MODULAR REPRESENTATIONS OF $\mathrm{GL}(n)$ DISTINGUISHED BY $\mathrm{GL}(n-1)$ OVER A p-ADIC FIELD

by

V. Sécherre & C. G. Venketasubramanian

Abstract. — Let F be a non-Archimedean locally compact field, q be the cardinality of its residue field, and R be an algebraically closed field of characteristic ℓ not dividing q. We classify all irreducible smooth R-representations of $\mathrm{GL}_n(\mathrm{F})$ having a nonzero $\mathrm{GL}_{n-1}(\mathrm{F})$-invariant linear form, when q is not congruent to 1 mod ℓ. Partial results in the case when q is 1 mod ℓ show that, unlike the complex case, the space of $\mathrm{GL}_{n-1}(\mathrm{F})$-invariant linear forms has dimension 2 for certain irreducible representations.

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1. Introduction

1.1.

Let F be a non-Archimedean locally compact field of residual characteristic p, let G denote the F-points of a connected reductive group over F together with a closed subgroup H of G, and let R be an algebraically closed field of characteristic different from p. Given irreducible smooth representations π of G and σ of H with coefficients in R, it is a question of general interest in representation theory, known as the branching problem, to understand whether π restricted to H has σ as a quotient. If R is the field of complex numbers, this question is classical and well understood is many situations (see for instance [**7, 8**]). A case of particular interest is when σ is the trivial representation. In this situation π is said to be H-*distinguished* if its restriction to H has the trivial representation as a quotient, that is, if π carries a nonzero H-invariant linear form.

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1.2.

In this article, we are interested in the case where G is the general linear group $\mathrm{GL}_n(\mathrm{F})$, with $n \geqslant 2$, and H is the group $\mathrm{GL}_{n-1}(\mathrm{F})$ embedded in G via:

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

When R is the field of complex numbers, it is a consequence of a result of Waldspurger [**26**] that, for $n = 2$, any infinite dimensional irreducible representation of G is H-distinguished. The classification of all H-distinguished irreducible representations of G for $n = 3$ is due to D. Prasad [**16**]. For any $n \geqslant 2$, Prasad [**16**] has also proved that any generic representation of G has every generic representation of H as a quotient, and Flicker [**6**] classified all H-distinguished irreducible unitary representations of G. The classification of all H-distinguished irreducible representations of G for any $n \geqslant 3$ has been obtained by Venketasubramanian [**20**], in terms of Langlands parameters. Thus, when R is the field of complex numbers, the question is well understood. In this paper we investigate the case where the field R has positive characteristic ℓ different from p.

1.3.

The representation theory of smooth representations of $\mathrm{GL}_n(\mathrm{F})$ with coefficients in any algebraically closed field R of characteristic $\ell \neq 0, p$ has been initiated by Vignéras [**22, 23**] in view to extend the local Langlands program to representations with coefficients in a field (or ring) as general as possible (see for instance [**24**]). It has then been pursued by Dat, Mínguez, Stevens and the first author [**4, 12, 13, 14, 15, 17**]. In many aspects, it is similar to the theory of complex representations of this group: the fact that ℓ is different from p ensures that there is an R-valued Haar measure on G, the functors of parabolic induction and restriction are exact and preserve finite length, there is a theory of derivatives, there is a notion of cuspidal support for irreducible representations and a classification of these representations by mutisegments. However, there are also important differences: the measure of a compact open subgroup may be zero, and the notions of cuspidal and supercuspidal representations do not coincide, that is, a representation whose all proper Jacquet modules are zero may occur as a subquotient of a proper parabolically induced representation. The combinatorics of multisegments is also much more involved, since the cardinality q of the residue field of F has finite order in $\mathrm{R}^\times$.

1.4.

We now come to the main theorem of this article. Let R denote an algebraically closed field of characteristic different from p (possibly 0) and write e for the order (possibly infinite) of q in $\mathrm{R}^\times$. Write ν for the normalized absolute value of F giving value q^{-1} to any uniformizer. Let us fix a square root of q in R, denoted $\sqrt{q}$. Given integers $k \in \mathbf{Z}$ and $n \geqslant 1$, we write:

$$\nu_n^{k/2} : g \mapsto (\sqrt{q})^{-k \cdot \mathrm{val}(\det(g))}$$

where val is the normalized valuation on F and det is the determinant from G to $\mathrm{F}^\times$. If π, σ are smooth representations of $\mathrm{GL}_u(\mathrm{F})$, $\mathrm{GL}_v(\mathrm{F})$ respectively, with $u + v = n$, we denote by $\pi \times \sigma$ the normalized parabolic induction of $\pi \otimes \sigma$ to G along the standard (upper triangular) parabolic subgroup. When $e > 1$, the induced representation:

$$\mathrm{V}_n = \nu_{n-1}^{1/2} \times \nu^{(n+1)/2} \tag{1.1}$$

has a unique irreducible quotient, denoted Λ_n (see Example 4.3). Note that, when e divides n, this representation is the trivial character. Let us write 1_n for the trivial representation of G.

Theorem 1.1. — *Suppose that $n \geqslant 2$ and $e > 1$. An irreducible representation of $\mathrm{GL}_n(\mathrm{F})$ is $\mathrm{GL}_{n-1}(\mathrm{F})$-distinguished if and only if it belongs to the following list:*

(1) *the trivial representation 1_n;*

(2) *an irreducible representation of the form $\nu_{n-1}^{\pm 1/2} \times \chi$ with χ a character of $\mathrm{GL}_1(\mathrm{F})$;*

(3) *an irreducible representation of the form $1_{n-2} \times \tau$ with τ an infinite dimensional irreducible representation of $\mathrm{GL}_2(\mathrm{F})$;*

(4) *the representation Λ_n and its contragredient.*

As in the complex case, the proof of Theorem 1.1 is by induction on n. There are two parts to the proof of Theorem 1.1: proving that the representations in the list offered by the theorem are H-distinguished is the easier part. The more difficult part is to show the converse, namely that all irreducible representations which are H-distinguished are in the list.

1.5.

Since our proof is by induction, we first treat the case when $n = 2$ and obtain a classification (see Theorem 3.8) of all the $\mathrm{GL}_1(\mathrm{F})$-distinguished irreducible representations of $\mathrm{GL}_2(\mathrm{F})$. When e is not 1, the result turns out to be the same as in the complex case: all infinite-dimensional irreducible representation of $\mathrm{GL}_2(\mathrm{F})$ are distinguished and their space of $\mathrm{GL}_1(\mathrm{F})$-invariant linear forms has dimension 1. When the characteristic of R divides $q - 1$ however, this dimension is 2 for certain representations.

1.6.

Assume now that $n \geqslant 3$. As in the complex case, one can show by restricting to the mirabolic subgroup that none of the cuspidal representations of G are distinguished (Theorem 8.2). Since any non-cuspidal irreducible smooth representation of G is a quotient of a parabolically induced representation of the form $\sigma \times \tau$ with σ, τ smooth irreducible representations of $\mathrm{GL}_u(\mathrm{F}), \mathrm{GL}_v(\mathrm{F})$ for some integers $u, v \geqslant 1$ such that $u + v = n$, it is natural to study the distinction of $\sigma \times \tau$. This was carried out in [**20**] in the complex case. In the modular case, it works as in the complex case: one gets a set of three necessary conditions for this induced representation to be distinguished by H, of which two are sufficient (see Lemma 8.9). This is attributed to the existence of three orbits for the action of H on the homogeneous space made of all subspaces of dimension u in F^n, out of which two are closed. The induced representations in (2) and (3) of Theorem 1.1 are shown to satisfy one of the sufficiency conditions coming from Lemma 8.9 (see Corollary 8.13). The contragredient of Λ_n, when non-trivial, is realized as a subrepresentation of a distinguished principal series of length 2, the quotient of which is a nontrivial character and is non-distinguished (see proof of Lemma 8.12).

1.7.

To prove the converse of Theorem 1.1, we first prove that any H-distinguished representation of G is a quotient of a representation of the form $\rho \times \chi$ where ρ is an irreducible representation of $\mathrm{GL}_{n-1}(\mathrm{F})$ and χ a character of $\mathrm{F}^\times$. *In particular, when $e > 1$, such a quotient is unique.* Using the conditions of Lemma 8.9 mentioned above and the induction hypothesis, we can specify ρ and χ to be in a list (see Proposition 8.18). Then, when $e > 1$, we analyze the unique irreducible quotient of all these $\rho \times \chi$. We show that if the quotient is distinguished, then it must be in our list. The case when $e = 1$ presents additional difficulties which we shall touch upon below.

1.8.

We now describe the contents of the article. In Section 2 we set some basic notation, and deal with the case $n = 2$ in Section 3. The complete classification for $n = 2$ is obtained in Theorem 3.8. We begin Section 4 by recalling some general results on ℓ-modular representations of $\mathrm{GL}_n(\mathrm{F})$ from [**22, 14**]. We get a complete description of the subquotients of representations of the form $\mathrm{Z}(\Delta) \times \mathrm{Z}(\Delta')$ where Δ, Δ' are segments and Δ' is of length at most 2 (see Propositions 4.10 and 4.13). In particular, Proposition 4.10 may be deemed to be a generalization of [**21**, Théorème 3]. Moreover, comparing with [**27**, Proposition 4.6], Propositions 4.10 and 4.13 highlight one of the essential differences between principal series representations in the complex and modular cases: a product of two characters has length at most 2 in the complex case, a fact which does not hold as such in the modular case. The representation Λ_n, which plays a essential role in the article, is defined in Example 4.3 for $e > 1$, and in Definition 5.4 in general. More generally, Section 5 is devoted to the case where $e = 1$. The avatar Π_n of Λ_n is defined in Example 4.11. In Section 6, we compute the derivatives of Λ_n and Π_n.

In Section 7, we prove a criterion for irreducibility of a product of the form $\mathrm{Z}(\Delta) \times \mathrm{L}(\Delta')$ where Δ' has length 2. This is a modular version of a result known in the complex case (Theorem 3.1 in [**3**]). We begin Section 8 with some basic results on H-distinguished representations of G. The first tool is to use Lemma 8.9, the conditions that we get from the three orbits that we mentioned above. This, along with some of its consequences, yields us Proposition 8.18 and we get a list of representations of the form $\rho \times \chi$ (see the list following Proposition 8.18): understanding the distinction of the quotients of representations in this list proves the difficult part of Theorem 1.1. The second tool in our proof is Proposition 8.8 using the Bernstein-Zelevinski filtration, which was available for the complex case [**6, 16**] and holds for R. The computation of the quotients of $\rho \times \chi$ in the list obtained from Proposition 8.18 is the content of Sections 9-12.

1.9.

We now explain some of the subtler ideas behind the proof of Theorem 1.1 in this article. Our proof is different from the one in [**20**] proved for complex representations. In [**20**], the main tool in analyzing the existence of a unique irreducible quotient is the Langlands Quotient Theorem and certain results of Zelevinski [**27**]. When these theorems fail to apply, [**20**] uses Theorem 7.1 of [**27**]. In fact, we use Lemma 4.2 which is sufficient for us to analyze the representations coming from the Lemma 8.9 when $e > 1$. Indeed, if one were to use Lemma 4.2 in the complex case, then the proof of Theorem 1.1 in [**20**] simplifies to some extent without having to resort to Theorem 7.1 of [**27**], because there we have to analyze all subquotients of a certain induced representation.

1.10.

However, in the modular case, even if Proposition 4.2 guarantees the existence of a unique irreducible quotient for the representations $\rho \times \chi$ arising from Proposition 8.18, to explicitly find this quotient is more difficult. This is due to the fact that, in order to determine whether the unique irreducible quotient of $\rho \times \chi$ is in the list offered by Theorem 1.1, we have to realize it as a quotient of a larger principal series and this larger principal series may not have a unique irreducible quotient. In such a situation, we had but no choice to use the analogue of Theorem 7.1 of

[**27**] for the larger principal series in hand. For our purposes, we reduce it to understand the subquotients of representations of the form $\mathrm{Z}(\Delta)\times\mathrm{Z}(\Delta')$ where the segment Δ' has length $\leqslant 2$. These subquotients have certain natural properties (see Section 4, **P1** to **P6**) proved in [**14**] which enables us to describe the subquotients. This result is obtained in Propositions 4.10 and 4.13. We then use Proposition 8.8 to rule out the subquotients in the larger principal series which are not in the list of Theorem 1.1.

1.11.

Let us mention that, when $e=1$, the list in Theorem 1.1 is not exhaustive. The first problem is that the representation (1.1) need not have a unique irreducible quotient. In particular, all its irreducible subquotients are H-distinguished (see Lemma 8.15), which is different behavior when we compare with the case when $e>1$. This forces us to consider more representations in the list offered by Proposition 8.18 and the tools that we use do not seem to be sufficient to understand the distinction of the quotients.

1.12.

In this last paragraph, we give a few remarks. First, the theory of p-modular representations of p-adic reductive groups is very different from the ℓ-modular theory. This is why we chosed to focus on the case where R has characteristic different from p.

In the complex case, the pair (G,H) is known to be a Gelfand pair [**1**], that is, the dimension:

$$d(\pi)=\dim\mathrm{Hom}_{\mathrm{H}}(\pi,1)$$

of the space of H-invariant linear forms on π is at most 1 for all irreducible complex representations π of G. This is no longer true in the modular case: when $e=1$ we have $d(\pi)=2$ for certain irreducible representations (Theorem 3.5 and Remark 8.16). When $e\geqslant 3$, it can be proved using the same methods as in [**20**] (see *ibid.* Remark 7.8) and our Theorem 1.1 that $d(\pi)\leqslant 1$ for all irreducible ℓ-modular representations π of G. When $e=2$ we expect $d(\pi)\leqslant 1$ still holds, but the proof in [**20**] fails (see Theorem 3.8 for $n=2$ and Remark 12.13 for more details).

When comparing the results in [**20**] with Mínguez [**10**], the classification of all H-distinguished irreducible complex representations of G turns out to be easily expressed in terms of the local theta correspondence from $\mathrm{GL}_2(\mathrm{F})$ to $\mathrm{GL}_n(\mathrm{F})$. It would be interesting to investigate this in the modular case, by developing an ℓ-modular theta correspondence (see [**11**]).

Our last remark is about reduction mod ℓ. It is not difficult to see that the reduction mod ℓ of an H-distinguished integral irreducible ℓ-adic representation of G contains at least one distinguished irreducible component, by reducing mod ℓ a nonzero invariant linear form. However this fact is not of much use here, and we do not say more about it.

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2. Notation and preliminaries

In all this article, we fix a locally compact non-Archimedean field F ; we write $\mathcal{O}$ for its ring of integers, $\mathfrak{p}$ for the maximal ideal of $\mathcal{O}$ and q for the cardinality of its residue field. We also fix an algebraically closed field R of characteristic not dividing q.

We write e for the order (possibly infinite) of the image of q in $\mathrm{R}^\times$ and define:

$$f = \begin{cases} 0 \text{ if R has characteristic } 0, \\ \text{the smallest positive integer } k \geqslant 2 \text{ such that } 1 + q + \cdots + q^{k-1} = 0 \text{ in R otherwise.} \end{cases}$$

When R has characteristic $\ell > 0$, we have $f = e$ if $e > 1$ and $f = \ell$ if $e = 1$.

Given a topological group G, a smooth R-representation (or representation for short) of G is a pair (π, V) made of an R-vector space V together with a group homomorphism $\pi : \mathrm{G} \to \mathrm{GL}(\mathrm{V})$ such that, for all $v \in \mathrm{V}$, there is an open subgroup of G fixing v. *In this article, all representations will be supposed to be smooth* R*-representations.*

A smooth R-character (or character for short) of G is a group homomorphism from G to $\mathrm{R}^\times$ with open kernel.

Given a representation π and a character χ of G, we write $\pi\chi$ for the twisted representation $g \mapsto \pi(g)\chi(g)$.

For $n \geqslant 1$, we write $\mathrm{G}_n = \mathrm{GL}_n(\mathrm{F})$, and $\widehat{\mathrm{G}}_n$ for the set of isomorphism classes of its irreducible representations. In particular, $\widehat{\mathrm{G}}_1$ will be identified with the group of characters of G_1.

Given a representation π of G_n, $n \geqslant 1$ and $\mu \in \widehat{\mathrm{G}}_1$, we write $\pi \cdot \mu = \pi(\mu \circ \det)$. If π has finite length, we write $[\pi]$ for its semi-simplification.

3. The pair $(\mathrm{GL}_2(\mathrm{F}), \mathrm{GL}_1(\mathrm{F}))$

Write $\mathrm{G} = \mathrm{GL}_2(\mathrm{F})$ and let:

$$\mathrm{H} = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \ ; \ x \in \mathrm{F}^\times \right\} \subseteq \mathrm{G}.$$

Let B denote the Borel subgroup of G made of upper triangular matrices, and write:

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{G}.$$

If X is a locally compact topological space and A is a commutative ring, let $\mathcal{C}_c^\infty(\mathrm{X}, \mathrm{A})$ denote the space of all locally constant and compactly supported functions from X to A.

We write dx for the R-valued Haar measure on $\mathrm{F}^\times$ giving measure 1 to the subgroup $1 + \mathfrak{p}$ of principal units (see [**22**, I.2]).

3.1. The principal series

Let α_1, α_2 be two smooth R-characters of $\mathrm{F}^\times$. Let:

$$\mathrm{V} = \mathrm{V}(\alpha_1, \alpha_2)$$

denote the (non-normalized) parabolic R-induction $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\alpha_1 \otimes \alpha_2)$, that is the space of all locally constant R-valued functions f on G such that $f(mng) = \alpha_1(m_1)\alpha_2(m_2)f(g)$ for all:

$$m = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \in \mathrm{G}, \quad n \in \begin{pmatrix} 1 & \mathrm{F} \\ 0 & 1 \end{pmatrix} \subseteq \mathrm{G}, \quad g \in \mathrm{G},$$

which is made into a smooth R-representation of G by making G act by right translations. Write W for the subspace of V made of all functions vanishing at 1 and s. The map:

$$\mathrm{W} \to \mathcal{C}_c^\infty(\mathrm{F}^\times, \mathrm{R})$$

which associates to $f \in \mathrm{W}$ the function:

$$\phi : x \mapsto f\left(s\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)$$

is an isomorphism of R-vector spaces, and becomes an isomorphism of representations of H if the right hand side is endowed with the action defined by:

$$a \cdot \phi : x \mapsto \alpha_2(a)\phi(xa^{-1}), \quad x, a \in \mathrm{F}^\times.$$

Up to a nonzero scalar, there is on W a unique nonzero H-invariant linear form, given by:

$$\mu : f \mapsto \int_{\mathrm{F}^\times} f\left(s\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\alpha_2(x)^{-1}\,dx.$$

Let α denote the character of B extending $\alpha_1 \otimes \alpha_2$. Fix an integer $i \geqslant 1$ such that α_1, α_2 are trivial on $1+\mathfrak{p}^i$, and let K_i be the subgroup of $\mathrm{GL}_2(\mathcal{O})$ made of matrices congruent to the identity mod $\mathfrak{p}^i$. We define two functions f_0 and f_∞ on G:

(1) f_0 is supported on $\mathrm{B}s\mathrm{K}_i$ and $f_0(bsx) = \alpha(b)$ for all $b \in \mathrm{B}$, $x \in \mathrm{K}_i$.
(2) f_∞ is supported on BK_i and $f_\infty(bx) = \alpha(b)$ for all $b \in \mathrm{B}$, $x \in \mathrm{K}_i$.

As α is trivial on $\mathrm{B} \cap \mathrm{K}_i$, these functions f_0, f_∞ are well defined. They are in V but not in W.

Lemma 3.1. — *Given $f \in \mathrm{V}$, there is a unique function $w(f) \in \mathrm{W}$ such that:*

$$f = f(s)f_0 + f(1)f_\infty + w(f).$$

This defines a projection $w : \mathrm{V} \to \mathrm{W}$ with kernel spanned by f_0 and f_∞.

Proof. — This follows from the fact that s does not belong to BK_i. □

Let λ be an H-invariant linear form on V. It is characterized by $\lambda(f_0)$, $\lambda(f_\infty) \in \mathrm{R}$ and its restriction to W. As this restriction is H-invariant, it is of the form $c\mu$ for a unique scalar $c \in \mathrm{R}$.

Corollary 3.2. — *The space $\mathrm{V}^{*\mathrm{H}}$ of H-invariant linear forms on V has dimension $\leqslant 3$.*

Now let λ be a linear form on V extending μ. We search for a necessary and sufficient condition on $\lambda(f_0)$, $\lambda(f_\infty) \in \mathrm{R}$ for λ to be H-invariant. By definition, this linear form is H-invariant if and only if:

$$\lambda\left(\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}\cdot f\right) = \lambda(f)$$

for all $x \in \mathrm{F}^\times$ and $f \in \mathrm{V}$, and it is enough to check this condition for all x of valuation 1 and $f = f_0, f_\infty$. Let $t \in \mathrm{F}^\times$ be of valuation 1. We have:

$$\begin{aligned}\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\cdot f_0 &= \alpha_2(t)f_0 + w\left(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\cdot f_0\right),\\ \begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\cdot f_\infty &= \alpha_1(t)f_\infty + w\left(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\cdot f_\infty\right).\end{aligned}$$

Thus the condition writes:

$$(1-\alpha_2(t))\lambda(f_0) = \mu_0(t) \quad \text{and} \quad (1-\alpha_1(t))\lambda(f_\infty) = \mu_\infty(t)$$

for all $t \in \mathrm{F}^\times$ of valuation 1, where:

$$\mu_0(t) = \mu\left(w\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \cdot f_0\right)\right) \quad \text{and} \quad \mu_\infty(t) = \mu\left(w\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \cdot f_\infty\right)\right).$$

Lemma 3.3. — *We have:*

$$\mu_0(t) = -\alpha_2(t)^{1-i} \int\limits_{\mathcal{O}^\times} \alpha_2(x)^{-1}\, dx \quad \textit{and} \quad \mu_\infty(t) = \alpha_1(-1)\alpha_1(t)^i \int\limits_{\mathcal{O}^\times} \alpha_1(x)^{-1}\, dx.$$

Proof. — Given $x \in \mathrm{F}^\times$, write $m \in \mathbf{Z}$ for the valuation of x (normalized in such a way that any uniformizer has valuation 1) and:

$$\iota(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

We have:

$$s\iota(x) = s\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \mathrm{B}s\mathrm{K}_i \quad \Leftrightarrow \quad m \geqslant i \tag{3.1}$$

and:

$$s\iota(x) = \begin{pmatrix} -x^{-1} & 1 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} \in \mathrm{BK}_i \quad \Leftrightarrow \quad m \leqslant -i. \tag{3.2}$$

Note that:

$$s\iota(x)\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = s\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = s\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & xt^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} s\iota(xt^{-1}).$$

We have:

$$\begin{aligned} \mu_0(t) &= \int\limits_{\mathrm{F}^\times} \left[f_0\left(s\iota(x)\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) - \alpha_2(t) f_0\left(s\iota(x)\right)\right] \alpha_2(x)^{-1}\, dx, \\ \mu_\infty(t) &= \int\limits_{\mathrm{F}^\times} \left[f_\infty\left(s\iota(x)\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) - \alpha_1(t) f_\infty\left(s\iota(x)\right)\right] \alpha_2(x)^{-1}\, dx. \end{aligned}$$

Let $\phi_0(x,t)$ and $\phi_\infty(x,t)$ denote the functions into brackets in the formulas above, respectively. We use formulas (3.1) and (3.2) above. For $\phi_0(x,t)$ we have the following:

(1) if $m \geqslant i+1$, then $\phi_0(x,t) = \alpha_2(t) - \alpha_2(t) = 0$;
(2) if $m = i$, then $\phi_0(x,t) = -\alpha_2(t)$;
(3) if $m \leqslant i-1$, then $\phi_0(x,t) = 0$.

For $\phi_\infty(x,t)$ we have:

(1) if $m \geqslant -i+2$, then $\phi_\infty(x,t) = 0$;
(2) if $m = -i+1$, then $\phi_\infty(x,t) = \alpha_1(-tx^{-1})\alpha_2(x)$;
(3) if $m \leqslant -i$, then $\phi_\infty(x,t) = \alpha_1(-tx^{-1})\alpha_2(x) - \alpha_1(-tx^{-1})\alpha_2(x) = 0$.

Therefore we have:

$$\mu_0(t) = -\alpha_2(t) \int_{\mathcal{O}^\times} \alpha_2(t^i x)^{-1}\ dx \quad \text{and} \quad \mu_\infty(t) = \int_{\mathcal{O}^\times} \alpha_1(-t^{1-(1-i)} x^{-1})\ dx.$$

This ends the proof of the lemma. □

We now have the following result.

Theorem 3.4. — *The linear form μ can be extended to an* H*-invariant linear form on* V *if and only if one of the two conditions below is satisfied:*

(1) *$q \neq 1$ in* R *and α_1, α_2 are nontrivial.*
(2) *$q = 1$ in* R.

Proof. — If α_1, α_2 are ramified (that is, nontrivial on $\mathcal{O}^\times$), then:

$$\int_{\mathcal{O}^\times} \alpha_1(x)^{-1}\ dx = \int_{\mathcal{O}^\times} \alpha_2(x)^{-1}\ dx = 0.$$

Thus μ can be extended uniquely to an H-invariant linear form λ on V, by setting $\lambda(f_0) = \lambda(f_\infty) = 0$. If α_i is unramified for some $i \in \{1,2\}$, then:

$$\int_{\mathcal{O}^\times} \alpha_i(x)^{-1}\ dx = q - 1.$$

Fix a uniformizer ϖ of F and put $z_i = \alpha_i(\varpi)$.

(1) If $i = 1$, the condition on $\lambda(f_\infty)$ writes:

$$(1 - z_1)\lambda(f_\infty) = z_1^i (q-1). \tag{3.3}$$

If $z_1 \neq 1$, then (3.3) has a unique solution:

$$\lambda(f_\infty) = z_1^i \cdot \frac{q-1}{1-z_1}.$$

If $z_1 = 1$, then (3.3) has a solution if and only if we have $q = 1$ in R, and in that case any value of $\lambda(f_\infty)$ in R is a solution.

(2) If $i = 2$, the condition on $\lambda(f_0)$ writes:

$$(1 - z_2)\lambda(f_0) = -z_2^{1-i} (q-1). \tag{3.4}$$

If $z_2 \neq 1$, then (3.4) has a unique solution:

$$\lambda(f_0) = -z_2^{1-i} \cdot \frac{q-1}{1-z_2}.$$

If $z_2 = 1$, then (3.4) has a solution if and only if we have $q = 1$ in R, and in that case any value of $\lambda(f_0)$ in R is a solution.

This ends the proof of the theorem. □

Write $d(\mathrm{V})$ for the dimension of $\mathrm{V}^{*\mathrm{H}}$ and $e(\mathrm{V})$ for that of the subspace of H-invariant linear forms which are trivial on W.

Theorem 3.5. — *Let n denote the number of trivial characters among α_1, α_2.*

(1) *If $n = 0$, then $d(\mathrm{V}) = 1$ and $e(\mathrm{V}) = 0$.*

(2) *If* $n \geqslant 1$ *and* $q \neq 1$ *in* R, *then* $d(\mathrm{V}) = e(\mathrm{V}) = n$.
(3) *If* $n \geqslant 1$ *and* $q = 1$ *in* R, *then* $d(\mathrm{V}) = n+1$ *and* $e(\mathrm{V}) = n$.

Proof. — If $q \neq 1$ in R, the result is as in the complex case. If $q = 1$ in R, then μ can always be extended to an H-invariant linear form on V, that is, we have an exact sequence:

$$0 \to (\mathrm{V}/\mathrm{W})^{*\mathrm{H}} \to \mathrm{V}^{*\mathrm{H}} \to \mathrm{W}^{*\mathrm{H}} \to 0$$

of R-vector spaces and the dimension of $\mathrm{W}^{*\mathrm{H}}$ is 1. One easily checks that $e(\mathrm{V}) = n$. The result follows. □

3.2. The classification of $\mathrm{GL}_1(\mathrm{F})$-distinguished irreducible representations of $\mathrm{GL}_2(\mathrm{F})$

For the definition of an H-distinguished representation of G, we refer to Definition 8.1.

For any irreducible (smooth) representation π of G, let $d(\pi)$ denote the dimension of its space of H-invariant linear forms.

Recall that f denotes the quantum characteristic:

$$f = \begin{cases} 0 \text{ if R has characteristic } 0, \\ \text{the smallest positive integer } k \geqslant 2 \text{ such that } 1+q+\cdots+q^{k-1} = 0 \text{ in R otherwise.} \end{cases}$$

An irreducible representation of G is said to be *cuspidal* if it does not embed in any $\mathrm{V}(\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 \in \widehat{\mathrm{G}}_1$. Just as in the complex case, we have the following result for cuspidal representations.

Proposition 3.6. — *All cuspidal irreducible representations* π *of* G *are* H*-distinguished, with* $d(\pi) = 1$.

Proof. — See Paragraph 8.1 for a proof, where we treat the more general case of G_n, $n \geqslant 2$. □

Now let St denote the Steinberg representation of G, that is the unique nondegenerate irreducible subquotient of $\mathrm{V} = \mathrm{Ind}_\mathrm{B}^\mathrm{G}(1 \otimes 1)$ (see [**22**, III.1]).

For the following lemma, see [**14**, §6].

Lemma 3.7. — *If* $f = 2$, *then* $\mathrm{St} \cdot \chi$ *is cuspidal for all* $\chi \in \widehat{\mathrm{G}}_1$.

If $f = 2$, then Proposition 3.6 implies that $\mathrm{St} \cdot \chi$ is H-distinguished with $d(\mathrm{St} \cdot \chi) = 1$ for all $\chi \in \widehat{\mathrm{G}}_1$. Assume now that $f \neq 2$. Thus V has length 2 and we have an exact sequence:

$$0 \to \chi \circ \det \to \mathrm{V} \cdot \chi = \mathrm{Ind}_\mathrm{B}^\mathrm{G}(\chi \otimes \chi) \to \mathrm{St} \cdot \chi \to 0$$

of representations of G. If χ is nontrivial, then any H-invariant linear form on $\mathrm{V} \cdot \chi$ is trivial on $\chi \circ \det$. We thus have $d(\mathrm{St} \cdot \chi) = d(\mathrm{V} \cdot \chi) = 1$. If $\chi = 1$, we have:

$$d(\mathrm{St}) \leqslant d(\mathrm{V}) \leqslant d(\mathrm{St}) + 1.$$

As λ_0 and λ_∞ are H-invariant linear form on V which are nonzero on the subspace of constant functions, we get $d(\mathrm{St}) = d(\mathrm{V}) - 1$. Finally, we have the following result.

Theorem 3.8. — (1) *An irreducible representation of* G *is* H*-distinguished if and only if it is not a nontrivial* 1*-dimensional representation.*

(2) *Let* π *be an* H*-distinguished irreducible representation of* G. *Then* $d(\pi) \leqslant 2$, *with equality if and only if* $q = 1$ *in* R *and we are in one of the following cases:*

(a) π *is the Steinberg representation* St *and* R *has characteristic* > 2 *;*
(b) π *is a principal series representation* $\mathrm{V}(1, \chi) = \mathrm{Ind}_\mathrm{B}^\mathrm{G}(1 \otimes \chi)$ *with* $\chi \in \widehat{\mathrm{G}}_1$ *nontrivial.*

4. General results on modulo ℓ representations of G_n

4.1. More notation

Let $\alpha = (n_1, \dots, n_r)$ be a composition of n, that is, a family of positive integer whose sum is n. We denote by M_α the subgroup of G_n of invertible matrices which are diagonal by blocks of size $n_1, \dots, n_r$ respectively (it is isomorphic to $\mathrm{G}_{n_1} \times \cdots \times \mathrm{G}_{n_r}$) and by P_α the subgroup of G_n generated by M_α and the upper triangular matrices.

We choose once and for all a square root of q in R. We write $\boldsymbol{r}_\alpha$ for the normalized Jacquet functor associated to $(\mathrm{M}_\alpha, \mathrm{P}_\alpha)$ and $\boldsymbol{i}_\alpha$ for its right adjoint functor, that is, normalized parabolic induction. If $\pi_1, \dots, \pi_r$ are smooth R-representations of $\mathrm{G}_{n_1}, \dots, \mathrm{G}_{n_r}$ respectively, we write:

$$\pi_1 \times \pi_2 \times \cdots \times \pi_r = \boldsymbol{i}_\alpha(\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_r). \tag{4.1}$$

Given a smooth representation π of finite length, we write $[\pi]$ for its semi-simplification and π^* for its contragredient.

We write ν for the normalized absolute value of F, giving value q^{-1} to any uniformizer. More generally, given integers $k \in \mathbf{Z}$ and $n \geqslant 1$, we write:

$$\nu_n^{k/2} : g \mapsto (\sqrt{q})^{-k \cdot \mathrm{val}(\det(g))}$$

where $\sqrt{q}$ is the square root of q in R that has been fixed above, val is the normalized valuation on F and det is the determinant map from G_n to $\mathrm{F}^\times$.

We also write 1_n for the trivial character of G_n, $n \geqslant 1$, and 1 for 1_1.

4.2. The Geometric Lemma

We give here a combinatorial version of Bernstein-Zelevinski's Geometric Lemma [**2**] (see also [**22**, II.2.19]). Let $\alpha = (n_1, \dots, n_r)$ and $\beta = (m_1, \dots, m_s)$ be two compositions of $n \geqslant 1$. For each $i \in \{1, \dots, r\}$, let $\pi_i \in \widehat{\mathrm{G}}_{n_i}$. Let $\mathscr{B}^{\alpha,\beta}$ be the set of all matrices $\mathrm{B} = (b_{i,j})$ whose coefficients are non-negative integers such that:

$$\sum_{j=1}^{s} b_{i,j} = n_i, \quad i \in \{1, \dots, r\}, \quad \sum_{i=1}^{r} b_{i,j} = m_j, \quad j \in \{1, \dots, s\}.$$

Fix $\mathrm{B} \in \mathscr{B}^{\alpha,\beta}$ and write $\alpha_i = (b_{i,1}, \dots, b_{i,s})$ and $\beta_j = (b_{1,j}, \dots, b_{r,j})$ which are compositions of n_i and m_j respectively. For all $i \in \{1, \dots, r\}$, the semi-simplification of $\boldsymbol{r}_{\alpha_i}(\pi_i)$ writes:

$$[\boldsymbol{r}_{\alpha_i}(\pi_i)] = \sum_{k=1}^{r_i} \sigma_{i,1}^{(k)} \otimes \cdots \otimes \sigma_{i,s}^{(k)}, \quad \sigma_{i,j}^{(k)} \in \widehat{\mathrm{G}}_{b_{i,j}}, \quad r_i \geqslant 1.$$

For all $j \in \{1, \dots, s\}$ and all r-tuples $\boldsymbol{k} = (k_1, \dots, k_r)$ with $1 \leqslant k_i \leqslant r_i$, we write:

$$\sigma_j^{(\boldsymbol{k})} = \sigma_{1,j}^{(k_1)} \times \cdots \times \sigma_{r,j}^{(k_r)},$$

which is a representation of G_{m_j}. Then we have:

$$[\boldsymbol{r}_\beta(\pi_1 \times \cdots \times \pi_r)] = \sum_{\mathrm{B}} \sum_{\boldsymbol{k}} \sigma_1^{(\boldsymbol{k})} \otimes \cdots \otimes \sigma_s^{(\boldsymbol{k})}$$

in the Grothendieck group of finite length representations of M_β.

4.3. Cuspidal support

An irreducible representation of G_n with $n \geqslant 1$ is said to be *cuspidal* if it does not embed in any representation of the form (4.1) with $r > 1$.

By [**14**, Theorem 2.1], for any irreducible representation $\pi \in \widehat{\mathrm{G}}_n$ with $n \geqslant 1$, there are positive integers $n_1, \dots, n_r$ and cuspidal irreducible representations $\rho_i \in \widehat{\mathrm{G}}_{n_i}$ with $i \in \{1, \dots, r\}$ such that $n = n_1 + \cdots + n_r$ and π embeds in $\rho_1 \times \cdots \times \rho_r$. Moreover, there is a permutation w of the set $\{1, 2, \dots, r\}$ such that π is a quotient of $\rho_{w(1)} \times \cdots \times \rho_{w(r)}$.

The family $(\rho_1, \dots, \rho_r)$, which depends on the choice of $\sqrt{q}$, is unique up to permutation. Its class up to permutation, denoted $[\rho_1] + \cdots + [\rho_r]$, is called the *cuspidal support* of π.

Proposition 4.1 **([14], Proposition 5.9)**. — *Let π and σ be irreducible representations of G_n and G_m, respectively. Write $[\pi_1]+\cdots+[\pi_r]$ and $[\sigma_1]+\cdots+[\sigma_s]$ for the cuspidal supports of π and σ, respectively. Assume that for all $i \in \{1, \dots, r\}$, $j \in \{1, \dots, s\}$ and $k \in \mathbf{Z}$, the representations $\pi_i \cdot \nu^k$ and σ_j are not isomorphic. Then $\pi \times \sigma$ is irreducible.*

4.4. Three lemmas about irreducibility

The following lemma is a particular case of [**13**, Lemma 6.1], which will be of crucial importance to us. Recall that e is the order (possibly infinite) of q in $\mathrm{R}^\times$.

Lemma 4.2. — *Assume that $e > 1$. Let $n \geqslant 2$, and let $\rho \in \widehat{\mathrm{G}}_{n-1}$, $\chi \in \widehat{\mathrm{G}}_1$. Then the representation $\pi = \rho \times \chi$ possesses a unique irreducible quotient, denoted $\mathbf{Q}(\pi)$, and a unique irreducible subrepresentation, denoted $\mathbf{S}(\pi)$. There is also a similar result for $\tau = \chi \times \rho$ and we have:*

$$\mathbf{Q}(\tau) = \mathbf{S}(\pi), \quad \mathbf{S}(\tau) = \mathbf{Q}(\pi).$$

Note that, by passing to the contragredient, we have:

$$\mathbf{Q}(\rho \times \chi)^* = \mathbf{S}(\rho^* \times \chi^{-1}), \quad \mathbf{S}(\rho \times \chi)^* = \mathbf{Q}(\rho^* \times \chi^{-1}).$$

From this lemma we deduce the following example.

Example 4.3. — Assume that $n \geqslant 2$, and write:

$$\mathrm{V}_n = \nu_{n-1}^{1/2} \times \nu^{(n+1)/2}. \tag{4.2}$$

If $e > 1$, Lemma 4.2 implies that V_n has a unique irreducible quotient, denoted Λ_n. We write:

$$\Lambda_n = \mathbf{Q}(\mathrm{V}_n) = \mathbf{Q}(\nu_{n-1}^{1/2} \times \nu^{(n+1)/2}), \quad \text{for } e > 1. \tag{4.3}$$

When e divides n, then Λ_n is the trivial character (see Proposition 4.10). By taking the contragredient, Λ_n^* is the unique irreducible subrepresentation of $\nu_{n-1}^{-1/2} \times \nu^{-(n+1)/2}$.

The following irreducibility criterion will also be very useful to us.

Lemma 4.4 **([14], Lemme 2.5)**. — *Let π be a smooth representation of G_n, $n \geqslant 2$. Suppose that there are two irreducible representations $\sigma \in \widehat{\mathrm{G}}_a$ and $\tau \in \widehat{\mathrm{G}}_b$ with $a, b \geqslant 1$ and $a + b = n$, such that:*

(1) *π is a subrepresentation of $\sigma \times \tau$ and a quotient of $\tau \times \sigma$;*
(2) *the multiplicity of $\sigma \otimes \tau$ in $\boldsymbol{r}_{(a,b)}(\sigma \times \tau)$ is 1.*

Then the representation π is irreducible.

Finally, we will use the following lemma (which follows from [**14**, Proposition 2.2]).

Lemma 4.5. — *Assume the induced representation* (4.1) *is irreducible. Then, for all permutation w of $\{1, 2, \dots, n\}$, there is an isomorphism $\pi_{w(1)} \times \pi_{w(2)} \times \cdots \times \pi_{w(r)} \simeq \pi_1 \times \pi_2 \times \cdots \times \pi_r$.*

4.5. Classification of $\widehat{\mathrm{G}}_n$ by multisegments

In [**14**] Mínguez and Sécherre give a classification of the union of all $\widehat{\mathrm{G}}_n$'s in terms of multisegments, that generalizes [**27, 19, 23**]. We will need some properties of this classification, that we recall below.

Given two half-integers $a, b \in \frac{1}{2}\mathbf{Z}$, we write:

$$a \equiv b \text{ if } \begin{cases} a - b \in e\mathbf{Z} & \text{if R has positive characteristic,} \\ a = b & \text{otherwise.} \end{cases}$$

We write $\mathbf{N}$ for the set of nonnegative integers.

Definition 4.6. — (1) A *segment* is a pair (a, b) of half-integers such that $b - a \in \mathbf{N}$.

(2) Two segments (a, b) and (c, d) are *equivalent* if $b - a = d - c$ and $a \equiv c$. The equivalence class of (a, b) will be denoted $[a, b]$ (and just $[a]$ if $b = a$).

(3) A *multisegment* is a formal finite sum of classes of segments, that is a element in the free semigroup generated by classes of segments.

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ be two partitions of a given integer n. We say that λ dominates μ, denoted $\lambda \trianglerighteq \mu$, if:

$$\lambda_1 + \cdots + \lambda_k \geqslant \mu_1 + \cdots + \mu_k$$

for all integers $k \geqslant 1$. We write $\lambda \triangleright \mu$ if we have in addition $\lambda \neq \mu$.

Given a nonzero multisegment $\mathfrak{m} = [a_1, b_1] + \cdots + [a_r, b_r]$, write $n_i = b_i - a_i + 1$ for all integer $i \in \{1, \dots, r\}$ and let $\lambda(\mathfrak{m})$ denote the partition associated with $(n_1, n_2, \dots, n_r)$. The *length* of $\mathfrak{m}$ is the sum $n = n_1 + n_2 + \cdots + n_r$.

One of the main results of [**14**] is the construction of a map $\mathfrak{m} \mapsto \mathrm{Z}(\mathfrak{m})$ that associates to any multisegment $\mathfrak{m}$ a class of irreducible representation $\mathrm{Z}(\mathfrak{m})$ with the following properties:

- **P1** If $\mathfrak{m}$ is a segment $[a, b]$ of length $n \geqslant 1$, then $\mathrm{Z}([a, b])$ is the character $\nu_n^{a+(n-1)/2} \in \widehat{\mathrm{G}}_n$.
- **P2** If $\mathfrak{m} = [a_1, b_1] + \cdots + [a_r, b_r]$, then $\mathrm{Z}(\mathfrak{m})$ occurs as a subquotient of the representation $\mathrm{Z}([a_1, b_1]) \times \cdots \times \mathrm{Z}([a_r, b_r])$ with multiplicity 1.
- **P3** If π is an irreducible subquotient of $\mathrm{Z}([a_1, b_1]) \times \cdots \times \mathrm{Z}([a_r, b_r])$, then there exists a unique multisegment $\mathfrak{n} = [c_1, d_1] + \cdots + [c_s, d_s]$ such that $\pi = \mathrm{Z}(\mathfrak{n})$. Moreover, we have $\lambda(\mathfrak{n}) \trianglerighteq \lambda(\mathfrak{m})$ and:
$$\sum_{i=1}^{s}([c_i] + \cdots + [d_i]) = \sum_{i=1}^{r}([a_i] + \cdots + [b_i]).$$
- **P4** If k is a half-integer, then $\mathrm{Z}([a_1 + k, b_1 + k] + \cdots + [a_r + k, b_r + k]) = \mathrm{Z}(\mathfrak{m}) \cdot \nu^k$.
- **P5** The contragredient of $\mathrm{Z}(\mathfrak{m})$ is $\mathrm{Z}(\mathfrak{m}^*)$ with $\mathfrak{m}^* = [-b_1, -a_1] + \cdots + [-b_r, -a_r]$.

We finally have the following definition and result.

Definition 4.7. — Two segments $[a, b]$ and $[c, d]$ are *linked* if $c - a \in \mathbf{Z}$ and at least one of the following two conditions holds:

(1) the length of $[a,b]$ is greater than or equal to that of $[c,d]$, and there exists a half-integer k such that $c \leqslant k \leqslant d$, and either $k \equiv b+1$ or $k \equiv a-1$;
(2) the length of $[c,d]$ is greater than or equal to that of $[a,b]$, and there exists a half-integer k such that $a \leqslant k \leqslant b$, and either $k \equiv d+1$ or $k \equiv c-1$.

***Proposition 4.8* ([14], Théorème 7.26).** — *Let $\Delta_1, \dots, \Delta_r$ be segments. The representation $\mathrm{Z}(\Delta_1) \times \cdots \times \mathrm{Z}(\Delta_r)$ is irreducible if and only if for all $i \neq j$, the segments Δ_i, Δ_j are not linked.*

4.6. Product of two characters

Here are some useful properties of the representation $\mathrm{Z}([a,b])$ for a segment $[a,b]$.

***Proposition 4.9*.** — *Let $[a,b]$ be a segment of length $n \geqslant 2$, and let $k \in \{1,\dots,n-1\}$.*
(1) We have $\boldsymbol{r}_{(k,n-k)}(\mathrm{Z}([a,b])) = \mathrm{Z}([a,a+k-1]) \otimes \mathrm{Z}([a+k,b])$.
(2) We have $\bar{\boldsymbol{r}}_{(k,n-k)}(\mathrm{Z}([a,b])) = \mathrm{Z}([a+n-k,b]) \otimes \mathrm{Z}([a,a+n-k-1])$ where $\bar{\boldsymbol{r}}_{(k,n-k)}$ denotes the Jacquet functor associated to $\mathrm{M}_{(k,n-k)}$ and the parabolic subgroup opposite to $\mathrm{P}_{(k,n-k)}$.
(3) Assume that $e > 1$. Then $\mathrm{Z}([a,b-1]) \times \nu^b$ has a unique irreducible subrepresentation and $\mathrm{Z}([a+1,b]) \times \nu^a$ has a unique irreducible quotient, both isomorphic to $\mathrm{Z}([a,b])$.

Proof. — See [**14**], Propositions 7.16 and 7.17. □

***Proposition 4.10*.** — *Assume $e > 1$. Let $a \leqslant b$ be integers and write $\pi(a,b) = \mathrm{Z}([a,b]) \times 1$.*
(1) If $a \not\equiv 1$ and $b \not\equiv -1$, then $\pi(a,b)$ is irreducible.
(2) If $a \equiv 1$ and $b \not\equiv -1$, then $\pi(a,b)$ has length 2 and we have an exact sequence:

$$0 \to \mathrm{Z}([a,b]+[0]) \to \pi(a,b) \to \mathrm{Z}([a-1,b]) \to 0.$$

(3) If $a \not\equiv 1$ and $b \equiv -1$, then $\pi(a,b)$ has length 2 and we have an exact sequence:

$$0 \to \mathrm{Z}([a,b+1]) \to \pi(a,b) \to \mathrm{Z}([a,b]+[0]) \to 0.$$

(4) If $a \equiv 1$ and $b \equiv -1$, then $\pi(a,b)$ has length 3 with irreducible subquotients $\mathrm{Z}([a-1,b])$, $\mathrm{Z}([a,b+1])$ and $\mathrm{Z}([a,b]+[0])$.

Proof. — Case 1 follows from Proposition 4.8. Moreover, the representation $\mathrm{Z}([a,b]+[0])$ always occurs as a subquotient with multiplicity 1 and the other irreducible subquotients of $\pi(a,b)$ are of the form $\mathrm{Z}(\mathfrak{n})$ with $\lambda(\mathfrak{n}) \rhd (n-1,1)$, where $n = b-a+2$. Therefore we have $\lambda(\mathfrak{n}) = (n)$, which implies that $\mathfrak{n}$ is a segment. Moreover, such an $\mathfrak{n}$ must be of the form $[a,b+1]$ with $b \equiv -1$ or $[a-1,b]$ with $a \equiv 1$.

Assume that $a \not\equiv 1$ and $b \equiv -1$. By the geometric lemma, the Jacquet module $\boldsymbol{r}_{(n-1,1)}(\pi(a,b))$ is made of the subquotients $\mathrm{Z}([a,b]) \otimes 1$ and $\pi(a,b-1) \otimes \nu^b$, and both are irreducible. Thus the representation $\pi(a,b)$ has length $\leqslant 2$. But Proposition 4.9 shows that $\mathrm{Z}([a,b+1])$ occurs as a subrepresentation of $\pi(a,b)$. The result follows.

The case where $a \equiv 1$ and $b \not\equiv -1$ is treated in a similar way. Thus it remains to study the case where $a \equiv 1$ and $b \equiv -1$. In this case, $\pi(a,b-1)$ has length 2, thus $\pi(a,b)$ has length $\leqslant 3$. By Proposition 4.9 we see that the length is actually 3 and we get the expected result. □

***Example 4.11*.** — Assume that $n \geqslant 2$ and write:

$$\Pi_n = \mathrm{Z}\left(\left[-\frac{n-3}{2}, \frac{n-1}{2}\right] + \left[\frac{n+1}{2}\right]\right). \tag{4.4}$$

Assume $e > 1$ and look at Example 4.3 for the definition of Λ_n. Then:

$$\Lambda_n = \begin{cases} \Pi_n & \text{if } e \text{ does not divide } n, \\ 1_n & \text{if } e \text{ divides } n. \end{cases} \tag{4.5}$$

We also get:

$$\Lambda_n^* = \begin{cases} \Pi_n^* = \mathrm{Z}([-\frac{n-1}{2}, \frac{n-3}{2}] + [-\frac{n+1}{2}]) & \text{if } e \text{ does not divide } n, \\ 1_n & \text{if } e \text{ divides } n. \end{cases}$$

If we want to go further, we need more properties of the representation $\mathrm{Z}(\mathfrak{m})$ for a multisegment $\mathfrak{m}$. Given $\chi_1, \chi_2 \in \widehat{\mathrm{G}}_1$, we write $\mathrm{St}(\chi_1, \chi_2)$ for the unique nondegenerate irreducible subquotient of $\chi_1 \times \chi_2$ (see [**14**, §8]). If St_2 is the Steinberg representation of G_2 as in Paragraph 3.2, then:

$$\mathrm{St}(\chi_1, \chi_2) = \begin{cases} \chi_1 \times \chi_2 & \text{if } \chi_1 \times \chi_2 \text{ is irreducible}, \\ \mathrm{St}_2 \cdot \chi_1 \nu^{1/2} & \text{if } \chi_2 = \chi_1 \nu, \\ \mathrm{St}_2 \cdot \chi_2 \nu^{1/2} & \text{if } \chi_1 = \chi_2 \nu. \end{cases}$$

Note that we have $\mathrm{St}(\chi_2, \chi_1) = \mathrm{St}(\chi_1, \chi_2)$. The following proposition follows from [**13**, §3.3.2].

Proposition 4.12. — *Let $\mathfrak{m}$ be a multisegment of length n and of the form $[a, b] + [c, d]$. Assume that $b - a \geqslant d - c$. Write $k = d - c + 1$ and:*

$$\begin{aligned} \mu(\mathfrak{m}) &= (1, \dots, 1, 2, \dots, 2) \text{ with } 1 \text{ occurring } n - 2k \text{ times and } 2 \text{ occurring } k \text{ times}, \\ \mathrm{St}(\mathfrak{m}) &= \nu^a \otimes \cdots \otimes \nu^{a+n-2k-1} \otimes \mathrm{St}(\nu^{a+n-2k}, \nu^c) \otimes \cdots \otimes \mathrm{St}(\nu^b, \nu^d). \end{aligned}$$

Then $\mathrm{Z}(\mathfrak{m})$ has the following property:

P6 $\mathrm{Z}(\mathfrak{m})$ *is the unique irreducible subquotient of* $\mathrm{Z}([a, b]) \times \mathrm{Z}([c, d])$ *whose Jacquet module with respect to* $\boldsymbol{r}_{\mu(\mathfrak{m})}$ *contains* $\mathrm{St}(\mathfrak{m})$ *as a subquotient.*

Proposition 4.13. — *Let $a, b \in \mathbf{Z}$ with $a \leqslant b$ and write $\pi(a, b) = \mathrm{Z}([a, b]) \times \mathrm{Z}([0, 1])$. Assume that $e > 1$.*

(1) $\mathrm{Z}([a, b] + [0, 1])$ *occurs as a subquotient of* $\pi(a, b)$ *with multiplicity* 1.
(2) *If* $b \equiv 0$, *then* $\mathrm{Z}([a, b + 1] + [0])$ *occurs as a subquotient of* $\pi(a, b)$ *with multiplicity* 1.
(3) *If* $a \equiv 1$, *then* $\mathrm{Z}([a - 1, b] + [1])$ *occurs as a subquotient of* $\pi(a, b)$ *with multiplicity* 1.
(4) *If* $b \equiv -1$, *then* $\mathrm{Z}([a, b + 2])$ *occurs as a subquotient of* $\pi(a, b)$.
(5) *If* $a \equiv 2$, *then* $\mathrm{Z}([a - 2, b])$ *occurs as a subquotient of* $\pi(a, b)$.
(6) *If* $b \equiv 0$ *and* $a \equiv 1$, *then* $\mathrm{Z}([a - 1, b + 1])$ *occurs as a subquotient of* $\pi(a, b)$.

Any irreducible subquotient of $\pi(a, b)$ is one of the representations occuring in Cases 1 to 6. Moreover, if $e > 2$, the multiplicities in Cases 4, 5 and 6 are equal to 1.

Proof. — Case 1 follows from **P2**. Write $n = b - a + 3$. The other irreducible subquotients of $\pi(a, b)$ are of the form $\mathrm{Z}(\mathfrak{n})$ with $\lambda(\mathfrak{n}) \rhd (n - 2, 2)$. Thus we have $\lambda(\mathfrak{n}) = (n - 1, 1)$ or $\lambda(\mathfrak{n}) = (n)$. If $b \equiv 0$ (resp. $a \equiv 1$) then $\mathrm{Z}([a, b + 1]) \times 1$ (resp. $\mathrm{Z}([a - 1, b]) \times \nu$) is a subquotient of $\pi(a, b)$ by [**14**, Lemme 7.34]. It follows from Proposition 4.10 that the representations in Cases 2, 3 and 6 occur in $\pi(a, b)$. Cases 4 and 5 are treated similarly. We now show that these are the only possible subquotients of $\pi(a, b)$ and they appear with the specified multiplicity.

Assume first that $\mathfrak{n} = [c, d] + [h]$ with $d - c + 1 = n - 1$. Then:

$$\begin{aligned} \mu(\mathfrak{n}) &= (1, \dots, 1, 2), \\ \mathrm{St}(\mathfrak{n}) &= \nu^c \otimes \nu^{c+1} \otimes \cdots \otimes \nu^{d-1} \otimes \mathrm{St}(\nu^d, \nu^h). \end{aligned}$$

By using the geometric lemma, the semi-simplification of $\boldsymbol{r}_{(n-2,2)}(\pi(a,b))$ is equal to:

$$\mathrm{Z}([a,b])\otimes\mathrm{Z}([0,1])+(\mathrm{Z}([a,b-1])\times 1)\otimes(\nu^b\times\nu)+\pi(a,b-2)\otimes\mathrm{Z}([b-1,b]).$$

If $\mathrm{Z}(\mathfrak{n})$ occurs as a subquotient of $\pi(a,b)$, then $\mathrm{St}(\mathfrak{n})$ occurs in $\boldsymbol{r}_{\mu(\mathfrak{n})}((\mathrm{Z}([a,b-1])\times 1)\otimes(\nu^b\times\nu))$ and $\mathrm{St}(\nu^d,\nu^h)$ occurs in $\nu^b\times\nu$ with multiplicity 1, which implies that $\mathrm{St}(\nu^d,\nu^h)=\mathrm{St}(\nu^b,\nu)$ and that $\mathrm{Z}(\mathfrak{n})$ occurs in $\pi(a,b)$ with multiplicity 1. By the geometric lemma, we get:

$$\boldsymbol{r}_{(1,\dots,1)}(\mathrm{Z}([a,b-1])\times 1)=\sum_{k=0}^{n-3}\nu^a\otimes\cdots\otimes\nu^{a+k-1}\otimes 1\otimes\nu^{a+k}\otimes\cdots\otimes\nu^{b-1}.$$

Thus there is a $k\in\{0,\dots,n\}$ such that:

$$\nu^c\otimes\nu^{c+1}\otimes\cdots\otimes\nu^{d-1}=\nu^a\otimes\cdots\otimes\nu^{a+k-1}\otimes 1\otimes\nu^{a+k}\otimes\cdots\otimes\nu^{b-1}.$$

Since $e>1$, comparing the exponents in the left hand side and the right hand side shows that k must be either 0 or $n-3=b-a$. If $k=0$, then:

$$\nu^c\otimes\nu^{c+1}\otimes\cdots\otimes\nu^{d-1}=1\otimes\nu^a\otimes\cdots\otimes\nu^{b-1}.$$

Thus we have $c\equiv 0$, $a\equiv 1$, $d\equiv b$ and $h\equiv 1$. It follows that $\mathfrak{n}=[a-1,b]+[1]$. If $k=n-3$, then:

$$\nu^c\otimes\nu^{c+1}\otimes\cdots\otimes\nu^{d-1}=\nu^a\otimes\cdots\otimes\nu^{b-1}\otimes 1.$$

Thus we have $c\equiv a$, $b\equiv 0$, $d\equiv 1$ and $h\equiv 0$. It follows that $\mathfrak{n}=[a,b+1]+[0]$.

Assume now that $\mathfrak{n}=[c,d]$ is a segment. Thus:

$$\begin{aligned}\mu(\mathfrak{n}) &= (1,\dots,1),\\ \mathrm{St}(\mathfrak{n}) &= \nu^c\otimes\nu^{c+1}\otimes\cdots\otimes\nu^d.\end{aligned}$$

By using the geometric lemma, we get:

$$\boldsymbol{r}_{\mu(\mathfrak{n})}(\pi(a,b))=\sum_{0\leqslant r\leqslant s\leqslant n}\nu^a\otimes\cdots\otimes\nu^{a+r-1}\otimes 1\otimes\nu^{a+r}\otimes\cdots\otimes\nu^{a+s-1}\otimes\nu\otimes\nu^{a+s}\otimes\cdots\otimes\nu^b.$$

If $\mathrm{Z}(\mathfrak{n})$ occurs as a subquotient of $\pi(a,b)$, there are integers $r\leqslant s$ in $\{0,\dots,n\}$ such that:

$$\nu^c\otimes\cdots\otimes\nu^d=\nu^a\otimes\cdots\otimes\nu^{a+r-1}\otimes 1\otimes\nu^{a+r}\otimes\cdots\otimes\nu^{a+s-1}\otimes\nu\otimes\nu^{a+s}\otimes\cdots\otimes\nu^b.$$

If $e>2$, comparing the exponents in the left hand side and the right hand side shows that the only possible values for r,s are:

(1) $r=s=0$ (thus $a\equiv 2$);
(2) $r=s=n$ (thus $b\equiv -1$);
(3) $r=0$ and $s=n$ (thus $a\equiv 1$ and $b\equiv 0$).

In all these cases, $\mathrm{St}(\mathfrak{n})$ occurs with multiplicity 1.

If $e=2$, there are more possible values for r,s (the condition is that $s-r$ is even) and $\mathrm{St}(\mathfrak{n})$ may occur with multiplicity greater than 1. □

4.7. Derivatives

By [**22**, III.1], there is a theory of derivatives for mod ℓ representations of G_n, $n \geqslant 1$ just as in the complex case. Given a smooth representation π of G_n, $n \geqslant 1$ and an integer $k \in \{0, \dots, n\}$, we will write $\pi^{(k)}$ for its k^{th} derivative, which is a smooth representation of G_{n-k} (where G_0 stands for the trivial group in the case $k = n$.)

The k^{th} derivative functor is exact from the category of smooth ℓ-modular representations of G_n to that of smooth ℓ-modular representations of G_{n-k}, for all $k \in \{0, \dots, n\}$. It is compatible with twisting by a character, that is, we have $(\pi \cdot \chi)^{(k)} = \pi^{(k)} \cdot \chi$ for any representation π of G_n, any character $\chi \in \widehat{\mathrm{G}}_1$ and any $k \in \{0, \dots, n\}$.

Recall that $[\pi]$ denotes the semi-simplification of a finite length representation π.

Lemma 4.14. — (1) *Given a cuspidal irreducible representation ρ of G_n, its k^{th} derivative is zero for all $k \in \{1, \dots, n-1\}$, and we have $\rho^{(n)} = 1$ for $k = n$.*

(2) *Given a segment $[a, b]$, the first derivative of $\mathrm{Z}([a, b])$ is $\mathrm{Z}([a, b-1])$, and its k^{th} derivative is zero for all $k \in \{2, \dots, n\}$.*

(3) *Let π, σ be finite length representations of $\mathrm{G}_n, \mathrm{G}_m$ respectively, with $m \geqslant n \geqslant 1$. Then :*

$$[(\pi \times \sigma)^{(k)}] = [\pi \times \sigma^{(k)}] + [\pi^{(1)} \times \sigma^{(k-1)}] + \cdots + [\pi^{(i)} \times \sigma^{(k-i)}]$$

for all $k \in \{0, \dots, n+m\}$, where $i = \min(n, k)$.

Proof. — Points (1) and (2) follows from V.9.1 (a) and (b) in [**23**]. For (3), see [**22**, III.1.10]. □

5. On the $e = 1$ case

In this section, we assume that $e = 1$ and $n \geqslant 2$. Write $\mathrm{K}_n = \mathrm{GL}_n(\mathcal{O})$ and let $\mathrm{K}_n(1)$ be the normal subgroup of K_n made of all matrices that are congruent to 1 mod $\mathfrak{p}$. Both are compact open subgroups of G_n, and the quotient $\mathrm{K}_n/\mathrm{K}_n(1)$ is canonically isomorphic to the finite group $\mathrm{GL}_n(q)$ of $n \times n$ invertible matrices with entries in the residue field of $\mathcal{O}$.

Given a smooth representation (π, W) of G_n, write $\overline{\mathrm{W}}$ for the space of $\mathrm{K}_n(1)$-fixed vectors of W and write $\overline{\pi}$ for the representation of $\mathrm{GL}_n(q)$ on $\overline{\mathrm{W}}$.

This defines an exact functor from the category of smooth R-representations of G_n to that of R-representations of $\mathrm{GL}_n(q)$.

We have defined two representations V_n and Π_n in (4.2) and (4.4). Note that $\mathrm{V}_n = \mathcal{C}_c^\infty(\mathrm{X}, \mathrm{R})$ with $\mathrm{X} = \mathrm{P}_{(n-1,1)} \backslash \mathrm{G}_n$. Its contains Π_n as a subquotient with multiplicity one, 1_n with some multiplicity and no other irreducible subquotient. It is a selfdual representation of G_n.

Thanks to the Iwasawa decomposition $\mathrm{G}_n = \mathrm{P}_{(n-1,1)}\mathrm{K}_n$, the restriction of V_n to K_n is $\mathrm{W}_n = \mathcal{C}_c^\infty(\mathrm{Y}, \mathrm{R})$ with $\mathrm{Y} = (\mathrm{K}_n \cap \mathrm{P}_{(n-1,1)}) \backslash \mathrm{K}_n$. Therefore we have:

$$\overline{\mathrm{V}}_n = \mathcal{C}_c^\infty(\mathrm{Y}/\mathrm{K}_n(1), \mathrm{R}),$$

which identifies with the space of R-valued functions on $\overline{\mathrm{X}} = \mathrm{P}_{(n-1,1)}(q) \backslash \mathrm{GL}_n(q)$, where we write $\mathrm{P}_{(n-1,1)}(q)$ for the standard maximal parabolic subgroup of $\mathrm{GL}_n(q)$ corresponding to $(n-1, 1)$.

Lemma 5.1. — *For $n \geqslant 2$, there exists a unique irreducible representation π_n of $\mathrm{GL}_n(q)$ having the following properties:*

(1) *If ℓ does not divide n, then $\overline{\mathrm{V}}_n$ is semisimple of length 2, with irreducible subquotients $\overline{1}_n$ and π_n.*

(2) *If ℓ divides n, then $\overline{\mathrm{V}}_n$ is indecomposable of length* 3*, with irreducible subquotients $\bar{1}_n$ (with multiplicity* 2*) and π_n.*

Proof. — Note that $\bar{1}_n$ occurs as a subrepresentation of $\overline{\mathrm{V}}_n$ (the space of R-valued constant functions on $\overline{\mathrm{X}}$). Write ψ for the $\mathrm{GL}_n(q)$-invariant linear form on $\overline{\mathrm{V}}_n$ that associates to a function the sum of its values on $\overline{\mathrm{X}}$. The set $\overline{\mathrm{X}}$ has cardinality:

$$(\mathrm{GL}_n(q) : \mathrm{P}_{(n-1,1)}(q)) = \frac{q^n-1}{q-1} = 1 + q + \cdots + q^{n-1}$$

which is 0 in R if and only if ℓ divides n. Thus the constant functions belong to the kernel of ψ if and only if ℓ divides n. According to [**9**, 11], we have the following properties:

(1) The kernel S_n of ψ (denoted $\mathrm{S}_{(n-1,1)}$ in [**9**], whereas $\overline{\mathrm{V}}_n$ is denoted $\mathrm{M}_{(n-1,1)}$) has a unique irreducible quotient π_n.

(2) The semi-simplification of $\overline{\mathrm{V}}_n$ contains π_n with multiplicity 1 and $\bar{1}_n$ with some multiplicity $\geqslant 1$, and no other irreducible subquotient.

By [**9**, 20.7], the multiplicity of $\bar{1}_n$ in $\overline{\mathrm{V}}_n$ is 1 if ℓ does not divide n, and 2 otherwise. It remains to prove that $\overline{\mathrm{V}}_n$ has the expected structure.

We first assume that ℓ does not divide n. Since $\bar{1}_n$ occurs as a subrepresentation of $\overline{\mathrm{V}}_n$, π_n must be a quotient of $\overline{\mathrm{V}}_n$. Since $\overline{\mathrm{V}}_n$ is selfdual, it follows that π_n is selfdual, thus it also occurs as a subrepresentation of $\overline{\mathrm{V}}_n$. We thus have two nonzero maps $\pi_n \to \overline{\mathrm{V}}_n$ and $\overline{\mathrm{V}}_n \to \pi_n$, whose composition is nonzero (or else it would contradict the fact that π_n occurs with multiplicity 1). Therefore $\overline{\mathrm{V}}_n$ is semisimple.

Assume now that ℓ divides n. By [**9**], the representation S_n is indecomposable (it has length 2 and a unique irreducible quotient). Since $\overline{\mathrm{V}}_n$ is selfdual, it implies that $\overline{\mathrm{V}}_n$ is indecomposable. □

Proposition 5.2. — (1) *The representation $\overline{\Pi}_n$ is irreducible and isomorphic to π_n.*

(2) *If ℓ does not divide n, the representation V_n is semisimple of length* 2*.*

(3) *If ℓ divides n, the representation V_n is indecomposable of length* 3*, with irreducible subquotients 1_n (with multiplicity* 2*) and Π_n.*

Proof. — By [**22**], II.5.8 and II.5.12, all irreducible subquotients of V_n have level 0, thus they are not killed by the functor $\pi \mapsto \overline{\pi}$.

We first assume that ℓ does not divide n. By Lemma 5.1, the representation V_n has length 2, with irreducible subquotients Π_n and 1_n, thus $\overline{\Pi}_n$ must be irreducible and isomorphic to π_n. The same argument as in the proof of Lemma 5.1 shows that V_n is semisimple.

Assume now that ℓ divides n. By Lemma 5.1 the representation V_n has length $\leqslant 3$. Assume it has length 2. Then the argument of the proof of Lemma 5.1 implies that $\mathrm{V}_n = 1_n \oplus \Pi_n$. Thus the one-dimensional space $\mathrm{Hom}_{\mathrm{G}_n}(\mathrm{V}_n, 1_n)$ is generated by a linear form λ which is nonzero on the subspace of constant functions. Since $\mathrm{K}_n(1)$ is a pro-p-group, $\mathrm{K}_n(1)$-invariant and $\mathrm{K}_n(1)$-co-invariant vectors of V_n are canonically identified. The K_n-invariant linear form λ thus induces a $\mathrm{GL}_n(q)$-invariant linear form on $\overline{\mathrm{V}}_n$, which is equal to ψ upto a nonzero scalar. But ψ is zero on constant functions, which contradicts the fact that λ is nonzero. This gives us a contradiction, and thus V_n has length 3. Now since $\overline{\mathrm{V}}_n$ is indecomposable, it follows that V_n is indecomposable. We also get that $\overline{\Pi}_n$ must be irreducible and isomorphic to π_n. □

Definition 5.3. — Assume $e = 1$ and let $n \geqslant 2$. In parallel with Example 4.11, we define:

$$\Lambda_n = \begin{cases} \Pi_n & \text{if } \ell \text{ does not divide } n, \\ 1_n & \text{if } \ell \text{ divides } n. \end{cases}$$

In conclusion, if we summarize Example 4.11 and Definition 5.3, we get the following definition of Λ_n.

Definition 5.4. — Assume e is arbitrary, and recall that f is the quantum characteristic (see Paragraph 3.2). For $n \geqslant 2$, we define:

$$\Lambda_n = \begin{cases} \Pi_n = \mathrm{Z}([-\frac{n-3}{2}, \frac{n-1}{2}] + [\frac{n+1}{2}]) & \text{if } f \text{ does not divide } n, \\ 1_n & \text{if } f \text{ divides } n. \end{cases} \tag{5.1}$$

Thanks to Example 4.11, note that we also have:

$$\Lambda_n^* = \begin{cases} \Pi_n^* = \mathrm{Z}([-\frac{n-1}{2}, \frac{n-3}{2}] + [-\frac{n+1}{2}]) & \text{if } f \text{ does not divide } n, \\ 1_n & \text{if } f \text{ divides } n. \end{cases} \tag{5.2}$$

If we look at Proposition 5.2, we also have the following property (for arbitrary $e \geqslant 1$).

Remark 5.5. — For $n \geqslant 2$, if f does not divide n, then Λ_n is an irreducible quotient of V_n.

6. Computing the derivatives of Λ_n and Π_n

In this section, we assume that e is arbitrary. Remind (see (4.2), (4.4) and (5.1)) that we have defined representations V_n, Π_n and Λ_n for all $n \geqslant 2$. By Propositions 4.10 and 5.2, we have:

$$[\mathrm{V}_n] = \begin{cases} \Pi_n + \nu_n & \text{if } f \text{ does not divide } n, \\ \Pi_n + \nu_n + 1_n & \text{if } f \text{ divides } n, \end{cases} \tag{6.1}$$

in the Grothendieck group of finite length representations of G_n. Let us compute the derivatives of Π_n.

Lemma 6.1. — *Suppose that $n \geqslant 2$.*

(1) *If $f = n = 2$, the derivative $\Pi_2^{(1)}$ is zero.*
(2) *Otherwise we have:*

$$\Pi_n^{(1)} = \begin{cases} 1_{n-2} \times \nu^{(n+1)/2} & \textit{if } f \textit{ does not divide } n, \\ \Lambda_{n-1}^* \cdot \nu^{1/2} & \textit{if } f \textit{ divides } n. \end{cases}$$

(3) *We have $\Pi_n^{(2)} = 1_{n-2}$ and $\Pi_n^{(k)}$ is zero for all $k \geqslant 3$.*

Proof. — By Leibniz's rule (see Lemma 4.14(3)), we have:

$$[\mathrm{V}_n^{(1)}] = [1_{n-2} \times \nu^{(n+1)/2}] + \nu_{n-1}^{1/2}$$

in the Grothendieck group of finite length representations of G_{n-1}. Since the k^{th} derivative of a character is zero for $k \geqslant 2$, we have $\mathrm{V}_n^{(2)} = 1_{n-2}$ and $\mathrm{V}_n^{(k)}$ is zero for all $k \geqslant 3$. The k^{th} derivative functors being exact, the expected result follows from (6.1) together with Propositions 4.10 and 5.2. □

Corollary 6.2. — *Suppose that $n \geqslant 2$.*

(1) *We have:*

$$\Lambda_n^{(1)} = \begin{cases} 1_{n-2} \times \nu^{(n+1)/2} & \text{if } f \text{ does not divide } n, \\ \nu_{n-1}^{-1/2} & \text{if } f \text{ divides } n. \end{cases}$$

(2) *The second derivative* $\Lambda_n^{(2)}$ *is equal to* 1_{n-2} *if* f *does not divide* n*, and is zero otherwise.*

(3) *The* k^{th} *derivative* $\Lambda_n^{(k)}$ *is zero for all* $k \geqslant 3$*.*

Remark 6.3. — Since $\Pi_n^* = \Pi_n \cdot \nu^{-1}$ by Properties **P4** and **P5**, we get the derivatives of Π_n^* and Λ_n^* from Lemma 6.1 and Corollary 6.2.

Example 6.4. — (1) We have $\mathrm{St}_2 = \mathrm{Z}([-1/2] + [1/2]) = \Pi_2 \cdot \nu^{-1}$. If $f = 2$, the representation St_2 is cuspidal thus its first derivative is zero. Otherwise, we have $(\mathrm{St}_2)^{(1)} = \nu^{1/2}$.

(2) Let St_3 denote the nondegenerate irreducible subquotient of $\nu^{-1} \times 1 \times \nu$, that is:

$$\mathrm{St}_3 = \mathrm{Z}([-1] + [0] + [1])$$

(see [**14**, §8]). If $f = 3$, then St_3 is cuspidal ([**14**, §6]) thus its first and second derivatives are zero. If $f \neq 3$, then:

$$[\nu^{-1} \times 1 \times \nu] = 1_3 + \Lambda_3 \cdot \nu^{-1} + (\Lambda_3)^* \cdot \nu + \mathrm{St}_3$$

in the Grothendieck group of finite length representations of G_3. We thus get $(\mathrm{St}_3)^{(1)} = \mathrm{St}_2 \cdot \nu^{1/2}$ and $(\mathrm{St}_3)^{(2)} = \nu$.

7. A modular version of Badulescu-Lapid-Mínguez's juxtaposition criterion

In Paragraph 4.5 we have defined $\mathrm{Z}(\Delta)$ for Δ a segment. In [**14**] an irreducible representation $\mathrm{L}(\Delta)$ is also introduced. We will need it only for segments of length $\leqslant 2$.

Definition 7.1. — Let a be a half-integer. Then $\mathrm{L}([a]) = \mathrm{Z}([a]) = \nu^a$ and:

$$\mathrm{L}([a, a+1]) = \begin{cases} \mathbf{Q}(\nu^a \times \nu^{a+1}) & \text{if } e > 1, \\ \Lambda_2 \cdot \nu^{a+1/2} & \text{if } e = 1. \end{cases}$$

Remark 7.2. — Note that we have $\boldsymbol{r}_{(1,1)}(\mathrm{L}([a, a+1])) = \nu^{a+1} \otimes \nu^a$ for all $a \in \frac{1}{2}\mathbf{Z}$.

If we write St_2 for the Steinberg representation of G_2 as in Paragraph 3.2, then we have:

$$\mathrm{L}([a, a+1]) = \begin{cases} \mathrm{St}_2 \cdot \nu^{a+1/2} & \text{if } f \neq 2, \\ \nu^{a-1/2} & \text{if } f = 2. \end{cases}$$

Note that $\Lambda_2 = \mathrm{St}_2 \cdot \nu$ if $f \neq 2$.

Lemma 7.3 **([14], Théorème 7.26)**. — *Let* Δ*,* Δ' *be two segments of length* $\leqslant 2$*. Then the representation* $\mathrm{L}(\Delta) \times \mathrm{L}(\Delta')$ *is irreducible if and only if* Δ *and* Δ' *are not linked.*

Following [**3**, Définition 2.1], say that two segments $[a, b]$ and $[c, d]$ are *juxtaposed* if we have $c \equiv b + 1$ or $a \equiv d + 1$ (see the notation of Paragraph 4.5).

Proposition 7.4. — *Assume that* $e > 2$*. Let* Δ, Δ' *be two segments, with* Δ' *of length* 2*. Then* $\mathrm{Z}(\Delta) \times \mathrm{L}(\Delta')$ *is reducible if and only if* Δ *and* Δ' *are juxtaposed.*

Remark 7.5. — If $e \leqslant 2$, we have $\mathrm{L}([a, a+1]) = \mathrm{Z}([a-1, a])$ for any half-integer a. It follows from Proposition 4.8 that $\mathrm{Z}(\Delta) \times \mathrm{L}(\Delta')$ is always reducible when $e \leqslant 2$.

Proof. — By twisting by a character, we may and will assume that $\Delta' = [0,1]$. We first assume that Δ and $[0,1]$ are juxtaposed. We thus have $\Delta = [a,b]$ with $a \leqslant b$ integers such that $b \equiv -1$ or $a \equiv 2$. Let us prove that $\pi = \mathrm{Z}(\Delta) \times \mathrm{L}([0,1])$ is reducible.

First note that π is a subquotient of $\xi = \mathrm{Z}([a,b]) \times 1 \times \nu$. Since $e > 2$ the representation $1 \times \nu$ has length 2, with irreducible subquotients $\mathrm{Z}([0,1])$ and $\mathrm{L}([0,1])$. By **P2** and Proposition 4.13, the irreducible representation $\mathrm{Z}([a,b]+[0]+[1])$ occurs in ξ but not in $\mathrm{Z}([a,b]) \times \mathrm{Z}([0,1])$, thus it occurs in π.

Let us assume that $a \not\equiv 2$ and $b \equiv -1$. Since $\mathrm{L}([0,1])$ is the unique irreducible subrepresentation of $\nu \times 1$, it follows that π embeds in $\xi' = \mathrm{Z}([a,b]) \times \nu \times 1$. Since $\mathrm{Z}([a,b]) \times \nu$ is irreducible, ξ' is isomorphic to $\nu \times \mathrm{Z}([a,b]) \times 1$ and has a unique irreducible subrepresentation by Lemma 4.2. By Proposition 4.9 (3), the unique irreducible subrepresentation of $\mathrm{Z}([a,b]) \times 1$ is $\mathrm{Z}([a,b+1])$. Thus we have:

$$\mathbf{S}(\xi') = \mathbf{S}(\nu \times \mathrm{Z}([a,b+1])) = \mathrm{Z}([a,b+1]+[1])$$

by Proposition 4.10. Since π embeds in ξ', it follows that π contains $\mathrm{Z}([a,b+1]+[1])$. Since it also contains $\mathrm{Z}([a,b]+[0]+[1])$, it cannot be irreducible.

The case where $a \equiv 2$ and $b \not\equiv -1$ is similar, using ξ instead of ξ'.

It remains to treat the case where $a \equiv 2$ and $b \equiv -1$. In that case, it follows from Proposition 4.13 that $\mathrm{Z}([a,b]) \times \mathrm{Z}([0,1])$ has length 3, with irreducible subquotients:

$$\mathrm{Z}([a,b]+[0,1]), \quad \mathrm{Z}([a,b+2]), \quad \mathrm{Z}([a-2,b]).$$

But ξ also contains $\mathrm{Z}([a,b]+[0]+[1])$ and $\mathrm{Z}([a-1,b+1])$, thus π has length at least 2. Thus, in any case, π is reducible when $[a,b]$ and $[0,1]$ are juxtaposed.

We now have to prove that π is irreducible when Δ and $[0,1]$ are not juxtaposed. Let us write $\Delta = [a,b]$ with $a \leqslant b$ and $2a \in \mathbf{Z}$. If $a \notin \mathbf{Z}$, then $\mathrm{Z}(\Delta) \times \mathrm{L}([0,1])$ is irreducible by Proposition 5.9 of [**14**]. We thus may assume that a,b are integers such that $b \not\equiv -1$ and $a \not\equiv 2$. The proof is by induction on $n = b-a+1$.

If $n = 1$ then $\pi = \nu^a \times \mathrm{L}([0,1])$ and the result follows from Lemma 7.3 since the segments $[a]$ and $[0,1]$ are not linked.

Assume now that $n \geqslant 2$. Our goal is to find irreducible representations σ, τ, of degree u, v respectively, such that π occurs as a subrepresentation of $\sigma \times \tau$ and as a quotient of $\tau \times \sigma$, and such that $\sigma \otimes \tau$ occurs with multiplicity 1 in $\boldsymbol{r}_{(u,v)}(\sigma \times \tau)$. We will distinguish the following cases:

(1) $a \not\equiv -1, 1$
(2) $a \equiv -1$
(3) $a \equiv 1$ and $b \not\equiv 0, 2$
(4) $a \equiv 1$ and $b \equiv 0, 2$ and $e > 3$
(5) $a \equiv 1$ and $b \equiv 0$ and $e = 3$

In Case 1, since $a \not\equiv 1$ and thanks to the inductive hypothesis, π embeds in:

$$\nu^a \times \mathrm{Z}([a+1,b]) \times \mathrm{L}([0,1]) \simeq \nu^a \times \mathrm{L}([0,1]) \times \mathrm{Z}([a+1,b]) \tag{7.1}$$

and $\nu^a \times \mathrm{L}([0,1])$ is irreducible because $a \not\equiv -1$. Since $\mathrm{Z}([a,b])$ is a quotient of $\mathrm{Z}([a+1,b]) \times \nu^a$, we can choose $\sigma = \nu^a \times \mathrm{L}([0,1])$ and $\tau = \mathrm{Z}([a+1,b])$. We compute the multiplicity of $\sigma \otimes \tau$ in $\boldsymbol{r}_{(3,n-1)}(\sigma \times \tau)$ by applying the geometric lemma. For this multiplicity to be 1, it is enough to prove that σ does not occur as a subquotient of the following representations:

(1.1) $\nu^a \times 1 \times \nu^{a+1}$;
(1.2) $\mathrm{L}([0,1]) \times \nu^{a+1}$;

(1.3) $\nu^a \times \mathrm{Z}([a+1, a+2])$;
(1.4) $\nu \times \mathrm{Z}([a+1, a+2])$;
(1.5) $\mathrm{Z}([a+1, a+3])$.
This follows from [**14**, Théorème 8.16].

In Case 2, Equation (7.1) in addition with the fact that $\mathrm{L}([0,1])$ embeds in $\nu \times 1$ implies that π is a subrepresentation of:

$$\nu^{-1} \times \nu \times 1 \times \mathrm{Z}([a+1, b]).$$

But π is also a quotient of:

$$\mathrm{Z}([a,b]) \times 1 \times \nu \simeq 1 \times \mathrm{Z}([a,b]) \times \nu$$

which itself is a quotient of the representation $1 \times \mathrm{Z}([a+1,b]) \times \nu^{-1} \times \nu$. We thus can choose $\sigma = \nu^{-1} \times \nu$ and $\tau = 1 \times \mathrm{Z}([a+1,b])$. Again, by the geometric lemma, it is enough to prove that σ does not occur as a subquotient of $\nu^{-1} \times 1$, $\nu \times 1$, $\mathrm{Z}([0,1])$ or 1×1, which follows easily.

In Case 3, we embed $\mathrm{Z}([a,b])$ into $\mathrm{Z}([a,b-1]) \times \nu^b$ and show by a similar argument that we can choose $\sigma = \mathrm{Z}([1,b-1]) \times \mathrm{L}([0,1])$ and $\tau = \nu^b$. By using the geometric lemma, it is enough to prove that ν^b is different from 1 and ν^{b-1}, which is immediate.

In Case 4, we prove the following more general lemma.

Lemma 7.6. — *Assume* $e > 3$. *Then* $\mathrm{Z}([1,b]) \times \mathrm{L}([0,1])$ *is irreducible for any* $b \geqslant 1$, $b \not\equiv -1$.

Proof. — We first treat the case where $b = 2$ (the case where $b = 1$ has already been done). We embed $\pi = \mathrm{Z}([1,2]) \times \mathrm{L}([0,1])$ in:

$$\mathrm{Z}([1,2]) \times \nu \times 1 \simeq \nu \times \mathrm{Z}([1,2]) \times 1 \hookrightarrow \nu \times \nu \times \nu^2 \times 1$$

and we choose $\sigma = \nu \times \nu$ and $\tau = \nu^2 \times 1$.

Now assume $b \geqslant 3$. We embed $\mathrm{Z}([1,b])$ in $\mathrm{Z}([1,2]) \times \mathrm{Z}([3,b])$ and then choose $\sigma = \mathrm{Z}([1,2])$ and $\tau = \mathrm{Z}([3,b]) \times \mathrm{L}([0,1])$. By the geometric lemma, it is enough to prove σ does not occur in:
(4.1) $\nu \times \nu^3$;
(4.2) $\nu \times \nu$;
(4.3) $\mathrm{L}([0,1])$;
(4.4) $\nu^3 \times \nu$;
(4.5) $\mathrm{Z}([3,4])$.
This is immediate. □

In Case 5, n is of the form $3k$ for some $k \geqslant 1$, and we write $\Omega_k = \mathrm{Z}([1,3k])$.

Lemma 7.7. — *The representation* $\Omega_1 \times \mathrm{L}([0,1])$ *is irreducible.*

Proof. — Let ξ be an irreducible subquotient of $\pi = \Omega_1 \times \mathrm{L}([0,1])$. It is thus a subquotient of the representation $\mathrm{Z}([1,3]) \times \nu \times 1$. By using Properties **P2** and **P3**, we deduce that ξ is of the form $\mathrm{Z}(\mathfrak{m})$ where $\mathfrak{m}$ is a multisegment in the following list:
(5.1) $\mathfrak{m} = [0,4]$;
(5.2) $\mathfrak{m} = [0,3] + [1]$;
(5.3) $\mathfrak{m} = [1,4] + [0]$;
(5.4) $\mathfrak{m} = [0,2] + [3,4]$;
(5.5) $\mathfrak{m} = [2,4] + [0,1]$;
(5.6) $\mathfrak{m} = [1,3] + [0,1]$;
(5.7) $\mathfrak{m} = [1,3] + [0] + [1]$.

We will prove that Case 5.7 is the only possible case, which implies that $\Omega_1 \times \mathrm{L}([0,1])$ is irreducible and equal to $\mathrm{Z}([1,3]+[0]+[1])$. By the geometric lemma, we get:

$$[\boldsymbol{r}_{(3,2)}(\pi)] = \mathrm{Z}([1,3]) \otimes \mathrm{L}([0,1]) + (\mathrm{Z}([1,2]) \times \nu) \otimes (1 \times 1) + (\nu \times \mathrm{L}([0,1])) \otimes \mathrm{Z}([2,3])$$

and each of these three subquotients is irreducible. Since $\boldsymbol{r}_{(3,2)}(\mathrm{Z}([0,4])) = \mathrm{Z}([0,2]) \otimes \mathrm{Z}([3,4])$, we see that $\mathrm{Z}([0,4])$ cannot occur as a subquotient of π.

Now the semi-simplification of $\boldsymbol{r}_{(1,2,2)}(\pi)$ is equal to:

$$\begin{aligned}\nu \otimes \mathrm{Z}([2,3]) \otimes \mathrm{L}([0,1]) + \nu \otimes \mathrm{L}([0,1]) \otimes \mathrm{Z}([2,3]) + \nu \otimes (\nu \times 1) \otimes \mathrm{Z}([2,3])\\ + \nu \otimes \mathrm{Z}([1,2]) \otimes (1 \times 1) + \nu \otimes (\nu^2 \times \nu) \otimes (1 \times 1).\end{aligned}$$

By using Proposition 4.12, we see that Cases 5.4, 5.5 and 5.6 cannot occur.

Now the semi-simplification of $\boldsymbol{r}_{(1,1,1,2)}(\pi)$ is equal to:

$$\begin{aligned}\nu \otimes 1 \otimes \nu \otimes \mathrm{Z}([2,3]) + \nu \otimes \nu^2 \otimes 1 \otimes \mathrm{L}([0,1]) + \nu \otimes \nu^2 \otimes \nu \otimes (1 \otimes 1)\\ + 2 \cdot \big(\nu \otimes \nu \otimes 1 \otimes \mathrm{Z}([2,3])\big) + 2 \cdot \big(\nu \otimes \nu \otimes \nu^2 \otimes (1 \times 1)\big).\end{aligned}$$

By using Proposition 4.12, we see that Case 5.2 cannot occur.

It remains to treat Case 5.3. The semi-simplification of $\boldsymbol{r}_{(1,1,3)}(\mathrm{Z}([1,4]) \times 1)$ is equal to:

$$\nu \otimes \nu^2 \otimes (\mathrm{Z}([0,1]) \times 1) + 1 \otimes \nu \otimes \mathrm{Z}([2,4]) + \nu \otimes 1 \otimes \mathrm{Z}([2,4]).$$

By Proposition 4.10(2) and the geometric lemma, we get:

$$[\boldsymbol{r}_{(1,1,3)}(\mathrm{Z}([1,4]) + [0])] = \nu \otimes \nu^2 \otimes (\mathrm{Z}([0,1]) \times 1) + \nu \otimes 1 \otimes \mathrm{Z}([2,4]).$$

On the other hand, the semisimplification of $\boldsymbol{r}_{(1,1,3)}(\pi)$ is equal to:

$$\nu \otimes 1 \otimes \mathrm{Z}([1,3]) + 2 \cdot (\nu \otimes \nu \otimes (\mathrm{Z}([2,3]) \times 1)) + \nu \otimes \nu^2 \otimes (1 \times \mathrm{L}([0,1]))$$

and each of the individual subquotients is irreducible. Therefore, Case 5.3 cannot occur. □

The proof is now by induction on k. We embed Ω_{k+1} into $\Omega_1 \times \Omega_k$ and choose $\sigma = \Omega_1$ and $\tau = \mathrm{L}([0,1]) \times \Omega_k$. By using the geometric lemma, we have to prove that, for all $0 \leqslant i \leqslant 2$, the factor $\sigma \otimes \tau$ does not occur as a subquotient of any of these three representations:

(5.A) $\mathrm{Z}([1,i]) \times \mathrm{L}([0,1]) \times \mathrm{Z}([1,1-i]) \otimes \mathrm{Z}([i+1,3]) \times \mathrm{Z}([2-i,3k])$;
(5.B) $\mathrm{Z}([1,i]) \times \nu \times \mathrm{Z}([1,2-i]) \otimes \mathrm{Z}([i+1,3]) \times 1 \times \mathrm{Z}([3-i,3k])$;
(5.C) $\mathrm{Z}([1,i]) \times \mathrm{Z}([1,3-i]) \otimes \mathrm{Z}([i+1,3]) \times \mathrm{L}([0,1]) \times \mathrm{Z}([4-i,3k])$.

This follows by using Property **P3**. (Notice that the term (5.A) does not appear if $i = 2$).

This ends the proof of Proposition 7.4. □

8. Distinguished representations

For $n \geqslant 2$, we write H_n for the subgroup of G_n made of all matrices of the form:

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, \quad g \in \mathrm{G}_{n-1}.$$

Definition 8.1. — Assume that $n \geqslant 2$. A smooth R-representation (π, V) of G_n is said to be H_n-distinguished if V possesses a nonzero H_n-invariant linear form.

If the space $\mathrm{Hom}_{\mathrm{H}_n}(\mathrm{V}, \mathrm{R})$ has finite dimension over R, we denote this dimension by $d(\pi)$.

8.1. Cuspidal representations

Just as in the complex case (see [**16**]), we have the following result.

Theorem 8.2. — *Let $n \geqslant 2$ and let $\rho \in \widehat{\mathrm{G}}_n$ be a cuspidal representation. Then ρ is distinguished if and only if $n = 2$. When it is the case, we have $d(\rho) = 1$.*

Proof. — Write P_n for the mirabolic subgroup of G_n, that is the subgroup made of all matrices with last row $(0, \dots, 0, 1)$. By [**22**, III,Theorem 1.1], the restriction of ρ to P_n is isomorphic, just as in the complex case, to the compact R-induction:

$$\mathrm{ind}_{\mathrm{U}_n}^{\mathrm{P}_n}(\psi_n)$$

of a generic character ψ_n of the standard maximal unipotent subgroup U_n of G_n. As $\mathrm{P}_n = \mathrm{H}_n\mathrm{U}_n$, the restriction of ρ to H_n is isomorphic to the compact R-induction $\mathrm{ind}_{\mathrm{H}_n\cap\mathrm{U}_n}^{\mathrm{H}_n}(\psi_n)$, which carries a nonzero H_n-fixed R-linear form if and only if ψ_n is trivial on $\mathrm{H}_n \cap \mathrm{U}_n$. This happens if and only if $n = 2$, in which case we have $d(\rho) = \dim \mathrm{Hom}_{\mathrm{H}_2\cap\mathrm{U}_2}(\psi_2, 1) = 1$. □

8.2. Distinction and contragredient

We have the very useful following result. Assume $n \geqslant 2$.

Proposition 8.3. — *Let $\pi \in \widehat{\mathrm{G}}_n$. Then π is H_n-distinguished if and only π^* is.*

This proposition will follow from the following one.

Proposition 8.4. — *Let us write σ for the involution on* G *defined by $g \mapsto$ transpose of g^{-1}. Let $\pi \in \widehat{\mathrm{G}}_n$. Then π^* is isomorphic to $\pi \circ \sigma$.*

Proof. — In the complex case, this is well-known and due to Gelfand and Kazhdan. When R has characteristic not 2, their argument still holds (see [**14**, Remarque 2.7]). We will need Proposition 8.4 when R has characteristic not 2 only, but we give below a proof in the general case, provided to us by the anonymous referee (whom we thank for this).

Let us write ℓ for the characteristic of R, and suppose that $\ell > 0$. It is enough to prove the proposition when R is an algebraic closure of a finite field with ℓ elements, denoted $\overline{\mathbf{F}}_\ell$. We thus have a reduction mod ℓ homomorphism:

$$\mathbf{r}_\ell : \mathcal{R}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)^{\mathrm{int}} \to \mathcal{R}(\mathrm{G}, \overline{\mathbf{F}}_\ell)$$

where $\mathcal{R}(\mathrm{G}, \overline{\mathbf{F}}_\ell)$ is the Grothendieck group of finite length $\overline{\mathbf{F}}_\ell$-representations of G and $\mathcal{R}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)^{\mathrm{int}}$ is the subgroup generated by integral representations in the Grothendieck group of finite length $\overline{\mathbf{Q}}_\ell$-representations of G (see [**22**]). Let us define an involutive group homomorphism $\pi \mapsto \pi^* \circ \sigma$ on $\mathcal{R}(\mathrm{G}, \overline{\mathbf{F}}_\ell)$, denoted α. Write $\widetilde{\alpha}$ for its analogue on $\mathcal{R}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)$. Since passing to the contragredient preserves integral representations and is compatible with reduction mod ℓ, we have:

$$\alpha \circ \mathbf{r}_\ell = \mathbf{r}_\ell \circ \widetilde{\alpha}.$$

Since $\widetilde{\alpha}$ is trivial by Gelfand-Kazhdan, and since $\mathbf{r}_\ell$ is surjective by [**5**] Corollaire 2.2.7 and [**14**] Théorème 9.40, it follows that α is trivial. □

Remark 8.5. — Note that the condition $e > 1$ implies that the characteristic of R is not 2.

Proposition 8.6. — *Write $n = n_1 + n_2$ where n_1, n_2 are positive integers, and let $\pi_i \in \widehat{\mathrm{G}}_{n_i}$ for $i = 1, 2$. Then $\pi_1 \times \pi_2$ is H_n-distinguished if and only if $\pi_2^* \times \pi_1^*$ is.*

Proof. — As in [**27**, 1.9] we define a group automorphism s of G_n by:

$$g \mapsto \mathrm{J}_n \cdot {}^{\mathrm{t}}g^{-1} \cdot \mathrm{J}_n^{-1}$$

where ${}^{\mathrm{t}}g$ is the transpose of g and J_n is the matrix whose $(i,j)^{\mathrm{th}}$ entry is $(-1)^i\delta_{i,n+1-j}$. As in the complex case, s maps $\mathrm{P}_{(n_1,n_2)}$ to $\mathrm{P}_{(n_2,n_1)}$ and $\pi_1 \times \pi_2$ to $s(\pi_2) \times s(\pi_1)$, and we have $s(\pi) \simeq \pi^*$ for all irreducible representations π of G_n by Proposition 8.4. Since s maps H_n to a conjugate of H_n, we get:

$$\mathrm{Hom}_{\mathrm{H}_n}(\pi_1 \times \pi_2, \mathrm{R}) \simeq \mathrm{Hom}_{\mathrm{H}_n}(\pi_2^* \times \pi_1^*, \mathrm{R})$$

and our claim follows. □

8.3. The Bernstein-Zelevinski filtration

For $i \in \{0, 1, \dots, n\}$, we write $\mathrm{R}_{i,n}$ for the subgroup of matrices of G_n of the form:

$$\begin{pmatrix} g & * \\ 0 & h \end{pmatrix}$$

such that $g \in \mathrm{G}_i$ and h is an upper triangular and unipotent matrix of G_{n-i}. In particular, $\mathrm{R}_{0,n}$ is the standard maximal unipotent subgroup U_n of G_n and $\mathrm{R}_{n-1,n}$ is the mirabolic subgroup P_n of G_n. Fix a nontrivial smooth character $\psi : \mathrm{F} \to \mathrm{R}^\times$ and, for $i \in \{0, 1, \dots, n-1\}$, write ψ_i for the generic character of U_i defined by:

$$\psi_i(h) = \psi(h_{1,2} + \cdots + h_{i-1,i})$$

for all $h \in \mathrm{U}_i$. From [**22**, III.1.3], we have the following result.

Theorem 8.7. — *Let* V *be a representation of* G_n. *There are* P_n*-stable subspaces* $\mathrm{V}_0, \dots, \mathrm{V}_n$ *of* V *such that* $\{0\} = \mathrm{V}_0 \subseteq \mathrm{V}_1 \subseteq \cdots \subseteq \mathrm{V}_n = \mathrm{V}$ *and:*

$$\mathrm{V}_{i+1}/\mathrm{V}_i \simeq \mathrm{ind}_{\mathrm{R}_{i,n}}^{\mathrm{P}_n}(\mathrm{V}^{(n-i)}\nu_i^{1/2} \otimes \psi_{n-i})$$

for all $i \in \{0, ..., n-1\}$.

As in the complex case (see page 54 of [**6**] and [**16**, Proposition 1]), we get the following result by using the Bernstein-Zelevinski filtration.

Lemma 8.8. — *Let* π *be a smooth representation of* G_n *with* $n \geqslant 3$, *and assume that:*

(1) $\pi^{(1)}$ *does not have any quotient isomorphic to* $\nu_{n-1}^{-1/2}$;

(2) $\pi^{(2)}$ *does not have any quotient isomorphic to* 1_{n-2}.

Then π *is not distinguished.*

8.4. The Three Orbits Lemma

As in the complex case [**20**], we have the following very useful lemma.

Lemma 8.9. — *Let* $n \geqslant 2$ *and* $k \in \{1, \dots, n-1\}$ *be integers, and let* $\rho \in \widehat{\mathrm{G}}_k$ *and* $\tau \in \widehat{\mathrm{G}}_{n-k}$. *Assumme* $\rho \times \tau$ *is* H_n*-distinguished. Then at least one of the following conditions is satisfied:*

(A) $\rho = \nu_k^{(n-2-k)/2}$ *and* $\tau \cdot \nu^{k/2}$ *is* H_{n-k}*-distinguished.*

(B) $\rho \cdot \nu^{-(n-k)/2}$ *is* H_k*-distinguished and* $\tau = \nu_{n-k}^{-(k-2)/2}$.

(C) $\rho^{(1)} \cdot \nu^{-(n-1-k)/2}$ *and* $\tau^{*(1)} \cdot \nu^{-(k-1)/2}$ *have a trivial quotient.*

Conversely, if $\rho \in \widehat{\mathrm{G}}_k$ *and* $\tau \in \widehat{\mathrm{G}}_{n-k}$ *satisfy* (A) *or* (B), *then* $\rho \times \tau$ *is* H_n*-distinguished.*

Proof. — The proof in just as in the complex case (see [**20**, Section 5]). □

Remark 8.10. — Notice that if τ is smooth (not necessarily irreducible), we still have conditions similar to Lemma 8.9. We will have the occasion to use this in the case where:

$$\rho = \nu_{n-3}^{1/2}, \quad \tau = \mathrm{St}_2 \cdot \nu^{-(n-2)/2} \times \chi, \quad \chi \in \widehat{\mathrm{G}}_1.$$

In this case, $\rho \times \tau$ is H_n-distinguished if and only if $\tau \cdot \nu^{(n-3)/2}$ is H_3-distinguished.

Corollary 8.11. — *Let $n \geqslant 3$, and let $\pi \in \widehat{\mathrm{G}}_n$ be H_n-distinguished. Then one of the following properties holds:*

(1) *$\pi = 1_{n-2} \times \tau$ for some irreducible cuspidal representation $\tau \in \widehat{\mathrm{G}}_2$.*
(2) *The cuspidal support of π is made of characters of G_1.*

Proof. — There are irreducible cuspidal representations $\tau_1, \dots, \tau_r$ such that π is a quotient of $\tau_1 \times \cdots \times \tau_r$. Since $n \geqslant 3$, Theorem 8.2 implies that π is not cuspidal, which implies that $r \geqslant 2$. Let k denote the largest integer among the $\deg(\tau_i)$'s and let τ_i have degree k with i maximal for this property. Then by [**15**] and Lemma 4.5, one may assume that $i = r$. Now write $\tau = \tau_r$ and let ρ be an irreducible subquotient of $\tau_1 \times \cdots \times \tau_{r-1}$ such that π is a quotient of $\rho \times \tau$. Since π is distinguished, so $\rho \times \tau$ is. Apply Lemma 8.9 to this product. According to Theorem 8.2, we obtain that k must be $\leqslant 2$. Moreover, if $k = 2$, then $\rho = 1_{n-2}$. □

8.5. Distinction of the twists of Λ_n and Π_n

We first determine which twists of Λ_n are distinguished.

Lemma 8.12. — *Let $n \geqslant 2$ and $\chi \in \widehat{\mathrm{G}}_1$. Then $\Lambda_n \cdot \chi$ is distinguished if and only if $\chi = 1$.*

Proof. — If f divides n, then Λ_n is the trivial character and the result is immediate. If f does not divide n, then we have the exact sequence:

$$0 \to \nu_n \cdot \chi \to \mathrm{V}_n \cdot \chi = (\nu_{n-1}^{1/2} \cdot \chi) \times \nu^{(n+1)/2}\chi \to \Lambda_n \cdot \chi \to 0$$

By Lemma 8.9 with $k = n-1$, the representation $\mathrm{V}_n \cdot \chi$, and hence $\Lambda_n \cdot \chi$, is non-distinguished for $\chi \notin \{1, \nu^{-1}\}$. If $\chi = \nu^{-1}$ is non-trivial (which forces $e > 1$), then Lemma 8.8 together with Corollary 6.2 imply that $\Lambda_n \cdot \nu^{-1}$ is not distinguished. Now assume that $\chi = 1$.

If $e > 1$, the contragredient V_n^* is distinguished by Lemma 8.9 (A) but ν_n^{-1} is not. Thus V_n^* carries a nonzero invariant linear form vanishing on ν_n^{-1}. It thus gives a nonzero invariant linear form on the subrepresentation Λ_n^*. By Proposition 8.3, the representation Λ_n is distinguished.

If $e = 1$, then $\mathrm{V}_n = 1_n \oplus \Lambda_n$ by Proposition 5.2. By Lemma 8.9, we have $d(\mathrm{V}_n) \geqslant 2$ since Conditions (A) and (B) are fulfilled. Thus Λ_n is distinguished with $d(\Lambda_n) = d(\mathrm{V}_n) - 1$. □

Corollary 8.13. — *Assume that $e > 1$. All the irreducible representations of G_n, $n \geqslant 3$ in the list given by Theorem 1.1 are distinguished.*

Proof. — When applied with $k = n-1$ and $k = n-2$ respectively, Lemma 8.9 gives the result for $\nu_{n-1}^{-1/2} \times \chi$ and $1_{n-2} \times \tau$. By passing to the contragredient (Proposition 8.3), we get the result for the representation $\nu_{n-1}^{1/2} \times \chi$ when $e > 1$.

By Lemma 8.12, Λ_n is distinguished. By passing to the contragredient, we get the result for Λ_n^* when $e > 1$. (Note that Λ_n is selfdual when $e = 1$.) This finishes the proof. □

We now determine which twists of Π_n are distinguished. This is done in Lemma 8.12 when f does not divide n. We now treat the case where f divides n.

Lemma 8.14. — *Assume that e is not* 1 *and divides n. For $\chi \in \widehat{\mathrm{G}}_1$, the representations $\Pi_n \cdot \chi$ and $\Pi_n^* \cdot \chi$ are not distinguished.*

Proof. — By Proposition 8.3, it is enough to prove it for $\Pi_n^* \cdot \chi$. By Lemma 8.8, for $\Pi_n^* \cdot \chi$ to be distinguished, it is necessary that at least one of the derivatives $(\Pi_n^* \cdot \chi)^{(i)}$ for $i = 1, 2$ has a character as a quotient. We have:

$$(\Pi_n^* \cdot \chi)^{(1)} = \Lambda_{n-1}^* \cdot \chi\nu^{-1/2}, \quad (\Pi_n^* \cdot \chi)^{(2)} = \nu_{n-2}^{-1} \cdot \chi.$$

By Lemma 8.8, we conclude that $\Pi_n^* \cdot \chi$ is not distinguished when $\chi \neq \nu$. It remains to consider $\Pi_n^* \cdot \nu$, or rather its contragredient $\Pi_n \cdot \nu^{-1}$. Its second derivative is ν_{n-2}^{-1}. By Lemma 8.8, our claim follows. □

Lemma 8.15. — *Assume that $e = 1$ and ℓ divides n. For $\chi \in \widehat{\mathrm{G}}_1$, the representation $\Pi_n \cdot \chi$ is distinguished if and only if $\chi = 1$.*

Proof. — When $e = 1$, the representation Π_n is selfdual thus the first part of the proof of Lemma 8.14 still holds. Thus $\Pi_n \cdot \chi$ is not distinguished for any $\chi \neq 1$. However, the second derivative of Π_n is 1_{n-2}, thus Lemma 8.8 is not sufficient to determine whether or not Π_n is distinguished.

Let H_n act on $\mathrm{X} = \mathrm{P}_{(n-1,1)}\backslash\mathrm{G}_n$. There are two closed orbits A, B in X, where A is reduced to a point and B is isomorphic to $\mathrm{P}_{(n-2,1)}\backslash\mathrm{G}_{n-1}$ (see [**20**, 5]). Since q is congruent to 1 mod the characteristic of R, the modulus R-character of $\mathrm{P}_{(n-1,1)}$ is trivial. By [**22**, Proposition I.2.8], there is a non-zero G_n-invariant linear form μ_X on V_n. Similarly, there is a non-zero H_n-invariant linear form on $\mathcal{C}_c^\infty(\mathrm{B}, \mathrm{R})$. Composition by the restriction from X to B gives us a non-zero H_n-invariant linear form μ_B on V_n. Finally, for $f \in \mathcal{C}_c^\infty(\mathrm{X}, \mathrm{R})$, we write $\mu_\mathrm{A}(f)$ for the value of f at A. We thus get three H_n-invariant linear forms on V_n.

The form μ_X is actually G_n-equivariant; its image is 1_n, and its kernel W_n has length 2, with socle 1_n (the space of R-valued constant functions on X) and irreducible quotient Π_n. We claim that these three linear forms are linearly independent. Granting the claim, there is no nontrivial linear combination of $\mu_\mathrm{A}, \mu_\mathrm{B}$ that vanishes on W_n. Moreover, if f_0 denotes the constant function taking value 1, and if μ_B is chosen so that $\mu_\mathrm{B}(f_0) = 1$, then:

$$(\mu_\mathrm{A} - \mu_\mathrm{B})(f_0) = 0.$$

Therefore, $\mu_\mathrm{A} - \mu_\mathrm{B}$ is a nonzero H_n-invariant linear form on W_n that vanishes on the space of constant functions; it thus induces a nonzero H_n-invariant linear form on Π_n. Thus, Π_n is H_n-distinguished when $e = 1$ and ℓ divides n.

It remains to prove the claim. Let U denote the unique open H_n-orbit in X, so that X is the disjoint union of A, B and U, and let $\widehat{\mathrm{U}}$ be its preimage in G_n. Let μ be a Haar measure on G_n. Since G_n is locally pro-p, there is a compact open subset $\Omega \subseteq \widehat{\mathrm{U}}$ with nonzero measure. Write ϕ for the characteristic function of the image of Ω in X. By [**22**, §2.8], there exists a $\alpha \in \mathrm{R}^\times$ such that:

$$\mu_\mathrm{X}(\phi) = \alpha \cdot \mu(1_\Omega) \neq 0.$$

On the other hand, we have $\mu_\mathrm{A}(\phi) = \mu_\mathrm{B}(\phi) = 0$ and hence the linear forms $\mu_\mathrm{X}, \mu_\mathrm{A}$ and μ_B are linearly independent. □

Remark 8.16. — Suppose $e = 1$ and ℓ does not divide n. It follows from the proof of Lemma 8.15 that $d(\mathrm{V}_n)$ is at least 3. On the other hand, the conditions of Lemma 8.9 implies that there is at most one H_n-invariant linear form upto scalars on each of three orbits A, B and U. Thus, $d(\mathrm{V}_n) = 3$. Since $\mathrm{V}_n = 1_n \oplus \Pi_n$, it follows that $d(\Pi_n) = 2$.

8.6. First reduction of the problem

Thanks to Corollary 8.11, we are already reduced to studying those H_n-distinguished irreducible representations of G_n, with $n \geqslant 3$, whose cuspidal support is made of characters.

Lemma 8.17. — *Let $\rho \in \widehat{\mathrm{G}}_k$ be such that $\rho^{(1)} \cdot \nu^{-(n-1-k)/2}$ has a trivial quotient. Then ρ is one of the following representations:*

(1) $\nu_{k-1}^{(n-k-1)/2} \times \mu$ *with* $\mu \in \widehat{\mathrm{G}}_1 - \{\nu^{(n-2k-1)/2}, \nu^{(n-1)/2}\}$*;*
(2) $\nu_k^{(n-k)/2}$*;*
(3) $\Lambda_k^* \cdot \nu^{(n-k)/2}$.

Proof. — We follow the proof given in the complex case in [**20**, Lemma 6.2]. The condition on ρ is equivalent to saying that ρ embeds into a representation of the form:

$$\mathrm{V}(\mu) = \nu_{k-1}^{(n-1-k)/2} \times \mu, \quad \mu \in \widehat{\mathrm{G}}_1.$$

If $\mu \notin \{\nu^{(n-2k-1)/2}, \nu^{(n-1)/2}\}$, this representation is irreducible (see Proposition 4.1) thus ρ is as in Case 1. Assume that $e > 1$. Thanks to Lemma 4.2, Proposition 4.10 and (5.2), we have:

(1) $\mathrm{V}(\nu^{(n-1)/2})$ has a unique irreducible subrepresentation, which is $\nu_k^{(n-k)/2}$. Thus ρ is as in Case 2.

(2) $\mathrm{V}(\nu^{(n-2k-1)/2})$ has a unique irreducible subrepresentation, which is $\Lambda_k^* \cdot \nu^{(n-k)/2}$. Thus ρ is as in Cases 2 or 3.

Assume now that $e = 1$. Then, by Proposition 5.2, any subrepresentation ρ of $\mathrm{V}(\nu^{(n-1)/2})$ is as in Case 2 or 3. Note that, in the case where f divides k, the representation V_k is indecomposable, thus ρ must be the character $\nu_k^{(n-k)/2}$. This finishes the proof of Lemma 8.17. □

In conclusion, we have the following result.

Proposition 8.18. — *Assume $n \geqslant 3$. Let $\pi \in \widehat{\mathrm{G}}_n$ be H_n-distinguished. Then there are $\rho \in \widehat{\mathrm{G}}_{n-1}$ and $\chi \in \widehat{\mathrm{G}}_1$ such that π is an irreducible quotient of $\rho \times \chi$ and at least one of the following conditions holds:*

(1) *One has $\rho = \nu_{n-1}^{-1/2}$ or $\rho = \nu_{n-1}^{1/2}$.*
(2) *One has $\rho = \Lambda_{n-1}^* \cdot \nu^{1/2}$.*
(3) *One has $\rho = 1_{n-2} \times \mu$ for some $\mu \in \widehat{\mathrm{G}}_1 - \{\nu^{-(n-1)/2}, \nu^{(n-1)/2}\}$.*
(4) *The representation $\rho \cdot \nu^{-1/2}$ is H_{n-1}-distinguished and $\chi = \nu^{-(n-3)/2}$.*

Moreover, if $e > 1$, then π is the unique irreducible quotient of $\rho \times \chi$.

In order to prove our main theorem 1.1, our strategy is to study, by induction on $n \geqslant 2$, the irreducible quotients of $\rho \times \chi$ in all these cases when $e > 1$, and to prove that they are either in the list of Theorem 1.1 or non-distinguished.

Assuming that Theorem 1.1 holds for G_{n-1} with $n \geqslant 3$, we thus have to study the distinction of the following representations:

Case 1: the irreducible quotients of $\nu_{n-1}^{-1/2} \times \chi$ and $\nu_{n-1}^{1/2} \times \chi$ for $\chi \in \widehat{\mathrm{G}}_1$;

Case 2: the irreducible quotients of $1_{n-2} \times \mu \times \chi$ for $\mu \in \widehat{\mathrm{G}}_1 - \{\nu^{-(n-1)/2}, \nu^{(n-1)/2}\}$, $\chi \in \widehat{\mathrm{G}}_1$;

Case 3: the irreducible quotients of $\Lambda_{n-1}^* \cdot \nu^{1/2} \times \chi$ for $\chi \in \widehat{\mathrm{G}}_1$;

Case 4: the irreducible quotients of $\rho \times \nu^{-(n-3)/2}$ where ρ is:

(4.a) the character $\nu_{n-1}^{1/2}$ (included in Case 1 above);
(4.b) a representation $1_{n-2} \times \mu$ with $\mu \in \widehat{\mathrm{G}}_1 - \{\nu^{-(n-1)/2}, \nu^{(n-1)/2}\}$ (included in Case 2);
(4.c) a representation $\nu_{n-2} \times \mu$ with $\mu \in \widehat{\mathrm{G}}_1 - \{\nu^{-(n-3)/2}, \nu^{(n+1)/2}\}$;
(4.d) a representation $\nu_{n-3}^{1/2} \times \tau$ with $\tau \in \widehat{\mathrm{G}}_2$ infinite-dimensional;
(4.e) one of the representations $\Lambda_{n-1} \cdot \nu^{1/2}$ or $\Lambda_{n-1}^* \cdot \nu^{1/2}$ (see Case 3 above);

Cases 1 and 4.c are treated in Section 9 for arbitrary $e \geqslant 1$, and Case 4.d is treated in Section 11 for $e > 1$.

We reduce Case 2 to studying $\mathbf{Q}(\mu \times \Lambda_{n-1} \cdot \nu^{-1/2})$ for $\mu \in \widehat{\mathrm{G}}_1 - \{\nu^{-(n-1)/2}, \nu^{(n-1)/2}\}$ in Section 10, when $e > 1$.

In Section 12, we do the remaining cases when $e > 1$.

9. Computing the irreducible quotients of $\nu_{n-1}^{1/2} \times \chi$ for $\chi \in \widehat{\mathrm{G}}_1$

Lemma 9.1. — *Assume $e > 1$. Let $a, b \in \mathbf{Z}$ with $a \leqslant b$. For $\chi \in \widehat{\mathrm{G}}_1$, write $\mathrm{V}(\chi) = \mathrm{Z}([a,b]) \times \chi$.*

(1) *If $\chi \notin \{\nu^{a-1}, \nu^{b+1}\}$, then $\mathrm{V}(\chi)$ is irreducible.*

(2) *Assume that $\chi = \nu^{b+1}$ and e does not divide n. Then $\mathrm{V}(\nu^{b+1})$ has length 2 and we have the following exact sequence:*

$$0 \to \mathrm{Z}([a, b+1]) \to \mathrm{V}(\nu^{b+1}) \to \mathrm{Z}([a,b] + [b+1]) \to 0.$$

(3) *Assume that $\chi = \nu^{a-1}$ and e does not divide n. Then $\mathrm{V}(\nu^{a-1})$ has length 2 and we have the following exact sequence:*

$$0 \to \mathrm{Z}([a,b] + [a-1]) \to \mathrm{V}(\nu^{a-1}) \to \mathrm{Z}([a-1,b]) \to 0.$$

(4) *If e divides n, then $\nu^{a-1} = \nu^{b+1}$ and $\mathrm{V}(\nu^{b+1})$ has length 3 with:*

$$\mathbf{S}(\mathrm{V}(\nu^{b+1})) = \mathrm{Z}([a, b+1]), \quad \mathbf{Q}(\mathrm{V}(\nu^{b+1})) = \mathrm{Z}([a-1, b]).$$

Proof. — Case 1 follows from Propositions 4.1 and 4.8. The other cases reduce to Proposition 4.10 by twisting by the character χ^{-1}, since $\mathrm{V}(\chi) \cdot \nu^{-c} = \mathrm{Z}([a-c, b-c]) \times \chi\nu^{-c}$ for $c \in \mathbf{Z}$. □

From Lemma 9.1 we get the following proposition.

Proposition 9.2. — *Assume $e > 1$. For all $n \geqslant 1$, we have:*

$$\mathbf{Q}(\nu_{n-1}^{1/2} \times \chi) = \begin{cases} \nu_{n-1}^{1/2} \times \chi & \text{if } \chi \notin \{\nu^{-(n-1)/2}, \nu^{(n+1)/2}\}, \\ 1_n & \text{if } \chi = \nu^{-(n-1)/2}, \\ \Lambda_n & \text{if } \chi = \nu^{(n+1)/2}. \end{cases}$$

Twisting by ν^{-1}, we get the following.

Proposition 9.3. — *Assume $e > 1$. For all $n \geqslant 1$, we have:*

$$\mathbf{Q}(\nu_{n-1}^{-1/2} \times \chi) = \begin{cases} \nu_{n-1}^{-1/2} \times \chi & \textit{if } \chi \notin \{\nu^{-(n+1)/2}, \nu^{(n-1)/2}\}, \\ \nu_n^{-1} & \textit{if } \chi = \nu^{-(n+1)/2}, \\ \Lambda_n \cdot \nu^{-1} & \textit{if } \chi = \nu^{(n-1)/2}. \end{cases}$$

By duality, we get the following.

Proposition 9.4. — *Assume $e > 1$. For all $n \geqslant 1$, we have:*

$$\mathbf{S}(\nu_{n-1}^{-1/2} \times \chi) = \begin{cases} \nu_{n-1}^{-1/2} \times \chi & \textit{if } \chi \notin \{\nu^{-(n+1)/2}, \nu^{(n-1)/2}\}, \\ 1_n & \textit{if } \chi = \nu^{(n-1)/2}, \\ \Lambda_n^* & \textit{if } \chi = \nu^{-(n+1)/2}. \end{cases}$$

Twisting by ν, we get the following.

Proposition 9.5. — *Assume $e > 1$. For all $n \geqslant 1$, we have:*

$$\mathbf{S}(\nu_{n-1}^{1/2} \times \chi) = \begin{cases} \nu_{n-1}^{1/2} \times \chi & \textit{if } \chi \notin \{\nu^{-(n-1)/2}, \nu^{(n+1)/2}\}, \\ \nu_n & \textit{if } \chi = \nu^{(n+1)/2}, \\ \Lambda_n^* \cdot \nu & \textit{if } \chi = \nu^{-(n-1)/2}. \end{cases}$$

In the case where $e = 1$, we summarize below the results obtained in Section 5.

Proposition 9.6. — *Assume $e = 1$.*

(1) *If $\chi \neq \nu^{(n+1)/2}$, then $\nu_{n-1}^{1/2} \times \chi$ is irreducible.*

(2) *If ℓ does not divide n, the irreducible quotients of $\nu_{n-1}^{1/2} \times \nu^{(n+1)/2}$ are 1_n and Π_n.*

(3) *If ℓ divides n, the irreducible quotient of $\nu_{n-1}^{1/2} \times \nu^{(n+1)/2}$ is 1_n.*

Thus we have treated Case 1 of Proposition 8.18.

Corollary 9.7. — *Let $e > 1$ and $\mu \in \widehat{\mathrm{G}}_1 - \{\nu^{-(n-3)/2}, \nu^{(n+1)/2}\}$. Then:*

$$\mathbf{Q}(\nu_{n-2} \times \mu \times \nu^{-(n-3)/2}) = \begin{cases} \mu \times \nu_{n-1}^{1/2} & \textit{if } \mu \neq \nu^{-(n-1)/2}, \\ \Lambda_n^* \cdot \nu & \textit{if } \mu = \nu^{-(n-1)/2}. \end{cases}$$

Proof. — By assumption on μ, the representation $\nu_{n-2} \times \mu$ is irreducible. It is thus isomorphic to $\mu \times \nu_{n-2}$. It thus suffices to consider the representation $\pi(\mu) = \mathbf{Q}(\mu \times \nu_{n-2} \times \nu^{-(n-3)/2})$. By Proposition 9.2, we have:

$$\mathbf{Q}(\nu_{n-2} \times \nu^{-(n-3)/2}) = \nu_{n-1}^{1/2}$$

thus $\pi(\mu)$ is equal to $\mathbf{Q}(\mu \times \nu_{n-1}^{1/2})$. By assumption on μ, the representation $\mu \times \nu_{n-1}^{1/2}$ is reducible if and only if $\mu = \nu^{-(n-1)/2}$. Finally, the representation:

$$\pi(\nu^{-(n-1)/2}) = \mathbf{Q}(\nu^{-(n-1)/2} \times \nu_{n-1}^{1/2}) = \mathbf{S}(\nu_{n-1}^{1/2} \times \nu^{-(n-1)/2})$$

is equal to $\Lambda_n^* \cdot \nu$ by Proposition 9.3. By Lemma 8.12, it is not distinguished. $\square$

Thus we have treated Case 4.c of Proposition 8.18.

10. Computing $\mathbf{Q}(1_{n-2} \times \mu \times \chi)$ for $\mu \in \widehat{\mathrm{G}}_1 - \{\nu^{-(n-1)/2}, \nu^{(n-1)/2}\}$ and $\chi \in \widehat{\mathrm{G}}_1$

In this section, we fix a character $\mu \in \widehat{\mathrm{G}}_1$ different from $\nu^{-(n-1)/2}$ and $\nu^{(n-1)/2}$, and we assume that $e > 1$. Note that this implies that $1_{n-2} \times \mu = \mu \times 1_{n-2}$ is irreducible. For $\chi \in \widehat{\mathrm{G}}_1$, write:

$$\mathrm{W}(\chi) = 1_{n-2} \times \mu \times \chi.$$

We record below two facts in the form of the following lemma which will be used repeatedly in what follows.

Lemma 10.1. — *The representation* $\mathrm{W}(\chi)$ *has unique irreducible subrepresentation and unique irreducible quotient. Moreover, one has:*

$$\mathbf{Q}(\chi \times \chi\nu) = \begin{cases} \mathrm{St}_2 \cdot \chi\nu^{1/2} & \text{if } e > 2, \\ 1_2 \cdot \chi\nu^{-1/2} & \text{if } e = 2. \end{cases}$$

In particular, when $e \geqslant 2$, *the representations* $1_{n-2} \times \mathrm{St}_2 \cdot \mu\nu^{1/2}$ *and* $1_{n-2} \times 1_2 \cdot \mu\nu^{-1/2}$ *have a unique irreducible quotient.*

Proof. — The first statement follows from Lemma 4.2 and the second one from Lemma 9.1. To prove the final statement, observe that $1_{n-2} \times \mathrm{St}_2 \cdot \mu\nu^{1/2}$ is a quotient of $\mathrm{W}(\mu\nu)$ if $e > 2$. Since $\mathrm{W}(\mu\nu)$ has a unique irreducible quotient, the claim follows. For $e = 2$, $1_{n-2} \times \mathrm{St}_2 \cdot \mu\nu^{1/2}$ is itself irreducible by Proposition 4.8. Similarly, for $e \geqslant 2$, $1_{n-2} \times 1_2 \cdot \mu\nu^{-1/2}$ is a quotient of $\mathrm{W}(\mu\nu^{-1})$, which has a unique irreducible quotient. This completes the proof of the proposition. □

Lemma 10.2. — *For any* $\chi \notin \{\mu\nu, \mu\nu^{-1}, \nu^{-(n-1)/2}, \nu^{(n-1)/2}\}$, *the representation* $\mathrm{W}(\chi)$ *is irreducible and distinguished.*

Proof. — By Proposition 4.8, $\mathrm{W}(\chi)$ is irreducible. It satifies Condition (A) of Lemma 8.9 with $k = n - 2$, thus it is distinguished. □

Lemma 10.3. — *One has:*

$$\mathbf{Q}(\mathrm{W}(\mu\nu)) = \begin{cases} \mathbf{Q}(1_{n-2} \times \mathrm{St}_2 \cdot \mu\nu^{1/2}) & \text{if } e > 2, \\ \mathbf{Q}(1_{n-2} \times 1_2 \cdot \mu\nu^{-1/2}) & \text{if } e = 2, \end{cases}$$

and $\mathbf{Q}(\mathrm{W}(\mu\nu^{-1})) = \mathbf{Q}(1_{n-2} \times 1_2 \cdot \mu\nu^{-1/2})$.

Proof. — First observe that, by Lemma 10.1, $\mathrm{W}(\mu\nu)$ has $1_{n-2} \times \mathrm{St}_2 \cdot \mu\nu^{1/2}$ as a quotient if $e > 2$ and $\mathrm{W}(\mu\nu^{-1})$ has $1_{n-2} \times 1_2 \cdot \mu\nu^{-1/2}$ as a quotient if $e \geqslant 2$. Once again, applying Lemma 10.1 the statement is proved. □

Proposition 10.4. — *Write* $\mathrm{Y}(\mu) = \mathbf{Q}(1_{n-2} \times \mathrm{St}_2 \cdot \mu\nu^{1/2})$. *Then:*

$$\mathrm{Y}(\mu) = \begin{cases} 1_{n-2} \times \mathrm{St}_2 \cdot \mu\nu^{1/2} & \text{if } \mu \neq \nu^{-(n+1)/2} \text{ or } e = 2, \\ \Lambda_n^* & \text{if } \mu = \nu^{-(n+1)/2} \text{ and } e \text{ does not divide } n \text{ and } e > 2. \end{cases}$$

Proof. — The statement follows from Proposition 4.8 if $e = 2$, and it follows from Proposition 7.4 if $\mu \neq \nu^{-(n+1)/2}$. Assume that $\mu = \nu^{-(n+1)/2}$ and e does not divide n and $e > 2$. We have:

$$\begin{aligned} \mathrm{Y}(\nu^{-(n+1)/2}) &= \mathbf{Q}(\mathrm{W}(\nu^{-(n-1)/2})) \\ &= \mathbf{Q}(\nu^{-(n+1)/2} \times 1_{n-2} \times \nu^{-(n-1)/2}) \\ &= \mathbf{Q}(\nu^{-(n+1)/2} \times \nu_{n-1}^{-1/2}) \end{aligned}$$

which is equal to Λ_n^* by applying respectively Lemma 10.3, Lemma 4.5 (since e does not divide n, the representation $1_{n-2} \times \nu^{-(n+1)/2}$ is irreducible by Proposition 4.8), Lemma 9.1 and (5.2). □

Proposition 10.5. — *Write* $\mathrm{P}(\mu) = \mathbf{Q}(1_{n-2} \times 1_2 \cdot \mu\nu^{-1/2})$. *For* $\mu \neq \nu^{-(n-3)/2}$, *the representation* $\mathrm{P}(\mu)$ *is not distinguished, and we have:*

$$\mathrm{P}(\nu^{-(n-3)/2}) = \begin{cases} \nu_{n-1}^{-1/2} \times \nu^{-(n-3)/2} & \text{if } e \text{ does not divide } n-2 \text{ and } e > 2, \\ \Lambda_n^* & \text{if } e = 2 \text{ and } n \text{ is odd.} \end{cases}$$

Proof. — The first assertion follows form Lemma 8.9. Assume now that $\mu = \nu^{-(n-3)/2}$. If $e > 2$ does not divide $n-2$, then $\mathrm{P}(\nu^{-(n-3)/2}) = \mathbf{Q}(\mathrm{W}(\nu^{-(n-1)/2})$ by Lemma 10.3. By Lemma 9.1 and Proposition 4.8, we have:

$$\mathbf{Q}(\mathrm{W}(\nu^{-(n-1)/2}) = \nu_{n-1}^{-1/2} \times \nu^{-(n-3)/2}.$$

Assume now that $e = 2$ and n is odd. By a similar argument as above, we deduce that:

$$\mathrm{P}(\nu^{-(n-3)/2}) = \mathbf{Q}(\nu^{-(n-3)/2} \times \nu_{n-1}^{-1/2}).$$

By Proposition 9.4 and the observation following Lemma 4.2, we get $\mathrm{P}(\nu^{-(n-3)/2}) = \Lambda_n^*$. □

Note that $1_{n-2} \times \mu \times \chi = \mu \times 1_{n-2} \times \chi$. Thus:

$$\begin{aligned} \mathbf{Q}(\mathrm{W}(\nu^{(n-1)/2})) &= \begin{cases} \mathbf{Q}(\mu \times \Lambda_{n-1} \cdot \nu^{-1/2}) & \text{if } e \text{ does not divide } n-1, \\ \mathbf{Q}(\mu \times \nu_{n-1}^{-1/2}) & \text{if } e \text{ divides } n-1, \end{cases} \\ \mathbf{Q}(\mathrm{W}(\nu^{-(n-1)/2})) &= \mathbf{Q}(\mu \times \nu_{n-1}^{-1/2}). \end{aligned}$$

We have the following proposition.

Proposition 10.6. — *One has:*

$$\mathbf{Q}(\mathrm{W}(\nu^{-(n-1)/2})) = \begin{cases} \nu_{n-1}^{-1/2} \times \mu & \text{if } \mu \neq \nu^{-(n+1)/2}, \\ \Lambda_n^* & \text{if } \mu = \nu^{-(n+1)/2} \text{ and } e \text{ does not divide } n. \end{cases}$$

Proof. — This follows from Propositions 9.4 and 9.5. □

It remains to study:

$$\mathbf{Q}(\mathrm{W}(\nu^{(n-1)/2})) = \mathbf{Q}(\mu \times \Lambda_{n-1} \cdot \nu^{-1/2})$$

when e does not divide $n-1$. This will be done in Section 12.

11. Computing $\mathbf{Q}(\nu_{n-3}^{1/2} \times \tau \times \nu^{-(n-3)/2})$ for $\tau \in \widehat{\mathrm{G}}_2$ infinite dimensional

In this section, we assume that $e > 1$. We consider all those infinite dimensional $\tau \in \widehat{\mathrm{G}}_2$ such that $\nu_{n-3}^{1/2} \times \tau$ is irreducible, that is:

(1) τ is cuspidal;

(2) τ is a Steinberg representation $\mathrm{St}_2 \cdot \mu\nu^{1/2}$ with $\mu \notin \{\nu^{-(n-1)/2}, \nu^{(n-1)/2}\}$ and $e > 2$;

(3) τ is a principal series $\lambda \times \mu$ with $\lambda\mu^{-1} \notin \{\nu^{-1}, \nu\}$ and $\lambda, \mu \notin \{\nu^{-(n-3)/2}, \nu^{(n-1)/2}\}$.

In all these cases, we study the unique irreducible quotient:

$$\mathrm{U}(\tau) = \mathbf{Q}(\nu_{n-3}^{1/2} \times \tau \times \nu^{-(n-3)/2}). \tag{11.1}$$

We first have the following results.

Lemma 11.1. — *For all these τ as above, we have* $\mathrm{U}(\tau) = \mathbf{Q}(\tau \times 1_{n-2})$.

Proof. — It follows from the fact that $\nu_{n-3}^{1/2} \times \tau = \tau \times \nu_{n-3}^{1/2}$ and $\mathbf{Q}(\nu_{n-3}^{1/2} \times \nu^{-(n-3)/2}) = 1_{n-2}$. □

Proposition 11.2. — *Assume that τ is cuspidal. Then* $\mathrm{U}(\tau) = \tau \times 1_{n-2}$.

Proof. — This follows from the fact that $\tau \times 1_{n-2}$ is irreducible when τ is cuspidal. □

We now treat the cases where τ is not cuspidal.

Proposition 11.3. — *Assume $\tau = \lambda \times \mu$ with $\lambda\mu^{-1} \notin \{\nu^{-1}, \nu\}$ and $\lambda, \mu \notin \{\nu^{-(n-3)/2}, \nu^{(n-1)/2}\}$. Then we have:*

$$\mathrm{U}(\tau) = \lambda \times \mu \times 1_{n-2} \quad \textit{for all } \lambda, \mu \neq \nu^{-(n-1)/2}$$

and, if $\mu = \nu^{-(n-1)/2}$ and e does not divide $n-1$, then $\mathrm{U}(\tau)$ is not distinguished.

Proof. — The first assertion follows from Proposition 4.8. Assume now that $\mu = \nu^{-(n-1)/2}$ and e does not divide $n-1$. It follows from Proposition 9.4 that:

$$\mathrm{U}(\tau) = \mathbf{Q}(\lambda \times \Lambda_{n-1}^{*} \cdot \nu^{1/2}),$$

which is not distinguished by Lemma 8.9 with $k = 1$. □

Proposition 11.4. — *Assume $e > 2$ and $\tau = \mathrm{St}_2 \cdot \mu\nu^{1/2}$ with $\mu \notin \{\nu^{-(n-1)/2}, \nu^{(n-1)/2}\}$. Then:*

$$\mathrm{U}(\tau) = \tau \times 1_{n-2} \quad \textit{for all } \mu \neq \nu^{-(n+1)/2}$$

and $\mathrm{U}(\tau)$ is not distinguished for $\mu = \nu^{-(n+1)/2}$.

Remark 11.5. — If we assume that $e = 2$ in Lemma 11.4, then τ is cuspidal and this case has already been done in Lemma 11.2.

Proof. — Write $\tau = \mathrm{L}([0,1]) \cdot \mu$. By Proposition 7.4, the representation $\tau \times 1_{n-2}$ is irreducible unless $\mu = \nu^k$ with k a half-integer and the segments $[-(n-3)/2, (n-3)/2]$ and $[k, k+1]$ are juxtaposed, that is $\mu = \nu^{(n-1)/2}$ (which is not allowed) or $\mu = \nu^{-(n+1)/2}$.

Assume $\mu = \nu^{-(n+1)/2}$ and e does not divide n (thus $\mu \neq \nu^{(n-1)/2}$). Let L be the unique irreducible quotient of $\mathrm{St}_2 \cdot \mu\nu^{1/2} \times \nu^{-(n-3)/2}$. If $e > 3$, note that $\mathrm{St}_3 \cdot \nu^{-1}$ is the unique irreducible quotient of $\mathrm{St}_2 \cdot \nu^{-3/2} \times 1$ (see p. 168 of [**16**] and the exact sequence (3.5) in [**20**]). Twisting by $\nu^{-(n-3)/2}$, we see that $\mathrm{L} = \mathrm{St}_3 \cdot \nu^{-(n-1)/2}$. Moreover, by [**16**, Theorem 2] or [**20**, Remark 6.7], no twist of L is distinguished. If $e = 3$, L is equal to a twist of Π_3, which is not distinguished by Lemma 8.14. Hence, no twist of L is distinguished. Applying Lemma 8.9 with $k = n-3$ to $\nu_{n-3}^{1/2} \times \mathrm{L}$ yields that it is not distinguished, and so $\mathrm{U}(\tau)$ is not distinguished. □

12. The remaining cases

In this section, we assume that $e > 1$ as in Sections 10 and 11. It remains for us to study the distinction of the following representations:

(1) the irreducible quotients of $\mu \times \Lambda_{n-1} \cdot \nu^{-1/2}$ for $\mu \in \widehat{\mathrm{G}}_1 - \{\nu^{-(n-1)/2}, \nu^{(n-1)/2}\}$;
(2) the irreducible quotients of $\Lambda_{n-1}^{*} \cdot \nu^{1/2} \times \chi$ for $\chi \in \widehat{\mathrm{G}}_1$;
(3) the irreducible quotient of $\Lambda_{n-1} \cdot \nu^{1/2} \times \nu^{-(n-3)/2}$.

Note that we may assume e does not divide $n-1$ (or else Λ_{n-1} would be the trivial character).

The first case is the one that remains from Section 10, the second one corresponds to Case 3 of Paragraph 8.6 and the third one corresponds to the part of Case 4.e of Paragraph 8.6 which does not belong to Case 3.

12.1. Distinction of $\mu \times \Lambda_{n-1} \cdot \nu^{-1/2}$ and $\Lambda^*_{n-1} \cdot \nu^{1/2} \times \chi$

In this paragraph, we show that, if $\Lambda^*_{n-1} \cdot \nu^{1/2} \times \chi$ is distinguished, then χ must be equal to $\nu^{-(n-3)/2}$. Given this, it will follow by Proposition 8.6 that $\mu \times \Lambda_{n-1} \cdot \nu^{-1/2}$ is distinguished if and only if $\mu = \nu^{(n-3)/2}$.

Lemma 12.1. — *Let $\chi \in \widehat{\mathrm{G}}_1$ and $e > 1$. Then the representation $\mathrm{St}_2 \cdot \nu^{-1/2} \times \chi$ is distinguished if and only if $\chi = 1$.*

Proof. — Write $\mathrm{B}(\chi) = \mathrm{St}_2 \cdot \nu^{-1/2} \times \chi$. If $\chi = 1$, then $\mathrm{B}(1)$ is distinguished as it satisfies (B) of Lemma 8.9 for $k = 2$.

Assume $\chi \notin \{1, \nu, \nu^{-2}\}$. Since χ is nontrivial, Lemma 8.9 implies that $\mathrm{B}(\chi)^*$ is not distinguished. Since $\chi \notin \{\nu, \nu^{-2}\}$, Lemma 7.3 shows that $\mathrm{B}(\chi)$ is irreducible. Thus, by Lemma 8.3, $\mathrm{B}(\chi)$ is not distinguished. It remains to consider the case when $\chi \in \{\nu, \nu^{-2}\}$.

If $e > 3$, then we remind that $[\mathrm{St}_2 \cdot \nu^{3/2} \times 1] = \mathrm{St}_3 \cdot \nu + \Lambda_3$ as in the complex case (see p. 168 of [**16**] or (3.5) in [**20**]). First we twist $\mathrm{St}_2 \cdot \nu^{3/2} \times 1$ by ν^{-2}. Secondly, we take the contragredient $\mathrm{St}_2 \cdot \nu^{-3/2} \times 1$ and twist by ν. These yield:

$$[\mathrm{B}(\nu^{-2})] = \Lambda_3 \cdot \nu^{-2} + \mathrm{St}_3 \cdot \nu^{-1}, \quad [\mathrm{B}(\nu)] = \Lambda_3^* \cdot \nu + \mathrm{St}_3$$

respectively. None of these subquotients are distinguished.

If $e = 2$, then St_2 is cuspidal, thus $\mathrm{B}(\nu)$ is irreducible and the result follows from Lemma 8.9.

We finally assume that $e = 3$. We first claim the principal series $\xi = \nu^{-1} \times 1 \times \nu$ has length 7, with subquotients:

$$1_3,\ \nu_3,\ \nu_3^{-1},\ \Pi_3,\ \Pi_3 \cdot \nu,\ \Pi_3 \cdot \nu^{-1} \text{ and the cuspidal representation } \mathrm{St}_3.$$

Indeed, ξ contains 1_3 and Π_3 as well as their twists by ν and ν^2, and it also contains the cuspidal (thus nondegenerate) representation St_3 with multiplicity 1. The Jacquet module $\boldsymbol{r}_{(1,1,1)}(\xi)$ has length 6, thus our claim follows. Now we have:

$$\begin{aligned} [\xi] &= [\nu_2^{-1/2} \times \nu] + [\mathrm{B}(\nu)] \\ &= (1_3 + \nu_3^{-1} + \Pi_3) + [\mathrm{B}(\nu)] \end{aligned}$$

by Proposition 4.10. It follows that:

$$[\mathrm{B}(\nu)] = \nu_3 + \Pi_3 + \Pi_3 \cdot \nu + \mathrm{St}_3$$

in the Grothendieck group of finite length representations of G_3.

By Lemma 8.14 and Theorem 8.2, none of these subquotients are distinguished. Since $\mathrm{B}(\nu^{-2})$ is equal to $\mathrm{B}(\nu)$, our lemma is proved. □

Given $\chi \in \widehat{\mathrm{G}}_1$, we now write:

$$\mathrm{A}(\chi) = \Lambda^*_{n-1} \cdot \nu^{1/2} \times \chi.$$

We study the distinction of $\mathrm{A}(\chi)$ in the following lemma.

Lemma 12.2. — *Assume that e does not divide $n-1$, and let $\chi \in \widehat{\mathrm{G}}_1$. Then $\mathrm{A}(\chi)$ is distinguished if and only if $\chi = \nu^{-(n-3)/2}$.*

Proof. — First, Lemma 8.9 with $k = n-1$ shows that $\mathrm{A}(\nu^{-(n-3)/2})$ is distinguished.

For the converse, we may assume that $n \geqslant 4$ since we have treated the case when $n = 3$ in Lemma 12.1. Assume first that $e > 2$. By Proposition 10.4, $\mathrm{A}(\chi)$ is a quotient of:

$$\nu_{n-3}^{1/2} \times \mathrm{St}_2 \cdot \nu^{-(n-2)/2} \times \chi,$$

which is distinguished by Remark 8.10 if and only if Condition (A) of Lemma 8.9 is satisfied with $k = n-3$. This is the case if and only if $\mathrm{St}_2 \cdot \nu^{-1/2} \times \chi\nu^{(n-3)/2}$ is distinguished. By Lemma 12.1, this happens if and only if $\chi = \nu^{-(n-3)/2}$.

Assume now that $e = 2$. Note that the characters $\nu^{(n-1)/2}$ and $\nu^{(n+1)/2}$ are the only ones that are obtained from $\nu^{-(n-3)/2}$ up to a translation of an integer power of ν.

Assume first that $\chi \notin \{\nu^{-(n-1)/2}, \nu^{(n-1)/2}\}$. Then $\mathrm{A}(\chi)$ is irreducible by Proposition 4.1, and Lemma 8.9 implies that $\mathrm{A}(\chi)^*$ is not distinguished. By Proposition 8.3, $\mathrm{A}(\chi)$ is not distinguished either.

It remains to consider the case where $\chi = \nu^{-(n-1)/2} = \nu^{(n+1)/2}$. We write $\mathrm{A} = \mathrm{A}(\nu^{(n+1)/2})$. By definition, $\Lambda_{n-1}^* \cdot \nu^{1/2}$ is the unique irreducible quotient of $\nu^{(n+1)/2} \times 1_{n-2}$. The representation A is thus a quotient of $\mathrm{V} = \nu^{(n+1)/2} \times 1_{n-2} \times \nu^{(n+1)/2}$. Now write the two exact sequences:

$$0 \to \nu_{n-1}^{-1/2} \to \nu^{(n+1)/2} \times 1_{n-2} \to \Lambda_{n-1}^* \cdot \nu^{1/2} \to 0 \tag{12.1}$$

and:

$$0 \to \Lambda_{n-1}^* \cdot \nu^{1/2} \to 1_{n-2} \times \nu^{(n+1)/2} \to \nu_{n-1}^{-1/2} \to 0. \tag{12.2}$$

Computing (12.1) $\times\, \nu^{(n+1)/2}$, we get:

$$0 \to \mathrm{W} \to \mathrm{V} \xrightarrow{\alpha} \mathrm{A} \to 0$$

where W is the representation $\nu_{n-1}^{-1/2} \times \nu^{(n+1)/2}$, which is irreducible since $\nu^{(n+1)/2} \neq \nu^{(n-1)/2}$ and $\nu^{(n+1)/2} \neq \nu^{-(n+1)/2}$. Thus W is isomorphic to $\nu^{(n+1)/2} \times \nu_{n-1}^{-1/2}$. Computing $\nu^{(n+1)/2} \times$ (12.2) we get:

$$0 \to \nu^{(n+1)/2} \times \Lambda_{n-1}^* \cdot \nu^{1/2} \to \mathrm{V} \xrightarrow{\beta} \mathrm{W} \to 0.$$

Observe that W is distinguished by Lemma 8.9, thus V is also distinguished. Lemma 8.9 (applied with $k = n-1$) also shows that the space of H_n-invariant forms on V is one-dimensional.

Now we claim A is not distinguished. Assume A is distinguished, and let T denote a nonzero invariant linear form on V which is trivial on $\mathrm{K}_1 = \mathrm{Ker}(\alpha)$. Since V has a one-dimensional space of invariant forms, T is proportional to any nonzero invariant linear form on V which is trivial on $\mathrm{K}_2 = \mathrm{Ker}(\beta)$. Thus, T is zero on $\mathrm{K}_1 + \mathrm{K}_2$. Since T is nonzero, $\mathrm{K}_1 + \mathrm{K}_2$ is different from the whole space V. Since K_1 is irreducible and isomorphic to W, we get that $\mathrm{K}_1 + \mathrm{K}_2 = \mathrm{K}_2$, thus:

$$\mathrm{K}_1 \subseteq \mathrm{K}_2 \simeq \nu^{(n+1)/2} \times \Lambda_{n-1}^* \cdot \nu^{1/2}.$$

It follows that:

$$\mathrm{W} = \mathbf{S}(\nu^{(n+1)/2} \times \Lambda_{n-1}^* \cdot \nu^{1/2}) = \mathbf{Q}(\mathrm{A}).$$

Thus $\mathrm{W} \cdot \nu$ is the unique irreducible quotient of $\mathrm{A} \cdot \nu$. Observe that $\mathrm{W} \cdot \nu = \nu_{n-1}^{1/2} \times \nu^{(n+3)/2}$ is isomorphic to W^* and hence is distinguished by Proposition 8.3. However, the representation $\mathrm{A} \cdot \nu = \Lambda_{n-1}^* \cdot \nu^{-1/2} \times \nu^{(n-1)/2}$ is not distinguished by Lemma 8.9, a contradiction. $\square$

12.2. Distinction of $\mathbf{Q}(\Lambda_{n-1}^{*} \cdot \nu^{1/2} \times \nu^{-(n-3)/2})$ and $\mathbf{Q}(\nu^{(n-3)/2} \times \Lambda_{n-1} \cdot \nu^{-1/2})$

By Lemma 12.2, in order to finish Cases 1 and 2 of Section 12 for $e > 1$, it remains to discuss the distinction of the irreducible quotients:

$$\mathbf{Q}(\Lambda_{n-1}^{*} \cdot \nu^{1/2} \times \nu^{-(n-3)/2}) \quad \text{and} \quad \mathbf{Q}(\nu^{(n-3)/2} \times \Lambda_{n-1} \cdot \nu^{-1/2}).$$

Note that the latter is the contragredient of the former, thus it is enough to study the distinction of the first one. Moreover, if $n = 3$, then:

$$\mathbf{Q}(\Lambda_{2}^{*} \cdot \nu^{1/2} \times 1) = \mathrm{St}_2 \cdot \nu^{-1/2} \times 1$$

is distinguished by Lemmas 7.3 and 12.1. So we will assume that $n \geqslant 4$ in the remainder of this Section. In what follows, the computation of distinguished quotients will fall into three cases:

(1) $e > 2$ and e does not divide $n - 2$;
(2) $e > 2$ and e divides $n - 2$ (this implies that e does not divide n);
(3) $e = 2$ (this implies that e divides $n - 2$ since e does not divide $n - 1$).

We start with the following lemma, which follows from Lemma 9.1.

Lemma 12.3. — *Assume $e > 1$. We have:*

$$[\nu_{n-1}^{-1/2} \times \nu^{-(n-3)/2}] = \begin{cases} \nu_{n-1}^{-1/2} \times \nu^{-(n-3)/2} & \text{if } e > 2 \text{ and } e \text{ does not divide } n-2, \\ 1_n + \Pi_n \cdot \nu^{-1} & \text{if } e > 2 \text{ and } e \text{ divides } n-2, \\ \nu_n^{-1} + \Pi_n^{*} & \text{if } e = 2 \text{ and } e \text{ does not divide } n-2, \\ 1_n + \nu_n^{-1} + \Pi_n^{*} & \text{if } e = 2 \text{ and } e \text{ divides } n-2. \end{cases}$$

We now define two irreducible representations of G_n.

Definition 12.4. — Assume $e > 1$ and $n \geqslant 4$. Define:

$$\begin{aligned} \Phi_n &= \mathrm{Z}\left(\left[-\frac{n-3}{2}, \frac{n-3}{2}\right] + \left[-\frac{n-1}{2}, -\frac{n-3}{2}\right]\right), \\ \Psi_n &= \mathrm{Z}\left(\left[-\frac{n-3}{2}, \frac{n-3}{2}\right] + \left[-\frac{n+1}{2}, -\frac{n-1}{2}\right]\right). \end{aligned}$$

Observe that Φ_n is selfdual if e divides $n-2$ and Ψ_n is selfdual if e divides n. We also recall that $\Pi_n^{*} = \Pi_n \cdot \nu^{-1}$ if e divides n.

Lemma 12.5. — *Assume $e > 1$ and $n \geqslant 4$, and suppose e does not divide $n-1$. The irreducible subquotients of:*

$$1_{n-2} \times 1_2 \cdot \nu^{-(n-2)/2} = \mathrm{Z}\left(\left[-\frac{n-3}{2}, \frac{n-3}{2}\right]\right) \times \mathrm{Z}\left(\left[-\frac{n-1}{2}, -\frac{n-3}{2}\right]\right) \tag{12.3}$$

are:

(1) *the representations $\nu_{n-1}^{-1/2} \times \nu^{-(n-3)/2}$ and Φ_n if e does not divide $n-2$,*
(2) *the representations 1_n, $\Pi_n^{*} \cdot \nu$, $\Pi_n \cdot \nu^{-1}$ and Φ_n if e divides $n-2$.*

Moreover, all subquotients appear with multiplicity 1 if $e > 2$. If $e = 2$, only 1_n may appear with multiplicity more than 1.

Proof. — We apply Proposition 4.13. The irreducible subquotients Φ_n and:

$$\mathrm{Z}\left(\left[-\frac{n-1}{2},\frac{n-3}{2}\right]+\left[-\frac{n-3}{2}\right]\right)=\begin{cases}\nu_{n-1}^{-1/2}\times\nu^{-(n-3)/2} & \text{if } e \text{ does not divide } n-2,\\ \Pi_n\cdot\nu^{-1} & \text{if } e \text{ divides } n-2,\end{cases}$$

always occur in (12.3). The irreducible subquotients $\mathrm{Z}([-(n-3)/2,(n-1)/2]+[-(n-1)/2])$ and $\mathrm{Z}([-(n-1)/2,(n-1)/2])=1_n$ occur if and only if e divides $n-2$. The irreducible subquotients $\mathrm{Z}([-(n-3)/2,(n+1)/2])=\nu_n$ and $\mathrm{Z}([-(n+1)/2,(n-3)/2])=\nu_n^{-1}$ do not occur, since e does not divide $n-1$ and $e>1$. □

Similarly, by applying Proposition 4.13, we have the following.

Lemma 12.6. — *Assume $e>1$ and $n\geqslant 4$, and suppose e does not divide $n-1$. The irreducible subquotients of $1_{n-2}\times 1_2\cdot\nu^{-n/2}$ are:*

(1) *the representations ν_n and Ψ_n if e does not divide n,*
(2) *the representations ν_n, ν_n^{-1} and Ψ_n if e divides n.*

Moreover, all subquotients appear with multiplicity 1 if $e>2$. If $e=2$, only ν_n may appear with multiplicity more than 1.

Lemma 12.7. — *Assume e does not divide $n-2$ nor $n-1$, thus $e>2$. Then:*

$$\mathbf{Q}(\Lambda_{n-1}^{*}\cdot\nu^{1/2}\times\nu^{-(n-3)/2})=1_{n-2}\times\mathrm{St}_2\cdot\nu^{-(n-2)/2}.$$

Proof. — Since e does not divide $n-2$, the product $1_{n-2}\times\nu^{-(n-3)/2}$ is irreducible, thus it is isomorphic to $\nu^{-(n-3)/2}\times 1_{n-2}$. Moreover, the representation $\nu^{-(n-1)/2}\times\nu^{-(n-3)/2}\times 1_{n-2}$ has a unique irreducible quotient by Proposition 4.2. It follows that this unique irreducible quotient is $\mathrm{St}_2\cdot\nu^{-(n-2)/2}\times 1_{n-2}$, which is irreducible by Proposition 7.4. □

Lemma 12.8. — *Assume $e>1$ and $n\geqslant 4$, and suppose e does not divide $n-1$. If the representation $\mathbf{Q}(\Lambda_{n-1}^{*}\cdot\nu^{1/2}\times\nu^{-(n-3)/2})$ is distinguished, then it is either 1_n or $1_{n-2}\times\mathrm{St}_2\cdot\nu^{-(n-2)/2}$.*

Proof. — If $e>2$ and does not divide $n-2$ we reduce to the case of Lemma 12.7. Therefore, we need only consider either $e=2$ or e divides $n-2$. The representation $\mathrm{B}=\Lambda_{n-1}^{*}\cdot\nu^{1/2}\times\nu^{-(n-3)/2}$ is a quotient of $\mathrm{U}=\nu^{-(n-1)/2}\times 1_{n-2}\times\nu^{-(n-3)/2}$. Observe that we have $[\mathrm{U}]=[\mathrm{P}]+[\mathrm{B}]$ where $\mathrm{P}=\nu_{n-1}^{-1/2}\times\nu^{-(n-3)/2}$. Now U has the same semisimplification as $1_{n-2}\times\nu^{-(n-1)/2}\times\nu^{-(n-3)/2}$, thus we have:

$$[\mathrm{U}]\;=\;\begin{cases}1_{n-2}\times\mathrm{St}_2\cdot\nu^{-(n-2)/2}+[1_{n-2}\times 1_2\cdot\nu^{-(n-2)/2}] & \text{if } e>2,\\ 1_{n-2}\times\mathrm{St}_2\cdot\nu^{-(n-2)/2}+[1_{n-2}\times 1_2\cdot\nu^{-(n-2)/2}]+[1_{n-2}\times 1_2\cdot\nu^{-n/2}] & \text{if } e=2,\end{cases}$$

since $1_{n-2}\times\mathrm{St}_2\cdot\nu^{-(n-2)/2}$ is irreducible (this follows from Proposition 7.4 if $e>2$, and from Proposition 4.1 together with the fact that St_2 is cuspidal when $e=2$.) Since e does not divide $n-1$, the irreducible subquotients occurring in $1_{n-2}\times 1_2\cdot\nu^{-(n-2)/2}$ by Lemma 12.5 are:

$$1_n,\ \nu_n,\ \Pi_n\cdot\nu^{-1},\ \Pi_n^{*}\cdot\nu,\ \Phi_n.$$

Moreover, all of them occur with multiplicity 1 except 1_n, which may appear with larger multiplicity if $e=2$. Also, By Lemma 12.6, since e does not divide $n-1$, the irreducible subquotients occurring in $1_{n-2}\times 1_2\cdot\nu^{-n/2}$ are ν_n, ν_n^{-1} and Ψ_n. Here Ψ_n always occurs with multiplicity one and if $e=2$ the other factors might appear with larger multiplicity. We will now obtain [B] by comparing [U] obtained from the two different expressions for [U] above.

Assume that $e > 2$ and e divides $n-2$. Then by Lemma 12.3 we have $[\mathrm{P}] = 1_n + \Pi_n \cdot \nu^{-1}$. Hence we have:

$$[\mathrm{B}] = \Phi_n + \Pi_n^* + 1_{n-2} \times \mathrm{St}_2 \cdot \nu^{-(n-2)/2}.$$

Next assume that $e = 2$. Then e necessarily divides $n-2$ (since e does not divide $n-1$). By Lemma 12.3, we have $[\mathrm{P}] = 1_n + \nu_n + \Pi_n^*$. Recall that if e divides n then $\Pi_n^* = \Pi_n \cdot \nu^{-1}$. Hence the only possible irreducible subquotients of B are:

$$1_n,\ \nu_n,\ 1_{n-2} \times \mathrm{St}_2 \cdot \nu^{-(n-2)/2},\ \Pi_n^* \cdot \nu,\ \Phi_n,\ \Psi_n.$$

The proof of Lemma 12.8 will be complete if we prove the following lemma.

Lemma 12.9. — *Assume $e > 1$ and $n \geqslant 4$, and suppose that e does not divide $n-1$. For any character $\chi \in \widehat{\mathrm{G}}_1$, the twists $\Phi_n \cdot \chi$ and $\Psi_n \cdot \chi$ are not distinguished.*

Proof. — Observe that $\Phi_n \cdot \chi$ and $\Psi_n \cdot \chi$ have only first and second derivatives which are nonzero. Thus we will use Lemma 8.8.

Assume the first derivative of $\Phi_n \cdot \chi$ has a quotient isomorphic to $\nu_{n-1}^{-1/2}$. By Lemma 8.17, this would imply that $\Phi_n \cdot \chi$ is a character, or that the multisegment that corresponds to it is made of one segment of length $n-1$ and one of length 1, which is not the case. The same argument holds for $\Psi_n \cdot \chi$.

From Lemma 12.6, we see that the second derivative of $\Psi_n \cdot \chi$ is $(\nu_{n-3}^{-1/2} \times \nu^{-(n+1)/2}) \cdot \chi$, and since e does not divide $n-1$ it is irreducible for all $\chi \in \widehat{\mathrm{G}}_1$ by Lemma 9.1. Thus it does not have any character as a quotient. Now we have:

$$\begin{aligned}
[1_{n-2} \times 1_2 \cdot \nu^{-(n-2)/2}]^{(2)} &= [\nu_{n-3}^{-1/2} \times \nu^{-(n-1)/2}] \\
&= \begin{cases} \nu_{n-2}^{-1} + 1_{n-2} + \Pi_{n-2}^* & \text{if } e \text{ divides } n-2. \\ \nu_{n-2}^{-1} + \Pi_{n-2}^* & \text{if } e \text{ does not divide } n-2. \end{cases}
\end{aligned}$$

By Lemma 6.1 and Corollary 6.3, we have $(\Pi_n \cdot \nu^{-1})^{(2)} = \nu_{n-2}$ and $(\Pi_n^* \cdot \nu)^{(2)} = 1_{n-2}$. Therefore, we conclude using Lemma 12.5 that the second derivative of Φ_n is Π_{n-2}^*. By Lemma 8.8, $\Phi_n \cdot \chi$ and $\Psi_n \cdot \chi$ are not distinguished. □

This ends the proof of Lemma 12.8. □

12.3. Distinction of $\mathbf{Q}(\Lambda_{n-1} \cdot \nu^{1/2} \times \nu^{-(n-3)/2})$

We begin this paragraph with a simple lemma which we will need in the sequel. We remind that $n \geqslant 4$ and e does not divide $n-1$.

Lemma 12.10. — *Let $n \geqslant 4$. Assume that $e > 1$ and let $\lambda, \mu \in \widehat{\mathrm{G}}_1 - \{\nu^{-(n-3)/2}\}$. Then the induced representation $1_{n-2} \times \lambda \times \mu$ has a unique irreducible quotient.*

Proof. — If $\lambda = \mu$, the result follows from [**13**, Lemma 6.1]. We thus assume that $\lambda \neq \mu$. By the geometric lemma, the semi-simplification of the Jacquet module $\boldsymbol{r}_{(n-2,1,1)}(1_{n-2} \times \lambda \times \mu)$ is the sum of the following representations:

(1) $1_{n-2} \otimes \lambda \otimes \mu$,
(2) $1_{n-2} \otimes \mu \otimes \lambda$,
(3) $[\nu_{n-3}^{-1/2} \times \lambda] \otimes \nu^{(n-3)/2} \otimes \mu$,
(4) $[\nu_{n-3}^{-1/2} \times \mu] \otimes \nu^{(n-3)/2} \otimes \lambda$,

(5) $[\nu_{n-3}^{-1/2} \times \lambda] \otimes \mu \otimes \nu^{(n-3)/2}$,
(6) $[\nu_{n-3}^{-1/2} \times \mu] \otimes \lambda \otimes \nu^{(n-3)/2}$,
(7) $[\nu_{n-4}^{-1} \times \lambda \times \mu] \otimes \nu^{(n-5)/2} \otimes \nu^{(n-3)/2}$,

in the Grothendieck group of finite length representations of the Levi subgroup $\mathrm{G}_{n-2} \times \mathrm{G}_1 \times \mathrm{G}_1$. If $\lambda, \mu \neq \nu^{(n-3)/2}$ then by [**14**, Lemme 2.4] the representation $1_{n-2} \times \lambda \times \mu$ has a unique irreducible subrepresentation. The result follows by taking contragredients. $\square$

Lemma 12.11. — *Assume that $e > 2$ and e does not divide $n-2$. Then:*

$$\mathbf{Q}(\Lambda_{n-1} \cdot \nu^{1/2} \times \nu^{-(n-3)/2}) = \Lambda_n.$$

Proof. — The representation $\mathrm{C} = \Lambda_{n-1} \cdot \nu^{1/2} \times \nu^{-(n-3)/2}$ is a quotient of:

$$\mathrm{W} = \nu_{n-2} \times \nu^{(n+1)/2} \times \nu^{-(n-3)/2}.$$

If we apply Lemma 12.10 with $\lambda = \nu^{(n-1)/2}$ and $\mu = \nu^{-(n-1)/2}$, which is possible since $e > 2$ and e does not divide $n-2$, we deduce that $\mathrm{W} \cdot \nu^{-1}$ (thus W) has a unique irreducible quotient. Since $\nu^{(n+1)/2} \times \nu^{-(n-3)/2}$ is irreducible, it is isomorphic to $\nu^{-(n-3)/2} \times \nu^{(n+1)/2}$. Thus $\nu_{n-1}^{1/2} \times \nu^{(n+1)/2}$ is a quotient of W, and it has the unique irreducible quotient Λ_n. $\square$

Lemma 12.12. — *Assume that $e > 1$ and $n \geqslant 4$. If $\mathbf{Q}(\Lambda_{n-1} \cdot \nu^{1/2} \times \nu^{-(n-3)/2})$ is distinguished, then it is Λ_n.*

Proof. — If $e > 2$ and does not divide $n-2$ we reduce to the case of Lemma 12.11. We may assume that $e = 2$ or e divides $n-2$. In this proof, W, C are as in Lemma 12.11 and U, P are as in Lemma 12.8. Assume that e divides $n-2$. Then $\nu^{(n-1)/2} = \nu^{-(n-3)/2}$ and therefore

$$\mathrm{W} \cdot \nu^{-1} = 1_{n-2} \times \nu^{-(n-3)/2} \times \nu^{-(n-1)/2}.$$

Therefore, we have

$$[\mathrm{W}] = [\mathrm{U} \cdot \nu] \quad \text{and} \quad [\mathrm{W}] = [\mathrm{P}^* \cdot \nu] + [\mathrm{C}]$$

where $\mathrm{P}^* \cdot \nu = \nu_{n-1}^{3/2} \times \nu^{(n-1)/2}$.

If $e > 2$ and e divides $n-2$, then we twist the subquotients of U in the proof of Lemma 12.8 by ν to get:

$$[\mathrm{W}] = \nu_{n-2} \times \mathrm{St}_2 \cdot \nu^{-(n-4)/2} + \Phi_n \cdot \nu + \Pi_n + \Pi_n \cdot \nu^2 + \nu_n$$

and $[\mathrm{P}^* \cdot \nu] = \nu_n + \Pi_n \cdot \nu^2$. It follows that:

$$[\mathrm{C}] = \nu_{n-2} \times \mathrm{St}_2 \cdot \nu^{-(n-4)/2} + \Phi_n \cdot \nu + \Pi_n.$$

Hence the only distinguished subquotient is Π_n which is the definition of Λ_n when e does not divide n.

Now $e = 2$, which necessarily divides $n-2$. Then $\mathrm{P}^* \cdot \nu$ is isomorphic to P. We twist the subquotients of U in the proof of Lemma 12.8 by ν to conclude that the only possible irreducible subquotients of W are:

$$1_n,\ \nu_n,\ \Pi_n,\ \Pi_n^*,\ \Phi_n \cdot \nu,\ \Psi_n \cdot \nu$$

with all representations except possibly 1_n and ν_n appearing with multiplicity 1. Since $[\mathrm{P}] = 1_n + \nu_n + \Pi_n^*$ it follows that the only possible irreducible subquotients of C are:

$$1_n,\ \nu_n,\ \Pi_n,\ \Phi_n \cdot \nu,\ \Psi_n \cdot \nu.$$

Hence the only distinguished subquotient is 1_n which is the definition of Λ_n when e divides n. This completes the proof of the Lemma. $\square$

Remark 12.13. — In the complex case, it has been proved in [**1**] that the dimension:

$$d(\pi) = \dim \mathrm{Hom}_{\mathrm{H}_n}(\pi, \mathrm{R})$$

satisfies $d(\pi) \leqslant 1$ for all $\pi \in \widehat{\mathrm{G}}_n$. This multiplicity one property does not hold in general when R has positive characteristic (see Paragraph 1.12). However, when $e > 1$, we expect that $d(\pi) \leqslant 1$ for all irreducible ℓ-modular representations π of G_n. When $e \geqslant 3$, this can be proved just as in [**20**]. But this approach fails when $e = 2$ for the irreducible principal series $1 \times 1 \times 1$ of $\mathrm{GL}_3(\mathrm{F})$. This is due the fact that the proof in [**20**] is by contradiction and relies on analyzing a reducible principal series representation of $\mathrm{GL}_4(\mathrm{F})$. When $e \geqslant 3$, this particular reducible principal series has at most one distinguished subquotient, whose multiplicity is one and the proof of that reduces to multiplicity one proved in Theorem 3.8. When $e = 2$, this is no longer true. The concerned principal series of $\mathrm{GL}_4(\mathrm{F})$ has more than one distinguished subquotient and the proof fails.

It is interesting to note the analogy of the situation in the case of $e = 1$ of Theorem 3.5 with [**18**, Corollary 3.3], where the author shows that $d(\pi) \leqslant 2$ for π an irreducible representation of $\mathrm{GL}_n(\mathbb{F}_q)$ and R an algebraically closed field of characteristic coprime to $2q$.

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V. Sécherre, Université de Versailles Saint-Quentin-en-Yvelines, Laboratoire de Mathématiques de Versailles, 45 avenue des Etats-Unis, 78035 Versailles cedex, France • *E-mail :* `vincent.secherre@math.uvsq.fr`

C. G. Venketasubramanian, Department of Mathematics, Ben-Gurion University of the Negev, P.O.B 653, Beer Sheva 84105, Israel. • *E-mail :* `coolimut@math.bgu.ac.il`