Nonlinear Klein-Gordon equation with damping - Exam March 27th of 2024 - 3h

All electronic devices are forbidden

Exercice 1. In this exercise, we consider the equation

$$\begin{cases} \partial_t^2 u + 2\alpha\partial_t u - \partial_x^2 u + \sin u = h, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $\alpha \in (0, 1/2)$ and $h \in \mathcal{C}([0,+\infty), L^2(\mathbb{R}))$.

(a) Justify briefly that the local Cauchy problem for (1) is well-posed for any initial data $(u_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. *Hint :* write $\sin u = u + (\sin u - u)$.

(b) For $(u, v) \in H^1 \times L^2$,

$$E(u,v) = \int_{\mathbb{R}} \left(\tfrac{1}{2}v^2 + \tfrac{1}{2}(\partial_x u)^2 + 1 - \cos u\right) \mathrm{d}x \tag{2}$$

and compute (formally)

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u(t), \partial_t u(t))$$

for $u(t,x)$ a solution of (1).

(c) Prove that any solution of (1) is global.

Exercice 2. In this exercise, we denote by C various positive constants that may change from one line to another.

We consider the following equation

$$\begin{cases} \partial_t u + \partial_x(\partial_x^2 u + u^7) = 0 \\ u_{|t=0} = u_0 \end{cases} \tag{gKdV}$$

Here (gKdV) means *generalized Korteweg de Vries equation*. In this exercise, we assume that for any initial data in $u_0 \in H^1(\mathbb{R})$, there exists a local in time solution $u \in \mathcal{C}([0, T_{\max}), H^1(\mathbb{R}))$. Moreover, in this exercise, all computations can be done formally, that is, without rigorous justification (usually based on density arguments and persistence of regularity).

(a) We define, for $v \in H^1(\mathbb{R})$, $E(v) = \frac{1}{2}\int(\partial_x v)^2 - \frac{1}{8}\int v^8$. Prove that for any solution u, the energy $E(u(t))$ is conserved for all time such that the solution is defined.

(b) Check that Q defined by $Q(x) = \left(\dfrac{4}{\cosh^2(3x)}\right)^{1/6}$ is solution of

$$Q \in H^1(\mathbb{R}), \quad Q'' + Q^7 = Q \text{ on } \mathbb{R}.$$

(c) Check that $q(t,x) = Q(x-t)$ is a solution of (gKdV).

(d) Check that

$$\int (Q')^2 = \frac{3}{8} \int Q^8.$$

Our objective in the next questions is to prove the *instability* of the solution q in the space H^1.

(e) Let $F(v) = E(v) + \frac{1}{2}\int v^2$. Prove that for any $\varepsilon \in H^1(\mathbb{R})$ such that $\|\varepsilon\|_{H^1} \leqslant 1$,

$$F(Q+\varepsilon) = F(Q) + \frac{1}{2}(\mathcal{L}\varepsilon, \varepsilon) + R(\varepsilon) \quad \text{where } |R(\varepsilon)| \leqslant C\|\varepsilon\|_{H^1}^3$$

and where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}\varepsilon = -\partial_x^2 \varepsilon + \varepsilon - 7Q^6\varepsilon.$$

(f) Define $\phi = Q^4$ and $\Lambda Q = \frac{1}{3}Q + xQ'$. Compute

$$\mathcal{L}\phi, \quad \mathcal{L}(\Lambda Q) \quad \text{and} \quad \mathcal{L}(Q').$$

We assume the following property : there exist $K_1, K_2 > 0$ such that

$$\textbf{for all } \varepsilon \in H^1(\mathbb{R}) \textbf{ with } (\varepsilon, Q') = 0, \quad (\mathcal{L}\varepsilon, \varepsilon) \geqslant K_1\|\varepsilon\|_{H^1}^2 - K_2(\varepsilon, \phi)^2. \quad (3)$$

(g) For $\alpha_0 > 0$ small, consider the set

$$U_{\alpha_0} = \left\{ u \in H^1(\mathbb{R}) \mid \inf_{y \in \mathbb{R}} \|u - Q(\cdot - y)\|_{H^1} \leqslant \alpha_0 \right\}.$$

Prove that for α_0 small enough, there exists a unique map $\sigma : U_{\alpha_0} \to \mathbb{R}$ such that for all $u \in U_{\alpha_0}$, we can decompose $u(x) = (Q+\varepsilon)(x+\sigma(u))$ with

$$(\varepsilon, Q') = 0 \text{ and } \|\varepsilon\|_{H^1} \leqslant C\alpha_0.$$

(h) Define the initial data $u_{0,\lambda}(x) = \lambda Q(\lambda^2 x)$, for $\lambda > 1$ close to 1. Prove

$$\|u_{0,\lambda}\|_{L^2} = \|Q\|_{L^2}, \quad \lim_{\lambda \to 1^+} \|u_{0,\lambda} - Q\|_{H^1} = 0, \quad \delta(\lambda) = E(Q) - E(u_{0,\lambda}) > 0.$$

Denote by $u_\lambda(t)$ the local H^1 solution of (gKdV) corresponding to $u_\lambda(0) = u_{0,\lambda}$.

For the sake of contradiction, we consider the following assertion

$$\begin{aligned} &\forall 0 < \alpha < \alpha_0, \exists \lambda > 1, \text{ such that } u_\lambda \text{ exists for all } t \geqslant 0 \\ &\qquad\qquad\qquad \text{and for all } t \geqslant 0, \ u_\lambda(t) \in U_\alpha. \end{aligned} \quad (4)$$

(i) Explain why contradicting assertion (4) will prove a form of instability of the solution $q(t,x)$.

From now on, we assume that (4) holds and we seek a contradiction. We take $0 < \alpha < \alpha_0$ (where α_0 can be taken smaller if needed) and we consider $\lambda > 1$ as in (4). For all $t \geqslant 0$, we decompose $u(t,x)$ as $u(t,x) = (Q+\varepsilon)(t, x+\sigma(t))$ where

$$(\varepsilon(t), Q') = 0 \textbf{ and } \|\varepsilon(t)\|_{H^1} \leqslant C\alpha_0.$$

We assume that the function $t \mapsto \sigma(t)$ is of class C^1.

(j) Using the conservation of the L^2 norm for $u(t)$, show that for all $t \geqslant 0$,

$$|(\varepsilon(t), Q)| \leqslant C\|\varepsilon(t)\|_{H^1}^2.$$

(k) Write the equation satisfied by ε and prove that, for all $t \geqslant 0$,

$$|\sigma'(t) - 1| \leqslant C\|\varepsilon(t)\|_{H^1}.$$

(l) Prove that there exists $K_3 > 0$ such that, for all $t \geqslant 0$,

$$|(\varepsilon(t), \phi)| \geqslant K_3\|\varepsilon(t)\|_{H^1} + K_3\sqrt{\delta(\lambda)}.$$

Let χ be a smooth function on $\mathbb{R}$ such that

$$\chi(x) \equiv 1 \text{ on } (-\infty, 1], \quad \chi(x) \equiv 0 \text{ on } [2, +\infty), \quad 0 \leqslant \chi(x) \leqslant 1 \text{ on } [1,2].$$

Let $\zeta(x)$ be defined as follows

$$\zeta(x) = \int_{-\infty}^{x} (\Lambda Q(y) + \beta\phi(y))dy, \quad \beta = -\frac{\int Q\Lambda Q}{\int Q^5}. \tag{5}$$

For $A > 1$ large to be defined later, for all $t \geqslant 0$, set

$$J(t) = \int \varepsilon(t,x)\zeta(x)\chi\left(\frac{x}{A}\right) dx.$$

(m) Prove that

$$\forall t \geqslant 0, \quad |J(t)| \leqslant C\alpha_0 A^{\frac{1}{2}}.$$

Check that $\int \zeta Q' = 0$.

(n) Using the equation of ε, prove (formally) that

$$\frac{d}{dt}J(t) = K_4(\varepsilon(t), \phi) + R_J(\varepsilon(t))$$

where $K_4 \neq 0$ is a constant to specify and where R_J satisfies

$$|R_J(\varepsilon)| \leqslant C\|\varepsilon\|_{H^1}^2 + CA^{-\frac{1}{2}}\|\varepsilon\|_{H^1}.$$

(o) Find a contradiction to (4) for $A > 1$ large enough and $\alpha_0 > 0$ small enough.