Geometry of webs: an introduction

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October 16th 2017

Plan

1. History
2. Webs
3. Examples
4. A classical theorem
5. Recent developments

History

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- **'Globalization':** (1990-...)

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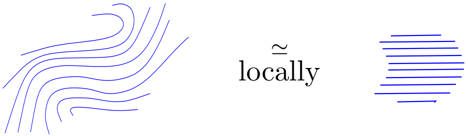

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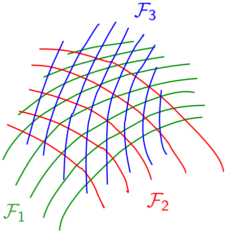

a planar 3-web

Webs

Definition : d-web $\mathcal{W}_d$ *of codimension r* on a domain $U \subset \mathbb{C}^N$ is

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- a *'normal'* to $\mathcal{F}_i$ = r-differential form $\Omega_i \in \Omega^r(U)$ such that

$$T_{\mathcal{F}_i} = \mathbf{ker}\big(\Omega_i\big) = \Big\{\, \xi \in T_U \;\Big|\; i_\xi(\Omega_i) = 0 \Big\}$$

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General position assumption : (case $N = nr$)

$$1 \leq i_1 < \cdots < i_n \leq d \quad \Longrightarrow \quad \Omega_{i_1} \wedge \cdots \wedge \Omega_{i_n} \neq 0$$

Webs

Definition : a d-web on a manifold M is $\mathcal{W}_d = \cup_i \mathcal{W}_d^i$ with

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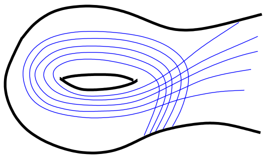

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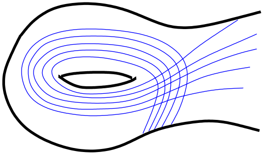

Definition : two webs $\mathcal{W}$ and $\mathcal{W}'$ are **equivalent** if

$$\exists\, \varphi \text{ local isomorphism such that } \quad \mathcal{W} = \varphi^*(\mathcal{W}')$$

Geometry of webs

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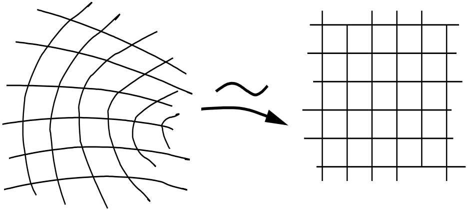

a planar 2-web is locally trivial

Geometry of webs

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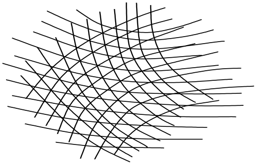

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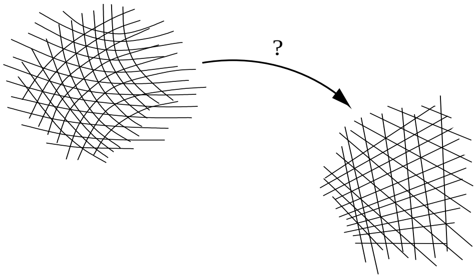

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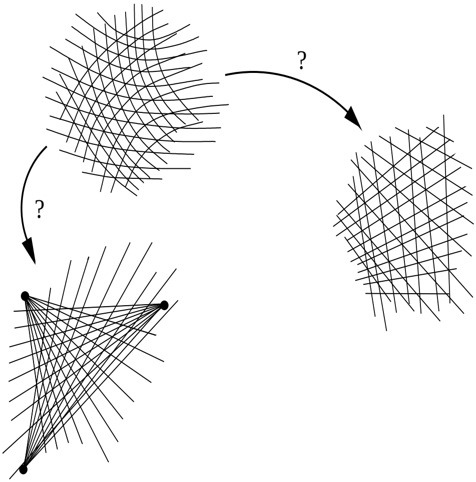

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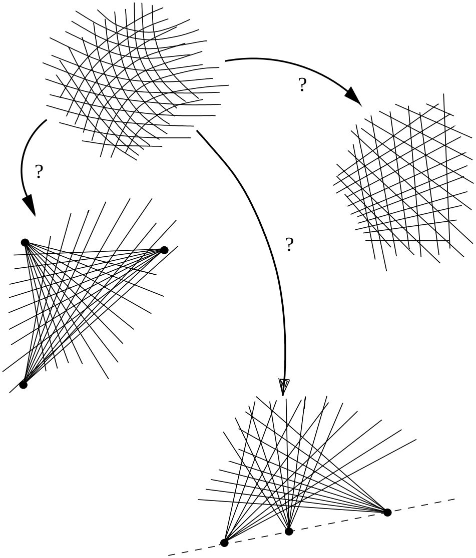

Examples of webs

Examples of webs : in classical function theory

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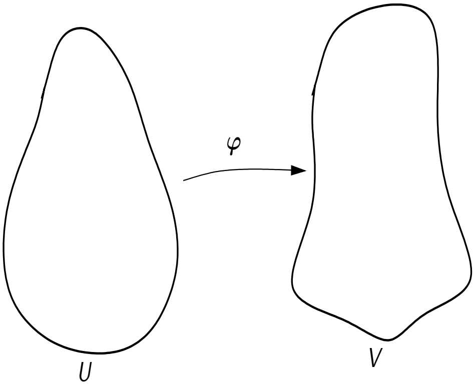

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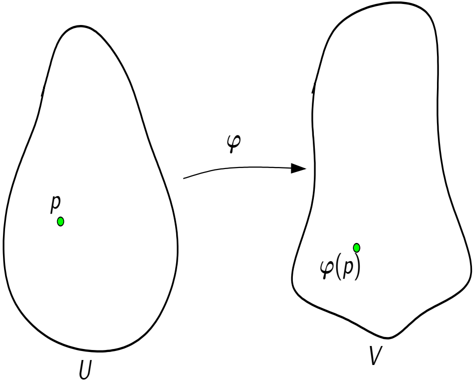

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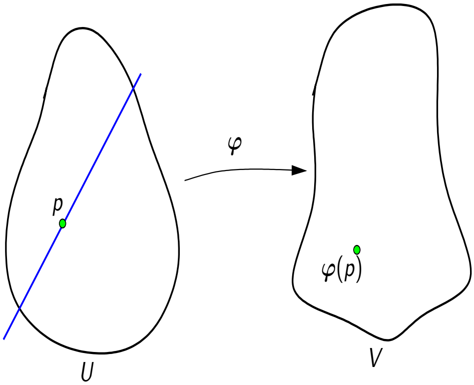

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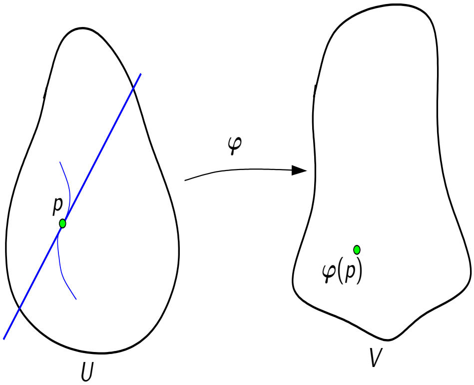

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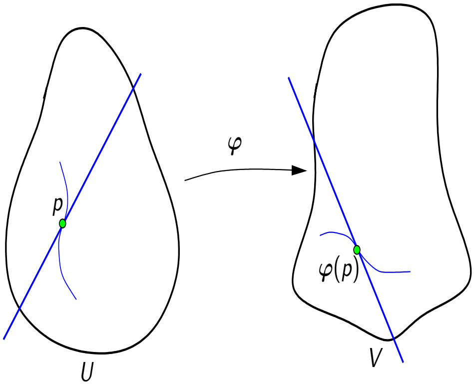

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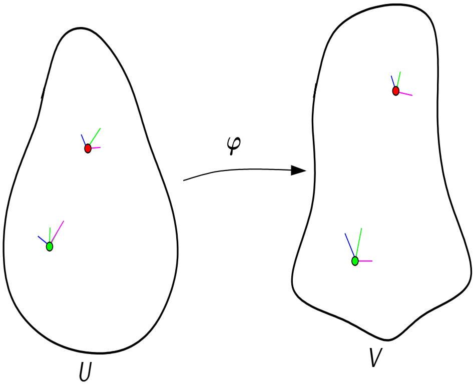

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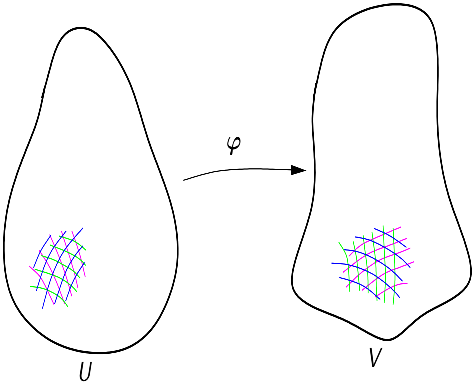

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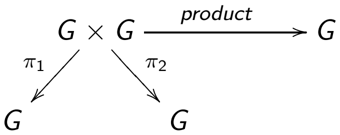

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$$\begin{array}{ccccc} & & G \times G & \xrightarrow{\;product\;} & G \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ G & & & & G \end{array}$$

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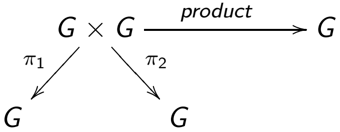

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- **Question** :

Algebraic properties of the Lie group G	$\overset{?}{\longleftrightarrow}$	Differential properties of the 3-web $\mathcal{W}_G$

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$$M_{0,5}$$

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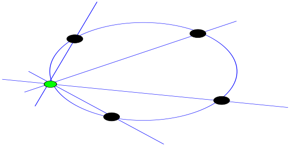

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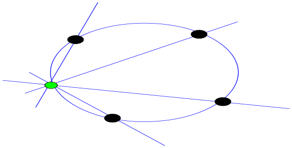
2. 

3. $$\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{W}}\left(x \, , \, y \, , \, \frac{x}{y} \, , \, \frac{1-x}{1-y} \, , \, \frac{x(1-y)}{y(1-x)} \right)$$

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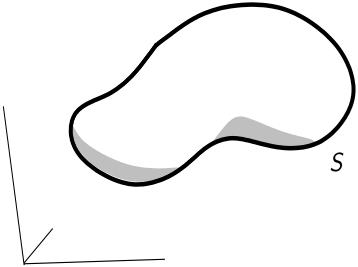

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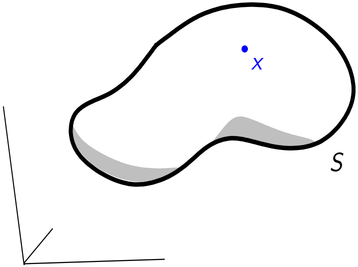

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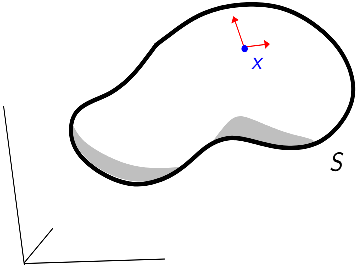

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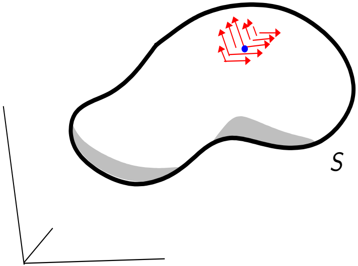

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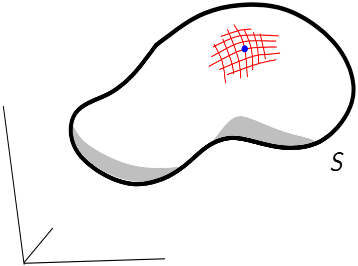

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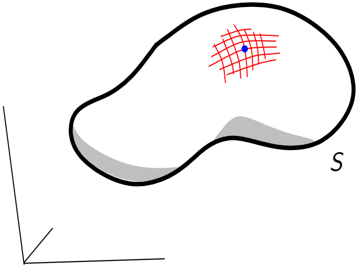

- $S \not\subset S^2 \quad \rightsquigarrow \quad$ 2-web $\mathcal{W}_S$ on S

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  - Darboux's web : $\mathcal{DW}_\Sigma = \left(\mathcal{P}_{L_1}, \ldots, \mathcal{P}_{L_{27}}\right)$

Examples of webs : in projective differential geometry

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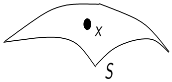

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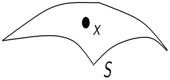
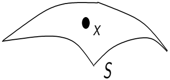
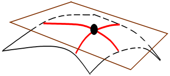
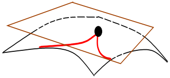
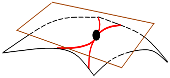
- Surface $S \subset \mathbb{P}^5$  (with regular 2-osculation at x)

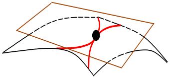
• Surface $S \subset \mathbb{P}^5$ 

(with regular 2-osculation at x)

Hyperplane H	H **and** S	**Curve** $H \cap S$
$H \supset T_{S,x}$		Node : $x^2 - y^2 = 0$
$H \in \mathcal{C}_x \simeq \mathbb{P}^1$		Cusp : $x^2 - y^3 = 0$
Definition : H principal		Tacnode : $x^2 - y^4 = 0$

Webs in projective differential geometry

- Let H = a principal hyperplane of S at x



Webs in projective differential geometry

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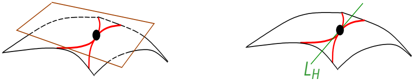

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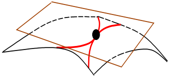

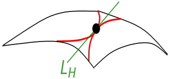

Definition : $L_H \in \mathbb{P}T_{S,x}$ is a *principal direction* of S at x

Webs in projective differential geometry

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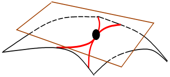

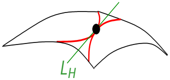

Definition : $L_H \in \mathbb{P}T_{S,x}$ is a *principal direction* of S at x

Proposition : [C. Segre] If x is not an umbilic :

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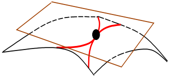

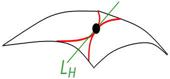

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$$S \text{ is totally umbilic} \iff S \subset v_2(\mathbb{P}^2) \subset \mathbb{P}^5$$

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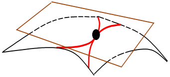
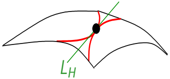

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S is totally umbilic $\iff$ $S \subset v_2(\mathbb{P}^2) \subset \mathbb{P}^5$

- $S \not\subset v_2(\mathbb{P}^2) \rightsquigarrow$ the principal curves form : *Segre's 5-web* $\mathcal{SW}_S$ on S

Example :

Webs in projective differential geometry

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- $P = \left\{ p_1 \quad p_2 \quad p_3 \quad p_4 \right\} \subset \mathbb{P}^2$ four points in general position

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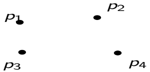
- $P = \Big\{$ 

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Remark : $\Sigma = \mathbf{Bl}_P(\mathbb{P}^2) \simeq \overline{M}_{0,5}$

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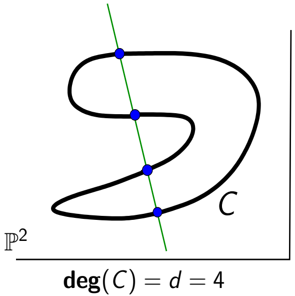

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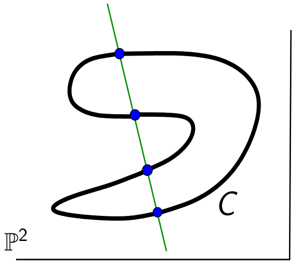
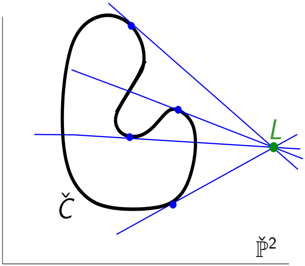


deg$(C) = d = 4$



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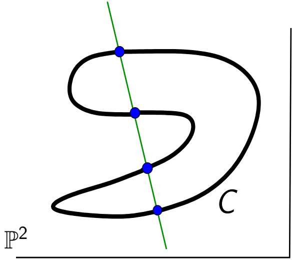

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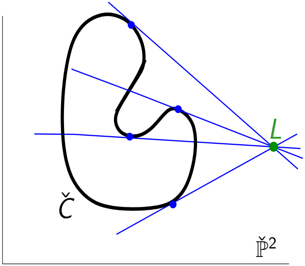

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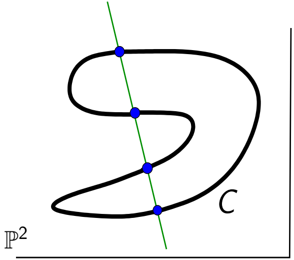

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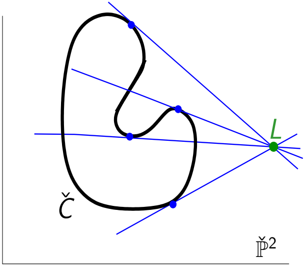

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A planar algebraic 3-web

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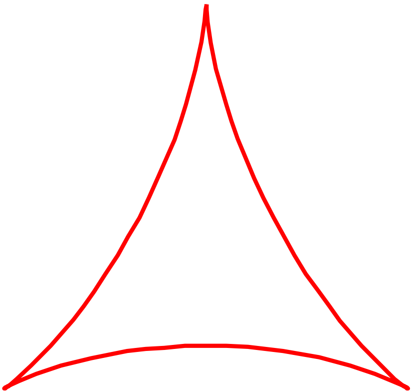

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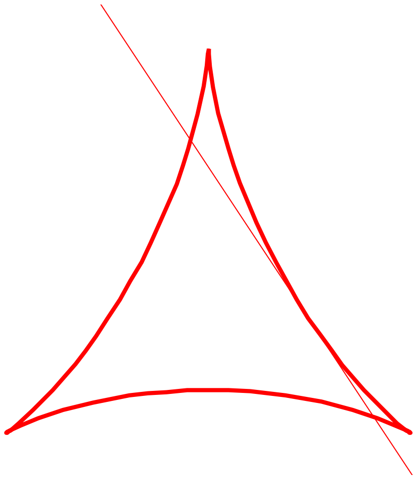

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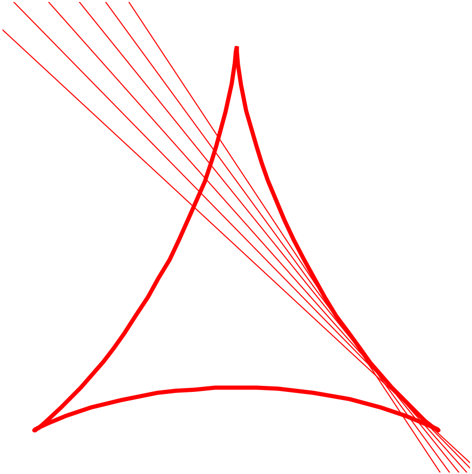

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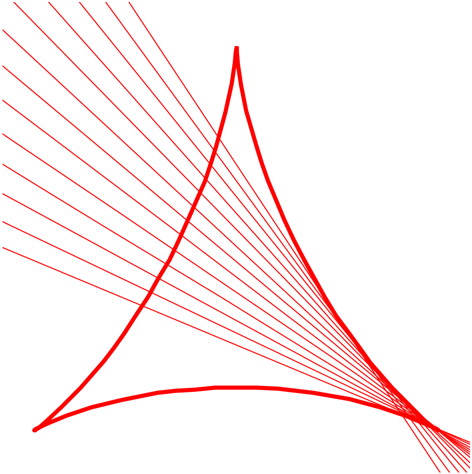

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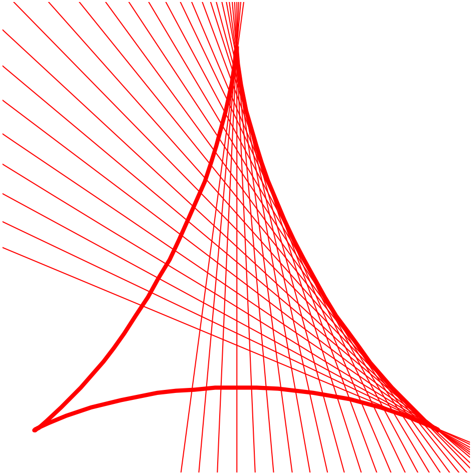

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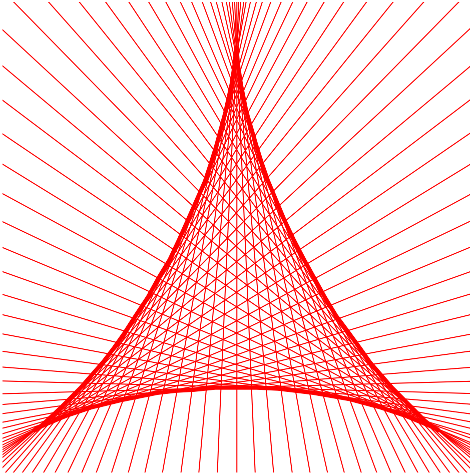

algebraic 3-web $\mathcal{W}_C$ associated to a plane cubic $C \subset \mathbb{P}^2$

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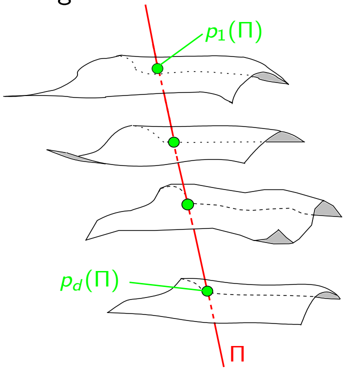

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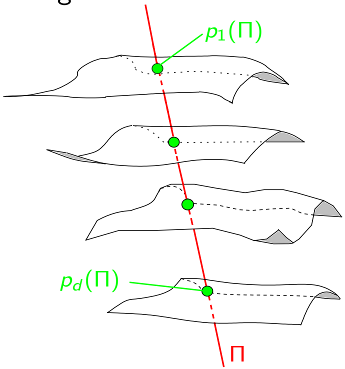

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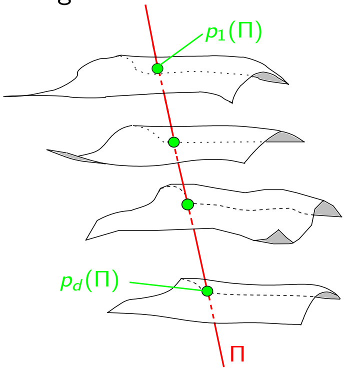

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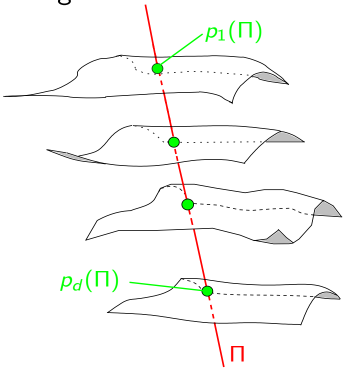

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[Chern 1982] :

"the subject is a wide generalization of the geometry of projective algebraic varieties. Just as intrinsic algebraic varieties are generalized to Kähler manifolds and complex manifolds, such a generalization to web geometry seems justifiable."

Webs are everywhere...

- Algebra
- Topology
- Geometry
- Theory of dynamical systems
- Theory of DEs & PDEs
- Mathematical Physics
- Economy

A classical theorem

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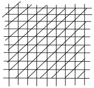
1. $\mathcal{W}_3$ is *parallelizable* $\simeq$  $= \mathcal{W}(x, y, x-y)$

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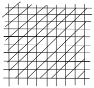
1. $\mathcal{W}_3$ is *parallelizable* $\simeq$  $= \mathcal{W}(x, y, x - y)$

2. $\mathcal{W}_3$ is *hexagonal*

A classical theorem

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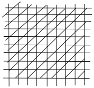
1. $\mathcal{W}_3$ is *parallelizable* $\simeq$  $= \mathcal{W}(x, y, x - y)$

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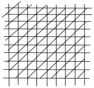
1. $\mathcal{W}_3$ is *parallelizable* $\simeq$  $= \mathcal{W}(x, y, x-y)$

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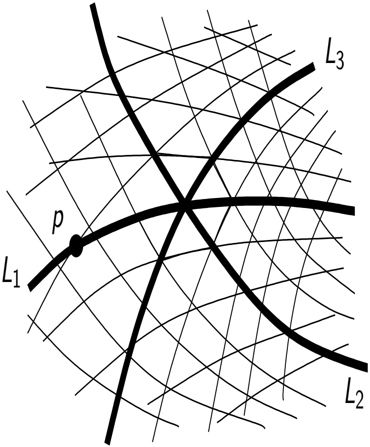

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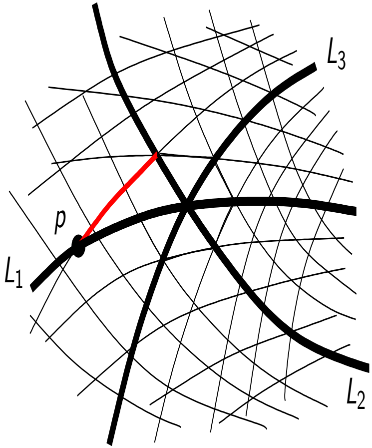

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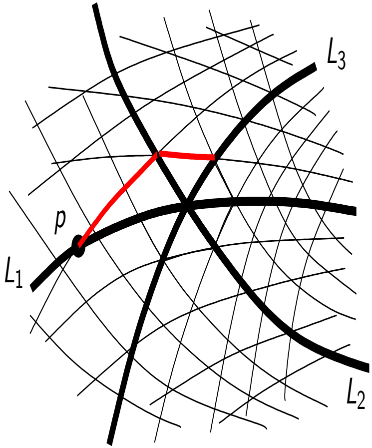

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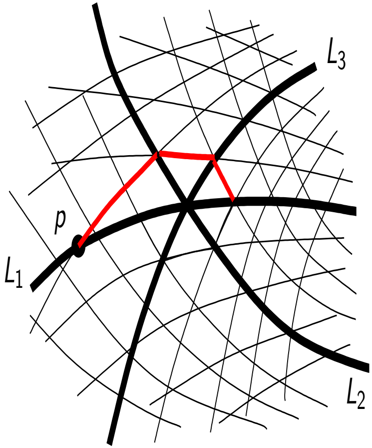

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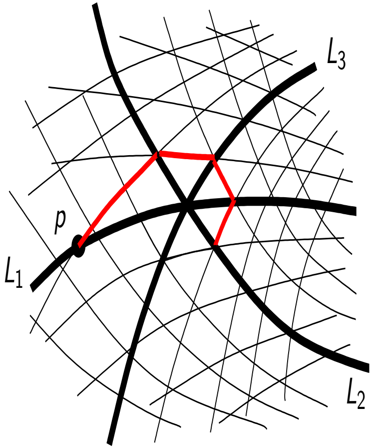

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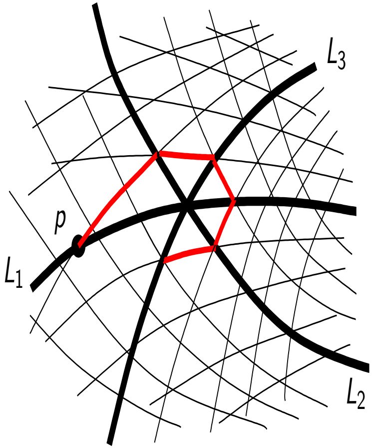

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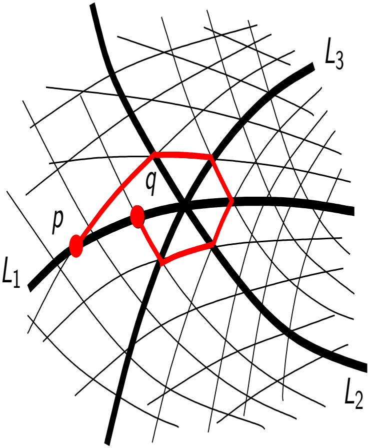

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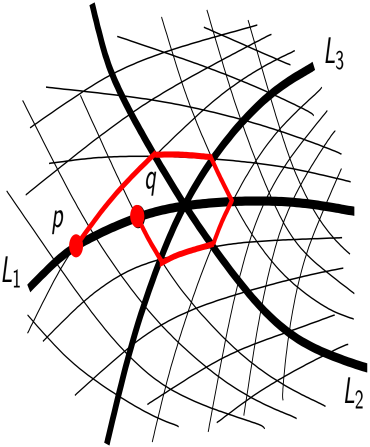

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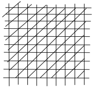
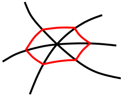
1. $\mathcal{W}_3$ is *parallelizable* $\simeq$  $= \mathcal{W}(x, y, x-y)$

2. $\mathcal{W}_3$ is *hexagonal* 

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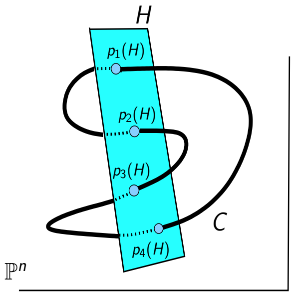

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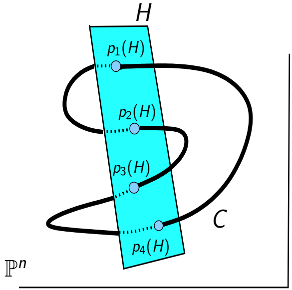

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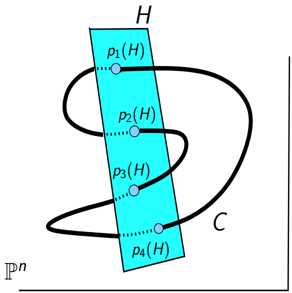

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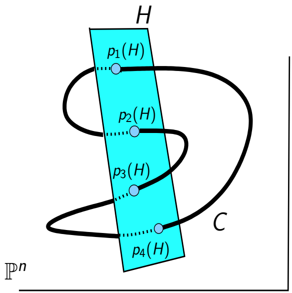

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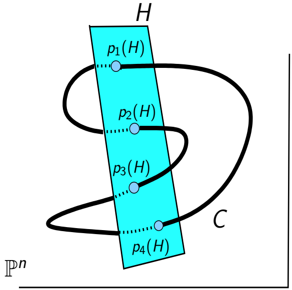

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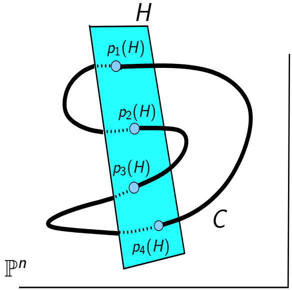

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Fact : $V^r \subset \mathbb{P}^{n+r-1}$ Castelnuovo $\Longrightarrow \mathcal{W}_V$ has maximal rank

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Theorem : [Blaschke-Howe 1932 ($n = 2$), Griffiths 1976]

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$$\Uparrow$$

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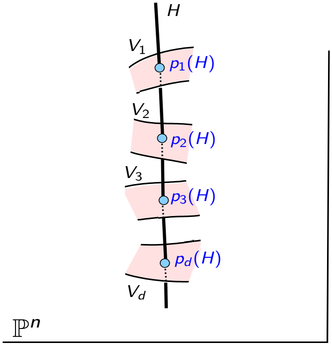

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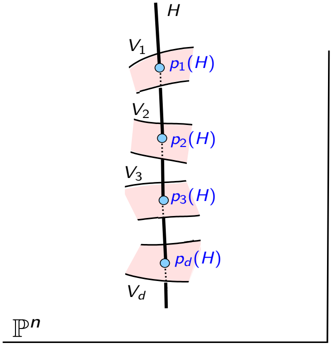

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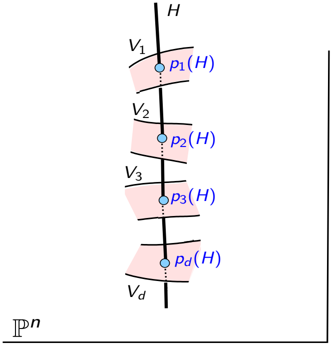

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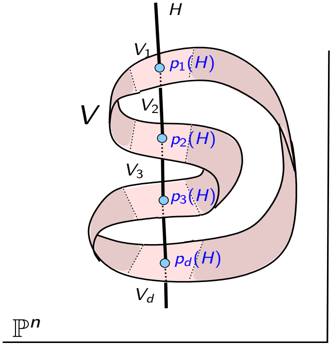

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$$\sum_i p_i^*(\omega_i) \equiv 0 \quad \Longrightarrow \quad \begin{array}{l} \exists\, V \subset \mathbb{P}^n \text{ alg. hypersurface} \\ \exists\, \omega \in \mathbf{H}^0(V, \omega_V^{n-1}) \quad \text{s. t.} \\ \forall\, i : V_i \subset V,\ \omega_i = \omega|_{V_i} \end{array}$$

Algebraization of webs of maximal rank

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Theorem : [Bol ($n = 3$), (Chern-Griffiths), Trépreau]

Let $\mathcal{W}_d$ be a d-web on $U \subset \mathbb{C}^n$ with $n \geq 3$:

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Theorem : [Pirio-Trépreau 2013]

For a d-web $\mathcal{W}_d$ of codimension $r > 1$ on $U \subset \mathbb{C}^{nr}$:

$\mathcal{W}_d$ has maximal r-rank $\big(\mathbf{rk}^r(\mathcal{W}_d) = \pi(d, n, r)\ \big)$ $\Longrightarrow$ $\mathcal{W}_d$ is 'algebraizable' $\big($generalized sense if $d = d_{n,r}\ \big)$

Algebraic geometry of webs

Algebraic curves	**Webs of codim 1**
degree d curve $C \subset \mathbb{P}^n$	
$\omega \in \mathbf{H}^0(C, \omega_C^1)$	
$\mathbf{p}_a(C) = \mathbf{h}^0(C, \omega_C^1)$	
$\mathbf{J}_C = \mathbf{H}^0(\omega_C^1)^\vee / \mathbf{H}_1(C, \mathbb{Z})$	
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Exceptional webs

Exceptional webs

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- **[Bol 1936] :** $\mathcal{B}$ is *'exceptional'* $\stackrel{def}{=\!=}$ $\begin{cases} \text{of maximal rank} \\ + \\ \text{non-algebraizable} \end{cases}$

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"The determination of all webs of maximal rank will remain a fundamental problem in web geometry and the non-algebraic ones, if there are any, will be most interesting."

Exceptional webs

Chern-Griffiths (1981) :

"(...) we cannot refrain from mentionning what we consider to be the fundamental problem on the subject, which is to determine the maximum rank non-linearizable webs. The strong conditions must imply that there are not many. It may not be unreasonable to compare the situation with the exceptional simple Lie groups."

Chern (1985) :

"The determination of all webs of maximal rank will remain a fundamental problem in web geometry and the non-algebraic ones, if there are any, will be most interesting."

Chern's problem : determine and classify the exceptional webs

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- The exceptional webs remain mysterious...

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- For a non-reduced curve $C \subset \mathbb{P}^2$:
  - is there a web-theoretic object $\mathcal{W}_C$ corresponding to it ?
  - what would be an abelian relation for such a 'web' $\mathcal{W}_C$?

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Algebraic geometry	**Web geometry**
Variety $V \subset \mathbb{P}^{n+r-1}$ degree d and dim r	$\mathcal{W}_d$ on $(\mathbb{C}^{nr}, 0)$ d-web of codim r
$\omega \in \mathbf{H}^0(V, \Omega^q_V) \quad q = 0, \ldots, r$	$\underline{\omega} = (\omega_i)_{i=1}^d \in \mathcal{A}^q(\mathcal{W}_d)$
$\mathbf{h}^{q,0}(V) = \mathbf{h}^0\big(V, \Omega^q_V\big)$	$\mathbf{rk}^q(\mathcal{W}_d) = \dim \mathcal{A}^q(\mathcal{W}_d)$
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- For $q < r$:

  determine the varieties $V^r \subset \mathbb{P}^{n+r-1}$ of *'maximal q-rank'*

  i.e. such that $\mathbf{h}^{q,0}(V) = \pi^q(d, n, r)$ where $d = \deg V$

끝났어

관심을 가져 주셔서 감사합니다

$\star$

THANK YOU FOR YOUR ATTENTION

THE END