The Cubic Klein-Gordon Equation with Damping (Master 2 Lecture Notes)

Yvan Martel*

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The objective of this course is to give a qualitative description of the asymptotic behavior in large time of all the global solutions of the one-dimensional focusing cubic Klein-Gordon equation with damping

$$\begin{cases} \partial_t^2 u + 2\alpha\partial_t u - \partial_x^2 u + u - u^3 = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

Here, $\alpha \in (0,1)$ is a fixed damping constant and $(u_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ is the initial data. We start by a study of the local and global Cauchy problem. Then, we introduce the key notion of solitary waves for this equation, and we study their stability properties. By variational techniques, it is then proved that in large time, any global solution converges strongly, at least for a subsequence, to the zero function or to a sum of decoupled solitary waves. Lastly, we describe a more detailed convergence result, for the whole sequence of time, with a characterization of all the possible asymptotic configurations and a precise convergence rate.

These lecture notes contain no new material and are entirely inspired by the references [1, 3, 4, 7, 9, 13, 14, 16].

1 The local Cauchy problem

1.1 The linear problem

A solution u of (1) will be seen as a solution of the first order system

$$\begin{cases} \partial_t u = v \\ \partial_t v = -2\alpha v + \partial_x^2 u - u + u^3, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{NLKG}$$

and we will use the notation $\vec{u} = (u, \partial_t u) = (u, v)$. We define the energy of $\vec{u}$ by

$$E(\vec{u}) = \int_{\mathbb{R}} \left(\tfrac{1}{2}v^2 + \tfrac{1}{2}(\partial_x u)^2 + \tfrac{1}{2}u^2 - \tfrac{1}{4}u^4\right) \mathrm{d}x. \tag{2}$$

*yvan.martel@uvsq.fr, Département de Mathématiques de l'UVSQ, Université Paris-Saclay

We check by integration by parts that it holds formally

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\vec{u}) = -2\alpha\|v\|^2$$

and thus, for $0 \le t_1 < t_2$,

$$E(\vec{u}(t_2)) - E(\vec{u}(t_1)) = -2\alpha \int_{t_1}^{t_2} \|v\|^2 \,\mathrm{d}t. \tag{3}$$

Since $\alpha > 0$, we obtain that the energy is nonincreasing for any solution for which (3) can be justified. This important qualitative property leads us to work for finite energy solutions, that is solutions such that $\vec{u}(t) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, for which the quantity E is well-defined. The space $H^1(\mathbb{R}) \times L^2(\mathbb{R})$, denoted simply by $H^1 \times L^2$ or by X, will be called the energy space. We also denote $Y = L^2 \times H^{-1}$.

The notation $\int$ will be used for $\int_{\mathbb{R}} \mathrm{d}x$. We denote $\langle \cdot, \cdot \rangle$ the L^2 scalar product for real-valued functions u_i or vector-valued functions $\vec{u}_i = (u_i, v_i)$ $(i = 1, 2)$

$$\|u\| := \|u\|, \quad \langle u_1, u_2 \rangle := \int u_1 u_2, \quad \langle \vec{u}_1, \vec{u}_2 \rangle := \int u_1 u_2 + \int v_1 v_2,$$

and we denote

$$\|\vec{u}\|_X := \sqrt{\|u\|_{H^1}^2 + \|v\|^2}, \quad \langle \vec{u}_1, \vec{u}_2 \rangle_X := \int (\partial_x u_1)(\partial_x u_2) + \int u_1 u_2 + \int v_1 v_2,$$

$$\|\vec{u}\|_Y := \sqrt{\|u\|^2 + \|v\|_{H^{-1}}^2}.$$

Lemma 1.1 ([4, Chapter 9.5]). *The linear problem*

$$\begin{cases} \partial_t u = v \\ \partial_t v = \partial_x^2 u - u - 2\alpha v, \end{cases} \tag{4}$$

generates a strongly continuous semigroup of contractions $(S_\alpha(t))_{t \ge 0}$ in X satisfying, for some $C_\alpha \ge 1$, $\gamma > 0$, for all $t \ge 0$,

$$\|S_\alpha(t)\|_{\mathcal{L}(X)} \le C_\alpha e^{-\gamma t}, \tag{5}$$

Moreover, $(S_\alpha(t))_{t \ge 0}$ extends to a strongly continuous semigroup of contraction in Y satisfying, for some $C'_\alpha \ge 1$, $\gamma' > 0$, for all $t \ge 0$,

$$\|S_\alpha(t)\|_{\mathcal{L}(Y)} \le C'_\alpha e^{-\gamma' t}.$$

Proof. We define the operator A_α on $H^1 \times L^2$ by

$$\begin{cases} D(A_\alpha) = H^2 \times H^1, \\ A_\alpha \vec{u} = (v, u'' - u - 2\alpha v), \text{ for any } \vec{u} \in D(A_\alpha). \end{cases}$$

We claim that the operator A_α is maximally dissipative in the sense that

- A_α is dissipative: for all $\vec{u} \in D(A_\alpha)$ and all $\lambda > 0$, $\|\vec{u} - \lambda A_\alpha \vec{u}\|_X \geq \|\vec{u}\|_X$,
- for all $\lambda > 0$ and all $\vec{f} \in X$, there exists $\vec{u} \in D(A_\alpha)$ such that $\vec{u} - \lambda A_\alpha \vec{u} = \vec{f}$.

Indeed, we have

$$\begin{aligned} \langle A_\alpha \vec{u}, \vec{w} \rangle_X &= \int v'w' + vw + (u'' - u - 2\alpha v)z \\ &= \int (v'w' + vw - u'z' - uz) - 2\alpha \int vz. \end{aligned}$$

In particular, $\langle A_\alpha \vec{u}, \vec{u} \rangle_X = -2\alpha \int v^2 \leq 0$. Moreover, for an operator on a Hilbert space, the property $\langle A_\alpha \vec{u}, \vec{u} \rangle_X \leq 0$ is known to be equivalent to the fact that A_α is dissipative. Then, we prove the surjectivity. It is enough to prove the surjectivity for $\lambda = 1$. Let $\vec{f} \in X$. We solve

$$\begin{cases} u - v = f \\ v - u'' + u + 2\alpha v = g \end{cases} \quad \Longleftrightarrow \quad \begin{cases} -u'' + 2(1+\alpha)u = g + (1+2\alpha)f \\ v = u - f \end{cases}$$

Using the Fourier transform, or the convolution product, or the Lax-Milgram theorem, it is easy to find $u \in H^2$, solution of $-u'' + 2(1+\alpha)u = g + (1+2\alpha)f$. Then, we set $v = u - f \in H^1$. Moreover, it is clear that the domain $D(A_\alpha)$ is dense in X. Therefore, by the Hille-Yosida-Phillips theorem, A_α generates a semigroup of contraction $(S_\alpha(t))_{t\geq 0}$ on X.

For a solution of (4), we set

$$N(t) = \int (v^2 + (\partial_x u)^2 + u^2 + 2\alpha uv)$$

and we compute $\frac{d}{dt}N = -2\alpha N$. Thus, $N(t) = N(0)e^{-2\alpha t}$ and since

$$(1-\alpha) \int (v^2 + (\partial_x u)^2 + u^2) \leq N(t) \leq (1+\alpha) \int (v^2 + (\partial_x u)^2 + u^2)$$

we obtain the result for the bound in $\mathcal{L}(X)$.

The theory in $Y = L^2 \times H^{-1}$ is done similarly. □

Remark 1.2 (The User Guide)**.** For $\vec{u}_0 \in D(A_\alpha)$, the function $\vec{u}(t) = S_\alpha(t)\vec{u}_0$ is the unique solution of the linear problem

$$\begin{cases} \vec{u} \in C([0,+\infty), D(A_\alpha)) \cap C^1([0,+\infty), X) \\ \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = A_\alpha \vec{u} \\ \vec{u}(0) = \vec{u}_0 \end{cases}$$

For $\vec{u}_0 \in X$, the function $\vec{u}(t) = S_\alpha(t)\vec{u}_0$ is unique solution of the linear problem

$$\begin{cases} \vec{u} \in C([0,+\infty), X) \cap C^1([0,+\infty), Y) \\ \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = A_\alpha \vec{u} \\ \vec{u}(0) = \vec{u}_0 \end{cases}$$

1.2 The nonlinear problem

The standard theory of semilinear evolution equations (see for instance [4, Chapter 4.3] or [19]) yields the following result.

Proposition 1.3. *For any initial data $\vec{u}_0 \in X$, there exists a unique maximal solution*

$$\vec{u} = (u, \partial_t u) \in C([0, T_{\max}), X) \cap C^1([0, T_{\max}), Y)$$

of (NLKG) *satisfying $\vec{u}(0) = \vec{u}_0$.*

If the maximal time of existence $T_{\max}$ is finite, then $\lim_{t \uparrow T_{\max}} \|\vec{u}(t)\|_X = \infty$.

If $\vec{u}_0 \in D(A_\alpha)$, then

$$\vec{u} = (u, \partial_t u) \in C([0, T_{\max}), D(A_\alpha)) \cap C^1([0, T_{\max}), X).$$

Moreover, the map $T_{\max} : \vec{u}_0 \in X \mapsto (0, \infty)$ is lower semicontinuous, and for any sequence $(\vec{u}_{0,n})$ in X, if $\lim_{n\to\infty} \vec{u}_{0,n} = \vec{u}_0$ in X then, for any $0 < T < T_{\max}$,

$$\lim_{n\to\infty} \vec{u}_n = \vec{u} \quad \text{in } C([0, T], X),$$

where $\vec{u}_n$ is the solution of (1) *corresponding to $\vec{u}_{0,n}$.*

Proof. Observe that the map $u \mapsto u^3$ is Lipschitz continuous from bounded sets of H^1 to L^2. Indeed, in dimension one, one has $\sup_{\mathbb{R}} |u| \le C\|u\|_{H^1}$ and thus

$$|u^3 - v^3| \le C(|u|^2 + |v|^2)|u - v| \quad \text{so that} \quad \|u^3 - v^3\| \le C(\|u\|_{H^1}^2 + \|v\|_{H^1}^2)\|u - v\|.$$

Let $\bar{B}_M$ denote the closed ball of X of center 0 and radius $M > 0$. It follows that there exists $C_L > 0$ such that for all $M > 0$ and for all $u, v \in \bar{B}_M$ it holds

$$\|u^3 - v^3\| \le C_L M^2 \|u - v\|. \tag{6}$$

We rewrite (NLKG) under the following equivalent Duhamel formulation

$$\vec{u}(t) = S_\alpha \vec{u}_0 + \int_0^t S_\alpha(t - s)(0, u^3(s))\,\mathrm{d}s. \tag{7}$$

Uniqueness. Let $T > 0$. Then there exists at most one solution of (7) on $[0, T]$. Indeed, let $\vec{u}_1, \vec{u}_2$ be two solutions of (7) with the same initial data. Set

$$M = \sup_{t\in[0,T]} \max\{\|\vec{u}_1(t)\|_X ; \|\vec{u}_2(t)\|_X\}.$$

We have by (7) and $\|S(t)\|_{\mathcal{L}(X)} \le C$,

$$\|\vec{u}_1(t) - \vec{u}_2(t)\|_X \le C \int_0^t \|u_1^3(s) - u_2^3(s)\|\,\mathrm{d}s \le CM^2 \int_0^t \|u_1(s) - u_2(s)\|\,\mathrm{d}s.$$

It follows from the Gronwall lemma that $\|\vec{u}_1(t) - \vec{u}_2(t)\|_X = 0$, for all $t \in [0, T]$. □

Existence of a local solution by contraction. Let $M > 0$ and fix

$$T_M = \frac{1}{2CM^2} > 0. \tag{8}$$

We claim that for any $\vec{u}_0 \in X$ such that $\|\vec{u}_0\|_X \leq M/2$, there exists a solution $\vec{u}$ of (7) on $[0, T]$. Define

$$E = \{\vec{u} \in \mathcal{C}([0, T_M], X) : \|\vec{u}(t)\|_X \leq M, \text{ for all } t \in [0, T_M]\}.$$

We equip E with the distance generated by norm of $\mathcal{C}([0, T_M], X)$, *i.e.*, for any $\vec{u}_1, \vec{u}_2 \in E$,

$$d(u, v) = \sup_{t \in [0, T_M]} \|\vec{u}_1(t) - \vec{u}_2(t)\|_X.$$

Since $\mathcal{C}([0, T_M], X)$ is a Banach space and E is closed in $\mathcal{C}([0, T_M], X)$, (E, d) is a complete metric space. For all $\vec{u} \in E$, we define $\Phi(\vec{u}) \in \mathcal{C}([0, T_M], X)$ by

$$\Phi(\vec{u})(t) = S(t)\vec{u}_0 + \int_0^t S(t - s)u^3(s)\, \mathrm{d}s,$$

for all $t \in [0, T_M]$.

First, we prove that $\Phi : E \to E$. Indeed, for any $s \in [0, T_M]$, by (6)

$$\|u^3\| \leq CM^2\|u\| \leq CM^3,$$

It follows from $\|S_\alpha(t)\| \leq C$ and the definition of T_M in (8) that for any $t \in [0, T_M]$,

$$\|\Phi(\vec{u})(t)\|_X \leq \|\vec{u}_0\|_X + \int_0^t \|u^3(s)\|\, \mathrm{d}s \leq M + C_L T_M M^3 \leq \frac{3}{2}M.$$

Second we prove that Φ is a contraction on (E, d). Indeed, for any $\vec{u}, \vec{v} \in E$, and for any $t \in [0, T_M]$,

$$\|\Phi(\vec{u})(t) - \Phi(\vec{v})(t)\| \leq \int_0^t \|u^3(s) - v^3(s)\|\, \mathrm{d}s \leq C_L T_M M^2 d(u, v) \leq \frac{1}{2}d(u, v).$$

By the Banach Fixed-Point Theorem, Φ has a unique fixed-point $\vec{u} \in E$, which is a solution of (7).

Maximal solution. We claim that there exists a function $T_{\max} : X \to (0, \infty]$ with the following properties. For any $\vec{u}_0 \in X$, there exists $u \in \mathcal{C}([0, T_{\max}(\vec{u}_0)), X)$, such that for all $T \in (0, T_{\max}(\vec{u}_0))$, u is the unique solution of (7). Moreover, the following alternative holds:

(i) Either $T_{\max}(\vec{u}_0) = \infty$;

(ii) Or $T_{\max}(\vec{u}_0) < \infty$ and then $\lim_{t \uparrow T_{\max}(\vec{u}_0)} \|\vec{u}(t)\|_X = \infty$.

When property (i) holds, one says that the solution is globally defined, or global. When property (ii) holds, one says that the solution blows up in finite time.

Proof. Let $\vec{u}_0 \in X$ and $M = 2\|\vec{u}_0\|_X$. We define

$$T_{\max}(\vec{u}_0) = \sup\{T > 0 : \text{there exists a solution } u \text{ of (7) on } [0,T]\}.$$

We have just proved that $T_{\max}$ is well-defined and $T_{\max} \geq T_M > 0$. Now, we define a function $\vec{u} \in \mathcal{C}([0, T_{\max}(\vec{u}_0)), X)$ which is solution of (7) on $[0,T]$ for any $T \in (0, T_{\max}(\vec{u}_0))$. Let $t \in [0, T_{\max}(\vec{u}_0))$. Let $T \in [t, T_{\max}(\vec{u}_0))$. By the definition of $T_{\max}(\vec{u}_0)$ as a supremum, there exists a solution $\vec{u}_T$ of (7) on $[0,T]$. Then, we set $\vec{u}(t) = \vec{u}_T(t)$ on $[0,T]$. By the uniqueness result, this definition does not depend on the choice of $T \in [t, T_{\max}(\vec{u}_0))$. Thus, it provides a function $\vec{u} \in \mathcal{C}([0, T_{\max}(\vec{u}_0)), X)$ which is indeed a solution of (7) on $[0,T]$ for any $T \in (0, T_{\max}(\vec{u}_0))$. Last, note that by the definition of $T_{\max}(\vec{u}_0)$, this solution cannot be extended beyond $T_{\max}(\vec{u}_0)$. This solution is called the *maximal solution* of (7).

Now, we prove the *blowup alternative.* Fix any $\tau \in [0, T_{\max}(\vec{u}_0))$, set $M = 2\|u(\tau)\|$ and consider $T_M > 0$ given by (8). There exists a solution $\vec{w}$ of

$$\begin{cases} \vec{w} \in \mathcal{C}([0, T_M], X), \\ \vec{w}(t) = S(t)\vec{u}(\tau) + \displaystyle\int_0^t S(t-s) w^3(s)\,\mathrm{d}s. \end{cases} \tag{9}$$

We extend the function $\vec{w} \in \mathcal{C}([0, \tau + T_M], X)$ by setting

$$\vec{w}(t) = \begin{cases} \vec{u}(t) & \text{if } t \in [0,\tau], \\ \vec{w}(t-\tau) & \text{if } t \in [\tau, \tau + T_M]. \end{cases}$$

We observe that $\vec{w}$ is now a solution of the problem (7) on the interval $[0,T]$, for $T = \tau + T_M$. By the definition of $T_{\max}(\vec{u}_0)$, this shows that

$$\tau + T_M < T_{\max}(\vec{u}_0).$$

Assume $T_{\max}(\vec{u}_0) < \infty$. By the general definition of T_M in (8) and the value of $M = 2\|u(\tau)\|$ in the present context, we obtain

$$\frac{1}{2C_L M^2} \leq T_{\max}(\vec{u}_0) - \tau.$$

This is equivalent to

$$2C_L \|u(\tau)\|^2 \geq \frac{1}{T_{\max}(\vec{u}_0) - \tau}, \tag{10}$$

which proves that if $T_{\max}(\vec{u}_0) < \infty$, then $\lim_{t \uparrow T_{\max}(\vec{u}_0)} \|\vec{u}(t)\|_X = \infty$.

Persistence of regularity. In the above framework, since $u^3 \in C([0, T_{\max}(\vec{u}_0), H^1)$, one has $(0, u^3) \in C([0, T_{\max}(\vec{u}_0), D(A_\alpha))$. Assume now in addition that $\vec{u}_0 \in D(A_\alpha)$. Using the Duhamel formulation (7) and the properties of S_α, we obtain $u \in C([0, T_{\max}), D(A_\alpha))$ and then $\partial_t u \in C([0, T_{\max}), X)$.

Continuous dependence on the initial data. Now, we claim that

(i) The function $T_{\max} : X \to (0, \infty]$ is lower semi-continuous;

(ii) If $\vec{u}_{0,n} \to \vec{u}_0$ as $n \to \infty$ in X, then for any $T \in (0, T_{\max}(\vec{u}_0))$, $\vec{u}_n \to \vec{u}$ in $\mathcal{C}([0,T], X)$ as $n \to \infty$, where $\vec{u}_n$ and $\vec{u}$ are the solutions of (7) corresponding respectively to $\vec{u}_{0,n}$ and $\vec{u}_0$.

Let $T \in (0, T_{\max}(\vec{u}_0))$. To prove (1)-(2), it suffices to show that if $\vec{u}_{0,n} \to \vec{u}_0$ then for n large enough $T_{\max}(\vec{u}_{0,n}) > T$ and $\vec{u}_n \to \vec{u}$ in $\mathcal{C}([0,T], X)$.

Set $M = 1 + 2\sup_{t\in[0,T]} \|\vec{u}(t)\|_X$ and define

$$\tau_n = \sup\{t \in [0, T_{\max}(\vec{u}_0)) : \|\vec{u}_n(s)\|_X \leq M \text{ for all } s \in [0,t]\}.$$

Since $\|\vec{u}_{0,n}\| < M/2$ for n large enough, $\tau_n > 0$ is well-defined. Moreover, by the well-posedness theory $\tau_n > T_M$. For any $t \in [0, \min(T; \tau_n)]$, we have

$$\|\vec{u}(t) - \vec{u}_n(t)\|_X \leq \|\vec{u}_0 - \vec{u}_{0,n}\|_X + C_L M^2 \int_0^t \|\vec{u}(s) - \vec{u}_n(s)\| \,\mathrm{d}s,$$

and thus by the Gronwall Lemma, for any $t \in [0, \min(T; \tau_n)]$,

$$\|\vec{u}(t) - \vec{u}_n(t)\|_X \leq \|\vec{u}_0 - \vec{u}_{0,n}\|_X \exp\left(C_L M^2 T\right). \tag{11}$$

This proves that for any $t \in [0, \min(T; \tau_n)]$,

$$\|\vec{u}_n(t)\|_X \leq \|\vec{u}(t)\|_X + \|\vec{u}(t) - \vec{u}_n(t)\|_X \leq \frac{M}{2} + \|\vec{u}_0 - \vec{u}_{0,n}\|_X \exp\left(C_L M^2 T\right) < \frac{3M}{4},$$

for n large enough. Therefore, $\tau_n > T$, which also justifies that $T_{\max}(\vec{u}_{0,n}) > T$.

Lastly, estimate (11) implies that $\vec{u}_n \to \vec{u}$ in $\mathcal{C}([0,T], X)$. □

In this course, we systematically work in the framework of such maximal finite energy solutions.

Corollary 1.4. *In the context of Proposition 1.3, the function $t \mapsto E(\vec{u}(t))$ is C^1 on $[0, T_{\max}(\vec{u}_0))$ and for all $t \in [0, T_{\max}(\vec{u}_0))$, it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\vec{u}(t)) = -2\alpha \|v(t)\|^2.$$

Proof. Let $\vec{u}_0 \in X$ and for all $n \geq 0$, let $\vec{u}_{0,n} \in D(A_\alpha)$ be such that $\vec{u}_{0,n} \to \vec{u}_0$ as $n \to \infty$ in X. It is known that for any $T \in (0, T_{\max}(\vec{u}_0))$, $\vec{u}_n \to \vec{u}$ in $\mathcal{C}([0,T], X)$ as $n \to \infty$.

For $\vec{u}(t)$, it is rigorously checked by using (NLKG) that

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\vec{u}_n(t)) = -2\alpha \|v_n(t)\|^2.$$

In particular, for all $t \in [0, T_{\max}(\vec{u}_0))$, and all n large,

$$E(\vec{u}_n(t)) - E(\vec{u}_{0,n}) = -2\alpha \int_0^t \|v_n(s)\|^2 \,\mathrm{d}s.$$

Passing to the limit $n \to +\infty$ $E(\vec{u}_n(t)) \to E(\vec{u}(t))$ and $\|v_n(s)\|^2 \to \|v(s)\|^2$. Thus, for all $t \in [0, T_{\max}(\vec{u}_0))$,

$$E(\vec{u}(t)) - E(\vec{u}_0) = -2\alpha \int_0^t \|v(s)\|^2 \,\mathrm{d}s.$$

This proves the result. □

2 The global Cauchy problem

2.1 On blowup in finite time

The negative sign in front of u^3 in equation (1) means that the equation is focusing. In particular, the sign of the quartic term in the definition of the energy prevents us to use the decay of energy to prove global wellposedness. On the contrary, we are going to prove that there exist blow up solutions for the equation.

Together with the energy functional $E(t) := E(\vec{u}(t))$ defined in (2) and satisfying (3), we will use the following quantities

$$
\begin{aligned}
M(t) &:= \frac{1}{2}\|u(t)\|^2 + \alpha \int_0^t \|u(s)\|^2 \, \mathrm{d}s, \\
W(t) &:= \frac{1}{2}\left(\|\partial_t u(t)\|^2 + \|\partial_x u(t)\|^2 + \|u(t)\|^2\right).
\end{aligned}
$$

Lemma 2.1. *It holds*

$$
\begin{aligned}
M'(t) &= \int u(t)\partial_t u(t) \, \mathrm{d}x + \alpha\|u(t)\|^2 && (12)\\
&= \int u(t)\partial_t u(t) \, \mathrm{d}x + 2\alpha \int_0^t \int u(s)\partial_t u(s) \, \mathrm{d}x \, \mathrm{d}s + \alpha\|u(0)\|^2, && (13)\\
M''(t) &= 3\|\partial_t u(t)\|^2 + \|\partial_x u(t)\|^2 + \|u(t)\|^2 - 4E(t), && (14)\\
W'(t) &= -2\alpha\|\partial_t u(t)\|^2 + \int u^3(t)\partial_t u(t) \, \mathrm{d}x. && (15)
\end{aligned}
$$

Proof. Direct computations using (1) and (2). Density arguments are used as in the proof of Corollary 1.4. □

Theorem 2.2. *Let* $0 < \alpha \leq \frac{1}{4}$*. If* $E(0) < 0$*, then the corresponding solution of* (1) *blows up in finite time.*

Proof. Assume that $E(0) < 0$. For the sake of contradiction, assume that the solution is global. Then, by (3), $E(t) \leq E(0) < 0$. In particular, by (14), we have $M''(t) \geq -4E(t) \geq -4E(0) > 0$. It follows that $\lim_{t\to+\infty} M(t) = +\infty$. Moreover, since $M''(t) \geq 3\|\partial_t u(t)\|^2 + \|u(t)\|^2$, we also have

$$
M(t)M''(t) \geq \frac{1}{2}\|u(t)\|^2(3\|\partial_t u(t)\|^2 + \|u\|^2) \geq \frac{3}{2}\left(\int u\partial_t u\right)^2 + \frac{1}{2}\|u\|^4.
$$

Using the inequality $(a+b)^2 \leq \frac{5}{4}a^2 + 5b^2$, and then $\alpha < \frac{1}{4}$, we have

$$
(M'(t))^2 \leq \frac{5}{4}\left(\int u\partial_t u\right)^2 + 5\alpha^2\|u\|^4 \leq \frac{5}{6}M(t)M''(t).
$$

This implies that for all $t \geq 0$,

$$
(M^{-\frac{1}{5}})''(t) \leq 0.
$$

Since $\lim_{t\to+\infty} M^{-\frac{1}{5}}(t) = 0$, there exists $t_1 > 0$ such that $(M^{-\frac{1}{5}})'(t_1) < 0$. Using the concavity, we obtain for $t \geq t_1$,

$$0 \leq M^{-\frac{1}{5}}(t) \leq M^{-\frac{1}{5}}(t_1) + (t - t_1)(M^{-\frac{1}{5}})'(t_1).$$

This is contradictory for t large. □

2.2 Global solutions are bounded

Using arguments of [3] and [2, Proof of Lemma 2.7] for the undamped Klein-Gordon equation, we prove a bound on global solutions of (1).

Theorem 2.3 ([2, 3]). *Any global solution of* (1) *is bounded in* X.

Proof. Let $\vec{u}$ be a global solution of (1). From (12) and the Cauchy-Schwarz inequality,

$$|M'(t)| \leq (1 + 2\alpha)W(t). \tag{16}$$

Moreover, by (3) and (14),

$$M''(t) \geq 2W(t) - 4E(0). \tag{17}$$

The proof of the global bound now proceeds in three steps.

Step 1. We prove that

$$\liminf_{t\to\infty} M'(t) < \infty. \tag{18}$$

Proof of (18). We argue by contradiction, proving that $\lim_\infty M' = \infty$ implies the following inequality, for all t large enough,

$$(1 + \epsilon)[M'(t)]^2 < M''(t)M(t) \quad \text{where } \epsilon > 0 \text{ is to be chosen.} \tag{19}$$

Then, we reach a contradiction by a standard argument. Indeed, remark that (19) implies $\frac{\mathrm{d}^2}{\mathrm{d}t^2}[M^{-\epsilon}(t)] < 0$, and $\lim_\infty M' = \infty$ also implies $\lim_\infty M^{-\epsilon} = 0$. Thus, there exists $t_1 > 0$ such that $\frac{\mathrm{d}}{\mathrm{d}t}[M^{-\epsilon}(t_1)] < 0$, and for all $t \geq t_1$,

$$0 \leq M^{-\epsilon}(t) \leq M^{-\epsilon}(t_1) + (t - t_1)\frac{\mathrm{d}}{\mathrm{d}t}[M^{-\epsilon}(t_1)],$$

which is absurd for $t \geq t_1$ large enough.

Thus, we only need to prove (19) assuming $\lim_\infty M' = \infty$. On the one hand, by (13) and the Cauchy-Schwarz inequality, it holds

$$|M'| \leq \|u\|\|\partial_t u\| + 2\alpha \left(\int_0^t \|u(s)\|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_0^t \|\partial_t u(s)\|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} + \alpha\|u(0)\|^2.$$

Let $\epsilon > 0$ to be chosen later, we estimate

$$\begin{aligned}|M'|^2 \leq (1 + \epsilon) &\left[\|u\|\|\partial_t u\| + 2\alpha \left(\int_0^t \|u(s)\|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_0^t \|\partial_t u(s)\|^2 \,\mathrm{d}s\right)^{\frac{1}{2}}\right]^2 \\ &+ \left(1 + \frac{1}{\epsilon}\right)\alpha^2\|u(0)\|^4.\end{aligned}$$

Thus, using the inequality $(AB+CD)^2 \le (A^2+C^2)(B^2+D^2)$, we obtain

$$
\begin{aligned}
|M'|^2 &\le (1+\epsilon)\left[\frac{1}{2}\|u\|^2 + \alpha\int_0^t \|u(s)\|^2\,\mathrm{d}s\right]\left[2\|\partial_t u\|^2 + 4\alpha\int_0^t \|\partial_t u(s)\|^2\,\mathrm{d}s\right] \\
&\quad + \left(1+\frac{1}{\epsilon}\right)\alpha^2\|u(0)\|^4 \\
&\le (1+\epsilon)M\left[2\|\partial_t u\|^2 + 4\alpha\int_0^t \|\partial_t u(s)\|^2\,\mathrm{d}s\right] + \left(1+\frac{1}{\epsilon}\right)\alpha^2\|u(0)\|^4.
\end{aligned}
$$

On the other hand, by (3) and (14),

$$
\begin{aligned}
M'' &= 2\|\partial_t u\|^2 + 2W + 8\alpha\int_0^t \|\partial_t u(s)\|^2\,\mathrm{d}s - 4E(0) \\
&\ge (1+\epsilon)^3\left[2\|\partial_t u\|^2 + 4\alpha\int_0^t \|\partial_t u(s)\|^2\,\mathrm{d}s\right] + W - 4E(0),
\end{aligned}
$$

by fixing any ϵ such that

$$
0 < \epsilon < \left(\frac{5}{4}\right)^{\frac{1}{3}} - 1.
$$

In particular, since $\lim_\infty W = \infty$ by (16) and the assumption $\lim_\infty M' = \infty$, we have for t large enough,

$$
M'' \ge (1+\epsilon)^3\left[2\|\partial_t u\|^2 + 4\alpha\int_0^t \|\partial_t u(s)\|^2\,\mathrm{d}s\right].
$$

Thus,

$$
(1+\epsilon)^2|M'|^2 \le MM'' + \left(1+\frac{1}{\epsilon}\right)\alpha^2\|u(0)\|^4,
$$

and using again $\lim_\infty M' = \infty$ we obtain (19) for any t large enough.

Step 2. We prove that

$$
\sup_{t\in[0,\infty)} |M'(t)| < \infty. \tag{20}
$$

Proof of (20). Combining (16) and (17), we obtain

$$
M''(t) \ge \frac{2}{1+2\alpha}|M'(t)| - 4E(0).
$$

Let

$$
\begin{aligned}
H_+(t) &= \frac{2}{1+2\alpha}M'(t) - 4E(0), \\
H_-(t) &= -\frac{2}{1+2\alpha}M'(t) - 4E(0).
\end{aligned}
$$

Then, $H'_+(t) = \frac{2}{1+2\alpha}M''(t) \geq \frac{p-1}{1+2\alpha} + (t)$. If there exists $t \geq 0$ such that $H_+(t) > 0$, then $\lim_\infty H_+ = \infty$, contradicting (18). It follows that for all $t \geq 0$,

$$M'(t) \leq 2(1+2\alpha)E(0).$$

Similarly, $H'_-(t) = -\frac{2}{1+2\alpha}M''(t) \leq -\frac{2}{1+2\alpha}H_-(t)$. It follows that $H_-(t) \leq e^{-\frac{2}{1+2\alpha}t}H_-(0)$, for all $t \geq 0$. Thus,

$$M'(t) \geq -\frac{1+2\alpha}{2}\left(4E(0) + |H_-(0)|\right).$$

and (20) is proved.

Step 3. Last, we prove the global bound

$$\sup_{t\in[0,\infty)} |W(t)| < \infty. \tag{21}$$

Proof of (21). We rewrite (17) as

$$W(t) \leq \frac{1}{2}M''(t) + 2E(0).$$

Integrating on $(t, t+1)$ and using (20), we observe that

$$\sup_{t\geq 0} \int_t^{t+1} W(s)\,\mathrm{d}s < \infty. \tag{22}$$

Moreover, by (15),

$$W' \leq -2\alpha\|\partial_t u\|^2 + \int |u|^3|\partial_t u| \leq \frac{1}{2}\|\partial_t u\|^2 + \frac{1}{2}\int |u|^6 \leq W + \frac{1}{2}\int |u|^6.$$

For $t \geq 1$ and $\tau \in (0,1)$, integrating on $(t-\tau, t)$, we find

$$\begin{aligned} W(t) &\leq W(t-\tau) + \int_{t-\tau}^t W(s)\,\mathrm{d}s + \frac{1}{2}\int_{t-\tau}^t \int |u(s)|^6\,\mathrm{d}x\,\mathrm{d}s \\ &\leq W(t-\tau) + \int_{t-1}^t W(s)\,\mathrm{d}s + \frac{1}{2}\int_{t-1}^t \int |u(s)|^6\,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

Using the Sobolev inequality (in space-time) for the last term, we obtain, for some constants $C > 0$,

$$\begin{aligned} W(t) &\leq W(t-\tau) + \int_{t-1}^t W(s)\,\mathrm{d}s + C\|u\|^6_{H^1((t-1,t)\times\mathbb{R})} \\ &\leq W(t-\tau) + \int_{t-1}^t W(s)\,\mathrm{d}s + C\left(\int_{t-1}^t W(s)\,\mathrm{d}s\right)^3. \end{aligned}$$

Integrating in $\tau \in (0,1)$ and using (22), we find (21). □

3 The solitary waves

It is also well-known that up to sign and translation, the only stationary solution of (1) is the solitary wave $(Q,0)$, where Q is the explicit ground state

$$Q(x)=\frac{\sqrt{2}}{\cosh(x)}=\sqrt{2}\,\mathrm{sech}(x) \tag{23}$$

which solves the equation

$$Q''-Q+Q^3=0 \quad \text{on } \mathbb{R}. \tag{24}$$

We see from the explicit expression of Q in (23) that, as $x\to\infty$,

$$Q(x)=c_Qe^{-x}+O(e^{-3x}),\quad Q'(x)=-c_Qe^{-x}+O(e^{-2x}) \tag{25}$$

where $c_Q=2\sqrt{2}$. Note that by (24), it holds $\int(\partial_xQ)^2+Q^2-Q^4=0$ and so

$$E(Q,0)=\frac{1}{4}\int Q^4>0. \tag{26}$$

Let

$$\begin{aligned}\mathcal{L}&=-\partial_x^2+1-3Q^2=-\partial_x^2+1-6\,\mathrm{sech}^2,\\ \langle\mathcal{L}\varepsilon,\varepsilon\rangle&=\int\left\{(\partial_x\varepsilon)^2+\varepsilon^2-3Q^2\varepsilon^2\right\}\mathrm{d}x.\end{aligned}$$

We recall some standard properties of the operator $\mathcal{L}$ (see *e.g.* [9, Lemma 1]).

Lemma 3.1. *The following properties hold.*

(i) Spectral properties. *The unbounded operator $\mathcal{L}$ on L^2 with domain H^2 is self-adjoint, its continuous spectrum is $[1,\infty)$, its kernel is* $\mathrm{span}\{Q'\}$ *and -3 is its unique negative eigenvalue with corresponding smooth normalized eigenfunction $Y=\frac{\sqrt{3}}{2}\,\mathrm{sech}^2(x)$.*

(ii) Coercivity property. *There exist $c_1,c_2>0$ such that, for all $\varepsilon\in H^1$,*

$$\langle\mathcal{L}\varepsilon,\varepsilon\rangle\geq c_1\|\varepsilon\|_{H^1}^2-c_2\left(\langle\varepsilon,Y\rangle^2+\langle\varepsilon,Q'\rangle^2\right).$$

Proof. The continuous spectrum of $\mathcal{L}$ is the same as the one of the operator $-\partial_x^2+1$, i.e. the interval $[1,+\infty)$, since the potential $-6\,\mathrm{sech}^2$ is a compact perturbation of $-\partial_x^2+1$.

We check by direct computations that $\mathcal{L}Y=-3Y$ and $\mathcal{L}Q'=0$. Since $Y>0$, it is a standard observation that -3 is the lowest eigenvalue of $\mathcal{L}$. Moreover, since Q' only has one zero, 0 is the second eigenvalue. Lastly, we check $R=1-\frac{3}{2}\,\mathrm{sech}^2$ satisfies $\mathcal{L}R=R$. Since $R\in L^\infty$ and $R'\in L^2$, but $R\notin L^2$ the bottom of the continuous spectrum 1 is called a resonance. Since R only vanishes twice on $\mathbb{R}$, there is no other discrete eigenvalue. In particular, by the spectral theorem, if $\langle\varepsilon,Y\rangle=0$ and $\langle\varepsilon,Q'\rangle=0$, then

$$\langle\mathcal{L}\varepsilon,\varepsilon\rangle\geq\|\varepsilon\|^2.$$

See also the Appendix.

For a general $\varepsilon \in H^1$, we decompose $\varepsilon = aY + bQ' + \eta$, where $\langle \eta, Y\rangle = 0$ and $\langle \eta, Q'\rangle = 0$. In particular, $a = \langle Y, \varepsilon\rangle$ and $b\|Q'\|^2 = \langle Q', \varepsilon\rangle$. We also have $\langle \mathcal{L}\eta, \eta\rangle \geq \|\eta\|^2$. Thus,

$$\begin{aligned}\langle \mathcal{L}\varepsilon, \varepsilon\rangle &= -3a^2 - b^2\|Q'\|^2 + \langle \mathcal{L}\eta, \eta\rangle \\ &\geq -3a^2 - b^2\|Q'\|^2 + \|\eta\|^2 \\ &\geq -4a^2 - 2b^2 + \|\varepsilon\|^2 \\ &\geq -4\langle \varepsilon, Y\rangle^2 - 2\langle \varepsilon, Q'\rangle^2 + \|\varepsilon\|^2\end{aligned}$$

Moreover, it is easy to see from the definition of $\mathcal{L}$ that

$$\langle \mathcal{L}\varepsilon, \varepsilon\rangle \geq \|\partial_x \varepsilon\|^2 - 5\|\varepsilon\|^2.$$

By taking a linear combinaison with coefficients 6/7 and 1/7 of the above inequalities, we find

$$\langle \mathcal{L}\varepsilon, \varepsilon\rangle \geq \frac{1}{7}\left(\|\partial_x\varepsilon\|^2 + \|\varepsilon\|^2\right) - \frac{24}{7}\langle \varepsilon, Y\rangle^2 - \frac{12}{7}\langle \varepsilon, Q'\rangle^2.$$

□

The unique negative eigenvalue of $\mathcal{L}$ is related to an instability of the solitary wave for the equation (1), described by the following functions:

$$\nu^\pm = -\alpha \pm \sqrt{\alpha^2 + 3}, \quad \vec{Y}^\pm = \begin{pmatrix} Y \\ \nu^\pm Y \end{pmatrix}, \tag{27}$$

$$\zeta^\pm = \alpha \pm \sqrt{\alpha^2 + 3}, \quad \vec{Z}^\pm = \begin{pmatrix} \zeta^\pm Y \\ Y \end{pmatrix}. \tag{28}$$

Indeed, it follows from explicit computations that the function

$$\vec{\varepsilon}^{\,\pm}(t, x) = \exp(\nu^\pm t)\vec{Y}^\pm(x)$$

is solution of the linearized problem

$$\begin{cases} \partial_t \varepsilon = \eta \\ \partial_t \eta = -\mathcal{L}\varepsilon - 2\alpha\eta. \end{cases} \tag{29}$$

Since $\nu^+ > 0$, the solution $\vec{\varepsilon}^{\,+}$ illustrates the exponential instability of the solitary wave in positive time. This means that the presence of the damping $\alpha > 0$ does not remove the exponential instability of the Klein-Gordon solitary wave. An equivalent formulation of instability is obtained by saying that the functions $\vec{Z}^\pm$ are the eigenfunctions of the adjoint linearized operator in (29):

$$\begin{pmatrix} 0 & -\mathcal{L} \\ 1 & -2\alpha \end{pmatrix} \vec{Z}^\pm = \nu^\pm \vec{Z}^\pm,$$

and as a consequence, for any solution $\vec{\varepsilon}$ of (29),

$$a^\pm = \langle \vec{\varepsilon}, \vec{Z}^\pm\rangle \quad \text{satisfies} \quad \frac{\mathrm{d}a^\pm}{\mathrm{d}t} = \nu^\pm a^\pm. \tag{30}$$

Remark 3.2. The existence of the solutions ε^+ is called linear exponential instability. More arguments are needed to prove that the solitary wave solution $(Q, 0)$ is actually nonlinearly unstable, in the following sense

$$\exists \delta_0 > 0, \forall \sigma > 0, \exists \vec{u}_0 \in X, \|\vec{u}_0 - (Q,0)\|_X \leq \sigma, \exists T > 0 : \inf_{a \in \mathbb{R}} \|\vec{u}(T) - (Q(\cdot - a), 0)\|_X \geq \delta_0$$

where $\vec{u}$ is the solution of (NLKG) with initial data $\vec{u}_0$. We will not address this question here, but it is an interesting exercise to prove this statement.

4 First decomposition result of any global solution

4.1 The Brezis-Lieb Lemma

The following is a particular case of the Brezis-Lieb lemma.

Lemma 4.1. *Let (f_n) be a sequence of functions in L^4 that converges a.e. to a function f and such that $\sup_n \|f_n\|_{L^4} < +\infty$. Then*

$$\lim_{n\to\infty} \int \left| f_n^4 - f^4 - (f - f_n)^4 \right| = 0.$$

In particular,

$$\lim_{n\to\infty} \left(\int f_n^4 - \int (f - f_n)^4 \right) = \int f^4.$$

Proof. Let $r_n = \left| f_n^4 - f^4 - (f - f_n)^4 \right|$. We have

$$\begin{aligned} r_n &= \left| (f_n - f + f)^4 - f^4 - (f - f_n)^4 \right| \\ &= \left| 4(f_n - f)^3 f + 6(f_n - f)^2 f^2 + 4(f_n - f) f^3 \right| \\ &\leq \epsilon (f_n - f)^4 + C_\epsilon f^4. \end{aligned}$$

for any $\epsilon > 0$. Thus, the nonnegative function[1] $s_{n,\epsilon} = (r_n - \epsilon(f_n - f)^4)_+$ converges a.e. to zero and is dominated by the integrable function $C_\epsilon f^4$. By the dominated convergence theorem, this proves that $\lim_{n\to+\infty} \int s_{n,\epsilon} = 0$. Now, $0 \leq r_n \leq s_{n,\epsilon} + \epsilon(f_n - f)^4$ and so

$$\limsup_{n\to+\infty} \int r_n \leq \epsilon \limsup_{n\to+\infty} \int (f_n - f)^4.$$

In particular,

$$\limsup_{n\to+\infty} \int r_n \leq C\epsilon \limsup_{n\to+\infty} \int (f_n^4 + f^4) \leq C\epsilon,$$

and ϵ being arbitrary, we obtain $\lim_{n\to+\infty} \int r_n = 0$. □

[1] The notation x_+ means $x_+ = \max(x, 0)$

4.2 A compactness result

Theorem 4.2. *Let (u_n) be a sequence of functions in $H^1(\mathbb{R})$ such that*

$$\text{(i)} \ \sup_n \|u_n\|_{H^1} < +\infty,$$
$$\text{(ii)} \ \lim_{n\to+\infty} u_n'' - u_n + u_n^3 = 0 \ \textit{in} \ H^{-1}.$$

Then, there exist a strictly increasing map $\varphi : \mathbb{N} \to \mathbb{N}$, $\xi_1 \in \mathbb{R}$, $\sigma_1 \in \{-1, 0, +1\}$, an integer $J \geq 1$ and if $J \geq 2$, $J - 1$ values $\sigma_j \in \{-1; +1\}$ and $J - 1$ sequences of points $(\xi_{j,n})_n$, $j = 2, \dots, J$ such that

$$\text{(i)} \ \lim_{n\to+\infty} |\xi_{j,n} - \xi_{k,n}| = \lim_{n\to+\infty} |\xi_{j,n}| = +\infty, \quad \textit{for } j \neq k, \ j \neq 1, \ k \neq 1$$

$$\text{(ii)} \ \lim_{n\to+\infty} \left\| u_{\varphi(n)} - \sigma_1 Q(x - \xi_1) - \sum_{j=2}^{J} \sigma_k Q(x - \xi_{j,n}) \right\|_{H^1(\mathbb{R})} = 0.$$

Remark 4.3. As usual, the sum $\sum_{j=2}^{J}$ is simply zero if $J = 1$.

Proof. The proof is taken from [1] (see also [14]).

Let

$$I(u) = \tfrac{1}{2} \int \left((u')^2 + u^2 - \tfrac{1}{2} u^4 \right)$$

and

$$\Omega(u) = -u'' + u - u^3$$

We extract subsequences of (u_n), but we will always denote them simply by (u_n). Firstly, by assumption (i), extracting a subsequence, we assume

$$\lim_{n\to+\infty} \|u_n\|_{H^1}^2 = c.$$

If $c = 0$, the conclusion of the theorem is true for $\sigma_1 = 0$ and $J = 1$.

Now, we assume $c > 0$. By assumption (ii), we also have

$$\lim_{n\to+\infty} \langle \Omega(u_n), u_n \rangle_{H^{-1},H^1} = \lim_{n\to+\infty} \int \left((u_n')^2 + u_n^2 - u_n^4 \right) = 0.$$

As a consequence,

$$\lim_{n\to+\infty} \int u_n^4 = c, \quad \lim_{n\to+\infty} I(u_n) = \tfrac{c}{4}.$$

Secondly, we extract a subsequence of (u_n) such that for some function $q_1 \in H^1(\mathbb{R})$,

$$u_n \rightharpoonup q_1 \text{ weakly in } H^1(\mathbb{R}) \text{ and } L^4(\mathbb{R}) \text{ as } n \to +\infty,$$
$$u_n \to q_1 \text{ a.e. in } \mathbb{R} \text{ as } n \to +\infty.$$

By passing to the limit in $\mathcal{D}'$, q_1 satisfies on $\mathbb{R}$

$$q_1'' - q_1 + q_1^3 = 0.$$

In particular, either $q_1 = 0$ or there exists $\sigma_1 \in \{-1, +1\}$ and $\xi_1 \in \mathbb{R}$ such that $q_1 = \sigma_1 Q(x - \xi_1)$. We summarize saying that $q_1 = \sigma_1 Q(x - \xi_1)$ for $\sigma_1 \in \{-1, 0, +1\}$.

Third, we set

$$v_{1,n} = u_n - q_1, \quad c_1 = \|q_1\|_{H^1}^2,$$

so that

$$v_{1,n} \rightharpoonup 0 \text{ as } n \to +\infty \text{ weakly in } H^1(\mathbb{R})$$
$$v_{1,n} \to 0 \text{ as } n \to +\infty \text{ in } L^2([-K, K]), \text{ for any } K > 0.$$

Moreover, by weak convergence

$$\lim_{n\to+\infty} \|v_{1,n}\|_{H^1}^2 = \lim_{n\to+\infty} \|u_n\|_{H^1}^2 - \|q_1\|_{H^1}^2 = c - c_1$$

and by a.e. convergence and the Brezis-Lieb lemma

$$\lim_{n\to+\infty} \int v_{1,n}^4 = \lim_{n\to+\infty} \int u_n^4 - \int q_1^4 = c - c_1.$$

Thus, $\lim_{n\to+\infty} I(v_{1,n}) = \frac{1}{4}(c - c_1)$. Then

$$\begin{aligned}\Omega(v_{1,n}) &= \Omega(u_n) - \Omega(q_1) - \big((u_n - q_1)^3 - u_n^3 + q_1^3\big) \\ &= \Omega(u_n) - \Omega(q_1) + 3u_n q_1 v_{1,n}.\end{aligned}$$

We have by hypothesis $\Omega(u_n) \to 0$ in H^{-1} and by the definition of q_1, $\Omega(q_1) = 0$. We now claim that $u_n q_1 v_{1,n} \to 0$ in L^2. This follows from $\|u_n\|_{L^\infty} + \|v_{1,n}\|_{L^\infty} \le C$, $v_{1,n} \to 0$ as $n \to +\infty$ in $L^2([-K, K])$, for any $K > 0$, and the fact that $\lim_{\pm\infty} q_1 = 0$. As a consequence

$$\Omega(v_{1,n}) \to 0 \text{ in } H^{-1} \text{ as } n \to +\infty.$$

If $c = c_1$ then $\sigma_1 = \pm 1$, the sequence $(v_{1,n})$ converges to 0 strongly in H^1 and the result is proved.

Now, we assume $c_1 < c$ and we want to prove that there exist a sequence $(\xi_{2,n})_n$ of $\mathbb{R}$ and $\sigma_2 = -1$ or $+1$ such that $|\xi_{2,n}| \to +\infty$ and $v_{1,n}(\cdot - \xi_{2,n}) \rightharpoonup \sigma_2 Q$ in H^1 weak. To begin with, we introduce $J[k] = [k, k+1]$ for any $k \in \mathbb{Z}$ and we set

$$\mu_n = \max_k \|v_{1,n}\|_{L^4(J_k)}.$$

We estimate

$$\begin{aligned}\int v_{1,n}^4 = \sum_{k\in\mathbb{Z}} \int_{J[k]} v_{1,n}^4 &\le \mu_n^2 \sum_{k\in\mathbb{Z}} \Big(\int_{J[k]} v_{1,n}^4\Big)^{1/2} \\ &\le C\mu_n^2 \sum_{k\in\mathbb{Z}} \int_{J[k]} \big((v_{1,n}')^2 + v_{1,n}^2\big) \le C\mu_n^2 \|v_{1,n}\|_{H^1}^2.\end{aligned}$$

Since $\lim_{n\to+\infty} \int v_{1,n}^4 = \lim_{n\to+\infty} \|v_{1,n}\|_{H^1}^2 = c - c_1 > 0$, for n large, $\mu_n \ge \gamma$ where $\gamma = \sqrt{1/2C} > 0$. We denote by k_n an integer such that $\mu_n = \|v_{1,n}\|_{L^4(J[k_n])}$. The sequence $(k_n)_n$ cannot be bounded in $\mathbb{Z}$. Indeed, otherwise, up to the extraction of a subsequence, the sequence $(k_n)_n$ would be constant, say $k_n = k_0$ for all $n \in \mathbb{N}$. Thus, $\|v_{1,n}\|_{L^4(J[k_0])} \geq \gamma$ for all n large. By the H^1 bound and Ascoli (or Rellich) theorem, $(v_{1,n})_n$ converges strongly in L^4 on $J[k_0]$ to a function $v \not\equiv 0$, up to a subsequence. But this is a contradiction with $v_{1,n} \rightharpoonup 0$ weakly in $H^1(\mathbb{R})$.

We denote by q_2 the weak limit of a subsequence of $(v_{1,n}(\cdot + k_n))_n$. Arguing as before, $q_2 \not\equiv 0$ and q_2 is a solution of $q_2'' - q_2 + q_2^3 = 0$ on $\mathbb{R}$. Therefore, there exists $\xi_{2,n}$ and $\sigma_2 \in \{-1, 1\}$ such that $|\xi_{2,n} - \xi_1| \to +\infty$ and $v_{1,n}(\cdot + \xi_{2,n}) \rightharpoonup \sigma_2 Q$ weakly in H^1. We set $v_{2,n} = v_{1,n} - \sigma_2 Q(\cdot - \xi_{2,n})$ so that, again by weak convergence

$$\lim_{n\to+\infty} \|v_{2,n}\|_{H^1}^2 = \lim_{n\to+\infty} \|v_{1,n}\|_{H^1}^2 - \|Q\|_{H^1}^2 = c - c_1 - \|Q\|_{H^1}^2.$$

Iterating this argument a finite number of times, we find that there exists $J \geq 2$ such that

$$c - c_1 = J\|Q\|_{H^1}^2$$

(in particular, we deduce that $c/\|Q\|_{H^1}^2$ is an integer), $J-1$ values $\sigma_j \in \{-1; +1\}$, and $J-1$ sequences of points $(\xi_{j,n})_n$ such that for $j = 2, \ldots, J$, it holds

$$\begin{aligned}
&\lim_{t\to+\infty} |\xi_{j,n} - \xi_{k,n}| = +\infty, \quad \text{for } j \neq k,\\
&v_{j,n}(\cdot + \xi_{j,n}) \rightharpoonup \sigma_j Q \text{ in } H^1(\mathbb{R}) \text{ weak, where } v_{j,n} = v_{j-1,n} - \sigma_{j-1} Q,\\
&w_n \to 0 \text{ in } H^1(\mathbb{R}) \text{ strong where } w_n = v_{J,n}(\cdot + \xi_{J,n}) - \sigma_J Q(\cdot - \xi_{J,n}).
\end{aligned}$$

In particular, we have obtained the decomposition

$$u_n = \sigma_1 Q(\cdot - \xi_1) + v_{1,n} = \cdots = \sigma_1 Q(\cdot - \xi_1) + \sum_{j=2}^{J} \sigma_j Q(\cdot - \xi_{j,n}) + w_n$$

with the requested properties on $(\xi_{j,n})_n$ and σ_j. □

4.3 First decomposition result on a subsequence of time

Theorem 4.4. *Any global solution $\vec{u}$ of* (1)

- *either converges to* 0, i.e. $\lim_{t\to\infty} \|\vec{u}(t)\|_{H^1\times L^2} = 0$;
- *or is asymptotically a single or multi-solitary wave along a subsequence of time in the following sense: there exist $K \geq 1$, a sequence $t_n \to \infty$, a sequence $(\xi_{k,n})_{k\in\{1,\ldots,K\}} \in \mathbb{R}^K$ and signs $\sigma_k = \pm 1$, for any $k \in \{1, \ldots, K\}$, such that*

$$\lim_{n\to\infty} \left\{ \left\| u(t_n) - \sum_{k=1}^{K} \sigma_k Q(\cdot - \xi_{k,n}) \right\|_{H^1} + \|\partial_t u(t_n)\|_{L^2} \right\} = 0. \tag{31}$$

Moreover, if $K \geq 2$ then

$$\lim_{n\to\infty} \xi_{k+1,n} - \xi_{k,n} = \infty \quad \textit{for any } 1 \leq k \leq K-1.$$

Remark 4.5. It is clear that if a global solution $\vec{u}$ satisfies (31) for two different sequences $(t_n)_n$ and $(t'_n)_n$, then the number $K \geq 1$ of solitary waves is the same for both sequences. Indeed, by monotonicity of the energy (3) and (26), it holds

$$\lim_{t\to\infty} E(\vec{u}(t)) = K\, E(Q,0) > 0. \tag{32}$$

Remark 4.6. The following stronger result holds in the framework of Theorem 4.4: for any sequence $(t_n)_n$ with $t_n \to \infty$, the multi-solitary wave behavior (31) is satisfied for a subsequence of $(t_n)_n$. This result, valid on any global solution of (1), is quite remarkable. However, it does not fully describe the asymptotic behavior of global solutions as $t \to \infty$, which is the objective of Theorem 5.1.

Proof. Let $\vec{u}$ be a global solution of (1); in particular, by Theorem 2.3, it is bounded in X. The proof proceeds in two steps.

Step 1. We prove that

$$\lim_{t\to\infty} \left\{ \|\partial_t u(t)\| + \|\partial_t^2 u(t)\|_{H^{-1}} \right\} = 0. \tag{33}$$

The function $\vec{v}(t) = (v(t), \partial_t v(t)) = (\partial_t u(t), \partial_t^2 u(t))$ satisfies

$$\vec{v}(t) = S_\alpha(t)\vec{v}(0) + \int_0^t S_\alpha(t-s)(0, 3u^2(s)v(s))\,\mathrm{d}s.$$

By the bound in X and (3), it follows that $v \in L^2((0,\infty)\times\mathbb{R})$. Moreover, using estimate (5), $\|\cdot\|_{H^{-1}} \lesssim \|\cdot\|$ and $\|\cdot\|_{L^\infty} \lesssim \|\cdot\|_{H^1}$, we have, for all $t \geq 0$,

$$\|\vec{v}(t)\|_{L^2\times H^{-1}} \lesssim e^{-\gamma t}\|\vec{v}(0)\|_{L^2\times H^{-1}} + \|u\|^2_{L^\infty([0,\infty),H^1)} \int_0^t e^{-\gamma(t-s)}\|v(s)\|\,\mathrm{d}s.$$

Splitting the integral $\int_0^t = \int_0^{t/2} + \int_{t/2}^t$ in the last term and using the Cauchy-Schwarz inequality

$$\int_0^t e^{-\gamma(t-s)}\|v(s)\|\,\mathrm{d}s \lesssim e^{-\gamma t/2}\|v\|_{L^2((0,\infty)\times\mathbb{R})} + \|v\|_{L^2((t/2,\infty)\times\mathbb{R})},$$

which implies $\lim_{t\to\infty}\|\vec{v}(t)\|_{L^2\times H^{-1}} = 0$ and thus (33).

Step 2. Let $(t_n)_n$ be any sequence such that $t_n \to \infty$ and let $u_n(x) = u(t_n, x)$. Then, by (33) and equation (1), it follows that

$$\lim_{n\to\infty} \|\partial_x^2 u_n - u_n + u_n^3\|_{H^{-1}} = 0.$$

Moreover, the sequence $(u_n)_n$ is bounded in H^1. Then, the alternative stated in the Theorem follows directly from Theorem 4.2.

In the case where $\lim_{n\to\infty}\|\vec{u}(s_n)\|_{H^1\times L^2} = 0$, for some sequence of time $(s_n)_n$, $s_n \to \infty$, then it follows from (3) that $\lim_{t\to\infty} E(\vec{u}(t)) = 0$. Thus, by (26) and the previous arguments applied to any sequence $(t_n)_n$, with $t_n \to \infty$, there exists a subsequence $(t_{n'})_{n'}$ such that $\lim_{n'\to\infty}\|\vec{u}(t_{n'})\|_{H^1\times L^2} = 0$. This implies that $\lim_{t\to\infty}\|\vec{u}(t)\|_{H^1\times L^2} = 0$ for the whole sequence of time, as stated in the first part of the alternative. □

In the next sections, we will go deeper into the analysis of global solutions of (1). However, because of limited time, most arguments will be formal and restricted to the soliton behavior. In particular, the perturbation term will not be rigorously estimated and only the interactions between the solitons will be studied. For rigorous proofs, we refer to the original articles [7, 8].

5 Refined convergence theorem

The objective of the rest of the course is to give a sketch of the proof of the following more refined convergence result.

Theorem 5.1. *For any global solution $\vec{u} \in C\big([0,\infty), X\big)$ of (1), one of the following three scenarios occurs:*

Vanishing *$\vec{u}(t)$ converges exponentially to 0 in X as $t \to \infty$.*

Single soliton *There exist $\sigma = \pm 1$, $\ell \in \mathbb{R}$ such that $\vec{u}(t)$ converges exponentially to $(\sigma Q(\cdot - \ell), 0)$ in X as $t \to \infty$.*

Multi-soliton *There exist $K \geq 2$, $\sigma = \pm 1$, $\ell \in \mathbb{R}$ and functions $z_k : [0,\infty) \to \mathbb{R}$, for all $k = 1, \ldots, K$ such that*

$$\lim_{t \to +\infty} \left\| u(t) - \sigma \sum_{k=1}^{K} (-1)^k Q(\cdot - z_k(t)) \right\|_{H^1} + \|\partial_t u(t)\|_{L^2} = 0 \tag{34}$$

and for any $k = 1, \ldots, K$,

$$\lim_{t \to +\infty} \left\{ z_k(t) - \left(k - \frac{K+1}{2} \right) \log t \right\} = \tau_k + \ell \tag{35}$$

where τ_k are given by (71).

6 Dynamics close to decoupled solitary waves

In this Section, we give general formal computation on solutions of (1) close to the sum of $K \geq 1$ decoupled solitary waves. For any $k \in \{1, \cdots, K\}$, let $\sigma_k = \pm 1$ and let $t \mapsto (z_k(t), \ell_k(t)) \in \mathbb{R}^2$ be $\mathcal{C}^1$ functions such that

$$\sum_{k=1}^{K} |\ell_k| \ll 1 \quad \text{and, if } K \geq 2, \text{ for any } k = 1, \ldots, K-1, \quad z_{k+1} - z_k \gg 1. \tag{36}$$

For $k \in \{1, \cdots, K\}$, define

$$Q_k = \sigma_k Q(\cdot - z_k), \quad \vec{Q}_k = \begin{pmatrix} Q_k \\ -\ell_k \partial_x Q_k \end{pmatrix}. \tag{37}$$

Set

$$R = \sum_{k=1}^{K} Q_k, \quad \vec{R} = \sum_{k=1}^{K} \vec{Q}_k, \quad G = \left(\sum_{k=1}^{K} Q_k \right)^3 - \sum_{k=1}^{K} Q_k^3. \tag{38}$$

We will also use the following notation $\sigma_{-1} = 0$ and $\sigma_{K+1} = 0$.

6.1 Leading order of the nonlinear interactions

Lemma 6.1. *Assume* (36). *For any $k, j \in \{1, \ldots, K\}$, $j \neq k$, it holds.*

(i) Asymptotics.

$$\langle Q_k^3, Q_j \rangle \sim \sigma_k \sigma_j c_1 \kappa e^{-|z_k - z_j|} \tag{39}$$

where

$$\kappa := \frac{2\sqrt{2}}{c_1} \int Q^3(x) e^x \, \mathrm{d}x > 0 \quad \text{and} \quad c_1 := \|Q'\|_{L^2}^2. \tag{40}$$

(ii) Leading order interactions.

- *If $K = 1$ then $G = 0$;*
- *If $K \geq 2$ then*

$$\langle G, \partial_x Q_k \rangle \sim c_1 \kappa \sigma_k \left(\sigma_{k-1} e^{-(z_k - z_{k-1})} - \sigma_{k+1} e^{-(z_{k+1} - z_k)} \right) \tag{41}$$

Proof. Assume that $z_j \gg z_k$. Then, using (25),

$$\begin{aligned} \langle Q_k^3, Q_j \rangle &= \sigma_k \sigma_j \int Q^3(x - z_k) Q(x - z_j) \, \mathrm{d}x \\ &= \sigma_k \sigma_j \int Q^3(x) Q(x + z_k - z_j) \, \mathrm{d}x \\ &\sim 2\sqrt{2} \sigma_k \sigma_j e^{z_k - z_j} \int Q^3(x) e^x \, \mathrm{d}x = \sigma_k \sigma_j c_1 \kappa e^{-|z_k - z_j|}. \end{aligned}$$

□

6.2 Decomposition close to the sum of solitary waves

Lemma 6.2. *Let $\vec{u} = (u, \partial_t u)$ be a solution of* (1) *such that for some $K \geq 1$*

$$\|u(t) - \sum_{k=1}^{K} \sigma_k Q(\cdot - \xi_k(t))\|_{H^1} \ll 1, \quad \|\partial_t u(t)\|_{L^2} \ll 1, \quad \xi_k(t) - \xi_{k+1}(t) \gg 1. \tag{42}$$

Then, there exist unique $\mathcal{C}^1$ functions $t \mapsto (z_k(t), \ell_k(t))_{k \in \{1,\ldots,K\}} \in \mathbb{R}^{2K}$, such that the solution $\vec{u}$ decomposes as

$$\vec{u} = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \sum_{k=1}^{K} \vec{Q}_k + \vec{\varepsilon}, \quad \vec{\varepsilon} = \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} \tag{43}$$

with the following properties.

(i) Orthogonality and smallness. *For any* $k=1,\dots,K$,

$$\langle \varepsilon, \partial_x Q_k\rangle = \langle \eta, \partial_x Q_k\rangle = 0 \tag{44}$$

and

$$\|\vec{\varepsilon}\|_{H^1\times L^2} \ll 1, \quad |\ell_l| \ll 1, \quad z_{k+1}-z_k \gg 1$$

(ii) Equation of $\vec{\varepsilon}$.

$$\begin{cases} \partial_t \varepsilon = \eta + \mathrm{Mod}_\varepsilon \\ \partial_t \eta = \partial_x^2 \varepsilon - \varepsilon + (R+\varepsilon)^3 - R^3 - 2\alpha\eta + \mathrm{Mod}_\eta + G \end{cases} \tag{45}$$

where

$$\mathrm{Mod}_\varepsilon = \sum_{k=1}^{K} (\dot{z}_k - \ell_k)\, \partial_x Q_k,$$
$$\mathrm{Mod}_\eta = \sum_{k=1}^{K} \big(\dot{\ell}_k + 2\alpha \ell_k\big)\partial_x Q_k - \sum_{k=1}^{K} \ell_k \dot{z}_k \partial_x^2 Q_k.$$

(iii) Control of the geometric parameters. *For* $k=1,\dots,K$,

$$|\dot{z}_k - \ell_k| \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{l=1}^{K} |\ell_l|^2, \tag{46}$$

$$\left|\dot{\ell}_k + 2\alpha\ell_k + \kappa\sigma_k \left(\sigma_{k-1} e^{-(z_k - z_{k-1})} - \sigma_{k+1} e^{-(z_{k+1}-z_k)}\right)\right| \tag{47}$$

$$\lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{l=1}^{K} |\ell_l|^2 + \sum_{l=1}^{K-1} e^{-2(z_{l+1}-z_l)}. \tag{48}$$

(iv) Control of the exponential directions. *For* $k=1,\cdots,K$, *if*

$$a_k^\pm = \langle \vec{\varepsilon}, \vec{Z}_k^\pm \rangle \tag{49}$$

then

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} a_k^\pm - \nu^\pm a_k^\pm\right| \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{l=1}^{K} |\ell_l|^2 + \sum_{l=1}^{K-1} e^{-(z_{l+1}-z_l)}. \tag{50}$$

Proof. Proof of (i). The existence and uniqueness of the geometric parameters (z_k,ℓ_k) is proved for a fixed time t and we set $u := u(T_1)$. Let $0<\gamma\ll 1$. First, for any $u\in H^1$ such that

$$\inf_{|\xi_1-\xi_2|>|\log\gamma|} \Big\| u - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_k)\Big\|_{H^1} \le \gamma, \tag{51}$$

we consider $z_1(u)$ and $z_2(u)$ achieving the infimum

$$\Big\|u - \sum_{k=1,2} \sigma_k Q(\cdot - z_k(u))\Big\|_{L^2} = \inf_{\xi_1,\xi_2\in\mathbb{R}} \Big\|u - \sum_{k=1,2} \sigma_k Q(\cdot - \xi_k)\Big\|_{L^2}.$$

Then, for $\gamma > 0$ small enough, the minimum is attained for $z_1(u)$ and $z_2(u)$ such that $|z_1(u) - z_2(u)| > |\log\gamma| - C$, for some $C > 0$. Let

$$\varepsilon(x) = u(x) - \sigma_1 Q(x - z_1(u)) - \sigma_2 Q(x - z_2(u)), \quad \|\varepsilon\|_{L^2} \le \gamma.$$

By the definition of $z_1(u)$ and $z_2(u)$, we have for $k = 1, 2$,

$$\frac{d}{d\xi_k} \int \Big[u - \sum_{k'=1,2} \sigma_{k'} Q(\cdot - \xi_{k'})\Big]^2 \Big|_{(\xi_1,\xi_2)=(z_1(u),z_2(u))} = 0$$

and so

$$\langle \varepsilon, Q'(\cdot - z_1(u))\rangle = \langle \varepsilon, Q'(\cdot - z_2(u))\rangle = 0. \tag{52}$$

For u and $\tilde u$ as in (51), we compare the corresponding z_k, $\tilde z_k$ and ε, $\tilde\varepsilon$. First, for ζ, $\tilde\zeta \in \mathbb{R}^N$, setting $\check\zeta = \zeta - \tilde\zeta$, we observe the following estimates

$$Q(\cdot - \zeta) - Q(\cdot - \tilde\zeta) = -\check\zeta Q'(\cdot - \zeta) + O_{H^1}(|\check\zeta|^2), \tag{53}$$

$$Q'(\cdot - \zeta) - Q'(\cdot - \tilde\zeta) = -\check\zeta Q''(\cdot - \zeta) + O_{H^1}(|\check\zeta|^2). \tag{54}$$

Thus, denoting $\check u = u - \tilde u$, $\check z_k = z_k - \tilde z_k$, $\check\varepsilon = \varepsilon - \tilde\varepsilon$, we obtain

$$\check u = -\sum_{k=1,2} \sigma_k \check z_k Q'(\cdot - z_k) + \check\varepsilon + O_{H^1}(|\check z_1|^2 + |\check z_2|^2).$$

(In the O_{H^1}, there is no dependence on $\check u$ or $\check\varepsilon$). Projecting on $Q'(\cdot - z_k)$, using (52) and the above estimates, we obtain

$$|\check z_1| + |\check z_2| \lesssim \|\check u\|_{L^2} + (|\check z_1| + |\check z_2|)(e^{-\frac12|z|} + \|\tilde\varepsilon\|_{L^2} + |\check z_1| + |\check z_2|)$$

and thus, for γ, $\check z_1$ and $\check z_2$ small,

$$\|\check\varepsilon\|_{H^1} \lesssim \|\check u\|_{H^1}, \quad |\check z_1| + |\check z_2| \lesssim \|\check u\|_{L^2}. \tag{55}$$

Therefore, for γ small enough, this proves uniqueness and Lipschitz continuity of z_1 and z_2 with respect to u in H^1.

Now, let $v \in L^2$ and z_1, z_2 be such that $\|v\|_{L^2} < \gamma$ and $|z_1 - z_2| \gg 1$. Set

$$\eta(x) = v(x) + \sigma_1\ell_1 Q'(x - z_1) + \sigma_2\ell_2 Q'(x - z_2).$$

Then, it is easy to check that the conditions

$$\langle \eta, Q'(\cdot - z_k)\rangle = 0 \tag{56}$$

are equivalent to a 2×2 linear system in the components of ℓ_1 and ℓ_2 whose matrix is a perturbation of the identity up to a multiplicative constant. In particular, it is invertible and the existence and uniqueness of parameters $\ell_1(v, z_1, z_2), \ell_2(v, z_1, z_2) \in \mathbb{R}^N$ satisfying (56) and $|\ell_1| + |\ell_2| \lesssim \|v\|_{L^2}$ is clear. Moroever, with similar notation as before, it holds

$$\|\check{\eta}\|_{L^2} + |\check{\ell}_1| + |\check{\ell}_2| \lesssim \|\check{v}\|_{L^2} + |\check{z}_1| + |\check{z}_2|. \tag{57}$$

The decomposition is thus achieved for $u(t)$ with any fixed $t \in [T_1, T_2]$. In the rest of this proof, we formally derive the equations of $\vec{\varepsilon}$ and the geometric parameters from the equation of u. This derivation is used to prove by the Cauchy-Lipschitz theorem that the parameters are $\mathcal{C}^1$ functions of time

The system of equations (45) follows from direct computations.

Now, we derive (46) from (44). We have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle \varepsilon, Q_1\rangle = \langle \partial_t \varepsilon, \partial_x Q_1\rangle + \langle \varepsilon, \partial_t(\partial_x Q_1)\rangle.$$

Thus (45) gives

$$\langle \eta, \partial_x Q_1\rangle + \langle \mathrm{Mod}_\varepsilon, \partial_x Q_1\rangle - \langle \varepsilon, \dot{z}_1 \partial_x^2 Q_1\rangle = 0.$$

The first term is zero due to the orthogonality (44). Hence, using the expression of Mod_ε

$$(\dot{z}_{1,j} - \ell_{1,j})\|\partial_x Q\|_{L^2}^2 = -\int (\dot{z}_2 - \ell_2)\partial_x Q_2 (\partial_x Q_1)\,\mathrm{d}x + \langle \varepsilon, \dot{z}_1 \partial_x^2 Q_1\rangle. \tag{58}$$

From this formula, we deduce

$$|\dot{z}_{1,j} - \ell_{1,j}| \lesssim |\dot{z}_2 - \ell_2| \int |\partial_x Q_2(x)||\partial_x Q_1(x)|\,\mathrm{d}x + |\dot{z}_1|\,\|\varepsilon\|.$$

Thus, we obtain

$$|\dot{z}_1 - \ell_1| \lesssim |\dot{z}_2 - \ell_2| e^{-\frac{1}{2}|z|} + |\dot{z}_1 - \ell_1|\,\|\vec{\varepsilon}\|_{H^1\times L^2} + |\ell_1|\,\|\vec{\varepsilon}\|_{H^1\times L^2}.$$

Since $\|\vec{\varepsilon}\|_{H^1\times L^2} \lesssim \gamma$, this yields

$$|\dot{z}_1 - \ell_1| \lesssim |\dot{z}_2 - \ell_2| e^{-\frac{1}{2}|z|} + |\ell_1|\,\|\vec{\varepsilon}\|_{H^1\times L^2}.$$

Similarly, it holds

$$|\dot{z}_2 - \ell_2| \lesssim |\dot{z}_1 - \ell_1| e^{-\frac{1}{2}|z|} + |\ell_2|\,\|\vec{\varepsilon}\|_{H^1\times L^2},$$

and thus, for large $|z|$,

$$\sum_{k=1,2} |\dot{z}_k - \ell_k| \lesssim (|\ell_1| + |\ell_2|)\,\|\vec{\varepsilon}\|_{H^1\times L^2},$$

which justifies (46).

Now, we prove (48) for $k \in \{2, \dots, K-1\}$ for $K \geq 3$. First,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \eta, \partial_x Q_k \rangle = \langle \partial_t \eta, \partial_x Q_k \rangle + \langle \eta, \partial_t \partial_x Q_k \rangle = 0.$$

Thus, using (44) and (45),

$$\begin{aligned} 0 =& \langle \partial_x^2 \varepsilon - \varepsilon + 3Q_k^2 \varepsilon, \partial_x Q_k \rangle + \langle (R+\varepsilon)^3 - R^3 - 3R^2\varepsilon, \partial_x Q_k \rangle \\ &+ 3\langle \big(R^2 - Q_k^2\big)\varepsilon, \partial_x Q_k \rangle + \langle G, \partial_x Q_k \rangle + \langle \mathrm{Mod}_\eta, \partial_x Q_k \rangle - \langle \eta, \dot{z}_k \partial_x^2 Q_k \rangle. \end{aligned}$$

Since $\partial_x Q_k$ satisfies $\partial_x^2 \partial_x Q_k - \partial_x Q_k + 3Q_k^2 \partial_x Q_k = 0$, by integration by parts, the first term is zero. Next, by Taylor expansion, we have the pointwise estimate

$$\big|(R+\varepsilon)^3 - R^3 - 3R^2\varepsilon\big| \lesssim |\varepsilon|^3 + |\varepsilon|^2 \sum_{l=1}^{K} |Q_l|$$

and

$$\big|R^2 - Q_k^2\big| \lesssim |Q_k| \sum_{j \neq k} |Q_j| + \sum_{j\neq k} \big|Q_j\big|^2.$$

Thus, using $\|\cdot\|_{L^\infty} \lesssim \|\cdot\|_{H^1}$,

$$\big|\langle (R+\varepsilon)^3 - R^3 - 3R^2\varepsilon, \partial_x Q_k \rangle\big| \lesssim \|\varepsilon\|_{H^1}^2$$

and by the Cauchy-Schwarz inequality

$$\big|\langle (R^2 - Q_k^2)\varepsilon, \partial_x Q_k \rangle\big| \lesssim \|\varepsilon\|_{H^1}^2 + \sum_{k=1}^{K-1} e^{-2(z_{k+1}-z_k)} .$$

By direct computation, we obtain

$$\begin{aligned} \langle \mathrm{Mod}_\eta, \partial_x Q_k \rangle = \big(\dot{\ell}_k + 2\alpha \ell_k\big)\|Q'\|_{L^2}^2 + \sum_{j\neq k} \big(\dot{\ell}_j + 2\alpha\ell_j\big)\langle \partial_x Q_j, \partial_x Q_k \rangle \\ - \sum_{j \neq k} \ell_j \dot{z}_j \langle \partial_x^2 Q_j, \partial_x Q_k \rangle. \end{aligned}$$

Thus, using the equation of Q, (40) and (46), we obtain

$$\begin{aligned} \langle \mathrm{Mod}_\eta, \partial_x Q_k \rangle = c_1\big(\dot{\ell}_k + 2\alpha\ell_k\big) + O\bigg(\sum_{j\neq k} |\dot{\ell}_j + 2\alpha \ell_j| e^{-\frac{3}{4}|z_j - z_k|}\bigg) \\ + O\bigg(\|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{l=1}^{K} |\ell_l|^2\bigg). \end{aligned}$$

Note that, by (46) and the Cauchy-Schwarz inequality, we obtain

$$\big|\langle \eta, \dot{z}_k \partial_x^2 Q_k \rangle\big| \lesssim \big(|\dot{z}_k - \ell_k| + |\ell_k|\big)\|\vec{\varepsilon}\|_{H^1\times L^2} \lesssim \|\vec{\varepsilon}\|_{H^1\times L^2}^2 + \sum_{k=1}^{K} |\ell_k|^2.$$

Gathering above estimates we obtain

$$\begin{aligned}&\left|\dot{\ell}_k+2\alpha\ell_k+\kappa\sigma_k\left(\sigma_{k-1}e^{-(z_k-z_{k-1})}-\sigma_{k+1}e^{-(z_{k+1}-z_k)}\right)\right|\\&\lesssim\|\vec{\varepsilon}\|_{H^1\times L^2}^2+\sum_{l=1}^{K}|\ell_l|^2+\sum_{j\neq k}|\dot{\ell}_j+2\alpha\ell_j|e^{-\frac{3}{4}|z_j-z_k|}+\sum_{l=1}^{K-1}e^{-2(z_{l+1}-z_l)}.\end{aligned}\tag{59}$$

We obtain (48) by combining the estimates (59) for all $k\in\{1,\dots,K\}$ and neglecting all second order terms. □

6.3 Energy estimates

For $\mu>0$ small to be chosen, we denote $\rho=2\alpha-\mu$. Consider the nonlinear energy functional

$$\mathcal{E}=\int\left\{(\partial_x\varepsilon)^2+(1-\rho\mu)\varepsilon^2+(\eta+\mu\varepsilon)^2-2[F(R+\varepsilon)-F(R)-f(R)\varepsilon]\right\}.$$

We recall the following energy estimates.

Lemma 6.3. *There exists $\mu>0$ such that in the context of Lemma 6.2, the following hold.*

(i) Coercivity and bound.

$$\mu\|\vec{\varepsilon}\|_{H^1\times L^2}^2-\frac{1}{2\mu}\sum_{k=1}^{K}\left((a_k^+)^2+(a_k^-)^2\right)\le\mathcal{E}\le\frac{1}{\mu}\|\vec{\varepsilon}\|_{H^1\times L^2}^2.\tag{60}$$

(ii) Time variation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}\le-2\mu\mathcal{E}+\frac{1}{\mu}\|\vec{\varepsilon}\|_{H^1\times L^2}\left[\|\vec{\varepsilon}\|_{H^1\times L^2}^2+\sum_{k=1}^{K}|\ell_k|^2+\sum_{k=1}^{K-1}e^{-(z_{k+1}-z_k)}\right].\tag{61}$$

Proof. Left as an exercise. See also [8]. □

6.4 Approximate transformed system

We introduce refined parameters and functionals to analyse the time evolution of solutions in the framework of Lemma 6.2. First, we set, for $k=1,\dots,K$,

$$y_k=z_k+\frac{\ell_k}{2\alpha},$$

and for $k=1,\dots,K-1$ (when $K\ge2$),

$$r_k=y_{k+1}-y_k\gg1.$$

Second, we define

$$\mathcal{K}_+ = \{k = 1, \dots, K-1 : \sigma_k = \sigma_{k+1}\}, \quad F_+ = \sum_{k \in \mathcal{K}_+} e^{-r_k},$$
$$\mathcal{K}_- = \{k = 1, \dots, K-1 : \sigma_k = -\sigma_{k+1}\}, \quad F_- = \sum_{k \in \mathcal{K}_-} e^{-r_k}.$$

Proposition 6.4. *Assume $K \geq 2$. The equation for the evolution of y_k is*

$$\dot{y}_k \sim -\frac{\kappa}{2\alpha}\sigma_{k-1}\sigma_k e^{-r_{k-1}} + \frac{\kappa}{2\alpha}\sigma_k\sigma_{k+1}e^{-r_k} \tag{62}$$

Moreover, there exists $\lambda > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{F_+}\right) \leq -\lambda. \tag{63}$$

Proof. By direct computation

$$\dot{y}_k = \dot{z}_k + \frac{\dot{\ell}_k}{2\alpha} \sim -\frac{\kappa}{2\alpha}\sigma_{k-1}\sigma_k e^{-r_{k-1}} + \frac{\kappa}{2\alpha}\sigma_k\sigma_{k+1}e^{-r_k}$$

It follows that, for any $k = 1, \dots, K-1$,

$$\dot{r}_k = \dot{y}_{k+1} - \dot{y}_k = -\frac{\kappa}{\alpha}\sigma_k\sigma_{k+1}e^{-r_k} + \frac{\kappa}{2\alpha}\sigma_{k+1}\sigma_{k+2}e^{-r_{k+1}} + \frac{\kappa}{2\alpha}\sigma_{k-1}\sigma_k e^{-r_{k-1}}$$

On the right-hand side of the above expression, the first term is always present for $k = 1, \dots, K-1$, while the second and third terms might be zero depending on the value of k. For $k \in \mathcal{K}_+$, it holds $\sigma_k = \sigma_{k+1}$ and one sees that

$$\dot{r}_k = -\frac{\kappa}{\alpha}e^{-r_k} + \frac{\kappa}{2\alpha}\sigma_{k+1}\sigma_{k+2}e^{-r_{k+1}} + \frac{\kappa}{2\alpha}\sigma_{k-1}\sigma_k e^{-r_{k-1}},$$

with the same observation concerning the second and third terms on the right-hand side. Thus,

$$-\dot{F}_+ = \sum_{k \in \mathcal{K}_+} \dot{r}_k e^{-r_k} = -\frac{\kappa}{2\alpha}S_+ \tag{64}$$

where S_+ denotes

$$S_+ = \sum_{k \in \mathcal{K}_+} \left(2e^{-2r_k} - \sigma_{k+1}\sigma_{k+2}e^{-(r_k + r_{k+1})} - \sigma_{k-1}\sigma_k e^{-(r_{k-1}+r_k)}\right).$$

We claim that there exists $\tilde{\lambda} > 0$ such that S_+ satisfies

$$S_+ \geq \tilde{\lambda} \sum_{k \in \mathcal{K}_+} e^{-2r_k}. \tag{65}$$

Indeed, first, recall that the symmetric matrix of size N

$$A_N = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

is definite positive by the Sylvester criterion since for any $j \in \{1, \dots N\}$, the jth leading principal minor of this matrix, *i.e.* the determinant of its upper-left $j \times j$ sub-matrix, is positive (its value is $j+1$).

Second, observe that in the sum defining S_+, for given $k \in \mathcal{K}_+$, if $k-1 \notin \mathcal{K}_+$, then $\sigma_{k-1}\sigma_k = 1$ (if $k \geq 2$) or 0 (if $k=1$) and thus the corresponding term is positive or zero and can be ignored in establishing a lower bound for S_+. The same property is true for the term corresponding to $\sigma_{k+1}\sigma_{k+2}$ if $k+1 \notin \mathcal{K}_+$. Letting $N = \mathrm{card}(\mathcal{K}_+) \geq 1$ (by the contradiction assumption), and calling ϕ the strictly monotone function such that $\phi : j \in \{1, \dots, N\} \mapsto \phi(j) \in \mathcal{K}_+$, we have

$$S_+ \geq \sum_{j \in \{1,\dots,N\}} \left(2e^{-2r_{\phi(j)}} - e^{-(r_{\phi(j)}+r_{\phi(j+1)})} - e^{-(r_{\phi(j-1)}+r_{\phi(j)})}\right) = f_N^t A_N f_N,$$

where

$$f_N = \begin{pmatrix} e^{-r(\phi(1))} \\ \vdots \\ e^{-r(\phi(N))} \end{pmatrix}.$$

Since $A_N > 0$, (65) holds for some $\tilde{\lambda} > 0$.

It follows from (64) and (65) that there exists $\lambda > 0$ such that

$$-\dot{F}_+ = \sum_{k \in \mathcal{K}_+} \dot{r}_k e^{-r_k} \leq -\lambda F_+^2.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{F_+}\right) = -\frac{1}{F_+^2}\dot{F}_+ \leq -\lambda.$$

□

6.5 Long-time energy asymptotics

Lemma 6.5. *It holds*

$$E(\vec{u}) \sim KE(Q,0) - c_1\kappa F_+ + c_1\kappa F_-. \tag{66}$$

Proof. Expanding $E(u, \partial_t u)$ using the decomposition (43), integration by parts, the equation $-\partial_x^2 Q + Q - f(Q) = 0$ and the definition of G in (38), we find

$$2E(u, \partial_t u) = \int (\partial_t u)^2 + 2E(R, 0) - 2\int G\varepsilon$$
$$+ \int \left((\partial_x \varepsilon)^2 + \varepsilon^2 - \tfrac{1}{2}(R+\varepsilon)^4 + \tfrac{1}{2}R^4 + \tfrac{1}{2}R^3\varepsilon\right).$$

Thus, using the Cauchy-Schwarz and Sobolev inequalities, it holds

$$2E(u, \partial_t u) \sim \int (\partial_t u)^2 + 2E(R, 0).$$

Note also that (43) implies

$$\int (\partial_t u)^2 \lesssim \int \left(|\eta|^2 + \sum_{k=1}^{K} |\ell_k \partial_x Q_k|^2\right) \sim 0.$$

Then, by direct computation, next $-\partial_x^2 Q + Q - f(Q) = 0$,

$$E(R, 0) = KE(Q, 0) + \sum_{k<j} \int \left[(\partial_x Q_k)(\partial_x Q_j) + Q_k Q_j - Q_k^3 Q_j - Q_j^3 Q_k\right]$$
$$- \frac{1}{4}\int \left(R^4 - \sum_{k=1}^{K} Q_k^4 - \sum_{k\neq j} Q_k^3 Q_j\right)$$
$$\sim KE(Q, 0) - \sum_{k<j} \langle Q_k^3, Q_j \rangle.$$

Last, by the definition of F_+ and F_-, we observe that

$$\sum_{k<j} \langle Q_k^3, Q_j \rangle \sim c_1 \kappa F_+ - c_1 \kappa F_-$$

Indeed, in the above double sum $\sum_{k<j}$ in k and j, the terms corresponding to $j = k+1$ contribute to $\pm c_1 \kappa F_\pm$ (depending on $k \in \mathcal{K}_\pm$) and the other terms (*i.e.* $j \geq k+2$) only contribute to the error term. □

Combining (66) with (3) and (32), we obtain the following result.

Corollary 6.6.

$$2\alpha \int_t^\infty \|\partial_t u(s)\|_{L^2}^2 \, \mathrm{d}s \sim -c_1 \kappa F_+(t) + c_1 \kappa F_-(t). \tag{67}$$

7 Description of long-time asymptotics

7.1 Alternate signs property

Proposition 7.1. *Let $\vec{u}$ be a global solution of* (1) *such that $K \geq 2$ in* (31) *of Theorem 4.4. Then,*

$$\sigma_k = -\sigma_{k+1} \quad \textit{for all } k \in \{1, \dots, K-1\}. \tag{68}$$

Proof. Assuming that $\mathcal{K}_+$ is not empty, we reach a contradiction. Indeed, recall that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{F_+}\right] \leq -\lambda,$$

By integrating the above estimate on $[T, t]$, we obtain

$$\frac{1}{F_+(t)} \leq \frac{1}{F_+(T)} - \lambda(t - T),$$

which is contradictory with $F_+(t) \geq 0$ for large t. This means that $\mathcal{K}_+ = \emptyset$ and so $\mathcal{K}_- = \{1, \dots, K-1\}$: the signs of the solitary waves are alternate. □

7.2 No soliton case

If $\vec{u}(t)$ converges to 0 as $t \to \infty$, using the energy functional (see the proof of Lemma 1.1)

$$N(t) = \int (v^2 + (\partial_x u)^2 + u^2 + 2\alpha uv - \tfrac{1}{2}u^4)$$

and we compute $\frac{d}{dt}N = -2\alpha N$. This proves the exponential convergence to 0 of $\vec{u}$ in X.

7.3 Multi-soliton case

In view of the alternate signs property (68), the system (62) is rewritten

$$\dot{y}_k = \frac{\kappa}{2\alpha}\left(e^{-(y_k - y_{k-1})} - e^{-(y_{k+1} - y_k)}\right) \tag{69}$$

This system of ODEs is studied in [16] and [10], where it appears in a different context (the description of characteristic blowup points of the semilinear wave equation).

We check that an explicit solution to the ODE system is given by

$$\bar{y}_k(t) := \left(k - \frac{K+1}{2}\right)\log t + \tau_k, \quad \text{for } k = 1, \dots, K, \tag{70}$$

where $(\tau_k)_{k=1,\dots,K}$ are constants uniquely defined by

$$\sum_{k=1}^{K} \tau_k = 0, \quad e^{-(\tau_{k+1} - \tau_k)} = \frac{2\alpha}{\kappa}\gamma_k \quad \text{where} \quad \gamma_k = \frac{k(K-k)}{2}. \tag{71}$$

(The first relation is choice made without loss of generality.)

A A positivity result

Lemma A.1. *Let* $P = \operatorname{sech}^2$ *and* $Z = \operatorname{sech}'$. *For any* $u \in H^1$, *set*

$$J(u) = \int (u')^2 - 6 \int Pu^2.$$

Let

$$\gamma = \inf \left\{ J(u) : u \in H^1 \text{ with } (u, P) = (u, Z) = 0, \|u\|_{L^2} = 1 \right\}. \tag{72}$$

Then $\gamma = 0$.

Proof. Set $R = 1 - \frac{3}{2}P$ and $Au = -u'' - 6Pu$. Observe that $P, Z, R' \in H^\infty$ and $R \in L^\infty$. Moreover, $AP = -4P$, $AZ = -Z$, $AR = 0$ and $(P, Z) = (P, R) = (Z, R) = 0$.

We first show that $\gamma \leq 0$. This is because $J(R) = 0$ and $(R, P) = (R, Z) = 0$. Since $R \notin L^2$, one needs an approximation argument. Let ζ be a smooth cut-off function ($\zeta(x) = 0$ for $|x| \geq 2$, $\zeta(x) = 1$ for $|x| \leq 1$), set $\zeta_\epsilon(x) = \zeta(\epsilon x)$ and let

$$R_\epsilon = \zeta_\epsilon R - \|P\|_{L^2}^{-2} (\zeta_\epsilon R, P) P - \|Z\|_{L^2}^{-2} (\zeta_\epsilon R, Z) Z,$$

for $\epsilon > 0$. It follows that $(\zeta_\epsilon, P) = (\zeta_\epsilon, Z) = 0$. Moreover,

$$|(\zeta_\epsilon R, P)| = |(\zeta_\epsilon R, P) + ((1 - \zeta_\epsilon) R, P)| = |((1 - \zeta_\epsilon) R, P)| \lesssim e^{-1/\epsilon},$$

and as well $|(\zeta_\epsilon R, Z)|| \lesssim e^{-1/\epsilon}$. Using similar estimates, it is not difficult to prove that

$$J(R_\epsilon) = J(R) + O(e^{-1/\epsilon}) = O(e^{-1/\epsilon}).$$

Since $\|R_\epsilon\|_{L^2} \sim \epsilon^{-\frac{1}{2}}$, it follows that $J(R_\epsilon / \|R_\epsilon\|_{L^2}) \to 0$ as $\epsilon \to 0$. Thus $\gamma \leq 0$.

Now, to how that $\gamma \geq 0$, we assume for the sake of contradiction that $\gamma < 0$.

Let $(u_n)_{n \geq 1}$ be a minimizing sequence for problem (72). Since $(u_n)_{n \geq 1}$ is clearly bounded in H^1, there exists $u \in H^1$ such that (after extracting a subsequence) $u_n \to u$ weakly in H^1 and strongly in $L^2(\{|x| < k\})$ for all $k > 0$. It follows that $(u, P) = (u, Z) = 0$ and that $J(u) \leq \liminf J(u_n) = \gamma$. In particular, $J(u) < 0$ so that $u \neq 0$. If $\|u\|_{L^2} < 1$, then $J(u / \|u\|_{L^2}) < \gamma$, which is absurd. Thus we see that u is a minimizer. It follows that there exist real constants λ, μ, ν (the Lagrange multipliers) such that

$$J'(y) = 2Au = \lambda u + \mu P + \nu Z.$$

Multiplying the above equation successively by P and Z and using the orthogonality properties and the equations for P and Z, we see that $\mu = \nu = 0$. Moreover, multiplying the equation by u we obtain $\lambda = 2\gamma$; and so

$$Au = \gamma u.$$

In particular, $u \in H^\infty$.

We next claim that $\gamma \neq -1$. Indeed, if $\gamma = -1$, then $u''Z - uZ'' = 0$ by the equations, so that $u'Z - uZ'$ is constant. Since $u'Z - uZ'$ vanishes at $\pm\infty$, we see that $u'Z = uZ'$, so that u and Z are proportional. This contradicts the orthogonality of u and Z and proves the claim.

Since $\gamma < 0$, u does not vanish for $|x|$ large. In particular, u has a finite number of zeroes. Also, since $(u, P) = 0$ and $P > 0$, u has at least one zero. We claim that $\gamma > -1$. Indeed, suppose to the contrary that $\gamma < -1$, and let x_0 be the smallest zero of u. Without loss of generality, we may assume that $u > 0$ on $(-\infty, x_0)$. On $(-\infty, \min\{0, x_0\})$, we have $u, Z > 0$ and $u''Z - uZ'' = (-\gamma - 1)uZ > 0$. Thus $u'Z - uZ'$ is increasing on $(-\infty, \min\{0, x_0\})$. Since $u'Z - uZ'$ vanishes at $-\infty$, we see that $u'Z - uZ' > 0$ at $x = \min\{0, x_0\}$. If $x_0 = \min\{0, x_0\}$, then $(u'Z - uZ')(x_0) = u'(x_0)Z(x_0) < 0$, which is a absurd. Thus $x_0 > 0$; and so $u > 0$ on $(-\infty, 0]$. One shows similarly that $u < 0$ on $[0, \infty)$, which is a contradiction and proves the claim $\gamma > -1$. Therefore $-\gamma - 1 < 0$ so that, arguing as above, $u'Z - uZ' < 0$ on $(-\infty, \min\{0, x_0\}]$. It easily follows that $x_0 < 0$, i.e. u has at least one zero on $(-\infty, 0)$. Similarly one shows that u has at least one zero on $(0, \infty)$, so that u has at least two zeroes.

We finally conclude. Let x_0 be the smallest zero of u, and let $-a < 0 < a$ be the two zeroes of R. Assuming $x_0 \le -a$, we have $u''R - uR'' = -\gamma uR > 0$ on $(-\infty, x_0)$ (recall that $\gamma < 0$). Since $u'R - uR'$ vanishes at $-\infty$, we conclude that $(u'R - uR')(x_0) > 0$. As $(u'R - uR')(x_0) = u'(x_0)R(x_0) < 0$, this is absurd. Therefore $-a < x_0$, i.e. R has a zero on $(-\infty, x_0)$. Similarly, if $x_1 > x_0$ is the largest zero of u, then R has a zero on (x_1, ∞). Finally, by a similar Wronskian argument, we see that between any two (finite) zeroes of u, there is a zero of R. Thus R has at least three zeroes. Since R has exactly two zeroes, we obtain a contradiction which completes the proof. □

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