pic}_{\mathbb{Z}}(X)$ is freely spanned over the integers by the e_i 's and h and the (classes of) lines in X are the following:

$$e_i, \quad h - e_i - e_j \quad \text{and} \quad 2h - e_{tot},$$

with i and j distinct and ranging in $\llbracket 5 \rrbracket$, where we use the notation $e_{tot} = \sum_{k=1}^5 e_k$. We will denote by $\mathcal{L}_X$ the set of lines contained in X , viewed as a subset of the Picard lattice of X .

As is well-known, the lines of X can be put in a 1-1 relation with the weights in $\mathfrak{W}^+$. Let us recall how it goes: let R be the orthogonal, for the intersection product, of the canonical class $K_X = -3h + e_{tot}$ in the Picard group $\mathbf{Pic}(X) = \mathbf{Pic}_{\mathbb{Z}}(X) \otimes \mathbf{R}$. Endowed with the opposite of the (restriction of the) intersection product, R becomes a 'root space of type D_5 ': it is Euclidean and admits as a basis the following five 'fundamental roots'

$$\rho_i = e_{i+1} - e_i \quad (i = 1, \dots, 4) \quad \text{and} \quad \rho_5 = h - e_1 - e_2 - e_3.$$

Denote by $\Pi : \mathbf{Pic}(X) \rightarrow R$ the orthogonal projection (whose kernel is $\langle K \rangle = R^\perp$). Let $(f_i)_{i=1}^5$ be the other basis of R determined by the relations $f_i - f_{i+1} = -\rho_i = e_i - e_{i+1}$ for $i = 1, \dots, 4$ and $f_4 + f_5 = \rho_5$, and let

$$(19) \quad \psi : R \longrightarrow \mathbf{R}^5, \quad \sum_{i=1}^5 u_i f_i \longmapsto (u_i)_{i=1}^5$$

be the associated isomorphism. Then the linear map $\mu = \psi \circ \Pi : \mathbf{Pic}(X) \rightarrow \mathbf{R}^5$ induces a bijection

$$(20) \quad \mu : \mathcal{L}_X \simeq \mathfrak{W}^+$$

from the set of lines in X onto the set of weights of S^+ which is explicitly given by

$$\mu(e_i) = w_{\llbracket 5 \rrbracket \setminus \{i\}}, \quad \mu(h - e_i - e_j) = w_{\{i, j\}} \quad \text{and} \quad \mu(2h - e_{tot}) = w_\emptyset,$$

for all $i, j \in \llbracket 5 \rrbracket$ distinct.

The previous bijection $\mathcal{L}_X \simeq \mathfrak{W}^+$ can be used to compute the action of the Weyl group W_{D_5} on the weights of S^+ . Indeed, building a graph with the i -th vertex corresponding to ρ_i and an arrow between two vertices with labels i and j if and only if the two corresponding roots satisfy $(\rho_i, \rho_j) = 1$, one exactly gets the labeled Dynkin diagram of Figure 1. Then for any $i = 1, \dots, 5$, one considers the map

$$(21) \quad s_i = s_{\rho_i} : d \mapsto d + (\rho_i, d) \rho_i.$$

One verifies that it is an involutive automorphism of $(\mathbf{Pic}(X), (\cdot, \cdot))$ letting K_X invariant, hence it gives rise to an orthogonal involution of R . The s_i 's satisfy the relations (6) so what we have obtained is a concrete realization of the Weyl group W_{D_5} , previously abstractly defined as a subgroup of $O(R)$ (in particular, this justifies having denoted by s_i the map (21)). This will allow us to compute in an explicit way the action of W_{D_5} on the set $\mathfrak{W}^+$ of weights of S^+ . As an example, we treat the case of s_1 . Via (21), it acts linearly on $\mathbf{Pic}(X)$ and is fully determined by indicating that it exchanges e_1 and e_2 and lets all the other generators e_3, e_4, e_5 and h unchanged. We deduce that its action on $\mathcal{L}_X \simeq \mathfrak{W}^+$ is given by

$$(22) \quad w_{\llbracket 5 \rrbracket \setminus \{1\}} \leftrightarrow w_{\llbracket 5 \rrbracket \setminus \{2\}} \quad \text{and} \quad w_{\{1, k\}} \leftrightarrow w_{\{2, k\}}$$

with $k = 3, 4, 5$, with the convention that the exchanges above (indicated by the double arrow $\leftrightarrow$) are the only non-trivial actions of s_i on the set of weights.

The 1-1 correspondence between $\mathcal{L}_X$ and $\mathbb{W}^+$ actually can be given a geometric origin. We denote by $L = \cup_{\ell \in \mathcal{L}} \ell$ with the lines viewed as subsets of X , and we set $X^* = X \setminus L$.

For any line $\ell \subset X$, let σ_ℓ be a generator of the 1-dimensional space of global sections $\mathbf{H}^0(X, \mathcal{O}_X(\ell))$ and let us consider the map $\hat{\sigma} : X \dashrightarrow S^+$ the w -component of which, for any weight $w \in \mathbb{W}^+$, is given by the section σ_ℓ for the line ℓ which corresponds to w up to the bijection $\mu : \mathcal{L}_X \simeq \mathbb{W}^+$ discussed above: $\hat{\sigma}_\mathcal{L} = (\sigma_{\mu^{-1}(w)})_{w \in \mathbb{W}^+}$. This map becomes a morphism when restricted to X^* and is well-defined up to post-composition by an element of the diagonal torus of $\mathrm{GL}(S^+)$. We claim that there is a way to chose the components σ_ℓ such that the projectivisation

$$(23) \quad F_\mathcal{L} = [\hat{\sigma}_\mathcal{L}] : X^* \rightarrow \mathbf{P}(S^+)$$

has values in $\mathbb{S}_5^*$. Moreover, this map extends to a well-defined morphism from X to $\mathbb{S}_5^{sf}$ denoted the same. For any weight $w \in \mathbb{W}^+$, let H_w be the coordinate hyperplane in $\mathbf{P}(S^+)$ corresponding to the vanishing of the affine w -coordinate (with respect to the decomposition in direct sum (7)). Then one has $F_\mathcal{L}^{-1}(H_{w(\ell)}) = \ell$ for any line $\ell \in \mathcal{L}$, which gives a geometric explanation to the bijection (20).

It can be proven that the composition $\nu \circ F_\mathcal{L}$ makes commutative the following diagram

$$(24) \quad \begin{array}{ccc} & & \mathbb{S}_5^{sf} \subset \mathbf{P}(S^+) \\ & \nearrow F_\mathcal{L} & \downarrow \nu \\ X & \xrightarrow{f_{SS}} & \mathcal{Y}_5 \end{array}$$

where the map $f_{SS} : X \rightarrow \mathcal{Y}_5$ is an embedding which has been introduced (in a non-constructive way) by Serganova and Skorobogatov in [SS] (hence the two S 's in the notation f_{SS}). In addition to being an embedding, f_{SS} satisfies several nice properties, such as the fact that the preimage of a weight divisor of $\mathcal{Y}_5$ is the line in X corresponding to the weight¹⁶ or that it induces an isomorphism of Picard lattices $f_{SS}^* : \mathrm{Pic}_\mathbb{Z}(\mathcal{Y}_5) \simeq \mathrm{Pic}_\mathbb{Z}(X) \simeq \mathbb{Z}^6$.

For our purpose, we need to make explicit the rational map $F_\mathcal{L} \circ \beta^{-1} : \mathbf{P}^2 \dashrightarrow \mathbb{S}_5$, or rather its birational model

$$G_\mathcal{L} = W^{-1} \circ F_\mathcal{L} \circ \beta^{-1} : \mathbf{P}^2 \dashrightarrow \mathrm{Asym}_5(\mathbb{C}).$$

Assume that X is the blow-up of $\mathbf{P}^2$ in five points p_i which, with respect to the standard homogeneous coordinates, are the vertices of the standard simplex for $p_1, \dots, p_4$, and with $p_5 = [a : b : 1]$ where a, b are two complex numbers only assumed to be such that $ab(a-1)(b-1)(a-b) \neq 0$ (this to ensure that the p_i 's are in general position in $\mathbf{P}^2$). In [Pi5], we have established that the expression of $G_\mathcal{L}$ in the affine coordinates x, y is given by

$$G_\mathcal{L}(x, y) = \frac{1}{C_{ab}} \begin{bmatrix} 0 & -b-a & -(a+1)y & (1-y)a & y-b \\ b+a & 0 & -x(1+b) & (1-x)b & x-a \\ (a+1)y & x(1+b) & 0 & y-x & bx-ay \\ (y-1)a & (x-1)b & x-y & 0 & (b-1)x + (1-a)y - b + a \\ b-y & a-x & ay-bx & (1-b)x + (a-1)y + b-a & 0 \end{bmatrix}$$

where $C_{ab} = (a-b)xy + b(1-a)x + a(b-1)y$ is an affine quadratic polynomial which cuts out the (affine part of the) conic passing through the points $p_1, \dots, p_5$.

¹⁶In mathematical terms: for any weight $w \in \mathbb{W}^+$, one has $f_{SS}^{-1}(D_w) = \ell \subset X$, where ℓ stands for the line such that $\mu(\ell) = w$.

In the considerations above, a and b are fixed parameters, but it makes sense and it is interesting to let them vary. Doing this, $G_{\mathcal{L}}$ is seen as a rational map from $\mathbf{C}^4$ (with affine coordinates x, y, a, b) onto $\text{Asym}_5(\mathbf{C})$, that we will denote simply by G . This map can be considered as a birational model of a putative rational map $\mathcal{G} : d\mathcal{P}_4 \dashrightarrow \mathbb{S}_5 \subset \mathbf{P}(S^+)$, where the source space is the ‘universal surface $d\mathcal{P}_4$ over the moduli space $\mathcal{M}_{d\mathcal{P}_4}$ (birational to $\mathbf{P}^2$) of quartic del Pezzo surfaces’, such that the restriction of $\mathcal{G}$ along a fiber of $d\mathcal{P}_4 \rightarrow \mathcal{M}_{d\mathcal{P}_4}$ birationally coincides with the map (23) of the corresponding del Pezzo quartic surface.

In the putative setting considered above, post-composing $\mathcal{G}$ with the quotient mapping $\mathbb{S}_5 \dashrightarrow \mathcal{Y}_5$ would give a map $\mathcal{F} : d\mathcal{P}_4 \dashrightarrow \mathcal{Y}_5$ whose restriction along the fibers of the universal del Pezzo quartic surface would coincide with Serganova-Skorobogatov’s embedding of the surface. The well-defined rational map

$$(25) \quad F = (F_i)_{i=1}^5 = \mathcal{P}_5 \circ G : \mathbf{C}^4 \dashrightarrow \mathbf{C}^5$$

can be thought of as a birational model of $\mathcal{F}$. Its components $F_i \in \mathbf{C}(x, y, a, b)$ are the following ones:

$$\begin{aligned} F_1 &= \frac{(ay - bx - a + b + x - y)(a + 1)y}{(b - y)(-y + x)} \\ F_2 &= \frac{x(b + 1)(y - 1)a}{(-y + x)(b + a)} \\ F_3 &= \frac{(b - y)(-1 + x)b}{(b + a)((1 - x)b + x + (y - 1)a - y)} \\ F_4 &= \frac{(-y + x)(a - x)}{x((1 - x)b + x + (y - 1)a - y)(b + 1)} \\ \text{and } F_5 &= \frac{(b + a)(ay - bx)}{(b - y)x(b + 1)}. \end{aligned}$$

Since for any del Pezzo quartic surface, the preimages by the associated map $f_{\mathbb{S}_5}$ of the 16 weight divisors $D_w = \nu(H_w \cap \mathbb{S}_5^{sf})$ are the 16 lines of the del Pezzo surface, one can expect something similar for the map F . And this does occur in a way. Indeed, the divisor Z in $\mathcal{Y}_5$ has only 10 components, but it can be verified that the preimage of $Z = \{\zeta_F = \prod_{i=1}^{10} \zeta_i = 0\}$ by F has 16 hypersurface components somehow. More precisely, one verifies that the support of the divisor cut out by $\zeta_Z(F) = \prod_{i=1}^{10} \zeta_i(F) = 0$ in $\mathbf{C}^4$ has 15 irreducible components, namely the irreducible affine hypersurfaces cut out by the following polynomials in the variables x, y, a and b :

$$\begin{aligned} x, y, x - 1, x - a, y - 1, y - b, x - y, bx - ay, a, a + 1, b, b + 1, b + a, \\ ab(x - y) - (a - b)xy + ay - bx, (a - 1)y - (b - 1)x - a + b. \end{aligned}$$

Together with the hyperplane at infinity in a compactification $\mathbf{C}^4 \subset \mathbf{P}^4$, this makes 16 irreducible divisors in total, which correspond to the 16 weight divisors D_w (with $w \in \mathbb{W}^+$) of $\mathcal{Y}_5$. We will not really use this below, but we wanted to stress this fact which indicates that the coordinates x, y, a, b and the map F are well-adapted to study $\mathcal{Y}_5$ from a birational point of view.

2.3.6. The birational action of W_{D_5} . Conjugating the isomorphism (13) by the birational identification (15) gives rise to a birational representation of the Weyl group W_{D_5} into the Cremona group $\text{Bir}(\mathbf{C}^5)$ of birational transformations of $\mathbf{C}^5$. Our goal here is to make this representation explicit.

Recall that $W_{D_5} = N/H_{D_5}$ where $N = N_{\text{Spin}_{10}(\mathbf{C})}(H_{D_5})$ stands for the normalizer of H_{D_5} in the spin group. From the proof of Theorem 2.2 in [Sk], we know that the action of an element $w \in W_{D_5}$ on $\mathcal{Y}_5$ is induced by the one of a lift $\hat{w}$ in the normalizer. Recall that w acts by permutations and transitively on the set of lines $\mathcal{L}$ and acts also naturally on the set of weights $\mathbb{B}^+$. Moreover, since the bijection (20) is W_{D_5} -equivariant (as an easy verification shows), one deduces a natural (but naive) lift for w , namely the linear map $\tilde{w}$ on S^+ defined requiring that it acts by permuting the element of the weight basis $(v_{\varpi})_{\varpi \in \mathbb{B}^+}$ of S^+ (see (7)) according to the weight, i.e. one has $\tilde{w}(v_{\varpi}) = v_{w(\varpi)}$ for any weight ϖ . However it is not the case that this lift belongs to the normalizer of the Cartan torus. Indeed, it is by no mean canonical, it actually depends on the chosen weight basis $(v_{\varpi})_{\varpi \in \mathbb{B}^+}$. Since each v_{ϖ} is only well-defined up to multiplication by a non-zero scalar, we look for a lift $\hat{w}$ acting as $v_{\varpi} \mapsto \lambda_{\varpi}^w \cdot v_{w(\varpi)}$ with $\lambda_{\varpi}^w \in \mathbf{C}^*$ for any ϖ .

Let us apply this strategy to the first generator s_1 of W_{D_5} . Let us first consider $\tilde{s}_i$, which stands for the ‘dumb lift’ of s_1 mentioned above. Given $\eta = \widehat{W}(X)$ for

$$X = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{13} & -x_{13} & 0 & x_{34} & x_{35} \\ -x_{14} & -x_{14} & -x_{34} & 0 & x_{45} \\ -x_{15} & -x_{15} & -x_{35} & -x_{45} & 0 \end{bmatrix} \in \text{Asym}_5(\mathbf{C}),$$

generic, one easily deduces from (22) the weight coordinates of $\tilde{L}_1(\eta) = (\tilde{s}_i \circ W)(X)$: one has

$$\eta = (1, x_{12}, x_{13}, x_{14}, x_{15}, x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45}, \text{Pf}(X_{\hat{1}}), \text{Pf}(X_{\hat{2}}), \text{Pf}(X_{\hat{3}}), \text{Pf}(X_{\hat{4}}), \text{Pf}(X_{\hat{5}}))$$

$$\text{and } \tilde{L}_1(\eta) = (1, x_{12}, \textcolor{blue}{x_{23}}, \textcolor{blue}{x_{24}}, \textcolor{blue}{x_{25}}, \textcolor{blue}{x_{13}}, \textcolor{blue}{x_{14}}, \textcolor{blue}{x_{15}}, x_{34}, x_{35}, x_{45}, \textcolor{blue}{\text{Pf}(X_{\hat{2}})}, \textcolor{blue}{\text{Pf}(X_{\hat{1}})}, \text{Pf}(X_{\hat{3}}), \text{Pf}(X_{\hat{4}}), \text{Pf}(X_{\hat{5}}))$$

(the coordinates which have been permuted in $\tilde{s}_1(\eta)$ are in blue). The transformation $\eta \mapsto \tilde{L}_1(\eta)$ is induced by an invertible linear map $\tilde{L}_1$ of S^+ (which permutes the coordinates in blue) which however is not an element of the image of the spin group in $\text{GL}(S^+)$ since it can be verified that $\tilde{L}_1(\mathbb{S}_5) \not\subset \mathbb{S}_5$. Then let $\overline{L}_1 \in \text{GL}(S^+)$ be obtained from $\tilde{L}_1$ by post-composing it with a diagonal matrix: there are non-zero complex numbers λ_k for $k = 0, \dots, 5$ and λ_{ij} for $(i, j) \in \llbracket 5 \rrbracket_{<}^2$ such that $\overline{L}_1$ is entirely characterized by the fact that one has

$$\begin{aligned} \overline{L}_1(\eta) = & (\lambda_0, \lambda_{12} x_{12}, \lambda_{13} \textcolor{blue}{x_{23}}, \lambda_{14} \textcolor{blue}{x_{24}}, \lambda_{15} \textcolor{blue}{x_{25}}, \lambda_{23} \textcolor{blue}{x_{13}}, \lambda_{24} \textcolor{blue}{x_{14}}, \lambda_{25} \textcolor{blue}{x_{15}}, \lambda_{34} x_{34}, \lambda_{35} x_{35}, \lambda_{45} x_{45}, \\ & \lambda_1 \textcolor{blue}{\text{Pf}(X_{\hat{2}})}, \lambda_2 \textcolor{blue}{\text{Pf}(X_{\hat{1}})}, \lambda_3 \text{Pf}(X_{\hat{3}}), \lambda_4 \text{Pf}(X_{\hat{4}}), \lambda_5 \text{Pf}(X_{\hat{5}})) \end{aligned}$$

for $\eta = \widehat{W}(X)$ with X generic. Since post-composing by an element of the Cartan torus is irrelevant, there is no loss in generality by assuming that $\lambda_k = 1$ for $k = 1, \dots, 5$. That $\overline{L}_1$ belongs to the normalizer of H_{D_5} in the spin group implies in particular that $\overline{L}_1$ stabilizes $\mathbb{S}_5$. Setting $\Lambda = \{\lambda_0\} \cup \{\lambda_{ij} \mid 1 \leq i < j \leq 5\}$ and

$$\overline{X} = \frac{1}{\lambda_0} \begin{bmatrix} 0 & \lambda_{12} x_{12} & \lambda_{13} \textcolor{blue}{x_{23}} & \lambda_{14} \textcolor{blue}{x_{24}} & \lambda_{15} \textcolor{blue}{x_{25}} \\ -\lambda_{12} x_{12} & 0 & \lambda_{23} \textcolor{blue}{x_{13}} & \lambda_{24} \textcolor{blue}{x_{14}} & \lambda_{25} \textcolor{blue}{x_{15}} \\ -\lambda_{13} \textcolor{blue}{x_{23}} & -\lambda_{23} \textcolor{blue}{x_{13}} & 0 & \lambda_{34} x_{34} & \lambda_{35} x_{35} \\ -\lambda_{14} \textcolor{blue}{x_{24}} & -\lambda_{24} \textcolor{blue}{x_{14}} & -\lambda_{34} x_{34} & 0 & \lambda_{45} x_{45} \\ -\lambda_{15} \textcolor{blue}{x_{25}} & -\lambda_{25} \textcolor{blue}{x_{15}} & -\lambda_{35} x_{35} & -\lambda_{45} x_{45} & 0 \end{bmatrix}$$

the condition $\bar{L}_1(\mathbb{S}_5) = \mathbb{S}_5$ is equivalent to the fact that the following pfaffian relations are satisfied for any generic (hence for any) matrix $X \in \text{Asym}_5(\mathbf{C})$:

$$(26) \quad \text{Pf}(\bar{X}_1) = \text{Pf}(X_2), \quad \text{Pf}(\bar{X}_2) = \text{Pf}(X_1) \quad \text{and} \quad \text{Pf}(\bar{X}_s) = \text{Pf}(X_s) \quad \text{for } s = 3, 4, 5.$$

These relations are polynomial identities in the indeterminates x_{ij} with $1 \leq i < j \leq 5$ whose coefficients are rational expressions in the elements of Λ . Assuming that all these coefficients are zero corresponds to a system of polynomial equations in the λ 's which is not difficult to solve. One obtains that the matrix $\bar{X}$ is necessarily the following one:

$$i \begin{bmatrix} 0 & x_{12} & x_{23} & x_{24} & x_{25} \\ -x_{12} & 0 & x_{13} & x_{14} & x_{15} \\ -x_{23} & -x_{13} & 0 & -x_{34} & -x_{35} \\ -x_{24} & -x_{14} & x_{34} & 0 & -x_{45} \\ -x_{25} & -x_{15} & x_{35} & x_{45} & 0 \end{bmatrix}.$$

The Cremona map σ_1 induced by s_1 is given by $\mathcal{P}_5(\bar{Y})$ where $\mathcal{P}_5$ and Y are given in Proposition 2.1. More explicitly, for $y = (y_{13}, y_{14}, y_{24}, y_{25}, y_{35})$, one has

$$\sigma_1(y) = \mathcal{P}_5(\bar{Y}(y)) = \mathcal{P}_5 \left(\begin{bmatrix} 0 & 1 & y_{13} & y_{14} & 1 \\ -1 & 0 & 1 & y_{24} & y_{25} \\ -y_{13} & -1 & 0 & 1 & y_{35} \\ -y_{14} & -y_{24} & -1 & 0 & 1 \\ -1 & -y_{25} & -y_{35} & -1 & 0 \end{bmatrix} \right) = \mathcal{P}_5 \left(i \begin{bmatrix} 0 & 1 & 1 & y_{24} & y_{25} \\ -1 & 0 & y_{13} & y_{14} & 1 \\ -1 & -y_{13} & 0 & -1 & -y_{35} \\ -y_{24} & -y_{14} & -1 & 0 & -1 \\ -y_{25} & -1 & y_{35} & 1 & 0 \end{bmatrix} \right).$$

One obtains eventually that σ_1 is the following involutive Cremona transformation:

$$(27) \quad \sigma_1 : y \longmapsto \left(\frac{1}{y_{25}}, -y_{24}y_{13}, -y_{14}y_{25}, \frac{1}{y_{13}}, -\frac{y_{35}}{y_{13}y_{25}} \right).$$

Proceeding in a similar way for each of the four other generators s_2, s_3, s_4 and s_5 of W_{D_5} , one obtains the following explicit formulas for the Cremona transformations σ_i induced by them:

$$(28) \quad \begin{aligned} \sigma_2 : y &\longmapsto \left(\frac{1}{y_{24}}, -\frac{y_{14}}{y_{24}y_{13}}, \frac{1}{y_{13}}, -y_{35}y_{24}, -y_{13}y_{25} \right) \\ \sigma_3 : y &\longmapsto \left(-y_{14}y_{35}, -y_{24}y_{13}, \frac{1}{y_{35}}, -\frac{y_{25}}{y_{35}y_{24}}, \frac{1}{y_{24}} \right) \\ \sigma_4 : y &\longmapsto \left(-\frac{y_{13}}{y_{35}y_{14}}, \frac{1}{y_{35}}, -y_{14}y_{25}, -y_{24}y_{35}, \frac{1}{y_{14}} \right) \\ \sigma_5 : y &\longmapsto \left(-P_2, y_{14}\frac{P_1}{P_3}, -\frac{y_{24}}{P_3}, -\frac{y_{25}}{P_1}, y_{35}\frac{P_3}{P_1} \right) \end{aligned}$$

(where in the formula for σ_5 , the P_i 's stand for the sub-pfaffians defined in (16)).

Proposition 2.3. *The map $s_i \longmapsto \sigma_i$ for $i = 1, \dots, 5$ gives rise to an embedding of groups*

$$(29) \quad W_{D_5} = \langle s_1, \dots, s_5 \rangle \longrightarrow \text{Bir}(\mathbf{C}^5)$$

which corresponds to Skorobogatov's isomorphism $W_{D_5} \simeq \text{Aut}(\mathcal{Y}_5)$ up to the birational identification (15).

Proof. Actually, this follows from [Sk, Theorem 2.2] since the formulas above for the σ_i 's have been obtained by following, in the specific case under scrutiny and working in a explicit manner, the proof given by Skorobogatov in his paper.

But that the proposition holds true can also be verified as follows: to ensure that (29) indeed induces a morphism of group, it suffices to verify that the Cremona transformations $\sigma_1, \dots, \sigma_5$ satisfy all the relations (6) satisfied by the generators s_i of W_{D_5} . Using the explicit expressions for the σ_i 's given above, this is something straightforward to check. Using a computer algebra system, there is no difficulty to show that the σ_i 's generate a finite subgroup of $\text{Bir}(\mathbf{C}^5)$ with 1920 elements. Since this is precisely the order of W_{D_5} , it follows that the morphism of groups (29) is injective. $\square$

We will denote by $\mathcal{W}_{D_5}$ the subgroup of $\text{Bir}(\mathbf{C}^5)$ generated by the σ_i 's:

$$(30) \quad W_{D_5} \simeq \mathcal{W}_{D_5} = \langle \sigma_1, \dots, \sigma_5 \rangle \subset \text{Bir}(\mathbf{C}^5).$$

Remark 2.4. *The birational realization $\mathcal{W}_{D_5}$ of the Weyl group W_{D_5} seems to be new.*

Because W_{D_5} acts by permutations (and transitively) on the set of weights $\mathbb{B}^+$ of S^+ , it acts also (in exactly the same way) on the set of coordinate hyperplanes $\{H_w\}_{w \in \mathbb{B}^+}$ hence on the set of weight divisors $\{D_w\}_{w \in \mathbb{B}^+} \subset \text{Pic}_{\mathbf{Z}}(\mathcal{Y})$ as well. For $w \in W_{D_5}$, denote by φ_w the corresponding automorphism of $\mathcal{Y}$. Then for any weight w and any $w \in W_{D_5}$, one has $\varphi_w(D_w) = D_{w(w)}$ from which it follows that φ_w induces an automorphism of $\mathcal{Y}_5^*$. Combined with Proposition 2.2, this gives us

Proposition 2.5. *For any $i = 1, \dots, 5$, the Cremona transformation σ_i is defined on $Y_5^* = \mathbf{C}^5 \setminus Z_5$ and gives rise to an automorphism of the pair $(\mathbf{C}^5, Z_5)$. Consequently, the image $\mathcal{W}_{D_5}$ of the group embedding (29) is a subgroup of $\text{Bir}(\mathbf{C}^5) \cap \text{Aut}(\mathbf{C}^5 \setminus Z_5)$.*

3. The Gelfand-MacPherson web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ and its abelian relations

This is the main section of the paper, in which we study the web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, in particular its abelian relations, which we make explicit and determine their invariance properties relative to the action of the Weyl group W_{D_5} .

3.1. The web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ in coordinates. We use here some results of [Pi5, §4.5] to which we refer the reader for more details.

The moment polytope of the spinor tenfold, denoted by Δ_{D_5} , is the 5-demihypercube, realized in $\mathfrak{h}_{D_5} \simeq \mathbf{R}^5$ as the convex envelope of $\mathbb{B}^+$. It has $10 + 2^4 = 26$ facets (= faces of codimension 1): 10 are 4-demihypercubes, the 16 other being 4-simplices. The former are facets of type D_4 , the latter of type A_4 (see [Pi5, Fig. 1]). Seeing Δ_{D_5} as the convex envelope of lines in the Picard lattice of a given smooth del Pezzo surface $X = \text{dP}_4$, and seeing the facets as determined by their set of vertices, the facets of type A_4 are in correspondence with the description of X as a blow-up of $\mathbf{P}^2$ in 5 points, the five corresponding exceptional divisors on dP_4 being the vertices of the corresponding facet. As for the facets which are demihypercubes, they correspond to the conic fibrations on X , the vertices of each such facet being the irreducible components (lines) of the non-irreducible fibers of the fibration.

Working in $\mathbf{R}^5$ by means of the isomorphism (19), then first one has that $\Psi(\mathbb{B}^+)$ is the set of 5-uplets $\frac{1}{2}\underline{\epsilon}$ with $\underline{\epsilon} = (\epsilon_i)_{i=1}^5 \in \{\pm 1\}^5$ even in the sense that $\epsilon_1 \cdots \epsilon_5 = 1$. Then it can be verified that the facets of type A_4 are given by intersecting Δ_{D_5} with the affine hyperplanes cut out by

$\sum_{i=1}^5 \varepsilon_i x_i = 3/2$ for all $\varepsilon = (\varepsilon_i)_{i=1}^5 \in \{\pm 1\}^5$ which are odd (that is such that $\varepsilon_1 \cdots \varepsilon_5 = -1$). As explained in [Pi5], the A_4 -facets of Δ_{D_5} are ‘web-irrelevant’ and have not to be considered further.

The 10 demihypercubical facets are the intersections $\Delta_{5,i}^\varepsilon = \Delta_{D_5} \cap \{x_i = \varepsilon/2\}$ for $i = 1, \dots, 5$ and $\varepsilon \in \{\pm 1\}$. These 10 facets are ‘web-relevant’ and from the material of [Pi5, §4.5] (in particular Figure 1 therein), we can see that the face map associated to the facet $\Delta_{5,i}^\varepsilon$ is a dominant rational map $w : \Pi_i^\varepsilon : \mathbb{S}_5 \dashrightarrow \mathbb{S}_4$. Here the target space $\mathbb{S}_4$ is the ‘spinor 6-fold’: it is the homogeneous space of type (D_4, ω_4) . Identifying $\mathbf{P}^7$ with the projectivization of $\mathbf{C} \oplus \text{Asym}_4(\mathbf{C}) \oplus \mathbf{C} \simeq \mathbf{C}^8$, the spinor 6-fold admits an affine parametrization *à la Wick* given by

$$W_4 : \text{Asym}_4(\mathbf{C}) \hookrightarrow \mathbb{S}_4 \subset \mathbf{P}^7$$

$$M \longmapsto [1 : M : \text{Pf}(M)],$$

from which it follows that $\mathbb{S}_4$ is isomorphic to the smooth hyperquadric $\mathbb{Q}^6 \subset \mathbf{P}^7$.

As for the expressions of the face maps Ψ_i^ε read in the antisymmetric matricial charts associated to Wick’s parametrizations of the two corresponding spinor manifolds, there is a simple (and nice) formula when $\varepsilon = +1$: for $i \in \llbracket 5 \rrbracket$, one has

$$\widetilde{\Phi}_i^+ = W_4^{-1} \circ \psi_i^+ \circ W_5 : \text{Asym}_5(\mathbf{C}) \dashrightarrow \text{Asym}_4(\mathbf{C})$$

$$A \longmapsto A_i$$

where A_i stands for the 4×4 antisymmetric matrix obtained from A by deleting its i -th line and its i -th column. One can give formulas for the maps $\widetilde{\Phi}_i^-$ read in Wick’s charts (see [Pi5, Prop. 4.17]), but none as nice as the one above for the $\widetilde{\Phi}_i^+$.¹⁷

From the explicit formulas in Wick’s charts for the face maps, one easily gets some formulas for some birational models of the H_{D_5} -equivariant quotient $\pi_i^\pm : \mathcal{Y}_5 \dashrightarrow \mathcal{Y}_4$ of the face maps $\Pi_i^\pm$. Let us denote by $\mathbf{W}_{Y_5}^{GM}$ the pull-back of $\mathcal{W}_{\mathcal{Y}_5}^{GM} = \mathcal{W}(\pi_i^\pm)_{i \in \llbracket 5 \rrbracket}$ under the birational map (15):

$$\mathbf{W}_{Y_5}^{GM} = \Theta^*(\mathcal{W}_{\mathcal{Y}_5}^{GM}).$$

Explicit first integrals for $\mathbf{W}_{Y_5}^{GM}$ have been given in [Pi5]. For $i = 1, \dots, 5$, let F_i (resp. F_{i+5}) be the foliation on Y_5 induced by $\pi_i^+ \circ \Theta$ (resp. by $\pi_i^- \circ \Theta$).

Proposition 3.1. *The following rational functions are first integrals of the foliations F_i :*

$$(31) \quad \begin{aligned} U_1 &= \left(y_{25}, y_{24}y_{35} \right) & U_6 &= \left(\frac{y_{14}y_{35} - y_{13} - 1}{y_{14}(y_{13}y_{25} - y_{35} - 1)}, \frac{y_{13}(y_{14}y_{25} - y_{24} - 1)}{y_{14}(y_{13}y_{25} - y_{35} - 1)} \right) \\ U_2 &= \left(\frac{1}{y_{13}}, \frac{y_{14}y_{35}}{y_{13}} \right) & U_7 &= \left(\frac{y_{24}y_{35} - y_{25} - 1}{y_{24}(y_{13}y_{25} - y_{35} - 1)}, \frac{y_{14}y_{25} - y_{24} - 1}{y_{24}(y_{13}y_{25} - y_{35} - 1)} \right) \\ U_3 &= \left(y_{24}, y_{14}y_{25} \right) & U_8 &= \left(\frac{y_{13}(y_{24}y_{35} - y_{25} - 1)}{y_{13}y_{25} - y_{35} - 1}, \frac{y_{14}y_{35} - y_{13} - 1}{y_{13}y_{25} - y_{35} - 1} \right) \\ U_4 &= \left(\frac{1}{y_{35}}, \frac{y_{13}y_{25}}{y_{35}} \right) & U_9 &= \left(\frac{y_{14}(y_{24}y_{35} - y_{25} - 1)}{y_{14}y_{25} - y_{24} - 1}, \frac{y_{24}(y_{14}y_{35} - y_{13} - 1)}{y_{14}y_{25} - y_{24} - 1} \right) \\ U_5 &= \left(y_{14}, y_{13}y_{24} \right) & U_{10} &= \left(\frac{y_{24}y_{35} - y_{25} - 1}{y_{35}(y_{14}y_{25} - y_{24} - 1)}, \frac{y_{25}(y_{14}y_{35} - y_{13} - 1)}{y_{35}(y_{14}y_{25} - y_{24} - 1)} \right). \end{aligned}$$

¹⁷The reason behind the dichotomy for the formulas of the face maps $\widetilde{\Phi}_i^\pm$ read in matricial charts is that the target spaces of the $\widetilde{\Phi}_i^+$ ’s are naturally identified with $\mathbb{S}_4$ embedded in $\mathbf{P}(S_4^+)$ whereas the images of the $\widetilde{\Phi}_i^-$ ’s naturally live in the projectivization $\mathbf{P}(S_4^-)$ of the other half-spin representation of $\text{Spin}_8(\mathbf{C})$. See [Pi5, §4.5.2] for more details.

The choice of the specific first integrals above is motivated by the fact that it will induce nice forms for the abelian relations of $\mathbf{W}_{Y_5}^{GM}$ we will deal with. For describing the ARs of $\mathbf{W}_{Y_5}^{GM}$, it is convenient to introduce a ‘third component’ to the first integral $U_i = (U_{i,1}, U_{i,2})$. One sets

$$(32) \quad U_{i,3} = 1 + U_{i,1} - U_{i,2}$$

for $i = 1, \dots, 10$ and via straightforward computations, one gets:

$$(33) \quad \begin{aligned} U_{1,3} &= -y_{24}y_{35} + y_{25} + 1 & U_{6,3} &= \frac{y_{13}y_{24} - y_{14} - 1}{(y_{13}y_{25} - y_{35} - 1)y_{14}} \\ U_{2,3} &= \frac{-y_{14}y_{35} + y_{13} + 1}{y_{13}} & U_{7,3} &= \frac{y_{25}(y_{13}y_{24} - y_{14} - 1)}{y_{24}(y_{13}y_{25} - y_{35} - 1)} \\ U_{3,3} &= -y_{14}y_{25} + y_{24} + 1 & U_{8,3} &= \frac{y_{35}(y_{13}y_{24} - y_{14} - 1)}{y_{13}y_{25} - y_{35} - 1} \\ U_{4,3} &= \frac{-y_{13}y_{25} + y_{35} + 1}{y_{35}} & U_{9,3} &= \frac{y_{13}y_{24} - y_{14} - 1}{y_{14}y_{25} - y_{24} - 1} \\ U_{5,3} &= -y_{13}y_{24} + y_{14} + 1 & U_{10,3} &= \frac{y_{13}y_{25} - y_{35} - 1}{y_{35}(y_{14}y_{25} - y_{24} - 1)}. \end{aligned}$$

The most important property which the $U_{i,s}$ ’s for $i = 1, \dots, 10$ and $s = 1, 2, 3$ satisfy is given by the

Lemma 3.2. *Up to a sign, any $U_{i,s}$ can be written as the product of some factors of the form $\zeta_k^{\pm 1}$, where the ζ_k ’s are the affine polynomials on $Y_5 = \mathbf{C}^5$ defined in (17).*

The five first functions U_i define a subweb of $\mathbf{W}_{Y_5}^{GM}$ which will be of great interest for us. This web will be denoted by $\mathbf{W}_{Y_5}^{GM}$, it is defined by simple monomial first integrals: one has

$$(34) \quad \mathbf{W}_{Y_5}^+ = \mathbf{W}\left((y_{25}, y_{24}y_{35}), \left(\frac{1}{y_{13}}, \frac{y_{14}y_{35}}{y_{13}}\right), (y_{24}, y_{14}y_{25}), \left(\frac{1}{y_{35}}, \frac{y_{13}y_{25}}{y_{35}}\right), (y_{14}, y_{13}y_{24})\right).$$

★

Essentially all the results to come have been obtained by means of explicit computations using the explicit expressions above for the first integrals of the webs we will consider. For this reason, all our results will be stated for the birational models $\mathbf{W}_{Y_5}^{GM}$ and $\mathbf{W}_{Y_5}^+$ but are of course valid for their geometric avatars $\mathbf{W}_{Y_5}^{GM}$ and $\mathbf{W}_{Y_5}^+ = \mathbf{W}(\pi_i^+)_{i \in \llbracket 5 \rrbracket}$ as well.

3.1.1. Ranks. Determining the virtual ranks of a c -codimensional web $\mathbf{W}$ defined by rational first integrals depending on n variables, amounts to computing the rank of certain vector spaces defined over $\mathbf{C}(u_1, \dots, u_n)$, where the u_i ’s are to be regarded as the affine coordinates of the generic point of $\mathbf{C}^n$.

As for the k -abelian relations of $\mathbf{W}$, for any $k = 0, \dots, c$, they correspond to the solutions of a linear differential system (which is not difficult to make explicit), and which is of finite type when the corresponding virtual rank is finite. In such cases, a computational approach similar to the one described in [?, §1.5] can be employed to determine the k -rank of the web.

The methods just outlined for computing the virtual or actual k -rank of a given web can be implemented in a computer algebra system. This provides computationally effective techniques

for determining these invariants.¹⁸ Using these, one easily computes all the ranks of the web $\mathbf{W}_{Y_5}^{GM}$.

Proposition 3.3. 1. *One has*

$$\begin{aligned} \rho_0^\bullet(\mathbf{W}_{Y_5}^{GM}) &= (15, 15, 10, 1) & \rho_1^\bullet(\mathbf{W}_{Y_5}^{GM}) &= (15, 20, 15, 2) & \rho_2^\bullet(\mathbf{W}_{Y_5}^{GM}) &= (5, 5, 1) \\ r_0(\mathbf{W}_{Y_5}^{GM}) &= 25 & r_1(\mathbf{W}_{Y_5}^{GM}) &= 35 & r_2(\mathbf{W}_{Y_5}^{GM}) &= 11. \end{aligned}$$

In particular, Gelfand-MacPherson's web $\mathbf{W}_{Y_5}^{GM}$ has AMP 2-rank.

2. *One has*

$$\begin{aligned} \rho_0^\bullet(\mathbf{W}_{Y_5}^+) &= (5) & \rho_1^\bullet(\mathbf{W}_{Y_5}^+) &= (5, 1) & \rho_2^\bullet(\mathbf{W}_{Y_5}^+) &= (1) \\ r_0(\mathbf{W}_{Y_5}^+) &= 5 & r_1(\mathbf{W}_{Y_5}^+) &= 6 & r_2(\mathbf{W}_{Y_5}^+) &= 1. \end{aligned}$$

In particular, the 5-web $\mathbf{W}_{Y_5}^+$ has AMP k -rank for $k = 0, 1, 2$.

3.1.2. The master 2-abelian relation $\mathbf{HLOG}_{Y_5}$ of $\mathbf{W}_{Y_5}^{GM}$. Given three variables u_1, u_2 and u_3 with the last one expressing in terms of the first two by $u_3 = 1 + u_1 - u_2$, one considers the following 'logarithmic' 2-form on $\mathbb{C}^3$

$$(35) \quad \Omega = \ln u_1 d \ln u_2 \wedge d \ln u_3 - \ln u_2 d \ln u_1 \wedge d \ln u_3 + \ln u_3 d \ln u_1 \wedge d \ln u_2$$

that is

$$(36) \quad \Omega = \ln u_1 \left(\frac{du_2 \wedge du_3}{u_2 u_3} \right) - \ln u_2 \left(\frac{du_1 \wedge du_3}{u_1 u_3} \right) + \ln u_3 \left(\frac{du_1 \wedge du_2}{u_1 u_2} \right),$$

or even more explicitly, setting $x = u_1$ and $u_2 = y$:

$$(37) \quad \begin{aligned} \Omega &= \left(-\frac{\ln u_1}{u_2 u_3} + \frac{\ln u_2}{u_1 u_3} + \frac{\ln u_3}{u_1 u_2} \right) du_1 \wedge du_2 \\ &= \left(\frac{\ln(1+x-y)}{xy} + \frac{\ln(y)}{x(1+x-y)} - \frac{\ln(x)}{y(1+x-y)} \right) dx \wedge dy. \end{aligned}$$

Let I be the hypercube formed by 5-tuples $(y_i)_{i=1}^5 \in]0, 1[^5$ and denote by $\mathcal{U}$ an arbitrary but fixed open domain containing it. Since the five pfaffian determinants $P_1, \dots, P_5$ (cf. (16)) are positive on I , it follows that the same occurs for the $U_{i,s}$ for all $i = 1, \dots, 10$ and $s = 1, 2, 3$.

One sets

$$(38) \quad (\epsilon_i)_{i=1}^{10} = (1, -1, 1, -1, 1, 1, -1, 1, -1, 1).$$

Proposition 3.4. *The following differential relation is identically satisfied on $\mathcal{U}$:*

$$(\mathbf{HLOG}_{Y_5}) \quad \sum_{i=1}^{10} \epsilon_i U_i^*(\Omega) = 0.$$

Consequently, the 10-tuple $(\epsilon_i U_i^*(\Omega))_{i=1}^{10}$ can be seen as a 2-abelian relation for $\mathbf{W}_{Y_5}^{GM}$, again denoted by $\mathbf{HLOG}_{Y_5}$.

¹⁸Maple worksheets for computing the k -ranks of webs in an arbitrary number of variables are available from the author upon request.

Proof. By formal elementary computations, one can express the scalar components of the sum $\Theta = \sum_{i=1}^{10} \epsilon_i U_i^*(\Omega)$ in the basis $dy_i \wedge dy_j$ with i, j such that $1 \leq i < j \leq 5$. For instance, the $dy_1 \wedge dy_2$ -component $\Theta_{12} = (\partial_{y_1} \wedge \partial_{y_2}) \lrcorner \Theta$ is a sum of terms of the form $R(y) \ln M$ with $R(y) \in \mathbf{C}(y)$ and where M is a monomial in the ζ_k 's defined in (17). Since these quantities are positive on I , using the functional equation of the logarithm, one can express Θ_{12} as a linear combination in the $\ln \zeta_k$'s with coefficients in $\mathbf{C}(y)$. Straightforward formal computations give that all these coefficients actually vanish, hence $\Theta_{12} \equiv 0$ on I . Proceeding similarly for all the other components of Θ , one gets that it vanishes identically on I hence on the complex domain $\mathcal{U}$. $\square$

From the presence of logarithms in the definition of Ω , it follows that $\mathbf{HLOG}_{Y_5}$ is not a global AR for $\mathbf{W}_{Y_5}^{GM}$ but a multivalued one, with additive monodromy. Hence one has to be a bit careful when wondering about the invariance property of 'the abelian relation' $\mathbf{HLOG}_{Y_5}$ under the birational action of W_{D_5} on Y_5 . There are two approaches for circumventing this non important technical issue.

The first one is to restrict to the reals and to deal with the real web $\widetilde{\mathbf{W}}_{Y_5}^{GM}$ on $\widetilde{Y}_5 = \mathbf{R}^5 \setminus Z$. One considers the following real-analytic version of Ω :

$$\begin{aligned} \Omega^\omega &= \ln |u_1| \left(\frac{du_2 \wedge du_3}{u_2 u_3} \right) - \ln |u_2| \left(\frac{du_1 \wedge du_3}{u_1 u_3} \right) + \ln |u_3| \left(\frac{du_1 \wedge du_2}{u_1 u_2} \right) \\ &= \left(\frac{\ln |1+x-y|}{xy} + \frac{\ln |y|}{x(1+x-y)} - \frac{\ln |x|}{y(1+x-y)} \right) dx \wedge dy. \end{aligned}$$

The scalar components of $\sum_{i=1}^{10} U_i^*(\Omega^\omega)$ are linear combinations of the quantities $\ln |\zeta_k|$'s with coefficients in $\mathbf{R}(y)$ which can easily be computed. One obtains the

Proposition 3.5. *The following differential relation*

$$\left(\mathbf{HLOG}_{Y_5}^\omega \right) \quad \sum_{i=1}^{10} \epsilon_i U_i^*(\Omega^\omega) = 0$$

is identically satisfied on Y_5 hence $(\epsilon_i U_i^(\Omega^\omega))_{i=1}^{10}$, which will be again denoted by $\mathbf{HLOG}_{Y_5}^\omega$, can be considered as a global real-analytic 2-AR either for $\mathbf{W}_{Y_5}^{GM}$ or for its real version $\widetilde{\mathbf{W}}_{Y_5}^{GM}$.*

Since it is global, it now does make sense to consider the invariance properties of $\mathbf{HLOG}_{Y_5}^\omega$ with respect to the action of $W_{D_5} \simeq \mathcal{W}_{D_5}$ on Y_5 or on $\widetilde{Y}_5$. But this is also possible when sticking to the holomorphic setting, following a similar approach to the one described in [Pi3, p. 96]. Indeed, as it follows from the description of the ARs of $\mathbf{W}_{Y_5}^{GM}$ (see further), the local system $\mathcal{AR}^2 = \mathcal{AR}^2(\mathbf{W}_{Y_5}^{GM})$ of 2-ARs of $\mathbf{W}_{Y_5}^{GM}$ (on Y_5^*) admits a 1-step 'weight filtration' $F^\bullet \mathcal{AR}^2$, with $F^1 \mathcal{AR}^2 = \mathcal{AR}^2$ and where $F^0 \mathcal{AR}^2 = \mathcal{AR}_{\text{Rat}}^2(\mathbf{W}_{Y_5}^{GM})$ is the vector space of rational ARs of $\mathbf{W}_{Y_5}^{GM}$. Then $\text{Gr}^1 \mathcal{AR}^2$ is of dimension 1 and the multivalued abelian relation $\mathbf{HLOG}_{Y_5}$ gives rise to a generator of this space. It is easy to see that the birational action of W_{D_5} on Y_5 induces a linear action on the associated graded space

$$\text{Gr}^\bullet \mathcal{AR}^2 = \mathcal{AR}_{\text{Rat}}^2(\mathbf{W}_{Y_5}^{GM}) \oplus \text{Gr}^1 \mathcal{AR}^2.$$

Let us now discuss how the Cremona transformations σ_i acts on $\mathbf{HLOG}_{Y_5}$

Proposition 3.6. 1. *As elements of $\mathcal{AR}^2(\mathcal{W}_{Y_5}^{GM})$, one has $\sigma_i^*(\mathbf{HLOG}_{Y_5}^\omega) = -\mathbf{HLOG}_{Y_5}^\omega$ for any $i = 1, \dots, 5$. Hence $\mathbf{HLOG}_{Y_5}^\omega$ spans a 1-dimensional non-trivial W_{D_5} -subrepresentation of $\mathcal{AR}^2(\mathcal{W}_{Y_5}^{GM})$ which necessarily is the signature representation.*

2. *There is a similar statement for the linear action of W_{D_5} on $\mathrm{Gr}^\bullet \mathcal{AR}^2$: the complex line $\mathrm{Gr}^1 \mathcal{AR}^2$ spanned by $\mathbf{HLOG}_{Y_5}$ is W_{D_5} -stable and is the signature representation.*

Proof. For $i = 1, \dots, 5$, one sets $\mathbf{HLOG}_{i,+}^\omega = (U_{i,1})^*(\Omega^\omega)$ and $\mathbf{HLOG}_{i,-}^\omega = (U_{i,2})^*(\Omega^\omega)$. Then by direct elementary computations, using the explicit expressions of the σ_i 's given above, one easily establish the following transformation formulas:

• for $k \in \{1, 2, 3, 4\}$, one sets $v_k = (k, k+1) \in \mathfrak{S}_5$. Then for $i = 1, \dots, 5$ and $\epsilon = \pm$, one has:

$$(39) \quad \sigma_k^*(\mathbf{HLOG}_{i,\epsilon}^\omega) = -\mathbf{HLOG}_{v_k(i),\epsilon}^\omega;$$

- let v_5 be the transposition exchanging 4 and 5 (i.e. $v_5 = v_4$). Then for $j \in \{1, 2, 3\}$, $\ell \in \{4, 5\}$ and $\epsilon = \pm$, one has:

$$(40) \quad \sigma_5^*(\mathbf{HLOG}_{j,\epsilon}^\omega) = -\mathbf{HLOG}_{j,\epsilon}^\omega \quad \text{and} \quad \sigma_5^*(\mathbf{HLOG}_{\ell,\epsilon}^\omega) = -\mathbf{HLOG}_{v_5(\ell),-\epsilon}^\omega$$

It follows that $\langle \mathbf{HLOG}_{Y_5}^\omega \rangle$ is W_{D_5} -stable. Since this 1-dimensional representation is not trivial, it has to be the signature.

The proof of the second point of the proposition is essentially similar. $\square$

Remark 3.7. *The components of $\mathbf{HLOG}_{Y_5}$ are linear combinations of logarithms of rational functions, multiplied by wedge products of total derivatives of terms of the same type. Such differential forms have already appeared in earlier works on polylogarithms and the functional equations they satisfy. For instance, see our previous work [Pi3] on the curvilinear webs defined by the $n+3$ forgetful maps on the moduli spaces $\mathcal{M}_{0,n+3}$ (in particular, refer to formulas (5.61) and (5.65) therein). See also [KLL]¹⁹ where polylogarithms are studied from the perspective of the theory of algebraic cycles and reciprocity laws.*

It would be of interest to gain a deeper understanding of the nature of such differential forms and to clarify why they naturally emerge in the context of functional identities and abelian relations.

3.2. The rational 2-abelian relations of $\mathcal{W}_{Y_5}^{GM}$. In this subsection, we give an explicit description of 10 linearly independent rational 2-abelian relations of $\mathcal{W}_{Y_5}^{GM}$ which are obtained as residues of $\mathbf{HLOG}_{Y_5}$. Then considering Proposition 3.3, it will follow that together with $\mathbf{HLOG}_{Y_5}$, these 10 rational ARs form a basis of $\mathcal{AR}^2(\mathcal{W}_{Y_5}^{GM})$.

3.2.1. Residues of $\mathbf{HLOG}_{Y_5}$. Recall that $\mathbf{C}(y)$ stands for the field of rational functions in the variables y_i for $i = 1, \dots, 5$, with $y_1 = y_{13}$, $y_2 = y_{14}$, $y_3 = y_{24}$, $y_4 = y_{25}$ and $y_5 = y_{35}$. Given F , an irreducible non-constant polynomial in the y_i 's, we denote by v_F the associated valuation $\mathbf{C}(y) \setminus \{0\} \rightarrow \mathbf{Z}$. Then for $V_1, V_2, V_3 \in \mathbf{C}(y)$, we set

$$\mathrm{Res}_F(\ln(V_1) dV_2 \wedge dV_3) = v_F(V_1) dV_2 \wedge dV_3 \in \Omega_{\mathbf{C}(y)}^2,$$

¹⁹More precisely, see the first line of page 157 in [KLL]

and we claim that this definition makes sense, that is that the RHS is independent of the determination of $\ln(V_1)$ taken in the LHS (its is an easy exercise, left to the reader). We call $\nu_F(V_1) dV_2 \wedge dV_3$ the residue of the logarithmic 2-form $\ln(V_1) dV_2 \wedge dV_3$ along the hypersurface cut out by F .²⁰

We aim to take residues of (the components of) $\mathbf{HLOG}_{Y_5}$ with respect to the 10 irreducible components of the divisor $Z \subset Y_5$ defined in (18). For any $i = 1, \dots, 10$, one denotes by $\mathbf{HLOG}_i$ the i -th component of $\mathbf{HLOG}_{Y_5}$, namely

$$(41) \quad \mathbf{HLOG}_i = \epsilon_i U_i^*(\Omega) = \epsilon_i \left(\ln U_{i,1} \left(\frac{dU_{i,2} \wedge dU_{i,3}}{U_{i,2}U_{i,3}} \right) - \ln U_{i,2} \left(\frac{dU_{i,1} \wedge dU_{i,3}}{U_{i,1}U_{i,3}} \right) + \ln U_{i,3} \left(\frac{dU_{i,1} \wedge dU_{i,2}}{U_{i,1}U_{i,2}} \right) \right).$$

From the topological definition of the residue (cf. the footnote below), it follows that for any non-constant irreducible polynomial F , the 10-tuple of residues

$$(42) \quad \mathbf{Res}_F(\mathbf{HLOG}_{Y_5}) = \left(\mathbf{Res}_F(\mathbf{HLOG}_i) \right)_{i=1}^{10}$$

with

$$\mathbf{Res}_F(\mathbf{HLOG}_i) = \nu_F(U_{i,1}) \left(\frac{dU_{i,2} \wedge dU_{i,3}}{U_{i,2}U_{i,3}} \right) - \nu_F(U_{i,2}) \left(\frac{dU_{i,1} \wedge dU_{i,3}}{U_{i,1}U_{i,3}} \right) + \nu_F(U_{i,3}) \left(\frac{dU_{i,1} \wedge dU_{i,2}}{U_{i,1}U_{i,2}} \right)$$

for any i , belongs to $\mathbf{AR}_{\text{Rat}}^2(\mathcal{W}_{Y_5}^{GM})$. Specializing F by taking for it one of the ten polynomials ζ_k defined in (17), one gets 10 rational abelian relations

$$\mathbf{Res}_i = \mathbf{Res}_{\zeta_i} = \mathbf{Res}_{\zeta_i}(\mathbf{HLOG}_{Y_5}), \quad i = 1, \dots, 10$$

which can be easily computed from the explicit expressions (31) and (33) for the $U_{i,s}$. For instance, one gets that $\mathbf{Res}_{y_1} = \mathbf{Res}_{y_1}(\mathbf{HLOG}_{Y_5})$ corresponds to the following differential relation

$$(43) \quad U_2^* \left(\left(\frac{du_2 \wedge du_3}{u_2 u_3} \right) - \left(\frac{du_3 \wedge du_1}{u_3 u_1} \right) - \left(\frac{du_1 \wedge du_2}{u_1 u_2} \right) \right) + U_4^* \left(\frac{du_3 \wedge du_1}{u_3 u_1} \right) \\ + U_5^* \left(\frac{du_3 \wedge du_1}{u_3 u_1} \right) + U_6^* \left(\frac{du_3 \wedge du_1}{u_3 u_1} \right) + U_8^* \left(\frac{du_2 \wedge du_3}{u_2 u_3} \right) = 0$$

which can be directly verified to be identically satisfied.

By straightforward computations, it can be verified that one has $\rho^2(\mathcal{W}_5) \leq 1$ for any 5-subweb $\mathcal{W}_5$ of $\mathcal{W}_{Y_5}^{GM}$. By analogy with the terminology introduced in [Da], any 2-abelian relation of such a 5-subweb $\mathcal{W}_5$ will be said to be ‘combinatorial’ and we will denote the space spanned by the combinatorial ARs by $\mathbf{AR}_C(\mathcal{W}_{Y_5}^{GM})$. From above, it follows that $\mathbf{Res}_{y_1}$ is combinatorial.

By straightforward computations, one makes the 10 residues $\mathbf{Res}_i$ entirely explicit from which one first deduces that all these ARs are combinatorial and rational. Using the explicit formulas obtained for the residues $\mathbf{Res}_i$ together with the ones for the automorphisms of webs σ_k ’s given in (28), it is straightforward (however a bit laborious) to compute all the pull-backs $\sigma_k^*(\mathbf{Res}_i)$ for $k = 1, \dots, 5$ and $i = 1, \dots, 10$. One obtains that any $\sigma_k^*(\mathbf{Res}_i)$ is a linear combination (with non-zero coefficients ± 1) of the ten residues of $\mathbf{HLOG}_{Y_5}$. For instance, one has

$$\sigma_1^*(\mathbf{Res}_{y_1}) = -\mathbf{Res}_{y_2} + \mathbf{Res}_{y_4} + \mathbf{Res}_{y_5} + \mathbf{Res}_{p_3} + \mathbf{Res}_{p_5}$$

²⁰The residue $\mathbf{Res}_F(\ln(V_1) dV_2 \wedge dV_3)$ can also be defined topologically, in terms of the monodromy of $\ln(V_1) dV_2 \wedge dV_3$ along a small loop around $\{F = 0\}$, see [Pi5, §2.2.6] for more details on this approach.

More generally, by means of straightforward explicit computations, we get the

Proposition 3.8. 1. *The residues $\mathbf{Res}_i(\mathbf{HLOG}_{Y_5})$ ($i = 1, \dots, 10$) all have exactly 5 non-trivial components hence belong to $\mathbf{AR}_C^2(\mathcal{W}_{Y_5}^{GM})$. Moreover, they form a basis of this space, which coincides with the space $\mathbf{AR}_{Rat}^2(\mathcal{W}_{Y_5}^{GM})$ of rational ARs of $\mathcal{W}_{Y_5}^{GM}$.*

2. *The birational maps σ_k 's induce linear automorphisms of $\mathbf{AR}_C^2(\mathcal{W}_{Y_5}^{GM})$ making of this space a W_{D_5} -representation. As such, it is irreducible and isomorphic to $V_{[11,111]}^{10}$.*

Proof. The proof goes by explicit computations.²¹ Since all the residues $\mathbf{Res}_i$'s have five non-zero terms which are rational, they are combinatorial 2-abelian relations which span a subspace of $\mathbf{AR}(\mathcal{W}_{Y_5}^{GM})$ in direct sum with the line spanned by $\mathbf{HLOG}_{Y_5}$. The $\mathbf{Res}_i$'s being linearly independent, it follows from Proposition 3.3 that $\mathbf{AR}_C(\mathcal{W}_{Y_5}^{GM})$ has dimension 10 and admits $\mathfrak{R} = (\mathbf{Res}_s)_{s=1}^{10}$ as a basis. This proves the first point of the proposition.

Computing all the pull-backs $\sigma_k^*(\mathbf{Res}_i)$, one first obtains that they all have exactly five non-zero terms hence are combinatorial 2-abelian relations hence are linear combinations of the $\mathbf{Res}_i$'s. It is then straightforward to get the explicit forms for the matrices of the $\sigma_k^* \in \mathrm{GL}(\mathbf{AR}_C^2(\mathcal{W}_{Y_5}^{GM}))$ expressed in the basis $\mathfrak{R}$. For instance, one gets that

$$\mathrm{Mat}_{\mathfrak{R}}(\sigma_1^*) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

Knowing explicitly the matrices $\mathrm{Mat}_{\mathfrak{R}}(\sigma_k^*)$ for $k = 1, \dots, 5$ and proceeding completely similarly as in the Appendix of [Pi4], one computes the character of the representation of W_{D_5} on $\mathbf{AR}_C^2(\mathcal{W}_{Y_5}^{GM})$: one get that this character is

$$\chi = (10, -2, 2, -4, 2, 0, -2, 2, -2, 0, 1, -1, 1, -1, 1, 0, 0, 0).$$

It corresponds to the first line of the characters table of the Weyl group of type D_5 given in [Pi4, Table 4], which gives us the second point of the proposition. $\square$

Remark 3.9. In [Pi5, §4.2], it has been established that one also has

$$\mathbf{HLogAR}_{\mathrm{asym}}^2 \simeq V_{[1^2, 1^3]}^{10}$$

as W_{D_5} -representations, where the left hand side $\mathbf{HLogAR}_{\mathrm{asym}}^2$ stands for the space of symbolic antisymmetric weight 2 AR of a del Pezzo web $\mathcal{W}_{\mathrm{dP}_4}$ of a quartic del Pezzo surface dP_4 . As it will be explained further, this is no mere coincidence.

From the results above completed by some computational checks, we deduce the following

²¹Another less computational proof could have been given, using the action of $W_{D_5} \simeq \mathcal{W}_{D_5}$ on the set of combinatorial abelian relations.

Corollary 3.10. 1. *One has $\rho^2(\mathcal{W}_5) \leq 1$ for any 5-subweb $\mathcal{W}_5$ of $\mathcal{W}_{Y_5}^{GM}$. Those for which the virtual 2-rank is 1 actually are of maximal 2-rank 1. These subwebs are exactly the $\mathcal{W}^\epsilon = \mathcal{W}(\mathcal{F}_1^{\epsilon_1}, \dots, \mathcal{F}_5^{\epsilon_5})$, for all 5-tuples $\underline{\epsilon} = (\epsilon_i)_{i=1}^5 \in \{\pm 1\}^5$ of even parity (i.e. such that $\epsilon_1 \cdots \epsilon_5 = +1$).*

For each even $\underline{\epsilon}$, the space of 2-ARs of $\mathcal{W}^\epsilon$ is spanned by a peculiar abelian relation $\mathbf{AR}^\epsilon$, uniquely defined up to sign, whose components with respect to each first integral $U_j = (U_{j,s})_{s=1}^3$ of $\mathcal{W}^\epsilon$ is a linear combination, with coefficients ± 1 , of the rational 2-forms

$$d \log U_{j,a} \wedge d \log U_{j,b} = \frac{dU_{j,a} \wedge dU_{j,b}}{U_{j,a}U_{j,b}} \quad (1 \leq a < b \leq 3).$$

Moreover, $\mathbf{AR}_C^2(\mathcal{W}_{Y_5}^{GM}) = \langle \mathbf{AR}^\epsilon \mid \underline{\epsilon} \in \{\pm 1\}^5 \text{ is even} \rangle$ coincides with the space $\mathbf{AR}_{\text{Rat}}^2(\mathcal{W}_{Y_5}^{GM})$ of rational 2-ARs of $\mathcal{W}_{Y_5}^{GM}$ and this space is 10-dimensional: $\dim \mathbf{AR}_C^2(\mathcal{W}_{Y_5}^{GM}) = 10$.

2. *One has*

$$(44) \quad \mathbf{AR}^2(\mathcal{W}_{Y_5}^{GM}) = \mathbf{AR}_C^2(\mathcal{W}_{Y_5}^{GM}) \oplus \langle \mathbf{HLOG}_{Y_5} \rangle.$$

from which it follows that the 2-rank of $\mathcal{W}_{Y_5}^{GM}$ is 11, that is is AMP.

3. *By residues/monodromy, the abelian relation $\mathbf{HLOG}_{Y_5}$ spans the subspace $\mathbf{AR}_C^2(\mathcal{W}_{Y_5}^{GM})$ of combinatorial ARs, which coincides with that of rational 2-ARs of $\mathcal{W}_{Y_5}^{GM}$: one has*

$$\text{Res}(\mathbf{HLOG}_{Y_5}) = \mathbf{AR}_C^2(\mathcal{W}_{Y_5}^{GM}) = \mathbf{AR}_{\text{Rat}}^2(\mathcal{W}_{Y_5}^{GM}).$$

4. *The decomposition in direct sum (44) actually is the decomposition of $\mathbf{AR}^2(\mathcal{W}_{Y_5}^{GM})$ into irreducible W_{D_5} -representations. The two pieces $\mathbf{AR}_C^2(\mathcal{W}_{Y_5}^{GM})$ and $\langle \mathbf{HLOG}_{Y_5} \rangle$ are isomorphic to $V_{[11,111]}^{10}$ and to the signature representation respectively.*

3.2.2. A specific combinatorial 2-abelian relation of $\mathcal{W}_{Y_5}^{GM}$. For what is to come further, it is interesting to consider the case of the 5-subweb $\mathcal{W}(U_1, \dots, U_5)$ of $\mathcal{W}_{Y_5}^{GM}$. This web will be denoted by $\mathcal{W}_{Y_5}^{GM,+}$ and is defined by simple monomial first integrals:

$$\mathcal{W}_{Y_5}^{GM,+} = \mathcal{W}\left(\left(y_{25}, y_{24}y_{35}\right), \left(\frac{1}{y_{13}}, \frac{y_{14}y_{35}}{y_{13}}\right), \left(y_{24}, y_{14}y_{25}\right), \left(\frac{1}{y_{35}}, \frac{y_{13}y_{25}}{y_{35}}\right), \left(y_{14}, y_{13}y_{24}\right)\right).$$

According to the fourth point of the above corollary, this web has maximal 2-rank (equal to 1). We aim below to describe very explicitly a generator of the complex line $\mathbf{AR}^2(\mathcal{W}_{Y_5}^{GM,+})$.

Setting

$$(45) \quad \eta = d \log x \wedge d \log y = \frac{dx}{x} \wedge \frac{dy}{y} = \frac{dx \wedge dy}{xy},$$

one verifies without any difficulty that the following proposition is satisfied: Recall (from (38)) that one has $\epsilon_1 = \epsilon_3 = \epsilon_5 = 1$ and $\epsilon_2 = \epsilon_4 = -1$.

Proposition 3.11. 1. *The 5-tuple $\mathbf{AR}_\eta^2 = (\epsilon_i U_i^*(\eta))_{i=1}^5$ is a rational 2-abelian relation for $\mathbf{W}_{Y_5}^+$, i.e. in the space $\Omega_{\mathbf{C}(y)}^2$ of rational 2-forms on $\mathbf{C}^5$, one has*

$$\sum_{i=1}^5 \epsilon_i U_i^* \left(\frac{dx \wedge dy}{xy} \right) = \sum_{i=1}^5 \epsilon_i \frac{dU_{i,1} \wedge dU_{i,2}}{U_{i,1} U_{i,2}} = 0.$$

Moreover, in terms of the residues of $\mathbf{HLOG}_{Y_5}$, one has $\mathbf{AR}_\eta^2 = \sum_{i=1}^5 \mathbf{Res}_{P_i}$.

2. *Consequently, one has $\mathbf{AR}^2(\mathbf{W}_{Y_5}^+) = \langle \mathbf{AR}_\eta^2 \rangle$.*

Since $(dx \wedge dy)/(xy)$ is closed, all the components $U_i^*((dx \wedge dy)/(xy))$ of $\mathbf{AR}_\eta^2$ are closed as well hence by Poincaré's lemma, the latter abelian relation admits a primitive, at least locally. Since $\mathbf{AR}_C(\mathbf{W}_{Y_5}^{GM})$ is irreducible as a W_{D_5} -representation, this space is spanned by the orbit $W_{D_5} \cdot \mathbf{AR}_\eta^2$ which is formed of closed 2-abelian relations. It follows that all the residues $\mathbf{Res}_i$ are closed (a fact which can also be verified directly) hence locally exact.

We will use this to construct specific primitives of the residues $\mathbf{Res}_i$ which will span an interesting subspace of $\mathbf{AR}^1(\mathbf{W}_{Y_5}^{GM})$ (see (49) in Theorem 3.14 below).

3.2.3. A more intrinsic and abstract approach. All the results of the previous subsection have been obtained by direct computations. Here we say a few words about a more abstract approach to the 2-ARs of $\mathbf{W}_{Y_5}^{GM}$, one of the key ingredient of which being the action of W_{D_5} on Y_5 .

Up to the birational identification Θ , the divisors $\{\zeta_k = 0\}$ in Y_5 correspond to 10 of the weight divisors D_w in $\mathbf{Y}_5$ hence to some weights and lines in $\mathbb{W}^+$ and $\mathcal{L}_r$ respectively (see the discussion following (15)). These correspondences are given in the table below, where we use the following notations: we denote by $(\frac{1}{2})^5 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ the dominant weight of $\mathbb{W}^+$. As elements of $\text{Pic}_{\mathbf{Z}}(\text{dP}_4)$, we set $\mathbf{e}_{tot} = \sum_{k=1}^5 \mathbf{e}_k$ and $\mathbf{e}_i = \mathbf{e}_{tot} - \mathbf{e}_i = \sum_{k \neq i} \mathbf{e}_k$ for $i \in \llbracket 5 \rrbracket$. Finally, we denote by $(\mathbf{v}_k)_{k=1}^5$ the standard basis of $\mathbf{R}^5$, that is $\mathbf{v}_i = (\delta_{ij})_{j=1}^5$ for $i = 1, \dots, 5$.

Line ℓ	Weight $\mathbf{w}_\ell$	Polynomial ζ_k
$\mathbf{e}_i$	$\mathbf{v}_i - (\frac{1}{2})^5$	P_i
$\mathbf{h} - \mathbf{e}_i - \mathbf{e}_j$	$(\frac{1}{2})^5 - \mathbf{v}_i - \mathbf{v}_j$	y_{ij}

Similarly, the correspondances between the W -relevant facets of the weight polytope, the corresponding conic classes and the first integrals of $\mathbf{W}_{Y_5}^{GM}$ are given in the second table just below.²²

index i	Facet	Conic class	First integral
$i \in \{1, \dots, 5\}$	$\Delta_{5,i}^+$	$2\mathbf{h} - \mathbf{e}_i$	U_i
$i \in \{6, \dots, 10\}$	$\Delta_{5,i}^-$	$\mathbf{h} - \mathbf{e}_i$	U_{i+5}

²²We recall the notation for the W -relevant facets of Δ_5 : one has $\Delta_{5,i}^\epsilon = \Delta_{D_5} \cap \{x_i = \epsilon/2\}$ for $i \in \llbracket 5 \rrbracket$ and $\epsilon = \pm 1$.

The variable $\zeta_1 = y_1 = y_{13}$ corresponds to the divisor associated to the line $\ell = \mathbf{h} - \mathbf{e}_1 - \mathbf{e}_3$. The conic classes adjacent to it are $\mathbf{h} - \mathbf{e}_1$, $\mathbf{h} - \mathbf{e}_3$ and the three classes $2\mathbf{h} - \mathbf{e}_k$ for $k = 2, 4, 5$. The two former classes correspond to the foliations induced by U_6 and U_8 , the three latter to the ones with U_2, U_4 and U_5 as first integrals. This is in accordance with the explicit expression (43) for the residue of $\mathbf{HLOG}_{\mathcal{Y}_5}$ along the divisor cut out by $y_1 = 0$. In terms of Gelfand-MacPherson's web on $\mathcal{Y}_5$, this translates as the fact that

$$(46) \quad \begin{aligned} & \text{'the residue of } \mathbf{HLOG}_{\mathcal{Y}_5} \text{ along the weight divisor associated to } \mathfrak{w}_\ell = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \text{ is} \\ & \text{the subweb defined by the face maps } \pi_F : \mathcal{Y}_5 \rightarrow \mathcal{Y}_F \text{ for all facets } F \text{ of } \Delta_5 \text{ adjacent to } \mathfrak{w}_\ell \text{'}. \end{aligned}$$

For a line $l \in \mathcal{L}$, let $\mathcal{W}_{\mathcal{Y}_5, l}^{GM}$ be the subweb of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ defined by the face maps $\pi_F : \mathcal{Y}_5 \rightarrow \mathcal{Y}_F$ for all facets $F \subset \Delta_5$ adjacent to l . More formally, setting $\mathcal{K}(l)$ for the subset of conic classes $\mathfrak{c} \in \mathcal{K}$ such that $\mathfrak{c} - l \in \mathcal{L}$, one has

$$\mathcal{W}_{\mathcal{Y}_5, l}^{GM} = \mathcal{W}(\pi_{F_c} \mid \mathfrak{c} \in \mathcal{K}(l)).$$

For instance, the web $\mathcal{W}_{\mathcal{Y}_5}^{GM, +}$ considered in the previous subsection coincides with $\mathcal{W}_{\mathcal{Y}_5, 2\mathbf{h} - \mathbf{e}_{tot}}^{GM}$.

Proposition 3.12. 1. *The sixteen 5-subwebs of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ with maximal 2-rank (equal to 1) are precisely the subwebs $\mathcal{W}_{\mathcal{Y}_5, \ell}^{GM}$'s for all lines $\ell \in \mathcal{L}$.*

2. *Moreover, for any line ℓ , the residue $\mathbf{Res}_\ell(\mathbf{HLOG}_{\mathcal{Y}_5})$ of $\mathbf{HLOG}_{\mathcal{Y}_5}$ along the divisor D_ℓ , denoted by $\mathbf{Res}_\ell$, is a non-trivial AR for $\mathcal{W}_{\mathcal{Y}_5, \ell}^{GM}$. In other terms, one has $\langle \mathbf{Res}_\ell \rangle = \mathbf{AR}^2(\mathcal{W}_{\mathcal{Y}_5, \ell}^{GM})$.*

3. *For any $\ell \in \mathcal{L}$, the following linear relation holds true in $\mathbf{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM})$:*

$$(\mathbf{Rel}_\ell) \quad \mathbf{Res}_\ell = \sum_{\mathfrak{c} \in \mathcal{K}(\ell)} \mathbf{Res}_{\mathfrak{c} - \ell}.$$

Moreover, for any exceptional collection²³ $\mathcal{E} \subset \mathcal{L}$, the set $\{\mathbf{Rel}_l\}_{l \in \mathcal{E}}$ is a basis of the space of linear relations between the 16 residues of $\mathbf{HLOG}_{\mathcal{Y}_5}$.

4. *For any exceptional collection $\mathcal{E} \subset \mathcal{L}$, there exists a unique line $\ell_{\mathcal{E}} \in \mathcal{L}$ which intersects all the elements of $\mathcal{E}$. Setting $\bar{\mathcal{E}} = \mathcal{E} \cup \{\ell_{\mathcal{E}}\}$, then the ten abelian relations $\mathbf{Res}_\ell$ for $\ell \in \mathcal{L} \setminus \bar{\mathcal{E}}$, form a basis of the space of combinatorial 2-ARs of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, i.e. there is an isomorphism*

$$\mathbf{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM}) \simeq \bigoplus_{\ell \in \mathcal{L} \setminus \bar{\mathcal{E}}} \mathbb{C} \cdot \mathbf{Res}_\ell.$$

Proof. This follows from explicit computations and from (46) combined with the facts that $W_{D_5} = \text{Aut}(\mathcal{Y}_5)$ acts as the signature on the complex line spanned by $\mathbf{HLOG}_{\mathcal{Y}_5}$ (cf. Proposition 3.6) and acts transitively on the set of weight divisors in $\mathcal{Y}_5$ as well as on the set of exceptional collections $\mathcal{E} \subset \mathcal{L}$ (see [Ma, Corollary 26.8]). $\square$

²³An 'exceptional collection' is a subset of $\mathcal{L}$ formed by five pairwise disjoint lines.

3.3. The 1-abelian relations of $\mathcal{W}_{Y_5}^{GM}$. We consider the following privileged primitive of the 2-form $\eta = d\text{Log}(x) \wedge d\text{Log}(y)$:

$$(47) \quad \delta = \frac{1}{2} \left(\text{Log}(x) d\text{Log}(y) - \text{Log}(y) d\text{Log}(x) \right) = \frac{1}{2} \left(\text{Log}(x) \frac{dy}{y} - \text{Log}(y) \frac{dx}{x} \right)$$

Via elementary computations, one gets

$$\begin{aligned} 2 U_1^*(\delta) &= \ln(y_{25}) \frac{dy_{24}}{y_{24}} - \ln(y_{24}y_{35}) \frac{dy_{25}}{y_{25}} + \ln(y_{25}) \frac{dy_{35}}{y_{35}} \\ 2 U_2^*(\delta) &= \ln\left(\frac{y_{14}y_{35}}{y_{13}}\right) \frac{dy_{13}}{y_{13}} - \ln(y_{13}) \frac{dy_{14}}{y_{14}} - \ln(y_{13}) \frac{dy_{35}}{y_{35}} \\ 2 U_3^*(\delta) &= \ln(y_{24}) \frac{dy_{14}}{y_{14}} - \ln(y_{14}y_{25}) \frac{dy_{24}}{y_{24}} + \ln(y_{24}) \frac{dy_{25}}{y_{25}} \\ 2 U_4^*(\delta) &= -\ln(y_{35}) \frac{dy_{13}}{y_{13}} - \ln(y_{35}) \frac{dy_{25}}{y_{25}} + \ln\left(\frac{y_{13}y_{25}}{y_{35}}\right) \frac{dy_{35}}{y_{35}} \\ \text{and } 2 U_5^*(\delta) &= \ln(y_{14}) \frac{dy_{13}}{y_{13}} - \ln(y_{13}y_{24}) \frac{dy_{14}}{y_{14}} + \ln(y_{14}) \frac{dy_{24}}{y_{24}}. \end{aligned}$$

Recall that $\epsilon_1 = \epsilon_3 = \epsilon_5 = 1$ and $\epsilon_2 = \epsilon_4 = -1$ (see (38)). The coefficient of the logarithmic differential dy_{24}/y_{24} in the sum $\sum_{i=1}^5 \epsilon_i U_i^*(\delta)$ is the sum of those of the terms $U_k^*(\delta)$ for $k = 1, 3, 5$, namely it is

$$\ln(y_{25}) + \left(-\ln(y_{14}y_{25}) \right) + \ln(y_{14}),$$

a quantity which vanishes identically on any complex domain in Y_5 containing $(\mathbf{R}_{>0})^5$. The same phenomenon occurs for all the logarithmic differential dy_i/y_i with $i = 1, \dots, 5$, which proves the

Proposition 3.13. *With $\epsilon_1 = \epsilon_3 = \epsilon_5 = 1$ and $\epsilon_2 = \epsilon_4 = -1$, one has identically*

$$(48) \quad \sum_{i=1}^5 \epsilon_i U_i^*(\delta) = \frac{1}{2} \sum_{i=1}^5 \epsilon_i U_i \left(\text{Log}(x) \frac{dy}{y} - \text{Log}(y) \frac{dx}{x} \right) = 0$$

hence $\mathbf{AR}_\delta^1 = \left(\epsilon_i U_i^*(\delta) \right)_{i=1}^5$ is a 1-AR of weight 1 for $\mathcal{W}_{Y_5}^+$. Moreover, one has $d^1(\mathbf{AR}_\delta^1) = \mathbf{AR}_\eta^2$.

For any $\ell \in \mathcal{L}$, the 2-abelian relation $\mathbf{AR}_\ell$ is exact and is the total derivative of an 1-abelian relation $\mathbf{AR}_\ell^1$ of weight 1, which is equivalent to (48). We denote by $\mathbf{AR}_C^1(\mathcal{W}_{Y_5}^{GM})$ (or just $\mathbf{AR}_C^1$ for short) the subspace of $W_1(\mathbf{AR}^2(\mathcal{W}_{Y_5}^{GM}))$ spanned by these ARs, a notation which is justified by the fact that d^1 sends $\mathbf{AR}_C^1(\mathcal{W}_{Y_5}^{GM})$ onto $\mathbf{AR}_C^2(\mathcal{W}_{Y_5}^{GM})$, a fact which in particular implies that $\dim \mathbf{AR}_C^1 \geq 10$. From $r_1(\mathcal{W}_{Y_5}^{GM}) = 35$ and essentially using explicit computations, one gets the following result which gives a quite detailed description of the structure of the space of 1-abelian relations of $\mathcal{W}_{Y_5}^{GM}$.

Theorem 3.14. 1. *One has $r^1(\mathcal{W}_5) \leq 6$ for any 5-subweb $\mathcal{W}_5$ of $\mathcal{W}_{Y_5}^{GM}$. Those for which the 1-rank is 6 are the subwebs $\mathcal{W}^\epsilon = \mathcal{W}(\mathcal{F}_1^{\epsilon_1}, \dots, \mathcal{F}_5^{\epsilon_5})$ for any 5-tuple $\underline{\epsilon} = (\epsilon_i)_{i=1}^5 \in \{\pm 1\}^5$.*

2. *When $\underline{\epsilon}$ is odd, the space of 1-abelian relations of $\mathcal{W}^\epsilon$ has a basis formed of ARs corresponding*

to identities of the form $\sum_{i \in I_\varepsilon} \sum_{s=1}^3 c_{i,s} d \log U_{i,s} = 0$ for some coefficients $c_{i,s} \in \{-1, 0, 1\}$. In particular, the 1-ARs of this web all are of weight 0:

$$\mathbf{AR}^1(\mathcal{W}^\varepsilon) = W_0(\mathbf{AR}^1(\mathcal{W}^\varepsilon)).$$

3. When ε is even, the situation is different since not all the ARs of $\mathcal{W}^\varepsilon$ are of weight 0. Indeed:

- the space 1-abelian relations of $\mathcal{W}^\varepsilon$ of weight 0 is of dimension 5 and has a basis formed of ARs of the same form as in **2**;
- but there exists an 1-AR of weight 1, denoted by $\mathbf{AR}_\varepsilon^1$ whose each component is a linear combination with coefficients in $\{0, \pm 1\}$ of the 1-forms with logarithmic coefficients

$$\frac{1}{2}(\log U_{j,a} d \log U_{j,b} - \log U_{j,b} d \log U_{j,a}) \quad (1 \leq a < b \leq 3).$$

and which is such that $d\mathbf{AR}_\varepsilon^1 = \mathbf{Res}_\varepsilon$.

4. The space of 1-abelian relations of weight 0 of $\mathcal{W}_{Y_5}^{GM}$ is of dimension 20 and has a basis of ARs as in **2**, that is which correspond to identities of the form $\sum_{i=1}^{10} \sum_{s=1}^3 c_{i,s} d \log U_{i,s} = 0$ for some coefficients $c_{i,s} \in \{-1, 0, 1\}$. Moreover, one has $W_0(\mathbf{AR}^1(\mathcal{W}_{Y_5}^{GM})) = \mathbf{AR}_{\text{Rat}}^1(\mathcal{W}_{Y_5}^{GM})$ and this space is a subspace of the space $\text{Ker}(d^1)$ of closed 1-ARs of $\mathcal{W}_{Y_5}^{GM}$.

5. The 1-ARs $\mathbf{AR}_\varepsilon^1$ for ε even span a space of dimension 10 on which the restriction of the derivative (5) induces an isomorphism onto $\mathbf{AR}_C^2(\mathcal{W}_{Y_5}^{GM})$: setting

$$(49) \quad \mathbf{AR}_C^1 = \mathbf{AR}_C^1(\mathcal{W}_{Y_5}^{GM}) = \langle \mathbf{AR}_\varepsilon^1 \mid \varepsilon \in \{\pm 1\}^5 \text{ is even} \rangle,$$

this isomorphism is given by $d^1 : \mathbf{AR}_C^1(\mathcal{W}_{Y_5}^{GM}) \xrightarrow{\sim} \mathbf{AR}_C^2(\mathcal{W}_{Y_5}^{GM})$, $\mathbf{AR}_\varepsilon^1 \mapsto \mathbf{Res}_\varepsilon$.

6. For any i, j, k such that $1 \leq i < j < k \leq 5$, the 6-subweb $\mathcal{W}_{ijki^*j^*k^*}$ of $\mathcal{W}_{Y_5}^{GM}$ defined by the first integrals U_l for $l \in \{i, j, k, i+5, j+5, k+5\}$ carries a complete and irreducible 1-AR of weight 1, denoted by $\mathbf{AR}_{ijk}^1$ (uniquely defined up to sign), the l -th component of which is a linear combination, with coefficients in $\{-1, 0, +1\}$, of the exact 1-forms

$$d(\log U_{l,a} \log U_{l,b}) = \log U_{l,a} d \log U_{l,b} + \log U_{l,b} d \log U_{l,a} \quad (1 \leq a < b \leq 3).$$

The ten 1-abelian relations $\mathbf{AR}_{ijk}^1$ span a subspace of $W_1(\mathbf{AR}_C^1(\mathcal{W}_{Y_5}^{GM})) \cap \text{Im}(d^0)$ of dimension 5, which will be denoted by $\mathbf{AR}_{\text{Sym}}^1(\mathcal{W}_{Y_5}^{GM})$ or just $\mathbf{AR}_{\text{Sym}}^1$ for short.

7. There is a decomposition in direct sum:

$$(50) \quad \mathbf{AR}^1(\mathcal{W}_{Y_5}^{GM}) = \underbrace{W_0(\mathbf{AR}^1(\mathcal{W}_{Y_5}^{GM}))}_{\text{Im}(d^0) = \text{Ker}(d^1)}^{20} \oplus \overbrace{\mathbf{AR}_{\text{Sym}}^1(\mathcal{W}_{Y_5}^{GM})}^{W_1(\mathbf{AR}^1(\mathcal{W}_{Y_5}^{GM}))}^5 \oplus \mathbf{AR}_C^1(\mathcal{W}_{Y_5}^{GM})^{10}.$$

where the upper exponents in red stand for the corresponding dimensions. It follows that the 1-rank of $\mathcal{W}_{Y_5}^{GM}$ is 35.

8. The decomposition in direct sum (50) actually is the decomposition of $\mathbf{AR}^1(\mathcal{W}_{Y_5}^{GM})$ into irreducible W_{D_5} -representations, with the following isomorphisms:

$$(51) \quad W_0(\mathbf{AR}^1) \simeq V_{[2,21]}^{20}, \quad \mathbf{AR}_{Sym}^1 \simeq V_{[-,221]}^5 \quad \text{and} \quad \mathbf{AR}_C^1 \simeq \mathbf{AR}_C^2 \simeq V_{[11,111]}^{10}.$$

Proof. The proof essentially goes by explicit computations that we do not reproduce here.²⁴

For the eighth point, we proceeded as follows: thanks to the explicit expressions (27) and (28) for the generators σ_i of $\mathcal{W}_{D_5}$ (see (30)), and working with an arbitrary but fixed basis of any one of the spaces $W_0(\mathbf{AR}^1)$, $\mathbf{AR}_{Sym}^1$ or $\mathbf{AR}_C^1$, noted here by S , there is no difficulty first to verify that the pull-back maps σ_i^* give rise to automorphisms of S , second to get explicit matrices of these automorphisms with respect to the chosen basis. Then proceeding as in the Appendix of [Pi4], there is no difficulty to compute the character of the representation $W_{D_5} \simeq \mathcal{W}_{D_5} \hookrightarrow \text{GL}(S)$. For instance, one obtains that the character of the action of W_{D_5} on $W_0(\mathbf{AR}^1(\mathcal{W}_{Y_5}^{GM}))$ is

$$\chi_{W_0(\mathbf{AR}^1)} = (20, -4, 4, 2, -2, 2, -2, 0, 0, 0, -1, 1, -1, -1, 1, 0, 0, 0).$$

Looking at the character table of W_{D_5} (cf. Table 4 in [Pi4] for instance), the first isomorphism in (51) follows. The two other cases are handled in the same way. $\square$

3.4. The 0-abelian relations of $\mathcal{W}_{Y_5}^{GM}$. One has $r_0(\mathcal{W}_{Y_5}^{GM}) = 25$ and because the differential d^0 induces an isomorphism $\mathbf{AR}^0 = \mathbf{AR}^0(\mathcal{W}_{Y_5}^{GM}) \simeq \text{Im}(d^0) = W_0(\mathbf{AR}^1) \oplus \mathbf{AR}_{Sym}^1$, the structure of $\mathbf{AR}^0$ essentially has already been given in the previous subsection.

For $i \in \llbracket 5 \rrbracket$, one sets $i^* = i + 5 \in \{6, \dots, 10\}$ and $l, m, n \in \llbracket 10 \rrbracket$ pairwise distinct, one denotes by $\mathcal{W}_{lmn}$ the 3-subweb of $\mathcal{W}_{Y_5}^{GM}$ defined by the first integrals U_l, U_m and U_n : $\mathcal{W}_{lmn} = \mathcal{W}(U_l, U_m, U_n)$. By direct explicit computations, we get the following result which just adds a few details to what can be deduced from Theorem 3.14.

Proposition 3.15. 1. *One has a direct sum*

$$\mathbf{AR}^0(\mathcal{W}_{Y_5}^{GM}) = W_1(\mathbf{AR}^0)^{20} \oplus W_2(\mathbf{AR}^0)^5$$

with the differential d^0 inducing isomorphisms of W_{D_5} -representations

$$W_1(\mathbf{AR}^0) \simeq W_0(\mathbf{AR}^1) \simeq V_{[2,21]}^{20} \quad \text{and} \quad W_2(\mathbf{AR}^0) \simeq \mathbf{AR}_{Sym}^1 \simeq V_{[-,221]}^5.$$

2. The 3-subwebs of $\mathcal{W}_{Y_5}^{GM}$ have 0-rank less than or equal to 1. Those with rank 1 are the 3-subwebs $\mathcal{W}_{ijk}$, $\mathcal{W}_{ijk^*}$, $\mathcal{W}_{ij^*k^*}$, $\mathcal{W}_{i^*j^*k^*}$, and are therefore 80 in number. Their 0-ARs all have weight 1 and they span the whole space $W_1(\mathbf{AR}^0)$ which is of dimension 20.

3. For any i, j, k such that $1 \leq i < j < k \leq 5$, the 1-AR $\mathbf{AR}_{ijk}^1$ of the 6-subweb $\mathcal{W}_{ijki^*j^*k^*}$ of $\mathcal{W}_{Y_5}^{GM}$ is exact, and is the derivative of a 0-AR denoted by $\mathbf{AR}_{ijk}^0$, which corresponds to the (unique up to sign) functional identity of the form $\sum_l \sum_{1 \leq a < b \leq 3} c_l^{a,b} \text{Log } U_{l,a} \text{Log } U_{l,b} = 0$ where the first sum is for l ranging in $\{i, j, k, i^*, j^*, k^*\}$, and where all the coefficients $c_l^{a,b}$ belong to $\{-1, 0, 1\}$.

The ten 0-abelian relations $\mathbf{AR}_{ijk}^0$ span the whole space $W_2(\mathbf{AR}^0)$ of 0-abelian relations of weight 2 of $\mathcal{W}_{Y_5}^{GM}$, which is of dimension 5. Moreover, the space $\text{Res}(W_2(\mathbf{AR}^0))$ spanned by the

²⁴Short Maple worksheets are available from the author upon request.

residues of the weight 2 abelian relations $\mathbf{AR}_{ijk}^0$ coincides with $W_1(\mathbf{AR}^0)$. One has:

$$\mathbf{Res}(W_2(\mathbf{AR}^0)) = \left\langle \mathbf{Res}_{\zeta_\ell}(\mathbf{AR}_{ijk}^1) \mid \begin{array}{l} \ell = 1, \dots, 10 \\ 1 \leq i < j < k \leq 5 \end{array} \right\rangle = W_1(\mathbf{AR}^0).$$

All the 0-ARs of $\mathcal{W}_{Y_5}^{GM}$ can be made explicit. For instance, the following weight 2 functional identity

$$\begin{aligned} & \left(\ln \left(\frac{U_{1,1}}{U_{1,2}} \right) \ln(U_{1,3}) \right) + \left(\ln(U_{2,1} U_{2,2}) \ln(U_{2,3}) \right) + \left(\ln \left(\frac{U_{3,1}}{U_{3,2}} \right) \ln(U_{3,3}) \right) \\ & + \left(\ln \left(\frac{U_{6,1}}{U_{6,2}} \right) \ln(U_{6,3}) \right) + \left(\ln(U_{7,1} U_{7,2}) \ln(U_{7,3}) \right) + \left(\ln \left(\frac{U_{8,1}}{U_{8,2}} \right) \ln(U_{8,3}) \right) = 0 \end{aligned}$$

corresponds to the weight 2 abelian relation $\mathbf{AR}_{123}^0$ of the web $\mathcal{W}_{123} = \mathcal{W}(U_1, U_2, U_3, U_6, U_7, U_8)$.

3.5. The various links between the spaces of ARs of $\mathcal{W}_{Y_5}^{GM}$ summarized in a table. We find interesting/enlightening as well as convenient to gather all our findings about the abelian relations of $\mathcal{W}_{Y_5}^{GM}$ in Table 2 below. In it:

- the columns are labelled by the weight of the ARs, the lines by their degree;
- the diagonal black arrows are isomorphisms of W_{D_5} -representations induced by the differentials $d^k : \mathbf{AR}^k \rightarrow \mathbf{AR}^{k+1}$ (with $k = 0, 1$);
- a dashed red arrow $T \dashleftarrow S$ means that the whole target space T is spanned by the residues of the elements of the source space S ;
- the upper exponents in red stand for the dimension of each space;
- **HLOG** stands for the complex line spanned by the ‘master 2-abelian relation’ **HLOG** $_{Y_5}$.

	$w = 0$	$w = 1$	$w = 2$
$\mathbf{AR}^0$		$W_1(\mathbf{AR}^0)^{20} \dashleftarrow W_2(\mathbf{AR}^0)^5$	
$\mathbf{AR}^1$	$W_0(\mathbf{AR}^1)^{20} \dashleftarrow \mathbf{AR}_C^1^{10} \oplus \mathbf{AR}_{Sym}^1^5$		
$\mathbf{AR}^2$	$\mathbf{AR}_C^2^{10} \dashleftarrow \mathbf{HLOG}^1$		

TABLE 1. The subspaces of the space of abelian relations of $\mathcal{W}_{Y_5}^{GM}$ and the many relations between them.

It may be useful to gather the structures as $W(D_5)$ -representations of the subspaces appearing in this table as well. This is given in the following Table 2.

	$w = 0$	$w = 1$	$w = 2$
AR^0		$V_{[2,21]}^{20}$	$V_{[-,221]}^5$
AR^1	$V_{[2,21]}^{20}$	$V_{[11,111]}^{10} \oplus V_{[-,221]}^5$	
AR^2	$V_{[11,111]}^{10}$	$V_{[-,11111]}^1$	

TABLE 2. The subspaces of $AR(\mathcal{W}_{Y_5}^{GM})$ as W_{D_5} -representations.

4. A cluster view on $\mathcal{W}_{\mathbb{S}_5}^{GM}$

A particularly interesting feature of Abel's identity of the dilogarithm is that it has a 'cluster nature', in the sense that it can be written in an equivalent nice form in terms of some cluster variables (see [Pi2] for details). Indeed, if X_ℓ (with $\ell \in \mathbf{Z}/5\mathbf{Z}$) stands for the $\mathcal{X}$ -cluster variables of type A_2 characterized by the 5-cyclic recurrence relations $X_{\ell-1}X_{\ell+1} = 1 + X_\ell$ for any ℓ , then setting $X_1 = u_1$ and $X_2 = u_2$, one has $X_3 = \frac{1+u_2}{u_1}$, $X_4 = \frac{1+u_1+u_2}{u_1u_2}$, $X_5 = \frac{1+u_1}{u_2}$ and $X_{\ell+5} = X_\ell$ for every $\ell \in \mathbf{Z}$.²⁵ Moreover it is well-known that $(\mathcal{A}b)$ is equivalent to the following functional identity

$$(52) \quad \sum_{i=1}^5 R(X_\ell) = \frac{\pi^2}{2}$$

which is satisfied for all $X_1, X_2 > 0$ by the 'cluster dilogarithm' R .²⁶ In addition to furnishing a very nice formal way to write this identity, the cluster perspective on it offers a more conceptual way to interpret it, since (52) (hence Abel's identity $(\mathcal{A}b)$) can be seen as the manifestation of a certain property of an important object associated to the $\mathcal{X}$ -cluster algebra of type A_2 , namely the associated scattering diagram SC_{A_2} . Then that (52) holds true follows from the fact that this scattering diagram is 'consistent' (see [Na2, Theorem 3.6]).

If one believes that $\mathbf{HLOG}_{Y_5}$ is a natural generalization of the five terms identity of the dilogarithm, it is natural to wonder whether the former differential identity may be explained in terms of a certain property of a putative scattering diagram SC_{Y_5} or not. Moreover, because $\mathbf{HLOG}_{Y_5}$ is given by the vanishing of a sum with finitely many terms, a naive expectation would be that SC_{Y_5} be of finite type. Thus if one is fool and dreamy enough to want explain $\mathbf{HLOG}_{Y_5}$ by means of a scattering diagram, a first step would be to find a cluster-like structure associated to the spinor 10-fold $\mathbb{S}_5$, which firstly is of finite type and secondly, is well suited to the web $\mathcal{W}_{Y_5}^{GM}$ under

²⁵It seems that, in a slightly different form, the cyclic recurrence of the X_ℓ 's was already known to Gauss, see here.

²⁶This is the function defined by $R(u) = \frac{1}{2} \int_0^u (\ln(1+t)/t - \ln(t)/(1+t)) dt$ for any $u > 0$ (see [Pi2, §2.2.2.1]). In terms of the classical bilogarithm, it is expressed as $R(u) = -\text{Li}_2(-u) - \frac{1}{2} \text{Log}(u) \text{Log}(1+u)$ for any $u \in \mathbf{R}_{>0}$.

consideration. There are many works on some cluster structures on some pieces (more precisely, some open domains of some peculiar subvarieties) of generalized flag manifolds, but except in very few known cases, these cluster structures are not of finite type. For the case under consideration, we are not aware of any classical cluster structure on (some dense open-subset of) $\mathbb{S}_5$ which is of finite type, which may temper our dream of explaining $\mathbf{HLOG}_{\mathcal{Y}_5}$ via a finite scattering diagram. However, some recent works by Ducat and Daisey-Ducat suggest that allowing to consider more general cluster-like structures than the classical ones might be the suitable path to reach our goal.

In [Du], for the case of $\mathcal{G}_5 = \mathbb{S}_5$, and in [DD] for the case of the Cayley plane $\mathcal{G}_6 = \mathbb{OP}^2$, Ducat then Ducat together with Daisey describe a cluster-like structure of finite type on the coordinate ring R_r of a certain Zariski open domain in $\mathcal{G}_r$ for $r = 5, 6$. These cluster-like structures are known as LPAs²⁷, a generalization of the classical notion of cluster algebra introduced by Lam and Pylyavskyy in [LP]. As the name suggests, the main feature of such an algebra is that any ‘cluster’ variable obtained by means of a finite sequence of ‘generalized mutations’ from the initial cluster variables is a Laurent polynomial in the latter. For $r = 4$, one has $\mathcal{G}_4 = G_2(\mathbb{C}^5)$ and this grassmannian carries a classical $\mathcal{A}$ -cluster structure of finite type A_2 which, after quotienting by the action of the rank 4 Cartan torus H_4 of $SL_5(\mathbb{C})$, gives rise to the $\mathcal{X}$ -cluster structure of type A_2 on $\mathcal{Y}_4^* = \mathcal{G}_4^* // H_4 = G_2(\mathbb{C}^5)^* // H_4 \simeq \mathcal{M}_{0,5}$.

In [Du], Ducat endows the coordinate ring of the complement $\mathbb{S}_5^\circ$ of a divisor in the spinor 10-fold with the structure of a finite LPA. Unfortunately, the theory of LPAs has not been developed that much so far. The notion of mutation for these algebras is a generalization of the binomial $\mathcal{A}$ -mutation of the classical theory of cluster algebras but as of the time of writing, an equivalent notion for LPAs of that of $\mathcal{X}$ -mutation has not been worked out yet. For that reason, we do not have a cluster description of Gelfand-MacPherson web on $\mathcal{Y}_5^*$ as nice as the one of $\mathcal{W}_{Y_4}^{GM}$ in terms of the $\mathcal{X}$ -cluster variable of type A_2 , which can be written in a concise mathematical form as

$$\mathcal{W}_{Y_4}^{GM} \simeq \mathcal{X}\mathcal{W}_{A_2} = \mathcal{W}(X_i)_{i=1}^5 = \mathcal{W}\left(u_1, u_2, \frac{1+u_2}{u_1}, \frac{1+u_1+u_2}{u_1u_2}, \frac{1+u_1}{u_2}\right).$$

Because there is no generalization to the case of LPAs of the notion of $\mathcal{X}$ -mutation of the classical theory of cluster algebras yet, we are going to deal with Gelfand-MacPherson’s web of the spinor tenfold $\mathbb{S}_5$ instead. In short, we prove that $\mathcal{W}_{\mathbb{S}_5}^{GM}$ is cluster with respect to the finite LPA structure on $\mathbb{S}_5$ constructed by Ducat in [Du].

Proposition 4.1. 1. *Wick’s parametrization (10) of $\mathbb{S}_5$ is cluster, i.e. the Wick coordinates $x_{ij} \circ W^{-1} : \mathbb{S}_5 \simeq \text{Asym}_5(\mathbb{C}) \dashrightarrow \mathbb{C}$ (for i, j such that $1 \leq i < j \leq 5$) naturally identify with some of the cluster variables of Ducat’s LPA of $\mathbb{S}_5$.*

2. *The ten face maps $\Pi_F : \mathbb{S}_5 \dashrightarrow \mathbb{S}_{4,F}$ are cluster in the sense that their components, when written in the initial cluster variables at the source and some Wick’s coordinates at the target, are cluster variables, possibly up to sign and up to multiplication by a monomial in the frozen variables.*

In the lines below, we explain how we proceeded to get this result.

²⁷LPA is the acronym for ‘Laurent Phenomenon Algebra’.

4.1. Ducat's LPA on $\mathbb{S}_5$. Ducat's LPA on $\mathcal{A}_5$ has rank 3 and 18 cluster variables in total, among which 8 are frozen. These latter are denoted by $a_1, \dots, a_8$. Denoting by x_1, x_2, x_3 the initial unfrozen cluster variables, all the other ones can be obtained using the following SageMath script written by Daisey and which can be run online at his webpage:

```
sage: var("x1,x2,x3")
sage: coeffs = [var("a%d" %i) for i in range(1, 9)] # coefficients
sage: F1 = a5*x2 + a8*x3 + a2*a3
sage: F2 = a6*x1*x3 + a3*a4*x1 + a8*a1*x3 + a1*a2*a3
sage: F3 = a4*x1 + a7*x2 + a1*a2
sage: S = LPASeed({x1:F1, x2:F2, x3:F3})
sage: show(S.variable_class())
```

In addition to the three initial cluster variables x_1, x_2, x_3 , the other unfrozen cluster variables are the following:

$$\begin{aligned} & \frac{a_2a_3 + a_5x_2 + a_8x_3}{x_1}, \frac{a_1a_2a_3 + a_1a_8x_3 + a_3a_4x_1 + a_6x_1x_3}{x_2}, \frac{a_1a_2 + a_4x_1 + a_7x_2}{x_3}, \\ & \frac{a_1a_2a_3 + a_1a_5x_2 + a_1a_8x_3 + a_3a_4x_1 + a_6x_1x_3}{x_1x_2}, \frac{a_1a_2a_3 + a_1a_8x_3 + a_3a_4x_1 + a_3a_7x_2 + a_6x_1x_3}{x_2x_3}, \\ & \frac{a_1a_2^2a_3 + a_1a_2a_5x_2 + a_1a_2a_8x_3 + a_2a_3a_4x_1 + a_2a_3a_7x_2 + a_4a_5x_1x_2 + a_5a_7x_2^2 + a_7a_8x_2x_3}{x_3x_1}, \\ & \frac{a_1a_2^2a_3 + a_1a_2a_5x_2 + a_1a_2a_8x_3 + a_2a_3a_4x_1 + a_2a_3a_7x_2 + a_2a_6x_1x_3 + a_4a_5x_1x_2 + a_5a_7x_2^2 + a_7a_8x_2x_3}{x_1x_2x_3}. \end{aligned}$$

These cluster variables are Laurent polynomials in the initial unfrozen cluster variables x_1, x_2 and x_3 , with coefficients in the monoid $\mathbb{N}_{>0}[a_1, \dots, a_8]$. The non initial unfrozen cluster variables are characterized by their denominators which are monomials in the x_i 's. Hence one can denote these cluster variables by X_i, X_{ij} and X_{123} for $i, j \in \{1, 2, 3\}$ distinct, where X_i (resp. X_{ij} or X_{123}) stands for the cluster variables with monomial denominator x_i (resp. x_ix_j or $x_1x_2x_3$), that is:

$$X_1 = \frac{a_2a_3 + a_5x_2 + a_8x_3}{x_1}, \quad X_2 = \frac{a_1a_2a_3 + \dots + a_6x_1x_3}{x_2}, \dots, \quad X_{123} = \frac{a_1a_2^2a_3 + \dots + a_7a_8x_2x_3}{x_1x_2x_3}.$$

4.2. The initial cluster on $\mathbb{S}_5$ and Wick's coordinates. The initial cluster of Ducat's LPA structure on (the affine cone $\widehat{\mathbb{S}_5}$ over) $\mathbb{S}_5$ is formed by the triple (x_1, x_2, x_3) together with the frozen variables $a_1, \dots, a_8$. These cluster variables are the restrictions to $\widehat{\mathbb{S}_5}$ of some of the weight coordinates on the spinor representation S_5^+ . So each of the initial cluster coordinates is associated to a well-defined weight of the spinor representation. The same holds for the Wick coordinates we use here for parametrizing birationnally $\mathbb{S}_5$ and it is by comparing the weights of all these coordinates that we are going to make explicit the way they are related.

In [DD], it is by means of a figure (namely Figure 3.(a)) that is indicated what are the weights of the 16 coordinates x_i, a_i for $i = 1, \dots, 8$ considered by Ducat. For the Wick coordinates, the analogous correspondence is given in [Pi5, Table 4.1]. Since [DD, Figure 3.(a)] pictures the images of the weights by 'the' Coxeter projection, it suffices to have an explicit expression for the latter in order to get how Ducat's cluster coordinates and Wick's ones are related. This is what we do below.

Recall that a useful description of the weights of S_5^+ is in terms of the sixteen lines of an arbitrary (but fixed) smooth del Pezzo surface dP_4 described as the blow-up of the projective plane in five points. Then $\text{Pic}_{\mathbf{Z}}(dP_4)$ is freely generated by the classes $e_1, \dots, e_5$ of the five exceptional divisors of the blow up plus the class h of the preimage under the blow-up map of a generic line in $\mathbf{P}^2$. With these notations, we use the following labels for (the classes of) the lines ℓ_i ($i = 1, \dots, 16$) of the considered del Pezzo surface:

$$\begin{aligned} (\ell_i)_{i=1}^{16} = & \left(e_1, e_2, e_3, e_4, e_5, h - e_1 - e_2, h - e_1 - e_3, h - e_1 - e_4, \right. \\ & h - e_1 - e_5, h - e_2 - e_3, h - e_2 - e_4, h - e_2 - e_5, \\ & \left. h - e_3 - e_4, h - e_3 - e_5, h - e_4 - e_5, 2h - e_1 - e_2 - e_3 - e_4 - e_5 \right). \end{aligned}$$

The canonical class is $\kappa = -3h + \sum_{i=1}^5 e_i$ and as a basis for its orthogonal $\kappa^\perp$, we take the one formed by the following classes:

$$\begin{aligned} f_1 = \frac{1}{2}(h + e_1 - e_2 - e_3 - e_4 - e_5) \quad f_2 = \frac{1}{2}(h - e_1 + e_2 - e_3 - e_4 - e_5) \quad f_3 = \frac{1}{2}(h - e_1 - e_2 + e_3 - e_4 - e_5) \\ f_4 = \frac{1}{2}(h - e_1 - e_2 - e_3 + e_4 - e_5) \quad \text{and} \quad f_5 = \frac{1}{2}(h - e_1 - e_2 - e_3 - e_4 + e_5). \end{aligned}$$

For $i = 1, \dots, 16$, we denote by $\omega_i = \omega_{\ell_i} \in \mathbf{R}^5$ the vector of coordinates of $\ell_i - (\kappa, \ell_i)\kappa \in \kappa^\perp$ expressed in the basis $f = (f_i)_{i=1}^5$. These sixteen 5-tuples are the weights of the spinor representation we are working with:

$$\begin{aligned} (53) \quad \omega_1 &= \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right) \quad \omega_2 = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right) \quad \omega_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right) \\ \omega_4 &= \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right) \quad \omega_5 = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right), \quad \omega_6 = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\ \omega_7 &= \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \quad \omega_8 = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right) \quad \omega_9 = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right) \\ \omega_{10} &= \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \quad \omega_{11} = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right) \quad \omega_{12} = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right) \\ \omega_{13} &= \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right) \quad \omega_{14} = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right) \quad \omega_{15} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right) \end{aligned}$$

$$\text{and} \quad \omega_{16} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right).$$

As generators of the Weyl group, we take the involutions S_i for $i = 1, \dots, 4$, which exchange f_i with f_{i+1} and let the others f_k 's fixed, and S_5 is the one such that $S_5(f_i) = f_i$ for $i = 1, 2, 3$ and $S_5(f_j) = -f_k$ for $\{j, k\} = \{4, 5\}$. The corresponding matrices in the basis f are denoted the same:

$$S_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad S_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad S_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

As a Coxeter element, we take

$$C = S_1 S_2 S_3 S_4 S_5 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

whose order is verified to be the Coxeter number $h = h_{D_5}$ of $W(D_5)$, namely $h = 8$. Let P_C be the Coxeter plane and $\Pi_C : \mathbf{R}^5 \rightarrow P_C$ be the corresponding Coxeter projection. By definition, P_C is the 2-plane in $\mathbf{R}^5$ which is fixed by C and such that $C|_{P_C}$ is a rotation of angle $2\pi/h = \pi/4$. Over the complex, the Coxeter element C admits $\lambda = e^{\frac{2i\pi}{h}}$ as an eigenvalue, with associated eigenspace of dimension 1 and spanned by the eigenvector $v = ((-1+i)/\sqrt{2}, i, (1+i)/\sqrt{2}, 1, 0)$. Then P_C is the plane spanned by the real and the imaginary parts of v . After normalization, we get that the following two vectors form a basis of P_C which is orthonormal with respect to the standard Euclidean structure on $\mathbf{R}^5$:

$$v_{\text{re}} = \left(-\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{\sqrt{2}}, 0\right), \quad v_{\text{im}} = \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}, 0, 0\right)$$

Then with respect to the bases f and $(v_{\text{re}}, v_{\text{im}})$, Coxeter projection is written $x \mapsto (\langle x, v_{\text{re}} \rangle, \langle x, v_{\text{im}} \rangle)$ and one can compute all points $\Pi_C(\omega_i) \in \mathbf{R}^2$ for $i = 1, \dots, 16$. One obtains the sixteen black dots in Figure 2 with the corresponding label indicated in green inside each. Then (up to an irrelevant rotation), one can identify the black dots of our figure with the vertices of [DD, Figure 3.(a)]. These latter being labeled by Ducat's notation $x_1, \dots, x_8, a_1, \dots, a_8$ for the weight variables, one can associate one of our weights ω_i to each of them. On the other hand, we indicated what are the weights of Wick's coordinates in [Pi5, Table 4.1]. We thus deduce the following relations between Ducat's labelling of the weight variables and the components of Wick's parametrization: there exist complex constants v_i 's (for $i = 1, \dots, 10$) such that the following relations (as rational functions) hold true:

$$\begin{aligned} \frac{x_1}{a_5} &= v_1 x_{14}, & \frac{x_2}{a_5} &= v_2 P_2, & \frac{x_3}{a_5} &= v_3 x_{13}, & \frac{a_1}{a_5} &= v_4 P_5, & \frac{a_2}{a_5} &= v_5 x_{15}, \\ \frac{a_3}{a_5} &= v_6 x_{34}, & \frac{a_4}{a_5} &= v_7 P_4, & \frac{a_6}{a_5} &= v_8 P_1, & \frac{a_7}{a_5} &= v_9 x_{12}, & \frac{a_8}{a_5} &= v_{10} x_{45}. \end{aligned}$$

Taking $(v_i)_{i=1}^{10} = (1, -1, 1, 1, 1, 1, -1, -1, 1, 1)$ (that is $v_i = -1$ for $i \in \{2, 7, 8\}$ and $v_i = 1$ otherwise), the relations above can be solved nicely. We obtain that $A = (x_{ij})_{i,j=1}^5 \in \text{Asym}_5(\mathbf{C})$ is such that $W(A) \in \mathbb{S}_5$ has Ducat's coordinates $x_1, \dots, x_8, a_1, \dots, a_8$ if and only if the following equality is satisfied:

$$(54) \quad \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{pmatrix} = \frac{1}{a_5} \begin{pmatrix} 0 & a_7 & x_3 & x_1 & a_2 \\ -a_7 & 0 & X_{12} & X_{23} & X_{123} \\ -x_3 & -X_{12} & 0 & a_3 & X_1 \\ -x_1 & -X_{23} & -a_3 & 0 & a_8 \\ -a_2 & -X_{123} & -X_1 & -a_8 & 0 \end{pmatrix}.$$

This proves the first point of Proposition 4.1.

With (54) at hand, proving the second point of Proposition 4.1 becomes straightforward using some formulas obtained in our previous article. In [Pi5], we gave explicit expressions for the 10 face maps expressed in Wick coordinates. The weight polytope $\Delta_{D_5} \subset \mathbf{R}^5$ is the convex envelope

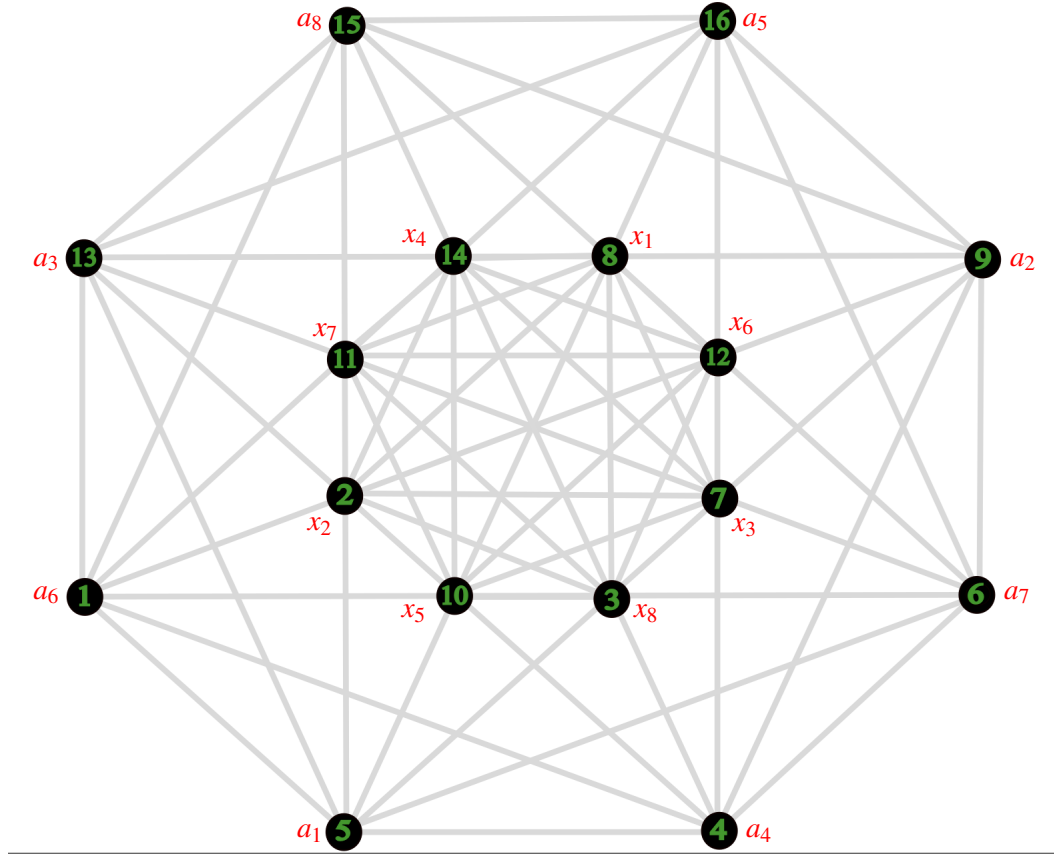


FIGURE 2. Projection of Gosset's graph²⁸ on the Coxeter plane and correspondence between the weights (with their label used in this paper in green) and the spinor coordinates of [DD, Figure 3.(a)] (in red).

of the 16 weights (53). It has 10 $\mathcal{W}$ -relevant facets, five 'positive' facets $\Delta_{D_5, i}^+ = \Delta_{D_5} \cap \{x_i = +1/2\}$ and five 'negative' ones $\Delta_{D_5, i}^- = \Delta_{D_5} \cap \{x_i = -1/2\}$ (with $i = 1, \dots, 5$).

The face maps $\mathbb{S}_5 \dashrightarrow \mathbb{S}_4$ associated with the five positive facets are written very simply in Wick coordinates since they correspond to the maps $\text{Asym}_5(\mathbb{C}) \ni A \mapsto A_i \in \text{Asym}_4(\mathbb{C})$ where A_i stands for the 4×4 matrix obtained by removing from A its i -th line and its i -th column (cf. Proposition 4.7 in [Pi5]). Viewed (54), we immediately get the second point of Proposition 4.1 for the five face maps associated to the positive facets of the weight polytope.

The expressions in Wick coordinates of the five face maps associated to the negative facets of Δ_{D_5} are more involved. Some birational models of these maps are the maps $A \mapsto A_{i,j} = \Psi_{i,j}(A)$ with $i \neq j$ where $\Psi_{i,j} : \text{Asym}_5(\mathbb{C}) \dashrightarrow \text{Asym}_4(\mathbb{C})$ is the rational map considered in (actually just above) Corollary 4.8 of [Pi5].

Using (54) and the explicit formulas for the $\Psi_{i,j}$'s, it is straightforward to get the following expressions for some models of the five 'negative face map' written in terms of the initial cluster

²⁸Gosset's graph is the 1-skeleton of Gosset's polytope 1_{21} , which here coincides with the weight polytope Δ_{D_5} .

coordinates:

$$\begin{aligned}
 A_{1,5} &= \frac{1}{a_2} \begin{bmatrix} 0 & -a_4a_5 & -a_5X_3 & a_7 \\ a_4a_5 & 0 & -a_5x_2 & x_3 \\ a_5X_3 & a_5x_2 & 0 & x_1 \\ -a_7 & -x_3 & -x_1 & 0 \end{bmatrix} & A_{2,1} &= \frac{1}{a_7} \begin{bmatrix} 0 & X_{12} & X_{23} & X_{123} \\ -X_{12} & 0 & -a_1a_5 & a_4a_5 \\ -X_{23} & a_1a_5 & 0 & a_5X_3 \\ -X_{123} & -a_4a_5 & -a_5X_3 & 0 \end{bmatrix} \\
 A_{3,4} &= \frac{1}{a_3} \begin{bmatrix} 0 & a_1a_5 & x_3 & -a_5x_2 \\ -a_1a_5 & 0 & X_{12} & -a_6a_5 \\ -x_3 & -X_{12} & 0 & -X_1 \\ a_5x_2 & a_6a_5 & X_1 & 0 \end{bmatrix} & A_{4,1} &= \frac{1}{x_1} \begin{bmatrix} 0 & X_{23} & a_3 & a_8 \\ -X_{23} & 0 & -a_1a_5 & a_5X_3 \\ -a_3 & a_1a_5 & 0 & a_5x_2 \\ -a_8 & -a_5X_3 & -a_5x_2 & 0 \end{bmatrix} \\
 \text{and } A_{5,4} &= \frac{1}{a_8} \begin{bmatrix} 0 & a_5X_3 & a_5x_2 & -a_2 \\ -a_5X_3 & 0 & a_6a_5 & -X_{123} \\ -a_5x_2 & -a_6a_5 & 0 & -X_1 \\ a_2 & X_{123} & X_1 & 0 \end{bmatrix}.
 \end{aligned}$$

Up to an irrelevant minus sign for some entries, the coefficients of these matrices are all either monomials in the frozen cluster variables or products of a cluster variable with such a monomial, which completes the proof of Proposition 4.1. $\star$

We finish this subsection by two remarks.

The first is that, if our results above are only about the cluster nature of Gelfand-MacPherson's web of the spinor tenfold $\mathbb{S}_5$, we are convinced that they must have a kind of $\mathcal{X}$ -counterparts for the web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ which is truly the one we are interested in. The lack of a theory of 'coefficients' and above all of their mutations is the reason why we didn't work with $\mathcal{W}_{\mathcal{Y}_5}^{GM}$. We hope to come back to this in the future.

Another remark concerning the above material is that while the approach used to prove Proposition 4.1 ultimately relies on some explicit calculations, it nevertheless points to a geometric way of constructing the LPA cluster structure on $\mathbb{S}_5$, namely by means of the face maps associated with the facets of the weight polytope. In an ongoing research project, we have verified that this geometric approach generalizes to several other homogeneous spaces. We hope to continue our researches in this direction in the near future as well.

5. The web $W_{Y_5}^+$, Bol's web, and their abelian relations

In this subsection, we study a certain 5-subweb $W_{Y_5}^+$ of $\mathcal{W}_{Y_5}^{GM}$ which carries interesting ARs which are in a 1-1 correspondance with those of Bol's web. In particular, one recovers Abel's 5-term identity from an 1-AR with logarithmic coefficients and monomial arguments, which is new as far we know. The latter abelian relation will be used further in section §6 for recovering the del Pezzo hypertrilogarithmic identities $\mathbf{HLog}_{\text{dP}_4}^3$ from $\mathbf{HLOG}_{Y_5}$.

5.1. The web $W_{Y_5}^+$ and its abelian relations. In this subsection, we aim to study the relations between the web

$$W_{Y_5}^+ = \mathcal{W} \left((y_{25}, y_{24}y_{35}), \left(\frac{1}{y_{13}}, \frac{y_{14}y_{35}}{y_{13}} \right), (y_{24}, y_{14}y_{25}), \left(\frac{1}{y_{35}}, \frac{y_{13}y_{25}}{y_{35}} \right), (y_{14}, y_{13}y_{24}) \right)$$

and its abelian relations to those of Bol's web.

The 1-AR $\mathbf{AR}_\delta^1$ of Proposition 3.13, which corresponds to the differential identity

$$0 = \sum_{i=1}^5 \epsilon_i U_i^*(\delta) = \frac{1}{2} \sum_{i=1}^5 \epsilon_i U_i^*(\text{Log}(x) dy/y - \text{Log}(y) dx/x)$$

spans the subspace of weight 1 abelian relations of $\mathbf{W}_{Y_5}^{GM,+}$: one has

$$W_1(\mathbf{AR}(\mathbf{W}_{Y_5}^{GM,+})) = \langle \mathbf{AR}_\delta^1 \rangle.$$

Determining the 1-ARs of weight 0 of $\mathbf{W}_{Y_5}^+$ amounts to find the scalar constants $c_{i,s}$ (with $i = 1, \dots, 5$ and $s = 1, 2$) such that the differential identity $\sum_{i=1}^5 (c_{i,1} dU_{i,1}/U_{i,1} + c_{i,2} dU_{i,2}/U_{i,2}) = 0$ is satisfied, hence reduces to elementary linear algebra. We find that any 3-subweb of $\mathbf{W}_{Y_5}^{GM,+}$ carries such an 1-AR (which moreover is unique up to multiplication by a non-zero scalar) which will be said 'combinatorial'. These ARs span a space $\mathbf{AR}_C(\mathbf{W}_{Y_5}^{GM,+})$ which is of dimension 5. From Proposition 3.3.2, it follows that we have the following decomposition in direct sum:

$$(55) \quad \mathbf{AR}^1(\mathbf{W}_{Y_5}^+) = \mathbf{AR}_C^1(\mathbf{W}_{Y_5}^+)^{\textcolor{red}{5}} \oplus \langle \mathbf{AR}_\delta^1 \rangle^{\textcolor{red}{1}}$$

with $\mathbf{AR}_C^1(\mathbf{W}_{Y_5}^+) = W_0(\mathbf{AR}^1(\mathbf{W}_{Y_5}^+))$ and $\langle \mathbf{AR}_\delta^1 \rangle = W_1(\mathbf{AR}^1(\mathbf{W}_{Y_5}^+))$.

Let $\mathcal{W}'$ be the subgroup of the Weyl group $\mathcal{W}_{D_5}$ spanned by the birational involutions σ_i for $i = 1, \dots, 4$ (see (28)). It is a group isomorphic to the Weyl group of type A_4 (hence to the symmetric group $\mathfrak{S}_5$) which lets the 5-web $\mathbf{W}_{Y_5}^+$ invariant. Moreover, it can be verified that $\mathcal{W}'$ acts by linear automorphisms on the space of 1-ARs of $\mathbf{W}_{Y_5}^+$ making of it a representation of $\mathfrak{S}_5$. By straightforward computations, we prove the

Proposition 5.1. *The decomposition in direct sum (55) actually is the decomposition of $\mathbf{AR}(\mathbf{W}_{Y_5}^+)$ into irreducible representations of $\mathfrak{S}_5$. Moreover, we have the following isomorphisms of $\mathfrak{S}_5$ -representations:*

$$(56) \quad \mathbf{AR}_C^1(\mathbf{W}_{Y_5}^+) \simeq V_{[221]}^{\textcolor{red}{5}} \quad \text{and} \quad \langle \mathbf{AR}_\delta^1 \rangle \simeq V_{[1^5]}^{\textcolor{red}{1}}.$$

Our aim is now to relate $\mathbf{W}_{Y_5}^+$ and its ARs to Bol's web and more specifically to its main abelian relation, the dilogarithmic AR given by Abel's relation.

5.2. The web $\mathbf{W}_{Y_5}^+$ and Bol's web. In §4.5.2 of [Pi5], we gave several birational models of Serganova-Skorobogatov's embedding $f_{SS} : X_r \hookrightarrow \mathcal{Y}_r$. Let $(a, b) \in \mathbb{C}^2$ be such that X_r identifies with the blow-up of the plane at the vertices of the standard quadrilateral in $\mathbb{P}^2$ plus the point $[a : b : 1]$. In [Pi5] (around Corollary 4.12 more precisely), we obtained that for generic complex constants $s_1, s_2, s_3 \in \mathbb{C}$ and setting $s_{ij} = s_i - s_j$ for any i, j and $\mathbf{y} = (y_{1,3}, y_{1,4}, y_{2,4}, y_{2,5}, y_{3,5})$, the following rational map

$$(x, y) \longmapsto \mathbf{y} = \left(\frac{(x-1)s_{13}}{(a-1)s_3}, \frac{x(y-1)(a-b)s_{123}}{(x-y)a(b-1)s_3s_{12}}, \frac{y(a-b)s_2}{(x-y)bs_{12}}, \frac{(b-1)s_3}{(y-1)s_{23}}, \frac{(x-y)(b-1)s_{12}}{(y-1)(a-b)s_{23}} \right)$$

is a birational model for f_{SS} . If one is no longer interested by taking track of the parameters a and b , by eliminating some parameters, one can consider the rational map

$$(x, y) \mapsto \mathbf{y} = \left(g_{13} (x-1), g_{14} \frac{x(y-1)}{(x-y)}, \frac{g_{24} y}{(x-y)}, \frac{g_{25}}{(y-1)}, g_{35} \frac{(x-y)}{(y-1)} \right)$$

where the g_{ij} 's are non zero complex parameters required to satisfy the following polynomial relation:

$$1 = g_{13}g_{14}g_{24}g_{25}g_{35} - g_{13}g_{14}g_{24} - g_{13}g_{14}g_{35} - g_{13}g_{25}g_{35} - g_{24}g_{25}g_{14} \\ - g_{24}g_{25}g_{35} + g_{13}g_{14} + g_{13}g_{35} + g_{14}g_{24} + g_{24}g_{25} + g_{25}g_{35}.$$

This relation admits $g_{13} = g_{14} = g_{24} = g_{25} = g_{35} = 1$ as a peculiar solution. We will work with the corresponding rational embedding of $\mathbf{C}^2$ into $\mathbf{Y}_5 = \mathbf{C}^5$, which is given by

$$(57) \quad \mathcal{D} : (x, y) \mapsto (y_{13}, y_{14}, y_{24}, y_{25}, y_{35}) = \left(x-1, \frac{x(y-1)}{x-y}, \frac{y}{x-y}, \frac{1}{y-1}, \frac{x-y}{y-1} \right).$$

One verifies that

$$\mathcal{D}^*(\mathbf{W}_{Y_5}^+) = \mathcal{W} \left(\left(\frac{1}{y-1}, \frac{y}{y-1} \right), \left(\frac{1}{x-1}, \frac{x}{x-1} \right), \left(\frac{y}{x-y}, \frac{x}{x-y} \right), \right. \\ \left. \left(\frac{y-1}{x-y}, \frac{x-1}{x-y} \right), \left(\frac{x(y-1)}{x-y}, \frac{y(x-1)}{x-y} \right) \right)$$

from which it follows that $d(U_{i,1} \circ \mathcal{D}) \wedge d(U_{i,2} \circ \mathcal{D}) = 0$ for $i = 1, \dots, 5$. This implies that the pull-back under $\mathcal{D}$ of the 2-codimensional web $\mathbf{W}_{Y_5}^+$ is only 1-codimensional. More precisely, setting $\varpi_i = U_{i,1} \circ \mathcal{D}$ for $i = 1, \dots, 5$, that is

$$\varpi_1 = \frac{1}{y-1}, \quad \varpi_2 = \frac{1}{x-1}, \quad \varpi_3 = \frac{y}{x-y}, \quad \varpi_4 = \frac{y-1}{x-y}, \quad \varpi_5 = \frac{x(y-1)}{x-y},$$

it comes that $\mathcal{D}^*(\mathbf{W}_{Y_5}^+)$ is the following web:

$$\mathbf{B} = \mathcal{W}(\varpi_i)_{i=1}^5 = \mathcal{W} \left(\frac{1}{y-1}, \frac{1}{x-1}, \frac{y}{x-y}, \frac{y-1}{x-y}, \frac{x(y-1)}{x-y} \right).$$

By using the criterion of [Pi5, §3.5.2], this web is easily seen to be a model of Bol's web and, as such, carries a version of Abel's 5-terms identity of the dilogarithm. We are going to prove that the latter can be obtained in a very simple way from the logarithmic identity (48).

We identify all the target affine spaces of the first integrals U_i to a same affine space with standard coordinates denoted by u_1, u_2 . Then one verifies that for any $i = 1, \dots, 5$, the image of $U_i \circ \mathcal{D}$ is the line cut out by $0 = 1 + u_1 - u_2$ which is parametrized by $L : z \mapsto (z, z+1)$. One has

$$L^*(\delta) = \frac{1}{2} \left(\text{Log}(z) \frac{dz}{1+z} - \text{Log}(z+1) \frac{dz}{z} \right) = -dR(z)$$

where R stands for the 'cluster dilogarithm' defined by

$$R(u) = \frac{1}{2} \int_0^u \left(\frac{\text{Log}(1+t)}{t} - \frac{\text{Log}(t)}{1+t} \right) dt \quad \text{for } u \geq 0.$$

Pulling-back the identity $0 = \frac{1}{2} \sum_{i=1}^5 \epsilon_i U_i^* (\text{Log}(x) \frac{dy}{y} - \text{Log}(y) \frac{dx}{x})$ under $\mathcal{D}$ is written

$$\omega_1^*(dR) - \omega_2^*(dR) + \omega_3^*(dR) - \omega_4^*(dR) + \omega_5^*(dR) = 0$$

or equivalently, in integrated form

$$(58) \quad R\left(\frac{1}{y-1}\right) - R\left(\frac{1}{x-1}\right) + R\left(\frac{y}{x-y}\right) - R\left(\frac{y-1}{x-y}\right) + R\left(\frac{x(y-1)}{x-y}\right) = \frac{\pi^2}{3}$$

a functional relation which is identically satisfied on any complex domain containing the ‘triangle’ $\{(x, y) \in \mathbf{R}^2 \mid 1 < y < x\}$. It is a version of Abel’s identity of the dilogarithm. We will denote this identity as well as the corresponding AR of $\mathbf{B}$ by $\mathbf{Ab}_R$.

More is true: it can be verified that given any 1-AR of a 3-subweb of $\mathbf{W}_{Y_5}^+$, its pull-back under $\mathcal{D}$ is an abelian relation of the corresponding 3-subweb of $\mathbf{B}$. With a few more computational checks, we get the following

Proposition 5.2. *The pull-back under $\mathcal{D}$ gives rise to two linear isomorphisms*

$$\langle \mathbf{AR}_\delta^1 \rangle \simeq \langle \mathbf{Ab}_R \rangle \quad \text{and} \quad \mathbf{AR}_C^1(\mathbf{W}_{Y_5}^+) \simeq \mathbf{AR}_C(\mathbf{B})$$

which actually turn out to be isomorphisms of $\mathfrak{S}_5$ -representations.

The last statement of this proposition requires a few lines of explanation. The point is that the group $\mathcal{W}' = \langle \sigma_1, \dots, \sigma_4 \rangle \subset \mathbf{Bir}(Y_5)$ lets invariant the image of the map (57) hence gives rise to a birational action on the source space of $\mathcal{D}$. More precisely, one verifies that setting

$$\tilde{\sigma}_1(x, y) = (x/y, 1/y) \quad \tilde{\sigma}_2(x, y) = (y, x) \quad \tilde{\sigma}_3(x, y) = (1-x, 1-y) \quad \tilde{\sigma}_4(x, y) = (1/x, 1/y)$$

then, as rational maps from $\mathbf{C}^2 \dashrightarrow \mathbf{C}^5 = Y_5$, one has $\mathcal{D} \circ \tilde{\sigma}_i = \sigma_i \circ \mathcal{D}$ for $i = 1, \dots, 5$. The birational action $\mathcal{W}' \rightarrow \mathbf{Bir}(\mathbf{C}^2)$ gives rise to a linear action of $\mathfrak{S}_5$ on $\mathbf{AR}(\mathbf{B})$. This gives a precise and rigorous meaning to the last assertion of Proposition 5.2.

Remark 5.3. 1. $\mathbf{B}$ is isomorphic to the web $\mathbf{W}_{\mathcal{M}_{0,5}}$ defined on the moduli space $\mathcal{M}_{0,5}$ by the five forgetful maps $\mathcal{M}_{0,5} \rightarrow \mathcal{M}_{0,4}$. Up to this isomorphism, the birational action of $\mathcal{W}' \simeq \mathfrak{S}_5$ on $\mathbf{C}^2$ corresponds to the one of $\mathbf{Aut}(\mathcal{M}_{0,5}) \simeq \mathfrak{S}_5$ on $\mathcal{M}_{0,5}$. That $\mathbf{AR}(\mathbf{W}_{\mathcal{M}_{0,5}})$ is isomorphic to $V_{[221]}^5 \oplus V_{[1^5]}^1$ as a $\mathfrak{S}_5$ -representation is due to Damiano (see [Pi5, Remark 3.1]).

2. We give above a new derivation of the five-term identity of the dilogarithm (in the form (58)) from the differential relation in five variables (48) which involves only logarithms and monomial functions. As far we know, this is new. It would be interesting to investigate whether the latter relation in five variables can be obtained formally from Abel’s identity in two variables or not.

3. The identities (48) and (58) admit real analytic versions which are globally satisfied. Setting $\delta^+ = \frac{1}{2}(\text{Log}|x|dy/y - \text{Log}|y|dx/x)$, one verifies that $\sum_{i=1}^5 \epsilon_i U_i^*(\delta^+) = 0$ holds true identically on Y_5 . Similarly, if one defines the ‘positive cluster dilogarithm’ by setting $R^+(u) = \frac{1}{2} \int_0^u (\text{Log}|1+t|/t - \text{Log}|t|/(1+t)) dt$, then for each connected component O of the complement in $\mathbf{R}^2$ of the arrangement of lines cut out by $xy(y-1)(x-1)(x-y) = 0$, there exists an integer $n_O \in \mathbf{Z}$ such that $\sum_{i=1}^5 \epsilon_i R^+(\varpi_i) \equiv \frac{\pi^2}{3} n_O$ on O .

6. Recovering the functional identities $\mathbf{HLog}_{\text{dP}_4}$ ’s from $\mathbf{HLog}_{Y_5}$

In this section, we state and prove a precised version of the seventh point of Theorem 1.1 which is about the fact that any weight 3 hyperlogarithmic functional identity $\mathbf{HLog}_{\text{dP}_4}$ of an arbitrary but fixed smooth del Pezzo quartic surface dP_4 can be recovered from the identity between 2-differential forms $\mathbf{HLog}_{Y_5}$.

We will first argue symbolically: we will associate a symbolic relation to $\mathbf{HLOG}_{Y_5}$ and will explain how to get from it the symbol corresponding to the hyperlogarithmic identity $\mathbf{HLog}(dP_d)$. Then we will argue more concretely, in terms of abelian relations.

6.1. Arguing symbolically. Let $\square$ stand for the quadrilateral cut out by $xyz(z + x - y) = 0$ in the homogeneous coordinates $[x : y : z]$ on $\mathbf{P}^2$. Then the space $\mathbf{H}_\square = \mathbf{H}^0(\mathbf{P}^2, \Omega_{\mathbf{P}^2}^1(\text{Log } \square))$ is of dimension 3 and admits the following logarithmic 1-forms as a basis:

$$\nu_1 = d\text{Log}(x) = \frac{dx}{x}, \quad \nu_2 = d\text{Log}(y) = \frac{dy}{y} \quad \text{and} \quad \nu_3 = d\text{Log}(1 + x - y) = \frac{dx - dy}{1 + x - y}.$$

Recall the 2-form Ω defined in (36) (see also). To it, we associate the symbol

$$\mathcal{S}(\Omega) = \nu_1 \otimes (\nu_2 \wedge \nu_3) - \nu_2 \otimes (\nu_1 \wedge \nu_3) + \nu_3 \otimes (\nu_1 \wedge \nu_2) \in \mathbf{H}_\square \otimes \wedge^2 \mathbf{H}_\square.$$

Let $\iota_\square : \mathbf{H}_\square \hookrightarrow \Omega_{\mathbf{C}(\mathbf{P}^2)}^1$ be the natural embedding into the space of rational 1-forms on the projective plane. For $p \in \mathbf{P}^2 \setminus \square$, one has a well-defined integration map $\Pi_p : \mathbf{H}_\square \rightarrow \mathcal{O}_{\mathbf{P}^2, p}$, $\nu \mapsto (\Pi_p : z \mapsto \int_p^z \nu)$ from which one gets a realization map

$$\mathcal{R}_p = \Pi_p \otimes \wedge^2 \iota_\square : \mathbf{H}_\square \otimes \wedge^2 \mathbf{H}_\square \rightarrow \Omega_{\mathbf{P}^2, p}^2$$

which associates to a tensor $\nu_a \otimes (\nu_b \wedge \nu_c)$ the germ of 2-form $(\int_p^\bullet \nu_a)(\nu_b \wedge \nu_c)$ at p . Then it is clear that the germ of Ω at p is the image of the symbol $\mathcal{S}(\Omega)$ by the realization map $\mathcal{R}_p$.

For any $i \in \llbracket 10 \rrbracket$, the pull-back $U_i^*(\Omega)$ has the same nature on Y_5 that Ω has: it is a linear combination of 2-forms of the type $\text{Log}(f)(d\text{Log}(g) \wedge \text{Log}(h))$ hence one can associate a symbol to it. For $i = 1, \dots, 10$, let η_i be the logarithmic derivative of the polynomial ζ_i defined in (17): one has

$$\eta_1 = \frac{dy_{13}}{y_{13}}, \quad \eta_2 = \frac{dy_{14}}{y_{14}}, \quad \eta_3 = \frac{dy_{24}}{y_{24}}, \quad \eta_4 = \frac{dy_{25}}{y_{25}}, \quad \eta_5 = \frac{dy_{35}}{y_{35}} \quad \text{and} \quad \eta_{k+5} = \frac{dP_k}{P_k}$$

for $k = 1, \dots, 5$. Let $\mathbf{H}_{Y_5}$ be the vector space admitting the η_i 's as a basis:²⁹

$$\mathbf{H}_{Y_5} = \bigoplus_{i=1}^{10} \mathbf{C} \eta_i.$$

For any $i \in \llbracket 10 \rrbracket$ and any $s = 1, 2, 3$, one can express the logarithmic derivative of $U_{i,s}$ as a linear combination in the η_k 's, with non-zero coefficients equal to ± 1 . For instance, for $i = 1$, one has

$$d\text{Log}(U_{1,1}) = \eta_4, \quad d\text{Log}(U_{1,2}) = \eta_3 + \eta_5 \quad \text{and} \quad d\text{Log}(U_{1,3}) = \eta_6.$$

Accordingly, the 'symbol' we associate to $U_1^*(\Omega)$ is the element of $\mathbf{H}_{Y_5} \otimes \wedge^2 \mathbf{H}_{Y_5}$ given by

$$\mathcal{S}(U_1^*(\Omega)) = \eta_4 \otimes ((\eta_3 + \eta_5) \wedge \eta_6) - (\eta_3 + \eta_5) \otimes (\eta_4 \wedge \eta_6) + \eta_6 \otimes (\eta_4 \wedge (\eta_3 + \eta_5)).$$

The η_i 's being closed, each is locally integrable on Y_5 . For any $z \in Y_5 \setminus Z$, one thus has a realization map

$$(59) \quad \mathcal{R}_z : \mathbf{H}_{Y_5} \otimes \wedge^2 \mathbf{H}_{Y_5} \longrightarrow \Omega_{Y_5, z}^2$$

²⁹For concreteness, one can take for $\mathbf{H}$ the subspace spanned by the η_k 's in $\Omega_{\mathbf{C}(Y_5)}^2$. But we believe it is better to deal with $\mathbf{H}$ as an abstract vector space and not as a subspace of a bigger space.

which associates to a tensor $\eta_a \otimes (\eta_b \wedge \eta_c)$ the germ of 2-form $(\int_z^\bullet \eta_a) \eta_b \wedge \eta_c$ at z .

The nine other symbols $\mathcal{S}(U_i^*(\Omega))$, $i = 2, \dots, 10$ can be computed easily. Recall (see (38)) that $\epsilon_i = -1$ for $i \in \{2, 4, 7, 9\}$, and $\epsilon_i = 1$ otherwise.

Lemma 6.1. *In $\mathbf{H}_{Y_5} \otimes \wedge^2 \mathbf{H}_{Y_5}$, one has $\sum_{i=1}^{10} \epsilon_i \mathcal{S}(U_i^*(\Omega)) = 0$.*

Proof. This follows from the fact that $\sum_{i=1}^{10} \epsilon_i U_i^*(\Omega) \equiv 0$ and that the realization map (59) is injective. But a more elementary proof is given by expressing the sum $S = \sum_{i=1}^{10} \epsilon_i \mathcal{S}(U_i^*(\Omega))$ in the basis $\eta_i \otimes (\eta_j \wedge \eta_k)$ of $\mathbf{H}_{Y_5} \otimes \wedge^2 \mathbf{H}_{Y_5}$, for $i \in \llbracket 10 \rrbracket$ and j, k such that $1 \leq j < k \leq 10$. Then one obtains that $S = 0$ by pure elementary linear algebra. $\square$

From the very definition of Ω (see the formulas at the beginning of §3.1.2), it can be easily verified that the symbol of Ω not only is an element of $\mathbf{H}_\square \otimes \wedge^2 \mathbf{H}_\square$ but actually lies in $\wedge^3 \mathbf{H}_\square$, the latter being seen as a subspace of $\mathbf{H}_\square \otimes \mathbf{H}_\square \otimes \mathbf{H}_\square$. It follows that for any $i \in \llbracket 10 \rrbracket$, $\mathcal{S}(U_i^*(\Omega))$ lies in $\wedge^3 \mathbf{H}_{Y_5}$ from what we deduce immediately the

Lemma 6.2. *The symbolic identity of Lemma 6.1 actually lies in $\wedge^3 \mathbf{H}_{Y_5}$. One has*

$$(60) \quad \sum_{i=1}^{10} \epsilon_i \mathcal{S}(U_i^*(\Omega)) = 0 \quad \text{in } \wedge^3 \mathbf{H}_{Y_5}.$$

Remark 6.3. *In [CP], that $\mathbf{HLog}(\mathrm{dP}_4)$ is satisfied is proven by showing that its symbol $\mathcal{S}(\mathbf{HLog}(\mathrm{dP}_4))$ vanishes in the third wedge product of the space of global 1-forms on dP_4 with poles of order at most 1 along the lines contained in dP_4 . Hence, the vanishing of $\mathcal{S}(\mathbf{HLog}(\mathrm{dP}_4)) = 0$ is formally analogous to the vanishing of $\sum_{i=1}^{10} \epsilon_i \mathcal{S}(U_i^*(\Omega))$. However, there is a significant distinction: $\mathcal{S}(\mathbf{HLog}(\mathrm{dP}_4))$ is a sum of 10 weight 3 tensors, all of which are integrable in the sense of Chen, meaning that the iterated integrals associated with each tensor are homotopy invariant (see [Br, §6] for further details). This is not the case for the symbols $\epsilon_i \mathcal{S}(U_i^*(\Omega))$ as some (if not all) of them are not integrable. Therefore, (60) represents an algebraic identity that does not have an analytic counterpart in functional form.*

From the material presented in §2.3.5, we get that the map

$$(61) \quad F_{SS} : (x, y) \longmapsto (Y_{13}, Y_{14}, Y_{24}, Y_{25}, Y_{35})$$

with

$$\begin{aligned} Y_{13} &= \frac{y(a+1)(ay-bx-a+b+x-y)}{(x-y)(b-y)} & Y_{14} &= \frac{ax(y-1)(b+1)}{(b+a)(x-y)} \\ Y_{24} &= \frac{(b-y)(-1+x)b}{(b+a)((1-x)b+x+(-1+y)a-y)} & Y_{25} &= \frac{(a-x)(x-y)}{(b+1)x((1-x)b+x+(-1+y)a-y)} \\ \text{and } Y_{35} &= \frac{(ay-bx)(b+a)}{x(b+1)(b-y)}. \end{aligned}$$

is a birational model of Serganova-Skorobogatov embedding $f_{SS} : \mathrm{dP}_4 \hookrightarrow \mathcal{Y}_5$.

By direct computation, we get that $F_{SS}^*(\mathcal{W}_{Y_5}^{GM})$ coincides with the 10-web on $\mathbf{P}^2$, denoted by $\mathcal{W}_{a,b}$, which is the direct image of del Pezzo's web $\mathcal{W}_{\mathrm{dP}_4}$ by the blow-up map $\beta = \beta_{a,b} : \mathrm{dP}_4 \rightarrow \mathbf{P}^2$ at the five points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, $[1 : 1 : 1]$ and $[a : b : 1]$: one has

$$(62) \quad \mathcal{W}_{a,b} = F_{SS}^*(\mathcal{W}_{Y_5}^{GM}) = \beta_*(\mathcal{W}_{\mathrm{dP}_4}).$$

Actually, we have that for any $i = 1, \dots, 10$, one has $F_{SS}^*(dU_{i,1} \wedge dU_{i,2}) = 0$ hence the map $U_i \circ F_{SS}$ has rank 1 (at the generic point of $\mathbf{C}^2$) and defines foliation $F_{SS}^*(\mathcal{F}_{U_i})$ which coincides with one of the web (62). More explicitly, setting $\mathcal{F}_f$ for the foliation defined by the level subsets of f , one has

$$F_{SS}^*(\mathcal{F}_{U_i}) = \mathcal{F}_{\phi_i}$$

for $i = 1, \dots, 10$, where the ϕ_i 's are the following rational functions

$$\begin{aligned} \phi_1 &= \frac{(x-1)(bx-ay)}{(x-a)(x-y)} & \phi_6 &= y \\ \phi_2 &= \frac{(y-1)(bx-ay)}{(y-b)(x-y)} & \phi_7 &= x \\ \phi_3 &= \frac{(y-1)(x-a)}{(x-1)(y-b)} & \phi_8 &= \frac{x}{y} \\ \phi_4 &= \frac{y(x-a)}{x(y-b)} & \phi_9 &= \frac{x-1}{y-1} \\ \phi_5 &= \frac{y(x-1)}{x(y-1)} & \phi_{10} &= \frac{b(x-a)}{a(y-b)}. \end{aligned}$$

(compare with the last pages of [Pi5, §4.5]).

The preimage by F_{SS} of the divisor Z of Y_5 defined in (18) is the divisor $\widetilde{Z}$ defined as the union of the ten irreducible divisors $\widetilde{Z}_i = \{\chi_i = 0\} \subset \mathbf{C}^2$ for $i = 1, \dots, 10$, where the χ_i 's are given by :

$$\begin{aligned} \chi_1 &= x & \chi_2 &= y & \chi_3 &= x-1 & \chi_4 &= y-1 & \chi_5 &= a-x & \chi_6 &= y-b & \chi_7 &= x-y \\ \chi_8 &= ay-bx & \chi_9 &= ay-bx-a+b+x-y & \chi_{10} &= ab(x-y) + (b-a)xy + ay-bx. \end{aligned}$$

The logarithmic derivatives $\lambda_i = d\text{Log}(\chi_i)$ for $i \in \llbracket 10 \rrbracket$ are linearly independant over $\mathbf{C}$ then span a subspace, denoted by $\mathbf{H}_{a,b}$, of the space of rational 1-forms on $\mathbf{C}^2$. All the pull-backs $\widetilde{\eta}_k = F_{SS}^*(\eta_k)$ of the rational 1-forms η_k 's under the map (61) are elements of $\mathbf{H}_{a,b}$ and it is straightforward to compute them explicitly but we will not even need that. Just for the record, we state the

Lemma 6.4. *The pull-back under F_{SS} induces a linear isomorphism $F_{SS}^* : \mathbf{H}_{Y_5} \xrightarrow{\sim} \mathbf{H}_{a,b}$ which is defined over $\mathbf{Z}$ with respect to the bases $(\eta_k)_{k=1}^{10}$ and $(\lambda_i)_{i=1}^{10}$ of $\mathbf{H}_{Y_5}$ and $\mathbf{H}_{a,b}$ respectively.*

Setting

$$u_{i,s} = F_{SS}^*(U_{i,s}) = U_{i,s} \circ F_{SS}$$

for $i \in \llbracket 10 \rrbracket$ and $s = 1, 2, 3$, taking the pull-back of (60) under F_{SS} , one gets that the following relation holds true in $\wedge^3 \mathbf{H}_{a,b}$:

$$(63) \quad \sum_{i=1}^{10} \epsilon_i \left(\frac{du_{i,1}}{u_{i,1}} \wedge \frac{du_{i,2}}{u_{i,2}} \wedge \frac{du_{i,3}}{u_{i,3}} \right) = 0.$$

Since everything is explicit, (63) can be made explicit as well. For instance, for $i = 6$, one has

$$U_6 \circ F_{SS} = \left(\frac{b+a}{(y-1)a}, \frac{(a+1)y}{(y-1)a}, \frac{b-y}{(y-1)a} \right)$$

hence

$$(64) \quad \frac{du_{6,1}}{u_{6,1}} = -\frac{dy}{y-1}, \quad \frac{du_{6,2}}{u_{6,2}} = \frac{dy}{y} - \frac{dy}{y-1} \quad \text{and} \quad \frac{du_{6,3}}{u_{6,3}} = \frac{dy}{y-b} - \frac{dy}{y-1}$$

from what it follows that

$$\frac{du_{6,1}}{u_{6,1}} \wedge \frac{du_{6,2}}{u_{6,2}} \wedge \frac{du_{6,3}}{u_{6,3}} = \frac{dy}{y} \wedge \frac{dy}{(y-1)} \wedge \frac{dy}{(y-b)}.$$

The computations of all the others wedge products

$$\varpi_i = d \ln(u_{i,1}) \wedge d \ln(u_{i,2}) \wedge d \ln(u_{i,3}) \in \wedge^3 \mathbf{H}_{a,b}$$

are as straightforward as the one above. Setting

$$r_1 = r_6 = b, \quad r_2 = r_7 = a, \quad r_3 = r_8 = \frac{a}{b}, \quad r_4 = r_9 = \frac{a-1}{b-1} \quad \text{and} \quad r_5 = r_{10} = \frac{b(a-1)}{a(b-1)},$$

one obtains that for any $i \in \llbracket 10 \rrbracket$, one has

$$\varpi_i = \left(\frac{d\phi_i}{\phi_i} \wedge \frac{d\phi_i}{(\phi_i - 1)} \wedge \frac{d\phi_i}{(\phi_i - r_i)} \right).$$

In other terms, for any i , ϖ_i is equal to the symbol of the weight 3 iterated integral $\mathbf{AH}_i^3(\phi_i)$ on $\mathbf{C}^2$, where $\mathbf{AH}_i^3$ stands for the complete antisymmetric weight 3 hyperlogarithm on $\mathbf{P}^1$ with respect to the 4-tuple of points $(0, 1, r_i, \infty)$ (see [CP, §2.1]). Injecting this in (63) gives

$$\sum_{i=1}^{10} \epsilon_i \left(\frac{d\phi_i}{\phi_i} \wedge \frac{d\phi_i}{(\phi_i - 1)} \wedge \frac{d\phi_i}{(\phi_i - r_i)} \right) = 0.$$

As explained in [CP], this is equivalent to the fact that the weight 3 hyperlogarithmic identity

$$\sum_{i=1}^{10} \epsilon_i \mathbf{AH}_i^3(\phi_i) = 0,$$

that is $\mathbf{HLog}(\mathbf{dP}_4)$, holds true.

6.2. From $\mathbf{HLog}_{Y_5}$ to $\mathbf{HLog}(\mathbf{dP}_4)$, by manipulating abelian relations. While we consider that the above symbolic derivation of $\mathbf{HLog}(\mathbf{dP}_4)$ from $\mathbf{HLog}_{Y_5}$ is pretty much clear, we have a feeling that it is somehow not very concrete. In this subsection, we describe succinctly how to obtain $\mathbf{HLog}(\mathbf{dP}_4)$ from $\mathbf{HLog}_{Y_5}$ by only working with genuine abelian relations. To simplify the notation, we write $\mathbf{HLog}$ instead of $\mathbf{HLog}_{Y_5}$ below. Non justified arguments are easy to check (and this is left to the reader...).

Let F be one of the polynomials ζ_k defined in (17), then $\mathbf{Res}_F = \mathbf{Res}_F(\mathbf{HLog})$ is a rational 2-AR of $\mathcal{W}_{Y_5}^{GM}$ carried by the 5-subweb $\mathcal{W}_{Y_5,F}^{GM} = \mathcal{W}(U_j \mid j \in J_F)$ for a certain subset $J_F \subset \llbracket 10 \rrbracket$ of cardinality 5 (see §3.2.1). Moreover, its non-trivial components, denoted by $\mathbf{Res}_{F,j}$ for $j \in J_F$, are linear combinations with coefficients 0, 1 or -1 of the wedge products $(dU_{j,a} \wedge dU_{j,b}) / (U_{j,a} dU_{j,b})$ for a, b such that $1 \leq a < b \leq 3$ (see Corollary 3.10). Replacing each such term by its primitive $\frac{1}{2}(\ln(U_{j,a})d \ln(U_{j,b}) - \ln(U_{j,b})d \ln(U_{j,a}))$, one obtains a 1-AR denoted by $\int \mathbf{Res}_F \in \mathbf{AR}^1(\mathcal{W}_{Y_5}^{GM})$, which is such that $d \int \mathbf{Res}_F = \mathbf{Res}_F$. The pull-back of it under F_{SS} is a 1-abelian relation for $F_{SS}^*(\mathcal{W}_{Y_5,F}) = \mathcal{W}(\phi_j \mid j \in J_F)$ which is closed hence admits a primitive, that we will denote by

$$\int F_{SS}^* \left[\int \mathbf{Res}_F \right].$$

As an explicit example, let us consider the case when $F = \zeta_{10} = P_5 = 1 + y_{14} - y_{13}y_{24}$. This case is completely similar³⁰ to the one of the web $\mathbf{W}_{Y_5}^+$ whose ARs have been considered in Proposition 3.11 and Proposition 3.13 above. When $F = P_5$:

- the polynomial $F = P_5$ is a factor of $U_{i,s}$ for $(i, s) = (j, 3)$ with $j \in J_F = \{5, 6, 7, 8, 9\}$.
- moreover for any $j \in J_F$, F appears in the numerator of $U_{j,3}$, with multiplicity 1. It follows that $\mathbf{Res}_F$ is the rational 2-abelian relation $0 = \sum_{j \in J_F} \epsilon_j U_j^*(\eta)$ (where η is the rational 2-form defined in (45));
- its primitive $\int \mathbf{Res}_F$ is the 1-abelian relation $0 = \sum_{j \in J_F} \epsilon_j U_j^*(\delta)$ where δ is the privileged primitive of η defined in (47);
- then it is not difficult to compute the pull-backs of the components $\int \mathbf{Res}_{F,j}$ under F_{SS} . One obtains that, for any $j \in J_F$, up to the addition of 1-forms of the type $d \ln(\phi_j - z_j)$ with $z_j \in \{0, 1, r_j\}$, one has

$$(65) \quad F_{SS}^* \left(\int \mathbf{Res}_{F,j} \right) = \frac{1}{2} \left(\frac{\ln(\phi_j)}{\phi_j - 1} - \frac{\ln(\phi_j - 1)}{\phi_j} \right) d\phi_j = \phi_j^*(d\tilde{R}).$$

where $\tilde{R} : x \mapsto \frac{1}{2} \int_1^x (\ln(u)/(u-1) - \ln(u-1)/u) du$ is a version of Rogers' dilogarithm defined on $[1, +\infty[$. After integrating the identity $0 = \sum_{j \in J_F} \epsilon_j F_{SS}^* \left(\int \mathbf{Res}_{F,j} \right)$, one obtains the following explicit form for the identity corresponding to the 0-AR $\int F_{SS}^* \left(\int \mathbf{Res}_F \right)$:

$$(66) \quad \tilde{R} \left(\frac{y(x-1)}{x(y-1)} \right) + \tilde{R}(y) - \tilde{R}(x) + \tilde{R} \left(\frac{x}{y} \right) - \tilde{R} \left(\frac{x-1}{y-1} \right) = 0,$$

a relation satisfied by all $x, y \in [1, +\infty[$ such that $1 < y < x$.

Given $\xi \in \mathbf{P}^1 \setminus \{0, 1, \infty\}$, one sets $\Sigma_\xi = \{0, 1, \xi, \infty\}$ and for $i \in \llbracket 10 \rrbracket$, one denotes by $\mathbf{H}_i$ the pull-back under ϕ_i of the space of holomorphic 1-form on $\mathbf{P}^1 \setminus \Sigma_{r_i}$ with logarithmic poles at the points of Σ_{r_i} :

$$\mathbf{H}_i = \phi_i^* \mathbf{H}^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}(\text{Log } \Sigma_{r_i})) \subset \mathbf{H}_{a,b}.$$

This space admits $d\phi_i/\phi_i$, $d\phi_i/(\phi_i - 1)$ and $d\phi_i/(\phi_i - r_i)$ as a basis and for any $\sigma, \sigma' \in \{0, 1, r_i\}$, one defines a weight 2 antisymmetric symbol by setting

$$\mathfrak{R}_{\sigma, \sigma'}^i = \left(\frac{d\phi_i}{\phi_i - \sigma} \right) \wedge \left(\frac{d\phi_i}{\phi_i - \sigma'} \right) = \phi_i^* \left(\left(\frac{dz}{z - \sigma} \right) \wedge \left(\frac{dz}{z - \sigma'} \right) \right) \in \wedge^2 \mathbf{H}_i \subset \wedge^2 \mathbf{H}_{a,b}.$$

Clearly, each $\mathfrak{R}_{\sigma, \sigma'}^i$ is the symbol of a dilogarithmic function composed with ϕ_i . For instance, it follows from (65) that $\mathfrak{R}_{0,1}^6$ is the symbol of the term $\tilde{R}(y)$ appearing in the analytic expression (66) of $\int F_{SS}^* \left(\int \mathbf{Res}_F \right)$.

For any i , the determination of the weight 2 symbol of the i -th components of the abelian relations $\int F_{SS}^* \left(\int \mathbf{Res}_{\zeta_k} \right)$ of $\mathbf{W}_{a,b} = \mathcal{W}(\phi_1, \dots, \phi_{10})$ for any ζ_k can be obtained by means of straightforward formal computations from the decompositions of the logarithmic derivatives of the components of the first integrals U_i of $\mathcal{W}_{Y_5}^{GM}$. As an example, below we give details in the case $i = 6$.

³⁰This not a coincidence. Indeed, since the Weyl group $W_{D_5} \simeq \text{Aut}(\mathcal{Y}_5)$ acts transitively on the set of weights of the half-spin representation, it acts similarly on the set of weight divisors $D_w \subset Y_5$ hence on the set of residues of $\mathbf{HLOG}_{\mathcal{Y}_5}$ along the latter (possibly up to rescaling but this is not relevant).

Setting $\Delta_i = (d \ln U_{i,s})_{s=1}^3$ for $i = 1, \dots, 10$, one has the following expressions of the $d \ln U_{i,s}$'s as linear combinations of the η_k 's:

$$\begin{aligned}
\Delta_1 &= (\eta_4, \eta_5 + \eta_3, \eta_6) \\
\Delta_2 &= (-\eta_1, \eta_5 - \eta_1 + \eta_2, \eta_7 - \eta_1) \\
\Delta_3 &= (\eta_3, \eta_4 + \eta_2, \eta_8) \\
\Delta_4 &= (-\eta_5, -\eta_5 + \eta_4 + \eta_1, -\eta_5 + \eta_9) \\
\Delta_5 &= (\eta_2, \eta_3 + \eta_1, \eta_{10}) \\
\Delta_6 &= (\eta_7 - \eta_2 - \eta_9, \eta_8 + \eta_1 - \eta_2 - \eta_9, \eta_{10} - \eta_2 - \eta_9) \\
\Delta_7 &= (\eta_6 - \eta_3 - \eta_9, -\eta_3 + \eta_8 - \eta_9, -\eta_3 + \eta_{10} + \eta_4 - \eta_9) \\
\Delta_8 &= (\eta_6 + \eta_1 - \eta_9, \eta_7 - \eta_9, \eta_5 + \eta_{10} - \eta_9) \\
\Delta_9 &= (\eta_6 - \eta_8 + \eta_2, \eta_7 + \eta_3 - \eta_8, \eta_{10} - \eta_8) \\
\Delta_{10} &= (\eta_6 - \eta_5 - \eta_8, \eta_7 - \eta_5 + \eta_4 - \eta_8, -\eta_5 - \eta_8 + \eta_9)
\end{aligned}$$

The indices k such that the 6-th component of the residue

$$\mathbf{Res}_k = \mathbf{Res}_{\zeta_k}(\mathbf{HLOG}_{Y_5}) \in AR^2(\mathcal{W}_{Y_5}^{GM})$$

is non-trivial are exactly the ones such that η_k appears in the expression of Δ_6 above. Namely, the set of such indices is $\mathfrak{R}_6 = \{1, 2, 7, 8, 9, 10\}$. For each $k \in \mathfrak{R}_6$, let $\kappa_k \in \{-1, 0, 1\}^3$ be the triplet obtained from Δ_6 by deriving it formally with respect to η_k (the latter hence being then considered as a symbolic indeterminate). One has

$$\begin{aligned}
\kappa_1 &= (0, 1, 0) & \kappa_2 &= (-1, -1, -1) & \kappa_7 &= (1, 0, 0) \\
\kappa_8 &= (0, 1, 0) & \kappa_9 &= (-1, -1, -1) & \kappa_{10} &= (0, 0, 1).
\end{aligned}$$

From the formula (35) for Ω , it follows that for any $\kappa_k = (\alpha_k, \beta_k, \gamma_k)$, the 6-th component $\mathbf{Res}_{k,6} =$ of the rational 2-abelian relation $\mathbf{Res}_k$ is

$$\mathbf{Res}_{k,6} = \alpha_k \left(\frac{dU_{6,2}}{U_{6,2}} \wedge \frac{dU_{6,3}}{U_{6,3}} \right) - \beta_k \left(\frac{dU_{6,1}}{U_{6,1}} \wedge \frac{dU_{6,3}}{U_{6,3}} \right) + \gamma_k \left(\frac{dU_{6,1}}{U_{6,1}} \wedge \frac{dU_{6,2}}{U_{6,2}} \right).$$

Each $\mathbf{Res}_{k,6}$ is a rational 2-form, but the expression above, up to a scaling factor (equal to 1/2) can also be seen as an element of $\wedge^2 \mathbf{H}_{Y_5}$, which will be denoted by $\frac{1}{2} \mathbf{Res}_{k,6}^{symb}$. Using (64), for any $k \in \mathfrak{R}_6$ one can compute the pull-back $\varrho_k^6 = F_{SS}^* (\frac{1}{2} \mathbf{Res}_{k,6}^{symb})$ which belongs to $\wedge^2 \mathbf{H}_6$. One has

$$\begin{aligned}
\varrho_k^6 &= \frac{\alpha_k}{2} \left(\left(\frac{dy}{y} - \frac{dy}{y-1} \right) \wedge \left(\frac{dy}{y-b} - \frac{dy}{y-1} \right) \right) - \frac{\beta_k}{2} \left(\left(-\frac{dy}{y-1} \right) \wedge \left(\frac{dy}{y-b} - \frac{dy}{y-1} \right) \right) \\
&\quad + \frac{\gamma_k}{2} \left(\left(-\frac{dy}{y-1} \right) \wedge \left(\frac{dy}{y} - \frac{dy}{y-1} \right) \right).
\end{aligned}$$

Expressed in terms of the dilogarithmic symbols $\mathfrak{R}_{\sigma, \sigma'}^6$ defined above and considering that $r_6 = b$, one obtains

$$\varrho_k^6 = \frac{\alpha_k}{2} (\mathfrak{R}_{0,b}^6 - \mathfrak{R}_{0,1}^6 - \mathfrak{R}_{1,b}^6) + \frac{\beta_k}{2} (\mathfrak{R}_{1,b}^6) + \frac{\gamma_k}{2} (\mathfrak{R}_{0,1}^6).$$

hence

$$(67) \quad \begin{aligned} \varrho_1^6 &= \Re_{1,b}^6 & \varrho_2^6 &= -\Re_{0,b}^6 & \varrho_7^6 &= -\Re_{0,1}^6 - \Re_{1,b}^6 + \Re_{0,b}^6 \\ \varrho_8^6 &= \Re_{1,b}^6 & \varrho_9^6 &= -\Re_{0,b}^6 & \varrho_{10}^6 &= \Re_{0,1}^6. \end{aligned}$$

One can verify that for any $k \in \Re_6$, ϱ_k^6 is also the symbol of the dilogarithmic function which is the 6-th component of the 0-AR $\int F_{SS}^*(\int \mathbf{Res}_{\zeta_k})$. Then using (67), straightforward computations give us that the symbol of the sixth component of the sum

$$(68) \quad \sum_{k=1}^{10} F_{SS}^*(\eta_k) \int F_{SS}^*(\int \mathbf{Res}_k)$$

is equal to

$$\left(\frac{dy}{y-b}\right) \otimes \Re_{0,1} - \left(\frac{dy}{y-1}\right) \otimes \Re_{0,b} + \left(\frac{dy}{y}\right) \otimes \Re_{1,b}.$$

This is the symbol of the complete antisymmetric weight-3 hyperlogarithm $AH_3(\phi_6)$ with respect to the set $\Sigma_{r_6} = \Sigma_b = \{0, 1, b, \infty\} \subset \mathbf{P}^1$ (cf. [CP, §2.1.3]). Through similar and straightforward computations, one can verify that the same result holds for each component of equation (68). It follows that the 1-abelian relation (68) of $\mathbf{W}_{a,b}$ corresponds to the derivative of the hypertrilogarithmic functional abelian relation $\mathbf{Hlog}(\mathrm{dP}_4)$. This leads to the justification of the formula:

$$(69) \quad \mathbf{Hlog}(\mathrm{dP}_4) = \sum_{k=1}^{10} \int \left[F_{SS}^*(\eta_k) \int F_{SS}^* \left(\int \mathbf{Res}_k(\mathbf{HLOG}_{Y_5}) \right) \right]$$

which summarizes the preceding discussion on how the weight-3 hyperlogarithmic functional identity $\mathbf{Hlog}(\mathrm{dP}_4)$ can be derived from the differential identity between differential 2-form with logarithmic coefficients $\mathbf{HLOG}_{Y_5}$.

7. The $(r-3)$ -rank and the $(r-3)$ -abelian relations of $\mathcal{W}_{\mathcal{Y}_r}^{GM}$ for $r = 4, \dots, 7$

In this section, we extend some of the results obtained above to the cases $r = 6$ and $r = 7$. The current proofs are based on explicit computations carried out using a computer algebra system. These computational details are omitted here, but we intend to provide conceptual proofs in a future work.

7.1. In addition to the notations introduced in the Introduction, we use also the following ones:

- we denote by $\mathbf{D}_r^\bullet$ the 1-marked Dynkin diagram associated to the simple Lie group G_r , with one marked vertex corresponding to the maximal parabolic subgroup $P_r \subset G_r$ such that $\mathcal{G}_r = G_r/P_r$. We denote by γ_r the dimension of $\mathcal{G}_r$. Then $\gamma_r - r$ is equal to the dimension of the quotient space $\mathcal{Y}_r$, denoted by y_r . One has

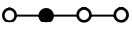
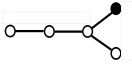
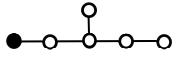
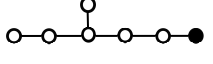
$$\gamma_r = \dim \mathcal{G}_r \quad \text{and} \quad y_r = \dim \mathcal{Y}_r = \gamma_r - r;$$

- V_r stands for the corresponding minuscule representation (of dimension v_r equal to 10, 16, 27 and 56 for $r = 4, 5, 6, 7$ respectively);

- Δ_r stands for the moment polytope of $\mathcal{G}_r$, which coincides with the weight polytope in $\mathfrak{h}_r^\vee(\mathbf{R})$ of the minuscule representation V_r . The series of Δ_r 's coincides with the one of Gosset's polytopes $(r-4)_{21}$. Each admits two kinds of facets, some $(r-1)$ -simplices and some $(r-1)$ -cross-polytopes (also known as 'orthoplexes'). Only the latter are $\mathcal{W}$ -relevant. The first two of the series are also known by other names: Δ_4 is the hypersimplex $\Delta_{2,5}$ and Δ_5 is the 5-demihypercube.
- $\mathcal{L}_r$ (resp. $\mathcal{K}_r$) denotes the number of lines (resp. of conic fibrations) on a fixed smooth del Pezzo surface dP_{9-r} ;
- we recall that there are natural bijections between $\mathcal{L}_r$ and the set of weights $\mathfrak{B}_r$ of V_r which is also the set of vertices $V_0(\Delta_r)$ of the weight polytope. Hence one has

$$\ell_r = |\mathcal{L}_r| = |V_0(\Delta_r)| = \dim V_r = \nu_r.$$

Similarly, the set $\mathcal{K}_r$ of conic classes on dP_{9-r} identifies with the one of cross-polytope facets of Δ_r .

r	$D_r^\bullet$	$\nu_r = \dim V_r$ $= \mathcal{L}_r = \ell_r$	$\mathcal{G}_r \subset \mathbf{P}V_r$	$\gamma_r = \dim \mathcal{G}_r$	$y_r = \dim \mathcal{Y}_r$	$\kappa_r = \mathcal{K}_r $
4		10	$G_2(\mathbf{C}^5) \subset \mathbf{P}^9$	5	2	5
5		16	$\mathbb{S}_5 \subset \mathbf{P}^{15}$	10	5	10
6		27	$\mathbb{O}_{\mathbf{C}}\mathbf{P}^2 \subset \mathbf{P}^{26}$	16	10	27
7		56	$\mathbf{Fr} \subset \mathbf{P}^{55}$	27	20	126

Many features and numerical quantities of the objects considered above are given in the table above in which $\mathbf{OP}^2 \subset \mathbf{P}^{26}$ stands for the Cayley plane over the complex octonions $\mathbf{O}$ and where $\mathbf{Fr} \subset \mathbf{P}^{55}$ denotes the 27-dimensional Freudenthal variety E_7/P_6 . Both varieties are considered in their fundamental embeddings which can also respectively be described as the projectivizations of the exceptional rank 3 complex Jordan algebra $\mathbf{J} = \mathrm{Herm}_3(\mathbf{O})$ and of the space $Z_2(\mathbf{J})$ of Zorn 2×2 matrices over $\mathbf{J}$.

The Gelfand-MacPherson web we are interested in lives on the quotient space $\mathcal{Y}_r = \mathcal{G}_r // H_r$. Given an orthoplex facet F of Δ_r , , there is a commutative diagram:

$$\begin{array}{ccc} \mathcal{G}_r & \xrightarrow{\Pi_F} & \mathcal{Q}_F \\ \downarrow & & \downarrow \\ \mathcal{Y}_r & \xrightarrow{\pi_F} & \mathcal{Y}_F \end{array}$$

in which the vertical maps are the quotient maps, and the horizontal maps are the corresponding *face maps*. It can be verified that Q_F is a smooth hyperquadric in $\mathbf{P}^{2r-3}$ and that $\mathcal{Y}_F$ embeds into $\mathbf{P}^{r-3}$ as a Zariski open subset. With respect to this embedding, one has $\mathcal{Y}_F^\times = \mathbf{P}^{r-3} \setminus \mathcal{H}_F$ where $\mathcal{H}_F$ is an arrangement of $r - 1$ hyperplanes in general position.

We have that the corresponding Gelfand-MacPherson web considered here

$$\mathcal{W}_{\mathcal{Y}_r}^{GM} = \mathcal{W}(\pi_F \mid F \in \mathcal{K}_r)$$

is a κ_r -web of codimension $r - 3$ on $\mathcal{Y}_r$, which is a rational variety of dimension y_r equal to 2, 5, 10 and 20 for $r = 4, 5, 6$ and 7 respectively. Since the hyperplanes of $\mathcal{H}_r$ are in general position, for some homogeneous coordinates $u_0, u_1, \dots, u_{r-3}$ on $\mathbf{P}^{r-3}$, one can assume that $\mathcal{H}_r$ is cut out by

$$u_0 u_1 u_2 \dots u_{r-3} (u_0 + u_1 + \dots + u_{r-3}) = 0.$$

7.1.1. Explicit birational models of $\mathcal{W}_{\mathcal{Y}_r}^{GM}$ for $r = 6, 7$. The approach used above to study $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ and its abelian relations relies on an explicit birational model of this web, obtained via a concrete birational parametrization of the spinor variety $\mathcal{S}_5$ embedded in the projectivization of the half-spin representation. A key observation is that, for $r = 6, 7$ as well, one can construct similar (birational) parametrizations of each corresponding homogeneous space $\mathcal{G}_r$, considered in their respective minuscule embeddings. This allows the same strategy, based on explicit formal computations, assisted by a computer algebra system, to be effectively extended to these higher-rank cases. In this subsection, we succinctly describe the parametrizations of $\mathcal{G}_6 = \mathbf{OP}^2$ and $\mathcal{G}_7 = \mathbf{Fr}$ that we have worked with. Our main reference for the material presented here is Yokota's remarkable book [Yo].

In order to obtain, in each case, explicit and well-behaved formulas (defined over $\mathbf{R}$, say) for the action of the Cartan torus, we considered a real rational parametrization of the split real form $\mathcal{G}_r^{\mathbf{R}}$ of the complex homogeneous space $\mathcal{G}_r$, embedded into the projectivization of the split real form $V_r^{\mathbf{R}}$ of the corresponding (complex) minuscule representation V_r . To this end, we worked with the algebra $\mathbb{O}_s$ of split (real) octonions rather than with the usual (Hurwitz) octonion algebra $\mathbb{O}$. This choice is ultimately inconsequential, since tensoring $\mathbb{O}_s$ and $\mathbb{O}$ with $\mathbf{C}$ yields two isomorphic complex algebras. In what follows, we will work with the complex algebra $\mathbf{O} = \mathbb{O}_s$ that we will call ‘the algebra of complex octonions’.

Let us consider the following two invertible linear maps from $\mathbf{R}^8$ into itself.

$$\begin{aligned} \phi : (x_i)_{i=0}^7 &\longmapsto (x_0 - x_7, x_1 - x_6, x_2 - x_5, x_3 - x_4, x_3 + x_4, x_2 + x_5, x_1 + x_6, x_0 + x_7) \\ \psi : (u_i)_{i=0}^7 &\longmapsto \frac{1}{2}(u_0 + u_7, u_1 + u_6, u_2 + u_5, u_3 + u_4, u_4 - u_3, u_5 - u_2, u_6 - u_1, u_7 - u_0) \end{aligned}$$

These maps are inverses of each other and denoting by $*$ the usual octonionic product on $\mathbb{O} = \mathbf{R}^8$, one defines $\mathbb{O}_s$ as $\mathbf{R}^8$ endowed with the product given by

$$(70) \quad u \cdot v = \phi(\psi(u) * \psi(v))$$

for any $u, v \in \mathbb{O}_s$. Then the unit element $\mathbf{1}$, the conjugate $\bar{u}$ and the norm $\|u\|^2$ of any element $u = (u_i)_{i=0}^7$ are given respectively by

$$\begin{aligned}\mathbf{1} &= (1, 0, 0, 0, 0, 0, 0, 1) \\ \bar{u} &= (u_7, -u_1, -u_2, -u_3, -u_4, -u_5, -u_6, u_0) \\ \text{and } \|u\|^2 &= u_0 u_7 + u_1 u_6 + u_2 u_5 + u_3 u_4.\end{aligned}$$

(We thus have $u \cdot \bar{u} = \bar{u} \cdot u = \|u\|^2 \mathbf{1}$ for every u).

For O one of the three algebras $\mathbb{O}$, $\mathbb{O}_s$ or $\mathbf{O}$, with definition field $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$, we define the space of hermitian 3×3 matrices with coefficients in $O \simeq \mathbf{k}^8$ by setting

$$\text{Herm}_3(O) = \left\{ \begin{bmatrix} s_1 & v_3 & \bar{v}_2 \\ \bar{v}_3 & s_2 & v_1 \\ v_2 & \bar{v}_1 & s_3 \end{bmatrix} \mid \begin{array}{l} s_1, s_2, s_3 \in \mathbf{k} \\ v_1, v_2, v_3 \in O \end{array} \right\} \simeq \mathbf{k}^3 \oplus O^3 \simeq \mathbf{k}^{27},$$

where the symbols $\simeq$ stand for isomorphisms of $\mathbf{k}$ -vector spaces.³¹ In this definition, the diagonal coefficients $s_i \in \mathbf{k}$ have to be understood as scalar elements of O , that is modulo the embedding $\mathbf{k} \hookrightarrow O$, $s \mapsto s \mathbf{1}$, where $\mathbf{1}$ is the unit element. In what follows, we set $\mathbb{J}$, $\mathbb{J}_s$ and $\mathbf{J}$ for $\text{Herm}_3(O)$ when $O = \mathbb{O}$, $\mathbb{O}_s$ and $\mathbf{O}$ respectively. When O is unspecified, we will denote J for $\text{Herm}_3(O)$ to shorten the notations. The algebra J being power-associative and of rank 3, there exist homogeneous forms $T, Q, N \in \text{Sym}(J^\vee)$, of degree 1, 2 and 3 respectively, such that $X^3 - T(X)X^2 + Q(X)X - N(X)\mathbf{1} = 0$ for any $X \in J$. The linear map T is the natural trace and when $J = \text{Herm}_3(O)$, the cubic form N is known as the ‘*determinant*’. As such, it is denoted by $\det : J \rightarrow \mathbf{k}$ and is explicitly given by

$$\det \left(\begin{bmatrix} s_1 & v_3 & \bar{v}_2 \\ \bar{v}_3 & s_2 & v_1 \\ v_2 & \bar{v}_1 & s_3 \end{bmatrix} \right) = T([v_1, v_2, v_3]) - \sum_{i=1}^3 s_i \|v_i\|^2 + s_1 s_2 s_3.$$

where $[\cdot, \cdot, \cdot] : J^3 \rightarrow \mathbf{k}$ denotes the associated ‘*triple product*’ given by $[u, v, w] = (uv)w - (uw)v + (vw)u$ for $u, v, w \in J$. Then one can define ‘*the adjoint* $X^\#$ ’ of an element of J by setting

$$X^\# = X^2 - T(X)X + Q(X)\mathbf{1}$$

for any $X \in J$. One has $XX^\# = X^\#X = \det(X)\mathbf{1}_J$ for any X .

To deal with the case $r = 7$ and the 27-dimensional Freudenthal variety $\mathbf{Fr} = E_6/P_6 = \mathcal{G}_7 \subset \mathbf{PV}_7 \simeq \mathbf{P}^{55}$, we need to go one step further and consider the *space of Zorn 2×2 matrices with coefficients in J* , defined by

$$\mathcal{Z}_2(J) = \left\{ \begin{bmatrix} \zeta_1 & Z_1 \\ Z_2 & \zeta_2 \end{bmatrix} \mid \begin{array}{l} \zeta_1, \zeta_2 \in \mathbf{k} \\ Z_1, Z_2 \in J \end{array} \right\} \simeq \mathbf{k} \oplus J \oplus J \oplus \mathbf{k} \simeq \mathbf{k}^{56}.$$

Then we consider the following Veronese maps

$$\begin{aligned} \nu_O^2 : O \oplus O &\longrightarrow J & \nu_J^3 : J &\longrightarrow \mathcal{Z}_2(J) \\ (u, v) &\longmapsto \begin{bmatrix} 1 & u & \bar{v} \\ \bar{u} & \|u\|^2 & \bar{v} \cdot u \\ v & v \cdot u & \|v\|^2 \end{bmatrix} & X &\longmapsto \begin{bmatrix} 1 & X \\ X^\# & \det(X) \end{bmatrix}. \end{aligned}$$

³¹Endowed with the symmetrization of the naive matricial product, $\text{Herm}_3(O)$ becomes a rank 3 Jordan algebra over $\mathbf{k}$, which is simple if O is non-degenerate (that is when $O \in \{\mathbb{O}, \mathbf{O}\}$). We will not really use these facts in this paper, they are mentioned only in passing.

These two Veronese maps are affine embeddings. The image of v_O^2 (resp. of v_J^3) lands into the algebraic subvariety of J (resp. of $\mathcal{Z}_2(J)$) cut out by the quadratic equation $X^2 = T(X)X$ (resp. the set of quadratic equations $Z_1 Z_2 = \zeta_1 \zeta_2 \mathbf{1}$, $Z_1^\# - \zeta_1 Z_2 = Z_2^\# - \zeta_2 Z_1 = 0$).

Proposition 7.1. *Over $\mathbf{C}$, that is for $O = \mathbf{O}$ and $J = \mathbf{J}$, the Zariski closures of the images of the two Veronese maps above are the minuscule homogeneous spaces $\mathcal{G}_6$ and $\mathcal{G}_7$ in their minuscule embeddings respectively. In other terms, one has :*

$$\overline{v_O^2(\mathbf{O} \oplus \mathbf{O})} = \mathbf{OP}^2 \subset \mathbf{P}(\mathbf{J}) \simeq \mathbf{P}^{26} \quad \text{and} \quad \overline{v_J^3(\mathbf{J})} = \mathbf{Fr} \subset \mathbf{P}(\mathcal{Z}_2(\mathbf{J})) \simeq \mathbf{P}^{55}.$$

The reason for our choice of the specific form(70) of the octonionic product lies in the fact that it yields particularly nice and fully explicit formulas– moreover defined over $\mathbb{R}$ – for the action of the maximal torus of E_6 and E_7 on the associated minuscule representations V_6 and V_7 .

Let us consider the following coordinates on the spaces of matrices introduced above:

- on $J = \text{Herm}_3(\mathbb{O})$, (ξ, v) with $\xi = (\xi_i)_{i=1}^3 \in \mathbf{k}^3$ and $v = (v_i)_{i=1}^3 \in \mathcal{O}^3$ will stand for the coordinates of the hermitian matrix $\begin{bmatrix} \xi_1 & v_3 & \overline{v_2} \\ \overline{v_3} & \xi_2 & v_1 \\ v_2 & \overline{v_1} & \xi_3 \end{bmatrix}$;
- on $\mathcal{Z}_2(J)$, the coordinates of the Zorn matrix $\begin{bmatrix} \xi_1 & Z_1 \\ Z_2 & \xi_2 \end{bmatrix}$ is the 4-tuple $(\zeta_1, Z_1, Z_2, \zeta_2)$ with $\zeta_i \in \mathbf{k}$ and $Z_i \in J$ for $i = 1, 2$.

Starting from now, we deal with $O = \mathbb{O}_s$ and $J = \mathbb{J}_s = \text{Herm}_3(\mathbb{O}_s)$.

We start by considering the action of $\mathbf{H}_{>0}^4 = \{ \mathbf{h} = (h_i)_{i=1}^4 \in (\mathbf{R}_{>0})^4 \}$ on $\mathbb{O}_s$ given by

$$(71) \quad \mathbf{h} \cdot u = (h_i)_{i=1}^4 \cdot (u_j)_{j=1}^8 = \left(h_1 u_1, h_2 u_2, h_3 u_3, h_4 u_4, h_4^{-1} u_5, h_3^{-1} u_6, h_2^{-1} u_7, h_1^{-1} u_8 \right)$$

for any $\mathbf{h} = (h_i)_{i=1}^4 \in \mathbf{H}_{>0}^4$ and any $u = (u_i)_{i=1}^8 \in \mathbb{O}_s$. Clearly one has $\|\mathbf{h} \cdot u\|^2 = \|u\|^2$ hence the complexification of (71) yields the action of the Cartan torus of $\text{SO}_8(\mathbf{C})$ on its standard representation.

The triality on $\text{SO}_8(\mathbf{C})$ gives us the two following order 3 automorphisms $\varphi : \mathbf{h} \mapsto \mathbf{h}^\varphi$ and $\psi = \varphi^2 : \mathbf{h} \mapsto \mathbf{h}^\psi$ of $\mathbf{H}_{>0}^4$, defined by (with $\mathbf{h} = (h_i)_{i=1}^4 \in \mathbf{H}_{>0}^4$):

$$\mathbf{h}^\varphi = \varphi(\mathbf{h}) = \left(\frac{\sqrt{h_3}}{\sqrt{h_1 h_2 h_4}}, \frac{\sqrt{h_1 h_2 h_3}}{\sqrt{h_4}}, \frac{\sqrt{h_2 h_3 h_4}}{\sqrt{h_1}}, \frac{\sqrt{h_1 h_3 h_4}}{\sqrt{h_2}} \right)$$

and $\mathbf{h}^\psi = \psi(\mathbf{h}) = \left(\frac{\sqrt{h_2 h_4}}{\sqrt{h_1 h_3}}, \frac{\sqrt{h_2 h_3}}{\sqrt{h_1 h_4}}, \sqrt{h_1 h_3 h_2 h_4}, \frac{\sqrt{h_4 h_3}}{\sqrt{h_1 h_2}} \right).$

The ‘positive torus’ $\mathbf{H}_{>0}^4$ acts on $\mathbb{J}_s = \text{Herm}_3(\mathbb{O}_s)$ by means of the following formula: one sets

$$(72) \quad \mathbf{h} \cdot (\xi, v) = \mathbf{h} \cdot \left(\xi, (v_j)_{j=1}^3 \right) = \left(\xi, (\mathbf{h} \cdot v_1, \mathbf{h}^\varphi \cdot v_2, \mathbf{h}^\psi \cdot v_3) \right)$$

for any $\mathbf{h} = (h_i)_{i=1}^4 \in \mathbf{H}_{>0}^4$ and any $(\xi, v) \in \mathbf{R}^3 \times \mathbb{O}_s^3$. We define another action of $\mathbf{L}_{>0}^2 = \{ \ell = (l_1, l_2, l_3) \in (\mathbf{R}_{>0})^3 \mid l_1 l_2 l_3 = 1 \} \simeq (\mathbf{R}_{>0})^2$ on $\mathbb{J}_s$ by setting

$$(73) \quad \ell \cdot (\xi, v) = \left((l_i^2 \xi_i)_{i=1}^3, (l_i^{-1} v_i)_{i=1}^3 \right).$$

The actions (72) and (73) commute hence give rise to an action of the ‘positive torus’

$$\mathbf{T}_{>0}^6 = \mathbf{L}^2 \times \mathbf{H}^4 \simeq (\mathbf{R}_{>0})^6$$

on $\mathbb{J}_s$ given by

$$(74) \quad (\ell, \mathbf{h}) \bullet \begin{bmatrix} \xi_1 & v_3 & \overline{v_2} \\ \overline{v_3} & \xi_2 & v_1 \\ v_2 & \overline{v_1} & \xi_3 \end{bmatrix} = \begin{bmatrix} (l_1)^2 \xi_1 & \mathbf{h}^\psi \cdot v_3 & \overline{\mathbf{h}^\varphi \cdot v_2} \\ \overline{\mathbf{h}^\psi \cdot v_3} & (l_2)^2 \xi_2 & \mathbf{h} \cdot v_1 \\ \mathbf{h}^\varphi \cdot v_2 & \overline{\mathbf{h} \cdot v_1} & (l_3)^2 \xi_3 \end{bmatrix}.$$

We extend the preceding action to an action of the larger group

$$\mathbf{T}_{>0}^7 = (\mathbf{R}_{>0}) \times \mathbf{L}^2 \times \mathbf{H}^4 \simeq (\mathbf{R}_{>0})^7$$

on $\mathcal{Z}_2(\mathbb{J}_s)$ by setting

$$(75) \quad (t, (\ell, \mathbf{h})) \cdot \begin{bmatrix} \zeta_1 & Z_1 \\ Z_2 & \zeta_2 \end{bmatrix} = \begin{bmatrix} t \zeta_1 & t^{1/3} (\ell, \mathbf{h}) \bullet Z_1 \\ t^{-1/3} (\ell^{-1}, \mathbf{h}) \bullet Z_2 & t^{-1} \zeta_2 \end{bmatrix}$$

for all $(t, (\ell, \mathbf{h}))$ in the group and all Zorn matrices $\begin{bmatrix} \zeta_1 & Z_1 \\ Z_2 & \zeta_2 \end{bmatrix} \in \mathcal{Z}_2(\mathbb{J}_s)$.

We have explicitly described several isomorphisms of $\mathbf{R}$ -vector spaces above: $\mathbb{O}_s \simeq \mathbf{R}^8$, $\mathbb{J}_s = \text{Herm}_3(\mathbb{O}_s) \simeq \mathbf{R}^3 \oplus \mathbb{O}_s^3$ and $\mathcal{Z}_2(\mathbb{J}_s) \simeq \mathbf{R} \oplus \mathbb{J}_s \oplus \mathbb{J}_s \oplus \mathbf{R}$. From these, one gets well-defined isomorphisms of real vector spaces

$$(76) \quad \mathbb{J}_s \simeq \mathbf{R}^{27} \quad \text{and} \quad \mathcal{Z}_2(\mathbb{J}_s) \simeq \mathbf{R}^{56}.$$

We denote by $(\mathfrak{E}_i)_{i=1}^{27}$ and $(\mathfrak{Z}_j)_{j=1}^{56}$ the bases of $\mathbb{J}_s$ and $\mathcal{Z}_2(\mathbb{J}_s)$ corresponding to the canonical bases of $\mathbf{R}^{27}$ and $\mathbf{R}^{56}$ respectively.

The genuine interest of working with the specific product (70) and the explicit actions (74) becomes apparent by considering the content of the following

Proposition 7.2. 1. *The images in $\text{GL}(\mathbf{J})$ and $\text{GL}(\mathcal{Z}_2(\mathbf{J}))$ of the ‘positive tori’ $\mathbf{T}_{>0}^6$ and $\mathbf{T}_{>0}^7$ land into E_6 and E_7 respectively. Consequently, up to isogenies, the actions (74) and (75) are induced by the ones of some Cartan subtori of these two simple complex Lie groups.*

2. *The bases $(\mathfrak{E}_i)_{i=1}^{27}$ and $(\mathfrak{Z}_j)_{j=1}^{56}$ are bases of weight vectors for $\mathbf{T}_{>0}^6$ and $\mathbf{T}_{>0}^7$ respectively.*

More precisely, for any $i \in \{1, \dots, 27\}$, there exists a weight $w_i = (w_{i,s})_{s=1}^6 \in \frac{1}{2}\mathbf{Z}^6$ such that for any $(\ell, \mathbf{h}) \in \mathbf{T}_{>0}^6$ with $\ell = (\ell_1, \ell_2) \in (\mathbf{R}_{>0})^2$ and $\mathbf{h} = (h_i)_{i=1}^4 \in (\mathbf{R}_{>0})^4$, one has

$$(\ell, \mathbf{h}) \bullet \mathfrak{E}_i = \omega_i(\ell, \mathbf{h}) \mathfrak{E}_i = \left(\ell_1^{\omega_{i,1}} \ell_2^{\omega_{i,2}} \prod_{k=1}^4 h_k^{\omega_{i,k+2}} \right) \mathfrak{E}_i.$$

And there is a similar statement for the action (75), but with weights in $\frac{1}{2}\mathbf{Z}^7 \cup \frac{1}{3}\mathbf{Z}^7$.

Proof. The first point follows from the explicit description of the Lie groups E_6 and E_7 as subgroups of $\text{GL}(\mathbf{J})$ and $\text{GL}(\mathcal{Z}_2(\mathbf{J}))$, respectively, as found in [Yo]. For instance, direct formal computations show that $\det((\ell, \mathbf{h}) \bullet Z) = \det(Z)$ for every $(\ell, \mathbf{h}) \in \mathbf{T}_{>0}^6$ and $Z \in \mathbf{J}$. Hence, the image of this torus in $\text{GL}(\mathbf{J})$ actually lies in E_6 , according to the definition of this group given in [Yo, §3.1].³²

³²Actually, it was by examining the descriptions of the real parts of the Lie algebras of the Cartan tori of the complex Lie groups E_6 and E_7 , as given on pages 94 and 139 of [Yo], that we derived the explicit formulas for the torus actions (74) and (75).

Direct formal computations, made possible by the completely explicit nature of the actions (74) and (75), yield the second part of the proposition $\square$

One can easily determine the weights of the representation for each case. For instance, when $r = 6$, we obtain the following set of 27 pairwise distinct weights

$$\begin{aligned} & (2, 0, 0, 0, 0, 0), (0, 2, 0, 0, 0, 0), (-2, -2, 0, 0, 0, 0), (-1, 0, 1, 0, 0, 0), \\ & (-1, 0, 0, 1, 0, 0), (-1, 0, 0, 0, 1, 0), (-1, 0, 0, 0, 0, 1), (-1, 0, 0, 0, 0, -1), \\ & (-1, 0, 0, 0, -1, 0), (-1, 0, 0, -1, 0, 0), (-1, 0, -1, 0, 0, 0), (0, -1, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), \\ & (0, -1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (0, -1, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (0, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (0, -1, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), \\ & (0, -1, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), (0, -1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), (0, -1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), (1, 1, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \\ & (1, 1, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (1, 1, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (1, 1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), \\ & (1, 1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), (1, 1, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), (1, 1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}). \end{aligned}$$

The weight polytope Δ_6 is the convex envelope of these 27 points of $\mathbf{R}^6$. With the above list of weights at hand, it is not difficult to obtain a new explicit list, of the sets of vertices of the facets of Δ_6 : for each facet F of Δ_6 , let I_F be the set of indices i such that $w_i \in F$. The $\mathcal{W}$ -relevant facets are 5-dimensional orthoplexes, with set of vertices of cardinality 10. For any such F , we set $\mathbf{C}_F^{10} = \oplus_{i \in I_F} \mathbf{C} \mathfrak{E}_i$ the linear projection $\Pi_F : \mathbf{J} \rightarrow \mathbf{C}_F^{10}$ is written $\tilde{\Pi}_F : (x_i)_{i=1}^{27} \mapsto (x_j)_{j \in I_F}$ in the linear coordinates associated to the basis $(\mathfrak{E}_i)_{i=1}^{27}$ of $\mathbf{J}$. Then the 27 rational maps

$$\tilde{\Pi}_F \circ \nu_{\mathbf{O}}^2 : \mathbf{O} \oplus \mathbf{O} \dashrightarrow \mathbf{P}(\mathbf{C}_F^{10}) \simeq \mathbf{P}_F^9$$

are the first integrals of Gelfand-MacPherson web on $\mathbf{OP}^2$, written in coordinates.

Given a web-relevant facet F associated to a conic class $\mathfrak{c} \in \mathcal{K}_6$, there is an involution on I_F when this set of indices is seen as a subset of $\mathcal{L}_6$, given by $\ell \mapsto \ell' = \mathfrak{c} - \ell$. Let $J_F \subset I_F$ be a fixed subset of cardinality 5 such that for any $\ell \in I_F$, either ℓ or ℓ' is in J_F . When seeing I_F as a subset of $\{1, \dots, 27\}$, we will denote this involution by $i \mapsto i'_F$. To simplify the formulas, we will often omit the subscript F in the lines below.

It can be verified that the closure in $\mathbf{P}^9$ of the image of $\tilde{\Pi}_F \circ \nu_{\mathbf{O}}^2$ is a smooth hyperquadric $Q_F \subset \mathbf{P}^9$ cut out by a quadric equation of the form

$$q_F = \sum_{j \in J_F} \epsilon_{F,j} x_j x_{j'} = 0$$

with $\epsilon_{F,j} \in \{\pm 1\}$ for any $j \in J_F$. The basis $(\mathfrak{E}_i)_{i \in I_F}$ actually is a weight basis for the action of the Lie group $\mathbf{SO}(q_F)$ of type D_5 on $\mathbf{C}_F^{10}$, with the action of a Cartan torus $H_F \simeq (\mathbf{C}^*)^{J_F}$ of $\mathbf{SO}(q_F)$ given by

$$(\lambda_j)_{j \in J_F} \cdot \mathfrak{E}_i = \begin{cases} \lambda_i \mathfrak{E}_i & \text{if } i \in J_F \\ \lambda_{i'}^{-1} \mathfrak{E}_i & \text{if } i \notin J_F. \end{cases}$$

It can be verified that the action induced by the Cartan torus of E_6 obtained by taking the complexification of $\mathbf{T}_{>0}^6$ is linearly equivalent to the one described just above. Let $Q_F^\star$ be Q_F with the union of the ten hyperplane sections $Q_F \cap \{x_i = 0\}$ removed. Then H_F acts nicely with finite kernel on $Q_F^\star$, and gives rise to a rational morphism $\tau_F : Q_F^\star \rightarrow \mathcal{Y}_F^\star = Q_F^\star/H_F$ which can be made explicit: fixing $j_0 \in J_F$, we set $J_F^\star = J_F \setminus \{j_0\}$. Then a birational model of τ_F is given by

$$(77) \quad \begin{aligned} \tilde{\tau}_F : \mathbf{P}_F^9 &\dashrightarrow \mathbf{P}^3 \\ [x_i]_{i \in I_F} &\longmapsto \left[\frac{x_j x_{j'}}{x_{j_0} x_{j'_0}} \right]_{j \in J_F^\star} . \end{aligned}$$

It follows that, possibly up to multiplying some of the monomials $x_i x_{j'}/(x_{i_0} x_{j'_0})$ by -1, one obtains a quotient morphism $\tau_F : Q_F^\star \rightarrow \mathcal{Y}_F^\star$ with $\mathcal{Y}_F^\star \subset \mathbf{P}^3$ equal to the complement of the hyperplane arrangement $\mathcal{H}_6 \subset \mathbf{P}^3$ cut out by $u_0 u_1 u_2 u_3 (u_0 + u_1 + u_2 + u_3) = 0$ for some homogeneous coordinates $u_0, \dots, u_3$ on $\mathbf{P}^3$.

The action of $\mathbf{T}_{>0}^6$ on $\mathbf{O} \oplus \mathbf{O}$ induced by the action (75) is given by

$$(a, b, \mathbf{h})(u, v) = \left(ab(\mathbf{h}^\psi \cdot u), b^{-1}(\mathbf{h}^\varphi \cdot v) \right).$$

For $(u, v) \in \mathbf{O} \oplus \mathbf{O}$ generic, one can find $(a, b, \mathbf{h}) \in (\mathbf{C}^*)^6$ such that $(\tilde{u}, \tilde{v}) = (a, b, \mathbf{h})(u, v)$ is in ‘normal form’, that is such that

$$(78) \quad \tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, 1, 1, 1, 1) \quad \text{and} \quad \tilde{v} = (1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4, \tilde{v}_5, \tilde{v}_6, \tilde{v}_7, \tilde{v}_8).$$

Moreover, the element $(a, b, \mathbf{h}) \in (\mathbf{C}^*)^6$ satisfying (78) is essentially unique; that is, it is unique up to the action of a certain finite subgroup of $(\mathbf{C}^*)^6$, which can be explicitly determined without difficulty. It follows that the ten algebraically independent variables $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ and $\tilde{v}_2, \dots, \tilde{v}_8$ define a rational chart on the quotient $\mathcal{Y}_6$ of Cayley octonionic projective plane $\mathbf{OP}^2$ by the action of the Cartan subgroup of E_6 we are working with. Thus: (1) $\mathcal{Y}_6$ is rational; (2) $Y_6 = \text{Spec}(\mathbf{C}[\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{v}_2, \dots, \tilde{v}_8])$ is a birational model of it; and (3) up to the birational identification $Y_6 \simeq \mathcal{Y}_6$, the following map corresponds to a rational section of the quotient map $\mathbf{OP}^2 \dashrightarrow \mathcal{Y}_6$:

$$(79) \quad \begin{aligned} \sigma_6 : \mathbf{C}^6 = Y_6 &\dashrightarrow \mathbf{OP}^2 \subset \mathbf{P}^{26} \\ ((\tilde{u}_i)_{i=1}^3, (\tilde{v}_j)_{j=2}^8) &\longmapsto \nu_{\mathbf{O}}^2(\tilde{u}, \tilde{v}) \end{aligned}$$

where $\tilde{u}$ and $\tilde{v}$ stand for the two elements of $\mathbf{O}$ respectively defined by the LHS of the two equalities in (78).

Post-composing σ_6 with the linear projections $\tilde{\Pi}_F$ then with the rational model $\tilde{\tau}_F$ (defined in (77)) of the quotient maps $\tau_F : Q_F \dashrightarrow \mathbf{P}^3$ for all web-relevant facets F of Δ_6 , we obtain 27 rational maps

$$(80) \quad U_F = \tilde{\tau}_F \circ \tilde{\Pi}_F \circ \sigma_6 : Y_6 \dashrightarrow \mathbf{P}^3$$

which define a web, noted by $\mathbf{W}_{Y_6}^{GM}$, which is the birational model of $\mathcal{W}_{\mathcal{Y}_6}^{GM}$ on Y_6 induced by the birational identification $Y_6 \simeq \mathcal{Y}_6$.

Thanks to our choice of the specific version (70) of the (split) octonionic product, the torus actions (74) and (75) are defined over $\mathbf{R}$, and so are the bases of weight vectors $(\mathfrak{E}_i)_{i=1}^{27}$ and $(\mathfrak{Z}_j)_{j=1}^{56}$ induced by the linear isomorphisms (76). This has the pleasant consequence that all the first integrals U_F are defined over $\mathbf{R}$ (in fact, over $\mathbf{Z}$), which makes the effective computations we

will perform from them – in order to study the ranks and abelian relations (ARs) of the web $\mathbf{W}_{Y_6}^{GM}$ – significantly more efficient.

★

All the considerations and constructions above are not truly specific to the case $r = 6$; they admit direct analogues in the case $r = 7$, which makes it possible to state the following result uniformly for all $r \in \{4, 5, 6, 7\}$:

Proposition 7.3. *Let V_r be the minuscule representation in whose projectivization the homogeneous space $\mathcal{G}_r = G_r/P_r$, of dimension γ_r , embeds.*

1. *One can explicitly describe:*

- *a basis $(e_i)_{1 \leq i \leq v_r}$ of V_r , yielding an isomorphism $V_r \simeq \bigoplus_{i=1}^{v_r} \mathbf{C} e_i$;*
- *an affine space $\mathbf{A}_r$ of dimension γ_r ;*
- *an affine embedding $v_r = (v_{r,i})_{1 \leq i \leq v_r} : \mathbf{A}_r \rightarrow V_r$,*

with the property that the image of the projectivization $[v_r] : \mathbf{A}_r \rightarrow \mathbf{P}V_r$ has Zariski closure equal to $\mathcal{G}_r$, that is,

$$\overline{[v_r](\mathbf{A}_r)} = \mathcal{G}_r \subset \mathbf{P}V_r \simeq \mathbf{P}^{v_r-1}.$$

2. *Moreover, one can describe just as explicitly an action of $H_r = (\mathbf{C}^*)^r$ on $V_r \oplus_{i=1}^{v_r} \mathbf{C} e_i$, satisfying the following properties:*

- *Each e_i is a weight vector with respect to this action, associated to a weight $\omega_i \in \mathbf{Q}^r$, which can be explicitly computed;*
- *The $\mathcal{W}$ -relevant facets of $\Delta_r = \text{Conv}(\{\omega_i\}_{i=1}^{v_r})$ are entirely determined by the set of their vertices: for each such facet F , let $I_F \subset \{1, \dots, v_r\}$ be the subset of cardinality $2r - 2$ consisting of the indices i such that the weight ω_i associated to e_i is a vertex of F ;*
- *The set of κ_r subsets I_F 's can be explicitly determined hence the linear projections*

$$\tilde{\Pi}_F : V_r = \bigoplus_{i=1}^{v_r} \mathbf{C} e_i \rightarrow \bigoplus_{i \in I_F} \mathbf{C} e_i =: V_{r,F}$$

as well. The projectivizations of the compositions $\tilde{\Pi}_F \circ v_r : \mathbf{A}_r \rightarrow V_F$ for all web-relevant facets F of the weight polytope Δ_r are rational first integrals of the birational model $v_r^(\mathbf{W}_{\mathcal{G}_r}^{GM})$ of Gelfand-MacPherson web on $\mathcal{G}_r$.*

3. *For a certain subset of indices $\mathcal{A}_r \subset \{1, \dots, \dim V_r\}$ of cardinality γ_r , the affine embedding v_r post-composed with the natural projection $\bigoplus_{i=1}^{v_r} \mathbf{C} e_i \rightarrow \bigoplus_{a \in \mathcal{A}_r} \mathbf{C} e_a$ is a linear isomorphism defined over $\mathbf{R}$ (actually, over $\mathbf{Z}$). Consequently the H_r -action on V_r induces an action of the same torus on $\mathbf{A}_r \simeq \bigoplus_{a \in \mathcal{A}_r} \mathbf{C} e_a$ which is a birational model of the action of the Cartan torus of G_r on $\mathcal{G}_r$.*

Moreover, one can provide an explicit list of algebraically independent monomials M_s for $s = 1, \dots, y_r$, which form an algebraic basis of the algebra of H_r -invariant rational functions on $\mathbf{A}_r$. Setting $Y_r = \text{Spec}(\mathbf{C}[M_1, \dots, M_{y_r}]) \simeq \mathbf{C}^{y_r}$, we obtain an explicit rational map $\mathcal{P}_r : \mathbf{A}_r \rightarrow Y_r$ which constitutes a birational model of the quotient map $\tau_r : \mathcal{G}_r \rightarrow \mathcal{Y}_r = \mathcal{G}_r/H_r$.

Finally, one constructs an explicit affine embedding $\sigma_r : Y_r \rightarrow \mathbf{A}_r$ which is a (rational) section of the quotient map $\mathbf{A}_r \rightarrow Y_r = \mathbf{A}_r/H_r$, and such that the composed map $\Xi_r = \tau_r \circ [v_r] \circ \sigma_r :$

$Y_r \dashrightarrow \mathcal{Y}_r$ is birational. This yields a birational identification $Y_r \simeq \mathcal{Y}_r$ and the construction fits into the following commutative diagram of rational maps:

$$(81) \quad \begin{array}{ccc} & A_r & \xrightarrow{[v_r]} \mathcal{G}_r \\ \sigma_r \curvearrowright & \downarrow \mathcal{P}_r & \downarrow \tau_r \\ & Y_r & \xrightarrow{\Xi_r} \mathcal{Y}_r. \end{array}$$

4. Given a $\mathcal{W}$ -relevant facet F , let $\mathbf{c} \in \mathcal{K}_r$ be the conic class associated to it. Identifying the basis $\{e_i\}_{i=1}^{v_r}$ with $\mathcal{L}_r$, the elements in the index set I_F appear in ‘associated pairs’, with $\{i, j\} \subset I_F$ (with $i \neq j$) being such a pair if and only if the sum of the two corresponding lines is equal to $\mathbf{c}$. For $i \in I_F$, we denote by i' the unique other element of I_F such that $\{i, i'\}$ is an associated pair. We then fix a subset $J_F \subset I_F$ such that, setting $J'_F = \{j'\}_{j \in J_F}$, one has $I_F = J_F \sqcup J'_F$.

Then the image of $\mathcal{G}_r$ by the projectivization $[\tilde{\Pi}_F] : \mathbf{P}V_r \dashrightarrow \mathbf{P}V_{r,F} = \mathbf{P}_F^{2r-3}$ of the linear projection $\tilde{\Pi}_F$ is a smooth hyperquadric $\mathcal{Q}_F$ cut out by an equation of the form $q_F = \sum_{i \in J_F} \varepsilon_{F,i} x_i x_{i'} = 0$ in the coordinate system on $V_{r,F}$ dual to the base $(e_i)_{i \in I_F}$, with $\varepsilon_{F,i} \in \{\pm 1\}$ for every $i \in J_F$. The torus action on $V_{r,F}$ induced by that of H_r on V_r is equivalent to the one defined by

$$\lambda \cdot e_i = \lambda_i e_i \quad \text{if } i \in J_F \quad \text{and} \quad \lambda \cdot e_i = \lambda_{i'}^{-1} e_i \quad \text{when } i \notin J_F$$

for any $\lambda = (\lambda_s)_{s=1}^{r-1} \in (\mathbf{C}^*)^{r-1}$ and any $i \in I_F$. We fix $j_0 \in J_F$ and we set $J_F^* = J_F \setminus \{j_0\}$. It follows that, as a rational map, the quotient map of $\mathbf{P}_F^{2r-3}$ by the action of the torus of rank $r-1$ induced by that of H_r is written

$$(82) \quad \tilde{\tau}_F : \mathbf{P}_F^{2r-3} \dashrightarrow \mathbf{P}^{r-3}, \quad [x_i]_{i \in I_F} \longmapsto [x_j x_{j'} / (x_{j_0} x_{j'_0})]_{j \in J_F^*}.$$

in the coordinates x_i , $i \in I_F$.

5. From the four points above, it comes that for any web-relevant facet F of Δ_r , the explicit rational map

$$(83) \quad U_F = \tilde{\tau}_F \circ [\tilde{\Pi}_F] \circ [v_r] \circ \sigma_r : Y_r \dashrightarrow \mathbf{P}^{r-3}$$

makes commutative the following diagram of rational maps

$$(84) \quad \begin{array}{ccccc} & A_r & \xrightarrow{[v_r]} \mathcal{G}_r & \xrightarrow{\Pi_F} \mathcal{Q}_F \subset \mathbf{P}_F^{2r-3} \\ \sigma_r \curvearrowright & \downarrow \mathcal{P}_r & \downarrow \tau_r & \downarrow \tau_F \\ & Y_r & \xrightarrow{\Xi_r} \mathcal{Y}_r & \xrightarrow{\pi_F} \mathbf{P}^{r-3} \\ & & & \uparrow U_F \end{array}$$

in which we have $\Pi_F = [\tilde{\Pi}_F]|_{\mathcal{G}_r}$ and $\tau_F = \tilde{\tau}_F|_{\mathcal{Q}_F}$. It follows that the U_F ’s are rational first integrals of a κ_r -web of codimension $r-3$ on $Y_r = \mathbf{C}^{y_r}$, that we will denote by $\mathbf{W}_{Y_r}^{GM}$, which is the

birational model of $\mathcal{W}_{\mathcal{Y}_r}^{GM}$ on Y_r induced by the birational identification $Y_r \simeq \mathcal{Y}_r$. One has:

$$\mathcal{W}_{Y_r}^{GM} = \mathcal{W} \left(U_F \mid \begin{array}{c} \mathcal{W}\text{-relevant} \\ \text{facet } F \subset \Delta_r \end{array} \right) = \Xi_r^* (\mathcal{W}_{\mathcal{Y}_r}^{GM}).$$

6. Given a web-relevant facet F , possibly up to multiplying certain components of the map $\tilde{\tau}_F$ by -1 , then the image of $\mathcal{Q}_F^*$ by $\tilde{\tau}_F$ in $\mathbf{P}^{r-3}$ is the complement of the arrangement of $r-1$ hyperplanes $\mathcal{H}_r \subset \mathbf{P}^{r-3}$ cut out by the equation

$$u_0 u_1 \cdots u_{r-3} (u_0 + u_1 + \cdots + u_{r-3}) = 0.$$

in the homogeneous coordinates $u_0, \dots, u_{r-3}$ corresponding to the explicit expression (82) of the map $\tilde{\tau}_F$. Moreover, the face map $\pi_F : \mathcal{Y}_r \dashrightarrow \mathbf{P}^{r-3}$ is defined on $\mathcal{Y}_r^*$ and one has $\pi_F(\mathcal{Y}_r^*) = \mathbf{P}^{r-3} \setminus \mathcal{H}_r$. Finally, on Y_r , there exists an explicit arrangement of hypersurfaces $\mathbf{H}_r \subset Y_r$ on the complement $Y_r^* = Y_r \setminus \mathbf{H}_r$ of which Ξ_r is defined and induces an isomorphism $\Xi_r : Y_r^* \simeq \mathcal{Y}_r^*$. It follows that the rational map $U_F = \pi_F \circ \Xi_r$ is defined on Y_r^* with image $U_F(Y_r^*) = \mathbf{P}^{r-3} \setminus \mathcal{H}_r$.

The result above has been established through direct computations and case-by-case verifications. It would nonetheless be of interest to provide a conceptual and genuinely uniform proof valid for all values of r .

It can be verified that all the objects considered in the above result are defined over $\mathbf{R}$. In particular, this is true for the rational maps (83) which actually can all be written as rational maps with non-zero coefficients ± 1 .

7.2. The virtual $(r-3)$ -ranks. Having explicit rational first integrals of the web $\mathcal{W}_{Y_r}^{GM}$ at our disposal, the determination of the virtual ranks of these webs becomes a matter of formal computation. After carrying out the necessary calculations, we obtain the following result for the virtual ranks of top-degree abelian relations:

Proposition 7.4. *One has*

$$\begin{aligned} \rho_3^\bullet(\mathcal{W}_{Y_6}^{GM}) &= (10, 10, 1) \quad \text{hence} \quad \rho_3(\mathcal{W}_{Y_6}^{GM}) = 21 \\ \text{and} \quad \rho_4^\bullet(\mathcal{W}_{Y_7}^{GM}) &= (28, 20, 1) \quad \text{hence} \quad \rho_4(\mathcal{W}_{Y_7}^{GM}) = 49. \end{aligned}$$

7.3. The $(r-3)$ -abelian relations. Most of the results concerning the (ranks and abelian relations of the) first two Gelfand-MacPherson webs $\mathcal{W}_{\mathcal{Y}_r}^{GM}$ appear to extend to the last two cases (i.e., for $r=6$ and $r=7$). In any case, this holds for abelian relations of maximal degree, as demonstrated below.

7.3.1. The master differential identity $\mathbf{HLOG}_{Y_r}$. In the affine coordinates $u_1, \dots, u_{r-3}$ on $U_0 = \{u_0 = 1\} \simeq \mathbf{C}^{r-3}$, we set $L_i = u_i$ for $i = 1, \dots, r-3$ and $L_{r-2} = \sum_{i=0}^{r-3} u_i = 1 + \sum_{i=1}^{r-3} u_i$. Then one defines a $(r-3)$ -differential form with logarithmic coefficients on $\mathbf{P}^{r-3} \setminus \mathcal{H}_r$ and with logarithmic poles along the components of $\mathcal{H}_r$ by setting

$$\begin{aligned} \Omega_r &= \sum_{i=1}^{r-2} (-1)^{i-1} \ln(L_i) \bigwedge_{\substack{k=1 \\ k \neq i}}^{r-2} \frac{dL_k}{L_k} \\ &= \sum_{i=1}^{r-2} (-1)^{i-1} \ln(L_i) d \ln(L_1) \wedge \cdots \wedge \widehat{d \ln(L_i)} \wedge \cdots \wedge d \ln(L_{r-2}). \end{aligned}$$

Up to a scaling factor, Ω_r can also be defined as the antisymmetrization of $\ln(L_1) \wedge_{k=2}^{r-2} d \ln(L_k)$ with respect to the L_i 's. Then we have the following result:

Proposition 7.5. *There exists $(\varepsilon_F)_{F \in \mathcal{K}_r} \in \{\pm 1\}^{\mathcal{K}_r}$, unique up to sign, such that the following differential relation is satisfied:*

$$(\mathbf{HLOG}_{Y_r}) \quad \sum_{F \in \mathcal{K}_r} \varepsilon_F U_F^*(\Omega_r) = 0.$$

Consequently, the κ_r -tuple $(\varepsilon_F U_F^*(\Omega_r))_{F \in \mathcal{K}_r}$ can be seen as a $(r-3)$ -abelian relation for $\mathcal{W}_{Y_r}^{GM}$, again denoted by $\mathbf{HLOG}_{Y_r}$.

Proof. The current proof relies on direct computations performed in MAPLE.³³ □

Remark 7.6. *The $(r-3)$ -form Ω_r is holomorphic and multivalued. It has a natural global univalued real-analytic version, namely $\Omega_r^\omega = \sum_{i=1}^{r-2} (-1)^{i-1} \ln |L_i| d \ln(L_1) \wedge \cdots \wedge \widehat{d \ln(L_i)} \wedge \cdots \wedge d \ln(L_{r-2})$. One can verify that the associated real-analytic identity*

$$(\mathbf{HLOG}_{Y_r}^\omega) \quad \sum_{F \in \mathcal{K}_r} \varepsilon_F U_F^*(\Omega_r^\omega) = 0.$$

holds true identically on Y_r^+ .

7.3.2. Combinatorial $(r-3)$ -abelian relations. It is more convenient to formulate the results of this subsection in terms of the web $\mathcal{W}_{Y_r}^{GM}$ on Y_r . Since the representation V_r , into whose projectivization $\mathcal{G}_r = G_r/P_r$ embeds, is minuscule, it admits a natural system of linear coordinates x_ℓ , indexed by $\mathcal{L}_r \simeq \mathfrak{B}_r$, each uniquely determined up to scaling.

It follows from [Sk] that the material described in §?? generalizes to all cases $r = 4, \dots, 7$. In particular, for each $\ell \in \mathcal{L}_r$, the image in Y_r - by the quotient map - of the H_r -invariant hyperplane section $\{x_\ell = 0\} \cap \mathcal{G}_r$ is an irreducible divisor $\mathcal{D}_\ell$, along which the residue of the abelian relation $\mathbf{HLOG}_{Y_r}$ can be taken. We denote this residue by

$$\mathbf{AR}_\ell^{r-3} = \text{Res}_{\mathcal{D}_\ell}(\mathbf{HLOG}_{Y_r}).$$

We will use the following fact which characterizes the adjacency between the vertices and the orthoplex facets of Δ_r in terms of their respective labelling by the set of lines $\mathcal{L}_r$ and the one of conic classes $\mathcal{K}_r$:

Lemma 7.7. *The vertex of Δ_r corresponding to $\ell \in \mathcal{L}_r$ is adjacent to the D_{r-1} -facet corresponding to $\mathfrak{c} \in \mathcal{K}_r$ if and only if $\mathfrak{c} \in \ell + \mathcal{L}_r$, that is there exists $\ell' \in \mathcal{L}_r$ such that $\mathfrak{c} = \ell + \ell'$.*

For any $\ell \in \mathcal{L}_r$, we define $\mathcal{K}_r(\ell) = \{\mathfrak{c} \in \mathcal{K}_r \mid \mathfrak{c} - \ell \in \mathcal{L}_r\}$. This set can be identified with the collection of (D_{r-1}) -facets of the polytope Δ_r that are adjacent to ℓ , viewed as a vertex of Δ_r . We denote by $\mathcal{W}_{Y_r, \ell}^{GM}$ the subweb of $\mathcal{W}_{Y_r}^{GM}$ corresponding to these facets; that is,

$$\mathcal{W}_{Y_r, \ell}^{GM} = \mathcal{W}(\pi_F \mid F \in \mathcal{K}_r(\ell)).$$

Since the vertex figure of Δ_r at ℓ is isomorphic to Δ_{r-1} , we obtain a natural bijection $\mathcal{K}_r(\ell) \simeq \mathcal{K}_{r-1}$. It follows that $\mathcal{W}_{Y_r, \ell}^{GM}$ is a κ_{r-1} -web.

With these notations in place, we obtain the following result by direct computation:

³³Maple worksheets are available from the author upon request.

Proposition 7.8. 1. *For any $\ell \in \mathcal{L}_r$, the residue $\mathbf{AR}_\ell^{r-3}$ is a rational $(r-3)$ -AR. Its non-trivial components are carried by the π_F foliations for all D_{r-1} -facets F of Δ_r adjacent to ℓ . In other terms, $\mathbf{AR}_\ell^{r-3}$ is a complete $(r-3)$ -AR of $\mathcal{W}_{\mathbf{y}_r, \ell}^{GM}$.*

2. *Actually, $\mathbf{AR}_\ell^{r-3}$ spans the space of $(r-3)$ -ARs of $\mathcal{W}_{\mathbf{y}_r, \ell}^{GM}$ which is $(r-3)$ -rank 1: one has*

$$\mathbf{AR}^{r-3}(\mathcal{W}_{\mathbf{y}_r, \ell}^{GM}) = \langle \mathbf{AR}_\ell^{r-3} \rangle.$$

3. *For $F \in \mathcal{K}_r(\ell)$, the π_F -component of $\mathbf{AR}_\ell^{r-3}$ is a linear combination with coefficients in $\{-1, 0, 1\}$ of the following $r-2$ wedge products for $i = 1, \dots, r-2$:*

$$d \ln(\pi_{F,1}) \wedge \cdots \wedge \widehat{d \ln(\pi_{F,i})} \wedge \cdots \wedge d \ln(\pi_{F,r-2}).$$

4. *The $\mathbf{AR}_\ell^{r-3}$'s for $\ell \in \mathcal{L}_r$ are called ‘combinatorial ARs’. They span a subspace of dimension $\rho_{r-3}(\mathcal{W}_{\mathbf{y}_r}^{GM}) - 1$ of $\mathbf{AR}^{r-3}(\mathcal{W}_{\mathbf{y}_r}^{GM})$ which we will denote by $\mathbf{AR}_C^{r-3}(\mathcal{W}_{\mathbf{y}_r}^{GM})$.*

This results shows that some properties of $\mathcal{W}_{\mathbf{y}_5}^{GM}$ admit natural generalizations to the cases $r = 5, 6$. It is natural to wonder if it might be the case of some other properties, e.g. the first one of Corollary 3.10. We believe that it is indeed the case and that for any $r \in \{4, \dots, 7\}$, the following holds true:

One has $\rho^{r-3}(\mathcal{W}) \leq 1$ for any κ_{r-1} -subweb $\mathcal{W}$ of $\mathcal{W}_{\mathbf{y}_r}^{GM}$. Those for which the virtual $(r-3)$ -rank is 1 actually are of maximal $(r-3)$ -rank 1. And these subwebs are exactly the ℓ_r subwebs $\mathcal{W}_{\mathbf{y}_r, \ell}^{GM}$ for all $\ell \in \mathcal{L}_r$.

7.3.3. The structure of $\mathbf{AR}^{r-3}(\mathcal{W}_{Y_r}^{GM})$. From the results above, we deduce the following

Theorem 7.9. 1. *One has*

$$(85) \quad \mathbf{AR}^{r-3}(\mathcal{W}_{Y_r}^{GM}) = \mathbf{AR}_C^{r-3}(\mathcal{W}_{Y_r}^{GM}) \oplus \langle \mathbf{HLOG}_{Y_r} \rangle.$$

Moreover, $\mathbf{AR}_C^{r-3}(\mathcal{W}_{Y_r}^{GM})$ is of dimension $\rho_{r-3}(\mathcal{W}_{Y_r}^{GM}) - 1$ hence the $(r-3)$ -rank of $\mathcal{W}_{Y_r}^{GM}$ is $\rho_{r-3}(\mathcal{W}_{Y_r}^{GM})$, that is, is AMP.

2. *By residues/monodromy, the abelian relation $\mathbf{HLOG}_{Y_r}$ spans the subspace $\mathbf{AR}_C^{r-3}(\mathcal{W}_{Y_r}^{GM})$ of combinatorial ARs, which coincides with that of rational $(r-3)$ -ARs of $\mathcal{W}_{Y_r}^{GM}$: one has*

$$\mathbf{Res}(\mathbf{HLOG}_{Y_r}) = \mathbf{AR}_C^{r-3}(\mathcal{W}_{Y_r}^{GM}) = \mathbf{AR}_{\text{Rat}}^{r-3}(\mathcal{W}_{Y_r}^{GM}).$$

3. *For any smooth del Pezzo surface $X_r = \text{dP}_d$ with $d = 9 - r$, using Serganova-Skorobogatov embedding $f_{SS} : \text{dP}_d \hookrightarrow \mathcal{Y}_r$ (cf. (24) above), one can recover in a natural way the weight $r-2$ hyperlogarithmic abelian relation $\mathbf{HLog}_{\text{dP}_d}^{r-2}$ of $\mathcal{W}_{\text{dP}_d}$ from the $(r-3)$ -AR $\mathbf{HLOG}_{\mathcal{Y}_r}$.*

We believe that the results known to hold for $r = 4, 5$ generalize straightforwardly to the cases $r = 6, 7$. For instance, we are convinced that the following statement remains valid for $r = 6, 7$, in the context of the natural linear action of the Weyl group W_r on the space of $(r-3)$ -abelian relations of the web $\mathcal{W}_{Y_r}^{GM}$

The decomposition in direct sum (85) is, in fact, the decomposition of $AR_C^{r-3}(\mathcal{W}_{Y_r}^{GM})$ into irreducible W_r -representations. The 1-dimensional subrepresentation spanned by $\mathbf{HLOG}_{Y_r}$ is (isomorphic to) the signature representation.

By analogy with the cases $r = 4, 5$, one may also conjecture that $AR_C^{r-3}(\mathcal{W}_{Y_r}^{GM})$ is irreducible as a W_r -representation. We will revisit this in a future work.

8. Perspectives and questions

In this final section, we begin by discussing natural generalizations to $\mathcal{W}_{Y_5}^{GM}$ of the many remarkable properties enjoyed by Bol's web. In §8.2, we offer some brief remarks on the possible interpretation of the differential identity $\mathbf{HLOG}_{Y_5}$ as the manifestation of a yet-to-be-determined property of a certain scattering diagram that might be associated to $\mathcal{Y}_5$.

8.1. Comparison between Bol's web $\mathcal{W}_{Y_4}^{GM}$ and $\mathcal{W}_{Y_5}^{GM}$. The main objective of our previous paper [Pi5] was to demonstrate that the web $\mathcal{W}_{dP_4}$, formed by the ten pencils of conics on a smooth del Pezzo surface of degree four dP_4 , shares many remarkable properties with Bol's classical web $\mathcal{B} \simeq \mathcal{W}_{dP_5} \simeq \mathcal{W}_{Y_4}^{GM}$. Moreover, in [Pi5, §4.5], we showed that any del Pezzo web $\mathcal{W}_{dP_4}$ can be recovered from the Gelfand–MacPherson web $\mathcal{W}_{Y_5}^{GM}$ as its pull-back under an embedding $dP_4 \hookrightarrow \mathcal{Y}_5$, first considered by Serganova and Skorobogatov in [SS].

One of the main results of the present paper is that, beyond this structural recovery, one can also retrieve the most significant abelian relation of $\mathcal{W}_{dP_4}$ – namely, its hyperlogarithmic abelian relation of weight three, $\mathbf{HLog}_{dP_4}$ – from the most notable $(r - 3)$ -abelian relation of the former web $\mathcal{W}_{Y_5}^{GM}$, namely $\mathbf{HLOG}_{Y_5}$. This naturally raises the question (at least for the author) of whether the many remarkable properties satisfied by Bol's web might also admit natural analogues in the context of the Gelfand–MacPherson web $\mathcal{W}_{Y_5}^{GM}$.

* * *

Below, we review the list of remarkable properties of Bol's web as presented in [Pi5, §1.1], and for each of them, we provide a brief comment regarding its possible generalization to the web $\mathcal{W}_{Y_5}^{GM}$.

1. Geometric definition. The del Pezzo webs are defined as the webs on del Pezzo surfaces whose foliations are the pencils of conics on these surfaces. It is plausible that a similar geometric interpretation may exist for the Gelfand–MacPherson web $\mathcal{W}_{Y_5}^{GM}$. Indeed, as established in [SS] (see the very end of the proof of Theorem 6.1 therein), for any del Pezzo quartic surface dP_4 , the associated Serganova–Skorobogatov embedding $F_{SS} : dP_4 \hookrightarrow \mathcal{Y}_5$ induces an isomorphism of Picard lattices $F_{SS}^* : \text{Pic}_{\mathbf{Z}}(\mathcal{Y}_5) \simeq \text{Pic}_{\mathbf{Z}}(dP_4)$. Let $\mathfrak{c} \in \text{Pic}_{\mathbf{Z}}(dP_4)$ be a conic class with associated facet $F \subset \Delta_5$ and corresponding face map $\pi_F : \mathcal{Y}_5 \dashrightarrow \mathbf{P}^2$. Then, in [Pi5], we proved that the pull-back of the linear system $|O_{\mathbf{P}^2}(1)|$ under the composition $\pi_F \circ F_{SS} : dP_4 \dashrightarrow \mathbf{P}^2$ coincides precisely with the complete linear system $|\mathfrak{c}|$. This observation suggests that the map $\pi_F : \mathcal{Y}_5 \dashrightarrow \mathbf{P}^2$ may in fact be induced by the complete linear system $|\mathfrak{c}|$ when $\mathfrak{c}$ is now seen as an element of $\text{Pic}_{\mathbf{Z}}(\mathcal{Y}_5)$.

2. Linearizability. Since it contains Bol's web as a subweb, any del Pezzo web $\mathcal{W}_{dP_4}$ is non-linearizable. It is currently unclear to what extent the (non-)linearizability of the web $\mathcal{W}_{Y_5}^{GM}$ is

relevant. We do believe that this web is not linearizable, but we also consider this fact to be non-essential. The approach discussed in [Pi1] should allow to address this question.

3. & 4. Structure of the space of top degree ARs & maximality of the rank. The spaces of top degree abelian relations of $\mathcal{B} \simeq \mathcal{W}_{\mathcal{Y}_4}^{GM}$ and $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ share the same structure. Indeed, in both cases, there exists a subspace of so-called *combinatorial* ARs (denoted by a capital subscript C), such that the following direct sum decompositions hold:

$$(86) \quad \begin{aligned} AR(\mathcal{W}_{\mathcal{Y}_4}^{GM}) &= AR_C(\mathcal{W}_{\mathcal{Y}_4}^{GM}) \oplus \langle \mathbf{Ab} \rangle \\ AR^{(2)}(\mathcal{W}_{\mathcal{Y}_5}^{GM}) &= AR_C^{(2)}(\mathcal{W}_{\mathcal{Y}_5}^{GM}) \oplus \langle \mathbf{HLOG}_{\mathcal{Y}_5} \rangle. \end{aligned}$$

Moreover, as is well known, Bol's web has maximal rank 6. As shown above, the genuine 2-rank of the web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ is equal to its virtual 2-rank. Therefore, both webs have AMP ranks. Regarding abelian relations and the rank structure of the webs, the similarity between $\mathcal{W}_{\mathcal{Y}_r}^{GM}$ for $r = 4$ and $r = 5$ is clearly observable.

5. 'Canonical algebraization'. In [Pi5, §3.5], we explained how Bol's web can be canonically recovered from the space of its combinatorial abelian relations. It is an interesting question to ask whether a similar phenomenon holds for the web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$. We believe that it might be the case.

Let $U\Omega^2$ be the subspace of rational 2-forms on $Y_5 = \mathbf{C}^5$ spanned by the 30 wedge products $d \ln U_{i,a} \wedge d \ln U_{i,b}$ for all $i \in \llbracket 10 \rrbracket$ and all a, b such that $1 \leq a < b \leq 3$. This is the space generated by all the components of all the combinatorial 2-abelian relations. It can be verified that $U\Omega^2$ is a 20-dimensional vector subspace of $\Omega_{\mathbf{C}(Y_5)}^2$. We wonder whether the following statements hold:

- for any $y \in Y_5^*$, the evaluation map at this point, $ev_y : U\Omega^2 \rightarrow \Omega_{Y_5, y}^2$, $\omega \mapsto \omega(y)$ is well-defined and surjective. Consequently, its kernel $K(y) = \text{Ker}(ev_y)$ is a vector subspace of $U\Omega^2$ of dimension 10;
- we thus obtain a map $Y_5^* \rightarrow G_{10}(U\Omega^2) \simeq G_{10}(\mathbf{C}^{20})$, $y \mapsto K(y)$ which, composed with the Plücker embedding, gives rise to a morphism

$$\mu : Y_5^* \longrightarrow \mathbf{P}(\wedge^{10} \mathbf{C}^{20}) = \mathbf{P}^{(20)-1}.$$

Composed by the birational identification $Y_5 \simeq \mathcal{Y}_5$ (see (15)), this yields an embedding $\mu \circ \Theta^{-1} : \mathcal{Y}_5 \hookrightarrow \mathbf{P}^{(20)-1}$ such that the closure of the image $\overline{\mathcal{Y}}_5 = \overline{\mu(Y_5^*)}$ provides a compactification of $\mathcal{Y}_5$ with 'nice properties'.

A similar construction for the web $\mathcal{W}_{\mathcal{Y}_4}^{GM}$ on $\mathcal{Y}_4^* \simeq \mathcal{M}_{0,5}$ gives rise to (the restriction to the open stratum of) the log canonical embedding $\overline{\mathcal{M}}_{0,5} \hookrightarrow \mathbf{P}^9$, that is the embedding induced by the complete linear system associated to the ample log-canonical divisor $K_{\overline{\mathcal{M}}_{0,5}} + \partial \overline{\mathcal{M}}_{0,5}$ (see [KT]). It is natural to ask whether an analogous result holds for the web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ on $\mathcal{Y}_5^*$.

In [Co], the author studied the Chow quotient $\mathbb{S}_5 // H$ of the spinor tenfold by the action of the Cartan torus $H \subset \text{Spin}(\mathbf{C}^{10})$. He proved that, up to normalization, this Chow quotient is smooth with a boundary having simple normal crossings, and described the log canonical model Y_5 of $\mathbb{S}_5 // H$. It would be interesting to investigate possible relationships between the conjectural compactification $\overline{\mathcal{Y}}_5$ discussed above and the proper varieties $\mathbb{S}_5 // H$ and Y_5 studied by Corey in his paper.

6. Weyl group action. Thanks to a result by Skorobogatov, for any $r = 4, \dots, 7$, the Weyl group W_r acts by automorphisms on $\mathcal{Y}_r$ and preserves Gelfand–MacPherson web $\mathcal{W}_{\mathcal{Y}_r}^{GM}$. From this, we deduce a linear action of W_r on $AR^{(r-3)}(\mathcal{W}_{\mathcal{Y}_r}^{GM})$. In both cases, the components $\langle \mathbf{Ab} \rangle$ and $\langle \mathbf{HLOG}_{\mathcal{Y}_5} \rangle$ are isomorphic to the signature representation of W_r .

7. Hexagonality and characterization. It has been known since Bol’s paper [Bo] that Bol’s web, that is $\mathcal{W}_{\text{dP}_5} \simeq \mathcal{W}_{\mathcal{Y}_4}^{GM}$, can be characterized – up to local analytic equivalence – as the unique planar hexagonal web that is not linearizable. It is natural, though perhaps somewhat naive, to ask whether a similar characterization holds for the Gelfand–MacPherson web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$.

Let us discuss briefly what could be the analog of Bol’s characterization for this 5-dimensional 10-web. First, the formal similarity between the direct sum decompositions given in (86) suggests that the appropriate analogue, for $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, of the notion of a hexagonal 3-subweb in planar web geometry might be the 5-subwebs of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ which carry a non-trivial irreducible and complete 2-abelian relation. These subwebs have been described above in Corollary 3.10. In particular, it follows that not every 5-subweb of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ carries a complete and irreducible 2-abelian relation. However, this does not prevent us from envisioning a Bol-type characterization of this web based on its 5-subwebs. We now outline a few more specific remarks on what would need to be established in order to derive such a characterization:

- *Given a 10-web $\mathcal{W}$ of codimension 2 on a 5-dimensional manifold M , one defines a function $R_{\mathcal{W}}$ on the set of its 5-subwebs by associating its 2-rank to any such subweb. The natural question is then whether the function $R_{\mathcal{W}_{\mathcal{Y}_5}^{GM}}$ characterizes $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ up to local analytic equivalence.*
- *Before attempting to investigate this question, it may be necessary to first study more thoroughly the analytic classification of 5-webs of codimension 2 on 5-dimensional manifolds. In particular, one may ask:*
 - *Is there a universal bound on the 2-rank of such a web? Could it be equal to 1?*
 - *What can be said about the webs that achieve this maximal 2-rank?*
 - *Can such webs be classified? Is there a unique normal form for them?*
 - *Does the class of webs attaining the maximal 2-rank share, in some meaningful sense, some ‘nice properties’ with the class of hexagonal planar 3-webs?*

8. Description à la Gelfand-MacPherson. That Bol’s web $\mathcal{B} \simeq \mathcal{W}_{\text{dP}_5}$ can be described as a Gelfand-MacPherson web is a result established in [GM]. This contrasts with $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, which is of this type by definition.

While the fact that $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ is of Gelfand–MacPherson type is immediate from its very definition, extending this perspective to its master 2-abelian relation $\mathbf{HLOG}_{\mathcal{Y}_5}$ is a much more subtle and interesting question, which deserves some further discussion. In their foundational work [GM], Gelfand and MacPherson not only provide a geometric construction of Bol’s web (namely as the web $\mathcal{W}_{\mathcal{Y}_4}^{GM}$), but also demonstrate how, within the same geometric framework, one can construct the dilogarithm function and derive its 5-term functional identity. Remarkably, this identity arises in their theory as a rather direct consequence of Stokes’ theorem, applied to the integration along the fibers of the action of the Cartan torus of $\text{SL}(\mathbf{R}^5)$ on the Grassmannian $G_2^{or}(\mathbf{R}^5)$ of oriented 2-planes in $\mathbf{R}^5$. The integrand is an invariant differential 4-form representing the first Pontrjagin class of the tautological bundle on $G_2^{or}(\mathbf{R}^5)$.

It appears natural to us to seek a similar geometric construction for the master 2-abelian relation. A question in this direction was already raised in [Pi5, §5.9], but for the weight 3 hyperlogarithmic abelian relation $\mathbf{HLog}_{\mathrm{dP}_4}$ associated with a smooth del Pezzo quartic surface dP_4 . In light of the results of the present paper – particularly the fact that any identity $\mathbf{HLog}_{\mathrm{dP}_4}$ can be obtained from $\mathbf{HLOG}_{\mathbf{y}_5}$ – we now believe that the following question should be preferred over Question 5.7 in [Pi5]:

Question.– *Can the differential identity $\mathbf{HLOG}_{\mathbf{y}_5}$ be obtained à la Gelfand-MacPherson by means of an invariant differential form $\Omega(P)$ on a real form S_5 of the spinor tenfold $\mathbb{S}_5$, representing a certain characteristic class $P \in \mathbf{H}^6(S_5, \mathbf{R})$?*

This is one of the questions suggested by our work that we find the most appealing and we plan to work on it in the near future.

9. Modularity. The web $\mathcal{W}_{\mathbf{y}_4}^{GM}$ can also be interpreted as the web defined on $\mathcal{M}_{0,5}$ by the five forgetful maps $\varphi_i : \mathcal{M}_{0,5} \rightarrow \mathcal{M}_{0,4} \simeq \mathbf{P}^1 \setminus \{0, 1, \infty\}$.

Since it is defined on a moduli space via morphisms with a modular interpretation, it is reasonable to describe $\mathcal{W}_{\mathbf{y}_4}^{GM}$ as a ‘modular web’. In [Pi5, §4.6], we proved that any del Pezzo web $\mathcal{W}_{\mathrm{dP}_4}$ can also be qualified as ‘modular web’. This naturally leads to the following question:

Question. *Does Gelfand-MacPherson’s web $\mathcal{W}_{\mathbf{y}_5}^{GM}$ admit an interpretation as a modular web? For instance, is it naturally defined on a subvariety of some moduli space of projective configurations?*

At this stage, we do not have any clear insight into a possible answer to this question.

10. Cluster nature. An interesting feature of the web $\mathcal{W}_{\mathrm{dP}_5} = \mathcal{W}_{\mathbf{y}_4}^{GM}$ is that it is a ‘cluster web’. In [Pi5, §4.7], we showed that this property also holds for any del Pezzo web $\mathcal{W}_{\mathrm{dP}_4}$. This raised the question whether this holds for $\mathcal{W}_{\mathbf{y}_4}^{GM}$ as well. This is precisely what we established in §4 above, provided one admits cluster-like structures that are more general than classical cluster structures.

8.2. An interpretation of $\mathbf{HLOG}_{\mathbf{y}_4}$ in terms of a scattering diagram? The author has long been fascinated by the functional equations of polylogarithms, among which Abel’s five-term identity ($\mathcal{Ab}$) for the dilogarithm stands out as particularly central and archetypal. In his view, there are two especially meaningful explanations for why ($\mathcal{Ab}$) holds: on the one hand, the cohomological-analytic and geometric approach developed by Gelfand and MacPherson discussed just above; on the other, the interpretation of this functional identity as a consequence of the consistency of the scattering diagram associated with the X -cluster variety of type A_2 .

Although Gelfand–MacPherson’s perspective appears to us as the most elegant – indeed, it is, in the author’s opinion, one of the most beautiful mathematical constructions he has ever encountered – the scattering diagram approach seems to offer a significantly broader scope. One particularly appealing consequence of the latter theory³⁴ is that any closed loop intersecting transversely a (possibly countable) collection of walls in a consistent scattering diagram gives rise to a dilogarithmic identity – which becomes a formal identity involving infinitely many terms if the loop meets infinitely many walls.

³⁴In the case of scattering diagrams arising from cluster algebras, this phenomenon has been thoroughly investigated by T. Nakanishi. For further details, the reader is referred to Nakanishi’s book and article [Na1] and [Na2].

The fact that the spinor tenfold $\mathbb{S}_5$, and more generally all the spaces $\mathcal{G}_r$ for $r = 4, \dots, 7^{35}$, carry a cluster-like structure of finite type invites us to dream of an interpretation of all the identities $\mathbf{HLOG}_{\mathcal{Y}_r}$ in terms of certain properties of scattering diagrams associated with the spaces $\mathcal{Y}_r$.

In [Du], Ducat constructed a finite type LPA structure on $\mathbb{S}_5$ and described an associated scattering diagram. It is natural to ask whether these objects admit versions defined on the torus quotient $\mathcal{Y}_5$. Addressing this question would require a theory of coefficients and their mutations for LPA algebras, which, to our knowledge, has not yet been developed.

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³⁵For $r = 4$ this is well-known; for $r = 5$ and $r = 6$, it has been established in [Du] and [DD], respectively. Only the case $r = 7$ remains conjectural at the time of writing.

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