Moduli spaces of flat tori and elliptic hypergeometric functions

Luc PIRIO

(CNRS & Université Versailles-Saint Quentin)

13 février 2017

Plan

I. Flat surfaces ($g \geq 0$)

Veech : "*Flat Surfaces*". Amer. J. Math. (1993)

II. Flat spheres ($g = 0$)

Thurston : "*Shapes of polyhedra*"

Deligne-Mostow : "*Monodromy of hypergeometric functions*"

III. Flat tori ($g = 1$)

Ghazouani-Pirio : $\begin{cases} [\mathrm{GP}_{\mathrm{geom}}] & = \mathrm{arXiv:1604.01812} \\ [\mathbf{GP}_{\mathbf{hypergeom}}] & = \mathrm{arXiv:1605.02356} \end{cases}$

Notations

g and n : integers such that $2g - 3 + n > 0$

Notations

g and n : integers such that $2g - 3 + n > 0$

- S : surface of genus g ;
- $p_1, \ldots, p_n$: n points on S
- $S^* = S \setminus \{p_1, \ldots, p_n\}$

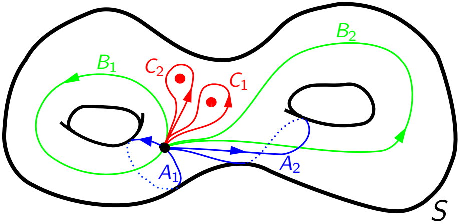
Notations

g and n : integers such that $2g - 3 + n > 0$

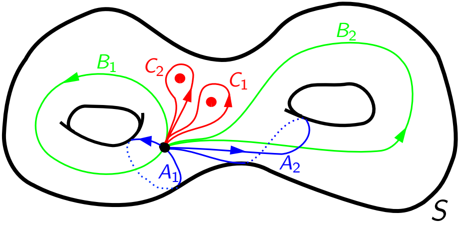
- S : surface of genus g ;
- $p_1, \ldots, p_n$: n points on S
- $S^* = S \setminus \{p_1, \ldots, p_n\}$

- X : Riemann surface of genus g
- $x = (x_1, \ldots, x_n)$: n-tuple of points on X
- $X^* = X \setminus \{x_1, \ldots, x_n\}$

- $\pi_1(g, n) = \pi_1(S^*) = \left\langle A_i, B_i, C_1, \ldots, C_n \mid \prod_i [A_i, B_i] = C_n \cdots C_1 \right\rangle$



- $\pi_1(g,n) = \pi_1(S^*) = \left\langle A_i, B_i, C_1, \ldots, C_n \mid \prod_i [A_i, B_i] = C_n \cdots C_1 \right\rangle$

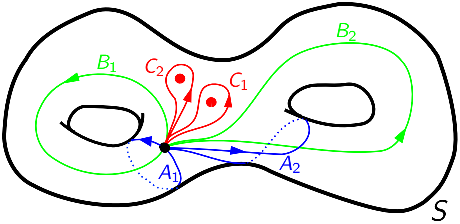


Definitions :

- a *marking of the* π_1 *of* (X, x) is an isomorphism

$$\varphi : \pi_1(g,n) \simeq \pi_1(X^*)$$

- $\pi_1(g,n) = \pi_1(S^*) = \left\langle A_i, B_i, C_1, \ldots, C_n \mid \prod_i [A_i, B_i] = C_n \cdots C_1 \right\rangle$

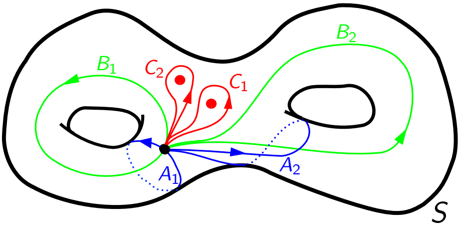


Definitions :

- a *marking of the π_1 of (X, x)* is an isomorphism

$$\varphi : \pi_1(g,n) \simeq \pi_1(X^*) \qquad \left(\text{modulo } \mathbf{Inn}\left(\pi_1(g,n)\right)^{\mathbf{C}}\right)$$

- $\pi_1(g,n) = \pi_1(S^*) = \left\langle A_i, B_i, C_1, \ldots, C_n \mid \prod_i [A_i, B_i] = C_n \cdots C_1 \right\rangle$



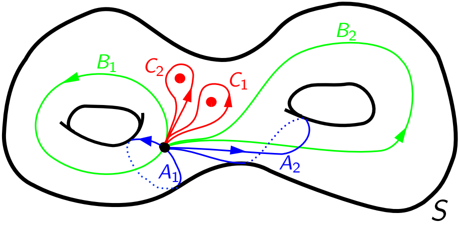
Definitions :

- a *marking of the π_1 of (X,x)* is an isomorphism

$$\varphi : \pi_1(g,n) \simeq \pi_1(X^*) \qquad \left(\text{modulo } \mathbf{Inn}\big(\pi_1(g,n)\big)^{\mathbf{C}}\right)$$

- $\mathcal{T}\mathbf{eich}_{g,n} = \left\{ \big((X,x),\varphi\big) \,\middle|\, \varphi \text{ marking of } \pi_1(X^*) \right\}$

- $\pi_1(g,n) = \pi_1(S^*) = \left\langle A_i, B_i, C_1, \ldots, C_n \mid \prod_i [A_i, B_i] = C_n \cdots C_1 \right\rangle$



Definitions :

- a *marking of the π_1 of (X,x)* is an isomorphism

$$\varphi : \pi_1(g,n) \simeq \pi_1(X^*) \qquad \left(\text{modulo } \mathbf{Inn}\big(\pi_1(g,n)\big)^{\mathbf{C}}\right)$$

- $\mathcal{T}\mathbf{eich}_{g,n} = \Big\{ \big((X,x),\varphi\big) \,\big|\, \varphi \text{ marking of } \pi_1(X^*) \Big\}$

- $\mathbf{PMCG}^{\pm}_{g,n} = \mathbf{Out}\big(\pi_1(g,n)\big)^{\mathbf{C}} = \mathbf{Aut}\big(\pi_1(g,n)\big)^{\mathbf{C}} \Big/ \mathbf{Inn}\big(\pi_1(g,n)\big)^{\mathbf{C}}$

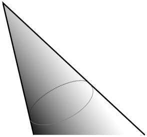
Teichmüller theory

Theorems :

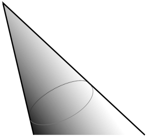
- $\mathcal{T}\mathbf{eich}_{g,n}$ is isomorphic to a bounded domain in $\mathbb{C}^{3g-3+n}$
- $\mathbf{PMCG}_{g,n}$ is isomorphic to $\mathbf{Bihol}(\mathcal{T}\mathbf{eich}_{g,n})$
- $\mathcal{T}\mathbf{eich}_{g,n}/\mathbf{PMCG}_{g,n}$ is isomorphic to $\mathbf{M}_{g,n}$
- $\mathbf{PMCG}_{g,n}$ is isomorphic to $\pi_{\mathbf{1}}^{\mathbf{orb}}(\mathbf{M}_{\mathbf{g,n}})$

Flat surfaces : definitions

Flat surfaces : definitions

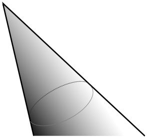
- For $\theta > 0$, C_θ = Euclidean cone of angle θ = 

Flat surfaces : definitions

- For $\theta > 0$, C_θ = Euclidean cone of angle θ = 

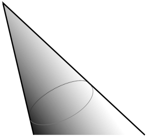
Def : a flat metric m on S^*

Flat surfaces : definitions

- For $\theta > 0$, C_θ = Euclidean cone of angle θ = 

Def : a flat metric m on S^* has a *conical singularity* at p_k

Flat surfaces : definitions

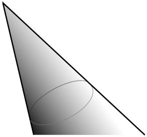
- For $\theta > 0$, C_θ = Euclidean cone of angle θ = 

Def : a flat metric m on S^* has a *conical singularity* at p_k if locally in the vicinity of this point :

- $(S, m) \simeq C_{\theta_k}$ for a certain *conical angle* $\theta_k > 0$

Flat surfaces : definitions

- For $\theta > 0$, C_θ = Euclidean cone of angle θ =

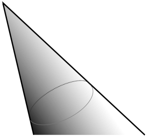


Def : a flat metric m on S^* has a *conical singularity* at p_k if locally in the vicinity of this point :

- $(S, m) \simeq C_{\theta_k}$ for a certain *conical angle* $\theta_k > 0$
- $m = |z^{\alpha_k} dz|^2$ for a certain *exponent* $\alpha_k > -1$

Flat surfaces : definitions

- For $\theta > 0$, C_θ = Euclidean cone of angle θ =



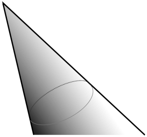
Def : a flat metric m on S^* has a *conical singularity* at p_k if locally in the vicinity of this point :

- $(S, m) \simeq C_{\theta_k}$ for a certain *conical angle* $\theta_k > 0$
- $m = |z^{\alpha_k} dz|^2$ for a certain *exponent* $\alpha_k > -1$

- $\theta_k = 2\pi(1 + \alpha_k)$;

Flat surfaces : definitions

- For $\theta > 0$, C_θ = Euclidean cone of angle θ =

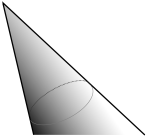


Def : a flat metric m on S^* has a *conical singularity* at p_k if locally in the vicinity of this point :

- $(S, m) \simeq C_{\theta_k}$ for a certain *conical angle* $\theta_k > 0$
- $m = |z^{\alpha_k} dz|^2$ for a certain *exponent* $\alpha_k > -1$

- $\theta_k = 2\pi(1 + \alpha_k)$; $\kappa(p_k) =$ *curvature at* $p_k = 2\pi - \theta_k$

Flat surfaces : definitions

- For $\theta > 0$, C_θ = Euclidean cone of angle θ = 

Def : a flat metric m on S^* has a *conical singularity* at p_k
if locally in the vicinity of this point :

- $(S, m) \simeq C_{\theta_k}$ for a certain *conical angle* $\theta_k > 0$
- $m = |z^{\alpha_k} dz|^2$ for a certain *exponent* $\alpha_k > -1$

- $\theta_k = 2\pi(1 + \alpha_k)$; $\kappa(p_k) =$ *curvature at* $p_k = 2\pi - \theta_k$

Gauß-Bonnet : p_k is a *conical point* $\forall\ k \implies \sum_{k=1}^{n} \alpha_k = 2g - 2$

Flat surfaces : examples

Flat surfaces : examples

- **Flat metrics on X :**

Flat surfaces : examples

- **Flat metrics on X :**
  - $m = |\omega|^2$ with $\omega \in H^0(X, K_X)$ translation surface

Flat surfaces : examples

- **Flat metrics on X :**
  - $m = |\omega|^2$ with $\omega \in H^0(X, K_X)$ translation surface
  - $m = |\eta|^{2/k}$ with $\eta \in H^0(X, K_X^{\otimes k})$ (1/k)-translation surface

Flat surfaces : examples

- **Flat metrics on X :**
  - $m = |\omega|^2$ with $\omega \in H^0(X, K_X)$ translation surface
  - $m = |\eta|^{2/k}$ with $\eta \in H^0(X, K_X^{\otimes k})$ (1/k)-translation surface
  - $m = |\mu|^{2/k}$ with $\mu \in H^0(X, K_X^{\otimes k} \otimes \mathcal{L}_\rho)$, $\rho : \pi_1(X) \to \mathbb{U}$

Flat surfaces : examples

- **Flat metrics on X :**
  - $m = |\omega|^2$ with $\omega \in H^0(X, K_X)$ translation surface
  - $m = |\eta|^{2/k}$ with $\eta \in H^0(X, K_X^{\otimes k})$ (1/k)-translation surface
  - $m = |\mu|^{2/k}$ with $\mu \in H^0(X, K_X^{\otimes k} \otimes \mathcal{L}_\rho)$, $\rho : \pi_1(X) \to \mathbb{U}$

- **Polygonal surface :**

Flat surfaces : examples

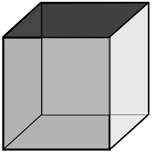
- **Flat metrics on X :**
  - $m = |\omega|^2$ with $\omega \in H^0(X, K_X)$ translation surface
  - $m = |\eta|^{2/k}$ with $\eta \in H^0(X, K_X^{\otimes k})$ (1/k)-translation surface
  - $m = |\mu|^{2/k}$ with $\mu \in H^0(X, K_X^{\otimes k} \otimes \mathcal{L}_\rho)$, $\rho : \pi_1(X) \to \mathbb{U}$

- **Polygonal surface :**
  obtained by gluing
  Euclidean polygons

Flat surfaces : examples

- **Flat metrics on X :**
  - $m = |\omega|^2$ with $\omega \in H^0(X, K_X)$ translation surface
  - $m = |\eta|^{2/k}$ with $\eta \in H^0(X, K_X^{\otimes k})$ (1/k)-translation surface
  - $m = |\mu|^{2/k}$ with $\mu \in H^0(X, K_X^{\otimes k} \otimes \mathcal{L}_\rho)$, $\rho : \pi_1(X) \to \mathbb{U}$

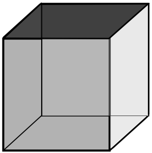
- **Polygonal surface :**
  obtained by gluing :
  Euclidean polygons



Flat surfaces : examples

- **Flat metrics on X :**
  - $m = |\omega|^2$ with $\omega \in H^0(X, K_X)$ translation surface
  - $m = |\eta|^{2/k}$ with $\eta \in H^0(X, K_X^{\otimes k})$ (1/k)-translation surface
  - $m = |\mu|^{2/k}$ with $\mu \in H^0(X, K_X^{\otimes k} \otimes \mathcal{L}_\rho)$, $\rho : \pi_1(X) \to \mathbb{U}$

- **Polygonal surface :**
  obtained by gluing :
  Euclidean polygons



Thm : **(Delaunay decomposition)**

A flat surface admits a canonical polygonal decomposition

Veech : '*Flat surfaces*'

Veech : '*Flat surfaces*'

- $\boldsymbol{\alpha} = (\alpha_k)_{k=1}^n$ fixed such that $\sum_k \alpha_k = 2g - 2$

Veech : '*Flat surfaces*'

- $\boldsymbol{\alpha} = (\alpha_k)_{k=1}^n$ fixed such that $\sum_k \alpha_k = 2g - 2$

- $$\mathcal{E}_{g,n}^{\boldsymbol{\alpha}} = \left\{ \begin{array}{l} \text{flat structures on } S \text{ with a} \\ \alpha_k\text{-conical singularity at } p_k \end{array} \right\} \Big/ \text{Isotopy}$$

Veech : '*Flat surfaces*'

- $\boldsymbol{\alpha} = (\alpha_k)_{k=1}^n$ fixed such that $\sum_k \alpha_k = 2g - 2$
- $\mathcal{E}_{g,n}^{\boldsymbol{\alpha}} = \left\{ \begin{array}{l} \text{flat structures on } S \text{ with a} \\ \alpha_k\text{-conical singularity at } p_k \end{array} \right\} \Big/ \text{Isotopy}$
- $(S, m) \in \mathcal{E}_{g,n}^{\boldsymbol{\alpha}}$:

Veech : '*Flat surfaces*'

- $\boldsymbol{\alpha} = (\alpha_k)_{k=1}^n$ fixed such that $\sum_k \alpha_k = 2g - 2$
- $\mathcal{E}_{g,n}^{\boldsymbol{\alpha}} = \left\{ \begin{array}{l} \text{flat structures on } S \text{ with a} \\ \alpha_k\text{-conical singularity at } p_k \end{array} \right\} \Big/ \text{Isotopy}$
- $(S, m) \in \mathcal{E}_{g,n}^{\boldsymbol{\alpha}} : \quad m = |z^{\alpha_k} dz|^2$ at p_k

Veech : '*Flat surfaces*'

- $\boldsymbol{\alpha} = (\alpha_k)_{k=1}^n$ fixed such that $\sum_k \alpha_k = 2g - 2$

- $\mathcal{E}_{g,n}^{\boldsymbol{\alpha}} = \left\{ \begin{array}{l} \text{flat structures on } S \text{ with a} \\ \alpha_k\text{-conical singularity at } p_k \end{array} \right\} \Big/ _{\text{Isotopy}}$

- $(S, m) \in \mathcal{E}_{g,n}^{\boldsymbol{\alpha}}$: $\quad m = |z^{\alpha_k} dz|^2$ at $p_k \Longrightarrow$ complex structure at p_k

Veech : '*Flat surfaces*'

- $\boldsymbol{\alpha} = (\alpha_k)_{k=1}^n$ fixed such that $\sum_k \alpha_k = 2g - 2$
- $\mathcal{E}_{g,n}^{\boldsymbol{\alpha}} = \left\{ \begin{array}{l} \text{flat structures on } S \text{ with a} \\ \alpha_k\text{-conical singularity at } p_k \end{array} \right\} \Big/ \text{Isotopy}$
- $(S, m) \in \mathcal{E}_{g,n}^{\boldsymbol{\alpha}}$: $\quad m = |z^{\alpha_k} dz|^2$ at $p_k \Longrightarrow$ complex structure at p_k
- Natural map

$$\begin{aligned} \mathcal{E}_{g,n}^{\boldsymbol{\alpha}} &\longrightarrow \mathcal{T}\mathbf{eich}_{g,n} \\ (S, m) &\longmapsto (X, x) \end{aligned}$$

Veech : '*Flat surfaces*'

- $\boldsymbol{\alpha} = (\alpha_k)_{k=1}^n$ fixed such that $\sum_k \alpha_k = 2g - 2$
- $\mathcal{E}_{g,n}^{\boldsymbol{\alpha}} = \left\{ \begin{array}{l} \text{flat structures on } S \text{ with a} \\ \alpha_k\text{-conical singularity at } p_k \end{array} \right\} \Big/ \text{Isotopy}$
- $(S, m) \in \mathcal{E}_{g,n}^{\boldsymbol{\alpha}}$: $m = |z^{\alpha_k} dz|^2$ at $p_k \Longrightarrow$ complex structure at p_k
- Natural map

$$\begin{aligned} \mathcal{E}_{g,n}^{\boldsymbol{\alpha}} &\longrightarrow \mathcal{T}\mathbf{eich}_{g,n} \\ (S, m) &\longmapsto (X, x) \end{aligned}$$

Thm : [Troyanov]

For any (X, x), there exists a unique flat metric $m_{X,x}^{\boldsymbol{\alpha}}$ on X with $m_{X,x}^{\boldsymbol{\alpha}} = |z^{\alpha_k} dz|^2$ in the vicinity of x_k, for every k

Veech : '*Flat surfaces*'

- $\boldsymbol{\alpha} = (\alpha_k)_{k=1}^n$ fixed such that $\sum_k \alpha_k = 2g - 2$
- $\mathcal{E}_{g,n}^{\boldsymbol{\alpha}} = \left\{ \begin{array}{l} \text{flat structures on } S \text{ with a} \\ \alpha_k\text{-conical singularity at } p_k \end{array} \right\} \Big/ \text{isotopy}$
- $(S, m) \in \mathcal{E}_{g,n}^{\boldsymbol{\alpha}}$: $\quad m = |z^{\alpha_k} dz|^2$ at $p_k \Rightarrow$ complex structure at p_k
- **Isomorphism**

$$\begin{aligned} \mathcal{E}_{g,n}^{\boldsymbol{\alpha}} &\xrightarrow{\sim} \mathcal{T}\mathbf{eich}_{g,n} \\ (S, m) &\longmapsto (X, x) \\ (S, m_{X,x}^{\boldsymbol{\alpha}}) &\longleftarrow (X, x) \end{aligned}$$

Thm : (Troyanov)

For any (X, x), there exists a unique flat metric $m_{X,x}^{\boldsymbol{\alpha}}$ on X with $m_{X,x}^{\boldsymbol{\alpha}} = |z^{\alpha_k} dz|^2$ at the vicinity of x_k, for every k

For $(X, x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \mathcal{E}^{\boldsymbol{\alpha}}_{g,n}$

For $(X, x) \in \boldsymbol{\mathcal{T}}\mathbf{eich}_{g,n} \simeq \mathcal{E}^{\boldsymbol{\alpha}}_{g,n} \rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

For $(X, x) \in \boldsymbol{\mathcal{T}}\mathbf{eich}_{g,n} \simeq \mathcal{E}^{\boldsymbol{\alpha}}_{g,n}$ $\rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- $\rho^{\boldsymbol{\alpha}}_{X,x} \; : \; \pi_1(g, n) \xrightarrow{\varphi} \pi_1(X^*) \xrightarrow{\text{holonomy of } m^{\boldsymbol{\alpha}}_{X,x}} \mathbb{U}$

For $(X,x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \mathcal{E}^{\boldsymbol{\alpha}}_{g,n} \rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- $\rho^{\boldsymbol{\alpha}}_{X,x} \; : \; \pi_1(g,n) \xrightarrow{\varphi} \pi_1(X^*) \xrightarrow{\text{holonomy of } m^{\boldsymbol{\alpha}}_{X,x}} \mathbb{U}$
- $K^{\boldsymbol{\alpha}} = \left\{ \rho : \pi_1(g,n) \longrightarrow \mathbb{U} \;\middle|\; \rho(C_k) = e^{i\theta_k} \right\} \simeq \mathbb{U}^{2g}$

For $(X,x) \in \mathcal{T}\!\mathbf{eich}_{g,n} \simeq \mathcal{E}^{\boldsymbol{\alpha}}_{g,n} \rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- $\rho^{\boldsymbol{\alpha}}_{X,x} \quad : \ \pi_1(g,n) \xrightarrow{\ \varphi\ } \pi_1(X^*) \xrightarrow{\text{holonomy of } m^{\boldsymbol{\alpha}}_{X,x}} \mathbb{U}$
- $K^{\boldsymbol{\alpha}} = \left\{ \rho : \pi_1(g,n) \longrightarrow \mathbb{U} \ \middle|\ \rho(C_k) = e^{i\,\theta_k} \right\} \simeq \mathbb{U}^{2g}$
- 'Holonomy map' $\quad \mathbf{H}^{\boldsymbol{\alpha}} : \mathcal{T}\!\mathbf{eich}_{g,n} \longrightarrow \mathbb{U}^{2g}$

For $(X,x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \mathcal{E}^{\boldsymbol{\alpha}}_{g,n} \rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- $\rho^{\boldsymbol{\alpha}}_{X,x} \; : \; \pi_1(g,n) \xrightarrow{\varphi} \pi_1(X^*) \xrightarrow{\text{holonomy of } m^{\boldsymbol{\alpha}}_{X,x}} \mathbb{U}$
- $K^{\boldsymbol{\alpha}} = \left\{ \rho : \pi_1(g,n) \longrightarrow \mathbb{U} \;\middle|\; \rho(C_k) = e^{i\theta_k} \right\} \simeq \mathbb{U}^{2g}$
- 'Holonomy map' $\quad \mathbf{H}^{\boldsymbol{\alpha}} : \mathcal{T}\mathbf{eich}_{g,n} \longrightarrow \mathbb{U}^{2g}, \; (X,x) \longmapsto \rho^{\boldsymbol{\alpha}}_{X,x}$

For $(X,x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \mathcal{E}^{\boldsymbol{\alpha}}_{g,n}$ $\rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- $\rho^{\boldsymbol{\alpha}}_{X,x} \ : \ \pi_1(g,n) \xrightarrow{\ \varphi\ } \pi_1(X^*) \xrightarrow{\text{holonomy of } m^{\boldsymbol{\alpha}}_{X,x}} \mathbb{U}$

- $K^{\boldsymbol{\alpha}} = \left\{ \rho : \pi_1(g,n) \longrightarrow \mathbb{U} \ \middle| \ \rho(C_k) = e^{i\theta_k} \right\} \simeq \mathbb{U}^{2g}$

- 'Holonomy map' $\quad \mathbf{H}^{\boldsymbol{\alpha}} : \mathcal{T}\mathbf{eich}_{g,n} \longrightarrow \mathbb{U}^{2g}, \ (X,x) \longmapsto \rho^{\boldsymbol{\alpha}}_{X,x}$

Thm : [Veech] Outside $\left(\mathbf{H}^{\boldsymbol{\alpha}}\right)^{-1}(\mathbf{1})$:

- $\mathbf{H}^{\boldsymbol{\alpha}}$ is a C^{ω}-submersion
- $\mathcal{F}^{\boldsymbol{\alpha}}_{\rho} = \left(\mathbf{H}^{\boldsymbol{\alpha}}\right)^{-1}(\rho)$:
  - complex subvariety of $\mathcal{T}\mathbf{eich}_{g,n}$
  - $\dim_{\mathbb{C}} \left(\mathcal{F}^{\boldsymbol{\alpha}}_{\rho}\right) = 2g - 3 + n$

$\rightsquigarrow$ **Def :** $\mathcal{F}^{\boldsymbol{\alpha}} = \left\{ \mathcal{F}^{\boldsymbol{\alpha}}_{\rho} \ \middle| \ \rho \in K^{\boldsymbol{\alpha}}, \rho \neq \mathbf{1} \right\}$ = *Veech's foliation*

- **"Veech's map $\mathbf{V}_{\rho}^{\alpha}$"** : for $\rho \in \mathbf{Im}\big(\mathbf{H}^{\alpha}\big) \setminus \{1\}$

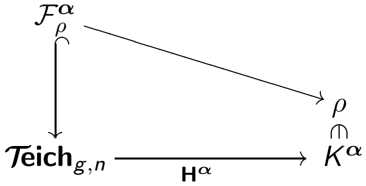
- **"Veech's map $\mathbf{V}^{\alpha}_{\rho}$"** : for $\rho \in \mathbf{Im}\big(\mathbf{H}^{\alpha}\big) \setminus \{1\}$

$$\mathcal{T}\!\mathbf{eich}_{g,n} \xrightarrow[\mathbf{H}^{\alpha}]{} K^{\alpha}$$

- **"Veech's map $\mathbf{V}_{\rho}^{\alpha}$"** : for $\boldsymbol{\rho} \in \mathbf{Im}\big(\mathbf{H}^{\boldsymbol{\alpha}}\big) \setminus \{\mathbf{1}\}$

$$\begin{array}{ccc} & & \rho \\ & & \cap \\ \mathcal{T}\mathbf{eich}_{g,n} & \xrightarrow[\mathbf{H}^{\alpha}]{} & K^{\boldsymbol{\alpha}} \end{array}$$

- **"Veech's map $\mathbf{V}^{\alpha}_{\rho}$"** : for $\boldsymbol{\rho} \in \mathbf{Im}\big(\mathbf{H}^{\alpha}\big) \setminus \{\mathbf{1}\}$



- **“Veech’s map $\mathbf{V}_{\boldsymbol{\rho}}^{\boldsymbol{\alpha}}$”** : for $\boldsymbol{\rho} \in \mathbf{Im}\big(\mathbf{H}^{\boldsymbol{\alpha}}\big) \setminus \{\mathbf{1}\}$

$$
\begin{array}{ccc}
\mathcal{F}_{\rho}^{\boldsymbol{\alpha}} & & \mathbb{P}H^1\big(S^*, \mathbb{C}_\rho\big) \\
\downarrow & \searrow & \downarrow \\
 & & \rho \\
 & & \cap \\
\mathcal{T}\mathbf{eich}_{g,n} & \xrightarrow[\mathbf{H}^{\boldsymbol{\alpha}}]{} & K^{\boldsymbol{\alpha}}
\end{array}
$$

- **"Veech's map $\mathbf{V}_{\boldsymbol{\rho}}^{\boldsymbol{\alpha}}$"** : for $\boldsymbol{\rho} \in \mathbf{Im}\big(\mathbf{H}^{\boldsymbol{\alpha}}\big) \setminus \{\mathbf{1}\}$

$$
\begin{array}{ccc}
\mathcal{F}_{\rho}^{\boldsymbol{\alpha}} & \searrow & \mathbb{P}H^1\big(S^*, \mathbb{C}_\rho\big) \simeq \mathbb{P}^{2g-3+n} \\
\big\downarrow & & \big\downarrow \\
 & & \rho \\
 & & \cap \\
\mathcal{T}\mathbf{eich}_{g,n} & \xrightarrow[\mathbf{H}^{\boldsymbol{\alpha}}]{} & K^{\boldsymbol{\alpha}}
\end{array}
$$

- **"Veech's map $\mathbf{V}^{\alpha}_{\rho}$"** : for $\rho \in \mathbf{Im}\big(\mathbf{H}^{\alpha}\big) \setminus \{\mathbf{1}\}$

$$\begin{array}{ccc}
\mathcal{F}^{\alpha}_{\rho} & \xrightarrow{\quad V^{\alpha}_{\rho} \quad} & \mathbb{P}H^{1}\big(S^{*}, \mathbb{C}_{\rho}\big) \simeq \mathbb{P}^{2g-3+n} \\
\cap\big\downarrow & \searrow & \big\downarrow \\
 & & \rho \\
 & & \cap \\
\mathcal{T}\mathbf{eich}_{g,n} & \xrightarrow[\quad \mathbf{H}^{\alpha} \quad]{} & K^{\alpha}
\end{array}$$

- **"Veech's map $\mathbf{V}^{\alpha}_{\rho}$"** : for $\rho \in \mathbf{Im}\big(\mathbf{H}^{\alpha}\big) \setminus \{\mathbf{1}\}$

$$\begin{array}{ccc} \mathcal{F}^{\alpha}_{\rho} & \xrightarrow{\;V^{\alpha}_{\rho}\;} & \mathbb{P}H^1\big(S^*, \mathbb{C}_{\rho}\big) \simeq \mathbb{P}^{2g-3+n} \\ \big\downarrow & \searrow & \big\downarrow \\ & & \rho \\ & & \cap \\ \mathcal{T}\mathbf{eich}_{g,n} & \xrightarrow[\;\mathbf{H}^{\alpha}\;]{} & K^{\alpha} \end{array}$$

Thm : [Veech]

- **"Veech's map $\mathbf{V}^{\alpha}_{\rho}$"** : for $\rho \in \mathbf{Im}(\mathbf{H}^{\alpha}) \setminus \{1\}$

$$\begin{array}{ccccc}
\mathcal{F}^{\alpha}_{\rho} & \xrightarrow{\;V^{\alpha}_{\rho}\;} & \mathbb{P}H^1(S^*, \mathbb{C}_\rho) & \simeq & \mathbb{P}^{2g-3+n} \\
\cap\!\!\downarrow & \searrow & \downarrow & & \\
 & & \rho & & \\
 & & \cap & & \\
\mathcal{T}\mathrm{eich}_{g,n} & \xrightarrow[\;\mathbf{H}^{\alpha}\;]{} & K^{\alpha} & &
\end{array}$$

Thm : [Veech]

- cup-product $\rightsquigarrow$ Hermitian form A^{α} on $H^1(S^*, \mathbb{C}_\rho)$
  of signature $(p, q) = (p^{\alpha}, q^{\alpha})$

- **“Veech’s map $\mathbf{V}^{\boldsymbol{\alpha}}_{\boldsymbol{\rho}}$”** : for $\boldsymbol{\rho} \in \mathbf{Im}\big(\mathbf{H}^{\boldsymbol{\alpha}}\big) \setminus \{\mathbf{1}\}$

$$
\begin{array}{ccccc}
\mathcal{F}^{\boldsymbol{\alpha}}_{\rho} & \xrightarrow{\;V^{\boldsymbol{\alpha}}_{\rho}\;} & \mathbb{P}H^1\big(S^*, \mathbb{C}_\rho\big) & \simeq & \mathbb{P}^{2g-3+n} \\
\cap & \searrow & \downarrow & & \\
\downarrow & & \rho & & \\
& & \cap & & \\
\mathcal{T}\mathbf{eich}_{g,n} & \xrightarrow[\;\mathbf{H}^{\boldsymbol{\alpha}}\;]{} & K^{\boldsymbol{\alpha}} & &
\end{array}
$$

Thm : [Veech]

- cup-product $\rightsquigarrow$ Hermitian form $A^{\boldsymbol{\alpha}}$ on $H^1\big(S^*, \mathbb{C}_\rho\big)$ of signature $(p, q) = \big(p^{\boldsymbol{\alpha}}, q^{\boldsymbol{\alpha}}\big)$
- $V^{\boldsymbol{\alpha}}_{\rho}$ is étale and $\mathbf{Im}\big(V^{\boldsymbol{\alpha}}_{\rho}\big) \subset \mathbb{P}\{0 < A^{\boldsymbol{\alpha}}\} = \mathbb{CH}^{p,q}$

- **"Veech's map $\mathbf{V}^{\alpha}_{\rho}$"** : for $\rho \in \mathbf{Im}(\mathbf{H}^{\alpha}) \setminus \{1\}$

$$\begin{array}{ccc}
\mathcal{F}^{\alpha}_{\rho} & \xrightarrow{\;V^{\alpha}_{\rho}\;} & \mathbb{P}H^1(S^*, \mathbb{C}_\rho) \simeq \mathbb{P}^{2g-3+n} \\
\downarrow & \searrow & \downarrow \\
 & & \rho \\
 & & \cap \\
\mathcal{T}\mathbf{eich}_{g,n} & \xrightarrow[\mathbf{H}^{\alpha}]{} & K^{\alpha}
\end{array}$$

Thm : [Veech]

- cup-product $\rightsquigarrow$ Hermitian form A^{α} on $H^1(S^*, \mathbb{C}_\rho)$ of signature $(p, q) = (p^{\alpha}, q^{\alpha})$
- V^{α}_{ρ} is étale and $\mathbf{Im}(V^{\alpha}_{\rho}) \subset \mathbb{P}\{0 < A^{\alpha}\} = \mathbb{CH}^{p,q}$
- $\mathcal{F}^{\alpha}_{\rho}$ carries a $\mathbb{CH}^{p,q}$-structure

- **"Veech's map $\mathbf{V}^{\alpha}_{\rho}$"** : for $\rho \in \mathbf{Im}\big(\mathbf{H}^{\alpha}\big) \setminus \{\mathbf{1}\}$

$$
\begin{array}{ccccc}
\mathcal{F}^{\alpha}_{\rho} & \xrightarrow{\;V^{\alpha}_{\rho}\;} & \mathbb{P}H^1\big(S^*, \mathbb{C}_{\rho}\big) & \simeq & \mathbb{P}^{2g-3+n} \\
\cap\!\!\downarrow & \searrow & \downarrow & & \\
 & & \rho & & \\
 & & \cap & & \\
\mathcal{T}\mathbf{eich}_{g,n} & \xrightarrow[\;\mathbf{H}^{\alpha}\;]{} & K^{\alpha} & &
\end{array}
$$

Thm : [Veech]

- cup-product $\rightsquigarrow$ Hermitian form A^{α} on $H^1\big(S^*, \mathbb{C}_{\rho}\big)$
  of signature $(p, q) = (p^{\alpha}, q^{\alpha})$
- V^{α}_{ρ} is étale and $\mathbf{Im}(V^{\alpha}_{\rho}) \subset \mathbb{P}\{0 < A^{\alpha}\} = \mathbb{CH}^{p,q}$
- $\mathcal{F}^{\alpha}_{\rho}$ carries a $\mathbb{CH}^{p,q}$-structure
- everything is $\mathbf{PMCG}_{g,n}$-invariant

- ⇝ foliation $\mathbf{F}^{\alpha}$ on $\mathbf{M}_{g,n}$:

$\bullet\!\rightsquigarrow$ foliation $\mathbf{F}^{\alpha}$ on $\mathbf{M}_{g,n}$: $\left\{ \begin{array}{l} - \text{ transversally modeled on } \mathbb{E}^{2g} \\ \\ \\ \end{array} \right.$

$\bullet\!\!\rightsquigarrow$ foliation $\mathbf{F}^{\boldsymbol{\alpha}}$ on $\mathbf{M}_{g,n}$: $\begin{cases} - \text{ transversally modeled on } \mathbb{E}^{2g} \\ - \dim_{\mathbb{C}}\left(\mathbf{F}^{\boldsymbol{\alpha}}_{\rho}\right) = 2g - 3 + n \\ \end{cases}$

$\rightsquigarrow$ foliation $\mathbf{F}^{\boldsymbol{\alpha}}$ on $\mathbf{M}_{g,n}$:

$$\begin{cases} -\text{ transversally modeled on } \mathbb{E}^{2g} \\ -\dim_{\mathbb{C}}\left(\mathbf{F}^{\boldsymbol{\alpha}}_{\rho}\right) = 2g-3+n \\ -\ \mathbf{F}^{\boldsymbol{\alpha}}_{\rho} \text{ locally modeled on } \mathbb{CH}^{p,q} \end{cases}$$

- $\rightsquigarrow$ foliation $\mathbf{F}^{\boldsymbol{\alpha}}$ on $\mathbf{M}_{g,n}$: $\begin{cases} - \text{ transversally modeled on } \mathbb{E}^{2g} \\ - \dim_{\mathbb{C}}\big(\mathbf{F}^{\boldsymbol{\alpha}}_{\rho}\big) = 2g-3+n \\ - \mathbf{F}^{\boldsymbol{\alpha}}_{\rho} \text{ locally modeled on } \mathbb{CH}^{p,q} \end{cases}$
- $\left\{ \begin{array}{c} ds^2_{\mathrm{Euc}} \text{ transversally} \\ + \ \Omega^{\boldsymbol{\alpha}}_{\rho} \text{ on } \mathcal{F}^{\boldsymbol{\alpha}}_{\rho} \end{array} \right\}$

- $\rightsquigarrow$ foliation $\mathbf{F}^{\boldsymbol{\alpha}}$ on $\mathbf{M}_{g,n}$: $\begin{cases} - \text{ transversally modeled on } \mathbb{E}^{2g} \\ - \dim_{\mathbb{C}}\big(\mathbf{F}^{\boldsymbol{\alpha}}_{\rho}\big) = 2g - 3 + n \\ - \mathbf{F}^{\boldsymbol{\alpha}}_{\rho} \text{ locally modeled on } \mathbb{CH}^{p,q} \end{cases}$

- $\left\{ \begin{array}{c} ds^2_{\mathrm{Euc}} \text{ transversally} \\ + \ \Omega^{\boldsymbol{\alpha}}_{\rho} \text{ on } \mathcal{F}^{\boldsymbol{\alpha}}_{\rho} \end{array} \right\} \rightsquigarrow$ volume form $\Omega^{\boldsymbol{\alpha}}$ on $\mathbf{M}_{g,n}$

- $\rightsquigarrow$ foliation $\mathbf{F}^{\boldsymbol{\alpha}}$ on $\mathbf{M}_{g,n}$: $\begin{cases} - \text{ transversally modeled on } \mathbb{E}^{2g} \\ - \dim_{\mathbb{C}}\left(\mathbf{F}^{\boldsymbol{\alpha}}_{\rho}\right) = 2g-3+n \\ - \mathbf{F}^{\boldsymbol{\alpha}}_{\rho} \text{ locally modeled on } \mathbb{CH}^{p,q} \end{cases}$

- $\left\{ \begin{array}{c} ds^2_{\rm Euc} \text{ transversally} \\ + \ \Omega^{\boldsymbol{\alpha}}_{\rho} \text{ on } \mathcal{F}^{\boldsymbol{\alpha}}_{\rho} \end{array} \right\} \rightsquigarrow$ volume form $\Omega^{\boldsymbol{\alpha}}$ on $\mathbf{M}_{g,n}$

Questions :

• $\rightsquigarrow$ foliation $\mathbf{F}^{\boldsymbol{\alpha}}$ on $\mathbf{M}_{g,n}$: $\begin{cases} -\text{ transversally modeled on } \mathbb{E}^{2g} \\ -\dim_{\mathbb{C}}\big(\mathbf{F}^{\boldsymbol{\alpha}}_{\rho}\big) = 2g-3+n \\ -\ \mathbf{F}^{\boldsymbol{\alpha}}_{\rho} \text{ locally modeled on } \mathbb{CH}^{p,q} \end{cases}$

• $\left\{ \begin{array}{c} ds^2_{\rm Euc} \text{ transversally} \\ +\ \Omega^{\boldsymbol{\alpha}}_{\rho} \text{ on } \mathcal{F}^{\boldsymbol{\alpha}}_{\rho} \end{array} \right\} \rightsquigarrow$ volume form $\Omega^{\boldsymbol{\alpha}}$ on $\mathbf{M}_{g,n}$

Questions :

- **About Veech's foliation $\mathbf{F}^{\alpha}$:**
  - dynamic ?
  - geometry ($\exists$ closed/algebraic leaves ?)

- $\rightsquigarrow$ foliation $\mathbf{F}^{\alpha}$ on $\mathbf{M}_{g,n}$: $\begin{cases} - \text{ transversally modeled on } \mathbb{E}^{2g} \\ - \dim_{\mathbb{C}}\big(\mathbf{F}^{\alpha}_{\rho}\big) = 2g-3+n \\ - \mathbf{F}^{\alpha}_{\rho} \text{ locally modeled on } \mathbb{CH}^{p,q} \end{cases}$

- $\left\{ \begin{array}{c} ds^2_{\mathrm{Euc}} \text{ transversally} \\ + \ \Omega^{\alpha}_{\rho} \text{ on } \mathcal{F}^{\alpha}_{\rho} \end{array} \right\}$ $\rightsquigarrow$ volume form Ω^{α} on $\mathbf{M}_{g,n}$

Questions :

- **About Veech's foliation $\mathbf{F}^{\alpha}$:**
  - dynamic ?
  - geometry ($\exists$ closed/algebraic leaves ?)

- **"Veech's volume conjecture" :**

$$\mathbf{Vol}^{\alpha}\big(\mathbf{M}_{g,n}\big) := \int_{\mathbf{M}_{g,n}} \Omega^{\alpha} < +\infty \qquad (?)$$

Let $(X, x) \in \mathcal{T}\textbf{eich}_{g,n} \simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n}$ $\rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

Let $(X, x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n}$ $\rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- **Developing map :** $D^{\boldsymbol{\alpha}}_{X,x} : \widetilde{X^*} \to \mathbb{C} \simeq \mathbb{E}^2$

Let $(X, x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n}$ $\rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- **Developing map :** $D^{\boldsymbol{\alpha}}_{X,x} : \widetilde{X^*} \to \mathbb{C} \simeq \mathbb{E}^2$ $\qquad m^{\boldsymbol{\alpha}}_{X,x} = \left| dD^{\boldsymbol{\alpha}}_{X,x} \right|^2$

Let $(X, x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n}$ $\rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- **Developing map :** $D^{\boldsymbol{\alpha}}_{X,x} : \widetilde{X^*} \to \mathbb{C} \simeq \mathbb{E}^2$ $m^{\boldsymbol{\alpha}}_{X,x} = \left| dD^{\boldsymbol{\alpha}}_{X,x} \right|^2$
- **Relative period :**

$$\boldsymbol{\xi}(\gamma) = \int_\gamma dD^{\boldsymbol{\alpha}}_{X,x}$$

Let $(X,x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n}$ $\rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- **Developing map :** $D^{\boldsymbol{\alpha}}_{X,x} : \widetilde{X^*} \to \mathbb{C} \simeq \mathbb{E}^2$ $\qquad m^{\boldsymbol{\alpha}}_{X,x} = \left| dD^{\boldsymbol{\alpha}}_{X,x} \right|^2$
- **Relative period :**

$$\boldsymbol{\xi}(\gamma) = \int_{\gamma} dD^{\boldsymbol{\alpha}}_{X,x} \qquad \text{with } \gamma \in H_1\big(X, \{x_k\}, \mathbb{C}_{\rho^{-1}}\big)$$

Let $(X,x) \in \mathcal{T}\!\mathbf{eich}_{g,n} \simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n}$ $\rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- **Developing map :** $D^{\boldsymbol{\alpha}}_{X,x} : \widetilde{X^*} \to \mathbb{C} \simeq \mathbb{E}^2$ $\qquad m^{\boldsymbol{\alpha}}_{X,x} = \left|dD^{\boldsymbol{\alpha}}_{X,x}\right|^2$

- **Relative period :**

$$\boldsymbol{\xi}(\gamma) = \int_\gamma dD^{\boldsymbol{\alpha}}_{X,x} \qquad \text{with } \gamma \in H_1\big(X, \{x_k\}, \mathbb{C}_{\rho^{-1}}\big)$$

Prop : Veech's map $V^{\boldsymbol{\alpha}}_\rho : \mathcal{F}^{\boldsymbol{\alpha}}_\rho \to \mathbb{P}H^1(S^*, \mathbb{C}_\rho)$ writes explicitly

$$\mathcal{F}^{\boldsymbol{\alpha}}_\rho \ni (X,x) \longmapsto \Big[\boldsymbol{\xi}(\gamma_\ell)\Big]_{\ell=0}^{2g-3+n} \in \mathbb{P}^{2g-3+n}$$

where $(\gamma_0, \ldots, \gamma_{2g-3+n})$ is a basis of $H_1\big(X, \{x_k\}, \mathbb{C}_{\rho^{-1}}\big)$

Let $(X,x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n}$ $\rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- **Developing map :** $D^{\boldsymbol{\alpha}}_{X,x} : \widetilde{X^*} \to \mathbb{C} \simeq \mathbb{E}^2$ $\quad m^{\boldsymbol{\alpha}}_{X,x} = \left| dD^{\boldsymbol{\alpha}}_{X,x} \right|^2$

- **Relative period :**

$$\boldsymbol{\xi}(\gamma) = \int_\gamma dD^{\boldsymbol{\alpha}}_{X,x} \qquad \text{with } \gamma \in H_1\big(X, \{x_k\}, \mathbb{C}_{\rho^{-1}}\big)$$

Prop : Veech's map $V^{\boldsymbol{\alpha}}_\rho : \mathcal{F}^{\boldsymbol{\alpha}}_\rho \to \mathbb{P}H^1(S^*, \mathbb{C}_\rho)$ writes explicitly

$$\mathcal{F}^{\boldsymbol{\alpha}}_\rho \ni (X,x) \longmapsto \Big[\boldsymbol{\xi}(\gamma_\ell)\Big]_{\ell=0}^{2g-3+n} \in \mathbb{P}^{2g-3+n}$$

where $(\gamma_0, \ldots, \gamma_{2g-3+n})$ is a basis of $H_1\big(X, \{x_k\}, \mathbb{C}_{\rho^{-1}}\big)$

- **Ex.** : $g = 0$: $X = \mathbb{P}^1$

Let $(X,x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n}$ $\rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- **Developing map :** $D^{\boldsymbol{\alpha}}_{X,x} : \widetilde{X^*} \to \mathbb{C} \simeq \mathbb{E}^2$ $\quad m^{\boldsymbol{\alpha}}_{X,x} = \left|dD^{\boldsymbol{\alpha}}_{X,x}\right|^2$
- **Relative period :**

$$\boldsymbol{\xi}(\gamma) = \int_\gamma dD^{\boldsymbol{\alpha}}_{X,x} \qquad \text{with } \gamma \in H_1\big(X, \{x_k\}, \mathbb{C}_{\rho^{-1}}\big)$$

Prop : Veech's map $V^{\boldsymbol{\alpha}}_\rho : \mathcal{F}^{\boldsymbol{\alpha}}_\rho \to \mathbb{P}H^1(S^*, \mathbb{C}_\rho)$ writes explicitly

$$\mathcal{F}^{\boldsymbol{\alpha}}_\rho \ni (X,x) \longmapsto \Big[\boldsymbol{\xi}(\gamma_\ell)\Big]_{\ell=0}^{2g-3+n} \in \mathbb{P}^{2g-3+n}$$

where $(\gamma_0, \ldots, \gamma_{2g-3+n})$ is a basis of $H_1\big(X, \{x_k\}, \mathbb{C}_{\rho^{-1}}\big)$

- **Ex.** : $g = 0$: $X = \mathbb{P}^1 \ni x_1, \ldots, x_n,$

Let $(X,x)\in \mathcal{T}\mathbf{eich}_{g,n}\simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n}$ $\rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- **Developing map :** $D^{\boldsymbol{\alpha}}_{X,x}:\widetilde{X^*}\to\mathbb{C}\simeq\mathbb{E}^2$ $\qquad m^{\boldsymbol{\alpha}}_{X,x}=\left|dD^{\boldsymbol{\alpha}}_{X,x}\right|^2$
- **Relative period :**

$$\boldsymbol{\xi}(\gamma)=\int_\gamma dD^{\boldsymbol{\alpha}}_{X,x} \qquad \text{with } \gamma\in H_1\big(X,\{x_k\},\mathbb{C}_{\rho^{-1}}\big)$$

Prop : Veech's map $V^{\boldsymbol{\alpha}}_\rho:\mathcal{F}^{\boldsymbol{\alpha}}_\rho\to\mathbb{P}H^1(S^*,\mathbb{C}_\rho)$ writes explicitly

$$\mathcal{F}^{\boldsymbol{\alpha}}_\rho\ni(X,x)\longmapsto\Big[\boldsymbol{\xi}(\gamma_\ell)\Big]_{\ell=0}^{2g-3+n}\in\mathbb{P}^{2g-3+n}$$

where $(\gamma_0,\ldots,\gamma_{2g-3+n})$ is a basis of $H_1\big(X,\{x_k\},\mathbb{C}_{\rho^{-1}}\big)$

- **Ex.** : $g=0$: $X=\mathbb{P}^1\ni x_1,\ldots,x_n,\quad m^{\boldsymbol{\alpha}}_{X,x}=\left|\prod_k(t-x_k)^{\alpha_k}dt\right|^2$

Let $(X,x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n}$ $\rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- **Developing map :** $D^{\boldsymbol{\alpha}}_{X,x} : \widetilde{X^*} \to \mathbb{C} \simeq \mathbb{E}^2$ $\qquad m^{\boldsymbol{\alpha}}_{X,x} = \left| dD^{\boldsymbol{\alpha}}_{X,x} \right|^2$

- **Relative period :**

$$\boldsymbol{\xi}(\gamma) = \int_\gamma dD^{\boldsymbol{\alpha}}_{X,x} \qquad \text{with } \gamma \in H_1\big(X, \{x_k\}, \mathbb{C}_{\rho^{-1}}\big)$$

Prop : Veech's map $V^{\boldsymbol{\alpha}}_\rho : \mathcal{F}^{\boldsymbol{\alpha}}_\rho \to \mathbb{P}H^1(S^*, \mathbb{C}_\rho)$ writes explicitly

$$\mathcal{F}^{\boldsymbol{\alpha}}_\rho \ni (X,x) \longmapsto \Big[\boldsymbol{\xi}(\gamma_\ell)\Big]_{\ell=0}^{2g-3+n} \in \mathbb{P}^{2g-3+n}$$

where $(\gamma_0, \ldots, \gamma_{2g-3+n})$ is a basis of $H_1\big(X, \{x_k\}, \mathbb{C}_{\rho^{-1}}\big)$

- **Ex.** : $g = 0$: $X = \mathbb{P}^1 \ni x_1, \ldots, x_n$, $\quad m^{\boldsymbol{\alpha}}_{X,x} = \left| \prod_k (t - x_k)^{\alpha_k} dt \right|^2$

$$\boldsymbol{\xi}(\gamma) = \int_\gamma \prod_k (t - x_k)^{\alpha_k} dt$$

Let $(X, x) \in \mathcal{T}\!\mathbf{eich}_{g,n} \simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n}$ $\rightsquigarrow$ flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- **Developing map :** $D^{\boldsymbol{\alpha}}_{X,x} : \widetilde{X^*} \to \mathbb{C} \simeq \mathbb{E}^2$ $\quad m^{\boldsymbol{\alpha}}_{X,x} = \left| dD^{\boldsymbol{\alpha}}_{X,x} \right|^2$
- **Relative period :**

$$\boldsymbol{\xi}(\gamma) = \int_\gamma dD^{\boldsymbol{\alpha}}_{X,x} \qquad \text{with } \gamma \in H_1\big(X, \{x_k\}, \mathbb{C}_{\rho^{-1}}\big)$$

Prop : Veech's map $V^{\boldsymbol{\alpha}}_\rho : \mathcal{F}^{\boldsymbol{\alpha}}_\rho \to \mathbb{P}H^1(S^*, \mathbb{C}_\rho)$ writes explicitly

$$\mathcal{F}^{\boldsymbol{\alpha}}_\rho \ni (X, x) \longmapsto \Big[\boldsymbol{\xi}(\gamma_\ell)\Big]_{\ell=0}^{2g-3+n} \in \mathbb{P}^{2g-3+n}$$

where $(\gamma_0, \ldots, \gamma_{2g-3+n})$ is a basis of $H_1\big(X, \{x_k\}, \mathbb{C}_{\rho^{-1}}\big)$

- **Ex.** : $g = 0 : X = \mathbb{P}^1 \ni x_1, \ldots, x_n, \quad m^{\boldsymbol{\alpha}}_{X,x} = \left| \prod_k (t - x_k)^{\alpha_k} dt \right|^2$

$$\boldsymbol{\xi}(\gamma) = \int_\gamma \prod_k (t - x_k)^{\alpha_k} dt = \textbf{hypergeometric integral}$$

II. $g = 0$: Thurston "*Shapes of polyhedra*" (1987)

II. $g = 0$: Thurston "*Shapes of polyhedra*" (1987)

- $K^{\boldsymbol{\alpha}} = \mathbb{U}^{2g}$

II. $g = 0$: Thurston "*Shapes of polyhedra*" (1987)

- $K^{\boldsymbol{\alpha}} = \mathbb{U}^{2g} = \mathbb{U}^0$

II. $g = 0$: Thurston "*Shapes of polyhedra*" (1987)

- $K^{\boldsymbol{\alpha}} = \mathbb{U}^{2g} = \mathbb{U}^0 \quad \Longrightarrow \quad \mathbf{F}^{\boldsymbol{\alpha}}$ has only one leaf : $\mathbf{M}_{0,n}$

II. $g = 0$: Thurston "*Shapes of polyhedra*" (1987)

- $K^{\boldsymbol{\alpha}} = \mathbb{U}^{2g} = \mathbb{U}^0 \quad \Longrightarrow \quad$ $\mathbf{F}^{\boldsymbol{\alpha}}$ has only one leaf : $\mathbf{M}_{0,n}$

- $\left\{ \begin{array}{c} 0 < \theta_k < 2\pi \\ \big(-1 < \alpha_k < 0 \big) \end{array} \right\}$

II. $g = 0$: Thurston "*Shapes of polyhedra*" (1987)

- $K^{\boldsymbol{\alpha}} = \mathbb{U}^{2g} = \mathbb{U}^0 \quad \Longrightarrow \quad$ $\mathbf{F}^{\boldsymbol{\alpha}}$ has only one leaf : $\mathbf{M}_{0,n}$

- $\left\{ \begin{array}{c} 0 < \theta_k < 2\pi \\ \big(-1 < \alpha_k < 0 \big) \end{array} \right\} \quad \Longrightarrow \quad$ $\mathbb{CH}^{n-3}$-structure on $\mathbf{M}_{0,n}$

II. $g = 0$: Thurston "*Shapes of polyhedra*" (1987)

- $K^{\boldsymbol{\alpha}} = \mathbb{U}^{2g} = \mathbb{U}^{0}$ $\Longrightarrow$ $\mathbf{F}^{\boldsymbol{\alpha}}$ has only one leaf : $\mathbf{M}_{0,n}$

- $\left\{ \begin{array}{c} 0 < \theta_k < 2\pi \\ \big(-1 < \alpha_k < 0 \big) \end{array} \right\}$ $\Longrightarrow$ $\mathbb{CH}^{n-3}$-structure on $\mathbf{M}_{0,n} \; = \; \mathbf{M}_{0,n}^{\boldsymbol{\alpha}}$

II. $g = 0$: Thurston "*Shapes of polyhedra*" (1987)

- $K^{\boldsymbol{\alpha}} = \mathbb{U}^{2g} = \mathbb{U}^0 \quad \Longrightarrow \quad$ $\mathbf{F}^{\boldsymbol{\alpha}}$ has only one leaf : $\mathbf{M}_{0,n}$

- $\left\{ \begin{array}{c} 0 < \theta_k < 2\pi \\ (-1 < \alpha_k < 0) \end{array} \right\} \Longrightarrow$ $\mathbb{CH}^{n-3}$-structure on $\mathbf{M}_{0,n} \; = \; \mathbf{M}_{0,n}^{\boldsymbol{\alpha}}$

Thm : [Thurston] The metric completion $\overline{\mathbf{M}_{0,n}^{\boldsymbol{\alpha}}}$:

- is a $\mathbb{CH}^{n-3}$-conifold of finite volume
- is obtained by attaching to $\mathbf{M}_{0,n}^{\boldsymbol{\alpha}}$ some "C-strata" $\mathbf{M}_{0,n'}^{\boldsymbol{\alpha}'}$
- $\exists$ formula for the conifold angle around $\mathbf{M}_{0,n'}^{\boldsymbol{\alpha}'}$ of codim 1
- is a $\mathbb{CH}^{n-3}$-orbifold if $\boldsymbol{\alpha} = (\alpha_k)_{k=1}^n$ satisfies

  (INT) $\quad \forall\, k < \ell \; : \; -1 < \alpha_k + \alpha_\ell \Longrightarrow \big(1 + \alpha_k + \alpha_\ell\big)^{-1} \in \mathbb{Z}$

II. $g = 0$: Thurston "*Shapes of polyhedra*" (1987)

- $K^{\boldsymbol{\alpha}} = \mathbb{U}^{2g} = \mathbb{U}^0 \quad \Longrightarrow \quad$ $\mathbf{F}^{\boldsymbol{\alpha}}$ has only one leaf : $\mathbf{M}_{0,n}$

- $\left\{ \begin{array}{c} 0 < \theta_k < 2\pi \\ (-1 < \alpha_k < 0) \end{array} \right\} \Longrightarrow$ $\mathbb{CH}^{n-3}$-structure on $\mathbf{M}_{0,n} = \mathbf{M}_{0,n}^{\alpha}$

Thm : [Thurston] The metric completion $\overline{\mathbf{M}_{0,n}^{\alpha}}$:

- is a $\mathbb{CH}^{n-3}$-conifold of finite volume
- is obtained by attaching to $\mathbf{M}_{0,n}^{\alpha}$ some "C-strata" $\mathbf{M}_{0,n'}^{\alpha'}$
- $\exists$ formula for the conifold angle around $\mathbf{M}_{0,n'}^{\alpha'}$ of codim 1
- is a $\mathbb{CH}^{n-3}$-orbifold if $\boldsymbol{\alpha} = (\alpha_k)_{k=1}^n$ satisfies

  (INT) $\quad \forall\, k < \ell \,:\, -1 < \alpha_k + \alpha_\ell \Longrightarrow \big(1 + \alpha_k + \alpha_\ell\big)^{-1} \in \mathbb{Z}$

$\Longrightarrow$ construction of AR & NAR lattices in $\mathbf{PU}(1, m)$, $m = 2, 3$

II. $g = 0$: Deligne-Mostow
"*Monodromy of hypergeometric functions*" (1986)

II. $g = 0$: Deligne-Mostow
"*Monodromy of hypergeometric functions*" (1986)

- Veech's map :

$$V^{\boldsymbol{\alpha}} \; : \; \widetilde{\mathbf{M}_{0,n}} \longrightarrow \mathbb{CH}^{n-3} \subset \mathbb{P}^{n-3}, \; x \mapsto \left[\int_{\gamma_\ell} \prod (t - x_k)^{\alpha_k} \, dt \right]_{\ell=0}^{n-3}$$

II. $g = 0$: Deligne-Mostow
"*Monodromy of hypergeometric functions*" (1986)

- Veech's map :

$$V^{\boldsymbol{\alpha}} \; : \; \widetilde{\mathbf{M}_{0,n}} \longrightarrow \mathbb{CH}^{n-3} \subset \mathbb{P}^{n-3}\,, \; x \mapsto \left[\int_{\gamma_\ell} \prod (t - x_k)^{\alpha_k} dt\right]_{\ell=0}^{n-3}$$

coincides with *'Schwarz's map'*

$$S^{\boldsymbol{\alpha}} \; : \; \widetilde{\mathbf{M}_{0,n}} \longrightarrow \mathbb{CH}^{n-3} \subset \mathbb{P}^{n-3}\,, \; x \mapsto \left[F_\ell(x)\right]_{\ell=0}^{n-3}$$

II. $g = 0$: Deligne-Mostow
"*Monodromy of hypergeometric functions*" (1986)

- Veech's map :

$$V^{\boldsymbol{\alpha}} \ : \ \widetilde{\mathbf{M}_{0,n}} \longrightarrow \mathbb{CH}^{n-3} \subset \mathbb{P}^{n-3}, \ x \mapsto \left[\int_{\gamma_\ell} \prod (t - x_k)^{\alpha_k} dt \right]_{\ell=0}^{n-3}$$

coincides with *'Schwarz's map'*

$$S^{\boldsymbol{\alpha}} \ : \ \widetilde{\mathbf{M}_{0,n}} \longrightarrow \mathbb{CH}^{n-3} \subset \mathbb{P}^{n-3}, \ x \mapsto \left[F_\ell(x) \right]_{\ell=0}^{n-3}$$

(F_ℓ) = basis of the space of sol$^{\mathbf{o}}$ of the *'hypergeometric system'*

$$(E^{\boldsymbol{\alpha}}) \qquad \frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{1}{x_i - x_j} \left(\alpha_i \frac{\partial F}{\partial x_j} - \alpha_j \frac{\partial F}{\partial x_i} \right), \quad 1 \leq i < j \leq n-3$$

II. $g = 0$: Deligne-Mostow
"*Monodromy of hypergeometric functions*" (1986)

- Veech's map :

$$V^{\boldsymbol{\alpha}} \ : \ \widetilde{\mathbf{M}_{0,n}} \longrightarrow \mathbb{CH}^{n-3} \subset \mathbb{P}^{n-3}, \ x \mapsto \left[\int_{\gamma_\ell} \prod (t - x_k)^{\alpha_k} dt\right]_{\ell=0}^{n-3}$$

coincides with *'Schwarz's map'*

$$S^{\boldsymbol{\alpha}} \ : \ \widetilde{\mathbf{M}_{0,n}} \longrightarrow \mathbb{CH}^{n-3} \subset \mathbb{P}^{n-3}, \ x \mapsto \left[F_\ell(x)\right]_{\ell=0}^{n-3}$$

(F_ℓ) = basis of the space of sol° of the *'hypergeometric system'*

$$(E^{\boldsymbol{\alpha}}) \qquad \frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{1}{x_i - x_j}\left(\alpha_i \frac{\partial F}{\partial x_j} - \alpha_j \frac{\partial F}{\partial x_i}\right), \quad 1 \leq i < j \leq n-3$$

Thm : (Schwarz (1873) - $\cdots$ -) **[Deligne-Mostow]**

α satisfies (**INT**) $\Longrightarrow$ **Monod**$\left(S^{\alpha}\right) \subset$ **PU**$(1, n-3)$ is a lattice

III. $g = 1$ [GPgeom]

III. $g = 1$ [GPgeom]

- $\mathbf{M}_{1,n+1}$ of dimension $n+1$;

III. $g = 1$ [GPgeom]

- $\mathbf{M}_{1,n+1}$ of dimension $n+1$;   leaf $\mathbf{F}_{\rho}^{\alpha}$ of dimension n

III. $g = 1$ [GPgeom]

- $\mathbf{M}_{1,n+1}$ of dimension $n+1$; leaf $\mathbf{F}_\rho^{\boldsymbol{\alpha}}$ of dimension n

- **[Veech] :** $\left[\begin{array}{c} 2\pi < \theta_1 < 4\pi \\ 0 < \theta_k < 2\pi \\ {\scriptstyle k=2,\ldots,n+1} \end{array} \right] \Longrightarrow$ $\mathbb{CH}^n$-structure on $\mathbf{F}_\rho^{\boldsymbol{\alpha}}$

III. $g = 1$ [GPgeom]

- $\mathbf{M}_{1,n+1}$ of dimension $n+1$; leaf $\mathbf{F}_\rho^{\boldsymbol{\alpha}}$ of dimension n

- **[Veech] :** $\left[\begin{array}{c} 2\pi < \theta_1 < 4\pi \\ 0 < \theta_k < 2\pi \\ {\scriptstyle k=2,\ldots,n+1} \end{array} \right] \implies$ $\mathbb{CH}^n$-structure on $\mathbf{F}_\rho^{\boldsymbol{\alpha}}$

Thm : [GP] For ρ torsion, the metric completion $\overline{\mathbf{F}_\rho^{\boldsymbol{\alpha}}}$:

- is a $\mathbb{CH}^n$-conifold with finite volume
- is obtained by adding to $\mathbf{F}_\rho^{\boldsymbol{\alpha}}$
  – "*C-strata*" $\mathbf{F}_{\rho'}^{\boldsymbol{\alpha}'}$ (genus 1)
  – "*P-strata*" $\mathbf{M}_{0,n'}^{\boldsymbol{\alpha}'}$ (genus 0)
- $\exists$ formulae for the conifold angles // strata of codim 1

III. $g = 1$ [GPgeom]

- $\mathbf{M}_{1,n+1}$ of dimension $n+1$; leaf $\mathbf{F}_\rho^{\boldsymbol\alpha}$ of dimension n

- **[Veech] :** $\left[\begin{array}{c} 2\pi < \theta_1 < 4\pi \\ 0 < \theta_k < 2\pi \\ {\scriptstyle k=2,\ldots,n+1} \end{array}\right] \Longrightarrow$ $\mathbb{CH}^n$-structure on $\mathbf{F}_\rho^{\boldsymbol\alpha}$

Thm : [GP] For ρ torsion, the metric completion $\overline{\mathbf{F}_\rho^{\boldsymbol\alpha}}$:

- is a $\mathbb{CH}^n$-conifold with finite volume
- is obtained by adding to $\mathbf{F}_\rho^{\boldsymbol\alpha}$
  – "*C-strata*" $\mathbf{F}_{\rho'}^{\boldsymbol\alpha'}$ (genus 1)
  – "*P-strata*" $\mathbf{M}_{0,n'}^{\boldsymbol\alpha'}$ (genus 0)
- $\exists$ formulae for the conifold angles // strata of codim 1

$\Longrightarrow$ AR lattices in $\mathbf{PU}(1, n)$ for $n \leq 5$

III. $g = 1$ [GPhypergeom]

III. $g = 1$ [GPhypergeom]

- $(X, x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n}$

III. $g = 1$ [GPhypergeom]

- $(X, x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \mathcal{E}^{\boldsymbol{\alpha}}_{g,n} \rightsquigarrow \boldsymbol{\alpha}$-flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

III. $g = 1$ [GPhypergeom]

- $(X, x) \in \boldsymbol{\mathcal{T}}\mathbf{eich}_{g,n} \simeq \mathcal{E}^{\boldsymbol{\alpha}}_{g,n} \rightsquigarrow \boldsymbol{\alpha}$-flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X
- $\mathbf{H}^{\boldsymbol{\alpha}}(X) : \pi_1(g, n) \longrightarrow \mathbb{U}$

III. $g = 1$ [GPhypergeom]

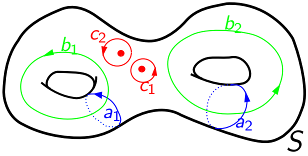
- $(X, x) \in \boldsymbol{\mathcal{T}}\mathbf{eich}_{g,n} \simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n} \rightsquigarrow \boldsymbol{\alpha}$-flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- $\mathbf{H}^{\boldsymbol{\alpha}}(X) : \pi_1(g, n) \longrightarrow \mathbb{U}$ fact. through $\pi_1(g, n)^{\mathbf{ab}} = H_1(g, n)$

III. $g = 1$ [GPhypergeom]

- $(X, x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \mathcal{E}^{\boldsymbol{\alpha}}_{g,n} \rightsquigarrow \boldsymbol{\alpha}$-flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

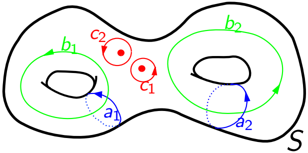
- $\mathbf{H}^{\boldsymbol{\alpha}}(X) : \pi_1(g, n) \longrightarrow \mathbb{U}$ fact. through $\pi_1(g, n)^{\mathbf{ab}} = H_1(g, n)$



III. $g = 1$ [GPhypergeom]

- $(X, x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \mathcal{E}^{\boldsymbol{\alpha}}_{g,n} \rightsquigarrow \boldsymbol{\alpha}$-flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- $\mathbf{H}^{\boldsymbol{\alpha}}(X) : \pi_1(g, n) \longrightarrow \mathbb{U}$ fact. through $\pi_1(g, n)^{\mathbf{ab}} = H_1(g, n)$

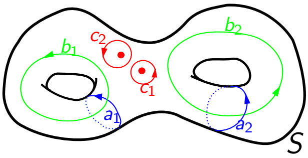


- $\mathcal{T}\mathbf{or}_{g,n} = \left\{ (X, x, \nu) \,\middle|\, \nu \text{ marking of } H_1(X^*) \right\}$

III. $g = 1$ [GPhypergeom]

- $(X, x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}_{g,n} \rightsquigarrow \boldsymbol{\alpha}$-flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- $\mathbf{H}^{\boldsymbol{\alpha}}(X) : \pi_1(g, n) \longrightarrow \mathbb{U}$ fact. through $\pi_1(g, n)^{\mathbf{ab}} = H_1(g, n)$

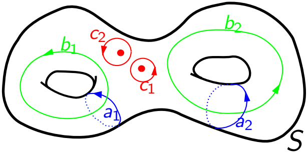


- $\mathcal{T}\mathbf{or}_{g,n} = \Big\{ (X, x, \nu) \,\Big|\, \nu \text{ marking of } H_1(X^*) \Big\} = \mathcal{T}\mathbf{eich}_{g,n} / \mathbf{Tor}_{g,n}$

III. $g = 1$ [GPhypergeom]

- $(X, x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \mathcal{E}^{\boldsymbol{\alpha}}_{g,n} \rightsquigarrow \boldsymbol{\alpha}$-flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- $\mathbf{H}^{\alpha}(X) : \pi_1(g,n) \longrightarrow \mathbb{U}$ fact. through $\pi_1(g,n)^{\mathbf{ab}} = H_1(g,n)$



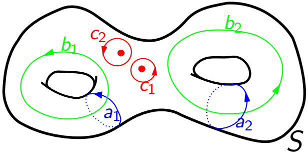
- $\mathcal{T}\mathbf{or}_{g,n} = \Big\{ (X, x, \nu) \,\Big|\, \nu \text{ marking of } H_1(X^*) \Big\} = \mathcal{T}\mathbf{eich}_{g,n} / \mathbf{Tor}_{g,n}$

$$\begin{array}{ccc} \mathcal{T}\mathbf{eich}_{g,n} & \xrightarrow{\quad \mathbf{H}^{\alpha} \quad} & \mathbb{U}^{2g} \\ \downarrow & \nearrow_{\mathbf{h}^{\alpha}} & \\ \mathcal{T}\mathbf{or}_{g,n} & & \end{array}$$

III. $g=1$ [GPhypergeom]

- $(X,x) \in \mathcal{T}\mathbf{eich}_{g,n} \simeq \mathcal{E}^{\boldsymbol{\alpha}}_{g,n} \rightsquigarrow \boldsymbol{\alpha}$-flat metric $m^{\boldsymbol{\alpha}}_{X,x}$ on X

- $\mathbf{H}^{\boldsymbol{\alpha}}(X) : \pi_1(g,n) \longrightarrow \mathbb{U}$ fact. through $\pi_1(g,n)^{\mathbf{ab}} = H_1(g,n)$



- $\mathcal{T}\mathbf{or}_{g,n} = \Big\{ (X,x,\nu) \,\Big|\, \nu \text{ marking of } H_1(X^*) \Big\} = \mathcal{T}\mathbf{eich}_{g,n} / \mathbf{Tor}_{g,n}$

$$\begin{array}{ccc} \mathcal{T}\mathbf{eich}_{g,n} & \xrightarrow{\mathbf{H}^{\boldsymbol{\alpha}}} & \mathbb{U}^{2g} \\ \downarrow & \nearrow_{\mathbf{h}^{\boldsymbol{\alpha}}} & \\ \mathcal{T}\mathbf{or}_{g,n} & & \end{array}$$

- $\rightsquigarrow$ natural to study $\mathcal{F}^{\boldsymbol{\alpha}}$ on $\mathcal{T}\mathbf{or}_{g,n}$

Thm : [Nag]

- $\mathcal{T}\mathbf{or}_{1,n} = \left\{ (\tau, z) \in \mathbb{H} \times \mathbb{C}^n \,\middle|\, \begin{array}{l} z = (z_1 = 0, z_2, \ldots, z_n) \\ z_k - z_\ell \notin \mathbb{Z}_\tau \,, \; \forall\, k < \ell \end{array} \right\}$
- $\mathbf{PMCG}_{1,n} / \mathbf{Tor}_{1,n} \simeq \mathrm{SL}_2(\mathbb{Z}) \ltimes \left(\mathbb{Z}^2\right)^{n-1}$ with standard action

Thm : [Nag]

- $$\mathcal{T}\mathbf{or}_{1,n} = \left\{ (\tau, z) \in \mathbb{H} \times \mathbb{C}^n \;\middle|\; \begin{array}{l} z = (z_1 = 0, z_2, \ldots, z_n) \\ z_k - z_\ell \notin \mathbb{Z}_\tau \,,\ \forall\, k < \ell \end{array} \right\}$$
- $\mathbf{PMCG}_{1,n} / \mathbf{Tor}_{1,n} \simeq \mathrm{SL}_2(\mathbb{Z}) \ltimes \left(\mathbb{Z}^2\right)^{n-1}$ with standard action

- for $(\tau, z) \in \mathcal{T}\mathbf{or}_{1,n}$: $m^{\boldsymbol{\alpha}}_{\tau,z} = \left\{ \begin{array}{l} \text{flat metric on } E_\tau = \mathbb{C}/\mathbb{Z}_\tau \\ \text{with } [z_k] = \alpha_k\text{-conical point} \end{array} \right.$

Thm : [Nag]

- $\mathcal{T}\mathbf{or}_{1,n} = \left\{ (\tau, z) \in \mathbb{H} \times \mathbb{C}^n \;\middle|\; \begin{array}{l} z = (z_1 = 0, z_2, \ldots, z_n) \\ z_k - z_\ell \notin \mathbb{Z}_\tau \,, \; \forall\, k < \ell \end{array} \right\}$
- $\mathbf{PMCG}_{1,n} / \mathbf{Tor}_{1,n} \simeq \mathrm{SL}_2(\mathbb{Z}) \ltimes \left(\mathbb{Z}^2\right)^{n-1}$ with standard action

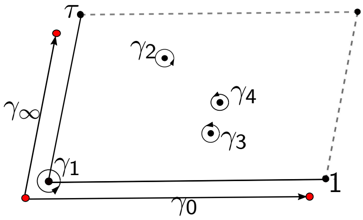
- for $(\tau, z) \in \mathcal{T}\mathbf{or}_{1,n}$: $m^{\boldsymbol{\alpha}}_{\tau,z} = \left\{ \begin{array}{l} \text{flat metric on } E_\tau = \mathbb{C}/\mathbb{Z}_\tau \\ \text{with } [z_k] = \alpha_k\text{-conical point} \end{array} \right.$

Prop : [GP] One has $m^{\boldsymbol{\alpha}}_{\tau,z} = \left| T(u)du \right|^2$ with

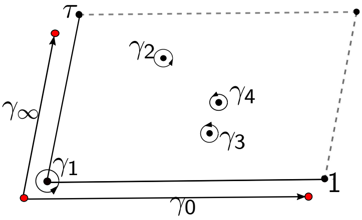
- $T(u) = T^{\boldsymbol{\alpha}}_{\tau,z}(u) = e^{2i\pi\alpha_0 u} \prod_k \theta\big(u - z_k \,,\, \tau\big)^{\alpha_k}$
- $\alpha_0 = \alpha_0(\tau, z) = -\mathfrak{Im}\big(\sum_k \alpha_k z_k\big) \Big/ \mathfrak{Im}(\tau) \in \mathbb{R}$

- $H_1(E^*_{\tau,z}, \mathbb{Z})$ spanned by $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_\infty$:

- $H_1(E^*_{\tau,z}, \mathbb{Z})$ spanned by $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_\infty$:

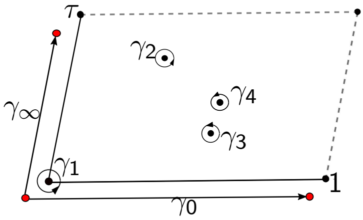


- $H_1(E^*_{\tau,z}, \mathbb{Z})$ spanned by $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_\infty$:



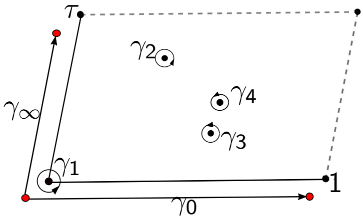
- $\mathcal{T}\mathbf{or}_{1,n} \ni (\tau, z)$

- $H_1(E^*_{\tau,z},\mathbb{Z})$ spanned by $\gamma_0,\gamma_1,\gamma_2,\dots,\gamma_n,\gamma_\infty$:



- $\mathcal{T}\mathbf{or}_{1,n}\ni(\tau,z)\leftrightarrow\left(E_{\tau,z},m^{\boldsymbol{\alpha}}_{\tau,z}\right),\ m^{\boldsymbol{\alpha}}_{\tau,z}=\left|e^{2i\pi\alpha_0 u}\prod_k\theta\left(u-z_k\right)^{\alpha_k}du\right|^2$

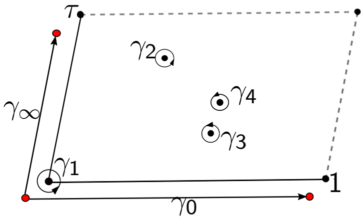
- $H_1(E^*_{\tau,z}, \mathbb{Z})$ spanned by $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_\infty$:



- $\mathcal{T}\mathbf{or}_{1,n} \ni (\tau, z) \leftrightarrow \left(E_{\tau,z}, m^{\boldsymbol{\alpha}}_{\tau,z}\right)\,,\ m^{\boldsymbol{\alpha}}_{\tau,z} = \left|e^{2i\pi\alpha_0 u}\prod_k \theta\left(u - z_k\right)^{\alpha_k} du\right|^2$

$$\mathbf{h}^{\boldsymbol{\alpha}}(\gamma_0) = e^{2i\pi\alpha_0}$$

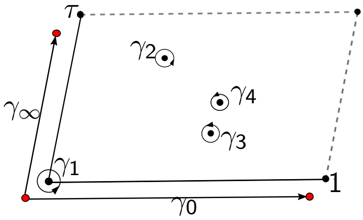
- $H_1(E^*_{\tau,z},\mathbb{Z})$ spanned by $\gamma_0,\gamma_1,\gamma_2,\ldots,\gamma_n,\gamma_\infty$:



- $\mathcal{T}\mathbf{or}_{1,n} \ni (\tau,z) \leftrightarrow \left(E_{\tau,z}, m^{\boldsymbol{\alpha}}_{\tau,z}\right),\ m^{\boldsymbol{\alpha}}_{\tau,z} = \left|e^{2i\pi\alpha_0 u}\prod_k \theta\left(u-z_k\right)^{\alpha_k} du\right|^2$

$$\mathbf{h}^{\boldsymbol{\alpha}}(\gamma_0) = e^{2i\pi\alpha_0} \qquad \text{with} \qquad \alpha_0 = -\Im\mathbf{m}\Big(\sum_k \alpha_k z_k\Big)/\Im\mathbf{m}(\tau) \in \mathbb{R}$$

- $H_1(E^*_{\tau,z},\mathbb{Z})$ spanned by $\gamma_0,\gamma_1,\gamma_2,\ldots,\gamma_n,\gamma_\infty$:

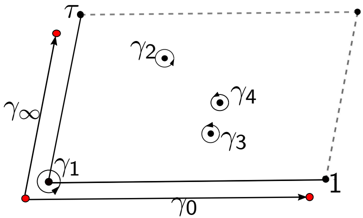


- $\mathcal{T}\mathbf{or}_{1,n} \ni (\tau,z) \leftrightarrow \left(E_{\tau,z}, m^{\boldsymbol{\alpha}}_{\tau,z}\right)$, $m^{\boldsymbol{\alpha}}_{\tau,z} = \left|e^{2i\pi\alpha_0 u}\prod_k \theta\left(u-z_k\right)^{\alpha_k} du\right|^2$

$$\mathbf{h}^{\boldsymbol{\alpha}}(\gamma_0) = e^{2i\pi\alpha_0} \qquad \text{with} \qquad \alpha_0 = -\Im\mathbf{m}\Big(\sum_k \alpha_k z_k\Big)/\Im\mathbf{m}(\tau) \in \mathbb{R}$$

$$\mathbf{h}^{\boldsymbol{\alpha}}(\gamma_\infty) = e^{2i\pi\alpha_\infty}$$

- $H_1(E^*_{\tau,z},\mathbb{Z})$ spanned by $\gamma_0,\gamma_1,\gamma_2,\dots,\gamma_n,\gamma_\infty$:

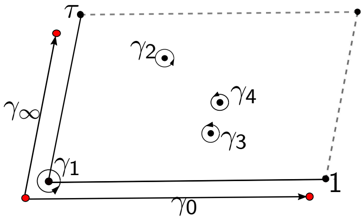


- $\mathcal{T}\mathbf{or}_{1,n} \ni (\tau,z) \leftrightarrow \left(E_{\tau,z}, m^{\boldsymbol{\alpha}}_{\tau,z}\right),\ m^{\boldsymbol{\alpha}}_{\tau,z} = \left|e^{2i\pi\alpha_0 u}\prod_k \theta\left(u-z_k\right)^{\alpha_k} du\right|^2$

$$\mathbf{h}^{\boldsymbol{\alpha}}(\gamma_0) = e^{2i\pi\alpha_0} \qquad \text{with} \qquad \alpha_0 = -\mathfrak{Im}\Big(\sum_k \alpha_k z_k\Big)/\mathfrak{Im}(\tau) \in \mathbb{R}$$

$$\mathbf{h}^{\boldsymbol{\alpha}}(\gamma_\infty) = e^{2i\pi\alpha_\infty} \qquad \text{with} \qquad \alpha_\infty = \alpha_0\tau + \sum_k \alpha_k z_k \in \mathbb{R}$$

- $H_1(E^*_{\tau,z}, \mathbb{Z})$ spanned by $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_\infty$:



- $\mathcal{T}\mathbf{or}_{1,n} \ni (\tau, z) \leftrightarrow \left(E_{\tau,z}, m^{\boldsymbol{\alpha}}_{\tau,z}\right), \; m^{\boldsymbol{\alpha}}_{\tau,z} = \left| e^{2i\pi\alpha_0 u} \prod_k \theta(u - z_k)^{\alpha_k} du \right|^2$

$$\mathbf{h}^{\boldsymbol{\alpha}}(\gamma_0) = e^{2i\pi\alpha_0} \qquad \text{with} \qquad \alpha_0 = -\mathfrak{Im}\Big(\sum_k \alpha_k z_k\Big) / \mathfrak{Im}(\tau) \in \mathbb{R}$$

$$\mathbf{h}^{\boldsymbol{\alpha}}(\gamma_\infty) = e^{2i\pi\alpha_\infty} \qquad \text{with} \qquad \alpha_\infty = \alpha_0 \tau + \sum_k \alpha_k z_k \in \mathbb{R}$$

- Holonomy map writes as follows :

$$\mathbf{h}^{\boldsymbol{\alpha}} \; : \; \mathcal{T}\mathbf{or}_{1,n} \longrightarrow \mathbb{U}^2, \; (\tau, z) \longmapsto \left(e^{2i\pi\alpha_0}, e^{2i\pi\alpha_\infty}\right)$$

"lifted holonomy"

- $$\begin{array}{ccc} & & \mathbb{R}^2 \\ & \nearrow^{\xi^{\boldsymbol{\alpha}}=(\alpha_0,\alpha_\infty)} & \Big\downarrow \exp(2i\pi\cdot) \\ \mathcal{T}\mathbf{or}_{1,n} & \xrightarrow[\mathbf{h}^{\boldsymbol{\alpha}}]{} & \mathbb{U}^2 \end{array}$$

"lifted holonomy"

- $$\begin{array}{ccc} & \xrightarrow{\xi^{\boldsymbol{\alpha}}=(\alpha_0,\alpha_\infty)} & \mathbb{R}^2 \\ & \nearrow & \big\downarrow \exp(2i\pi\cdot) \\ \mathcal{T}\mathbf{or}_{1,n} & \xrightarrow[\mathbf{h}^{\boldsymbol{\alpha}}]{} & \mathbb{U}^2 \end{array}$$

Thm : [GP] In $\mathcal{T}\mathbf{or}_{1,n}$:

- the leaves of $\mathcal{F}^{\boldsymbol{\alpha}}$ are the affine subvarieties

$$\mathcal{F}^{\boldsymbol{\alpha}}_a \;:\qquad a_0\tau + \sum_k \alpha_k z_k = a_\infty \;, \qquad a = (a_0, a_\infty) \in \mathbb{R}^2$$

- $\mathcal{F}^{\boldsymbol{\alpha}}$ depends only on $[\boldsymbol{\alpha}] \in \mathbb{P}(\mathbb{R}^n)$

- *"lifted holonomy"* $\xi^{\boldsymbol{\alpha}} = (\alpha_0, \alpha_\infty) : \mathcal{T}\mathbf{or}_{1,n} \to \mathbb{R}^2$

$$\mathcal{T}\mathbf{or}_{1,n} \xrightarrow{\mathbf{h}^{\boldsymbol{\alpha}}} \mathbb{U}^2, \qquad \mathbb{R}^2 \xrightarrow{\exp(2i\pi\cdot)} \mathbb{U}^2$$

Thm : [GP] In $\mathcal{T}\mathbf{or}_{1,n}$:

- the leaves of $\mathcal{F}^{\boldsymbol{\alpha}}$ are the affine subvarieties

$$\mathcal{F}_a^{\boldsymbol{\alpha}} \quad : \qquad a_0\tau + \sum_k \alpha_k z_k = a_\infty \ , \qquad a = (a_0, a_\infty) \in \mathbb{R}^2$$

- $\mathcal{F}^{\boldsymbol{\alpha}}$ depends only on $[\boldsymbol{\alpha}] \in \mathbb{P}(\mathbb{R}^n)$

- **n = 2 :**

- *"lifted holonomy"*

$$\begin{array}{ccc} & \xrightarrow{\xi^{\boldsymbol{\alpha}}=(\alpha_0,\alpha_\infty)} & \mathbb{R}^2 \\ & \nearrow & \downarrow \exp(2i\pi\cdot) \\ \mathcal{T}\mathbf{or}_{1,n} & \xrightarrow[\mathbf{h}^{\boldsymbol{\alpha}}]{} & \mathbb{U}^2 \end{array}$$

Thm : [GP] In $\mathcal{T}\mathbf{or}_{1,n}$:

- the leaves of $\mathcal{F}^{\boldsymbol{\alpha}}$ are the affine subvarieties

$$\mathcal{F}^{\boldsymbol{\alpha}}_a \ : \qquad a_0\tau + \sum_k \alpha_k z_k = a_\infty \ , \qquad a = (a_0, a_\infty) \in \mathbb{R}^2$$

- $\mathcal{F}^{\boldsymbol{\alpha}}$ depends only on $[\boldsymbol{\alpha}] \in \mathbb{P}(\mathbb{R}^n)$

- $\mathbf{n = 2}$ **:** $\boldsymbol{\alpha} = (\alpha_1, -\alpha_1)$ with $0 < \alpha_1 < 1$

"lifted holonomy"

- $$\begin{array}{ccc} & \xrightarrow{\xi^{\boldsymbol{\alpha}}=(\alpha_0,\alpha_\infty)} & \mathbb{R}^2 \\ & & \big\downarrow \exp(2i\pi\cdot) \\ \mathcal{T}\mathbf{or}_{1,n} & \xrightarrow[\mathbf{h}^{\boldsymbol{\alpha}}]{} & \mathbb{U}^2 \end{array}$$

Thm : [GP] In $\mathcal{T}\mathbf{or}_{1,n}$:

- the leaves of $\mathcal{F}^{\boldsymbol{\alpha}}$ are the affine subvarieties

$$\mathcal{F}^{\boldsymbol{\alpha}}_a \ : \qquad a_0\tau + \sum_k \alpha_k z_k = a_\infty \ , \qquad a = (a_0, a_\infty) \in \mathbb{R}^2$$

- $\mathcal{F}^{\boldsymbol{\alpha}}$ depends only on $[\boldsymbol{\alpha}] \in \mathbb{P}(\mathbb{R}^n)$

- $\mathbf{n = 2 : \boldsymbol{\alpha}} = (\alpha_1, -\alpha_1)$ with $0 < \alpha_1 < 1$

$$\mathbf{F}^{\boldsymbol{\alpha}} = \text{ foliation on } \mathbf{M}_{1,2} \begin{cases} \text{– canonical} \\ \text{– by } \mathbb{CH}^1\text{-curves} \end{cases}$$

Thm : [GP]

- In $\mathcal{T}\mathbf{or}_{1,2}$, for any $a = (a_0, a_\infty) \in \mathbb{R}^2 \setminus \alpha_1\mathbb{Z}^2$:

$$\begin{aligned} \mathbb{H} &\xrightarrow{\quad\sim\quad} \mathcal{F}_a^{\boldsymbol{\alpha}} \\ \tau &\longmapsto \Big(\tau, \tfrac{1}{\alpha_1}\big(a_0\tau - a_\infty\big)\Big) \end{aligned}$$

- The restriction $\mathcal{F}_a^{\boldsymbol{\alpha}} \to \mathbf{F}_a^{\boldsymbol{\alpha}}$ of $\mathcal{T}\mathbf{or}_{1,2} \longrightarrow \mathbf{M}_{1,2}$ is isomorphic to the quotient of $\mathbb{H}$ by
  - $\mathbf{Id} : \tau \mapsto \tau \quad \Longrightarrow \quad \mathbf{F}_a^{\boldsymbol{\alpha}} \simeq \mathbb{H}$
  - $\mathbf{T} : \tau \mapsto \tau + 1 \quad \Longrightarrow \quad \mathbf{F}_a^{\boldsymbol{\alpha}} \simeq$ infinite cylinder
  - $\Gamma_1(N) \quad \Longrightarrow \quad \mathbf{F}_a^{\boldsymbol{\alpha}} \simeq \mathbb{H}/_{\Gamma_1(N)} = Y_1(N)\,,\ N \geq 2$

- Closed leaves of $\mathbf{F}^{\boldsymbol{\alpha}}$: the $\mathbf{F}^{\boldsymbol{\alpha}}_{(0,1/N)} \simeq Y_1(N)$ with $N \geq 2$

- $N \geq 2$:

- $N \geq 2$: uniformization of $Y_1(N)^{\alpha_1} = \mathbf{F}^{\alpha}_{(0,1/N)} \subset \mathbf{M}_{1,2}$:

- $N \geq 2$: uniformization of $Y_1(N)^{\alpha_1} = \mathbf{F}^{\alpha}_{(0,1/N)} \subset \mathbf{M}_{1,2}$:

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & Y_1(N)^{\alpha_1} & \hookrightarrow & \mathbf{M}_{1,2} \\ \tau & \longmapsto & \left(E_\tau, \left[\frac{1}{N}\right]\right) & \longmapsto & \left(E_\tau, [0], \left[\frac{1}{N}\right]\right) \end{array}$$

- $N \geq 2$: uniformization of $Y_1(N)^{\alpha_1} = \mathbf{F}^{\alpha}_{(0,1/N)} \subset \mathbf{M}_{1,2}$:

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & Y_1(N)^{\alpha_1} & \hookrightarrow & \mathbf{M}_{1,2} \\ \tau & \longmapsto & \left(E_\tau, \left[\frac{1}{N}\right]\right) & \longmapsto & \left(E_\tau, [0], \left[\frac{1}{N}\right]\right) \end{array}$$

- In terms of flat surfaces :

- $N \geq 2$: uniformization of $Y_1(N)^{\alpha_1} = \mathbf{F}^{\alpha}_{(0,1/N)} \subset \mathbf{M}_{1,2}$:

$$\mathbb{H} \longrightarrow Y_1(N)^{\alpha_1} \hookrightarrow \mathbf{M}_{1,2}$$

$$\tau \longmapsto \left(E_\tau, \left[\tfrac{1}{N}\right]\right) \longmapsto \left(E_\tau, [0], \left[\tfrac{1}{N}\right]\right)$$

- In terms of flat surfaces :

$$\mathbb{H} \longrightarrow Y_1(N)^{\alpha_1} \hookrightarrow \mathbf{M}^{\boldsymbol{\alpha}}_{1,2}$$

$$\tau \longmapsto \left(E_\tau, m^{\alpha_1}_{N,\tau}\right)$$

- $N \geq 2$: uniformization of $Y_1(N)^{\alpha_1} = \mathbf{F}^{\alpha}_{(0,1/N)} \subset \mathbf{M}_{1,2}$:

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & Y_1(N)^{\alpha_1} & \hookrightarrow & \mathbf{M}_{1,2} \\ \tau & \longmapsto & \left(E_\tau, \left[\frac{1}{N}\right]\right) & \longmapsto & \left(E_\tau, [0], \left[\frac{1}{N}\right]\right) \end{array}$$

- In terms of flat surfaces :

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & Y_1(N)^{\alpha_1} & \hookrightarrow & \mathbf{M}^{\boldsymbol{\alpha}}_{1,2} \\ \tau & & \longmapsto & & \left(E_\tau, m^{\alpha_1}_{N,\tau}\right) \end{array}$$

where $m^{\alpha_1}_{N,\tau} = \left|T_\tau(u)du\right|^2$ with $T_\tau(u) = \frac{\theta(u,\tau)^{\alpha_1}}{\theta\left(u-\frac{1}{N},\tau\right)^{\alpha_1}}$

- $N \geq 2$: uniformization of $Y_1(N)^{\alpha_1} = \mathbf{F}^{\alpha}_{(0,1/N)} \subset \mathbf{M}_{1,2}$:

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & Y_1(N)^{\alpha_1} & \hookrightarrow & \mathbf{M}_{1,2} \\ \tau & \longmapsto & \left(E_\tau, \left[\frac{1}{N}\right]\right) & \longmapsto & \Big(E_\tau, [0], \left[\frac{1}{N}\right]\Big) \end{array}$$

- In terms of flat surfaces :

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & Y_1(N)^{\alpha_1} & \hookrightarrow & \mathbf{M}^{\boldsymbol{\alpha}}_{1,2} \\ \tau & & \longmapsto & & \left(E_\tau, m^{\alpha_1}_{N,\tau}\right) \end{array}$$

where $m^{\alpha_1}_{N,\tau} = \left|T_\tau(u)du\right|^2$ with $T_\tau(u) = \frac{\theta(u,\tau)^{\alpha_1}}{\theta\left(u-\frac{1}{N},\tau\right)^{\alpha_1}}$

- $N \geq 2$: uniformization of $Y_1(N)^{\alpha_1} = \mathbf{F}^{\alpha}_{(0,1/N)} \subset \mathbf{M}_{1,2}$:

$$\mathbb{H} \longrightarrow Y_1(N)^{\alpha_1} \hookrightarrow \mathbf{M}_{1,2}$$

$$\tau \longmapsto \left(E_\tau, \left[\tfrac{1}{N}\right]\right) \longmapsto \left(E_\tau, [0], \left[\tfrac{1}{N}\right]\right)$$

- In terms of flat surfaces :

$$\mathbb{H} \longrightarrow Y_1(N)^{\alpha_1} \hookrightarrow \mathbf{M}^{\boldsymbol{\alpha}}_{1,2}$$

$$\tau \longmapsto \left(E_\tau, m^{\alpha_1}_{N,\tau}\right)$$

where $m^{\alpha_1}_{N,\tau} = \left|T_\tau(u)du\right|^2$ with $T_\tau(u) = \frac{\theta(u,\tau)^{\alpha_1}}{\theta\left(u-\frac{1}{N},\tau\right)^{\alpha_1}}$

- Veech's map : $\mathcal{F}^{\alpha_1}_N \simeq \mathbb{H} \xrightarrow{V^{\alpha_1}_N} \mathbb{CH}^1 \subset \mathbb{P}^1$

- $N \geq 2$: uniformization of $Y_1(N)^{\alpha_1} = \mathbf{F}^{\alpha}_{(0,1/N)} \subset \mathbf{M}_{1,2}$:

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & Y_1(N)^{\alpha_1} & \hookrightarrow & \mathbf{M}_{1,2} \\ \tau & \longmapsto & \left(E_\tau, \left[\frac{1}{N}\right]\right) & \longmapsto & \left(E_\tau, [0], \left[\frac{1}{N}\right]\right) \end{array}$$

- In terms of flat surfaces :

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & Y_1(N)^{\alpha_1} & \hookrightarrow & \mathbf{M}^{\boldsymbol{\alpha}}_{1,2} \\ \tau & \longmapsto & & & \left(E_\tau, m^{\alpha_1}_{N,\tau}\right) \end{array}$$

where $m^{\alpha_1}_{N,\tau} = \left|T_\tau(u)du\right|^2$ with $T_\tau(u) = \frac{\theta(u,\tau)^{\alpha_1}}{\theta\left(u-\frac{1}{N},\tau\right)^{\alpha_1}}$

- Veech's map : $\mathcal{F}^{\alpha_1}_N \simeq \mathbb{H} \xrightarrow{V^{\alpha_1}_N} \mathbb{CH}^1 \subset \mathbb{P}^1$

- $N \geq 2$: uniformization of $Y_1(N)^{\alpha_1} = \mathbf{F}^{\alpha}_{(0,1/N)} \subset \mathbf{M}_{1,2}$:

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & Y_1(N)^{\alpha_1} & \hookrightarrow & \mathbf{M}_{1,2} \\ \tau & \longmapsto & \left(E_\tau, \left[\frac{1}{N}\right]\right) & \longmapsto & \left(E_\tau, [0], \left[\frac{1}{N}\right]\right) \end{array}$$

- In terms of flat surfaces :

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & Y_1(N)^{\alpha_1} & \hookrightarrow & \mathbf{M}^{\boldsymbol{\alpha}}_{1,2} \\ \tau & \longmapsto & & & \left(E_\tau, m^{\alpha_1}_{N,\tau}\right) \end{array}$$

where $m^{\alpha_1}_{N,\tau} = \left|T_\tau(u)du\right|^2$ with $T_\tau(u) = \frac{\theta(u,\tau)^{\alpha_1}}{\theta\left(u-\frac{1}{N},\tau\right)^{\alpha_1}}$

- Veech's map : $\mathcal{F}^{\alpha_1}_N \simeq \mathbb{H} \xrightarrow{V^{\alpha_1}_N} \mathbb{CH}^1 \subset \mathbb{P}^1$

- $N \geq 2$: uniformization of $Y_1(N)^{\alpha_1} = \mathbf{F}^{\alpha}_{(0,1/N)} \subset \mathbf{M}_{1,2}$:

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & Y_1(N)^{\alpha_1} & \hookrightarrow & \mathbf{M}_{1,2} \\ \tau & \longmapsto & \left(E_\tau, \left[\tfrac{1}{N}\right]\right) & \longmapsto & \left(E_\tau, [0], \left[\tfrac{1}{N}\right]\right) \end{array}$$

- In terms of flat surfaces :

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & Y_1(N)^{\alpha_1} & \hookrightarrow & \mathbf{M}^{\boldsymbol{\alpha}}_{1,2} \\ \tau & & \longmapsto & & \left(E_\tau, m^{\alpha_1}_{N,\tau}\right) \end{array}$$

where $m^{\alpha_1}_{N,\tau} = \left|T_\tau(u)du\right|^2$ with $T_\tau(u) = \frac{\theta(u,\tau)^{\alpha_1}}{\theta\left(u-\frac{1}{N},\tau\right)^{\alpha_1}}$

- Veech's map : $\mathcal{F}^{\alpha_1}_N \simeq \mathbb{H} \xrightarrow{V^{\alpha_1}_N} \mathbb{CH}^1 \subset \mathbb{P}^1$

$$\tau \longmapsto \left[\int_0^1 T_\tau(u)du : \int_0^\tau T_\tau(u)du\right]$$

Work of Mano & Watanabe :

Work of Mano & Watanabe:

- Cohomological/analytic (='hypergeometric') interpretation of

$$F_0(\tau) = \int_{[0,1]} T_\tau(u)du$$

Work of Mano & Watanabe:

- Cohomological/analytic (='hypergeometric') interpretation of

$$F_0(\tau) = \int_{[0,1]} T_\tau(u)du = \left\langle \mathbf{[0,1]} \, , \, \mathbf{[T_\tau(u)du]} \right\rangle$$

Work of Mano & Watanabe :

- Cohomological/analytic (='hypergeometric') interpretation of

$$F_0(\tau) = \int_{[0,1]} T_\tau(u)du = \Big\langle \mathbf{[0,1]}\,,\, \mathbf{[T_\tau(u)du]} \Big\rangle$$

with $\mathbf{[0,1]} \in H_1\big(E_\tau^*, L_\tau\big)$

Work of Mano & Watanabe :

- Cohomological/analytic (='hypergeometric') interpretation of

$$F_0(\tau) = \int_{[0,1]} T_\tau(u)du = \Big\langle \mathbf{[0,1]}\, , \, \mathbf{[T_\tau(u)du]} \Big\rangle$$

with $\mathbf{[0,1]} \in H_1\big(E^*_\tau, L_\tau\big)$ and $\mathbf{[T_\tau(u)du]} \in H^1\big(E^*_\tau, L^\vee_\tau\big)$

Work of Mano & Watanabe:

- Cohomological/analytic (='hypergeometric') interpretation of

$$F_0(\tau) = \int_{[0,1]} T_\tau(u)du = \Big\langle \mathbf{[0,1]}\,,\, \mathbf{[T_\tau(u)du]} \Big\rangle$$

with $\mathbf{[0,1]} \in H_1\big(E_\tau^*, L_\tau\big)$ and $\mathbf{[T_\tau(u)du]} \in H^1\big(E_\tau^*, L_\tau^\vee\big)$

- Letting τ varies in $\mathbb{H}$ $\rightsquigarrow$ Gauß-Manin connection

Work of Mano & Watanabe :

- Cohomological/analytic (='hypergeometric') interpretation of

$$F_0(\tau) = \int_{[0,1]} T_\tau(u)du = \Big\langle \mathbf{[0,1]}\,,\,\mathbf{[T_\tau(u)du]} \Big\rangle$$

with $\mathbf{[0,1]} \in H_1\big(E_\tau^*, L_\tau\big)$ and $\mathbf{[T_\tau(u)du]} \in H^1\big(E_\tau^*, L_\tau^\vee\big)$

- Letting τ varies in $\mathbb{H}$ $\rightsquigarrow$ Gauß-Manin connection

- $\rightsquigarrow$ Construction of the *"elliptic hypergeometric equation"*

Work of Mano & Watanabe :

- Cohomological/analytic (='hypergeometric') interpretation of

$$F_0(\tau) = \int_{[0,1]} T_\tau(u)du = \Big\langle \mathbf{[0,1]}\,,\, \mathbf{[T_\tau(u)du]} \Big\rangle$$

with $\mathbf{[0,1]} \in H_1\big(E_\tau^*, L_\tau\big)$ and $\mathbf{[T_\tau(u)du]} \in H^1\big(E_\tau^*, L_\tau^\vee\big)$

- Letting τ varies in $\mathbb{H}$ ⇝ Gauß-Manin connection

- ⇝ Construction of the *"elliptic hypergeometric equation"*

$$\big(Ell_N^{\alpha_1}\big) \quad : \quad \ddot{F}(\tau) + \mathbf{P}(\tau)\,\dot{F}(\tau) + \mathbf{Q}(\tau)F(\tau) = 0$$

Work of Mano & Watanabe:

- Cohomological/analytic (='hypergeometric') interpretation of

$$F_0(\tau) = \int_{[0,1]} T_\tau(u)du = \Big\langle \mathbf{[0,1]}\,,\, \mathbf{[T_\tau(u)du]} \Big\rangle$$

with $\mathbf{[0,1]} \in H_1\big(E_\tau^*, L_\tau\big)$ and $\mathbf{[T_\tau(u)du]} \in H^1\big(E_\tau^*, L_\tau^\vee\big)$

- Letting τ varies in $\mathbb{H}$ $\rightsquigarrow$ Gauß-Manin connection

- $\rightsquigarrow$ Construction of the *"elliptic hypergeometric equation"*

$$\big(Ell_N^{\alpha_1}\big) \quad : \quad \ddot{F}(\tau) + \mathbf{P}(\tau)\,\dot{F}(\tau) + \mathbf{Q}(\tau)F(\tau) = 0$$

such that $\Big\langle \int_0^1 T_\tau(u)du\,,\, \int_0^\tau T_\tau(u)du \Big\rangle = \mathbf{Sol}\,\big(Ell_N^{\alpha_1}\big)$

Work of Mano & Watanabe :

- Cohomological/analytic (='hypergeometric') interpretation of

$$F_0(\tau) = \int_{[0,1]} T_\tau(u)du = \Big\langle \mathbf{[0,1]}\,,\,\mathbf{[T_\tau(u)du]} \Big\rangle$$

with $\mathbf{[0,1]} \in H_1\big(E_\tau^*, L_\tau\big)$ and $\mathbf{[T_\tau(u)du]} \in H^1\big(E_\tau^*, L_\tau^\vee\big)$

- Letting τ varies in $\mathbb{H}$ $\rightsquigarrow$ Gauß-Manin connection

- $\rightsquigarrow$ Construction of the *"elliptic hypergeometric equation"*

$$\big(Ell_N^{\alpha_1}\big) \quad : \quad \ddot{F}(\tau) + \mathbf{P}(\tau)\,\dot{F}(\tau) + \mathbf{Q}(\tau)F(\tau) = 0$$

such that $\Big\langle \int_0^1 T_\tau(u)du\,,\, \int_0^\tau T_\tau(u)du \Big\rangle = \mathbf{Sol}\,\big(Ell_N^{\alpha_1}\big)$

Prop : $\big(Ell_N^{\alpha_1}\big)$ Fuchsian at $i\infty$ + formula for the projective index

Thm : [GP] For any $\alpha_1 \in]0,1[$:

- Veech's $\mathbb{CH}^1$-structure of $Y_1(N)^{\alpha_1}$ extends as a conifold structure $X_1(N)^{\alpha_1}$ on the compactification $X_1(N)$

- The conifold angle of $X_1(N)^{\alpha_1}$ at $\mathfrak{c} = [a/c] \in \mathbb{P}^1(\mathbb{Q})$ is

$$\theta_{\mathfrak{c}} = 2\pi \frac{c(N-c)}{N \cdot \gcd\big(c, N\big)} \cdot \alpha_1$$

- $\alpha_1 = \frac{N}{\ell N^*}$ with $\ell \geq 1 \implies X_1(N)^{\alpha_1}$ is a $\mathbb{CH}^1$-orbifold

---

- p prime : $\mathbf{Area}\Big(Y_1(p)^{\alpha_1}\Big) = \frac{\pi}{6}(1-\alpha_1)(p^2-1)$

- One has $\mathbf{Vol}\Big(\mathbf{M}_{1,2}^{\alpha_1}\Big) = \frac{\pi}{6}(1-\alpha_1) < \infty$