CUSPIDAL REPRESENTATIONS OF QUATERNIONIC $\mathrm{GL}_n(D)$ WITH SYMPLECTIC PERIODS

by

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Abstract. — We prove a conjecture of Prasad predicting that a cuspidal representation of $\mathrm{GL}_n(D)$, for an integer $n \geqslant 1$ and a non-split quaternion algebra D over a non-Archimedean locally compact field of odd residue characteristic, has a symplectic period if and only if its Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ is non-cuspidal.

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1. Introduction

1.1. Let F be a non-Archimedean locally compact field of residue characteristic $p \neq 2$, and D be a non-split quaternion algebra of centre F. Fix a positive integer $n \geqslant 1$ and set $G = \mathrm{GL}_n(D)$. This is an inner form of $\mathrm{GL}_{2n}(F)$, which can be equipped with an involution σ whose fixed point subgroup G^σ is equal to $\mathrm{Sp}_n(D)$, the non-quasi-split inner form of the symplectic group $\mathrm{Sp}_{2n}(F)$. One is interested in the classification of the irreducible (smooth, complex) representations of G which are distinguished by G^σ, that is, which admit non-zero G^σ-invariant linear forms. For discrete series representations of G, Dipendra Prasad proposed the following conjecture, stated in terms of the local Jacquet–Langlands correspondence, which is a bijection between the discrete series of G and $\mathrm{GL}_{2n}(F)$ (see [**33**] Conjecture 7.1).

Conjecture 1.1. — (1) *There is a discrete series representation of* $\mathrm{GL}_n(D)$ *distinguished by* $\mathrm{Sp}_n(D)$ *if and only if n is odd.*

(2) *Suppose that the integer n is odd. The discrete series representations of* $\mathrm{GL}_n(D)$ *which are distinguished by* $\mathrm{Sp}_n(D)$ *are exactly the cuspidal representations of* $\mathrm{GL}_n(D)$ *whose Jacquet–Langlands transfer to* $\mathrm{GL}_{2n}(F)$ *is of the form*[(1)] $\mathrm{St}_2(\tau)$ *for some cuspidal representation τ of* $\mathrm{GL}_n(F)$.

Note that, if one replaces the groups G and G^σ by their split forms $\mathrm{GL}_{2n}(F)$ and $\mathrm{Sp}_{2n}(F)$, then [**19**] Theorem 1 implies that there is no generic (in particular, no discrete series) representation of $\mathrm{GL}_{2n}(F)$ distinguished by $\mathrm{Sp}_{2n}(F)$, whatever the parity of n.

(1) The notation $\mathrm{St}_2(\tau)$ is defined in §2.3 below.

1.2. In this article, we prove Conjecture 1.1 for cuspidal representations of G, that is:

Theorem 1.2. — *A cuspidal representation of* $\mathrm{GL}_n(D)$ *is distinguished by* $\mathrm{Sp}_n(D)$ *if and only if its Jacquet–Langlands transfer to* $\mathrm{GL}_{2n}(F)$ *is non-cuspidal.*

It then follows from well-known properties of the Jacquet–Langlands correspondence (see §2.3) that

– there is a cuspidal representation of $\mathrm{GL}_n(D)$ distinguished by $\mathrm{Sp}_n(D)$ if and only if n is odd,

– if n is odd, a cuspidal representation π of $\mathrm{GL}_n(D)$ is distinguished by $\mathrm{Sp}_n(D)$ if and only if its Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ is a discrete series representation of the form $\mathrm{St}_2(\tau)$, where τ is a cuspidal representation of $\mathrm{GL}_n(F)$ uniquely determined by π.

1.3. Let us first consider the forward direction of the theorem: any cuspidal representation of G distinguished by G^σ has a non-cuspidal Jacquet–Langlands transfer. This has been proved by Verma [**33**] in the case when F has characteristic 0 via a globalization argument. We adapt Verma's argument to the case when F has arbitrary characteristic (see Section 2).

1.4. Let us now concentrate on the converse: any cuspidal representation of G whose Jacquet–Langlands transfer is non-cuspidal is distinguished by G^σ. The strategy of our proof is, given a cuspidal representation π of G whose Jacquet–Langlands transfer is non-cuspidal, to produce a pair $(\boldsymbol{J},\boldsymbol{\lambda})$ made of a compact mod centre, open subgroup $\boldsymbol{J}$ of G and an irreducible representation $\boldsymbol{\lambda}$ of $\boldsymbol{J}$ such that:

– $\boldsymbol{\lambda}$ is distinguished by $\boldsymbol{J}\cap G^\sigma$,

– the compact induction of $\boldsymbol{\lambda}$ to G is isomorphic to π.

By a simple application of Mackey's formula, this will imply that π is distinguished by G^σ. The construction of a suitable pair $(\boldsymbol{J},\boldsymbol{\lambda})$ is based on Bushnell–Kutzko's theory of types, as we explain below.

1.5. Start with a cuspidal irreducible representation π of G. By [**10, 27**], it is compactly induced from a Bushnell–Kutzko type: this is a pair $(\boldsymbol{J},\boldsymbol{\lambda})$ with the following properties:

– the group $\boldsymbol{J}$ is open and compact mod centre, it has a unique maximal compact subgroup $\boldsymbol{J}^0$ and a unique maximal normal pro-p-subgroup $\boldsymbol{J}^1$,

– the representation $\boldsymbol{\lambda}$ of $\boldsymbol{J}$ is irreducible and factors (non-canonically) as $\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$, where $\boldsymbol{\kappa}$ is a representation of $\boldsymbol{J}$ whose restriction to $\boldsymbol{J}^1$ is irreducible and $\boldsymbol{\rho}$ is an irreducible representation of $\boldsymbol{J}$ whose restriction to $\boldsymbol{J}^1$ is trivial,

– the quotient $\boldsymbol{J}^0/\boldsymbol{J}^1$ is isomorphic to $\mathrm{GL}_m(\boldsymbol{l})$ for some integer m dividing n and some finite extension $\boldsymbol{l}$ of the residue field of F, and the restriction of $\boldsymbol{\rho}$ to $\boldsymbol{J}^0$ is the inflation of a cuspidal representation ϱ of $\mathrm{GL}_m(\boldsymbol{l})$.

Our first task is to prove that, if the Jacquet–Langlands transfer of π to $\mathrm{GL}_{2n}(F)$ is non-cuspidal, then, among all possible Bushnell–Kutzko types $(\boldsymbol{J},\boldsymbol{\lambda})$ whose compact induction to G is isomorphic to π (they form a single G-conjugacy class), there is one such that $\boldsymbol{J}$ is stable by σ and $\boldsymbol{\kappa}$ can be chosen to be distinguished by $\boldsymbol{J}\cap G^\sigma$.

1.6. Assuming this has been done, our argument is as follows:

(1) The fact that $\boldsymbol{\kappa}$ is distinguished by $\boldsymbol{J} \cap G^\sigma$, together with the decomposition

$$\mathrm{Hom}_{\boldsymbol{J}\cap G^\sigma}(\boldsymbol{\kappa}\otimes\boldsymbol{\rho},\mathbb{C}) \simeq \mathrm{Hom}_{\boldsymbol{J}\cap G^\sigma}(\boldsymbol{\kappa},\mathbb{C})\otimes\mathrm{Hom}_{\boldsymbol{J}\cap G^\sigma}(\boldsymbol{\rho},\mathbb{C}), \tag{1.1}$$

implies that $\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$ is distinguished by $\boldsymbol{J} \cap G^\sigma$ if and only if $\boldsymbol{\rho}$ is distinguished by $\boldsymbol{J} \cap G^\sigma$.

(2) The representation $\boldsymbol{\rho}$ is distinguished by $\boldsymbol{J} \cap G^\sigma$ if and only if the cuspidal representation ϱ of $\mathrm{GL}_m(\boldsymbol{l})$ is distinguished by a unitary group, or equivalently, ϱ is invariant by the non-trivial automorphism of $\boldsymbol{l}/\boldsymbol{l}_0$, where $\boldsymbol{l}_0$ is a subfield of $\boldsymbol{l}$ over which $\boldsymbol{l}$ is quadratic.

(3) The fact that the Jacquet–Langlands transfer of π is non-cuspidal implies that ϱ is invariant by $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$.

Note that (2) is reminiscent of [**33**] Section 5. See Section 6 below for more details.

1.7. It remains to prove that $\boldsymbol{J}$ and $\boldsymbol{\kappa}$ can be chosen as in §1.5. The construction of $\boldsymbol{\kappa}$ relies on the notion of simple character, which is the core of Bushnell–Kutzko's type theory. The cuspidal representation π of §1.5 contains a simple character, and the set of all simple characters contained in π form a single G-conjugacy class. We first prove that, if the Jacquet–Langlands transfer of π to $\mathrm{GL}_{2n}(F)$ is non-cuspidal, then, among all simple characters contained in π, there is a simple character θ such that $\theta\circ\sigma = \theta^{-1}$. Zou [**34**] proved a similar result for cuspidal representations of $\mathrm{GL}_n(F)$ with respect to an orthogonal involution, and we explain how to transfer it to G in an appropriate manner. (Note that, if the Jacquet–Langlands transfer of π is cuspidal, such a θ may not exist.)

Next, fix a simple character θ as above, and let $\boldsymbol{J}_\theta$ denote its normalizer in G, which is stable by σ. A standard construction (see for instance [**26, 18**]) provides us with:

– a representation $\boldsymbol{\kappa}$ of $\boldsymbol{J}_\theta$ such that the contragredient of $\boldsymbol{\kappa}\circ\sigma$ is isomorphic to $\boldsymbol{\kappa}$,

– a quadratic character χ of $\boldsymbol{J}_\theta \cap G^\sigma$ such that the vector space $\mathrm{Hom}_{\boldsymbol{J}\cap G^\sigma}(\boldsymbol{\kappa},\chi)$ is non-zero.

To prove that the character χ is trivial, we show that, if χ were non-trivial, one could construct a G^σ-distinguished cuspidal representation of G with cuspidal transfer to $\mathrm{GL}_{2n}(F)$, thus contradicting the first part of Theorem 1.2 (see §1.3). Together with the argument of §1.6, this finishes the proof of Theorem 1.2.

1.8. Now let us go back to §1.2, assuming that n is odd. We defined a map

$$\pi \mapsto \tau \tag{1.2}$$

from cuspidal representations of $\mathrm{GL}_n(D)$ whose Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ is non-cuspidal to cuspidal representations of $\mathrm{GL}_n(F)$, and this map is a bijection (see Remark 2.1 below). As suggested by Prasad, the inverse of this map can be thought of as a 'non-abelian' base change, denoted $\mathbf{b}_{D/F}$, from cuspidal representations of $\mathrm{GL}_n(F)$ to those of $\mathrm{GL}_n(D)$, whose image is made of those cuspidal representations which are distinguished by $\mathrm{Sp}_n(D)$. For instance, if $n = 1$, the map $\mathbf{b}_{D/F}$ is just $\chi \mapsto \chi\circ\mathrm{Nrd}$, where Nrd is the reduced norm from $D^\times$ to $F^\times$, and χ ranges over the set of all characters of $F^\times$.

A type theoretic, explicit description of $\mathbf{b}_{D/F}$ can be extracted from [**7, 28, 12**], at least up to inertia, that is: given a cuspidal representation τ of $\mathrm{GL}_n(F)$, described as the compact induction of a Bushnell–Kutzko type, one has an explicit description of the type of the inertial class of the cuspidal representation $\mathbf{b}_{D/F}(\tau)$ in terms of the type of τ.

The case of cuspidal representations of depth 0 has been considered in [**33**] Section 5. The explicit description of $\mathbf{b}_{D/F}$ provided by [**33**] Proposition 5.1, Remark 5.2 is somewhat incomplete (see Remarks 7.1 and 7.4 below). In Section 7, thanks to [**29, 30, 8**], we provide a full description of $\mathbf{b}_{D/F}$ for cuspidal representations of depth 0.

1.9. Finally, let us comment on the assumption "$p \neq 2$". The only places where we use this assumption are: Proposition 4.1, Lemma 5.2 and Proposition 6.2. Proposition 4.1 is the main difficulty: this proposition might not hold when $p = 2$.

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2.

2.1. Let F be a non-Archimedean locally compact field of residue characteristic $p \neq 2$. We will denote by $\mathcal{O}_F$ its ring of integers, by $\mathfrak{p}_F$ the maximal ideal of $\mathcal{O}_F$, by $\boldsymbol{k}_F$ its residue field and by $|\cdot|_F$ the absolute value on F sending any uniformizer to the inverse of the cardinality of $\boldsymbol{k}_F$.

Let D be a non-split quaternion algebra with centre F. Let us denote by $\mathcal{O}_D$ its ring of integers, by $\mathfrak{p}_D$ the maximal ideal of $\mathcal{O}_D$ and by $\boldsymbol{k}_D$ its residue field. The algebra D is equipped with the anti-involution

$$x \mapsto \overline{x} = \mathrm{trd}_{D/F}(x) - x \tag{2.1}$$

where $\mathrm{trd}_{D/F}$ denotes the reduced trace of D over F.

Given an $a \in A = \mathbf{M}_n(D)$, for an $n \geqslant 1$, we write ${}^{\mathsf{t}}a$ for the transpose of a with respect to the antidiagonal and $\overline{a}$ for the matrix obtained by applying (2.1) to each entry of a. We define an anti-involution

$$a \mapsto a^* = {}^{\mathsf{t}}\overline{a} \tag{2.2}$$

on the F-algebra A. The group $G = \mathrm{GL}_n(D)$ is an inner form of $\mathrm{GL}_{2n}(F)$, equipped with the involution $\sigma : x \mapsto (x^*)^{-1}$.

The subgroup G^σ made of all elements of G that are fixed by σ is denoted $\mathrm{Sp}_n(D)$. This is an inner form of the symplectic group $\mathrm{Sp}_{2n}(F)$.

We also write $\mathrm{Nrd}_{A/F}$ for the reduced norm on A, and $G' = \mathrm{SL}_n(D)$ for the kernel of the restriction to G of the reduced norm.

2.2. By *representation* of a closed subgroup H of G, we mean a smooth, complex representation of H. By *character* of H, we mean a group homomorphism from H to $\mathbb{C}^\times$ with open kernel. If π is a representation of H, we denote by $\pi^\vee$ its contragredient and by π^σ the representation $\pi \circ \sigma$ of $\sigma(H)$. If χ is a character of H, we denote by $\pi\chi$ the representation $h \mapsto \chi(h)\pi(h)$. If K is a subgroup of H, the representation π is said to be *distinguished by* K if its vector space V carries a non-zero linear form Λ such that $\Lambda(\pi(x)v) = \Lambda(v)$ for all $x \in K$ and all $v \in V$.

2.3. The Jacquet–Langlands correspondence [**22, 11, 1**] is a bijection between the discrete series of G and that of $\mathrm{GL}_{2n}(F)$. If π is a discrete series representation of G, we will write ${}^{\mathrm{JL}}\pi$ for its Jacquet–Langlands transfer. According to [**32, 3**], there are a uniquely determined integer r dividing $2n$ and a cuspidal representation τ of $\mathrm{GL}_{2n/r}(F)$, uniquely determined upto isomorphism, such that ${}^{\mathrm{JL}}\pi$ is the unique irreducible quotient of the (normalized) parabolically induced representation

$$\mathrm{Ind}_P^G \left(\tau\nu^{(1-r)/2} \otimes \tau\nu^{(3-r)/2} \otimes \cdots \otimes \tau\nu^{(r-1)/2}\right)$$

where, for any real number a, we denote by ν^a the character $g \mapsto |\det(g)|_F^a$ and by P the parabolic subgroup of G generated by the standard Levi subgroup $\mathrm{GL}_{2n/r}(F) \times \cdots \times \mathrm{GL}_{2n/r}(F)$ together with upper triangular unipotent matrices. This irreducible quotient is denoted $\mathrm{St}_r(\tau)$.

Remark 2.1. — When π is a cuspidal representation of G, the integer r associated with it (it is denoted $s(\pi)$ in [**26**] §3.5) has the following properties:

- it divides the reduced degree of D ([**26**] Remark 3.15(1)),
- it is prime to n ([**26**] Remark 3.15(2)).

Since r divides 2 and is prime to n, we immediately deduce that, if the cuspidal representation π has a non-cuspidal Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$, then $r = 2$ and n is odd. Conversely, if n is odd, and if τ is any cuspidal representation of $\mathrm{GL}_n(F)$, the unique discrete series representation of G whose Jacquet–Langlands transfer is $\mathrm{St}_2(\tau)$ is cuspidal ([**26**] Remark 3.15(3)).

2.4. In this paragraph, we prove:

Theorem 2.2. — *Let π be a cuspidal (irreducible) representation of G. If π is distinguished by G^σ, then ${}^{\mathrm{JL}}\pi$ is non-cuspidal.*

Proof. — In the case when F has characteristic 0, this is a consequence of a theorem of Verma ([**33**] Theorem 1.2). We give a proof which is valid in any characteristic.

Our argument is inspired by [**33**] Section 6, in particular the proof of Theorem 6.3.

Let π be a cuspidal irreducible representation of G. Assume that π is distinguished by G^σ and ${}^{\mathrm{JL}}\pi$ is cuspidal. Let Z denote the centre of G, which is isomorphic to $F^\times$, and $G' = \mathrm{SL}_n(D)$. By [**16**] Theorem 4.2, the restriction of π to the normal, cocompact, closed subgroup $G_1 = ZG'$ is semisimple of finite length. Let π_1 be an irreducible summand of this restriction. The centre Z acts on it through ω, the central character of π. The restriction of π_1 to G', denoted π', is thus irreducible.

Now let k be a global field with a finite place w dividing p such that k_w, the completion of k at w, is isomorphic to F. Thus k is a finite extension of $\mathbb{Q}$ when F has characteristic 0, and the field of rational functions over a smooth irreducible projective curve defined over a finite field of characteristic p when F has characteristic p.

Let $\mathbb{D}$ be a quaternion algebra over k such that $\mathbb{D}\otimes_k k_w$ is non-split (it is thus isomorphic to D). Let $\mathbb{G}'$ be the group $\mathrm{SL}_n(\mathbb{D})$. It is an inner form of SL_{2n} over k which contains $\mathbb{H}=\mathrm{Sp}_n(\mathbb{D})$. Note that the connected component of the centre of $\mathbb{G}'$ is trivial and that $\mathbb{H}$ is a closed algebraic k-subgroup of $\mathbb{G}'$ with no non-trivial character. Let V denote the k-vector space made of all matrices $a\in\mathbf{M}_n(\mathbb{D})$ such that $a^*=a$ and consider the algebraic representation of $\mathbb{G}'$ on V defined by $(g,a)\mapsto\sigma(g)ag^{-1}$. This representation is semisimple (it is even irreducible) and the $\mathbb{G}'$-stabilizer of the identity matrix (on V) is $\mathbb{H}$.

We now apply either [**21**] Theorem 4.1 (if F has characteristic 0) or [**13**] Theorem 1.3 (if F has characteristic p): there exists a cuspidal automorphic representation Π' of $\mathbb{G}'(\mathbb{A})$ with a non-zero $\mathbb{H}(\mathbb{A})$-period and such that the local component of Π' at w is isomorphic to π'. (Here $\mathbb{A}$ denotes the ring of adèles of k.)

By [**17**] Theorem 5.2.2, the representation Π' occurs as a subrepresentation in the restriction to $\mathbb{G}'(\mathbb{A})$ of a cuspidal automorphic representation Π of $\mathbb{G}(\mathbb{A})$. Since $\mathbb{G}'$ contains $\mathbb{H}$, the representation Π has a non-zero $\mathbb{H}(\mathbb{A})$-period. It follows that:

(1) for any finite place v, the local component Π_v of Π at v is distinguished by $\mathbb{H}(k_v)$,
(2) the restriction to $\mathbb{H}(k_w)$ of the local component Π_w of Π at w contains π'.

More precisely, let us prove that Π_w is isomorphic to $\pi\otimes(\chi\circ\mathrm{Nrd}_{A/F})$ for some character χ of the group $F^\times$. Arguing as at the beginning of the proof of the theorem, the restriction of Π_w to G_1 is semisimple of finite length and contains an irreducible summand π_2 whose restriction to G' is isomorphic to π'. If μ denotes the central character of Π_w, we thus have $\pi_2(zx)=\mu(z)\pi'(x)$ for all $z\in Z$ and $x\in G'$. Similarly, we have $\pi_1(zx)=\omega(z)\pi'(x)$ for all $z\in Z$ and $x\in G'$. By twisting π by a character of G, we may assume that $\omega=\mu$. The representation Π_w is unitary as a local component of the unitary representation Π. The character μ is thus unitary, which implies that π is unitary. By [**17**] Proposition 2.2.2, we get the expected result.

Let Θ be a character of $\mathbb{A}^\times/k^\times$ whose local component at w is χ. By twisting Π by Θ^{-1} composed with the reduced norm from $\mathbb{G}(\mathbb{A})$ to $\mathbb{A}^\times$, we may and will assume that Π_w is isomorphic to π.

Now let ${}^{\mathrm{JL}}\Pi$ be the Jacquet–Langlands transfer of Π to $\mathrm{GL}_{2n}(\mathbb{A})$ (see [**2**] Theorem 5.1 and [**4**] Theorem 3.2). This is an automorphic representation in the discrete spectrum of $\mathrm{GL}_{2n}(\mathbb{A})$, with the following properties:

(3) its local component at w is cuspidal, isomorphic to ${}^{\mathrm{JL}}\pi$,
(4) for any finite place v such that $\mathbb{D}\otimes_k k_v$ is split, the local components of ${}^{\mathrm{JL}}\Pi$ and Π at v are isomorphic.

It follows from (3) that ${}^{\mathrm{JL}}\Pi$ is cuspidal, thus generic. Therefore:

(5) for any finite place v, the local component of ${}^{\mathrm{JL}}\Pi$ at v is generic.

It follows from (1), (4) and (5) that, if v is a finite place of k such that $\mathbb{D}\otimes_k k_v$ is split, Π_v is an irreducible representation of $\mathrm{GL}_{2n}(k_v)$ which is generic and distinguished by $\mathbb{H}(k_v)\simeq\mathrm{Sp}_{2n}(k_v)$. This contradicts [**19**] Theorem 1. □

Remark 2.3. — Verma ([**33**] Theorem 1.1) proves that, when F has characteristic 0, the vector space $\mathrm{Hom}_{G^\sigma}(\pi, \mathbb{C})$ has dimension $\leqslant 1$ for any irreducible representation π of G, but we will not use this fact.

The remaining sections are devoted to the proof of the converse of Theorem 2.2: any cuspidal representation of G with non-cuspidal transfer to $\mathrm{GL}_{2n}(F)$ is distinguished by G^σ.

3.

In this section, we introduce the type theoretical material which we will need in Sections 4–8. Let $\varDelta$ be *any finite dimensional central division F-algebra* (this extra generality will be useful in §5.3.) Let A be the central simple F-algebra $\mathsf{M}_n(\varDelta)$ of $n \times n$ matrices with coefficients in $\varDelta$ for some integer $n \geqslant 1$, and $G = A^\times = \mathrm{GL}_n(\varDelta)$. Let us fix a character

$$\psi : F \to \mathbb{C}^\times \tag{3.1}$$

trivial on $\mathfrak{p}_F$ but not on $\mathcal{O}_F$. For the definitions and main results stated in this section, we refer the reader to [**10, 9**] (see also [**26, 18**]).

3.1. A *simple stratum* in A is a pair $[\mathfrak{a}, \beta]$ made of a hereditary $\mathcal{O}_F$-order $\mathfrak{a}$ of A and an element $\beta \in A$ such that the F-algebra $E = F[\beta]$ is a field, and the multiplicative group $E^\times$ normalizes $\mathfrak{a}$ (plus an extra technical condition on β which it is not necessary to recall here). The centralizer B of E in A is a central simple E-algebra, and $\mathfrak{b} = \mathfrak{a} \cap B$ is a hereditary $\mathcal{O}_E$-order in B.

Associated to a simple stratum $[\mathfrak{a}, \beta]$, there are a pro-p-subgroup $H^1(\mathfrak{a}, \beta)$ of G and a non-empty finite set $\mathcal{C}(\mathfrak{a}, \beta)$ of characters of $H^1(\mathfrak{a}, \beta)$ called *simple characters*, depending on ψ.

Remark 3.1. — This includes the case where $\beta = 0$. The simple stratum $[\mathfrak{a}, 0]$ is then said to be *null*. One has $H^1(\mathfrak{a}, 0) = 1 + \mathfrak{p}_\mathfrak{a}$ (where $\mathfrak{p}_\mathfrak{a}$ is the Jacobson radical of $\mathfrak{a}$) and the set $\mathcal{C}(\mathfrak{a}, 0)$ is reduced to the trivial character of $1 + \mathfrak{p}_\mathfrak{a}$.

When the order $\mathfrak{b}$ is maximal in B, the simple stratum $[\mathfrak{a}, \beta]$ is said to be *maximal*, and the simple characters in $\mathcal{C}(\mathfrak{a}, \beta)$ are said to be *maximal*. If this is the case, and if $[\mathfrak{a}', \beta']$ is another simple stratum in A such that $\mathcal{C}(\mathfrak{a}, \beta) \cap \mathcal{C}(\mathfrak{a}', \beta')$ is non-empty, then

$$\mathcal{C}(\mathfrak{a}', \beta') = \mathcal{C}(\mathfrak{a}, \beta), \quad \mathfrak{a}' = \mathfrak{a}, \quad [F[\beta'] : F] = [F[\beta] : F], \tag{3.2}$$

and the simple stratum $[\mathfrak{a}', \beta']$ is maximal ([**26**] Proposition 3.6).

3.2. Let $\varDelta'$ be a finite dimensional central division F-algebra and $[\mathfrak{a}', \beta']$ be a simple stratum in $\mathsf{M}_{n'}(\varDelta')$ for some $n' \geqslant 1$. Assume that there is a morphism of F-algebras $\varphi : F[\beta] \to F[\beta']$ such that $\varphi(\beta) = \beta'$. Then there is a natural bijection from $\mathcal{C}(\mathfrak{a}, \beta)$ to $\mathcal{C}(\mathfrak{a}', \beta')$ called transfer.

3.3. Let $\mathcal{C}$ denote the union of the sets $\mathcal{C}(\mathfrak{a}', \beta')$ for all maximal simple strata $[\mathfrak{a}', \beta']$ of $\mathbf{M}_{n'}(\varDelta')$, for all $n' \geqslant 1$ and all finite dimensional central division F-algebras $\varDelta'$. Any two maximal simple characters $\theta_1, \theta_2 \in \mathcal{C}$ are said to be *endo-equivalent* if they are transfers of each other, that is, if there exist

– maximal simple strata $[\mathfrak{a}_1, \beta_1]$ and $[\mathfrak{a}_2, \beta_2]$,
– a morphism of F-algebras $\varphi : F[\beta_1] \to F[\beta_2]$ such that $\varphi(\beta_1) = \beta_2$,

such that $\theta_i \in \mathcal{C}(\mathfrak{a}_i, \beta_i)$ for $i = 1, 2$, and θ_2 is the image of θ_1 by the transfer map from $\mathcal{C}(\mathfrak{a}_1, \beta_1)$ to $\mathcal{C}(\mathfrak{a}_2, \beta_2)$. This defines an equivalence relation on $\mathcal{C}$, called *endo-equivalence*. An equivalence class for this equivalence relation is called an *endoclass*.

The *degree* of an endoclass $\boldsymbol{\Theta}$ is the degree of $F[\beta']$ over F, for any choice of $[\mathfrak{a}', \beta']$ such that $\mathcal{C}(\mathfrak{a}', \beta') \cap \boldsymbol{\Theta}$ is non-empty. (It is well defined thanks to (3.2).)

3.4. Let $\mathcal{C}(G)$ be the union of the sets $\mathcal{C}(\mathfrak{a}, \beta)$ for all maximal simple strata $[\mathfrak{a}, \beta]$ of A. Any two maximal simple characters $\theta_1, \theta_2 \in \mathcal{C}(G)$ are endo-equivalent if and only if they are G-conjugate.

Given a cuspidal representation π of G, there exists a maximal simple character $\theta \in \mathcal{C}(G)$ contained in π, and any two maximal simple characters contained in π are G-conjugate. The maximal simple characters contained in π thus all belong to the same endoclass $\boldsymbol{\Theta}$, called the endoclass of π. Conversely, any maximal simple character $\theta \in \mathcal{C}(G)$ of endoclass $\boldsymbol{\Theta}$ is contained in π ([**26**] Corollaire 3.23).

3.5. Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a simple character with respect to a maximal simple stratum $[\mathfrak{a}, \beta]$ in A as in §3.1. There are a divisor m of n and a finite dimensional central division E-algebra $\varGamma$ such that B is isomorphic to $\mathbf{M}_m(\varGamma)$. Let $\boldsymbol{J}_\theta$ be the normalizer of θ in G. Then

(1) the group $\boldsymbol{J}_\theta$ has a unique maximal compact subgroup $\boldsymbol{J}^0 = \boldsymbol{J}^0_\theta$ and a unique maximal normal pro-p-subgroup $\boldsymbol{J}^1 = \boldsymbol{J}^1_\theta$,
(2) the group $\boldsymbol{J}_\theta \cap B^\times$ is the normalizer of $\mathfrak{b}$ in $B^\times$ and $\boldsymbol{J}^0 \cap B^\times = \mathfrak{b}^\times$, $\boldsymbol{J}^1 \cap B^\times = 1 + \mathfrak{p}_\mathfrak{b}$,
(3) one has $\boldsymbol{J}_\theta = (\boldsymbol{J}_\theta \cap B^\times)\boldsymbol{J}^0$ and $\boldsymbol{J}^0 = (\boldsymbol{J}^0 \cap B^\times)\boldsymbol{J}^1$.

Since $\mathfrak{b}$ is a maximal order in B, it follows from (2) and (3) that there is a group isomorphism

$$\boldsymbol{J}^0/\boldsymbol{J}^1 \simeq \mathrm{GL}_m(\boldsymbol{l}) \tag{3.3}$$

where $\boldsymbol{l}$ is the residue field of $\varGamma$, and an element $\varpi \in B^\times$ normalizing $\mathfrak{b}$ such that $\boldsymbol{J}_\theta$ is generated by $\boldsymbol{J}^0$ and ϖ.

There is an irreducible representation $\eta = \eta_\theta$ of $\boldsymbol{J}^1$ whose restriction to $H^1(\mathfrak{a}, \beta)$ contains θ. It is unique up to isomorphism, and it is called the *Heisenberg representation* associated with θ. It extends to the group $\boldsymbol{J}_\theta$ (thus its normalizer in G is equal to $\boldsymbol{J}_\theta$).

If $\boldsymbol{\kappa}$ is a representation of $\boldsymbol{J}_\theta$ extending η, any other extension of η to $\boldsymbol{J}_\theta$ has the form $\boldsymbol{\kappa\xi}$ for a unique character $\boldsymbol{\xi}$ of $\boldsymbol{J}_\theta$ trivial on $\boldsymbol{J}^1$. More generally, the map

$$\boldsymbol{\tau} \mapsto \boldsymbol{\kappa} \otimes \boldsymbol{\tau} \tag{3.4}$$

is a bijection between isomorphism classes of irreducible representations of $\boldsymbol{J}_\theta$ trivial on $\boldsymbol{J}^1$ and isomorphism classes of irreducible representations of $\boldsymbol{J}_\theta$ whose restriction to $\boldsymbol{J}^1$ contains η.

4.

In this section, we go back to the group $G = \mathrm{GL}_n(D)$ of §2.1 and fix a cuspidal representation π of G with non-cuspidal transfer to $\mathrm{GL}_{2n}(F)$. In particular, as explained in Remark 2.1, the integer n is odd. The main result of this section is the following proposition.

Proposition 4.1. — *There are a maximal simple stratum $[\mathfrak{a}, \beta]$ in A and a maximal simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ contained in π such that*

(1) *the group $H^1(\mathfrak{a}, \beta)$ is stable by σ and $\theta \circ \sigma = \theta^{-1}$,*
(2) *the order $\mathfrak{a}$ is stable by $*$ and β is invariant by $*$.*

4.1. Let $\boldsymbol{\Theta}$ denote the endoclass of π. Since π contains any maximal simple character of $\mathcal{C}(G)$ of endoclass $\boldsymbol{\Theta}$, it suffices to prove the existence of a maximal simple stratum $[\mathfrak{a}, \beta]$ in A and a character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ of endoclass $\boldsymbol{\Theta}$ satisfying Conditions (1) and (2) of Proposition 4.1.

Since the Jacquet–Langlands transfer of π is non-cuspidal, it follows from §2.3 that this transfer is of the form $\mathrm{St}_2(\tau)$ for a cuspidal irreducible representation τ of $\mathrm{GL}_n(F)$. By Dotto [**12**], the representations π and τ have the same endoclass. It follows that the degree of $\boldsymbol{\Theta}$ divides n.

4.2. Let d denote the degree of $\boldsymbol{\Theta}$. Thanks to §4.1, it is an odd integer dividing n.

Let σ_0 be the involution $x \mapsto {}^{\natural}x^{-1}$ on $\mathrm{GL}_d(F)$ where $\natural$ denotes the transpose with respect to the antidiagonal. The fixed points of $\mathrm{GL}_d(F)$ by σ_0 is a split orthogonal group.

By [**34**] Theorem 4.1, there are a maximal simple stratum $[\mathfrak{a}_0, \beta]$ in $\mathbf{M}_d(F)$ and a maximal simple character $\theta_0 \in \mathcal{C}(\mathfrak{a}_0, \beta)$ of endoclass $\boldsymbol{\Theta}$ such that

– the group $H^1(\mathfrak{a}_0, \beta)$ is stable by σ_0 and $\theta_0 \circ \sigma_0 = \theta_0^{-1}$,
– the order $\mathfrak{a}_0$ is stable by $\natural$ and β is invariant by $\natural$.

Write E for the sub-F-algebra $F[\beta] \subseteq \mathbf{M}_d(F)$. It is made of $\natural$-invariant matrices. Its centralizer in $\mathbf{M}_d(F)$ is equal to E itself. The intersection $\mathfrak{a}_0 \cap E$ is $\mathcal{O}_E$, the ring of integers of E.

4.3. Let us write $n = md$. We identify $\mathbf{M}_n(F)$ with $\mathbf{M}_m(\mathbf{M}_d(F))$ and E with its diagonal image in $\mathbf{M}_n(F)$. The centralizer of E in $\mathbf{M}_n(F)$ is thus $\mathbf{M}_m(E)$.

Now consider $\mathbf{M}_n(F)$ as a sub-F-algebra of A. The centralizer B of E in A is equal to $\mathbf{M}_m(C)$ where C is an E-algebra isomorphic to $E \otimes_F D$. Since the degree d of E over F is odd, C is a non-split quaternion E-algebra. Denote by $*_B$ the anti-involution on B analogous to (2.2).

Proposition 4.2. — *The restriction of $*$ to B is equal to $*_B$.*

Proof. — It suffices to treat the case where $m = 1$. We will thus assume that $m = 1$, in which case we have $B = C$. We thus have to prove that

$$c^* = \mathrm{trd}_{C/E}(c) - c \tag{4.1}$$

for all $c \in C \subseteq \mathbf{M}_d(D)$. Let us identify $\mathbf{M}_d(D)$ with $\mathbf{M}_d(F) \otimes_F D$. Then $(a \otimes x)^* = {}^{\mathsf{t}}a \otimes \overline{x}$ for all $a \in \mathbf{M}_n(F)$ and $x \in D$, and C identifies with $E \otimes_F D$. Thus (4.1) is equivalent to

$$e \otimes \overline{x} = \mathrm{trd}_{C/E}(e \otimes x) - e \otimes x \tag{4.2}$$

for all $e \in E$ and $x \in D$. Thanks to (2.1), we are thus reduced to proving that

$$\mathrm{trd}_{C/E}(x) = \mathrm{trd}_{D/F}(x) \tag{4.3}$$

for all $x \in D$, where the F-algebra D is embedded in C via $x \mapsto 1 \otimes_F x$.

Let L be a quadratic unramified extension of F. Since the degree of E over F is odd, $E \otimes_F L$ is a field, denoted EL. The reduced trace is invariant by extension of scalars (see [**5**] §17.3, Proposition 4). Thus $\mathrm{trd}_{D/F}(x)$ is the trace of x in $D \otimes_F EL \simeq \mathsf{M}_2(EL)$. (By the Skolem-Noether theorem, the computation of this trace does not depend on the choice of the isomorphism.) Similarly, $\mathrm{trd}_{C/E}(x)$ is the trace of x in $C \otimes_E EL \simeq \mathsf{M}_2(EL)$. The proposition is proven. □

Let $\mathfrak{b}$ denote the standard maximal order $\mathsf{M}_m(\mathcal{O}_C)$ in B. Then $\mathfrak{b}^\times$ is a maximal open compact subgroup of $B^\times$ which is stable by σ. Let $\mathfrak{a}$ denote the unique $\mathcal{O}_F$-order in A normalized by $E^\times$ such that $\mathfrak{a} \cap B = \mathfrak{b}$ (see [**24**] Lemme 1.6). We thus obtain a maximal simple stratum $[\mathfrak{a}, \beta]$ in A where $\mathfrak{a}$ is stable by $*$ and $\beta^* = \beta$, and E is made of $*$-invariant matrices.

Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be the transfer of θ_0. We are going to prove that the group $H^1(\mathfrak{a}, \beta)$ is stable by σ and $\theta \circ \sigma = \theta^{-1}$, which will finish the proof of Proposition 4.1. For this, set $\theta^* = \theta^{-1} \circ \sigma$. This is a character of $\sigma(H^1(\mathfrak{a}, \beta))$. We thus have to prove that $\theta^* = \theta$.

Let ϑ_0 be an arbitrary character of $\mathcal{C}(\mathfrak{a}_0, \beta)$ and let ϑ be its transfer to $\mathcal{C}(\mathfrak{a}, \beta)$. Let us define the characters ${}^\natural\vartheta_0 = \vartheta_0^{-1} \circ \sigma_0$ and $\vartheta^* = \vartheta^{-1} \circ \sigma$. By Lemma 8.1 (which we will prove in a separate section since its proof requires techniques which are not used anywhere else in the paper), we have

$$\vartheta^* \in \mathcal{C}(\mathfrak{a}^*, \beta^*), \quad {}^\natural\vartheta_0 \in \mathcal{C}({}^\natural\mathfrak{a}_0, {}^\natural\beta).$$

On the one hand, by [**31**] Proposition 6.3, the transfer of ${}^\natural\vartheta_0 \in \mathcal{C}({}^\natural\mathfrak{a}_0, {}^\natural\beta)$ to $\mathcal{C}(\mathfrak{a}^*, \beta^*)$ is equal to ϑ^*. On the other hand, we have $\mathcal{C}(\mathfrak{a}^*, \beta^*) = \mathcal{C}(\mathfrak{a}, \beta)$ since $\mathfrak{a}$ is stable by $*$ and $\beta^* = \beta$, and likewise $\mathcal{C}({}^\natural\mathfrak{a}_0, {}^\natural\beta) = \mathcal{C}(\mathfrak{a}_0, \beta)$. Now choose $\vartheta_0 = \theta_0$. Since ${}^\natural\theta_0 = \theta_0$, we deduce that $\theta^* = \theta$.

5.

In this section, we focus on the maximal simple stratum $[\mathfrak{a}, \beta]$ and the maximal simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ constructed in Section 4, forgetting temporarily the cuspidal representation π. We thus have $\mathfrak{a}^* = \mathfrak{a}$, $\beta^* = \beta$ and $\theta^{-1} \circ \sigma = \theta$. Recall that the centralizer B of E in A is equal to $\mathsf{M}_m(C)$ where $m[E:F] = n$ and C is a quaternion E-algebra isomorphic to $E \otimes_F D$, and $\mathfrak{b}$ is the standard maximal order $\mathsf{M}_m(\mathcal{O}_C)$ in B. Let $\boldsymbol{\Theta}$ denote the endoclass of θ.

5.1. Let $\boldsymbol{J}_\theta$ be the normalizer of θ in G. According to §3.5, it has a unique maximal compact subgroup $\boldsymbol{J}^0 = \boldsymbol{J}^0_\theta$ and a unique maximal normal pro-p-subgroup $\boldsymbol{J}^1 = \boldsymbol{J}^1_\theta$. One has the identity $\boldsymbol{J}_\theta = C^\times \boldsymbol{J}^0$, where $C^\times$ is diagonally embedded in $\mathrm{GL}_m(C) = B^\times \subseteq G$, and a group isomorphism

$$\boldsymbol{J}^0/\boldsymbol{J}^1 \simeq \mathrm{GL}_m(\boldsymbol{l}) \tag{5.1}$$

where $\boldsymbol{l}$ is the residue field of C, coming from the identities $\boldsymbol{J}^0 = (\boldsymbol{J}^0 \cap B^\times)\boldsymbol{J}^1$, $\boldsymbol{J}^0 \cap B^\times = \mathfrak{b}^\times$ and $\boldsymbol{J}^1 \cap B^\times = 1 + \mathfrak{p}_\mathfrak{b}$.

5.2. Let η denote the Heisenberg representation associated with θ and $\boldsymbol{\kappa}$ be a representation of $\boldsymbol{J}_\theta$ extending η. Let ϱ be a cuspidal irreducible representation of $\boldsymbol{J}^0/\boldsymbol{J}^1$. Its inflation to $\boldsymbol{J}^0$ will still be denoted by ϱ. The normalizer $\boldsymbol{J}$ of ϱ in $\boldsymbol{J}_\theta$ satisfies

$$E^\times \boldsymbol{J}^0 \subseteq \boldsymbol{J} \subseteq \boldsymbol{J}_\theta.$$

Since $E^\times \boldsymbol{J}^0$ has index 2 in $\boldsymbol{J}_\theta = C^\times \boldsymbol{J}^0$, there are only two possible values for $\boldsymbol{J}$, namely $E^\times \boldsymbol{J}^0$ and $\boldsymbol{J}_\theta$. More precisely, $\boldsymbol{J}_\theta$ is generated by $\boldsymbol{J}^0$ and a uniformizer ϖ of C, and the action of ϖ on $\boldsymbol{J}^0$ by conjugacy identifies through (5.1) with the action on $\mathrm{GL}_m(\boldsymbol{l})$ of the generator of $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$, where $\boldsymbol{l}_0$ is the residue field of E. It follows that $\boldsymbol{J} = \boldsymbol{J}_\theta$ if and only if ϱ is $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$-stable. For the following three assertions, see for instance [**26**] 3.5.

Let $\boldsymbol{\rho}$ be a representation of $\boldsymbol{J}$ extending ϱ. Then the representation of G compactly induced from $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$ is irreducible and cuspidal, of endoclass $\boldsymbol{\Theta}$.

Conversely, any cuspidal representation of G of endoclass $\boldsymbol{\Theta}$ is obtained this way, for a suitable choice of ϱ and of an extension $\boldsymbol{\rho}$ to $\boldsymbol{J}$.

Two pairs $(\boldsymbol{J}, \boldsymbol{\kappa} \otimes \boldsymbol{\rho})$ and $(\boldsymbol{J}', \boldsymbol{\kappa} \otimes \boldsymbol{\rho}')$ constructed as above give rise to the same cuspidal representation of G if and only if they are $\boldsymbol{J}_\theta$-conjugate, that is, if and only if $\boldsymbol{J}' = \boldsymbol{J}$ and $\boldsymbol{\rho}'$ is $\boldsymbol{J}_\theta$-conjugate to $\boldsymbol{\rho}$.

The following theorem will be crucial in our proof of Theorem 1.2.

Theorem 5.1. — *The Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ of the cuspidal representation of G compactly induced from $(\boldsymbol{J}, \boldsymbol{\kappa} \otimes \boldsymbol{\rho})$ is cuspidal if and only if $\boldsymbol{J} = E^\times \boldsymbol{J}^0$.*

Proof. — The Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ of the cuspidal representation compactly induced from $(\boldsymbol{J}, \boldsymbol{\kappa} \otimes \boldsymbol{\rho})$ is cuspidal if and only if the integer r associated to it (in §2.3) is 1. By [**26**] Remarque 3.15(1) (which is based on [**7**]), this integer is equal to the order of the stabilizer of ϱ in $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$, that is, to the index of $E^\times \boldsymbol{J}^0$ in $\boldsymbol{J}$. □

5.3. Let us prove that there exists a representation $\boldsymbol{\kappa}$ of $\boldsymbol{J}_\theta$ extending η such that $\boldsymbol{\kappa}^{\sigma\vee}$ is isomorphic to $\boldsymbol{\kappa}$. As in [**26**] Lemme 3.28, we prove it in a more general context (see Section 3).

Lemma 5.2. — *Let Δ be a finite dimensional central division F-algebra, let τ be a continuous automorphism of $\mathrm{GL}_r(\Delta)$ for some integer $r \geqslant 1$, let ϑ be a maximal simple character of $\mathrm{GL}_r(\Delta)$ such that $\vartheta \circ \tau = \vartheta^{-1}$, let $\boldsymbol{J}_\vartheta$ its normalizer in $\mathrm{GL}_r(\Delta)$ and η be its Heisenberg representation.*

(1) *The representation $\eta^{\tau\vee}$ is isomorphic to η.*

(2) *For any representation $\boldsymbol{\kappa}$ of $\boldsymbol{J}_\vartheta$ extending η, there exists a unique character $\boldsymbol{\xi}$ of $\boldsymbol{J}_\vartheta$ trivial on $\boldsymbol{J}^1$ such that $\boldsymbol{\kappa}^{\tau\vee}$ is isomorphic to $\boldsymbol{\kappa}\boldsymbol{\xi}$.*

(3) *Assume that the order of τ is finite and prime to p. There exists a representation $\boldsymbol{\kappa}$ of $\boldsymbol{J}_\vartheta$ extending η such that $\boldsymbol{\kappa}^{\tau\vee}$ is isomorphic to $\boldsymbol{\kappa}$.*

Proof. — The first two assertions are given by [**26**] Lemme 3.28. For the third one, note that

$$\mathrm{val}_F \circ \mathrm{Nrd} \circ \tau = \epsilon(\tau) \cdot \mathrm{val}_F \circ \mathrm{Nrd}$$

where val_F is any valuation on F, Nrd is the reduced norm on $\mathbf{M}_r(\Delta)$ and $\epsilon(\tau)$ is a sign uniquely determined by τ. Indeed, the left hand side is a morphism from $\mathrm{GL}_r(\Delta)$ to $\mathbb{Z}$. As τ is continuous, it stabilizes the kernel of $\mathrm{val}_F \circ \mathrm{Nrd}$, which is generated by compact subgroups. The left hand side thus factors through $\mathrm{val}_F \circ \mathrm{Nrd}$, and the surjective morphisms from $\mathbb{Z}$ to $\mathbb{Z}$ are the identity and $x \mapsto -x$.

If $\epsilon(\tau) = 1$, the result is given by [**26**] Lemme 3.28. (Note that, it this case, the assumption on the order of τ is unnecessary.) We thus assume that $\epsilon(\tau) = -1$. Let $\boldsymbol{\kappa}$ be such that $\det(\boldsymbol{\kappa})$ has p-power order on $\boldsymbol{J}_\vartheta$ (whose existence is granted by [**26**] Lemme 3.12). The representation $\boldsymbol{\kappa}^{\tau\vee}$ is then isomorphic to $\boldsymbol{\kappa\xi}$ for some character $\boldsymbol{\xi}$ of $\boldsymbol{J}_\vartheta$ trivial on $\boldsymbol{J}^1$. As in the proof of [**26**] Lemma 3.28, since p is odd, this $\boldsymbol{\xi}$ is trivial on $\boldsymbol{J}^0$ and it has p-power order.

The group $\boldsymbol{J}_\vartheta$ is generated by $\boldsymbol{J}^0$ and an element ϖ whose reduced norm has non-zero valuation (see §3.5). Since $\epsilon(\tau) = -1$ and $\boldsymbol{J}_\vartheta$ is stable by τ, we have $\tau(\varpi) \in \varpi^{-1}\boldsymbol{J}^0$. And since $\boldsymbol{\xi}$ is trivial on $\boldsymbol{J}^0$, we deduce that $\boldsymbol{\xi} \circ \tau = \boldsymbol{\xi}^{-1}$.

Now write a for the order of τ, which we assume to be prime to p. Then the identity $\boldsymbol{\kappa}^{\tau\vee} \simeq \boldsymbol{\kappa\xi}$ applied $2a$ times shows that $\boldsymbol{\kappa\xi}^{2a} \simeq \boldsymbol{\kappa}$ so that $\boldsymbol{\xi}^{2a} = 1$. But since $\boldsymbol{\xi}$ has p-power order, and $2a$ is prime to p, we deduce that $\boldsymbol{\xi}$ is trivial. □

Remark 5.3. — In the case when $\epsilon(\tau) = -1$ and the order of τ is finite and prime to p, we even proved that any $\boldsymbol{\kappa}$ such that $\det(\boldsymbol{\kappa})$ has p-power order satisfies $\boldsymbol{\kappa}^{\tau\vee} \simeq \boldsymbol{\kappa}$. We also have

$$\boldsymbol{J}_\vartheta \cap \mathrm{GL}_r(\varDelta)^\tau = \boldsymbol{J}^0 \cap \mathrm{GL}_r(\varDelta)^\tau. \tag{5.2}$$

Indeed, if $x \in \boldsymbol{J}_\vartheta$ is τ-invariant, its valuation has to be equal to its opposite: it is thus 0.

5.4. Now let us go back to the situation of §5.2 with the group $G = \mathrm{GL}_n(D)$ equipped with the involution σ. Note that $\epsilon(\sigma) = -1$ (in the notation of the proof of Lemma 5.2) and the order of σ is prime to p, so Lemma 5.2 and Remark 5.3 apply. We will need the following lemma, which is [**26**] Lemme 3.30.

Lemma 5.4. — *Let $\boldsymbol{\kappa}$ be a representation of $\boldsymbol{J}_\theta$ extending η such that $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}$.*

(1) *There is a unique character χ of $\boldsymbol{J}_\theta \cap G^\sigma = \boldsymbol{J}^0 \cap G^\sigma$ trivial on $\boldsymbol{J}^1 \cap G^\sigma$ such that*

$$\mathrm{Hom}_{\boldsymbol{J}_\theta \cap G^\sigma}(\boldsymbol{\kappa}, \chi) \neq \{0\}$$

and this χ is quadratic (that is, $\chi^2 = 1$).

(2) *Let $\boldsymbol{\rho}$ be an irreducible representation of $\boldsymbol{J}_\theta$ trivial on $\boldsymbol{J}^1$. The canonical linear map:*

$$\mathrm{Hom}_{\boldsymbol{J}^1 \cap G^\sigma}(\eta, \mathbb{C}) \otimes \mathrm{Hom}_{\boldsymbol{J}_\theta \cap G^\sigma}(\boldsymbol{\rho}, \chi) \to \mathrm{Hom}_{\boldsymbol{J}_\theta \cap G^\sigma}(\boldsymbol{\kappa} \otimes \boldsymbol{\rho}, \mathbb{C})$$

is an isomorphism.

6.

Let π be a cuspidal irreducible representation of G with non-cuspidal transfer to $\mathrm{GL}_{2n}(F)$, as in Section 4. By Proposition 4.1, there are a maximal simple stratum $[\mathfrak{a}, \beta]$ in A and a maximal simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ such that $\mathfrak{a}^* = \mathfrak{a}$, $\beta^* = \beta$ and $\theta^{-1} \circ \sigma = \theta$. We use the notation of Section 5. In particular, we have groups $\boldsymbol{J}_\theta$, $\boldsymbol{J}^0$ and $\boldsymbol{J}^1$.

6.1. Identify $\boldsymbol{J}^0/\boldsymbol{J}^1$ with $\mathrm{GL}_m(\boldsymbol{l})$ thanks to the group isomorphism (5.1). Through this identification, and thanks to Proposition 4.2, the involution σ on $\boldsymbol{J}^0/\boldsymbol{J}^1$ identifies with the unitary involution

$$x \mapsto {}^{\mathsf{t}}\overline{x}^{-1} \tag{6.1}$$

on $\mathrm{GL}_m(\boldsymbol{l})$, where $x \mapsto \overline{x}$ is the action of the non-trivial element of $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$ componentwise. It follows that $(\boldsymbol{J}^0 \cap G^\sigma)/(\boldsymbol{J}^1 \cap G^\sigma)$ identifies with the unitary group $\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)$. The following lemma will be useful. Note that m is odd since it divides n, which is odd by Remark 2.1.

Lemma 6.1. — *Let ϱ be a cuspidal irreducible representation of $\mathrm{GL}_m(\boldsymbol{l})$. The following assertions are equivalent.*

(1) *The representation ϱ is $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$-invariant.*
(2) *The representation ϱ is distinguished by $\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)$.*

Moreover, there exist $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$-invariant cuspidal irreducible representations of $\mathrm{GL}_m(\boldsymbol{l})$.

Proof. — For the equivalence between (1) and (2), see for instance [**20**] Theorem 2 or [**14**] Theorem 2.4. For the last assertion, see for instance [**25**] Lemma 2.3. □

Let $\boldsymbol{\kappa}$ be a representation of $\boldsymbol{J}_\theta$ extending η such that $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}$ (whose existence is given by Lemma 5.2) and χ be the quadratic character of the group $\boldsymbol{J}_\theta \cap G^\sigma$ (which is equal to $\boldsymbol{J}^0 \cap G^\sigma$ by (5.2)) given by Lemma 5.4.

Proposition 6.2. — *The character χ is trivial.*

Proof. — Assume this is not the case. Then χ, considered as a character of $\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)$, is trivial on unipotent elements because these elements have p-power order and $p \neq 2$. Thus χ is trivial on the subgroup generated by all transvections. By [**15**] Theorem 11.15, this subgroup is $\mathrm{SU}_m(\boldsymbol{l}/\boldsymbol{l}_0)$. Thus $\chi = \alpha \circ \det$ for some quadratic character α of $\boldsymbol{l}^1$, where det is the determinant on $\mathrm{GL}_m(\boldsymbol{l})$ and $\boldsymbol{l}^1$ is the subgroup of $\boldsymbol{l}^\times$ made of elements of $\boldsymbol{l}/\boldsymbol{l}_0$-norm 1. Let β extend α to $\boldsymbol{l}^\times$, and let $\varkappa$ be the character of $\boldsymbol{J}^0$ inflated from $\beta \circ \det$. It extends χ.

Since m is odd, there is a cuspidal irreducible representation ϱ of $\mathrm{GL}_m(\boldsymbol{l})$ which is invariant by $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$ (equivalently, which is distinguished by $\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)$), thanks to Lemma 6.1.

Let ϱ' be the cuspidal representation $\varrho\varkappa$. Let us prove that it is not $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$-invariant. Let γ denote the generator of $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$. If ϱ' were $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$-invariant, $\varrho\varkappa^\gamma$ would be isomorphic to $\varrho\varkappa$. Comparing the central characters, one would get $(\beta^\gamma\beta^{-1})^m = 1$, that is, $\alpha(x^\gamma x^{-1})^m = 1$ for all $x \in \boldsymbol{l}^\times$, or equivalently $\alpha^m = 1$. But α is quadratic and m is odd: contradiction.

Let $\boldsymbol{\rho}'$ be a representation of $\boldsymbol{J}' = E^\times \boldsymbol{J}^0$ whose restriction to $\boldsymbol{J}^0$ is the inflation of ϱ'. Since ϱ' is not $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$-invariant, the normalizer of $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}'$ in $\boldsymbol{J}_\theta$ is $\boldsymbol{J}'$ (which has index 2 in $\boldsymbol{J}_\theta$).

On the one hand, the representation π' compactly induced by $(\boldsymbol{J}', \boldsymbol{\kappa} \otimes \boldsymbol{\rho}')$ is irreducible and cuspidal, and its Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ is cuspidal by Theorem 5.1.

On the other hand, the map

$$\mathrm{Hom}_{\boldsymbol{J}^1 \cap G^\sigma}(\eta, \mathbb{C}) \otimes \mathrm{Hom}_{\boldsymbol{J}' \cap G^\sigma}(\boldsymbol{\rho}', \chi) \to \mathrm{Hom}_{\boldsymbol{J}' \cap G^\sigma}(\boldsymbol{\kappa} \otimes \boldsymbol{\rho}', \mathbb{C})$$

is an isomorphism (by Lemma 5.4) and the space $\mathrm{Hom}_{\boldsymbol{J}'\cap G^\sigma}(\boldsymbol{\rho}',\chi)$ is non-zero by construction. This implies that $\boldsymbol{\kappa}\otimes\boldsymbol{\rho}'$ is $\boldsymbol{J}'\cap G^\sigma$-distinguished. Thus π' is distinguished, contradicting Theorem 2.2. Thus χ is trivial. □

6.2. According to §5.2, our cuspidal representation π of G contains a representation of the form $(\boldsymbol{J},\boldsymbol{\kappa}\otimes\boldsymbol{\rho})$, where

– the group $\boldsymbol{J}$ satisfies $E^\times\boldsymbol{J}^0\subseteq\boldsymbol{J}\subseteq\boldsymbol{J}_\theta$,

– the representation $\boldsymbol{\kappa}$ is the restriction to $\boldsymbol{J}$ of a representation of $\boldsymbol{J}_\theta$ extending η,

– the representation $\boldsymbol{\rho}$ of $\boldsymbol{J}$ is trivial on $\boldsymbol{J}^1$ and its restriction to $\boldsymbol{J}^0$ is the inflation of a cuspidal representation ϱ of $\boldsymbol{J}^0/\boldsymbol{J}^1\simeq\mathrm{GL}_m(\boldsymbol{l})$ whose normalizer in $\boldsymbol{J}_\theta$ is equal to $\boldsymbol{J}$.

Thanks to Lemma 5.2 and Proposition 6.2, we may and will assume that $\boldsymbol{\kappa}^{\sigma\vee}\simeq\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}$ is distinguished by $\boldsymbol{J}_\theta\cap G^\sigma$.

Thanks to Theorem 5.1, the fact that the Jacquet–Langlands transfer of π is non-cuspidal implies that $\boldsymbol{J}=\boldsymbol{J}_\theta$. By §5.2, this implies that ϱ is $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$-invariant. It follows from Lemma 6.1 that ϱ is distinguished by $\mathrm{U}_m(\boldsymbol{l}/\boldsymbol{l}_0)$, thus $\boldsymbol{\rho}$ is distinguished by $\boldsymbol{J}^0\cap G^\sigma=\boldsymbol{J}_\theta\cap G^\sigma$. By Lemma 5.4, the representation $\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$ is distinguished by $\boldsymbol{J}_\theta\cap G^\sigma$. It follows from Mackey's formula

$$\mathrm{Hom}_{G^\sigma}(\pi,\mathbb{C})\simeq\prod_g\mathrm{Hom}_{\boldsymbol{J}_\theta\cap gG^\sigma g^{-1}}(\boldsymbol{\kappa}\otimes\boldsymbol{\rho},\mathbb{C})$$

(where g ranges over a set of representatives of $(\boldsymbol{J}_\theta,G^\sigma)$-double cosets of G) that π is distinguished by G^σ. This finishes the proof of Theorem 1.2.

7.

In this section, we discuss in more detail the case of representations of depth 0. We assume throughout the section that n is odd.

7.1. Let π be a cuspidal irreducible representation of G with non-cuspidal transfer to $\mathrm{GL}_{2n}(F)$ as in Section 6. Assume moreover that π has depth 0. In that case, we are in the situation described by Remark 3.1. In this situation, we have $m=n$ and $\boldsymbol{l}$ is the residue field of D (thus $\boldsymbol{l}_0$ is that of F). We have $\boldsymbol{J}=D^\times\mathrm{GL}_n(\mathcal{O}_D)$ and $\boldsymbol{J}^0=\mathrm{GL}_n(\mathcal{O}_D)$, and one can choose for $\boldsymbol{\kappa}$ the trivial character of $\boldsymbol{J}$. The representation π is compactly induced from an irreducible representation $\boldsymbol{\rho}$ of $\boldsymbol{J}$ whose restriction to $\boldsymbol{J}^0$ is the inflation of a $\mathrm{Gal}(\boldsymbol{k}_D/\boldsymbol{k}_F)$-invariant, cuspidal representation ϱ of $\mathrm{GL}_n(\boldsymbol{k}_D)$.

Remark 7.1. — In [**33**] Proposition 5.1, the inducing subgroup should be $D^\times\mathrm{GL}_n(\mathcal{O}_D)$ and not $F^\times\mathrm{GL}_n(\mathcal{O}_D)$. Inducing from the latter subgroup gives a representation which is not irreducible. The same comment applies to [**33**] Remark 5.2(1). See also Remark 7.4 below.

7.2. Let us now consider the map $\mathbf{b}_{D/F}$ defined in §1.8. This is a bijection from cuspidal representations of $\mathrm{GL}_n(F)$ to those cuspidal representations of $\mathrm{GL}_n(D)$ which are distinguished by the subgroup $\mathrm{Sp}_n(D)$. In this paragraph, given a cuspidal representation τ of level 0 of $\mathrm{GL}_n(F)$, we describe explicitly the cuspidal representation $\pi=\mathbf{b}_{D/F}(\tau)$, that is, the unique cuspidal representation of G whose Jacquet–Langlands transfer to $\mathrm{GL}_{2n}(F)$ is $\mathrm{St}_2(\tau)$.

On the one hand, it follows from [**29**] Proposition 3.2 that the representation π has depth 0. It can thus be described as in §7.1, that is, it is compactly induced from a representation $\boldsymbol{\rho}$ of the group $\boldsymbol{J} = D^\times\mathrm{GL}_n(\mathcal{O}_D)$ whose restriction to $\boldsymbol{J}^0$ is the inflation of ϱ.

On the other hand, the representation τ can be described in a similar way: it is compactly induced from a representation of $F^\times\mathrm{GL}_n(\mathcal{O}_F)$ whose restriction to $\mathrm{GL}_n(\mathcal{O}_F)$ is the inflation of a cuspidal representation ϱ_0 of $\mathrm{GL}_n(\boldsymbol{k}_F)$. (See for instance [**6**] 1.2.)

[**29**] Theorem 4.1 provides a simple and natural relation between ϱ and ϱ_0: the representation ϱ is the base change (that is, the Shintani lift) of ϱ_0. (This relation was pointed out in [**33**] Remark 5.2(1) without reference to [**29**].)

The knowledge of ϱ does not quite determine the representation π. In order to completely determine it, fix a uniformizer ϖ_F of F and a uniformizer $\varpi = \varpi_D$ of D such that $\varpi^2 = \varpi_F$. As the group $\boldsymbol{J}$ is generated by $\boldsymbol{J}^0$ and ϖ, it remains to compute the operator $A = \boldsymbol{\rho}(\varpi)$, which intertwines ϱ with ϱ^γ, where γ is the non-trivial element of $\mathrm{Gal}(\boldsymbol{k}_D/\boldsymbol{k}_F)$, that is, one has

$$A \circ \varrho(x) = \varrho(x^\gamma) \circ A, \quad x \in \mathrm{GL}_n(\boldsymbol{k}_D).$$

The space of intertwining operators between ϱ and ϱ^γ has dimension 1. To go further, we have to identify this space with $\mathbb{C}$ in a natural way.

Fix a non-trivial character ψ_0 of $\boldsymbol{k}_F$, and let ψ be the character of $\boldsymbol{k}_D$ obtained by composing ψ_0 with the trace of $\boldsymbol{k}_D/\boldsymbol{k}_F$. Let U denote the subgroup of $\mathrm{GL}_n(\boldsymbol{k}_D)$ made of all unipotent upper triangular matrices, and consider ψ as the character $u \mapsto \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n})$ of U. It is well-known that, if V is the underlying vector space of ϱ, then

$$\varrho^\psi = \{v \in V \mid \varrho(u)(v) = \psi(u)v,\ u \in U\}$$

has dimension 1. Since the character ψ is $\mathrm{Gal}(\boldsymbol{k}_D/\boldsymbol{k}_F)$-invariant, this 1-dimensional space is stable by A. There is thus a non-zero scalar $\alpha \in \mathbb{C}^\times$ such that $A(v) = \alpha v$, and A is uniquely determined by α. Let ω_0 denote the central character of τ. (Note that the representation τ is entirely determined by ϱ_0 and ω_0.)

Proposition 7.2. — *One has $\alpha = \omega_0(-\varpi_F)$.*

We now have completely determined π from the knowledge of τ. The proof of this proposition, based on [**30**] and [**8**], will be done in the next paragraph.

Remark 7.3. — The proposition thus implies that the result does not depend on the choice of a $\varpi \in D$ such that $\varpi^2 = \varpi_F$. Replacing ϖ by $-\varpi$ should thus lead to the same result, that is, one should have $\boldsymbol{\rho}(-\varpi) = \boldsymbol{\rho}(\varpi)$, or equivalently, the central character of π should be trivial at -1. This is the case indeed, since π is distinguished by $\mathrm{Sp}_n(D)$, which contains -1.

Remark 7.4. — This paragraph corrects the description made in [**33**] Remark 5.2(1), which is incorrect due to the error pointed out in Remark 7.1. Note that [**33**] Remark 5.2(2) is correct: it follows from [**29**] Theorem 4.1 or [**7**] Theorem 6.1.

7.3. We now proceed to the proof of Proposition 7.2, which is essentially an exercice of translation into the language of [**30**] and [**8**]. Fix a separable closure $\overline{F}$ of F. A *tame admissible pair* is a pair $(K/F,\xi)$ made of an unramified finite extension K of F contained in $\overline{F}$ together with a tamely ramified character $\xi : K^\times \to \mathbb{C}^\times$ all of whose $\mathrm{Gal}(K/F)$-conjugate ξ^γ, $\gamma \in \mathrm{Gal}(K/F)$, are pairwise distinct. The *degree* of such a pair is the degree of K over F.

Given any integer $m \geqslant 1$ and any inner form H of $\mathrm{GL}_m(F)$, Silberger–Zink [**30**] have defined a bijection $\varPi^H$ between:

(1) the set of Galois conjugacy classes of tame admissible pairs of degree dividing m,
(2) the set of isomorphism classes of discrete series representations of depth 0 of H.

They have also described (in [**30**] Theorem 3) the behavior of this parametrization of the discrete series of inner forms of $\mathrm{GL}_m(F)$ with respect to the Jacquet–Langlands correspondence: if H is isomorphic to $\mathrm{GL}_r(\varDelta)$ for some divisor r of m and some central division F-algebra $\varDelta$ of reduced degree m/r, and if $(K/F,\xi)$ is a tame admissible pair of degree f dividing m, then the Jacquet–Langlands transfer of $\varPi^H(K/F,\xi)$ to $\mathrm{GL}_m(F)$ is equal to

$$\varPi^{\mathrm{GL}_m(F)}\left(K/F,\xi\mu_K^{m-r+(f,r)-f}\right) \tag{7.1}$$

where μ_K is the unique unramified character of $K^\times$ of order 2 and (a,b) denotes the greatest common divisor of two integers $a,b \geqslant 1$. (Silberger–Zink state their result by using the multiplicative group of a central division F-algebra of reduced degree m as an inner form of reference, but it is more convenient for us to use $\mathrm{GL}_m(F)$ as the inner form of reference.)

Let us start with our cuspidal representation τ of depth 0 of $\mathrm{GL}_n(F)$. Let $(K/F,\xi)$ be a tame admissible pair associated with it by the bijection $\varPi^{\mathrm{GL}_n(F)}$. It follows from [**8**] 5.1 that this pair has degree n, and that the central character ω_0 of τ is equal to the restriction of ξ to $F^\times$.

Now form the discrete series representation $\mathrm{St}_2(\tau)$ of $\mathrm{GL}_{2n}(F)$. By [**8**] 5.2, the tame admissible pair associated with it by the bijection $\varPi^{\mathrm{GL}_{2n}(F)}$ is of the form $(K/F,\xi')$ where ξ' coincides with ξ on the units of $\mathcal{O}_K$. By [**8**] 6.4, one has

$$\xi'(\varpi_F) = -\omega_0(\varpi_F) = -\xi(\varpi_F).$$

One thus has $\xi' = \xi\mu_K$. We now claim that the representation π is parametrized, through the bijection $\varPi^G$, by the tame admissible pair $(K/F,\xi)$. Indeed, by (7.1), and since n is odd and μ_K is quadratic, we have

$${}^{\mathrm{JL}}\varPi^G(K/F,\xi) = \varPi^{\mathrm{GL}_{2n}(F)}(K/F,\xi\mu_K^{2n-n+(n,n)-n}) = \varPi^{\mathrm{GL}_{2n}(F)}(K/F,\xi\mu_K)$$

which is equal to $\mathrm{St}_2(\tau)$. It now follows from [**30**] (8), p. 196, that the scalar α by which A acts on the line ϱ^ψ is equal to $\xi(-\varpi_F) = \omega_0(-\varpi_F)$ as expected.

8. Appendix

In this section, A is as in Section 2. Let $[\mathfrak{a},\beta]$ be a simple stratum in A. Let ψ^A denote the character $x \mapsto \psi(\mathrm{trd}_{A/F}(x))$ of A, where $\mathrm{trd}_{A/F}$ is the reduced trace of A over F.

We prove Lemma 8.1 (which has been used in Section 4), whose proof has been postponed to this last section since it requires techniques which are not used anywhere else in the paper.

Lemma 8.1. — *Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a simple character. Then*

(1) $[\mathfrak{a}^*, \beta^*]$ *is a simple stratum realizing β, and*
(2) θ^* *is a simple character in $\mathcal{C}(\mathfrak{a}^*, \beta^*)$.*

Proof. — If $[\mathfrak{a}, \beta]$ is the null stratum, there is nothing to prove. We will thus assume that $[\mathfrak{a}, \beta]$ has positive level.

The map $\iota : x \mapsto x^*$ is an F-linear involution of A such that $\iota(xy) = \iota(y)\iota(x)$. Restricting to the commutative F-algebra $E = F[\beta]$, it is thus an embedding of F-algebras from E to A. This proves (1). Note that, if B is the centralizer of E in A, then the centralizer of E^* in A is B^*.

Lemma 8.2. — *One has $\mathrm{Nrd}_{B/E}(x^*) = \mathrm{Nrd}_{B^*/E^*}(x)^*$ for all $x \in B^*$.*

Proof. — The proof is similar to [**26**] Lemme 5.15. □

We will prove (2) by induction on the integer $q = -k_0(\mathfrak{a}, \beta)$. (See [**23**] §2.1 for the definition of $k_0(\mathfrak{a}, \beta)$.) Define $r = \lfloor q/2 \rfloor + 1$. First note that $k_0(\mathfrak{a}^*, \beta^*) = k_0(\mathfrak{a}, \beta)$ and θ^* is normalized by $\mathcal{K}(\mathfrak{a}^*) \cap B^{*\times}$ as θ is normalized by $\mathcal{K}(\mathfrak{a}) \cap B^\times$. (Here, $\mathcal{K}(\mathfrak{a})$ denotes the normalizer in G of the order $\mathfrak{a}$.) For any integer $i \geqslant 1$, let us write $\mathrm{U}^i(\mathfrak{a}) = 1 + \mathfrak{p}_{\mathfrak{a}}^i$.

Assume first that β is minimal over F (see [**23**] §2.3.3). In this case, we have

– $H^1(\mathfrak{a}, \beta) = \mathrm{U}^1(\mathfrak{b})\mathrm{U}^r(\mathfrak{a})$,
– the restriction of θ to $\mathrm{U}^r(\mathfrak{a})$ is the character $\psi^A_\beta : 1 + x \mapsto \psi^A(\beta x)$,
– the restriction of θ to $\mathrm{U}^1(\mathfrak{b})$ is equal to $\xi \circ \mathrm{Nrd}_{B/E}$ for some character ξ of $1 + \mathfrak{p}_E$.

The character θ^* is defined on the group $\sigma(H^1(\mathfrak{a}, \beta)) = \mathrm{U}^1(\mathfrak{b}^*)\mathrm{U}^r(\mathfrak{a}^*) = H^1(\mathfrak{a}^*, \beta^*)$. Its restriction to $\mathrm{U}^r(\mathfrak{a}^*)$ is the character

$$1 + y \mapsto \psi^A(\beta y^*) = \psi^A(\beta^* y) = \psi^A_{\beta^*}(1 + y)$$

since ψ^A is invariant by $*$. By Lemma 8.2, its restriction to $\mathrm{U}^1(\mathfrak{b}^*)$ is $\xi^* \circ \mathrm{Nrd}_{B^*/E^*}$ where ξ^* is the character $x \mapsto \xi(x^*)$ of $1 + \mathfrak{p}_{E^*}$. It follows from [**23**] Proposition 3.47 that θ^* is a simple character in $\mathcal{C}(\mathfrak{a}^*, \beta^*)$.

Now assume that β is not minimal over F, and that γ is an approximation of β with respect to $\mathfrak{a}$ (see [**23**] §2.1). We have

– $H^1(\mathfrak{a}, \beta) = \mathrm{U}^1(\mathfrak{b})H^r(\mathfrak{a}, \gamma)$,
– the restriction of θ to $H^r(\mathfrak{a}, \gamma)$ is equal to $\psi^A_{\beta-\gamma}\theta'$ for some simple character $\theta' \in \mathcal{C}(\mathfrak{a}, \gamma)$,
– the restriction of θ to $\mathrm{U}^1(\mathfrak{b})$ is equal to $\xi \circ \mathrm{Nrd}_{B/E}$ for some character ξ of $1 + \mathfrak{p}_E$.

The character θ^* is defined on the group

$$\sigma(H^1(\mathfrak{a}, \beta)) = \mathrm{U}^1(\mathfrak{b}^*)\sigma(H^1(\mathfrak{a}, \gamma)) = \mathrm{U}^1(\mathfrak{b}^*)H^1(\mathfrak{a}^*, \gamma^*) = H^1(\mathfrak{a}^*, \beta^*)$$

since γ^* is an approximation of β^* with respect to $\mathfrak{a}^*$. By induction, its restriction to $H^1(\mathfrak{a}^*, \gamma^*)$ is the character $\psi^A_{\beta^*-\gamma^*}\theta'^*$ where $\theta'^* \in \mathcal{C}(\mathfrak{a}^*, \gamma^*)$ is the transfer of θ'. Its restriction to $\mathrm{U}^1(\mathfrak{b}^*)$ is the character $\xi^* \circ \mathrm{Nrd}_{B^*/E^*}$. It follows from [**23**] Proposition 3.47 that $\theta^* \in \mathcal{C}(\mathfrak{a}^*, \beta^*)$. □

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