(Webs of maximal rank)
n-covered varieties
and Jordan algebras

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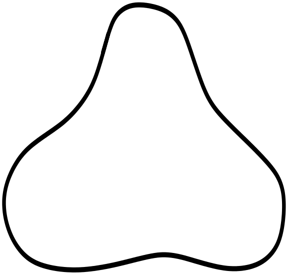
Plan of the talk

0. Motivation (Webs of maximal rank)
1. Varieties n-covered by curves
2. Case $n = 3$ (Jordan algebra)
3. The XJC-correspondence
4. Questions

Introduction

- X irreducible projective variety (over $\mathbb{C}$)



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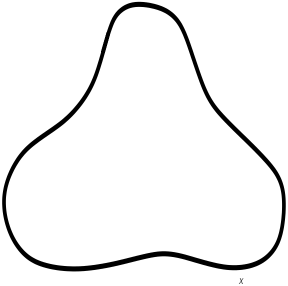

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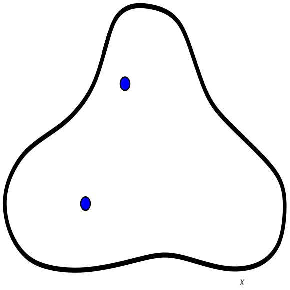

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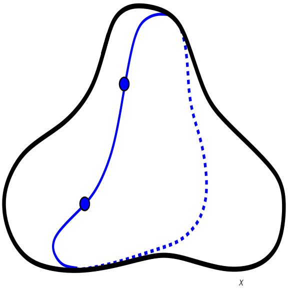

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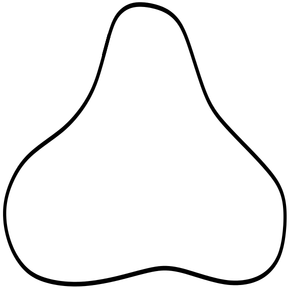

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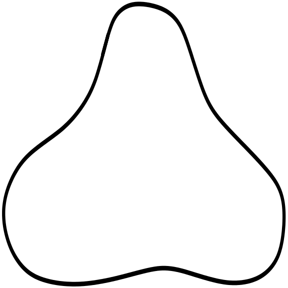

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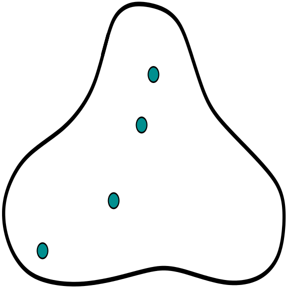

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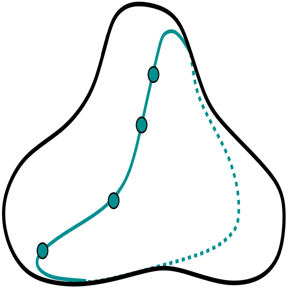

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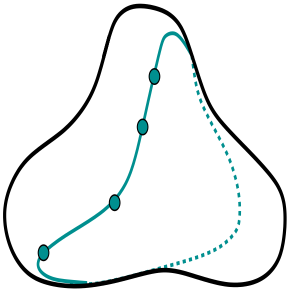

Proposition : X RC $\iff$ X n-RC for all $n \geq 2$

Notations

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- over $\mathbb{C}$
- X irreducible projective variety
- $X \subset \mathbb{P}^N$ fixed embedding
- $r = \dim X \geq 1$
- n number of points ≥ 2
- $\delta =$ degree with respect to $\mathcal{O}_X(1) \geq n-1$

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Remarks :

- 1. is sharp
- $\pi(r,n,\delta)$ is '*Castelnuovo-Harris bound*' on the genus
- 1. implies 2.
- 2. is due to Fano when $n = 2$ and $\Sigma \subset \mathrm{RatCurves}(X)$

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Proposition : Let $X = \mathbf{X}^r(n, \delta) \subset \mathbb{P}^N$

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Definition : an *'osculating projection'* is a linear projection

$$\Pi_S : X \subset \mathbb{P}^{\pi-1} \dashrightarrow T^{(\rho)}_{X,x_1} \simeq \mathbb{P}^{\binom{r+\rho}{r}-1}$$

$$\text{from } S = \Big(\oplus_{i=2}^{m} T^{(\rho)}_{X,x_i}\Big) \oplus \Big(\oplus_{j=1}^{m'} T^{(\rho-1)}_{X,x_{m+j}}\Big)$$

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- **[Harris] :** $\big|I_V(2)\big|$ cut out $Y_V \subset \mathbb{P}^{n+r-2}$: $\begin{cases} \dim Y_V = r \\ \deg Y_V = n-1 \end{cases}$

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- **[Harris] :** $\big|I_V(2)\big|$ cut out $Y_V \subset \mathbb{P}^{n+r-2}$: $\begin{cases} \dim Y_V = r \\ \deg Y_V = n-1 \end{cases}$

- For some $L_V \in \textbf{Pic}(Y_V)$, the following diagram commutes

$$\begin{array}{ccc} V & \overset{\phi_{|K_V|}}{\dashrightarrow} & \mathbb{P}^{\pi-1} \\ \big\downarrow & & \big\| \\ Y_V & \underset{\psi_{|L_V|}}{\dashrightarrow} & \mathbb{P}^{\pi-1} \end{array}$$

Varieties $X(n, \delta)$ of Castelnuovo type

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Facts :

- for all r, n, δ : $\exists\; X = \mathbf{X}(n,\delta)$ of C-type
- $X = \mathbf{X}^r(2,\delta)$ C-type $\Longrightarrow$ $X = v_\delta(\mathbb{P}^r)$
- $X = \mathbf{X}^r\big(n,\rho(n-1)\big)$ C-type $\Longrightarrow$ $X = v_\rho(Y)$, deg $Y = n-1$

A classical problem

Problem : *classify the* $X = X(n, \delta)$ *: is* X *of C-type ?*

Classical results :

Darboux (1880), C. Segre (1920), Bompiani (1921), ...

More recent ones :

..., Lanteri-Palleschi (1987), Andreatta-Ballico-Wisniewski (1993)
Kachi-Sato (1999), Kachi (1999), Ionescu-Russo (2009), ...

Pirio-Trépreau (2013), **Pirio-Russo** (2013 - 2016)

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$$\text{Let } X = \mathbf{X}^r(2,\delta) : \begin{cases} \bullet\ X \subset \mathbb{P}^{\binom{r+\delta}{r}-1} \text{ non degenerate} \\ \bullet\ \text{2-covered by a family } \Sigma \text{ of} \\ \quad \text{irred. curves of degree } \delta \end{cases}$$

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1. $X = v_\delta\big(\mathbb{P}^r\big)$
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– $x \in X$ general : $S = T^{(\delta-1)}_{X,x}$ has dimension $\binom{r+\delta-1}{r} - 1$

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– $\begin{cases} \bullet\ r' = \dim X' \leq r-1 \\ \bullet\ \langle X' \rangle = \mathbb{P}^{\binom{r+\delta-1}{r-1}-1} \\ \bullet\ \underbrace{\Sigma' = \left\{ \Pi_S(C) \middle| C \in \Sigma \right\}}_{2-covering} \end{cases} \Longrightarrow X' = \mathbf{X}^{r-1}(2,\delta) = v_\delta(\mathbb{P}^{r-1})$

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Main tool : induction on n using osculating projections

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- Let $X = \mathbf{X}^r(n,\delta)$: $\langle X\rangle = \mathbb{P}^{\pi-1} + \begin{cases} X \text{ } n\text{-covered by family } \Sigma \\ \text{of irred. curves of degree } \delta \end{cases}$

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- $\left. \begin{array}{l} \deg C' = \rho \quad \forall C' \in \Sigma' \\ + \ \Sigma' \text{ is 2-covering} \end{array} \right\} \Longrightarrow \quad X' = \mathbf{X}^r(2,\rho) = v_\rho(\mathbb{P}^r)$

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Theorem : [P.-Trépreau] For $X = \mathbf{X}^r(n,\delta)$ and $C \in \Sigma$ general

1. C is a RNC of degree δ
2. X smooth along C & $N_{C/X} = \mathcal{O}_C(n-1)^{\oplus(r-1)}$
3. $\exists\,!$ RNC of degree δ through $x_1, \ldots, x_n \in X$ general
4. a general $\Pi_S : X \dashrightarrow \Pi_S(X) \subset T^{(\rho)}_{X,x}$ is birational
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- 3. $\Longrightarrow$ n-covering family Σ is unique

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Varieties $X = \mathbf{X}(n, \delta)$: general case $n \geq 3$

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- $\boldsymbol{\mathcal{H}}_X = \cup_{\mathbf{x} \in X^{n-2}} \boldsymbol{\mathcal{H}}_{\mathbf{x}}$: algebraic system of divisors on X

Theorem : [Pirio-Trépreau] For $X = \mathbf{X}^r(n, \delta)$:

I. The following assertions are equivalent

1. X is of C-type
2. $\mathcal{H}_X$ is a linear system
3. AG_{Σ_X} is flat

II. This is the case if

- $n = 2$ [Bompiani]
- $r = 2$ [(Enriques-)Bol-Bompiani]
- $n \geq 3$ and $\delta \neq 2n - 3$

III. $\exists\ X = \mathbf{X}(n, 2n-3)$'s not of C-type for $n = 3, 4, 5, 6$

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Remark : $\mathbf{X}(3,3)$ of C-type $\implies$ $S_{1\ldots13}$ or $S_{1\ldots122}$

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Question : are there others $X = \mathbf{X}(3,3)$'s ?

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Examples :

- $\mathbf{A}$ associative non-commutative (e.g. $\mathbf{A} = M_n(\mathbb{C})$)
  $x * y = \frac{xy+yx}{2} \quad \Longrightarrow \quad \mathbf{A}^+ = (\mathbf{A}, *)$ Jordan algebra
- $\forall\ q \in \mathbf{Sym}^2(W^*)$, $\mathbf{J}_q = \mathbb{C} \oplus W$ is Jordan with the product
  $$(\lambda, w) \bullet (\lambda', w') = \Big(\lambda\lambda' - q(w, w'), \lambda\, w' + \lambda'\, w\Big)$$
- $B = (\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}) \otimes \mathbb{C} \Longrightarrow \mathbf{Herm}_3(B)$ with
  $M * N = \frac{MN+NM}{2}$ is a Jordan algebra

Jordan algebra of rank 3

- **J** Jordan

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- $\mathbf{J}$ Jordan $\Longrightarrow$
  - x^k well defined $\forall\, k \in \mathbb{N}, \forall x \in \mathbf{J}$
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Remark : $\mathbb{P}\mathbf{J} \circlearrowleft : [x] \longmapsto [x^{-1}] = \left[x^{\#}\right]$ belongs to $\mathbf{Bir}_{2,2}\big(\mathbb{P}\mathbf{J}\big)$

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$\Longrightarrow X_{\mathbf{J}}$ 3-covered by twisted cubics

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$\Longrightarrow$ $X_{\mathbf{J}}$ 3-covered by twisted cubics

– $X_{\mathbf{J}}$ not a scroll $\Longrightarrow$ not of C-type

Semi-simple Jordan algebras

- There is a notion of (semi-)simplicity for Jordan algebras
- Simple Jordan algebras are classified [Albert 1947]

Jordan algebra $\mathbf{J}$	Cubic curve $X_{\mathbf{J}}$
$\mathbb{C} \times \mathbf{J}'$ with $\mathbf{J}'$ ssimple $\mathbf{rk}(\mathbf{J}') = 2$ dim $\mathbf{J}' = r - 1$	$\mathbf{Seg}\left(\mathbb{P}^1 \times Q^{r-1}\right) \subset \mathbb{P}^{2r+1}$
$\mathbf{Herm}_3\left(\mathbb{R}_{\mathbb{C}}\right)$	$LG_3\left(\mathbb{C}^6\right) \subset \mathbb{P}^{13}$
$\mathbf{Herm}_3\left(\mathbb{C}_{\mathbb{C}}\right)$	$G_3\left(\mathbb{C}^6\right) \subset \mathbb{P}^{19}$
$\mathbf{Herm}_3\left(\mathbb{H}_{\mathbb{C}}\right)$	$OG_6\left(\mathbb{C}^{12}\right) \subset \mathbb{P}^{31}$
$\mathbf{Herm}_3\left(\mathbb{O}_{\mathbb{C}}\right)$	$E_7/P_7 \subset \mathbb{P}^{55}$

TABLE : s-simple rank 3 Jordan algebras and their associated cubic curves

$\mathbf{X}(3,3)$: towards a classification

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5. $E' = \varphi_x(E)$ hyperplane in $\mathbb{P}^r$ $\Longrightarrow$ $\varphi_x \in \mathbf{Bir}_2(E, E')$
6. In fact $\varphi_x \in \mathbf{Bir}_{22}\big(E, E'\big)$ $\rightsquigarrow$ $\varphi_x \in \mathbf{Bir}_{22}\big(\mathbb{P}^{r-1}\big)$

$\mathbf{X}(3,3)$: towards a classification

$$\varphi_x \in \mathbf{Bir}_{22}(\mathbb{P}^{r-1}) \quad \rightsquigarrow \begin{cases} \mathcal{B}_x = \mathbf{Baseloc}(\varphi_x) \quad \subset E = \mathbb{P}\big(T_x X\big) \\ \mathcal{B}'_x = \mathbf{Baseloc}(\varphi_x^{-1}) \subset E' = \mathbb{P}^{r-1} \end{cases}$$

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Theorem : [Pirio-Russo]

A. X of C-type $\iff$ $\varphi_x \in \mathbf{Lin} \subset \mathbf{Bir}_{22}(\mathbb{P}^{r-1})$

B. If X not of C-type then

1. $\Pi_x^{-1} : \mathbb{P}^r \overset{bir.}{\dashrightarrow} X$ induced by $\big|3H' - 2\boldsymbol{\mathcal{B}}'_x\big|$
2. $\boldsymbol{\mathcal{B}}_x = \mathbf{Hilb}^{t+1}(X,x) \sim_{proj} \boldsymbol{\mathcal{B}}'_x$
3. X smooth $\iff$ $\boldsymbol{\mathcal{B}}_x$ and $\boldsymbol{\mathcal{B}}'_x$ smooth

$\mathbf{X}(3,3)$: classification

$\mathbf{X}(3,3)$: classification

Theorem : [Pirio-Russo]

If $X = \mathbf{X}(3,3)$ is smooth then

- either X is of Castelnuovo type

$$X = S_{1\ldots 13}\ ,\ S_{1\ldots 122}$$

- either $X = X_{\mathbf{J}}$ for a semi-simple Jordan algebra $\mathbf{J}$

$$X = \mathbf{Seg}\big(\mathbb{P}^1 \times Q^{r-1}\big),\ LG_3\big(\mathbb{C}^6\big),\ G_3\big(\mathbb{C}^6\big),\ OG_6\big(\mathbb{C}^{12}\big),\ E_7/P_7$$

Three worlds

$\mathbf{X}^r(3,3)$

$\mathbf{Jordan}_3^r$

$\mathbf{Bir}_{22}\left(\mathbb{P}^{r-1}\right)$

Three worlds

$\mathbf{X}^r(3,3) \Big/ \text{\textit{projective equivalence}}$

$\mathbf{Jordan}_3^r \Big/ \text{\textit{isotopy}}$

$\mathbf{Bir}_{22}(\mathbb{P}^{r-1}) \Big/ \text{\textit{linear equivalence}}$

Three worlds

$$\left[\mathbf{X}^r(3,3)\right]$$

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$$\left[\mathbf{X}^r(3,3)\right] \xleftarrow{\quad X_J \longleftarrow J \quad} \left[\mathbf{Jordan}_3^r\right]$$

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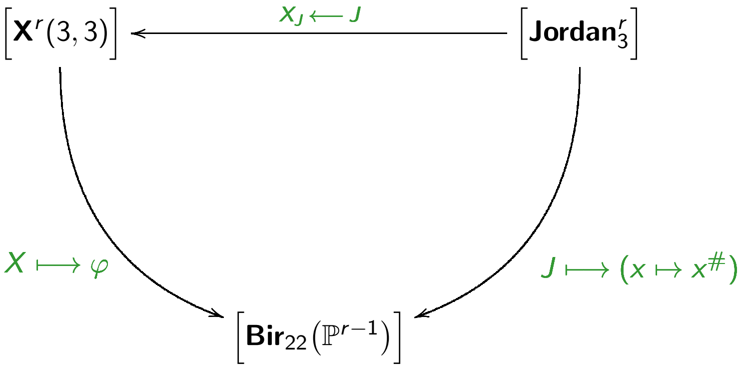
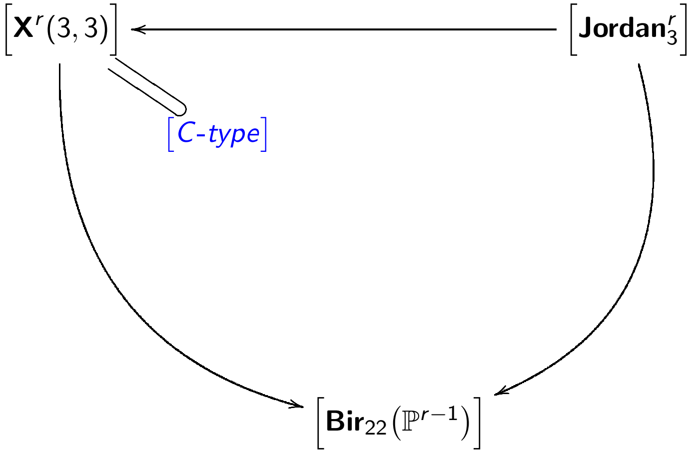
$$\left[\mathbf{X}^r(3,3)\right] \xrightarrow{\;X \longmapsto \varphi_X\;} \left[\mathbf{Bir}_{22}\left(\mathbb{P}^{r-1}\right)\right]$$

Three worlds



Three worlds



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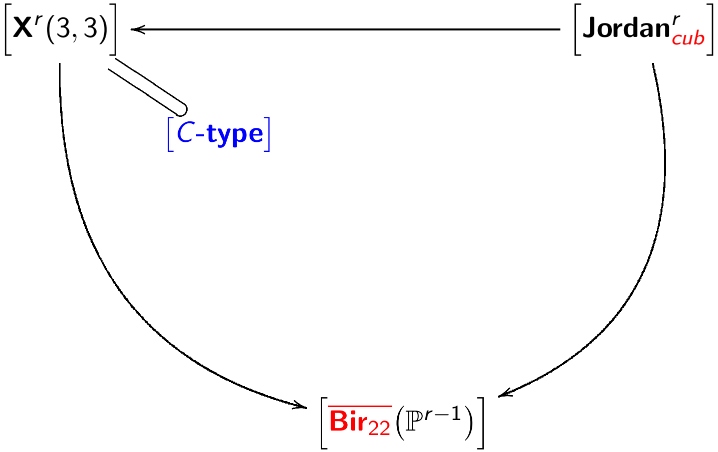

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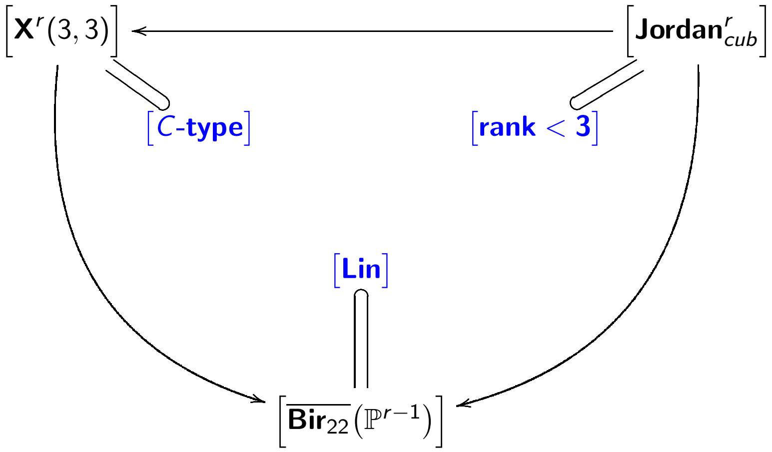

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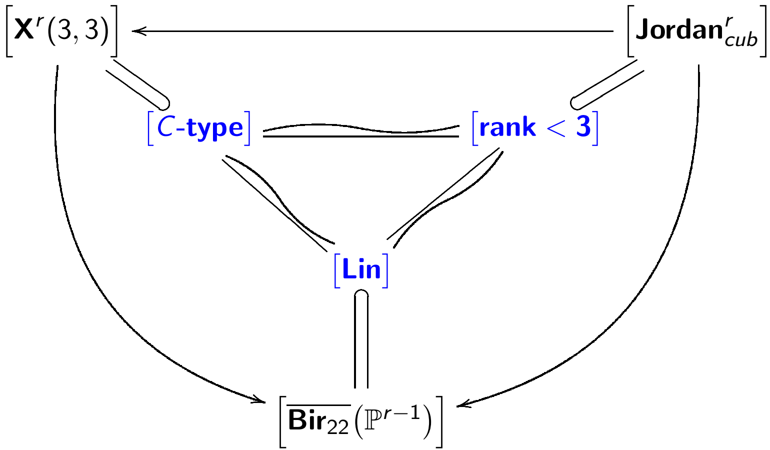

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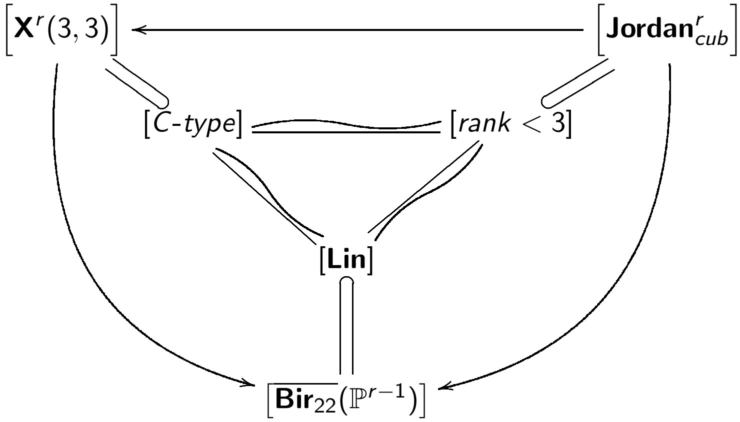

Theorem :

- the *'XJC-diagram'* below is commutative
- all maps are bijections
- any composition with source $=$ target is the identity

$$
\begin{array}{ccc}
\Big[\mathbf{X}^r(3,3)\Big] & \longleftrightarrow & \Big[\mathbf{Jordan}^r_{cub}\Big] \\
 & \nwarrow\!\!\searrow \qquad \swarrow\!\!\nearrow & \\
 & \Big[\overline{\mathbf{Bir}_{22}}(\mathbb{P}^{r-1})\Big] &
\end{array}
$$

The XJC-correspondence [Pirio-Russo]

Theorem :

- the *'XJC-diagram'* below is commutative
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$$\left[\mathbf{X}^r(3,3)\right] \longleftrightarrow \left[\mathbf{Jordan}^r_{cub}\right]$$

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The XJC-principle : any notion / construction / result concerning the **X**, **J** or **C**-world admits counterparts in the other two

The XJC-principle I

Let $\left\{ \begin{array}{l} X \in \mathbf{X}(3,3) \\ J \text{ cubic Jordan algebra} \\ \varphi \text{ q-quadric Cremona map} \end{array} \right\}$ be corresponding objects

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Let $\left\{ \begin{array}{l} X \in \mathbf{X}(3,3) \\ J \text{ cubic Jordan algebra} \\ \varphi \text{ q-quadric Cremona map} \end{array} \right\}$ be corresponding objects

Theorem : the following assertions are equivalent :

- X is smooth
- J is semi-simple
- φ is semi-special

The XJC-principle II

The *XJC*-principle II

Let $\mathbf{J}$ be a Jordan algebra

- Radical $\mathbf{R} = \mathbf{Rad}(\mathbf{J}) < \mathbf{J}$
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Wedderburn's type theorem : [Albert-Penico $\sim$ 1950]

1. $\mathbf{J}_{ss} = \mathbf{J}/\mathbf{R}$ semi-simple
2. $(\mathscr{R})$ splits : $\mathbf{J} \simeq \mathbf{R} \ltimes \mathbf{J}_{ss}$
3. $\mathbf{R}$ solvable : $0 = \mathbf{R}^{(s)} \subsetneq \mathbf{R}^{(s-1)} \subsetneq \cdots \subsetneq \mathbf{R}^{(2)} \subsetneq \mathbf{R}^{(1)} = \mathbf{R}$

$$\text{with } \left(\frac{\mathbf{R}^{(i-1)}}{\mathbf{R}^{(i)}}\right)^2 = 0 \text{ for } i = 1, \ldots, s$$

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Theorem :

1. $X_{ss} = \Pi_{R_X}(X) = \mathbf{X}^{r_{ss}}(3,3)$ is semi-simple
2. $\Pi_{R_X} : X \dashrightarrow X_{ss}$ splits :

$$\exists\, \sigma : \mathbb{P}^{2r_{ss}+1} \overset{linear}{\hookrightarrow} \mathbb{P}^{2r+1} \quad \textit{such that} \quad \big(\Pi_{R_X} \circ \sigma\big)\big|_{X_{ss}} = \mathbf{Id}_{X_{ss}}$$

Theorem :

1. $X_{ss} = \Pi_{R_X}\big(X\big) \in \Big\{ \mathbf{Seg}(\mathbb{P}^1 \times Q),\, LG_3(\mathbb{C}^6),\, G_3(\mathbb{C}^6),\, OG_6(\mathbb{C}^{12}),\, E_7/P_7 \Big\}$

2. Π_{R_X} splits :

$$X \xleftarrow{\text{linear embedding}} X_{ss}, \qquad \Pi_{R_X} : X \dashrightarrow X_{ss}$$

3. Π_{R_X} is *'solvable'* :

$$X = X^{(s)} \dashrightarrow X^{(s-1)} \dashrightarrow \cdots \dashrightarrow X^{(2)} \dashrightarrow X^{(1)} = X_{ss}, \qquad \Pi_{R_X} : X \dashrightarrow X_{ss}$$

with for $i = 1, \ldots, s$:

- $X^{(i)} = \mathbf{X}^{r_i}(3,3)$
- $X^{(i)} \dashrightarrow X^{(i-1)}$ lin. projection, linear fibers, admissible

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- For $n \geq 4$:
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- For $n = 3$: given a 'Jordan cubic curve' $X_{\mathbf{J}} \in \mathbf{X}(3,3)$:
  - relation between $\mathbf{Rad}(\mathbf{J})$ and $\mathbf{sing}(X_{\mathbf{J}})$?
  - desingularization of $X_{\mathbf{J}}$?
  - description of $\mathbf{Pic}(X_{\mathbf{J}})$?

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- Same question for $n \geq 3$, *e.g.* $n = 6$: $v_3(\mathbb{P}^3) = \mathbf{X}(6, 9)$