Nonlinear Klein-Gordon equation with damping - Exam March 27th of 2024 - 3h

All electronic devices are forbidden

Exercice 1. In this exercise, we consider the equation

$$\begin{cases} \partial_t^2 u + 2\alpha\partial_t u - \partial_x^2 u + \sin u = h, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $\alpha \in (0, 1/2)$ and $h \in \mathcal{C}([0,+\infty), L^2(\mathbb{R}))$.

(a) Justify briefly that the local Cauchy problem for (1) is well-posed for any initial data $(u_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. *Hint :* write $\sin u = u + (\sin u - u)$.

We rewrite the equation as

$$\begin{cases} \partial_t u = v \\ \partial_t v = -2\alpha v + \partial_x^2 u - \sin u + h, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{1'}$$

We define as in the course the operator A_α on $H^1 \times L^2$ by

$$\begin{cases} D(A_\alpha) = H^2 \times H^1, \\ A_\alpha \vec{u} = (v, u'' - u - 2\alpha v), \text{ for any } \vec{u} \in D(A_\alpha) \end{cases}$$

and the strongly continuous semigroup of contractions $(S_\alpha(t))_{t\geqslant 0}$ in X generated by A_α. We rewrite the equation in the following equivalent Duhamel formulation

$$\vec{u}(t) = S_\alpha \vec{u}_0 + \int_0^t S_\alpha(t-s)(0, u(s) - \sin u(s) + h(s))\,\mathrm{d}s.$$

Using the same arguments as in the course, we prove the local existence and uniqueness of a solution $\vec{u}$.

(b) For $(u,v) \in H^1 \times L^2$,

$$E(u,v) = \int_{\mathbb{R}} \left(\tfrac{1}{2}v^2 + \tfrac{1}{2}(\partial_x u)^2 + 1 - \cos u\right) \mathrm{d}x \tag{2}$$

and compute (formally)

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u(t), \partial_t u(t))$$

for $u(t,x)$ a solution of (1).

We compute

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u(t), \partial_t u(t)) = -2\alpha \int v^2 + \int hv.$$

(c) Prove that any solution of (1) is global.

Note that $E(u(t)) \geqslant \int_{\mathbb{R}} \left(\frac{1}{2}v^2 + \frac{1}{2}(\partial_x u)^2\right)$. We introduce

$$\varphi(t) = E(u(t)) + \frac{1}{2}\int u^2(t).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = -2\alpha \int v^2 + \int hv + \int uv \leqslant C\varphi(t) + C\int h^2.$$

Assuming that the solution is not global, there exists $T > 0$ such that $\vec{u}$ exists on $[0, T)$ but $\lim_{t\to T^-} \|\vec{u}(t)\|_{H^1\times L^2} = +\infty$. But by the previous computation, we have

$$\int_{\mathbb{R}} \left(\tfrac{1}{2}v^2 + \tfrac{1}{2}(\partial_x u)^2 + \tfrac{1}{2}u^2\right) \leqslant \varphi(t) \leqslant Ce^{Ct}\int_0^T \int h^2 + e^{Ct}\varphi(0),$$

which provides a uniform bound on $[0, T]$ and thus a contradiction.

Exercice 2. In this exercise, we denote by C various positive constants that may change from one line to another.

We consider the following equation

$$\begin{cases} \partial_t u + \partial_x(\partial_x^2 u + u^7) = 0 \\ u_{|t=0} = u_0 \end{cases} \qquad \text{(gKdV)}$$

Here (gKdV) means *generalized Korteweg de Vries equation*. In this exercise, we assume that for any initial data in $u_0 \in H^1(\mathbb{R})$, there exists a local in time solution $u \in \mathcal{C}([0, T_{\max}), H^1(\mathbb{R}))$. Moreover, in this exercise, all computations can be done formally, that is, without rigorous justification (usually based on density arguments and persistence of regularity).

(a) We define, for $v \in H^1(\mathbb{R})$, $E(v) = \frac{1}{2}\int(\partial_x v)^2 - \frac{1}{8}\int v^8$. Prove that for any solution u, the energy $E(u(t))$ is conserved for all time such that the solution is defined.

We formally find the energy conservation by multiplying (gKdV) by $\partial_x^2 u + u^7$ and using

$$\int \partial_x(\partial_x^2 u + u^7)(\partial_x^2 u + u^7) = 0$$

so that

$$\int (\partial_t u)(\partial_x^2 u + u^7) = 0,$$

which provides after integration by parts

$$\frac{d}{dt}E(u(t)) = 0.$$

(b) Check that Q defined by $Q(x) = \left(\frac{4}{\cosh^2(3x)}\right)^{1/6}$ is solution of

$$Q \in H^1(\mathbb{R}), \quad Q'' + Q^7 = Q \text{ on } \mathbb{R}.$$

We have

$$Q(x) = 4^{\frac{1}{3}} \cosh^{-\frac{1}{3}}(3x)$$

and so

$$\begin{aligned} Q''(x) &= -4^{\frac{1}{3}} \left(\sinh(3x)\cosh^{-\frac{4}{3}}(3x)\right)' \\ &= 4^{\frac{1}{3}} \left(-3\cosh^{-\frac{1}{3}}(3x) + 4\sinh^2(3x)\cosh^{-\frac{7}{3}}(3x)\right) \\ &= 4^{\frac{1}{3}} \left(\cosh^{-\frac{1}{3}}(3x) - 4\cosh^{-\frac{7}{3}}(3x)\right) = Q(x) - Q^7(x) \end{aligned}$$

(c) Check that $q(t, x) = Q(x - t)$ is a solution of (gKdV).

We have

$$\partial_t q + \partial_x(\partial_x^2 q + q^7) = -Q'(x-t) + (Q'' + Q^7)'(x-t) = 0$$

(d) Check that

$$\int (Q')^2 = \frac{3}{8}\int Q^8.$$

Multiplying the equation of Q by Q and integrating by parts, we find

$$-\int (Q')^2 + \int Q^8 = \int Q^2.$$

Multiplying the equation of Q by Q' and integrating on $(-\infty, x]$, we find

$$\frac{1}{2}(Q')^2 + \frac{1}{8}Q^8 = \frac{1}{2}Q^2.$$

Integrating on $\mathbb{R}$, we find

$$\int (Q')^2 + \frac{1}{4}\int Q^8 = \int Q^2$$

Combining the two identities, we find the desired result.

Our objective in the next questions is to prove the *instability* of the solution q in the space H^1.

(e) Let $F(v) = E(v) + \frac{1}{2}\int v^2$. Prove that for any $\varepsilon \in H^1(\mathbb{R})$ such that $\|\varepsilon\|_{H^1} \leqslant 1$,

$$F(Q + \varepsilon) = F(Q) + \frac{1}{2}(\mathcal{L}\varepsilon, \varepsilon) + R(\varepsilon) \quad \text{where } |R(\varepsilon)| \leqslant C\|\varepsilon\|_{H^1}^3$$

and where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}\varepsilon = -\partial_x^2\varepsilon + \varepsilon - 7Q^6\varepsilon.$$

We compute

$$\begin{aligned}E(Q+\varepsilon) &= \frac{1}{2}\int(\partial_x(Q+\varepsilon))^2 - \frac{1}{8}\int(Q+\varepsilon)^8 \\ &= E(Q) + \int Q'\partial_x\varepsilon - \int Q^7\varepsilon + \frac{1}{2}\int(\partial_x\varepsilon)^2 + \frac{7}{2}\int Q^6\varepsilon^2 \\ &\quad - \frac{1}{8}\int\left((Q+\varepsilon)^8 - 8Q^7\varepsilon - 28Q^6\varepsilon^2\right) \\ &= E(Q) - \int Q\varepsilon + \frac{1}{2}\int(\partial_x\varepsilon)^2 + \frac{7}{2}\int Q^6\varepsilon^2 + R(\varepsilon)\end{aligned}$$

where

$$|R(\varepsilon)| \leqslant \int |\varepsilon|^3 + |\varepsilon|^7 \leqslant C\|\varepsilon\|_{H^1}^3.$$

Second

$$\frac{1}{2}\int(Q+\varepsilon)^2 = \frac{1}{2}\int Q^2 + \int Q\varepsilon + \int \varepsilon^2.$$

Thus,

$$F(Q+\varepsilon) = F(Q) + \frac{1}{2}(\mathcal{L}\varepsilon,\varepsilon) + R(\varepsilon).$$

(f) Define $\phi = Q^4$ and $\Lambda Q = \frac{1}{3}Q + xQ'$. Compute

$$\mathcal{L}\phi, \quad \mathcal{L}(\Lambda Q) \quad \text{and} \quad \mathcal{L}(Q').$$

We compute

$$Q^4(x) = 4^{\frac{2}{3}}\cosh^{-\frac{4}{3}}(3x),$$

and thus

$$\begin{aligned}(Q^4)'' &= -4^{\frac{5}{3}}\left(\sinh(3x)\cosh^{-\frac{7}{3}}(3x)\right)' \\ &= 4^{\frac{5}{3}}\left(-3\cosh^{-\frac{4}{3}}(3x) + 7\sinh^2(3x)\cosh^{-\frac{10}{3}}(3x)\right) \\ &= 4^{\frac{5}{3}}\left(4\cosh^{-\frac{4}{3}}(3x) - 7\cosh^{-\frac{10}{3}}(3x)\right) \\ &= 16Q^4 - 7Q^6Q^4.\end{aligned}$$

Therefore

$$\mathcal{L}Q^4 = -15Q^4.$$

Then

$$\mathcal{L}Q' = 0$$

by differentiating $Q'' + Q^7 = Q$ with respect to x. Lastly,

$$\mathcal{L}(\Lambda Q) = \frac{1}{3}\mathcal{L}Q + \mathcal{L}(xQ') = -2Q^7 - 2Q'' = -2Q.$$

We assume the following property : there exist $K_1, K_2 > 0$ such that for all $\varepsilon \in H^1(\mathbb{R})$ with $(\varepsilon, Q') = 0$, $(\mathcal{L}\varepsilon, \varepsilon) \geqslant K_1\|\varepsilon\|_{H^1}^2 - K_2(\varepsilon, \phi)^2$. (3)

(g) For $\alpha_0 > 0$ small, consider the set

$$U_{\alpha_0} = \{u \in H^1(\mathbb{R}) \mid \inf_{y\in\mathbb{R}} \|u - Q(\cdot - y)\|_{H^1} \leqslant \alpha_0\}.$$

Prove that for α_0 small enough, there exists a unique map $\sigma : U_{\alpha_0} \to \mathbb{R}$ such that for all $u \in U_{\alpha_0}$, we can decompose $u(x) = (Q + \varepsilon)(x + \sigma(u))$ with

$$(\varepsilon, Q') = 0 \text{ and } \|\varepsilon\|_{H^1} \leqslant C\alpha_0.$$

Let $u \in U_{\alpha_0}$. We fix y_1 such that $\|u(\cdot + y_1) - Q\|_{H^1} \leqslant 2\alpha_0$ Then, we define $v = u(\cdot + y_1)$ and we apply the result from the course (based on the Implicit Function Theorem). There exists

$$v \mapsto z(v)$$

such that $\varepsilon(x) = v(x) - Q(x - z(v))$ satisfies $\int \varepsilon Q'(x - z(v)) = 0$.

(h) Define the initial data $u_{0,\lambda}(x) = \lambda Q(\lambda^2 x)$, for $\lambda > 1$ close to 1. Prove

$$\|u_{0,\lambda}\|_{L^2} = \|Q\|_{L^2}, \quad \lim_{\lambda\to1^+} \|u_{0,\lambda} - Q\|_{H^1} = 0, \quad \delta(\lambda) = E(Q) - E(u_{0,\lambda}) > 0.$$

The computations follow from change of variable

$$\int u_{0,\lambda}^2 = \lambda^2 \int Q^2(\lambda^2 x)dx = \int Q^2.$$

Then

$$\int (\lambda Q(\lambda^2 x) - Q(x))^2 dx \to 0, \quad \int (\lambda^3 Q'(\lambda^2 x) - Q'(x))^2 dx \to 0$$

as $\lambda \to 1$ by the dominated convergence theorem.
Last, we have

$$E(Q) = \frac{1}{2}\int (Q')^2 - \frac{1}{8}\int Q^8 = \frac{1}{8}\left(\frac{3}{2} - 1\right)\int Q^8 = \frac{1}{16}\int Q^8$$

and

$$\begin{aligned} E(u_{0,\lambda}) &= \frac{1}{2}\lambda^4 \int (Q')^2 - \frac{1}{8}\lambda^6 \int Q^8 \\ &= \frac{3}{16}\lambda^4 \int Q^8 - \frac{1}{8}\lambda^6 \int Q^8 \\ &= \frac{1}{16}\lambda^4(3 - 2\lambda^2)\int Q^8. \end{aligned}$$

Thus,

$$\delta(\lambda) = \frac{1}{16}(1 - 3\lambda^4 + 2\lambda^6)\int Q^8$$

Setting $f(\lambda) = 1 - 3\lambda^4 + 2\lambda^6$ we have $f(1) = 0$, $f'(1) = 0$ but $f''(1) = 24 > 0$.

Denote by $u_\lambda(t)$ the local H^1 solution of (gKdV) corresponding to $u_\lambda(0) = u_{0,\lambda}$.

For the sake of contradiction, we consider the following assertion

$$\forall 0 < \alpha < \alpha_0,\ \exists \lambda > 1, \text{ such that } u_\lambda \text{ exists for all } t \geqslant 0 \\ \text{and for all } t \geqslant 0,\ \ u_\lambda(t) \in U_\alpha. \tag{4}$$

(i) Explain why contradicting assertion (4) will prove a form of instability of the solution $q(t,x)$.

Taking the negation, we have

$$\exists 0 < \alpha < \alpha_0,\ \forall \lambda > 1, \text{ such that } \ \exists t \geqslant 0,\ \ u_\lambda(t) \notin U_\alpha.$$

This means that even taking $\lambda > 1$ arbitrarily close to 1, the corresponding solution u_λ exits the neighbourhood U_α.

From now on, we assume that (4) **holds and we seek a contradiction. We take** $0 < \alpha < \alpha_0$ **(where** α_0 **can be taken smaller if needed) and we consider** $\lambda > 1$ **as in** (4)**. For all** $t \geqslant 0$**, we decompose** $u(t,x)$ **as** $u(t,x) = (Q+\varepsilon)(t, x+\sigma(t))$ **where**

$$(\varepsilon(t), Q') = 0 \ \textbf{and}\ \|\varepsilon(t)\|_{H^1} \leqslant C\alpha_0.$$

We assume that the function $t \mapsto \sigma(t)$ **is of class** C^1**.**

(j) Using the conservation of the L^2 norm for $u(t)$, show that for all $t \geqslant 0$,

$$|(\varepsilon(t), Q)| \leqslant C\|\varepsilon(t)\|_{H^1}^2.$$

Indeed, we check that

$$\frac{d}{dt}\int u^2(t) = \int u\partial_t u = \int (\partial_x^2 u + u^7)\partial_x u = 0$$

by integration by parts. Thus,

$$\int Q^2 = \int u_{0,\lambda}^2 = \int u^2(t) = \int Q^2 + 2\int Q\varepsilon(t) + \int \varepsilon^2(t).$$

This proves

$$|(\varepsilon(t), Q)| \leqslant \frac{1}{2}\|\varepsilon(t)\|_{L^2}^2.$$

(k) Write the equation satisfied by ε and prove that, for all $t \geqslant 0$,

$$|\sigma'(t) - 1| \leqslant C\|\varepsilon(t)\|_{H^1}.$$

We have

$$\partial_t u = \partial_t\varepsilon(x+\sigma(t)) + \sigma'(Q' + \partial_x\varepsilon)(x+\sigma(t)),$$
$$\partial_x(\partial_x^2 u + u^7) = (Q''' + \partial_y^3\varepsilon + (Q^7)' + 7\partial_y(Q^6\varepsilon) + \partial_y((Q+\varepsilon)^7 - Q^7 - 7Q^6\varepsilon))(x+\sigma(t)).$$

Thus, the equation of u is rewritten as follows

$$\partial_t\varepsilon + \partial_y(\mathcal{L}\varepsilon + (Q+\varepsilon)^7 - Q^7 - 7Q^6\varepsilon) + (\sigma' - 1)(Q' + \partial_x\varepsilon) = 0$$

Multiplying this equation by Q' and using $(\varepsilon, Q') = 0$, we find the bound $|\sigma'(t) - 1| \leqslant C\|\varepsilon(t)\|_{H^1}$.

(l) Prove that there exists $K_3 > 0$ such that, for all $t \geqslant 0$,

$$|(\varepsilon(t), \phi)| \geqslant K_3\|\varepsilon(t)\|_{H^1} + K_3\sqrt{\delta(\lambda)}.$$

We have

$$(\mathcal{L}\varepsilon, \varepsilon) \geqslant K_1\|\varepsilon\|_{H^1}^2 - K_2(\varepsilon, \phi)^2$$

and by $F(u_{0,\lambda}) = F(Q+\varepsilon)$,

$$-2\delta(\lambda) = (\mathcal{L}\varepsilon, \varepsilon) + 2R(\varepsilon) \geqslant \|\varepsilon\|_{H^1}^2 - K_2(\varepsilon, \phi)^2$$

Thus,

$$(\varepsilon, \phi)^2 \geqslant C\|\varepsilon\|_{H^1}^2 + C\delta(\lambda).$$

for a constant $C > 0$, which implies the result.

Let χ be a smooth function on $\mathbb{R}$ such that

$$\chi(x) \equiv 1 \text{ on } (-\infty, 1], \quad \chi(x) \equiv 0 \text{ on } [2, +\infty), \quad 0 \leqslant \chi(x) \leqslant 1 \text{ on } [1,2].$$

Let $\zeta(x)$ be defined as follows

$$\zeta(x) = \int_{-\infty}^{x} (\Lambda Q(y) + \beta\phi(y))dy, \quad \beta = -\frac{\int Q\Lambda Q}{\int Q^5}. \tag{5}$$

For $A > 1$ large to be defined later, for all $t \geqslant 0$, set

$$J(t) = \int \varepsilon(t,x)\zeta(x)\chi\left(\frac{x}{A}\right)dx.$$

(m) Prove that

$$\forall t \geqslant 0, \quad |J(t)| \leqslant C\alpha_0 A^{\frac{1}{2}}.$$

Check that $\int \zeta Q' = 0$.

This follows from the Cauchy-Schwarz inequality

$$|J| \leqslant C\|\varepsilon\|_{L^2} \left(\int_0^{2A} dx \right)^{\frac{1}{2}} \leqslant C\|\varepsilon\|_{L^2}\sqrt{A} \leqslant C\alpha_0\sqrt{A}.$$

(n) Using the equation of ε, prove (formally) that

$$\frac{d}{dt}J(t) = K_4(\varepsilon(t), \phi) + R_J(\varepsilon(t))$$

where $K_4 \neq 0$ is a constant to specify and where R_J satisfies

$$|R_J(\varepsilon)| \leqslant C\|\varepsilon\|_{H^1}^2 + CA^{-\frac{1}{2}}\|\varepsilon\|_{H^1}.$$

(o) Find a contradiction to (4) for $A > 1$ large enough and $\alpha_0 > 0$ small enough.