SUPERCUSPIDAL REPRESENTATIONS OF $\mathrm{GL}_n(\mathrm{F})$ DISTINGUISHED BY A GALOIS INVOLUTION

by

Vincent Sécherre

Abstract. — Let F/F_0 be a quadratic extension of non-Archimedean locally compact fields of residual characteristic $p \neq 2$, and let σ denote its non-trivial automorphism. Let R be an algebraically closed field of characteristic different from p. To any cuspidal representation π of $\mathrm{GL}_n(\mathrm{F})$, with coefficients in R, such that $\pi^\sigma \simeq \pi^\vee$ (such a representation is said to be σ-selfdual) we associate a quadratic extension D/D_0, where D is a tamely ramified extension of F and D_0 is a tamely ramified extension of F_0, together with a quadratic character of $\mathrm{D}_0^\times$. When π is supercuspidal, we give a necessary and sufficient condition, in terms of these data, for π to be $\mathrm{GL}_n(\mathrm{F}_0)$-distinguished. When the characteristic ℓ of R is not 2, denoting by ω the non-trivial R-character of $\mathrm{F}_0^\times$ trivial on F/F_0-norms, we prove that any σ-selfdual supercuspidal R-representation is either distinguished or ω-distinguished, but not both. In the modular case, that is when $\ell > 0$, we give examples of σ-selfdual cuspidal non-supercuspidal representations which are not distinguished nor ω-distinguished. In the particular case where R is the field of complex numbers, in which case all cuspidal representations are supercuspidal, this gives a complete distinction criterion for arbitrary complex cuspidal representations, as well as a purely local proof, for cuspidal representations, of the dichotomy and disjunction theorem due to Kable and Anandavardhanan–Kable–Tandon, when $p \neq 2$.

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1. Introduction

1.1.

Let F/F_0 be a separable quadratic extension of non-Archimedean locally compact fields of residual characteristic p, and let σ denote its non-trivial automorphism. Let $\mathbf{G}$ be a connected reductive group defined over F_0, let G denote the locally profinite group $\mathbf{G}(\mathrm{F})$ equipped with the natural action of σ and $\mathrm{G}^\sigma = \mathbf{G}(\mathrm{F}_0)$ be the σ-fixed points subgroup. The study of those irreducible (smooth) complex representations of G which are G^σ-distinguished, that is which carry a non-zero G^σ-invariant linear form, goes back to the 1980's. We refer to [**22, 28**] for the initial motivation for distinguished representations in a global context, and to [**23, 16**] in a non-Archimedean context.

1.2.

In this work, we will consider the case where **G** is the general linear group GL_n for $n \geqslant 1$. We thus have $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ and $\mathrm{G}^\sigma = \mathrm{GL}_n(\mathrm{F}_0)$. In this case, it is well-known (see [**41, 16, 42**]) that any distinguished irreducible complex representation π of G is *σ-selfdual*, that is, the contragredient $\pi^\vee$ of π is isomorphic to $\pi^\sigma = \pi \circ \sigma$, and the space $\mathrm{Hom}_{\mathrm{G}^\sigma}(\pi, 1)$ of all G^σ-invariant linear forms on π has dimension 1. Also, the central character of π is trivial on $\mathrm{F}_0^\times$. This gives us two necessary conditions for an irreducible complex representation of G to be distinguished, and it is natural to ask whether or not they are sufficient.

1.3.

First, let us consider the case where F/F_0 is replaced by a quadratic extension $\boldsymbol{k}/\boldsymbol{k}_0$ of finite fields of characteristic p. In this case, Gow [**18**] proved that an irreducible complex representation of $\mathrm{GL}_n(\boldsymbol{k})$ is $\mathrm{GL}_n(\boldsymbol{k}_0)$-distinguished if and only if π is σ-selfdual. (Note that the latter condition automatically implies that the central character is trivial on $\boldsymbol{k}_0^\times$.) Besides, if $p \neq 2$, the σ-selfdual irreducible representations of $\mathrm{GL}_n(\boldsymbol{k})$ are those which arise from some irreducible representation of the unitary group $\mathrm{U}_n(\boldsymbol{k}/\boldsymbol{k}_0)$ by base change (see Kawanaka [**31**]).

1.4.

We now go back to the non-Archimedean setting of §1.2 and consider a σ-selfdual irreducible complex representation π of G whose central character is trivial on $\mathrm{F}_0^\times$. When π is cuspidal and F/F_0 is unramified, Prasad [**42**] proved that, if $\omega = \omega_{\mathrm{F}/\mathrm{F}_0}$ denotes the non-trivial character of $\mathrm{F}_0^\times$ trivial on F/F_0-norms, then π is either distinguished or ω-distinguished, the latter case meaning that the complex vector space $\mathrm{Hom}_{\mathrm{G}^\sigma}(\pi, \omega \circ \det)$ is non-zero.

When $p \neq 2$ and π is an *essentially tame* cuspidal representation, that is, when the number of unramified characters χ of G such that $\pi\chi \simeq \pi$ is prime to p, Hakim and Murnaghan [**24**] gave sufficient conditions for π to be distinguished. These conditions are stated in terms of admissible pairs [**27**], which parametrize essentially tame cuspidal complex representations of G ([**27, 9**]). Note that they assume F has characteristic 0, but their approach also works in characteristic p.

When π is a discrete series representation and F has characteristic 0, Kable [**30**] proved that if π is σ-selfdual, then it is either distinguished or ω-distinguished: this is the *Dichotomy Theorem.* In addition, Anandavardhanan, Kable and Tandon [**2**] proved that π can't be both distinguished and ω-distinguished: this is the *Disjunction Theorem.* The proofs use global arguments, which is why F were assumed to have characteristic 0, and the Asai L-function of π. However, these results still hold when F has characteristic $p \neq 2$, as is explained in [**3**] Appendix A. Note that:

– the disjunction theorem implies that the sufficient conditions given by Hakim and Murnaghan in [**24**] in the essentially tame cuspidal case are necessary conditions as well;

– the dichotomy theorem implies that, when n is odd, any σ-selfdual discrete series representation of G with central character trivial on $\mathrm{F}_0^\times$ is automatically distinguished. Indeed, an ω-distinguished irreducible representation has a central character whose restriction to $\mathrm{F}_0^\times$ is ω^n.

When $p \neq 2$ and π is cuspidal of level zero – in particular π is essentially tame – Coniglio [**11**] gave a necessary and sufficient condition of distinction in terms of admissible pairs. Her proof is purely local, and does not use the disjunction theorem. (In fact, she considers the more general case where $\mathbf{G}$ is an inner form of GL_n over F_0, and representations of level zero of $\mathbf{G}(\mathrm{F})$ whose local Jacquet-Langlands transfer to $\mathrm{GL}_n(\mathrm{F})$ is cuspidal.)

If one takes the classification of distinguished cuspidal representations of general linear groups for granted, and assuming F has characteristic 0, Anandavardhanan and Rajan [**5, 1**] classified all distinguished discrete series representations of G in terms of the distinction of their cuspidal support (see also [**32**]) and Matringe [**33, 35**] classified distinguished generic, as well as distinguished unitary, representations of G in terms of the Langlands classification. This provides a class of representations for which the dichotomy and disjunction theorems do not hold: some σ-selfdual generic irreducible representations are nor distinguished nor ω-distinguished, and some are both.

In [**20**] Max Gurevich extended the dichotomy theorem to the class of *ladder* representations, which contains all discrete series representations: a σ-selfdual ladder representation of G is either distinguished or ω-distinguished, but it may be both (see [**20**] Theorem 4.6). Here again, F is assumed to have characteristic 0.

Finally, one can deduce from these works the connection between distinction for generic irreducible representations of G and base change from a quasi-split unitary group: see [**17**] Theorem 6.2, [**4**] Theorem 2.3 or [**21**].

1.5.

The discussion above leaves us with an open problem about cuspidal representations: find a distinction criterion for an *arbitrary* σ-selfdual cuspidal representation π, with no assumption on the characteristic of F, on the ramification of F/F_0, on n nor on the torsion number of π (that is, the number of unramified characters χ such that $\pi\chi \simeq \pi$).

In this paper, *assuming that* $p \neq 2$, we propose an approach which allows us to generalize both Hakim-Murnaghan's and Coniglio's distinction criterions to *all* cuspidal irreducible complex representations of G and which works:

– with no assumption on the characteristic of F (apart from the assumption "$p \neq 2$"),
– with purely local methods,
– not only for complex representations, but more generally for representations with coefficients in an algebraically closed field of arbitrary characteristic $\ell \neq p$.

We thus give a complete solution to the problem above for cuspidal complex representations when $p \neq 2$. We actually do more: we solve this problem in the larger context of *supercuspidal* representations with coefficients in an algebraically closed field of arbitrary characteristic $\ell \neq p$.

1.6.

First, let us say a word about the third item above. The theory of smooth representations of $\mathrm{GL}_n(\mathrm{F})$ with coefficients in an algebraically closed field of characteristic $\ell \neq p$ has been initiated by Vignéras [**47, 48**] in view to extend the local Langlands programme to representations with coefficients in a field – or a ring – as general as possible (see for instance [**49, 26**]). Inner forms have also been taken into account ([**37, 43**]) and the congruence properties of the local Jacquet–Langlands correspondence have been studied in [**13, 39**]. It is thus natural to extend the study of distinguished representations to this wider context, where the field of complex numbers is replaced by a more general field. Very little has been done about distinction of modular representations so far: a first study can be found in [**44**].

An important phenomenon in the modular case, that is when $\ell > 0$, is that a cuspidal representation π may occur as a subquotient of a proper parabolically induced representation (see [**46**] Corollaire 5). When this is not the case, that is when π does not occur as a subquotient of a proper parabolically induced representation, π is said to be *supercuspidal*.

1.7.

From now on, we fix an algebraically closed field R *of arbitrary characteristic* $\ell \neq p$*, and consider irreducible smooth representations of* G *with coefficients in* R. Note that ℓ can be 0.

We first notice that, as in the complex case, any distinguished irreducible representation π of G with coefficients in R is σ-selfdual and $\mathrm{Hom}_{\mathrm{G}^\sigma}(\pi, 1)$ is 1-dimensional (see Theorem 4.1).

We prove that, if $\ell \neq 2 \neq p$, the dichotomy and disjunction theorems hold for all *supercuspidal* representations with coefficients in R. In particular, when R is the field of complex numbers, in which case any cuspidal representation is supercuspidal, we get a purely local proof of the dichotomy and disjunction theorems for cuspidal representations in the case where $p \neq 2$.

When $\ell = 2 \neq p$, in which case there is no character of order 2 on $\mathrm{F}_0^\times$, the dichotomy theorem takes a simplified form: any σ-selfdual supercuspidal representation is distinguished. Let us summarize this first series of results in the theorem below.

***Theorem 1.1* (Theorem 10.8).** — *Suppose that* $p \neq 2$*, and let* π *be a* σ*-selfdual supercuspidal irreducible* R*-representation of* G.

(1) *If* $\ell = 2$*, then* π *is distinguished.*
(2) *If* $\ell \neq 2$*, then* π *is either distinguished or* ω*-distinguished, but not both.*

In the modular case, for $\ell > 2$, we give examples of σ-selfdual cuspidal, non-supercuspidal representations which are not distinguished nor ω-distinguished (see Remarks 7.5 and 2.8).

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From now on, until the end of this introduction, we will assume that $p \neq 2$.

1.8.

The dichotomy and disjunction theorem stated in Theorem 1.1 relies on a distinction criterion, which we state in Theorem 1.2. The basic idea is that we canonically associate to any σ-selfdual supercuspidal representation π of G a finite extension D of F equipped with an F_0-involution extending σ and a quadratic character δ_0 of the fixed points of $\mathrm{D}^\times$; it is these data which govern the distinction of π. The character δ_0 refines the information given by the central character of π in the sense that they coincide on $\mathrm{F}_0^\times$, the latter one being not enough in general to determine whether π is distinguished or not. To state our distinction criterion, let us write D_0 for the fixed points subfield of D.

***Theorem 1.2* (Theorem 10.9)**. — *A σ-selfdual supercuspidal* R*-representation of* G *is distinguished if and only if:*

(1) *either $\ell = 2$,*

(2) *or $\ell \neq 2$ and:*

(a) *if* F/F_0 *is ramified,* D/D_0 *is unramified and* $\mathrm{D}_0/\mathrm{F}_0$ *has odd ramification order, then the character δ_0 is non-trivial,*

(b) *otherwise, δ_0 is trivial.*

Even in the complex case, this is the first time a necessary and sufficient distinction criterion is exhibited for an *arbitrary* cuspidal representation of $\mathrm{GL}_n(\mathrm{F})$ in odd residual characteristic, in the spirit of [**24, 25, 11**].

1.9.

The starting point of our strategy for proving Theorem 1.2 is to use Bushnell-Kutzko's construction of cuspidal representations of G via compact induction. This construction, elaborated in the complex case by Bushnell and Kutzko [**10**], has been extended to the modular case by Vignéras [**47**] and Minguez–Sécherre [**36**]. There is a family of pairs $(\mathbf{J}, \boldsymbol{\lambda})$, made of certain compact mod centre open subgroups $\mathbf{J}$ of G and certain irreducible representations $\boldsymbol{\lambda}$ of $\mathbf{J}$, such that:

– for any such pair $(\mathbf{J}, \boldsymbol{\lambda})$, the compact induction of $\boldsymbol{\lambda}$ to G is irreducible and cuspidal;

– any irreducible cuspidal representation of G occurs in this way, for a pair $(\mathbf{J}, \boldsymbol{\lambda})$ uniquely determined up to G-conjugacy.

Such pairs are called *extended maximal simple types* in [**10**], which we will abbreviate to *types* for simplicity. We need to give more details about the structure of these types:

(1) The group $\mathbf{J}$ has a unique maximal compact subgroup J, and a unique maximal normal pro-p-group J^1.

(2) There is a group isomorphism $\mathrm{J}/\mathrm{J}^1 \simeq \mathrm{GL}_m(\boldsymbol{l})$ for some divisor m of n and finite extension $\boldsymbol{l}$ of the residual field $\boldsymbol{k}$ of F.

(3) The restriction of $\boldsymbol{\lambda}$ to J^1 is made of copies of a single irreducible representation η, which extends (not uniquely, nor canonically) to $\mathbf{J}$.

(4) Given a representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η, there is a unique irreducible representation $\boldsymbol{\rho}$ of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\lambda}$ is isomorphic to $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$, and $\boldsymbol{\rho}$ restricts irreducibly to J.

(5) The representation of $\mathrm{GL}_m(\boldsymbol{l})$ obtained by restricting $\boldsymbol{\rho}$ to J is cuspidal.

The integer m, called the *relative degree* of π, is uniquely determined by π. There is another type-theoretical invariant called the *tame parameter field* of π: this is a tamely ramified extension T of F, uniquely determined up to F-isomorphism, whose degree divides n/m and whose residual field is $\boldsymbol{l}$ (see [**8**] for more details). Note that π if essentially tame if and only if $[\mathrm{T} : \mathrm{F}] = n/m$.

1.10.

Now consider a σ-selfdual cuspidal R-representation π of G. The starting point of all our work is [**3**] Theorem 4.1, which asserts that among all the types contained in π, there is a type $(\mathbf{J}, \boldsymbol{\lambda})$

which is σ-selfdual, that is $\mathbf{J}$ is σ-stable and $\boldsymbol{\lambda}^\vee$ is isomorphic to $\boldsymbol{\lambda}^\sigma$. Moreover, the tame parameter field T of π is equipped with an F_0-involution. If T_0 denotes the fixed points subfield of T, then T/T_0 is a quadratic extension, uniquely determined up to F_0-isomorphism. The invariants m and T/T_0 associated with π will play a central role in what follows.

First, the following result says that the distinction of π can be detected by a σ-selfdual type.

***Theorem 1.3* (Theorem 6.1).** — *Let π be a σ-selfdual cuspidal representation of* G. *Then π is distinguished if and only if it contains a distinguished σ-selfdual type, that is a σ-selfdual type* $(\mathbf{J},\boldsymbol{\lambda})$ *such that* $\mathrm{Hom}_{\mathbf{J}\cap G^\sigma}(\boldsymbol{\lambda},1)$ *is non-zero.*

The proof of this theorem – which occupies Section 6 – is the most technical part of the paper: starting with a σ-selfdual type $(\mathbf{J},\boldsymbol{\lambda})$ contained in π and $g \in G$, one has to prove that, if the type $(\mathbf{J}^g,\boldsymbol{\lambda}^g)$ is distinguished, then it is σ-selfdual, that is $\sigma(g)g^{-1} \in \mathbf{J}$. First, one determines the set of the $g \in G$ such that η^g is distinguished, as well as the dimension of the space of invariant linear forms (Paragraphs 6.1 to 6.3); then, one analyzes the distinction of $\boldsymbol{\kappa}^g$ (Paragraphs 6.4 and 6.5); one obtains the final statement by using the cuspidality of the representation of $\mathrm{GL}_m(\boldsymbol{l})$ induced by $\boldsymbol{\rho}$ (see Theorem 6.21).

1.11.

When T is unramified over T_0, the σ-selfdual types contained in π form a single G^σ-conjugacy class. When T is ramified over T_0, the σ-selfdual types contained in π form $\lfloor m/2\rfloor+1$ different G^σ-conjugacy classes, characterized by an integer $i \in \{0,\dots,\lfloor m/2\rfloor\}$ called the *index* of the class. Since the space $\mathrm{Hom}_{G^\sigma}(\pi,1)$ has dimension 1, only one of these conjugacy classes can contribute to distinction: we prove that it is the one with maximal index. This gives us the following refinement of Theorem 1.3.

***Proposition 1.4* (Corollary 6.24 and Proposition 7.1).** — *Let π be a σ-selfdual cuspidal representation of* G. *Let m be its relative degree and* T/T_0 *be its associated quadratic extension.*

(1) *If* T *is unramified over* T_0, *then π is distinguished if and only if any of its σ-selfdual types is distinguished.*

(2) *If* T *is ramified over* T_0, *then π is distinguished if and only if any of its σ-selfdual types of index $\lfloor m/2\rfloor$ is distinguished.*

Note that this proposition is proved in [**3**] in a different manner, based on a result of Ok [**40**]. However, the proof given in the present article is more likely to generalize to other situations.

When T/T_0 is ramified, one can be more precise (see Proposition 7.1): if π is distinguished, m is either even or equal to 1. It is not difficult to construct σ-selfdual cuspidal representations of G such that T/T_0 is ramified and $m > 1$ is odd: such cuspidal representations are not distinguished nor ω-distinguished (see Remark 7.5).

In the case where T/T_0 is unramified, m is odd for any supercuspidal σ-selfdual representation (see Proposition 9.8). This does not hold for σ-selfdual cuspidal representations (which is easy to see), and this does not even hold for distinguished cuspidal representations: Kurinczuk and Matringe recently proved that, when F/F_0 is unramified and $n = \ell = 2$, any σ-selfdual non-supercuspidal cuspidal representation of $\mathrm{GL}_2(\mathrm{F})$ of level zero (thus of relative degree 2) is distinguished.

1.12.

As in the previous paragraph, π is a σ-selfdual cuspidal R-representation of G. The following definition will be convenient to us (see Remark 10.2 for the connection with the usual notion of a generic representation).

***Definition 1.5* (Definition 10.1).** — A σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$ in π is *generic* if either T/T_0 is unramified, or T/T_0 is ramified and this type has index $\lfloor m/2 \rfloor$.

Proposition 1.4 thus says that, up to G^σ-conjugacy, a σ-selfdual cuspidal representation π contains a unique generic σ-selfdual type, and that π is distinguished if and only if such a type is distinguished (see Theorem 10.3). This uniqueness property is crucial to the proof of the disjunction part of Theorem 1.1.

Let us fix a generic σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$ in π. Recall that, by construction, $\boldsymbol{\lambda}$ can be decomposed (non canonically) as $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$. However, not any of these decompositions are suitable for our purpose. It is not difficult to prove that $\boldsymbol{\kappa}$ can be chosen to be σ-selfdual, but this is not enough: we need to prove that $\boldsymbol{\kappa}$ can be chosen to be both σ-selfdual and distinguished. The strategy of the proof depends on the ramification of T over T_0. This is why we treat separately the ramified and the unramified cases, in Sections 7 and 9, respectively.

The easiest case is when T/T_0 is ramified. Using the fact that m is either even or equal to 1, we prove that $\boldsymbol{\kappa}$ can be chosen to be distinguished by adapting a result of Matringe [**34**] (which we do in Paragraph 2.3).

When T/T_0 is unramified, the existence of a distinguished $\boldsymbol{\kappa}$ is more difficult to establish. Our proof requires π to be supercuspidal, since in that case m is known to be odd, thus $\mathrm{GL}_m(\boldsymbol{l})$ has $\mathrm{GL}_m(\boldsymbol{l}_0)$-distinguished supercuspidal representations in characteristic 0, where $\boldsymbol{l}_0$ is the residual field of T_0 (see the proof of Proposition 9.4).

In both cases, a distinguished $\boldsymbol{\kappa}$ is automatically σ-selfdual, and π is distinguished if and only if $\boldsymbol{\rho}$ is distinguished. Considering $\boldsymbol{\rho}$ as a (σ-selfdual) level zero type, we are then reduced to the level zero case, which has been treated by Coniglio in the complex case. We thus have to extend her results to the modular case, which we know how to do when π is supercuspidal only.

To summarize, we need the assumption that π is supercuspidal in Theorems 1.1 and 1.2 for two reasons: for the existence of a distinguished $\boldsymbol{\kappa}$ in the case when T/T_0 is unramified, and for the level zero case.

1.13.

To study the distinction of $\boldsymbol{\rho}$ when π is supercuspidal, we use admissible pairs of level zero as in Coniglio [**11**]. We attach to $\boldsymbol{\rho}$ a pair $(\mathrm{D}/\mathrm{T}, \delta)$ made of an unramified extension of degree m equipped with an involutive T_0-algebra homomorphism, non-trivial on T, denoted by σ, together with a character δ of $D^\times$ such that $\delta \circ \sigma = \delta^{-1}$. (See Paragraphs 5.3, 5.5 although the result is presented in a different way there.)

However, the distinguished representation $\boldsymbol{\kappa}$ of Paragraph 1.12 is not unique in general, thus neither $\boldsymbol{\rho}$ nor δ are. Write D_0 for the σ-fixed points of D, and δ_0 for the restriction of δ to $D_0^\times$. This is a quadratic character, trivial on D/D_0-norms. We prove in Proposition 10.5 that the pair $(D/D_0, \delta_0)$ is uniquely determined by π up to F_0-isomorphism. This is the one occuring in our distinction criterion theorem 1.2.

It remains to explain our strategy to prove the distinction criterion for $\boldsymbol{\rho}$, in the modular case, in terms of the character δ_0, as well as the dimension of the space of invariant linear forms. This depends on the ramification of T/T_0.

The easiest case is when T/T_0 is unramified. In this case, we are reduced to studying the distinction of supercuspidal representations of $GL_m(\boldsymbol{l})$ by $GL_m(\boldsymbol{l}_0)$. That any distinguished irreducible representation is σ-selfdual follows from a finite and ℓ-modular version of Theorem 4.1 (see Remark 4.3). For the converse statement, we use a lifting argument to characteristic 0, based on the fact that any σ-selfdual supercuspidal $\overline{\mathbf{F}}_\ell$-representation has a σ-selfdual $\overline{\mathbf{Q}}_\ell$-lift, where $\overline{\mathbf{Q}}_\ell$ is an algebraic closure of the field of ℓ-adic numbers and $\overline{\mathbf{F}}_\ell$ is its residual field. This does not hold for σ-selfdual non-supercuspidal representations: by Remark 2.8, there are σ-selfdual cuspidal representations, with m even, which are not distinguished.

In the case where T/T_0 is ramified, we are reduced to studying the distinction of supercuspidal representations of $GL_m(\boldsymbol{l})$ by either $GL_1(\boldsymbol{l})$ if $m = 1$, or $GL_r(\boldsymbol{l}) \times GL_r(\boldsymbol{l})$ if $m = 2r$ is even. It is more difficult, as we do not have an analogue of Theorem 4.1. Our proof relies on the structure of the projective envelope of a supercuspidal representation of $GL_m(\boldsymbol{l})$, as well as a lifting argument to characteristic 0. We prove that a supercuspidal representation is distinguished if and only if it is selfdual. Unlike the complex case, one can find σ-selfdual cuspidal representations, with $m > 1$ odd, which are not distinguished (see Remark 2.18).

In both cases, we prove that a σ-selfdual supercuspidal representation of $GL_m(\boldsymbol{l})$ is distinguished if any only if it admits a distinguished lift to characteristic 0. We conclude by the following theorem.

***Theorem 1.6* (Theorem 10.11).** — *Let π be a σ-selfdual supercuspidal representation of the group* G *with coefficients in $\overline{\mathbf{F}}_\ell$.*

(1) *The representation π admits a σ-selfdual supercuspidal lift to $\overline{\mathbf{Q}}_\ell$.*

(2) *Let $\widetilde{\pi}$ be a σ-selfdual lift of π, and suppose that $\ell \neq 2$. Then $\widetilde{\pi}$ is distinguished if and only if π is distinguished.*

1.14.

In this paragraph, we discuss the assumption $p \neq 2$. In Section 2, in which we study the finite field case, we assume that $p \neq 2$ in Paragraphs 2.3 and 2.4 only: see Remark 2.11. Note that nor Gow's results [**18**] nor their modular version established in §2.2 require p to be odd. The same is true of the results of Prasad and Flicker – as well as their modular version proved in Section 4 – asserting that any distinguished irreducible representation π of G is σ-selfdual and $\mathrm{Hom}_{G^\sigma}(\pi, 1)$ has dimension 1.

From Paragraph 5.4 the assumption $p \neq 2$ is crucial (as in [**24, 25**] and to a lesser extent [**11**]). We use at many places, in particular in Section 6 and in the proof of the σ-selfdual type theorem 5.10, the fact that the first cohomology set of $\mathrm{Gal}(\mathrm{F}/\mathrm{F}_0)$ in a pro-p-group is trivial. I do not know whether or not Theorem 5.10 still holds when $p = 2$.

I also do not now whether the dichotomy and disjunction theorems hold when F has characteristic 2. The only exception is Prasad's dichotomy theorem [**42**] for cuspidal complex representations when F/F_0 is unramified, which remains the only known distinction criterion for cuspidal representations in arbitrary residual characterictic. Note that Prasad's approach does not work in the modular case, for [**42**] Theorem 1 does not hold in characteristic $\ell > 0$.

1.15.

Finally, let us mention that the methods developed in this paper are expected to generalize to other groups. The distinction of supercuspidal representations of $\mathrm{GL}_n(\mathrm{F})$ by a unitary group is currently explored by Jiandi Zou in his ongoing PhD thesis at Université de Versailles St-Quentin.

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2. The finite field case

The aim of this section is to extend to the modular case some results about distinction of cuspidal representations of GL_n over a finite field which are know in the complex case only. These results will be needed in Sections 8 and 9, but this section can also be read independently from the rest of the article.

In this section, $\boldsymbol{k}$ is a finite field whose characteristic is an arbitrary prime number p. In Paragraphs 2.3 and 2.4, we will assume that p is odd. Let q denote the cardinality of $\boldsymbol{k}$.

Let R be an algebraically closed field of characteristic different from p, denoted ℓ. (Note that ℓ can be 0.) We say that we are in the "modular case" when we consider the case where $\ell > 0$. By *representation* of a finite group, we mean a representation on an R-vector space.

Given a representation π of a finite group G, we write $\pi^\vee$ for the contragredient of π. Given a subgroup H of G, we say that π is H-*distinguished* if the space $\mathrm{Hom}_{\mathrm{H}}(\pi, 1)$ is non-zero, where 1 denotes the trivial character of H.

Given $n \geqslant 1$, an irreducible representation of $\mathrm{GL}_n(\boldsymbol{k})$ is said to be *cuspidal* if it has no non-zero invariant vector under the unipotent radical of any proper parabolic subgroup or, equivalently, if it does not occur as a subrepresentation of any proper parabolically induced representation. It is *supercuspidal* if it does not occur as a subquotient of a proper parabolically induced representation. When R has characteristic 0, any cuspidal representation is supercuspidal.

When $\ell > 0$, we denote by $\overline{\mathbf{Q}}_\ell$ an algebraic closure of the field of ℓ-adic numbers, by $\overline{\mathbf{Z}}_\ell$ the ring of ℓ-adic integers in $\overline{\mathbf{Q}}_\ell$ and by $\overline{\mathbf{F}}_\ell$ the residue field of $\overline{\mathbf{Z}}_\ell$. We refer to [**45**] §15 for a definition of the reduction mod ℓ of a $\overline{\mathbf{Q}}_\ell$-representation of a finite group.

2.1. Parametrisation of supercuspidal representations

For the results stated in this paragraph, we refer to [**19, 14, 15, 29**] (see also [**47**] III.2, and [**38**] §2.6).

Let $\boldsymbol{t}/\boldsymbol{k}$ be an extension of degree $n \geqslant 1$ of finite fields of characteristic p. Fix a $\boldsymbol{k}$-embedding of $\boldsymbol{t}$ in the matrix algebra $\mathrm{M}_n(\boldsymbol{k})$, and consider $\boldsymbol{t}^\times$ as a maximal torus in $\mathrm{GL}_n(\boldsymbol{k})$. An R-character ξ of $\boldsymbol{t}^\times$ is said to be $\boldsymbol{k}$-*regular* if the characters $\xi, \xi^q, \dots, \xi^{q^{n-1}}$ are all distinct.

Let ξ be a $\boldsymbol{k}$-regular R-character of $\boldsymbol{t}^\times$. By Green [**19**] when R has characteristic 0 and James [**29**] when R has positive characteristic $\ell \neq p$, there is a supercuspidal irreducible representation ρ_ξ of $\mathrm{GL}_n(\boldsymbol{k})$, uniquely determined up to isomorphism, such that:

$$\mathrm{tr}\ \rho_\xi(g) = (-1)^{n-1} \cdot \sum_\gamma \xi^\gamma(g) \tag{2.1}$$

for all $g \in \boldsymbol{t}^\times$ with irreducible characteristic polynomial, where γ runs over $\mathrm{Gal}(\boldsymbol{t}/\boldsymbol{k})$. This induces a surjective map:

$$\xi \mapsto \rho_\xi \tag{2.2}$$

between $\boldsymbol{k}$-regular characters of $\boldsymbol{t}^\times$ and isomorphism classes of supercuspidal irreducible representations of $\mathrm{GL}_n(\boldsymbol{k})$, whose fibers are $\mathrm{Gal}(\boldsymbol{t}/\boldsymbol{k})$-orbits.

Suppose that R is the field $\overline{\mathbf{Q}}_\ell$. In the next proposition, we record the main properties of the reduction mod ℓ of supercuspidal $\overline{\mathbf{Q}}_\ell$-representations of $\mathrm{GL}_n(\boldsymbol{k})$.

Proposition 2.1 **([14, 15, 29])**. — *Let ξ be a $\boldsymbol{k}$-regular $\overline{\mathbf{Q}}_\ell$-character of $\boldsymbol{t}^\times$ and ρ be the supercuspidal irreducible representation which corresponds to it.*

(1) *The reduction mod ℓ of ρ, denoted $\overline{\rho}$, is irreducible and cuspidal.*

(2) *The representation $\overline{\rho}$ is supercuspidal if and only if the reduction mod ℓ of ξ, denoted $\overline{\xi}$, is $\boldsymbol{k}$-regular.*

(3) *When $\overline{\xi}$ is $\boldsymbol{k}$-regular, the supercuspidal irreducible $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_n(\boldsymbol{k})$ which corresponds to it is $\overline{\rho}$.*

Moreover, for any cuspidal irreducible $\overline{\mathbf{F}}_\ell$-representation π of $\mathrm{GL}_n(\boldsymbol{k})$, there is a supercuspidal irreducible $\overline{\mathbf{Q}}_\ell$-representation ρ of $\mathrm{GL}_n(\boldsymbol{k})$ whose reduction mod ℓ is π. Such a representation ρ is said to be a *lift* of π.

Remark 2.2. — Here are a couple of additional properties which we will use at various places.

(1) The identity (2.1) shows that, if ξ is a $\boldsymbol{k}$-regular character of $\boldsymbol{t}^\times$ and ρ is the supercuspidal representation corresponding to ξ by (2.2), then its contragredient $\rho^\vee$ corresponds to ξ^{-1}.

(2) Let $\iota : \mathrm{R} \to \mathrm{R}'$ be an embedding of algebraically closed fields of characteristic $\ell \neq p$. Then any irreducible R′-representation π' of $\mathrm{GL}_n(\boldsymbol{k})$ is isomorphic to $\pi \otimes \mathrm{R}'$ for a uniquely determined irreducible R-representation π of $\mathrm{GL}_n(\boldsymbol{k})$, which follows from the fact that, since $\mathrm{GL}_n(\boldsymbol{k})$ is finite, the trace of π' takes values in $\iota(\mathrm{R})$. Given a subgroup H of $\mathrm{GL}_n(\boldsymbol{k})$, the representation π is

cuspidal (respectively supercuspidal, H-distinguished) if and only if π' is cuspidal (respectively supercuspidal, H-distinguished). Moreover, by (2.1), if π is supercuspidal and corresponds to the $\boldsymbol{k}$-regular R-character ξ, then π' corresponds to the $\boldsymbol{k}$-regular R$'$-character $\xi' = \iota \circ \xi$.

2.2. The Galois case

Recall that p is an arbitrary prime number. Let $\boldsymbol{k}/\boldsymbol{k}_0$ be a quadratic extension of finite fields of characteristic p. Write σ for the non-trivial $\boldsymbol{k}_0$-automorphism of $\boldsymbol{k}$, and q_0 for the cardinality of $\boldsymbol{k}_0$. We thus have $q_0^2 = q$.

If π is an irreducible representation of $\mathrm{GL}_n(\boldsymbol{k})$, we write π^σ for the representation $\pi \circ \sigma$, and we say that π is *σ-selfdual* if π^σ, $\pi^\vee$ are isomorphic.

Lemma 2.3. — *Let $n \geqslant 1$ be a positive integer.*

(1) *If there is a σ-selfdual supercuspidal irreducible representation of* $\mathrm{GL}_n(\boldsymbol{k})$*, then n is odd.*

(2) *Suppose that* R *has characteristic* 0 *and n is odd. Then there is a σ-selfdual supercuspidal irreducible representation of* $\mathrm{GL}_n(\boldsymbol{k})$*.*

Proof. — Let ξ be a $\boldsymbol{k}$-regular character of $\boldsymbol{t}^\times$, and let ρ denote the supercuspidal irreducible representation of $\mathrm{GL}_n(\boldsymbol{k})$ corresponding to it by (2.2). The identity (2.1) shows that ρ^σ corresponds to ξ^{q_0}. Indeed, for all $g \in \boldsymbol{t}^\times \subseteq \mathrm{GL}_n(\boldsymbol{k})$ with irreducible characteristic polynomial, $\sigma(g)$ and g^{q_0} have the same characteristic polynomial, thus they are conjugate in $\mathrm{GL}_n(\boldsymbol{k})$. It follows that:

$$\mathrm{tr}\ \rho^\sigma(g) = \mathrm{tr}\ \rho(g^{q_0}) = (-1)^{n-1} \cdot \sum_\gamma \xi^\gamma(g^{q_0}) = (-1)^{n-1} \cdot \sum_\gamma (\xi^{q_0})^\gamma(g)$$

for all $g \in \boldsymbol{t}^\times$ with irreducible characteristic polynomial. Thus ρ is σ-selfdual if and only if:

$$\xi^{-q_0} = \xi^{q_0^{2i}}, \quad \text{for some } i \in \{0, \dots, n-1\}. \tag{2.3}$$

Exponentiating to $-q_0$ again gives us the equality $\xi^q = \xi^{q^{2i}}$. The $\boldsymbol{k}$-regularity assumption on ξ implies that n divides $2i-1$, thus n is odd. Besides, since $0 \leqslant i \leqslant n-1$, we have $n = 2i-1$. It follows that:

$$\rho \text{ is } \sigma\text{-selfdual} \ \Leftrightarrow\ \xi^{-1} = \xi^{q_0^n}. \tag{2.4}$$

This is also equivalent to ξ being trivial on $\boldsymbol{t}_0^\times$, where $\boldsymbol{t}_0$ is the subfield of $\boldsymbol{t}$ with q_0^n elements.

Conversely, suppose that R has characteristic 0 and n is odd. Let ξ be an R-character of $\boldsymbol{t}^\times$ of order $q_0^n + 1$, which exists since $\boldsymbol{t}^\times$ has order $q^n - 1 = (q_0^n - 1)(q_0^n + 1)$. It is thus trivial on $\boldsymbol{t}_0^\times$. On ther other hand, q_0 has order $2n$ mod $q_0^n + 1$, thus q has order n mod $q_0^n + 1$. It follows that the character ξ is $\boldsymbol{k}$-regular. ☐

Remark 2.4. — When R has characteristic $\ell > 0$, the group $\mathrm{GL}_n(\boldsymbol{k})$ may have σ-selfdual cuspidal (non supercuspidal) representations for n even, and it may have no σ-selfdual supercuspidal representation for n odd.

(1) For an example of σ-selfdual cuspidal non supercuspidal representation for n even, let e be the order of q mod ℓ, and suppose that $n = e\ell^u$ for some integer $u \geqslant 0$. Then, by [**38**] Théorème 2.4, the unique generic subquotient π of the representation induced from the trivial character of

a Borel subgroup of $\mathrm{GL}_n(\boldsymbol{k})$ is cuspidal and σ-selfdual. One may choose q, ℓ and u such that n is even. For instance, this is the case when $n=2$ and $\ell\neq 2$ divides $q+1$.

(2) The group $\mathrm{GL}_n(\boldsymbol{k})$ may even have no supercuspidal representation at all: this is the case, for instance, when $n=3$, $q=2$ and $\ell=7$.

Lemma 2.5. — *Let ρ be a supercuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$ for some odd integer $n\geqslant 1$. The following assertions are equivalent:*

(1) *The representation ρ is σ-selfdual.*
(2) *The representation ρ is $\mathrm{GL}_n(\boldsymbol{k}_0)$-distinguished.*
(3) *The space $\mathrm{Hom}_{\mathrm{GL}_n(\boldsymbol{k}_0)}(\rho,1)$ has dimension 1.*

Proof. — When R has characteristic 0, this is due to Gow [**18**]. Suppose that R has characteristic $\ell>0$ prime to q. We postpone to Section 4 the proof of the fact that (2) implies (1) and is equivalent to (3), since the proof of Theorem 4.1 works in both the finite and non-Archimedean cases (see Remark 4.3). Here we prove that (1) implies (2). For this, we use the following general lemma.

Lemma 2.6. — *Let G be a finite group and H be a subgroup of G. Let π be an irreducible representation of G on a $\overline{\mathbf{Q}}_\ell$-vector space V such that $\mathrm{Hom}_{\mathsf{H}}(\pi,1)$ is non-zero. Let $\mathrm{L}\subseteq\mathrm{V}$ be a G-stable $\overline{\mathbf{Z}}_\ell$-lattice. Then $\mathrm{L}\otimes\overline{\mathbf{F}}_\ell$ has at least one irreducible G-subquotient τ such that $\mathrm{Hom}_{\mathsf{H}}(\tau,1)\neq\{0\}$.*

Proof. — Let φ be a non-zero H-invariant linear form on V. The image of L by φ, denoted M, is a $\overline{\mathbf{Z}}_\ell$-lattice in $\overline{\mathbf{Q}}_\ell$. Reducing mod the maximal ideal of $\overline{\mathbf{Z}}_\ell$ gives a non-zero H-invariant $\overline{\mathbf{F}}_\ell$-linear map $\overline{\varphi}$ from $\mathrm{L}\otimes\overline{\mathbf{F}}_\ell$ to $\mathrm{M}\otimes\overline{\mathbf{F}}_\ell\simeq\overline{\mathbf{F}}_\ell$. Let W be the largest subrepresentation of $\mathrm{L}\otimes\overline{\mathbf{F}}_\ell$ contained in the kernel of $\overline{\varphi}$. Then any irreducible subrepresentation τ of $(\mathrm{L}\otimes\overline{\mathbf{F}}_\ell)/\mathrm{W}$ satisfies the required condition. □

Let ξ be a $\boldsymbol{k}$-regular character of $\boldsymbol{t}^\times$ parametrizing some σ-selfdual supercuspidal representation ρ of $\mathrm{GL}_n(\boldsymbol{k})$. By (2.4), we have $\xi^{-1}=\xi^{q_0^n}$. Let us fix a field embedding $\iota:\overline{\mathbf{F}}_\ell\to\mathrm{R}$. Since ξ has finite image, there is a $\boldsymbol{k}$-regular character $\widetilde{\xi}$ of $\boldsymbol{t}^\times$ with values in $\overline{\mathbf{Z}}_\ell$ such that:

– the character $\widetilde{\xi}$ satifies the identity $\widetilde{\xi}^{-1}=\widetilde{\xi}^{q_0^n}$, and
– one has $\xi=\iota\circ\xi_0$ where ξ_0 is the reduction mod ℓ of $\widetilde{\xi}$.

The character $\widetilde{\xi}$ corresponds to a σ-selfdual supercuspidal $\overline{\mathbf{Q}}_\ell$-representation $\widetilde{\rho}$ of $\mathrm{GL}_n(\boldsymbol{k})$. Let V denote the vector space of $\widetilde{\rho}$ and fix a $\mathrm{GL}_n(\boldsymbol{k})$-stable $\overline{\mathbf{Z}}_\ell$-lattice L in V. By Proposition 2.1, the representation of $\mathrm{GL}_n(\boldsymbol{k})$ on the $\overline{\mathbf{F}}_\ell$-vector space $\mathrm{L}\otimes\overline{\mathbf{F}}_\ell$ is isomorphic to the supercuspidal representation ρ_0 corresponding to ξ_0, and it is distinguished by Lemma 2.6. The result now follows from Remark 2.2, which tells us that $\rho_0\otimes\mathrm{R}$ is distinguished and isomorphic to ρ. □

Remark 2.7. — If R is the field $\overline{\mathbf{F}}_\ell$, we proved that the representation ρ is distinguished if and only if it has a distinguished lift to $\overline{\mathbf{Q}}_\ell$.

Remark 2.8. — We give an example of a σ-selfdual cuspidal non supercuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$ which is not distinguished. With the notation of Remark 2.4, assume that $n=e=2$.

Thus π is a σ-selfdual cuspidal (non supercuspidal) representation of $\mathrm{GL}_2(\boldsymbol{k})$. Let $\widetilde{\pi}$ be an ℓ-adic lift of π (see Remark 2.2), and decompose its restriction to $\mathrm{GL}_2(\boldsymbol{k}_0)$ as a direct sum:

$$\mathrm{V}_1 \oplus \cdots \oplus \mathrm{V}_r$$

of irreducible components. Since the order of $\mathrm{GL}_2(\boldsymbol{k}_0)$ is prime to ℓ, reduction mod ℓ preserves irreducibility, and the restriction of π to $\mathrm{GL}_2(\boldsymbol{k}_0)$ is semisimple. It follows that π decomposes as $\mathrm{W}_1 \oplus \cdots \oplus \mathrm{W}_r$, where W_i is irreducible and is the reduction mod ℓ of V_i for each $i = 1, \ldots, r$. Suppose that π is distinguished. Then W_i is the trivial character for some i. Thus V_i is a character, and it must be trivial since $\mathrm{GL}_2(\boldsymbol{k}_0)$ has no non-trivial character of order a power of ℓ, which implies that $\widetilde{\pi}$ is distinguished. This is impossible, since $n = 2$ is even.

Remark 2.9. — More generally, the argument of Remark 2.8 shows that, if H is a subgroup of $\mathrm{GL}_n(\boldsymbol{k})$ whose order is prime to ℓ, then a cuspidal representation π of $\mathrm{GL}_n(\boldsymbol{k})$ is H-distinguished if and only if any ℓ-adic lift of π is H-distinguished.

2.3. A mirabolic interlude

This paragraph is inspired from Matringe [**34**]. We assume that $p \neq 2$. Let G denote the group $\mathrm{GL}_n(\boldsymbol{k})$ for some $n \geqslant 2$. Write P for the mirabolic subgroup of G, which is made of all matrices in G whose last row is $(0\ \ldots\ 0\ 1)$. Let U be the unipotent radical of P, and G' be the image of $\mathrm{GL}_{n-1}(\boldsymbol{k})$ in G under the group homomorphism:

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

We thus have $\mathrm{P} = \mathrm{G}'\mathrm{U}$, and we write P' for the mirabolic subgroup of G'. Let N be the maximal unipotent subgroup of G made of all upper trangular unipotent matrices, and ψ be a non-trivial character of $\boldsymbol{k}$. We still write ψ for the non-degenerate character

$$x \mapsto \psi(x_{1,2} + \cdots + x_{n-1,n})$$

of N. We have a functor:

$$\pi \mapsto \mathrm{Ind}_{\mathrm{P}'\mathrm{U}}^{\mathrm{P}}(\pi \otimes \psi)$$

denoted Φ^+, from R-representations of P' to R-representations of P, where $\pi \otimes \psi$ is the representation of $\mathrm{P}'\mathrm{U}$ defined by $xu \mapsto \pi(x)\psi(u)$ for all $x \in \mathrm{P}'$ and $u \in \mathrm{U}$.

Given integers $r \geqslant s \geqslant 0$ such that $r + s = n$, let $\mathrm{H}_{r,s}$ be the subgroup of G defined in [**34**]. It is the conjugate of the Levi subgroup $\mathrm{GL}_r(\boldsymbol{k}) \times \mathrm{GL}_s(\boldsymbol{k})$ of G under the permutation matrix $w_{r,s}$ defined by the permutation:

$$\begin{pmatrix} 1 & \cdots & t & \cdots & t+i & \cdots & r & r+1 & \cdots & r+1+j & \cdots & n-1 & n \\ 1 & \cdots & t & \cdots & t+2i & \cdots & n-1 & t+1 & \cdots & t+1+2j & \cdots & n-2 & n \end{pmatrix}$$

where $t = r - s + 1$. If $s \geqslant 1$, let $\mathrm{H}'_{r,s}$ be the subgroup $\mathrm{G}' \cap \mathrm{H}_{r,s}$ (denoted $\mathrm{H}_{r,s-1}$ in [**34**]).

Lemma 2.10. — *Let π be a representation of P', and χ be a character of $\mathrm{H}_{r,s}$. Suppose that the vector space $\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}(\Phi^+(\pi), \chi)$ is non-zero. Then it is isomorphic to $\mathrm{Hom}_{\mathrm{P}'\cap\mathrm{H}'_{r,s}}(\pi, \chi)$.*

Proof. — Given $g \in \mathrm{G}$ and a representation τ of a subgroup H of G, we will write $\mathrm{H}^g = g^{-1}\mathrm{H}g$, and τ^g for the representation $x \mapsto \tau(gxg^{-1})$ of H^g. Applying the Mackey formula, and since G′ normalizes U, the restriction of $\Phi^+(\pi)$ to $\mathrm{P} \cap \mathrm{H}_{r,s}$ decomposes as the direct sum:

$$\bigoplus_g \mathrm{Ind}_{\mathrm{P}\cap\mathrm{H}_{r,s}\cap\mathrm{P}'^g\mathrm{U}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}(\pi^g \otimes \psi^g)$$

where g ranges over a set of representatives of $(\mathrm{P}'\mathrm{U}, \mathrm{P}\cap\mathrm{H}_{r,s})$-double cosets in P. Since $\mathrm{P} = \mathrm{G}'\mathrm{U}$, we may assume g ranges over a set of representatives of $(\mathrm{P}', \mathrm{H}'_{r,s})$-double cosets in G′. For each g, let us write the following isomorphism of representations of $\mathrm{P} \cap \mathrm{H}_{r,s}$:

$$\mathrm{Ind}_{\mathrm{U}\cap\mathrm{H}_{r,s}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}(\pi^g \otimes \psi^g) \simeq \mathrm{Ind}_{\mathrm{P}\cap\mathrm{H}_{r,s}\cap\mathrm{P}'^g\mathrm{U}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}\left((\pi^g \otimes \psi^g) \otimes \mathrm{Ind}_{\mathrm{U}\cap\mathrm{H}_{r,s}}^{\mathrm{P}\cap\mathrm{H}_{r,s}\cap\mathrm{P}'^g\mathrm{U}}(1)\right). \tag{2.5}$$

Since the induced representation $\mathrm{Ind}_{\mathrm{U}\cap\mathrm{H}_{r,s}}^{\mathrm{P}\cap\mathrm{H}_{r,s}\cap\mathrm{P}'^g\mathrm{U}}(1)$ canonically surjects onto the trivial character of $\mathrm{P} \cap \mathrm{H}_{r,s} \cap \mathrm{P}'^g\mathrm{U}$ by Frobenius reciprocity, the right hand side of (2.5) surjects onto:

$$\mathrm{Ind}_{\mathrm{P}\cap\mathrm{H}_{r,s}\cap\mathrm{P}'^g\mathrm{U}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}(\pi^g \otimes \psi^g).$$

On the other hand, since π^g is trivial on $\mathrm{U} \cap \mathrm{H}_{r,s}$, the left hand side of (2.5) is a sum of finitely many copies of $\mathrm{Ind}_{\mathrm{U}\cap\mathrm{H}_{r,s}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}(\psi^g)$. It follows that, if $\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}(\Phi^+(\pi), \chi)$ is non-zero, then there is a $g \in \mathrm{G}'$ such that:

$$\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}\left(\mathrm{Ind}_{\mathrm{U}\cap\mathrm{H}_{r,s}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}(\psi^g), \chi\right) \neq \{0\}.$$

By Frobenius reciprocity, this implies that $\psi^g = \chi$ on $\mathrm{U} \cap \mathrm{H}_{r,s}$. Since we assumed that $p \neq 2$, the field $\boldsymbol{k}$ has at least three elements, thus any character of $\mathrm{H}_{r,s} \simeq \mathrm{GL}_r(\boldsymbol{k}) \times \mathrm{GL}_s(\boldsymbol{k})$ is trivial on unipotent elements. We thus get $\psi^g = 1$ on $\mathrm{U} \cap \mathrm{H}_{r,s}$. By [**34**] Lemma 3.1, this implies that $g \in \mathrm{P}'\mathrm{H}'_{r,s}$, that is, we may assume that $g = 1$. We thus have:

$$\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}(\Phi^+(\pi), \chi) = \mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}\left(\mathrm{Ind}_{\mathrm{P}'\mathrm{U}\cap\mathrm{H}_{r,s}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}(\pi \otimes \psi), \chi\right) \simeq \mathrm{Hom}_{\mathrm{P}'\mathrm{U}\cap\mathrm{H}_{r,s}}(\pi \otimes \psi, \chi).$$

The result now follows from the fact that $\mathrm{P}'\mathrm{U} \cap \mathrm{H}_{r,s} = \mathrm{P}' \cap \mathrm{H}'_{r,s}$. (This latter equality has been pointed out to me by N. Matringe.) □

Remark 2.11. — (1) If χ is assumed to be trivial on unipotent elements, or if $\boldsymbol{k}$ has at least three elements, then Lemma 2.10 holds without assuming that $p \neq 2$.

(2) If $\boldsymbol{k}$ has cardinality 2, the group $\mathrm{GL}_2(\boldsymbol{k})$ has a cuspidal character. Thus, when $r \geqslant 2$, the group $\mathrm{H}_{r,2}$ has a character which is non-trivial on $\mathrm{U} \cap \mathrm{H}_{r,2}$.

Now write G″ for the copy of $\mathrm{GL}_{n-2}(\boldsymbol{k})$ in the upper left block of G′ and P″ for the mirabolic subgroup of G″.

Lemma 2.12. — *Let π' be a representation of* P″*, and χ' be a character of* $\mathrm{H}'_{r,s}$*. Suppose that $s \geqslant 1$. If $\mathrm{Hom}_{\mathrm{P}'\cap\mathrm{H}'_{r,s}}(\Phi^+(\pi'), \chi')$ is non-zero, then it is isomorphic to $\mathrm{Hom}_{\mathrm{P}''\cap\mathrm{H}_{r-1,s-1}}(\pi', \chi')$.*

Proof. — It is similar to that of Lemma 2.10, replacing [**34**] Lemma 3.1 by [**34**] Lemma 3.2. □

Let Γ denote the mirabolic representation of P. Recall that it is defined as the representation of P induced from the character ψ of N.

Lemma 2.13. — *Let $n \geqslant 2$ and $r \geqslant s \geqslant 0$ be such that $r+s=n$. Let χ be a character of $\mathrm{H}_{r,s}$. If the space $\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}(\Gamma,\chi)$ is non-zero, then $r=s$.*

Proof. — If $s=0$, then $\mathrm{H}_{r,s}=\mathrm{G}$ and the result follows from the fact that the representation Γ is irreducible of dimension greater than 1.

Suppose that $s \geqslant 1$ and that the space $\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}(\Gamma,\chi)$ is non-zero. The mirabolic representation Γ is isomorphic to $\Phi^+(\Gamma')$, where Γ' denotes the mirabolic representation of P'. By Lemma 2.10, the space $\mathrm{Hom}_{\mathrm{P}'\cap\mathrm{H}'_{r,s}}(\Gamma',\chi')$ is non-zero, where χ' is the restriction of χ to $\mathrm{H}'_{r,s}$. Now identify Γ' with $\Phi^+(\Gamma'')$, where Γ'' is the mirabolic representation of P''. By Lemma 2.12, the space $\mathrm{Hom}_{\mathrm{P}''\cap\mathrm{H}_{r-1,s-1}}(\Gamma'',\chi'')$ is non-zero, where χ'' is the restriction of χ' to $\mathrm{H}_{r-1,s-1}$. By induction on n, the fact that $\mathrm{Hom}_{\mathrm{P}''\cap\mathrm{H}_{r-1,s-1}}(\Gamma'',\chi'')$ is non-zero implies that $r-1=s-1$, thus $r=s$. ☐

Proposition 2.14. — *Let $n \geqslant 2$ and $r,s \geqslant 0$ be such that $r+s=n$. Let ρ be a cuspidal representation of G, and χ be a character of $\mathrm{M}=(\mathrm{GL}_r\times\mathrm{GL}_s)(\boldsymbol{k})$. Suppose $\mathrm{Hom}_{\mathrm{M}}(\rho,\chi)$ is non-zero. Then $r=s$.*

Proof. — Conjugating M and χ if necessary, we may assume that $r \geqslant s$. The result then follows from Lemma 2.13 and the fact that the restriction of ρ to P is isomorphic to Γ. This latter fact is well-known when R has characteristic 0, and is given by [**47**] III.1 when R is equal to $\overline{\mathbf{F}}_\ell$. For an arbitrary R of characteristic $\ell>0$, fix a field embedding of $\overline{\mathbf{F}}_\ell$ in R and write ρ as $\rho_0\otimes\mathrm{R}$ for some cuspidal irreducible $\overline{\mathbf{F}}_\ell$-representation ρ_0 of G as in Remark 2.2. Since the restriction of ρ_0 to P is isomorphic to Γ_0, the mirabolic $\overline{\mathbf{F}}_\ell$-representation of P, the restriction of ρ to P is isomorphic to $\Gamma_0\otimes\mathrm{R}\simeq\Gamma$. ☐

Remark 2.15. — Suppose that $r=s\geqslant 1$. Putting Lemmas 2.10 and 2.12 together, we get:

$$\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,r}}((\Phi^+)^2(\pi),1)\simeq\mathrm{Hom}_{\mathrm{P}''\cap\mathrm{H}_{r-1,r-1}}(\pi,1)$$

for any representation π of P''. By induction, we get an isomorphism $\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,r}}(\Gamma,1)\simeq\mathrm{R}$.

Corollary 2.16. — *Suppose that $n=2r$ for some $r\geqslant 1$, and let ρ be a cuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$. Then the* R*-vector space $\mathrm{Hom}_{(\mathrm{GL}_r\times\mathrm{GL}_r)(\boldsymbol{k})}(\rho,1)$ has dimension at most* 1.

Proof. — This follows from Remark 2.15, together with the fact that the restriction of ρ to P is isomorphic to Γ. ☐

2.4. The Levi case

In this paragraph, we consider the supercuspidal irreducible representations of $\mathrm{GL}_n(\boldsymbol{k})$ distinguished by some maximal Levi subgroup. As in Paragraph 2.3, we assume that $p\neq 2$.

Lemma 2.17. — *Let $n\geqslant 1$ be a positive integer.*

(1) *If there is a selfdual supercuspidal irreducible representation of $\mathrm{GL}_n(\boldsymbol{k})$, then either $n=1$ or n is even.*

(2) *Suppose that* R *has characteristic* 0, *and that either $n=1$ or n is even. Then there exists a selfdual supercuspidal irreducible representation of $\mathrm{GL}_n(\boldsymbol{k})$.*

Proof. — If $n = 1$, the trivial character of $\boldsymbol{k}^\times$ is selfdual and supercuspidal. Suppose that $n \geqslant 2$. Let us fix an extension $\boldsymbol{t}$ of $\boldsymbol{k}$ of degree n, and identify $\boldsymbol{t}^\times$ with a maximal torus in $\mathrm{GL}_n(\boldsymbol{k})$. We consider the Green-James parametrisation (2.2) of isomorphism classes of supercuspidal irreducible representations of $\mathrm{GL}_n(\boldsymbol{k})$ by $\boldsymbol{k}$-regular characters of $\boldsymbol{t}^\times$.

Given a $\boldsymbol{k}$-regular character ξ of $\boldsymbol{t}^\times$, let ρ denote the representation corresponding to it. Recall (see Remark 2.2) that $\rho^\vee$ corresponds to ξ^{-1}. Thus ρ is selfdual if and only if:

$$\xi^{-1} = \xi^{q^i}, \quad \text{for some } i \in \{0, \dots, n-1\}. \tag{2.6}$$

Taking the contragredient again gives us the equality $\xi = \xi^{q^{2i}}$. The regularity assumption on ξ implies that n divides $2i$ and, since $0 \leqslant i \leqslant n-1$, we get $n = 2i$. It follows that:

$$\rho_\xi \text{ is selfdual} \quad \Leftrightarrow \quad \xi^{-1} = \xi^{q^{n/2}}. \tag{2.7}$$

This is also equivalent to ξ being trivial on $\boldsymbol{t}'^\times$, where $\boldsymbol{t}'$ is the subfield of $\boldsymbol{t}$ with cardinality $q^{n/2}$.

Conversely, suppose that R has characteristic 0 and $n = 2r$ for some $r \geqslant 1$. Let us consider an R-character ξ of $\boldsymbol{t}^\times$ of order $q^r + 1$, which exists since $\boldsymbol{t}^\times$ has order $q^n - 1 = (q^r - 1)(q^r + 1)$. It is trivial on $\boldsymbol{t}'^\times$. On ther other hand, q has order $n = 2r$ mod $q^r + 1$, which implies that ξ is $\boldsymbol{k}$-regular. □

Remark 2.18. — When R has characteristic $\ell > 0$, the group $\mathrm{GL}_n(\boldsymbol{k})$ may have selfdual cuspidal (non supercuspidal) representations even if n is odd and > 1. Indeed, let e be the order of q mod ℓ, and suppose that $n = e\ell^u$ for some $u \geqslant 0$. The unique generic irreducible subquotient of the representation induced from the trivial character of a Borel subgroup of $\mathrm{GL}_n(\boldsymbol{k})$ is then both cuspidal (see Remark 2.4) and selfdual. One then can choose q, ℓ and u such that n is odd and > 1. For instance, this is the case when $\ell \neq 2$ divides $q - 1$ and $n = \ell$.

Also, as in Remark 2.4, the group $\mathrm{GL}_n(\boldsymbol{k})$ may even have no supercuspidal representation at all, which is the case, for instance, when $n = q = 2$ and $\ell = 3$.

Lemma 2.19. — *Suppose that $n = 2r$ with $r \geqslant 1$, and let ρ be a supercuspidal representation of* $\mathrm{GL}_n(\boldsymbol{k})$. *The following assertions are equivalent:*

(1) *The representation ρ is selfdual.*
(2) *The representation ρ is* $(\mathrm{GL}_r \times \mathrm{GL}_r)(\boldsymbol{k})$*-distinguished.*
(3) *The* R*-vector space* $\mathrm{Hom}_{(\mathrm{GL}_r \times \mathrm{GL}_r)(\boldsymbol{k})}(\rho, 1)$ *has dimension* 1.

Proof. — When R has characteristic 0, this is [**24**] Proposition 6.1 (see also [**11**] Lemme 3.4.10). Suppose now R has prime characteristic ℓ not dividing q. To prove that (1) implies (2), we follow the same lifting argument as in the proof of Lemma 2.5.

We now prove that (2) implies (1). Let us write $\mathrm{G} = \mathrm{GL}_n(\boldsymbol{k})$ and $\mathrm{H} = (\mathrm{GL}_r \times \mathrm{GL}_r)(\boldsymbol{k})$. Let ρ be an H-distinguished supercuspidal representation of G. If one fixes a field embedding of $\overline{\mathbf{F}}_\ell$ in R, then Remark 2.2 tells us that ρ is isomorphic to $\rho_0 \otimes \mathrm{R}$ for some distinguished supercuspidal irreducible $\overline{\mathbf{F}}_\ell$-representation ρ_0 of G. Since ρ is selfdual if and only if ρ_0 is, we are thus reduced to proving the result in the case where R is equal to $\overline{\mathbf{F}}_\ell$, which we assume now.

Since ρ is distinguished, its contragredient $\rho^\vee$ has a non-zero H-invariant vector. We thus have a non-zero homomorphism $i : \overline{\mathbf{Z}}_\ell[\mathrm{H}\backslash\mathrm{G}] \to \rho^\vee$. Let us consider a projective envelope $f : \mathrm{P} \to \rho^\vee$

of $\rho^\vee$ in the category of $\overline{\mathbf{Z}}_\ell[\mathrm{G}]$-modules. Since $\rho^\vee$ is supercuspidal, it has the following properties (see for instance [**47**] III.2.9):

– the representation $\mathrm{P}\otimes\overline{\mathbf{Q}}_\ell$ is isomorphic to the direct sum of all the $\overline{\mathbf{Q}}_\ell$-lifts of $\rho^\vee$;

– there are ℓ^a such lifts, where a is the ℓ-adic valuation of q^n-1;

– the representation $\mathrm{P}\otimes\overline{\mathbf{F}}_\ell$ is indecomposable of length ℓ^a and has a unique irreducible quotient, and all its irreducible components are isomorphic to $\rho^\vee$.

By projectivity of P, the homomorphism i gives rise to a non-zero homomorphism:

$$j:\mathrm{P}\to\overline{\mathbf{Z}}_\ell[\mathrm{H}\backslash\mathrm{G}] \tag{2.8}$$

such that $i\circ j=f$. Inverting ℓ, we get a non-zero homomorphism from $\mathrm{P}\otimes\overline{\mathbf{Q}}_\ell$ to $\overline{\mathbf{Q}}_\ell[\mathrm{H}\backslash\mathrm{G}]$. It follows that $\rho^\vee$ has at least one H-distinguished lift. Thanks to the characteristic 0 case, such a lift is selfdual. Reducing mod ℓ, it follows that ρ is selfdual.

We now go back to the case of a general R. The fact that (2) implies (3) is a particular case of Corollary 2.16. However, we are going to give another proof here, which works for supercuspidal representations only but is more likely to generalize to other situations.

Let V be the maximal direct summand of $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}]$ in the block of ρ. This means that $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}]$ decomposes as a direct sum $\mathrm{V}\oplus\mathrm{V}'$ where all irreducible subquotients of V are isomorphic to ρ, and no irreducible subquotient of V' is isomorphic to ρ. Besides, since ρ is selfdual, we have:

$$\dim\mathrm{Hom}_\mathrm{H}(\rho,1)=\dim\mathrm{Hom}_\mathrm{H}(1,\rho)=\dim\mathrm{Hom}_\mathrm{G}(\mathrm{R}[\mathrm{H}\backslash\mathrm{G}],\rho)=\dim\mathrm{Hom}_\mathrm{G}(\mathrm{V},\rho).$$

We thus have to prove that the cosocle of V is isomorphic to ρ.

Lemma 2.20. — *The* R*-algebra* $\mathrm{A}=\mathrm{End}_\mathrm{G}(\mathrm{V})$ *is commutative.*

Proof. — Since the convolution algebra $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}/\mathrm{H}]$ decomposes as $\mathrm{End}_\mathrm{G}(\mathrm{V})\oplus\mathrm{End}_\mathrm{G}(\mathrm{V}')$, it suffices to prove that $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}/\mathrm{H}]$ is commutative. For $x\in\mathrm{G}$, let f_x be the characteristic function in $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}/\mathrm{H}]$ of the double coset $\mathrm{H}x\mathrm{H}$. For $x,y\in\mathrm{G}$, one has:

$$f_x*f_y=\sum_{z\in\mathrm{H}\backslash\mathrm{G}/\mathrm{H}}a(x,y,z)f_z$$

where $a(x,y,z)\in\mathrm{R}$ is the image of the cardinality of $(\mathrm{H}x\mathrm{H}\cap z\mathrm{H}y^{-1}\mathrm{H})/\mathrm{H}$ in R. When R has characteristic 0, the algebra $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}/\mathrm{H}]$ is known to be commutative since $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}]$ is multiplicity free as an $\mathrm{R}[\mathrm{G}]$-module, thus:

$$\mathrm{card}\ (\mathrm{H}x\mathrm{H}\cap z\mathrm{H}y^{-1}\mathrm{H})/\mathrm{H}=\mathrm{card}\ (\mathrm{H}y\mathrm{H}\cap z\mathrm{H}x^{-1}\mathrm{H})/\mathrm{H} \tag{2.9}$$

for all $x,y,z\in\mathrm{G}$. Now if R has characteristic $\ell>0$, reducing (2.9) mod ℓ gives us a congruence relation which tells us that the algebra $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}/\mathrm{H}]$ is commutative. □

It remains to prove that the cosocle of V is multiplicity free. Let $m\geqslant1$ be the multiplicity of ρ in the cosocle of V and Q be the projective indecomposable $\mathrm{R}[\mathrm{G}]$-module associated with ρ. It has length ℓ^a, has a unique irreducible quotient, and all its irreducible components are isomorphic to ρ. Write $\mathrm{V}=\mathrm{V}_1\oplus\cdots\oplus\mathrm{V}_m$ where $\mathrm{V}_1,\dots,\mathrm{V}_m$ are indecomposable $\mathrm{R}[\mathrm{G}]$-modules with cosocle isomorphic to ρ. There is a nilpotent endomorphism $\mathrm{N}\in\mathrm{End}_\mathrm{G}(\mathrm{Q})$ such that:

$$\mathrm{End}_\mathrm{G}(\mathrm{Q})=\mathrm{R}[\mathrm{N}]$$

with $N^{\ell^a} = 0$ and $N^{\ell^a - 1} \neq 0$. Therefore each V_i is isomorphic to the quotient of Q by the image of N^{k_i} for some $k_i \geqslant 0$. Reordering if necessary, we may assume that $\mathrm{Hom}(V_1, V_i)$ is non-zero for all $i \geqslant 1$. Suppose that $m \geqslant 2$, and define two endomorphisms $u, u' \in A$ by:

(1) the endomorphisms u, u' are trivial on V_i for all $i \geqslant 2$,
(2) the restriction of u to V_1 is the identity on V_1,
(3) the restriction of u' to V_1 coincides with some non-zero homomorphism in $\mathrm{Hom}(V_1, V_2)$.

Then $uu' = 0$ and $u'u \neq 0$, thus A is not commutative. Thus $m = 1$. □

Remark 2.21. — If R is the field $\overline{\mathbf{F}}_\ell$, we proved that ρ is distinguished if and only if it has a distinguished lift to $\overline{\mathbf{Q}}_\ell$ (see Remark 2.7).

Remark 2.22. — If we only assume ρ to be cuspidal in Lemmas 2.17 and 2.19, then the lifting argument may not work, that is, ξ may not have a σ-selfdual $\boldsymbol{k}$-regular lift $\widetilde{\xi}$. Besides, the structure of the projective envelope of ρ is more complicated when ρ is cuspidal non-supercuspidal.

3. Notation and basic definitions in the non-Archimedean case

Let F/F_0 be a separable quadratic extension of locally compact non-archimedean local fields of residual characteristic p. Apart from Section 4, we will assume that $p \neq 2$.

Write σ for the non-trivial F_0-automorphism of F. Write $\mathcal{O}$ for the ring of integers of F and $\mathcal{O}_0$ for that of F_0. Write $\boldsymbol{k}$ for the residue field of F and $\boldsymbol{k}_0$ for that of F_0. The involution σ induces a $\boldsymbol{k}_0$-automorphism of $\boldsymbol{k}$, still denoted σ, which generates $\mathrm{Gal}(\boldsymbol{k}/\boldsymbol{k}_0)$.

As in Section 2, let R be an algebraically closed field of characteristic ℓ different from p. (Note that ℓ can be 0.) We say we are in the "modular case" when we consider the case where $\ell > 0$.

We fix once and for all a character:

$$\psi_0 : F_0 \to R^\times \tag{3.1}$$

trivial on the maximal ideal of $\mathcal{O}_0$ but not on $\mathcal{O}_0$, and define $\psi = \psi_0 \circ \mathrm{tr}_{F/F_0}$.

When $\ell \neq 2$, we write:

$$\omega = \omega_{F/F_0} : F_0^\times \to R^\times \tag{3.2}$$

for the character of $F_0^\times$ whose kernel is the subgroup of F/F_0-norms.

Let G be the locally profinite group $G = \mathrm{GL}_n(F)$, with $n \geqslant 1$, equipped with the involution σ acting componentwise. Its σ-fixed points is the closed subgroup $G^\sigma = \mathrm{GL}_n(F_0)$. We will identify the centre of G with $F^\times$, and that of G^σ with $F_0^\times$.

By *representation* of a locally profinite group, we always mean a smooth representation on an R-module. Given a representation π of a closed subgroup H of G, we write $\pi^\vee$ for the smooth contragredient of π, and π^σ for the representation $\pi \circ \sigma$ of the subgroup $\sigma(H)$. We say that π is *σ-selfdual* if H is σ-stable and π^σ, $\pi^\vee$ are isomorphic. If $g \in G$, we write $H^g = \{g^{-1}hg \mid h \in H\}$ and π^g for the representation $x \mapsto \pi(gxg^{-1})$ of H^g. If χ is a character of H, we write $\pi\chi$ for the representation $g \mapsto \chi(g)\pi(g)$.

If μ is a character of $H \cap G^\sigma$, we say that π is *μ-distinguished* if the space $\mathrm{Hom}_{H \cap G^\sigma}(\pi, \mu)$ is non-zero. If μ is the trivial character, we will simply say that π is $H \cap G^\sigma$-*distinguished*, or just *distinguished*. If $\mathrm{H} = \mathrm{G}$ and ϕ is a character of $\mathrm{F}_0^\times$, we will abbreviate $\phi \circ \det$-*distinguished* to ϕ-*distinguished*.

An irreducible representation of G is said to be *cuspidal* if all its proper Jacquet modules are trivial or, equivalently, if it does not occur as a subrepresentation of a proper parabolically induced representation. It is said to be *supercuspidal* if it does not occur as a subquotient of a proper parabolically induced representation (by [**12**] Corollaire B.1.3, this is equivalent to not occuring as a subquotient of the parabolic induction of any *irreducible* representation of a proper Levi subgroup of G). When R has characteristic 0, any cuspidal representation is supercuspidal.

4. A modular version of theorems of Prasad and Flicker

In this section, the residue characteristic p is arbitrary. We prove the following theorem, which is well-known in the complex case. Note that, in the modular case, any irreducible representation of G has a central character by [**47**] II.2.8.

Theorem 4.1. — *Let π be a distinguished irreducible representation of* G*. Then:*

(1) *The central character of π is trivial on $\mathrm{F}_0^\times$.*
(2) *The contragredient representation $\pi^\vee$ is distinguished.*
(3) *The space $\mathrm{Hom}_{\mathrm{G}^\sigma}(\pi, 1)$ has dimension* 1*.*
(4) *The representations π^σ and $\pi^\vee$ are isomorphic, that is, π is σ-selfdual.*

Remark 4.2. — In the complex case, this theorem was first proved under the assumption that the characteristic of F is not 2, which was required in the proof of [**16**] Proposition 10. Later, in [**42**] Section 4, Prasad gave an argument which only requires F/F_0 to be separable quadratic.

Proof. — Property 1 is straightforward. Property 2 follows from an argument of Gelfand-Kazhdan (see [**44**] Proposition 8.4 in the modular case). For Properties 3 and 4, we follow the proofs of Prasad [**41**] and Flicker [**16**]. The reference for the basic results in the theory of modular representations of p-adic reductive groups which we use in the proof is [**47**].

Write $\mathcal{C}_c^\infty(\mathrm{G})$ for the space of locally constant, compactly supported R-valued functions on G, and fix an R-valued Haar measure on G, that is, a non-zero R-linear form on $\mathcal{C}_c^\infty(\mathrm{G})$ invariant under left translation by G.

Let W denote the vector space of π, and $l : \mathrm{W} \to \mathrm{R}$ be a non-zero G^σ-invariant linear form. For any $f \in \mathcal{C}_c^\infty(\mathrm{G})$, define a linear form on W by:

$$\pi(f)l : w \mapsto \int_{\mathrm{G}} f(x) l(\pi(x)w)\ dx.$$

Since f is smooth, the linear form $\pi(f)l$ on W is smooth. This defines a non-zero homomorphism $\mathrm{L} : \mathcal{C}_c^\infty(\mathrm{G}) \to \mathrm{W}^\vee$. It is G-equivariant under right translation and G^σ-invariant under left translation. Since W is irreducible, it is surjective. Similarly, given a non-zero G^σ-invariant linear form $m : \mathrm{W}^\vee \to \mathrm{R}$, we obtain a surjective right G-equivariant and left G^σ-invariant homomorphism M from $\mathcal{C}_c^\infty(\mathrm{G})$ to $\mathrm{W}^{\vee\vee} \simeq \mathrm{W}$ (see [**47**] Proposition I.4.18 for the latter isomorphism). We now define:

$$\mathrm{B}(f, g) = \langle \mathrm{M}(f), \mathrm{L}(g) \rangle \in \mathrm{R}$$

for all $f, g \in \mathcal{C}_c^\infty(G)$. This defines a right G-invariant and left $G^\sigma \times G^\sigma$-invariant linear form B on the space $\mathcal{C}_c^\infty(G) \otimes \mathcal{C}_c^\infty(G) \simeq \mathcal{C}_c^\infty(G \times G)$. As in [**41**] Lemma 4.2 (and with [**42**] Lemma 4.1, which extends the result of [**16**] Proposition 10 to the case where F has arbitrary characteristic) we have:

$$B(f, g) = B(g \circ \sigma, f \circ \sigma) \tag{4.1}$$

for all $f, g \in \mathcal{C}_c^\infty(G)$. It follows that the kernel of L is equal to $\{f \circ \sigma \mid f \in \mathrm{Ker}(M)\}$. Thus, if l' is any non-zero G^σ-invariant linear form on W, with associated homomorphism L′, then L, L′ have the same kernel. Since $\pi^\vee$ is admissible (by [**47**] II.2.8), Schur's Lemma applies (see [**47**] I.6.9) thus one has $l' = cl$ for some $c \in R^\times$. Thus $\mathrm{Hom}_{G^\sigma}(W, 1)$ has dimension 1.

As in [**41**] Lemma 4.2, the bilinear form B corresponds to the G^σ-bi-invariant linear form D on $\mathcal{C}_c^\infty(G)$ defined by:

$$D(h) = m(\pi(h)l)$$

for all $h \in \mathcal{C}_c^\infty(G)$. The correspondence between B and D is given by:

$$D(h) = B(k), \quad \text{with } k : (x, y) \mapsto h(xy^{-1}).$$

Note that (4.1) gives $D(h) = D(h \circ \sigma \circ \iota)$ with $\iota : x \mapsto x^{-1}$ on G. Replacing π by $\pi^* = \pi^{\vee\sigma}$ and exchanging the roles played by l, m we get a linear form:

$$D^* : h \mapsto l(\pi^*(h)m).$$

Since we have $l(\pi^*(h)m) = m(\pi(h \circ \sigma \circ \iota)l)$, it follows that $D^* = D$. In order to deduce Property 4, it remains to prove that D determines π entirely. For any $\xi \in W^\vee$ we define the function:

$$c_\xi : x \mapsto m(\pi^\vee(x)\xi) = m(\xi \circ \pi(x^{-1}))$$

on G. Then $\xi \mapsto c_\xi$ is an embedding of $W^\vee$ in the space $\mathcal{C}^\infty(G^\sigma \backslash G)$ of smooth R-valued functions on $G^\sigma \backslash G$. For $y \in G$ and $h \in \mathcal{C}_c^\infty(G)$, let ${}^y h$ denote the function $x \mapsto h(xy)$. Since L and M are surjective, there is a function h such that $\pi(h)l$ is non-zero. Then $y \mapsto D({}^y h)$ is a non-zero function in the space $\mathcal{C}^\infty(G^\sigma \backslash G)$, generating a subrepresentation isomorphic to $W^\vee$. Indeed, it is equal to $c_{\pi(h)l}$. It thus follows from the equality $D^* = D$ that we have $\pi^\sigma \simeq \pi^\vee$, as expected. □

Remark 4.3. — The same results hold – and the same argument works – when F/F_0 is replaced by a quadratic extension of finite fields of arbitrary characteristic. It suffices to replace [**42**] Lemma 4.1 by [**18**] Lemma 3.5.

5. Preliminaries on simple types

From now on, until the end of this article, we will assume that $p \neq 2$. This assumption is not needed in Paragraphs 5.1 to 5.3, but we assume it from now on for simplicity.

We assume the reader is familiar with the language of simple types. We recall the main results on simple strata, characters and types [**10, 6, 8, 36**] that we will need. Part of these preliminaries can also be found in [**3**].

5.1. Simple strata and characters

Let $[\mathfrak{a},\beta]$ be a simple stratum in the F-algebra $\mathrm{M}_n(\mathrm{F})$ of $n\times n$ matrices with entries in F for some $n\geqslant 1$. Recall that $\mathfrak{a}$ is a hereditary order in $\mathrm{M}_n(\mathrm{F})$ and β is a matrix in $\mathrm{M}_n(\mathrm{F})$ such that:

(1) the F-algebra $\mathrm{E}=\mathrm{F}[\beta]$ is a field, whose degree over F is denoted d,

(2) the multiplicative group $\mathrm{E}^\times$ normalizes $\mathfrak{a}$.

The centralizer of E in $\mathrm{M}_n(\mathrm{F})$, denoted B, is an E-algebra isomorphic to $\mathrm{M}_m(\mathrm{E})$, with $n=md$. The intersection $\mathfrak{b}=\mathfrak{a}\cap\mathrm{B}$ is a hereditary order in B.

Write $\mathfrak{p}_\mathfrak{a}$ for the Jacobson radical of $\mathfrak{a}$, and $\mathrm{U}^1(\mathfrak{a})$ for the compact open pro-p-subgroup $1+\mathfrak{p}_\mathfrak{a}$ of $\mathrm{G}=\mathrm{GL}_n(\mathrm{F})$. We recall the following useful *simple intersection property* ([**10**] Theorem 1.6.1): for all $x\in\mathrm{B}^\times$, we have:

$$\mathrm{U}^1(\mathfrak{a})x\mathrm{U}^1(\mathfrak{a})\cap\mathrm{B}^\times=\mathrm{U}^1(\mathfrak{b})x\mathrm{U}^1(\mathfrak{b}). \tag{5.1}$$

Associated with $[\mathfrak{a},\beta]$, there are compact open subgroups:

$$\mathrm{H}^1(\mathfrak{a},\beta)\subseteq\mathrm{J}^1(\mathfrak{a},\beta)\subseteq\mathrm{J}(\mathfrak{a},\beta)$$

of $\mathfrak{a}^\times$ and a finite set $\mathcal{C}(\mathfrak{a},\beta)$ of characters of $\mathrm{H}^1(\mathfrak{a},\beta)$ called *simple characters*, depending on the choice of the character ψ fixed in Section 3. Write $\mathbf{J}(\mathfrak{a},\beta)$ for the compact mod centre subgroup generated by $\mathrm{J}(\mathfrak{a},\beta)$ and the normalizer of $\mathfrak{b}$ in $\mathrm{B}^\times$.

Proposition 5.1 **(**[**8**] 2.1**)**. — *We have the following properties:*

(1) *The group* $\mathrm{J}(\mathfrak{a},\beta)$ *is the unique maximal compact subgroup of* $\mathbf{J}(\mathfrak{a},\beta)$.

(2) *The group* $\mathrm{J}^1(\mathfrak{a},\beta)$ *is the unique maximal normal pro-p-subgroup of* $\mathrm{J}(\mathfrak{a},\beta)$.

(3) *The group* $\mathrm{J}(\mathfrak{a},\beta)$ *is generated by* $\mathrm{J}^1(\mathfrak{a},\beta)$ *and* $\mathfrak{b}^\times$*, and we have:*

$$\mathrm{J}(\mathfrak{a},\beta)\cap\mathrm{B}^\times=\mathfrak{b}^\times,\quad \mathrm{J}^1(\mathfrak{a},\beta)\cap\mathrm{B}^\times=\mathrm{U}^1(\mathfrak{b}). \tag{5.2}$$

(4) *The normalizer of any simple character* $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ *in* G *is equal to* $\mathbf{J}(\mathfrak{a},\beta)$.

(5) *The intertwining set of any* $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ *in* G *is equal to* $\mathrm{J}^1(\mathfrak{a},\beta)\mathrm{B}^\times\mathrm{J}^1(\mathfrak{a},\beta)$.

By [**10**] Theorem 3.4.1, the quotient $\mathrm{J}^1(\mathfrak{a},\beta)/\mathrm{H}^1(\mathfrak{a},\beta)$ is a finite $\boldsymbol{k}$-vector space, and the map:

$$(x,y)\mapsto\langle x,y\rangle=\theta(xyx^{-1}y^{-1}) \tag{5.3}$$

makes it into a non-degenerate symplectic space. More precisely, if $\mathfrak{h}^1(\mathfrak{a},\beta)$ and $\mathfrak{j}^1(\mathfrak{a},\beta)$ are the sub-$\mathcal{O}$-lattices of $\mathfrak{a}$ such that $\mathrm{H}^1(\mathfrak{a},\beta)=1+\mathfrak{h}^1(\mathfrak{a},\beta)$ and $\mathrm{J}^1(\mathfrak{a},\beta)=1+\mathfrak{j}^1(\mathfrak{a},\beta)$, then we have:

$$\langle 1+u,1+v\rangle=\psi\circ\mathrm{tr}(\beta(uv-vu)) \tag{5.4}$$

for all $u,v\in\mathfrak{j}^1(\mathfrak{a},\beta)$ ([**7**] Proposition 6.1), where tr denotes the trace map of $\mathrm{M}_n(\mathrm{F})$.

Let $[\mathfrak{a}',\beta']$ be another simple stratum in $\mathrm{M}_{n'}(\mathrm{F})$ for some $n'\geqslant 1$, and suppose that there is an F-algebra isomorphism $\varphi:\mathrm{F}[\beta]\to\mathrm{F}[\beta']$ such that $\varphi(\beta)=\beta'$. Then there is a canonical bijective map:

$$\mathcal{C}(\mathfrak{a},\beta)\to\mathcal{C}(\mathfrak{a}',\beta') \tag{5.5}$$

called the *transfer map* ([**10**] Theorem 3.6.14).

When the hereditary order $\mathfrak{b}=\mathfrak{a}\cap\mathrm{B}$ is a maximal order in B, we say that the simple stratum $[\mathfrak{a},\beta]$ and the simple characters in $\mathcal{C}(\mathfrak{a},\beta)$ are *maximal*. When this is the case, then, given a homomorphism of E-algebras $B \simeq M_m(E)$ identifying $\mathfrak{b}$ with the standard maximal order, there are group isomorphisms:

$$J(\mathfrak{a},\beta)/J^1(\mathfrak{a},\beta) \simeq \mathfrak{b}^\times/U^1(\mathfrak{b}) \simeq GL_m(\boldsymbol{l}) \tag{5.6}$$

where $\boldsymbol{l}$ is the residue field of E.

5.2. Types and cuspidal representations

Let us write $G = GL_n(F)$ for some $n \geqslant 1$. A family of pairs $(\mathbf{J},\boldsymbol{\lambda})$ called *extended maximal simple types*, made of a compact mod centre, open subgroup $\mathbf{J}$ of G and an irreducible representation $\boldsymbol{\lambda}$ of $\mathbf{J}$, has been constructed in [**10**] (see also [**36**] in the modular case).

Given an extended maximal simple type $(\mathbf{J},\boldsymbol{\lambda})$ in G, there are a maximal simple stratum $[\mathfrak{a},\beta]$ in $M_n(F)$ and a maximal simple character $\theta \in \mathcal{C}(\mathfrak{a},\beta)$ such that $\mathbf{J}(\mathfrak{a},\beta) = \mathbf{J}$ and θ is contained in the restriction of $\boldsymbol{\lambda}$ to $H^1(\mathfrak{a},\beta)$. Such a simple character is said to be *attached to* $\boldsymbol{\lambda}$. By [**10**] Proposition 5.1.1 (or [**36**] Proposition 2.1 in the modular case), the group $J^1(\mathfrak{a},\beta)$ carries, up to isomorphism, a unique irreductible representation η whose restriction to $H^1(\mathfrak{a},\beta)$ contains θ. It is called the *Heisenberg representation* associated to θ and has the following properties:

(1) the restriction of η to $H^1(\mathfrak{a},\beta)$ is made of $(J^1(\mathfrak{a},\beta) : H^1(\mathfrak{a},\beta))^{1/2}$ copies of θ,

(2) the representation η extends to $\mathbf{J}$.

For any representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η, there is, up to isomorphism, a unique irreducible representation $\boldsymbol{\rho}$ of $\mathbf{J}$ trivial on $J^1(\mathfrak{a},\beta)$ such that $\boldsymbol{\lambda} \simeq \boldsymbol{\kappa}\otimes\boldsymbol{\rho}$. Through (5.6), the restriction of $\boldsymbol{\rho}$ to the maximal compact subgroup $J = J(\mathfrak{a},\beta)$ identifies with a cuspidal representation of $GL_m(\boldsymbol{l})$.

Remark 5.2. — The reader familiar with the theory of simple types will have noticed that we did not introduce the notion of beta-extension. Since $GL_m(\boldsymbol{l})$ is not isomorphic to $GL_2(\mathbb{F}_2)$ (as p is not 2), any character of $GL_m(\boldsymbol{l})$ factors through the determinant. It follows that, if $[\mathfrak{a},\beta]$ is a maximal simple stratum, any representation of J extending η is a beta-extension.

We have the following additional property, which follows from [**36**] Lemme 2.6.

Proposition 5.3. — *Let $\boldsymbol{\kappa}$ be a representation of $\mathbf{J}$ extending η, and write J^1 for the maximal normal pro-p-subgroup of $\mathbf{J}$. The map:*

$$\boldsymbol{\xi} \mapsto \boldsymbol{\kappa} \otimes \boldsymbol{\xi}$$

induces a bijection between isomorphism classes of irreducible representations $\boldsymbol{\xi}$ of $\mathbf{J}$ trivial on J^1 and isomorphism classes of irreducible representations of $\mathbf{J}$ whose restriction to J^1 contains η.

We now give the classification of cuspidal irreducible representations of G in terms of extended maximal simple types (see [**10**] 6.2, 8.4 and [**36**] Section 3 in the modular case).

Proposition 5.4 **([10, 36])**. — *Let π be a cuspidal representation of* G.

(1) *There is an extended maximal simple type $(\mathbf{J},\boldsymbol{\lambda})$ such that $\boldsymbol{\lambda}$ occurs as a subrepresentation of the restriction of π to $\mathbf{J}$. It is uniquely determined up to* G*-conjugacy.*

(2) *Compact induction defines a bijection between the* G*-conjugacy classes of extended maximal simple types and the isomorphism classes of cuspidal representations of* G.

From now on, we will abbreviate *extended maximal simple type* to *type*.

5.3. Supercuspidal representations

Let π be a cuspidal representation of G. By Proposition 5.4, it contains a type $(\mathbf{J}, \boldsymbol{\lambda})$. Fix an irreducible representation $\boldsymbol{\kappa}$ as in Proposition 5.3 and let $\boldsymbol{\rho}$ be the corresponding representation of $\mathbf{J}$ trivial on its maximal normal pro-p-subgroup J^1.

Fix a maximal simple stratum $[\mathfrak{a}, \beta]$ such that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$. Write $\mathrm{E} = \mathrm{F}[\beta]$ and let ρ be the cuspidal representation of $\mathrm{J}/\mathrm{J}^1 \simeq \mathrm{GL}_m(\boldsymbol{l})$ induced by $\boldsymbol{\rho}$. We record the following fact.

Fact 5.5 **([37] Proposition 6.10)**. — *The representation π is supercuspidal if and only if ρ is supercuspidal.*

Now suppose that π is supercuspidal, thus ρ is also supercuspidal. We show how to parametrize $\boldsymbol{\rho}$ by an "admissible pair of level zero". This will be needed in Sections 8 and 10.

First, let $\boldsymbol{t}$ be an extension of degree m of $\boldsymbol{l}$, and identify $\boldsymbol{t}^\times$ with a maximal torus in $\mathrm{GL}_m(\boldsymbol{l})$. We have the correspondence (2.2) between $\boldsymbol{l}$-regular characters of $\boldsymbol{t}^\times$ and isomorphism classes of supercuspidal irreducible representations of $\mathrm{GL}_m(\boldsymbol{l})$.

Definition 5.6 **([27, 9])**. — An *admissible pair of level zero* over E is a pair $(\mathrm{K}/\mathrm{E}, \xi)$ made of a finite unramified extension K of E and a tamely ramified character $\xi : \mathrm{K}^\times \to \mathrm{R}^\times$ which does not factor through $\mathrm{N}_{\mathrm{K}/\mathrm{L}}$ for any field L such that $\mathrm{E} \subseteq \mathrm{L} \subsetneq \mathrm{K}$. Its *degree* is $[\mathrm{K} : \mathrm{E}]$.

If $(\mathrm{K}'/\mathrm{E}, \xi')$ is another admissible pair of level zero over E, it is said to be *isomorphic* to $(\mathrm{K}/\mathrm{E}, \xi)$ if there is an isomorphism of E-algebras $\varphi : \mathrm{K}' \to \mathrm{K}$ such that $\xi' = \xi \circ \varphi$.

Recall (see §5.1) that, if we write $\mathrm{B}^\times$ for the centralizer of E in G, then $\mathbf{J} = (\mathbf{J} \cap \mathrm{B}^\times)\mathrm{J}^1$, thus the group $\mathbf{J}/\mathrm{J}^1$ is isomorphic to $(\mathbf{J} \cap \mathrm{B}^\times)/(\mathrm{J}^1 \cap \mathrm{B}^\times)$. In particular, the image of $\mathrm{E}^\times$ in $\mathbf{J}/\mathrm{J}^1$ is central. Since $\boldsymbol{\rho}$ is trivial on J^1, the automorphism $\boldsymbol{\rho}(x)$ is thus a scalar for all $x \in \mathrm{E}^\times$.

Definition 5.7. — An admissible pair $(\mathrm{K}/\mathrm{E}, \xi)$ of level zero and degree m is *attached to* $\boldsymbol{\rho}$ if:

(1) writing $\boldsymbol{t}$ for the residue field of K, the $\boldsymbol{l}$-regular character of $\boldsymbol{t}^\times$ induced by the restriction of ξ to the units of the ring of integers of K corresponds to ρ via (2.2),

(2) one has $\boldsymbol{\rho}(x) = \xi(x) \cdot \mathrm{id}$ for all $x \in \mathrm{E}^\times$, where id is the identity on the space of $\boldsymbol{\rho}$.

The following proposition is a refinement of the property of the map (2.2).

Proposition 5.8. — *The attachment relation defines a bijection:*

$$(\mathrm{K}/\mathrm{E}, \xi) \mapsto \boldsymbol{\rho}(\mathrm{K}/\mathrm{E}, \xi) \tag{5.7}$$

between isomorphism classes of admissible pairs of level zero over E *and isomorphism classes of irreducible representations of* $\mathbf{J}$ *trivial on* J^1 *whose restriction to* J *defines a supercuspidal representation of* $\mathrm{GL}_m(\boldsymbol{l})$ *through* (5.6).

Remark 5.9. — As in Remark 2.2, let us fix an embedding $\iota : \mathrm{R} \to \mathrm{R}'$ of algebraically closed fields of characteristic ℓ, and let $(\mathrm{K}/\mathrm{E}, \xi)$ be an admissible pair of level zero over E such that ξ takes values in R. Then $(\mathrm{K}/\mathrm{E}, \iota \circ \xi)$ is an admissible pair of level zero over E, and:

$$\boldsymbol{\rho}(\mathrm{K}/\mathrm{E}, \iota \circ \xi) = \boldsymbol{\rho}(\mathrm{K}/\mathrm{E}, \xi) \otimes \mathrm{R}'.$$

This refines the last assertion of Remark 2.2.

5.4. The σ-selfdual type theorem

Let us fix an integer $n \geqslant 1$ and write $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$. We recall the first main result of [**3**].

Theorem 5.10 **([3] Theorem 4.1).** — *Let* π *be a cuspidal representation of* G*. It is* σ*-selfdual if and only if it contains a type* $(\mathbf{J}, \boldsymbol{\lambda})$ *such that* $\mathbf{J}$ *is* σ*-stable and* $\boldsymbol{\lambda}^{\sigma} \simeq \boldsymbol{\lambda}^{\vee}$*.*

Remark 5.11. — More precisely (see [**3**] Corollary 4.21), any σ-selfdual cuspidal representation contain a σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$ with the additional property that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ for some maximal simple stratum $[\mathfrak{a}, \beta]$ in $\mathrm{M}_n(\mathrm{F})$ such that:

(1) the hereditary order $\mathfrak{a}$ is σ-stable and $\sigma(\beta) = -\beta$;
(2) the element β has the block diagonal form:

$$\beta = \begin{pmatrix} \beta_0 & & \\ & \ddots & \\ & & \beta_0 \end{pmatrix} = \beta_0 \otimes 1 \in \mathrm{M}_d(\mathrm{F}) \otimes_{\mathrm{F}} \mathrm{M}_m(\mathrm{F}) = \mathrm{M}_n(\mathrm{F})$$

for some $\beta_0 \in \mathrm{M}_d(\mathrm{F})$, where d is the degree of β over F and $n = md$; the centralizer B of $\mathrm{E} = \mathrm{F}[\beta]$ in $\mathrm{M}_n(\mathrm{F})$ thus identifies with $\mathrm{M}_m(\mathrm{E})$, equipped with the involution σ acting componentwise;
(3) the order $\mathfrak{b} = \mathfrak{a} \cap \mathrm{B}$ is the standard maximal order of $\mathrm{M}_m(\mathrm{E})$.

Such a type will be useful in the discussion following Proposition 5.17.

Remark 5.12. — If $(\mathbf{J}, \boldsymbol{\lambda})$ is any σ-selfdual type, then there is a maximal simple stratum $[\mathfrak{a}, \beta]$ in $\mathrm{M}_n(\mathrm{F})$ such that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$, the order $\mathfrak{a}$ is σ-stable and $\sigma(\beta) = -\beta$ (see [**3**] Corollary 4.24). Such a maximal simple stratum will be said to be *σ-selfdual*.

Remark 5.13. — Let π be a σ-selfdual cuspidal representation of G. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type in π, let $[\mathfrak{a}, \beta]$ be a simple stratum such that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ and let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be the maximal simple character attached to $\boldsymbol{\lambda}$. Then $\mathrm{H}^1(\mathfrak{a}, \beta)$ is σ-stable and $\theta \circ \sigma = \theta^{-1}$.

Let π be a σ-selfdual cuspidal representation of G. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type in π and fix a σ-selfdual simple stratum $[\mathfrak{a}, \beta]$ as in Remark 5.12. Then $\mathrm{E} = \mathrm{F}[\beta]$ is σ-stable. We denote by E_0 the field of σ-fixed points in E, by T the maximal tamely ramified sub-extension of E over F and by T_0 the intersection $\mathrm{T} \cap \mathrm{E}_0$. Also write $d = [\mathrm{E} : \mathrm{F}]$ and $n = md$.

Proposition 5.14 **([3] Proposition 4.30).** — *The integer:*

$$m(\pi) = m = n/d \tag{5.8}$$

and the F_0*-isomorphism class of the quadratic extension* T/T_0 *only depend on* π*, and not on the choice of the* σ*-selfdual simple stratum* $[\mathfrak{a}, \beta]$ *as in Remark 5.12.*

The integer m defined by (5.8) it called the *relative degree* of π. We record a list of properties of the field extension T/F.

Lemma 5.15. — (1) *The canonical homomorphism of* $T_0 \otimes_{F_0} F$*-modules:*

$$T_0 \otimes_{F_0} F \to T$$

is an isomorphism.

(2) *If* F/F_0 *is unramified, then* T/T_0 *is unramified and* T/F *has odd residual degree.*

(3) *The extension* T/T_0 *is ramified if and only if* F/F_0 *is ramified and* T_0/F_0 *has odd ramification order.*

Proof. — Assertion (1) is [**3**] Lemma 4.10. We now prove (2) and (3).

First, suppose that F/F_0 is unramified. We remark that:

$$f(T/F) \cdot f(F/F_0) = f(T/T_0) \cdot f(T_0/F_0)$$

is even. Since F does not embed in T_0 as an F_0-algebra, T_0 has odd residue degree over F_0. It follows that $f(T/T_0) = 2$ and that T has odd residue degree over F.

Suppose F/F_0 is ramified, and let ϖ be a uniformizer of F such that $\varpi_0 = \varpi^2$ is a uniformizer of F_0. Let e_0 be the ramification order of T_0/F_0, and let t_0 be a uniformizer of T_0 such that:

$$\varpi_0 = t_0^{e_0}\zeta_0$$

for some root of unity $\zeta_0 \in F_0^\times$ of order prime to p. Let a be the greatest integer smaller than or equal to $e_0/2$, and write $x = \varpi t_0^{-a}$. We have $\sigma(x) = -x$, thus $x \notin T_0$ and $x^2 \in T_0$.

If e_0 is odd, then $x^2 = \zeta_0 t_0$ is a uniformizer of T_0, whereas x is a uniformizer of T, thus T is ramified over T_0.

If e_0 is even, then $x^2 = \zeta_0$. It follows that x is a root of unity of order prime to p which is in T but not in T_0, thus T is unramified over T_0. ☐

Remark 5.16. — (1) The extensions E/E_0 and T/T_0 have the same ramification order.

(2) The extension E/E_0 is ramified if and only if F/F_0 is ramified and E_0/F_0 has odd ramification order.

The first property comes from [**3**] Remark 4.22, and the second one follows from the first one together with Lemma 5.15.

We now recall the classification of all σ-selfdual types contained in a given σ-selfdual cuspidal representation of G (see [**3**] Proposition 4.31).

Proposition 5.17. — *Let* π *be a* σ*-selfdual cuspidal representation of* G*, and let* T/T_0 *denote the quadratic extension associated to it.*

(1) *If* T *is unramified over* T_0*, the* σ*-selfdual types contained in* π *form a single* G^σ*-conjugacy class.*

(2) *If* T *is ramified over* T_0*, the* σ*-selfdual types contained in* π *form exactly* $\lfloor m/2\rfloor+1$ *different* G^σ*-conjugacy classes.*

One can give a more precise description in the ramified case. Suppose that T is ramified over T_0, and let $(\mathbf{J}_0, \boldsymbol{\lambda}_0)$ be a σ-selfdual type in π satisfying the conditions of Remark 5.11. Let us fix a uniformizer t of E. For $i = 0, \dots, \lfloor m/2\rfloor$, let t_i denote the diagonal matrix:

$$\mathrm{diag}(t, \dots, t, 1, \dots, 1) \in B^\times = GL_m(E)$$

where t occurs i times. Then the pairs $(\mathbf{J}_i, \boldsymbol{\lambda}_i) = (\mathbf{J}_0^{t_i}, \boldsymbol{\lambda}_0^{t_i})$, for $i = 0, \dots, \lfloor m/2 \rfloor$, form a set of representatives of the G^σ-conjugacy classes of σ-selfdual types in π.

Definition 5.18. — The integer i is called the *index* of the G^σ-conjugacy class of $(\mathbf{J}_i, \boldsymbol{\lambda}_i)$. It does not depend on the choice of $(\mathbf{J}_0, \boldsymbol{\lambda}_0)$, nor on that of t.

Let $[\mathfrak{a}, \beta]$ be a simple stratum as in Remark 5.11 such that $\mathbf{J}_0 = \mathbf{J}(\mathfrak{a}, \beta)$. If one identifies the quotient $J(\mathfrak{a}, \beta)^{t_i}/J^1(\mathfrak{a}, \beta)^{t_i}$ with $GL_m(\boldsymbol{l})$ via:

$$J(\mathfrak{a}, \beta)^{t_i}/J^1(\mathfrak{a}, \beta)^{t_i} \simeq J(\mathfrak{a}, \beta)/J^1(\mathfrak{a}, \beta) \simeq U(\mathfrak{b})/U^1(\mathfrak{b}) \simeq GL_m(\boldsymbol{l})$$

then σ acts on $GL_m(\boldsymbol{l})$ by conjugacy by the diagonal element:

$$\delta_i = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1) \in GL_m(\boldsymbol{l})$$

where -1 occurs i times, and $(J(\mathfrak{a}, \beta)^{t_i} \cap G^\sigma)/(J^1(\mathfrak{a}, \beta)^{t_i} \cap G^\sigma)$ identifies with the Levi subgroup $(GL_i \times GL_{m-i})(\boldsymbol{l})$ of $GL_m(\boldsymbol{l})$.

5.5. Admissible pairs and σ-selfduality

Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type in G. Fix a σ-selfdual maximal simple stratum $[\mathfrak{a}, \beta]$ such that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ as in Remark 5.12, and a decomposition of $\boldsymbol{\lambda}$ of the form $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$ as in Paragraph 5.2. Write $E = F[\beta]$ as usual.

Proposition 5.19. — *Suppose that the representation $\boldsymbol{\rho}$ is σ-selfdual, and let $(K/E, \xi)$ be an admissible pair of level zero attached to it. There is a unique involutive E_0-automorphism $\widehat{\sigma}$ of K such that $\xi \circ \widehat{\sigma} = \xi^{-1}$ and $\widehat{\sigma}$ coincides with σ on E.*

Proof. — Let K′ denote the extension of E given by the field K equipped with the map $x \mapsto \sigma(x)$ from E to K. Then the pair $(K'/E, \xi)$ is admissible of level zero, and it is attached to $\boldsymbol{\rho}^\sigma$. On the other hand, $(K/E, \xi^{-1})$ is admissible of level zero, attached to $\boldsymbol{\rho}^\vee$. Since $\boldsymbol{\rho}$ is σ-selfdual, there is an E-algebra isomorphism $\widehat{\sigma} : K \to K'$ such that $\xi \circ \widehat{\sigma} = \xi^{-1}$. We thus have:

$$\xi \circ \widehat{\sigma}^2 = \xi^{-1} \circ \widehat{\sigma} = \xi$$

and $\widehat{\sigma}^2$ is an E-algebra automorphism of K. By admissibility of $(K/E, \xi)$, the latter automorphism is trivial, thus $\widehat{\sigma}$ satisfies the required conditions. Uniqueness follows by admissibility again. ☐

For simplicity, we will write σ for the involutive automorphism given by Proposition 5.19. Let K_0 be the field of σ-fixed points of K. The following lemma will be useful in Section 10.

Lemma 5.20. — *If E/E_0 is ramified and m is even, then K/K_0 is unramified.*

Proof. — Write $m = 2r$ for some $r \geqslant 1$. Let t be a uniformizer of E such that $\sigma(t) = -t$ and let $\zeta \in K$ be a root of unity of order $c^m - 1$, where c is the cardinality of $\boldsymbol{l}$. We thus have $E = E_0[t]$ and $K = E[\zeta]$. Since σ is involutive, it induces an involutive $\boldsymbol{l}$-automorphism of $\boldsymbol{t}$, the residual field of K. If the latter were trivial, the relation $\xi \circ \sigma = \xi^{-1}$ would imply that the character $\overline{\xi}$ of $\boldsymbol{t}^\times$ induced by ξ is quadratic, contradicting the fact that it is $\boldsymbol{l}$-regular. The automorphism of $\boldsymbol{t}$ induced by σ is thus the rth power of the Frobenius automorphism. Now consider the element:

$$\alpha = \zeta^{(c^r+1)/2}.$$

It has order $2(c^r-1)$, thus $\sigma(\alpha) = -\alpha$. Since α^2 has order c^r-1, the extension of E_0 it generates is unramified and has degree r. We thus have $\mathrm{E}_0[\alpha^2, t\alpha] \subseteq \mathrm{K}_0$ and their degrees are equal. Now we deduce that $\mathrm{K} = \mathrm{K}_0[\alpha] = \mathrm{K}_0[\zeta]$ is unramified over K_0. □

5.6.

The following lemma will be useful in Sections 7 and 9, when we investigate decompositions of σ-selfdual types of the form $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$ which behave well under σ.

Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a maximal simple character such that $\mathrm{H}^1(\mathfrak{a}, \beta)$ is σ-stable and $\theta \circ \sigma = \theta^{-1}$. Let $\mathbf{J}$ be its normalizer in G, let J^1 be the maximal normal pro-p-subgroup of $\mathbf{J}$ and η be the irreducible representation of J^1 containing θ.

Lemma 5.21. — *Let $\boldsymbol{\kappa}$ be a representation of $\mathbf{J}$ extending η. There is a unique character μ of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}\mu$. It satisfies the identity $\mu \circ \sigma = \mu$.*

Proof. — Let $\boldsymbol{\kappa}$ be an irreducible representation of $\mathbf{J}$ extending η. By uniqueness of the Heisenberg representation, the fact that $\theta \circ \sigma = \theta^{-1}$ implies that $\eta^{\sigma\vee}$ is isomorphic to η. Thus $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}^{\sigma\vee}$ are representations of $\mathbf{J}$ extending η. There is a character μ of $\mathbf{J}$ trivial on J^1 such that we have $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}\mu$. It satisfies $\mu \circ \sigma = \mu$. It is unique by Proposition 5.3. □

6. The distinguished type theorem

In this section we prove the following result, which is our first main theorem. It will be refined by Theorem 10.3 in Section 10. Recall that $p \neq 2$ until the end of this article.

Theorem 6.1. — *Let π be a σ-selfdual cuspidal representation of* G. *Then π is distinguished if and only if it contains a σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$ such that $\mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}, 1)$ is non-zero.*

Remark 6.2. — If π is distinguished, it follows easily from the multiplicity 1 property in Theorem 4.1 that the distinguished σ-selfdual types $(\mathbf{J}, \boldsymbol{\lambda})$ occurring in π form a single G^σ-conjugacy class (see Remark 6.23).

Remark 6.3. — Theorem 6.1 is proved in [**3**] in a different manner than the one we give here, although both proofs use the σ-selfdual type theorem 5.10. The proof given in [**3**] is based on a result of Ok [**40**], proved by Ok for complex representations and extended to the modular case in [**3**] Appendix B. However, the proof we give here is more likely to generalize to other situations.

Let π be a σ-selfdual cuspidal representation. Theorem 5.10 tells us that it contains a σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$, and Proposition 5.4 tells us that π is compactly induced from $\boldsymbol{\lambda}$. A simple application of the Mackey formula gives us:

$$\mathrm{Hom}_{\mathrm{G}^\sigma}(\pi, 1) \simeq \prod_g \mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}^g, 1) \tag{6.1}$$

where g ranges over a set of representatives of $(\mathbf{J}, \mathrm{G}^\sigma)$-double cosets in G.

Remark 6.4. — It follows from Theorem 4.1 that there is at most one double coset $\mathbf{J}g\mathrm{G}^\sigma$ such that the space $\mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}^g,1)$ is non-zero, and that this space has dimension at most 1. Thus the product in (6.1) is actually a direct sum.

In this section, our main task (achieved in Paragraph 6.5) is to prove that, if $\mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}^g,1)$ is non-zero, then $\sigma(g)g^{-1}\in\mathbf{J}$. Theorem 6.1 will follow easily from there (see Paragraph 6.6).

We may assume that $\mathbf{J}=\mathbf{J}(\mathfrak{a},\beta)$ for a maximal simple stratum $[\mathfrak{a},\beta]$ satisfying the conditions of Remark 5.11. The extension $\mathrm{E}=\mathrm{F}[\beta]$, its centralizer B and the maximal order $\mathfrak{b}=\mathfrak{a}\cap\mathrm{B}$ are thus stable by σ. We write $d=[\mathrm{E}:\mathrm{F}]$ and $n=md$. We identify B with the E-algebra $\mathrm{M}_m(\mathrm{E})$ equipped with the involution σ acting componentwise, and $\mathfrak{b}$ with its standard maximal order.

We write $\mathrm{E}_0=\mathrm{E}^\sigma$, the field of σ-invariant elements of E, and fix once and for all a uniformizer t of E such that:

$$\sigma(t)=\begin{cases} t & \text{if E is unramified over } \mathrm{E}_0,\\ -t & \text{if E is ramified over } \mathrm{E}_0.\end{cases} \tag{6.2}$$

We also write $\mathrm{J}=\mathrm{J}(\mathfrak{a},\beta)$, $\mathrm{J}^1=\mathrm{J}^1(\mathfrak{a},\beta)$ and $\mathrm{H}^1=\mathrm{H}^1(\mathfrak{a},\beta)$. Recall that $\mathbf{J}=\mathrm{E}^\times\mathrm{J}$.

We denote by T the maximal tamely ramified sub-extension of E over F, and set $\mathrm{T}_0=\mathrm{T}\cap\mathrm{E}_0$.

We insist on the fact that, throughout this section, we assume that the stratum $[\mathfrak{a},\beta]$ satisfies the conditions of Remark 5.11.

6.1. Double cosets contributing to the distinction of θ

Let $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ be the maximal simple character occurring in the restriction of $\boldsymbol{\lambda}$ to H^1. Suppose that $\mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}^g,1)$ is non-zero for some double coset $\mathbf{J}g\mathrm{G}^\sigma$. Restricting to $\mathrm{H}^{1g}\cap\mathrm{G}^\sigma$, we deduce that the character θ^g is trivial on $\mathrm{H}^{1g}\cap\mathrm{G}^\sigma$.

In this paragraph, we look for the double cosets $\mathbf{J}g\mathrm{G}^\sigma\subseteq\mathrm{G}$ such that the character θ^g is trivial on $\mathrm{H}^{1g}\cap\mathrm{G}^\sigma$. For this, let us introduce the following general lemma.

Lemma 6.5. — *Let τ be an involution of* G*, let* H *be a τ-stable open pro-p-subgroup of* G *and let χ be a character of* H *such that $\chi\circ\tau=\chi^{-1}$. For any $g\in\mathrm{G}$, the character χ^g is trivial on $\mathrm{H}^g\cap\mathrm{G}^\tau$ if and only if $\tau(g)g^{-1}$ intertwines χ.*

Proof. — Write K for the τ-stable subgroup $\mathrm{H}^g\cap\tau(\mathrm{H}^g)$, which contains $\mathrm{H}^g\cap\mathrm{G}^\tau$. Let A be the quotient of K by $\overline{[\mathrm{K},\mathrm{K}]}$, the closure of the derived subgroup of K. This is a τ-stable commutative pro-p-group. Given $x\in\mathrm{K}$, write x' for its image in A. For any $b\in\mathrm{A}$, we have:

$$b=\sqrt{b\tau(b)}\cdot\sqrt{b\tau(b)^{-1}}$$

where $b\mapsto\sqrt{b}$ is the inverse of the automorphism $b\mapsto b^2$ of A. Thus, for any $x\in\mathrm{K}$, there are $y,z\in\mathrm{K}$ such that $x=yz$ and $\tau(y')=y'$ and $\tau(z')=z'^{-1}$.

Since $\tau(z)=z^{-1}h$ for some $h\in\overline{[\mathrm{K},\mathrm{K}]}$, we have:

$$\chi^g(\tau(z))=\chi^g(z^{-1}h)=\chi^g(z)^{-1}. \tag{6.3}$$

On the other hand, since $\tau(y)=yk$ for some $k\in\overline{[\mathrm{K},\mathrm{K}]}$, the element $y^{-1}\tau(y)$ defines a 1-cocycle in the τ-stable pro-p-group $\overline{[\mathrm{K},\mathrm{K}]}$. Since $p\neq2$, this cocycle is a coboundary, which implies:

$$y\in(\mathrm{H}^g\cap\mathrm{G}^\tau)\overline{[\mathrm{K},\mathrm{K}]}. \tag{6.4}$$

Now suppose that χ^g is trivial on $\mathrm{H}^g \cap \mathrm{G}^\tau$. Then (6.3) and (6.4) imply that:

$$\chi^g(\tau(x)) = \chi^g(\tau(z)) = \chi^g(z)^{-1} = \chi^g(x)^{-1}, \quad \text{for all } x \in \mathrm{K}. \tag{6.5}$$

Besides, (6.5) is *equivalent* to χ^g being trivial on $\mathrm{H}^g \cap \mathrm{G}^\tau$. On the other hand, we have:

$$\chi^g \circ \tau = (\chi \circ \tau)^{\tau(g)} = (\chi^{-1})^{\tau(g)} = (\chi^{\tau(g)})^{-1} \tag{6.6}$$

on K by assumption on χ. If we set $\gamma = \tau(g)g^{-1}$, then (6.5) is equivalent to:

$$\chi(h) = \chi^\gamma(h) \quad \text{for all } h \in \mathrm{H} \cap \gamma^{-1}\mathrm{H}\gamma.$$

This amounts to saying that γ intertwines χ. □

Proposition 6.6. — *Let $g \in \mathrm{G}$. Then the character θ^g is trivial on $\mathrm{H}^{1g} \cap \mathrm{G}^\sigma$ if and only if we have $\sigma(g)g^{-1} \in \mathrm{J}\mathrm{B}^\times\mathrm{J}$.*

Proof. — This follows from Lemma 6.5 applied to the simple character θ of H^1 and the involution σ, together with the fact that the intertwining set of θ is $\mathrm{J}\mathrm{B}^\times\mathrm{J}$ by Proposition 5.1(5). □

6.2. The double coset lemma

We now prove the following fundamental lemma.

Lemma 6.7. — *Let $g \in \mathrm{G}$. Then $\sigma(g)g^{-1} \in \mathrm{J}\mathrm{B}^\times\mathrm{J}$ if and only if $g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$.*

Proof. — Write $\gamma = \sigma(g)g^{-1}$. If $g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$, one verifies immediately that $\gamma \in \mathrm{J}\mathrm{B}^\times\mathrm{J}$. Conversely, suppose that $\gamma \in \mathrm{J}c\mathrm{J}$ for some $c \in \mathrm{B}^\times$. We will first show that the double coset representative c can be chosen nicely.

Lemma 6.8. — *There is a $b \in \mathrm{B}^\times$ such that $\gamma \in \mathrm{J}b\mathrm{J}$ and $b\sigma(b) = 1$.*

Proof. — Recall that $\mathrm{B}^\times$ has been identified with $\mathrm{GL}_m(\mathrm{E})$ and $\mathrm{U} = \mathrm{J} \cap \mathrm{B}^\times = \mathfrak{b}^\times$ is its standard maximal compact subgroup. By the Cartan decomposition, $\mathrm{B}^\times$ decomposes as the disjoint union of the double cosets:

$$\mathrm{U} \cdot \mathrm{diag}(t^{a_1}, \ldots, t^{a_m}) \cdot \mathrm{U}$$

where $a_1 \geqslant \ldots \geqslant a_m$ ranges over non-increasing sequences of m integers, and $\mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ denotes the diagonal matrix of $\mathrm{B}^\times$ with eigenvalues $\lambda_1, \ldots, \lambda_m \in \mathrm{E}^\times$. We thus may assume that $c = \mathrm{diag}(t^{a_1}, \ldots, t^{a_m})$ for a uniquely determined sequence of integers $a_1 \geqslant \ldots \geqslant a_m$.

The fact that $\sigma(\gamma) = \gamma^{-1}$ implies that we have $c \in \mathrm{J}c^{-1}\mathrm{J} \cap \mathrm{B}^\times$. Using the simple intersection property (5.1) together with the fact that $\mathrm{J} = \mathrm{U}\mathrm{J}^1$ and $\mathrm{J}^1 \subseteq \mathrm{U}^1(\mathfrak{a})$, we have $\mathrm{J}c^{-1}\mathrm{J} \cap \mathrm{B}^\times = \mathrm{U}c^{-1}\mathrm{U}$. The uniqueness of the Cartan decomposition of $\mathrm{B}^\times$ thus implies that the sequences $a_1 \geqslant \ldots \geqslant a_m$ and $-a_m \geqslant \ldots \geqslant -a_1$ are equal. We thus have $a_i + a_{m+1-i} = 0$ for all $i \in \{1, \ldots, m\}$. Now write $\kappa = \sigma(t)t^{-1} \in \{-1, 1\}$ and choose signs $\kappa_1, \ldots, \kappa_m \in \{-1, 1\}$ such that $\kappa_i\kappa_{m+1-i} = \kappa^{a_i}$ for all i. This is always possible since $a_{(m+1)/2} = 0$ when m is odd. Then the antidiagonal element:

$$b = \begin{pmatrix} & & \kappa_1 t^{a_1} \\ & \iddots & \\ \kappa_m t^{a_m} & & \end{pmatrix} \in \mathrm{B}^\times \tag{6.7}$$

satisfies the required conditions $b\sigma(b) = 1$ and $\gamma \in \mathrm{J}b\mathrm{J}$. ☐

Now write $\gamma = x'bx$ with $x, x' \in \mathrm{J}$ and $b \in \mathrm{B}^\times$. Replacing g by $\sigma(x')^{-1}g$ does not change the double coset $\mathrm{J}g\mathrm{G}^\sigma$ but changes γ into $bx\sigma(x')$. From now on, we will thus assume that:

$$\gamma = bx, \quad b\sigma(b) = 1, \quad x \in \mathrm{J}, \quad b \text{ is of the form (6.7).} \tag{6.8}$$

Write K for the group $\mathrm{J} \cap b^{-1}\mathrm{J}b$. Since $\sigma(b) = b^{-1}$ and J is σ-stable, we have $x \in \mathrm{K}$.

Lemma 6.9. — *The map $\delta : k \mapsto b^{-1}\sigma(k)b$ is an involutive group automorphism of* K.

Proof. — This follows from an easy calculation using the fact that $b\sigma(b) = 1$. ☐

Let $b_1 > \cdots > b_r$ be the unique decreasing sequence of integers such that:

$$\{a_1, \dots, a_m\} = \{b_1, \dots, b_r\}$$

and m_j denote the multiplicity of b_j in $(a_1, \dots, a_m)$, for $j \in \{1, \dots, r\}$. We have $m_j = m_{r+1-j}$ for all j, and $m_1 + \cdots + m_r = m$. These integers define a standard Levi subgroup:

$$\mathrm{M} = \mathrm{GL}_{m_1 d}(\mathrm{F}) \times \cdots \times \mathrm{GL}_{m_r d}(\mathrm{F}) \subseteq \mathrm{G}. \tag{6.9}$$

Write P for the standard parabolic subgroup of G generated by M and upper triangular matrices. Let N and N^- denote the unipotent radicals of P and its opposite parabolic subgroup with respect to M, respectively. Since b has the form (6.7), it normalizes M and we have:

$$\begin{aligned} \mathrm{K} &= (\mathrm{K} \cap \mathrm{N}^-) \cdot (\mathrm{K} \cap \mathrm{M}) \cdot (\mathrm{K} \cap \mathrm{N}), \\ \mathrm{K} \cap \mathrm{P} &= \mathrm{J} \cap \mathrm{P}, \\ \mathrm{K} \cap \mathrm{N}^- &\subseteq \mathrm{J}^1 \cap \mathrm{N}^-. \end{aligned}$$

We have similar properties for the subgroup $\mathrm{V} = \mathrm{K} \cap \mathrm{B}^\times = \mathrm{U} \cap b^{-1}\mathrm{U}b$ of $\mathrm{B}^\times$, that is:

$$\begin{aligned} \mathrm{V} &= (\mathrm{V} \cap \mathrm{N}^-) \cdot (\mathrm{V} \cap \mathrm{M}) \cdot (\mathrm{V} \cap \mathrm{N}), \\ \mathrm{V} \cap \mathrm{P} &= \mathrm{U} \cap \mathrm{P}, \\ \mathrm{V} \cap \mathrm{N}^- &\subseteq \mathrm{U}^1 \cap \mathrm{N}^-, \end{aligned}$$

where $\mathrm{U}^1 = \mathrm{J}^1 \cap \mathrm{B}^\times = \mathrm{U}^1(\mathfrak{b})$. Note that this subgroup V is stable by δ.

Lemma 6.10. — *The subset:*

$$\mathrm{K}^1 = (\mathrm{K} \cap \mathrm{N}^-) \cdot (\mathrm{J}^1 \cap \mathrm{M}) \cdot (\mathrm{K} \cap \mathrm{N})$$

is a δ-stable normal pro-p-subgroup of K, *and we have* $\mathrm{K} = \mathrm{V}\mathrm{K}^1$.

Proof. — To prove that K^1 is a subgroup of K, it is enough to prove that one has the containment $(\mathrm{K} \cap \mathrm{N}) \cdot (\mathrm{K} \cap \mathrm{N}^-) \subseteq \mathrm{K}^1$. Let $\mathfrak{j}^1$ be the sub-$\mathcal{O}$-lattice of $\mathfrak{a}$ such that $\mathrm{J}^1 = 1 + \mathfrak{j}^1$ and let $\mathfrak{j} = \mathfrak{b} + \mathfrak{j}^1$, thus $\mathrm{J} = \mathfrak{j}^\times$. A simple computation shows that $\mathrm{K} \cap \mathrm{N}^- \subseteq (1 + t\mathfrak{j}) \cap \mathrm{N}^-$ and:

$$(\mathrm{K} \cap \mathrm{N}) \cdot (\mathrm{K} \cap \mathrm{N}^-) \subseteq (\mathrm{K} \cap \mathrm{N}^-) \cdot ((1 + t\mathfrak{j}) \cap \mathrm{M}) \cdot (\mathrm{K} \cap \mathrm{N}).$$

The expected result thus follows from the fact that $t\mathfrak{j} \subseteq \mathfrak{j}^1$. Besides, K^1 is a δ-stable pro-p-group.

Since $\mathrm{V} \cap \mathrm{M}$ normalizes $\mathrm{K} \cap \mathrm{N}^-$, $\mathrm{K} \cap \mathrm{N}$ and $\mathrm{K}^1 \cap \mathrm{M}$, we have $(\mathrm{V} \cap \mathrm{M})\mathrm{K}^1 = \mathrm{K}$ whence K^1 is normal in K and $\mathrm{K} = \mathrm{V}\mathrm{K}^1$, as expected. ☐

The subgroup K^1 is useful in the following lemma. Note that we have $x\delta(x) = 1$.

Lemma 6.11. — *Let $y \in \mathrm{K}$ be such that $y\delta(y) = 1$. There are $k \in \mathrm{K}$ and $v \in \mathrm{V}$ such that:*

(1) *the element v is diagonal in $\mathrm{B}^\times$ with eigenvalues in $\{-1, 1\}$ and it satisfies $v\delta(v) = 1$,*
(2) *one has $\delta(k)yk^{-1} \in v\mathrm{K}^1$.*

Proof. — Let $\mathrm{V}^1 = \mathrm{V} \cap \mathrm{K}^1 = \mathrm{K}^1 \cap \mathrm{B}^\times$. We have:

$$\mathrm{V}^1 = (\mathrm{V} \cap \mathrm{N}^-) \cdot (\mathrm{U}^1 \cap \mathrm{M}) \cdot (\mathrm{U} \cap \mathrm{N}).$$

We thus have canonical δ-equivariant group isomorphisms:

$$\mathrm{K}/\mathrm{K}^1 \simeq \mathrm{V}/\mathrm{V}^1 \simeq (\mathrm{U} \cap \mathrm{M})/(\mathrm{U}^1 \cap \mathrm{M}). \tag{6.10}$$

By (6.9), we have $\mathrm{M} \cap \mathrm{B}^\times = \mathrm{GL}_{m_1}(\mathrm{E}) \times \cdots \times \mathrm{GL}_{m_r}(\mathrm{E})$, thus the right hand side of (6.10) identifies with $\mathcal{M} = \mathrm{GL}_{m_1}(\boldsymbol{l}) \times \cdots \times \mathrm{GL}_{m_r}(\boldsymbol{l})$, where $\boldsymbol{l}$ denotes the residue field of E. Besides, since b is given by (6.7), the involution δ acts on $\mathcal{M}$ as:

$$(g_1, \dots, g_r) \mapsto (\sigma(g_r), \dots, \sigma(g_1)).$$

Write $y = vy'$ for some $v \in \mathrm{V}$ and $y' \in \mathrm{K}^1$. The simple intersection property (5.1) gives us:

$$\delta(v)^{-1} = \delta(y')vy' \in \mathrm{V} \cap \mathrm{K}^1v\mathrm{K}^1 = \mathrm{V}^1v\mathrm{V}^1.$$

Thus there is $u \in \mathrm{V}^1$ such that $vu\delta(vu) \in \mathrm{V}^1$. Replacing (v, y') by $(vu, u^{-1}y')$, we may and will assume that $y = vy'$ with $v\delta(v) \in \mathrm{V}^1$.

We now compute the first cohomology set of δ in $\mathcal{M}$. Let $w = (w_1, \dots, w_r)$ denote the image of y in $\mathcal{M}$. We have $w\delta(w) = 1$, that is:

$$\sigma(w_j) = w_{r+1-j}^{-1}, \qquad \text{for all } j \in \{1, \dots, r\}.$$

If r is even, one can find an element $a \in \mathcal{M}$ such that $w = \delta(a)a^{-1}$. If r is odd, say $r = 2s-1$, one can find an element $a \in \mathcal{M}$ such that:

$$\delta(a)wa^{-1} = (1, \dots, 1, w_s, 1, \dots, 1)$$

and we have $w_s\sigma(w_s) = 1$. If E/E_0 is unramified, then $\boldsymbol{l}$ is quadratic over the residue field of E_0, and it follows from the triviality of the first cohomology set of σ in $\mathrm{GL}_{m_s}(\boldsymbol{l})$ that $w = \sigma(c)c^{-1}$ for some $c \in \mathcal{M}$. In these two cases, we thus may find $k \in \mathrm{K}$ such that $\delta(k)xk^{-1} \in \mathrm{K}^1$.

It remains to treat the case where r is odd and E/E_0 is ramified. In this case, we have $w_s^2 = 1$, thus w_s is conjugate in $\mathrm{GL}_{m_s}(\boldsymbol{l})$ to a diagonal element with eigenvalues 1 and -1. Let i denote the multiplicity of -1. Let:

$$v \in \mathrm{U} \cap \mathrm{M} = \mathrm{GL}_{m_1}(\mathcal{O}_\mathrm{E}) \times \cdots \times \mathrm{GL}_{m_r}(\mathcal{O}_\mathrm{E})$$

(here $\mathcal{O}_\mathrm{E}$ is the ring of integers of E) be a diagonal matrix with eigenvalues 1 and -1, such that -1 occurs with multiplicity i and only in the sth block. Then $v\delta(v) = 1$ and there is $k \in \mathrm{K}$ such that $\delta(k)yk^{-1} \in v\mathrm{K}^1$. □

Applying Lemma 6.11 to x gives us $k \in \mathrm{K}$, $v \in \mathrm{V}$ such that $bv\sigma(bv) = 1$ and $\delta(k)xk^{-1} \in v\mathrm{K}^1$. Besides, bv is antidiagonal of the form (6.7) and $\sigma(k)\gamma k^{-1} \in bv\mathrm{K}^1$. Therefore, replacing g by kg, which does not change the double coset $\mathrm{J}g\mathrm{G}^\sigma$, we will assume that γ can be written:

$$\gamma = bx, \quad b\sigma(b) = 1, \quad x \in \mathrm{J}^1, \quad b \text{ is of the form (6.7).} \tag{6.11}$$

Comparing with (6.8), we now have a stronger condition on x, that is $x\delta(x) = 1$ and $x \in \mathrm{K}^1$.

Since K^1 is a δ-stable pro-p-group and p is odd, the first cohomology set of δ in K^1 is trivial. Thus $x = \delta(y)y^{-1}$ for some $y \in \mathrm{K}^1$, hence $\gamma = \sigma(y)by^{-1}$. Since $b\sigma(b) = 1$ and the first cohomology set of σ in $\mathrm{B}^\times$ is trivial, one has $b = \sigma(h)h^{-1}$ for some $h \in \mathrm{B}^\times$. Thus $g \in yh\mathrm{G}^\sigma \subseteq \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$, and Lemma 6.7 is proved. □

6.3. Contribution of the Heisenberg representation

Let η be the Heisenberg representation of J^1 associated to θ (see §5.2). In this paragraph, we prove the following result.

Proposition 6.12. — *Given $g \in \mathrm{G}$, we have:*

$$\dim \mathrm{Hom}_{\mathrm{J}^{1g}\cap\mathrm{G}^\sigma}(\eta^g, 1) = \begin{cases} 1 & \text{if } g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — Suppose that $\mathrm{Hom}_{\mathrm{J}^{1g}\cap\mathrm{G}^\sigma}(\eta^g, 1)$ is non-zero. Restricting to $\mathrm{H}^{1g} \cap \mathrm{G}^\sigma$, the character θ^g is trivial on $\mathrm{H}^{1g} \cap \mathrm{G}^\sigma$, and Proposition 6.6 together with Lemma 6.7 give us $g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$. Conversely, assume that $g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$. Since the dimension of $\mathrm{Hom}_{\mathrm{J}^{1g}\cap\mathrm{G}^\sigma}(\eta^g, 1)$ does not change when g varies in a given $(\mathbf{J}, \mathrm{G}^\sigma)$-double coset, we may and will assume that we have $g \in \mathrm{B}^\times$. Thus we have $\gamma = \sigma(g)g^{-1} \in \mathrm{B}^\times$ as well.

Lemma 6.13. — *The map $\tau : x \mapsto \gamma^{-1}\sigma(x)\gamma$ is an involutive automorphism of* G *and, for any subgroup $\mathrm{H} \subseteq \mathrm{G}$, we have $\mathrm{H}^g \cap \mathrm{G}^\sigma = (\mathrm{H} \cap \mathrm{G}^\tau)^g$.*

Proof. — This follows from an easy calculation using the fact that $\sigma(\gamma) = \gamma^{-1}$. □

Our goal is thus to prove that the space $\mathrm{Hom}_{\mathrm{J}^1\cap\mathrm{G}^\tau}(\eta, 1)$ has dimension 1. By Paragraph 5.2, the representation of J^1 induced from θ decomposes as the direct sum of $(\mathrm{J}^1 : \mathrm{H}^1)^{1/2}$ copies of the representation η. The space:

$$\mathrm{Hom}_{\mathrm{J}^1\cap\mathrm{G}^\tau}\left(\mathrm{Ind}_{\mathrm{H}^1}^{\mathrm{J}^1}(\theta), 1\right) \tag{6.12}$$

thus decomposes as the direct sum of $(\mathrm{J}^1 : \mathrm{H}^1)^{1/2}$ copies of $\mathrm{Hom}_{\mathrm{J}^1\cap\mathrm{G}^\tau}(\eta, 1)$. Applying the Frobenius reciprocity and the Mackey formula, the space (6.12) is isomorphic to:

$$\mathrm{Hom}_{\mathrm{J}^1}\left(\mathrm{Ind}_{\mathrm{H}^1}^{\mathrm{J}^1}(\theta), \mathrm{Ind}_{\mathrm{J}^1\cap\mathrm{G}^\tau}^{\mathrm{J}^1}(1)\right) \simeq \bigoplus_{x\in\mathrm{X}} \mathrm{Hom}_{\mathrm{H}^1}\left(\theta, \mathrm{Ind}_{\mathrm{H}^1\cap(\mathrm{J}^1\cap\mathrm{G}^\tau)^x}^{\mathrm{H}^1}(1)\right)$$

where X is equal to $\mathrm{J}^1/(\mathrm{J}^1 \cap \mathrm{G}^\tau)\mathrm{H}^1$ (recall that H^1 is normal in J^1 and $\mathrm{J}^1/\mathrm{H}^1$ is abelian). Since J^1 normalizes θ, this is isomorphic to:

$$\bigoplus_{x\in\mathrm{X}} \mathrm{Hom}_{\mathrm{H}^1}\left(\theta, \mathrm{Ind}_{\mathrm{H}^1\cap\mathrm{G}^\tau}^{\mathrm{H}^1}(1)\right) \simeq \bigoplus_{x\in\mathrm{X}} \mathrm{Hom}_{\mathrm{H}^1\cap\mathrm{G}^\tau}(\theta, 1). \tag{6.13}$$

Since $\mathrm{Hom}_{\mathrm{H}^1\cap\mathrm{G}^\tau}(\theta,1)$ has dimension 1, the right hand side of (6.13) has dimension the cardinality of X. It thus remains to prove that X has cardinality $(\mathrm{J}^1:\mathrm{H}^1)^{1/2}$, or equivalently:

$$(\mathrm{J}^1\cap\mathrm{G}^\tau:\mathrm{H}^1\cap\mathrm{G}^\tau)=(\mathrm{J}^1:\mathrm{H}^1)^{\frac{1}{2}}. \tag{6.14}$$

Now consider the groups $\mathrm{J}^1\cap\mathrm{J}^{1\gamma}$ and $\mathrm{H}^1\cap\mathrm{H}^{1\gamma}$, which are both stable by τ.

Lemma 6.14. — *We have $\theta(\tau(x))=\theta(x)^{-1}$ for all $x\in\mathrm{H}^1\cap\mathrm{H}^{1\gamma}$.*

Proof. — Given $x\in\mathrm{H}^1\cap\mathrm{H}^{1\gamma}$, and using the fact that $\theta\circ\sigma=\theta^{-1}$ on H^1, we have:

$$\theta(\tau(x))^{-1}=\theta\circ\sigma(\tau(x))=\theta^\gamma(x)=\theta(x)$$

since $\gamma\in\mathrm{B}^\times$ intertwines θ. □

Let us write $\mathbb{V}$ for the $\boldsymbol{k}$-vector space $(\mathrm{J}^1\cap\mathrm{J}^{1\gamma})/(\mathrm{H}^1\cap\mathrm{H}^{1\gamma})$ equipped with both the involution τ and the symplectic form $(x,y)\mapsto\langle x,y\rangle$ induced by (5.3). We write $\mathbb{V}^+=\{v\in\mathbb{V}\mid\tau(v)=v\}$ and $\mathbb{V}^-=\{v\in\mathbb{V}\mid\tau(v)=-v\}$. We have the decomposition $\mathbb{V}=\mathbb{V}^+\oplus\mathbb{V}^-$ since $p\neq2$.

Lemma 6.15. — *There is a group isomorphism $\mathbb{V}^+\simeq(\mathrm{J}^1\cap\mathrm{G}^\tau)/(\mathrm{H}^1\cap\mathrm{G}^\tau)$.*

Proof. — First note that we have the containment $(\mathrm{J}^1\cap\mathrm{G}^\tau)(\mathrm{H}^1\cap\mathrm{H}^{1\gamma})/(\mathrm{H}^1\cap\mathrm{H}^{1\gamma})\subseteq\mathbb{V}^+$. The lemma will follow if we prove that this containment is an equality. Let $x\in\mathrm{J}^1\cap\mathrm{J}^{1\gamma}$ be such that $x(\mathrm{H}^1\cap\mathrm{H}^{1\gamma})\in\mathbb{V}^+$. One thus has $x^{-1}\tau(x)\in\mathrm{H}^1\cap\mathrm{H}^{1\gamma}$. Since $\mathrm{H}^1\cap\mathrm{H}^{1\gamma}$ is a τ-stable pro-p-group and $p\neq2$, there is an $h\in\mathrm{H}^1\cap\mathrm{H}^{1\gamma}$ such that $x^{-1}\tau(x)=h^{-1}\tau(h)$. The expected result follows. □

We are now going to prove that $\mathbb{V}^+$ and $\mathbb{V}^-$ have the same dimension.

Lemma 6.16. — *The subspaces $\mathbb{V}^+$ and $\mathbb{V}^-$ are totally isotropic.*

Proof. — Indeed, thanks to Lemma 6.14, first note that:

$$\langle\tau(x),y\rangle=\langle\tau(y),x\rangle,\quad x,y\in\mathbb{V}. \tag{6.15}$$

If $x,y\in\mathbb{V}^+$, then we get $\langle x,y\rangle=\langle x,y\rangle^{-1}$, thus $\langle x,y\rangle=1$ since $p\neq2$. If $x,y\in\mathbb{V}^-$, then we get $\langle x^{-1},y\rangle=\langle x,y^{-1}\rangle^{-1}$. But $\langle x^{-1},y\rangle=\langle x,y\rangle^{-1}=\langle x,y^{-1}\rangle$. It follows again that $\langle x,y\rangle=1$. □

Let $\mathbb{W}$ denote the kernel of the symplectic form $(x,y)\mapsto\langle x,y\rangle$ on $\mathbb{V}$, that is:

$$\mathbb{W}=\{w\in\mathbb{V}\mid\langle w,v\rangle=1\text{ for all }v\in\mathbb{V}\}.$$

Let $\mathbb{Y}$ and $\mathbb{Y}'$ denote the images of $\mathrm{H}^1\cap\mathrm{J}^{1\gamma}$ and $\mathrm{J}^1\cap\mathrm{H}^{1\gamma}$ in $\mathbb{V}$, respectively.

Lemma 6.17. — *The subspaces $\mathbb{Y}$ and $\mathbb{Y}'$ are both contained in $\mathbb{W}$, and we have $\mathbb{W}=\mathbb{Y}\oplus\mathbb{Y}'$.*

Proof. — One easily verifies that τ stabilizes $\mathbb{W}$ and exchanges $\mathbb{Y}$ and $\mathbb{Y}'$. First note that $\mathbb{Y}\subseteq\mathbb{W}$, since $\langle x,y\rangle=1$ for any $x\in\mathrm{H}^1$ and $y\in\mathrm{J}^1$. By applying τ, and thanks to (6.15), we deduce that $\mathbb{Y}'$ is also contained in $\mathbb{W}$. Now, thanks to (5.4), we have:

$$\langle1+x,1+y\rangle=\psi\circ\mathrm{tr}(\beta(xy-yx))$$

for all $x,y\in\mathrm{j}^1\cap\mathrm{j}^{1\gamma}$, where j^1 is the sub-$\mathcal{O}$-lattice of $\mathfrak{a}$ such that $\mathrm{J}^1=1+\mathrm{j}^1$. Let a_β denote the endomorphism of F-algebras $x\mapsto\beta x-x\beta$ of $\mathrm{M}_n(\mathrm{F})$. Given a subset $\mathrm{S}\subseteq\mathrm{M}_n(\mathrm{F})$, write S^* for the set of $a \in \mathrm{M}_n(\mathrm{F})$ such that $\psi(\mathrm{tr}(as)) = 1$ for all $s \in \mathrm{S}$. Then the set of $x \in \mathfrak{j}^1 \cap \mathfrak{j}^{1\gamma}$ such that $\langle 1+x, 1+y\rangle = 1$ for all $y \in \mathfrak{j}^1 \cap \mathfrak{j}^{1\gamma}$ is equal to:

$$\begin{aligned} \mathfrak{j}^1 \cap \mathfrak{j}^{1\gamma} \cap a_\beta(\mathfrak{j}^1 \cap \mathfrak{j}^{1\gamma})^* &= \mathfrak{j}^1 \cap \mathfrak{j}^{1\gamma} \cap (a_\beta(\mathfrak{j}^1) \cap a_\beta(\mathfrak{j}^1)^\gamma)^* \\ &= \mathfrak{j}^1 \cap \mathfrak{j}^{1\gamma} \cap (a_\beta(\mathfrak{j}^1)^* + a_\beta(\mathfrak{j}^1)^{*\gamma}) \\ &= \mathfrak{j}^{1\gamma} \cap (\mathfrak{j}^1 \cap a_\beta(\mathfrak{j}^1)^*) + \mathfrak{j}^1 \cap (\mathfrak{j}^1 \cap a_\beta(\mathfrak{j}^1)^*)^\gamma. \end{aligned}$$

We now claim that:

$$\mathfrak{j}^1 \cap a_\beta(\mathfrak{j}^1)^* = \mathfrak{h}^1. \tag{6.16}$$

To see this, look at the case where $g = 1$. On the one hand, for $x \in \mathfrak{j}^1$, we have $\langle 1+x, 1+y\rangle = 1$ for all $y \in \mathfrak{j}^1$ if and only if $x \in \mathfrak{j}^1 \cap a_\beta(\mathfrak{j}^1)^*$. On the other hand, the symplectic form (5.3) on the space $\mathrm{J}^1/\mathrm{H}^1$ is non-degenerate. We thus have $\mathfrak{j}^1 \cap a_\beta(\mathfrak{j}^1)^* \subseteq \mathfrak{h}^1$ and the other containment follows from the fact that ψ is trivial on the maximal ideal of $\mathcal{O}$.

We now go back to our general situation with $g \in \mathrm{B}^\times$. Applying (6.16) to $\mathfrak{j}^1$ and $\mathfrak{j}^{1\gamma}$, we get:

$$\mathfrak{j}^1 \cap \mathfrak{j}^{1\gamma} \cap a_\beta(\mathfrak{j}^1 \cap \mathfrak{j}^{1\gamma})^* = \mathfrak{j}^{1\gamma} \cap \mathfrak{h}^1 + \mathfrak{h}^{1\gamma} \cap \mathfrak{j}^1.$$

The result follows. □

Corollary 6.18. — *The subspaces $\mathbb{W}^+ = \mathbb{W} \cap \mathbb{V}^+$ and $\mathbb{W}^- = \mathbb{W} \cap \mathbb{V}^-$ have the same dimension and we have $\mathbb{W} = \mathbb{W}^+ \oplus \mathbb{W}^-$.*

Proof. — The map $x \mapsto x + \tau(x)$ is an isomorphism from $\mathbb{Y}$ to $\mathbb{W}^+$, and the map $x \mapsto x - \tau(x)$ is an isomorphism from $\mathbb{Y}$ to $\mathbb{W}^-$. Thanks to Lemma 6.17 and the fact that $\mathbb{Y}$, $\mathbb{Y}'$ have the same dimension, we thus have:

$$\dim \mathbb{W}^+ + \dim \mathbb{W}^- = 2 \cdot \dim \mathbb{Y} = \dim \mathbb{W},$$

which ends the proof of the corollary. □

Now consider the non-degenerate symplectic space $\mathbb{V}/\mathbb{W}$. It decomposes into the direct sum of two totally isotropic subspaces $(\mathbb{V}^+ + \mathbb{W})/\mathbb{W}$ and $(\mathbb{V}^- + \mathbb{W})/\mathbb{W}$. We thus have:

$$\begin{gathered} \max(\dim((\mathbb{V}^+ + \mathbb{W})/\mathbb{W}), \dim((\mathbb{V}^- + \mathbb{W})/\mathbb{W})) \leqslant \frac{1}{2} \cdot \dim(\mathbb{V}/\mathbb{W}), \\ \dim((\mathbb{V}^+ + \mathbb{W})/\mathbb{W}) + \dim((\mathbb{V}^- + \mathbb{W})/\mathbb{W}) = \dim(\mathbb{V}/\mathbb{W}). \end{gathered}$$

These spaces thus have the same dimension and are maximal totally isotropic. Corollary 6.18 now implies that $\mathbb{V}^+$ and $\mathbb{V}^-$ have the same dimension.

In order to deduce the equality (6.14), and thanks to Lemma 6.15, it remains to prove[(1)] that:

$$(\mathrm{J}^1 \cap \mathrm{J}^{1\gamma} : \mathrm{H}^1 \cap \mathrm{H}^{1\gamma}) = (\mathrm{J}^1 : \mathrm{H}^1),$$

which follows from [**10**] Lemma 5.1.10. This ends the proof of Proposition 6.12. □

(1) I thank Jiandi Zou for pointing out this argument to me, which was missing in a former version of the proof.

6.4. Contribution of the mixed Heisenberg representation

Let $g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$. We have seen in Paragraph 6.2 (see (6.11)) that, changing g without changing the double coset $\mathrm{J}g\mathrm{G}^\sigma$, we may assume that $g \in \mathrm{B}^\times$ and that $\gamma = \sigma(g)g^{-1}$ can be written $\gamma = bu$ with b of the form (6.7) and $u \in \mathrm{U}^1 = \mathrm{J}^1 \cap \mathrm{B}^\times$. We write τ for the involution defined by Lemma 6.13 and $\mathrm{U} = \mathrm{J} \cap \mathrm{B}^\times$.

We have a standard Levi subgroup M of G defined by (6.9) and parabolic subgroups P, P^- of G with Levi component M, opposite to each other and with unipotent radicals N, N^- respectively. There is a unique standard hereditary order $\mathfrak{b}_\mathrm{m} \subseteq \mathfrak{b}$ such that:

$$\mathfrak{b}_\mathrm{m}^\times = (\mathrm{U}^1 \cap \mathrm{N}^-) \cdot (\mathrm{U} \cap \mathrm{P}).$$

Since $u \in \mathrm{U}^1 \subseteq \mathrm{U}^1(\mathfrak{b}_\mathrm{m})$ and thanks to the specific form of b, one verifies that:

$$\mathrm{U}^1(\mathfrak{b}_\mathrm{m}) = (\mathrm{U}^1 \cap \mathrm{P}^-) \cdot (\mathrm{U} \cap \mathrm{N}) = (\mathrm{U} \cap \mathrm{U}^{1\gamma})\mathrm{U}^1. \tag{6.17}$$

Let $\mathfrak{a}_\mathrm{m} \subseteq \mathfrak{a}$ be the unique hereditary order of $\mathrm{M}_n(\mathrm{F})$ normalized by $\mathrm{E}^\times$ such that $\mathfrak{a}_\mathrm{m} \cap \mathrm{B} = \mathfrak{b}_\mathrm{m}$. This gives us a simple stratum $[\mathfrak{a}_\mathrm{m}, \beta]$. Let $\theta_\mathrm{m} \in \mathcal{C}(\mathfrak{a}_\mathrm{m}, \beta)$ be the transfer of θ (see (5.5)) and η_m be the Heisenberg representation on $\mathrm{J}^1_\mathrm{m} = \mathrm{J}^1(\mathfrak{a}_\mathrm{m}, \beta)$ associated with θ_m (by [**10**] Proposition 5.1.1 and [**36**] Proposition 2.1).

Let S^1 be the pro-p-subgroup $\mathrm{U}^1(\mathfrak{b}_\mathrm{m})\mathrm{J}^1 \subseteq \mathrm{J}$. By [**10**] Proposition 5.1.15, there is an irreducible representation μ of the group S^1, unique up to isomorphism, extending η and such that:

$$\mathrm{Ind}_{\mathrm{S}^1}^{\mathrm{U}^1(\mathfrak{a}_\mathrm{m})}(\mu) \simeq \mathrm{Ind}_{\mathrm{J}^1_\mathrm{m}}^{\mathrm{U}^1(\mathfrak{a}_\mathrm{m})}(\eta_\mathrm{m}). \tag{6.18}$$

In this paragraph, we prove the following result.

Proposition 6.19. — *We have* $\dim \mathrm{Hom}_{\mathrm{S}^{1g} \cap \mathrm{G}^\sigma}(\mu^g, 1) = 1$.

Proof. — Since μ extends η, the space $\mathrm{Hom}_{\mathrm{S}^{1g} \cap \mathrm{G}^\sigma}(\mu^g, 1)$ is contained in the 1-dimensional space $\mathrm{Hom}_{\mathrm{J}^{1g} \cap \mathrm{G}^\sigma}(\eta^g, 1)$. It is thus enough to prove that $\mathrm{Hom}_{\mathrm{S}^{1g} \cap \mathrm{G}^\sigma}(\mu^g, 1)$ is non-zero. Equivalently, by Lemma 6.13, it is enough to prove that $\mathrm{Hom}_{\mathrm{S}^1 \cap \mathrm{G}^\tau}(\mu, 1)$ is non-zero.

First note that, since $\mathfrak{b}_\mathrm{m}$ is σ-stable, $\mathfrak{a}_\mathrm{m}$ is σ-stable as well. We have:

$$\sigma(\mathrm{H}^1(\mathfrak{a}_\mathrm{m}, \beta)) = \mathrm{H}^1(\sigma(\mathfrak{a}_\mathrm{m}), \sigma(\beta)) = \mathrm{H}^1(\mathfrak{a}_\mathrm{m}, -\beta) = \mathrm{H}^1(\mathfrak{a}_\mathrm{m}, \beta)$$

thus $\mathrm{H}^1_\mathrm{m} = \mathrm{H}^1(\mathfrak{a}_\mathrm{m}, \beta)$ is σ-stable. By an argument similar to the one used in [**3**] Paragraph 4.6, it then follows that $\theta_\mathrm{m} \circ \sigma = (\theta_\mathrm{m})^{-1}$.

Since γ intertwines θ_m by Proposition 5.1(5), it follows from Proposition 6.6 that the character θ_m^g is trivial on $\mathrm{H}^{1g}_\mathrm{m} \cap \mathrm{G}^\sigma$, thus $\mathrm{Hom}_{\mathrm{J}^{1g}_\mathrm{m} \cap \mathrm{G}^\sigma}(\eta^g_\mathrm{m}, 1) = \mathrm{Hom}_{\mathrm{J}^1_\mathrm{m} \cap \mathrm{G}^\tau}(\eta_\mathrm{m}, 1)$ is non-zero. Inducing to $\mathrm{U}^1(\mathfrak{a}_\mathrm{m})$, we get:

$$\mathrm{Hom}_{\mathrm{U}^1(\mathfrak{a}_\mathrm{m}) \cap \mathrm{G}^\tau}\left(\mathrm{Ind}_{\mathrm{J}^1_\mathrm{m}}^{\mathrm{U}^1(\mathfrak{a}_\mathrm{m})}(\eta_\mathrm{m}), 1\right) \neq \{0\}.$$

Applying the Frobenius reciprocity and the Mackey formula, it follows that there is a $x \in \mathrm{U}^1(\mathfrak{a}_\mathrm{m})$ such that:

$$\mathrm{Hom}_{\mathrm{S}^{1x} \cap \mathrm{G}^\tau}(\mu^x, 1) \neq \{0\}. \tag{6.19}$$

We claim that $x \in \mathrm{S}^1(\mathrm{U}^1(\mathfrak{a}_\mathrm{m}) \cap \mathrm{G}^\tau)$. Restricting (6.19) to the subgroup $\mathrm{H}^{1x} \cap \mathrm{G}^\tau$ and applying Proposition 6.6, we get:

$$\sigma(xg)g^{-1}x^{-1} = \sigma(x)\gamma x^{-1} \in \mathrm{J}^1\mathrm{B}^\times\mathrm{J}^1 \cap \mathrm{U}^1(\mathfrak{a}_\mathrm{m})\gamma\mathrm{U}^1(\mathfrak{a}_\mathrm{m}).$$

Write $\sigma(x)\gamma x^{-1} = jcj'$ for some $j, j' \in \mathrm{J}^1$ and $c \in \mathrm{B}^\times$. Since $\gamma \in \mathrm{B}^\times$ and $\mathrm{J}^1 \subseteq \mathrm{U}^1(\mathfrak{a}_\mathrm{m})$, the simple intersection property (5.1) implies that $c \in \mathrm{U}^1(\mathfrak{a}_\mathrm{m})\gamma\mathrm{U}^1(\mathfrak{a}_\mathrm{m}) \cap \mathrm{B}^\times = \mathrm{U}^1(\mathfrak{b}_\mathrm{m})\gamma\mathrm{U}^1(\mathfrak{b}_\mathrm{m})$. Therefore we have $\sigma(x)\gamma x^{-1} = \sigma(s)\gamma s'$ for some $s, s' \in \mathrm{S}^1$. If we let $y = s^{-1}x$, then we have $\sigma(y)\gamma y^{-1} = \gamma l$ for some $l \in \mathrm{S}^1$, that is $\tau(y)y^{-1} = l$. Since the first cohomology set of τ in $\mathrm{S}^1 \cap \mathrm{S}^{1\gamma}$ is trivial, we get $l = \tau(h)h^{-1}$ for some $h \in \mathrm{S}^1$. This gives us:

$$x \in \mathrm{U}^1(\mathfrak{a}_\mathrm{m}) \cap \mathrm{S}^1(\mathrm{G}^\sigma)^{g^{-1}}$$

and the claim follows from the fact that $\mathrm{S}^1 \subseteq \mathrm{U}^1(\mathfrak{a}_\mathrm{m})$.

Putting the claim and (6.19) together, we deduce that $\mathrm{Hom}_{\mathrm{S}^1 \cap \mathrm{G}^\tau}(\mu, 1)$ is non-zero. □

6.5. The double coset theorem

Let $\boldsymbol{\kappa}$ be an irreducible representation of $\mathbf{J}$ extending η as in Paragraph 5.2. There is an irreducible representation $\boldsymbol{\rho}$ of $\mathbf{J}$, unique up to isomorphism, which is trivial on the subgroup J^1 and satisfies $\boldsymbol{\lambda} \simeq \boldsymbol{\kappa} \otimes \boldsymbol{\rho}$. We have the following lemma.

Lemma 6.20. — *Let $g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$.*

(1) *There is a unique character χ of $\mathbf{J}^g \cap \mathrm{G}^\sigma$ trivial on $\mathrm{J}^{1g} \cap \mathrm{G}^\sigma$ such that:*

$$\mathrm{Hom}_{\mathrm{J}^{1g} \cap \mathrm{G}^\sigma}(\eta^g, 1) = \mathrm{Hom}_{\mathbf{J}^g \cap \mathrm{G}^\sigma}(\boldsymbol{\kappa}^g, \chi^{-1}).$$

(2) *The canonical linear map:*

$$\mathrm{Hom}_{\mathrm{J}^{1g} \cap \mathrm{G}^\sigma}(\eta^g, 1) \otimes \mathrm{Hom}_{\mathbf{J}^g \cap \mathrm{G}^\sigma}(\boldsymbol{\rho}^g, \chi) \to \mathrm{Hom}_{\mathbf{J}^g \cap \mathrm{G}^\sigma}(\boldsymbol{\lambda}^g, 1)$$

is an isomorphism.

Proof. — Let us fix a non-zero linear form $\mathscr{E} \in \mathrm{Hom}_{\mathrm{J}^{1g} \cap \mathrm{G}^\sigma}(\eta^g, 1)$. The choice of $\boldsymbol{\kappa}$ defines an action of $\mathbf{J}^g \cap \mathrm{G}^\sigma$ on the space $\mathrm{Hom}_{\mathrm{J}^{1g} \cap \mathrm{G}^\sigma}(\eta^g, 1)$, which has dimension 1 by Proposition 6.12. This determines a unique character χ of $\mathbf{J}^g \cap \mathrm{G}^\sigma$ trivial on $\mathrm{J}^{1g} \cap \mathrm{G}^\sigma$ such that:

$$\mathscr{E} \circ \boldsymbol{\kappa}^g(x) = \chi(x)^{-1} \cdot \mathscr{E}$$

for all $x \in \mathbf{J}^g \cap \mathrm{G}^\sigma$. This gives us the first part of the lemma.

Given $\mathscr{L} \in \mathrm{Hom}_{\mathbf{J}^g \cap \mathrm{G}^\sigma}(\boldsymbol{\lambda}^g, 1)$ and w in the space of $\boldsymbol{\rho}$, the linear form $v \mapsto \mathscr{L}(v \otimes w)$ defined on the space of η is in $\mathrm{Hom}_{\mathrm{J}^{1g} \cap \mathrm{G}^\sigma}(\eta^g, 1)$. By Proposition 6.12 it is thus of the form $\mathscr{F}(w)\mathscr{E}$ for a unique $\mathscr{F}(w) \in \mathrm{R}$. We have $\mathscr{L} = \mathscr{E} \otimes \mathscr{F}$ and $\mathscr{F} \in \mathrm{Hom}_{\mathbf{J}^g \cap \mathrm{G}^\sigma}(\boldsymbol{\rho}^g, \chi)$. □

Theorem 6.21. — *Let $g \in \mathrm{G}$ and suppose $\mathrm{Hom}_{\mathbf{J}^g \cap \mathrm{G}^\sigma}(\boldsymbol{\lambda}^g, 1)$ is non-zero. Then $\sigma(g)g^{-1} \in \mathbf{J}$.*

Proof. — We know from Proposition 6.6 and Lemma 6.7 that $g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$. We thus may assume that $g \in \mathrm{B}^\times$ and $\gamma = \sigma(g)g^{-1}$ is as in Paragraph 6.4. In particular, we have a standard hereditary order $\mathfrak{b}_\mathrm{m} \subseteq \mathfrak{b}$ and an involution τ.

Let us fix an irreducible representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η, and let χ be the character given by Lemma 6.20. The restriction of $\boldsymbol{\kappa}$ to J, denoted κ, is an irreducible representation of J extending

η. It follows from Remark 5.2 that κ is a beta-extension of η, and from [**10**] Theorem 5.2.3 that κ extends μ. Proposition 6.19 thus implies:

$$\mathrm{Hom}_{\mathrm{S}^{1g}\cap\mathrm{G}^\sigma}(\mu^g,1)=\mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\kappa}^g,\chi^{-1})$$

and χ is trivial on $\mathrm{U}^1(\mathfrak{b}_\mathrm{m})^g\cap\mathrm{G}^\sigma$. By Lemma 6.20, the space $\mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\rho}^g,\chi)$ is non-zero.

Write $\mathrm{U}=\mathrm{J}\cap\mathrm{B}^\times$ and $\mathrm{U}^1=\mathrm{J}^1\cap\mathrm{B}^\times$. Since $g\in\mathrm{B}^\times$, we have $\mathrm{J}^g\cap\mathrm{G}^\sigma=(\mathrm{U}^g\cap\mathrm{G}^\sigma)(\mathrm{J}^{1g}\cap\mathrm{G}^\sigma)$. Let ρ be the restriction of $\boldsymbol{\rho}$ to J. Then $\mathrm{Hom}_{\mathrm{U}^g\cap\mathrm{G}^\sigma}(\rho^g,\chi)$ is non-zero. Lemma 6.13 implies:

$$\mathrm{Hom}_{\mathrm{U}^1(\mathfrak{b}_\mathrm{m})\cap\mathrm{G}^\tau}(\rho,1)=\mathrm{Hom}_{\mathrm{U}^1(\mathfrak{b}_\mathrm{m})^g\cap\mathrm{G}^\sigma}(\rho^g,1)\neq\{0\}. \tag{6.20}$$

We now describe more carefuly the subgroup $\mathrm{U}^1(\mathfrak{b}_\mathrm{m})$.

Lemma 6.22. — *We have* $\mathrm{U}^1(\mathfrak{b}_\mathrm{m})=(\mathrm{U}^1(\mathfrak{b}_\mathrm{m})\cap\mathrm{G}^\tau)\mathrm{U}^1$.

Proof. — We follow the proof of [**25**] Proposition 5.20. According to (6.17) it is enough to prove that $\mathrm{U}\cap\mathrm{U}^{1\gamma}$ is contained in $(\mathrm{U}^1(\mathfrak{b}_\mathrm{m})\cap\mathrm{G}^\tau)\mathrm{U}^1$. Let $x\in\mathrm{U}\cap\mathrm{U}^{1\gamma}$ and define $y=x^{-1}\tau(x)^{-1}x\tau(x)$. Then $y\in\mathrm{U}^1\cap\mathrm{U}^{1\gamma}$ and $y\tau(y)=1$. Since the first cohomology set of τ in $\mathrm{U}^1\cap\mathrm{U}^{1\gamma}$ is trivial, we get $y=z\tau(z)^{-1}$ for some $z\in\mathrm{U}^1\cap\mathrm{U}^{1\gamma}$. Define $x'=x\tau(x)\tau(z)$. Then $x'\in\mathrm{U}^1(\mathfrak{b}_\mathrm{m})\cap\mathrm{G}^\tau$ and we have $x\in x'\mathrm{U}^1$. □

Since ρ is trivial on U^1, Lemma 6.22 and (6.20) together imply that $\mathrm{Hom}_{\mathrm{U}^1(\mathfrak{b}_\mathrm{m})}(\rho,1)$ is non-zero. Since $\mathrm{U}^1(\mathfrak{b}_\mathrm{m})/\mathrm{U}^1$ is a unipotent subgroup of $\mathrm{U}/\mathrm{U}^1\simeq\mathrm{GL}_m(\boldsymbol{l})$, the fact that the representation ρ is cuspidal (see Paragraph 5.2) implies that $\mathfrak{b}_\mathrm{m}=\mathfrak{b}$, that is $\gamma\in\mathrm{U}\subseteq\mathbf{J}$. □

Lemma 4.25 of [**3**] gives a detailed account of the elements $g\in\mathrm{G}$ such that $\sigma(g)g^{-1}\in\mathbf{J}$.

6.6. Proof of Theorem 6.1

Let π be a σ-selfdual cuspidal representation of G, and $(\mathbf{J},\boldsymbol{\lambda})$ be a σ-selfdual type in π given by Theorem 5.10. If the space $\mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda},1)$ is non-zero, then (6.1) implies that π is distinguished.

Conversely, suppose that π is distinguished and that $(\mathbf{J},\boldsymbol{\lambda})$ has been chosen as in Remark 5.11 as it may be. Then the space $\mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}^g,1)$ is non-zero for some $g\in\mathrm{G}$. By Theorem 6.21, one has $\sigma(g)g^{-1}\in\mathbf{J}$. Thus $\mathbf{J}^g$ is σ-stable, and:

$$(\boldsymbol{\lambda}^g)^\sigma=(\boldsymbol{\lambda}^\sigma)^{\sigma(g)}\simeq(\boldsymbol{\lambda}^\vee)^g=(\boldsymbol{\lambda}^g)^\vee$$

thus the type $(\mathbf{J}^g,\boldsymbol{\lambda}^g)$ is σ-selfdual.

Remark 6.23. — Let $(\mathbf{J},\boldsymbol{\lambda})$ and $(\mathbf{J}',\boldsymbol{\lambda}')$ be two distinguished σ-selfdual types in π. Since they both occur in π, there is a $g\in\mathrm{G}$ such that $\mathbf{J}'=\mathbf{J}^g$ and $\boldsymbol{\lambda}'\simeq\boldsymbol{\lambda}^g$. Thanks to the multiplicity 1 property of Theorem 4.1, the formula (6.1) tells us that the double cosets $\mathbf{J}\mathrm{G}^\sigma$ and $\mathbf{J}g\mathrm{G}^\sigma$ are equal, which implies that $g\in\mathbf{J}\mathrm{G}^\sigma$. Thus a distinguished cuspidal representation π contains, up to G^σ-conjugacy, a unique distinguished σ-selfdual type.

Recall that Proposition 5.14 associates to any σ-selfdual cuspidal representation of G a quadratic extension T/T_0.

Corollary 6.24. — *Let π be a σ-selfdual cuspidal representation of* G, *and suppose that* T/T_0 *is unramified. Then π is distinguished if and only if any σ-selfdual type in π is distinguished.*

Proof. — This follows from Theorem 6.1 together with Proposition 5.17, which says that the representation π contains, up to G^σ-conjugacy, a unique σ-selfdual type. □

When T/T_0 is ramified, Proposition 5.17 tells us that π contains more than one G^σ-conjugacy class of σ-selfdual types as soon as its relative degree m is at least 2. In the next section, we will see that the G^σ-conjugacy class of index $\lfloor m/2 \rfloor$ (see Definition 5.18) is the only one which may contribute to the distinction of π.

7. The cuspidal ramified case

As usual, write $G = GL_n(F)$ for some $n \geqslant 1$. To any σ-selfdual cuspidal representation of G, one can associate a quadratic extension T/T_0 and its relative degree m (see Proposition 5.14). In this section, we will consider the case where T/T_0 is ramified.

7.1.

The first main result of this section is the following proposition, which we will prove in Paragraph 7.3.

Proposition 7.1. — *Let π be a σ-selfdual cuspidal representation of* G *with quadratic extension* T/T_0 *and relative degree m. Suppose* T/T_0 *is ramified. Then π is distinguished if and only if:*

(1) *either $m = 1$ or m is even, and*
(2) *any σ-selfdual type of index $\lfloor m/2 \rfloor$ contained in π is distinguished.*

Remark 7.2. — Proposition 7.1 refines Theorem 6.1 by saying that, if T/T_0 is ramified, then the G^σ-conjugacy class of σ-selfdual types of index $\lfloor m/2 \rfloor$ contained in π is the only one which may contribute to the distinction of π. See [**3**] Proposition 5.5 for a characterization of this class in terms of Whittaker data. See also Definition 10.1 and Remark 10.2 below.

Remark 7.3. — Proposition 7.1 is proved in [**3**] in a different manner from the one we give here (see Remark 6.3 above and [**3**] Corollary 6.6, Remark 6.7).

Remark 7.4. — If we assume π to be *supercuspidal* in Proposition 7.1, then m is automatically either even or equal to 1, even if π is not distinguished (see Proposition 8.1).

Remark 7.5. — However, if π is non-supercuspidal in Proposition 7.1, then its relative degree m need not be either even nor equal to 1. Let k be a divisor of n, and τ be a σ-selfdual supercuspidal representation of $GL_{n/k}(F)$. Assume R has characteristic $\ell > 0$, let ν be the unramified character "absolute value of the determinant" and let $e(\tau)$ be the smallest integer $i \geqslant 1$ such that $\tau\nu^i \simeq \tau$. Suppose that $k = e(\tau)\ell^u$ for some $u \geqslant 0$. Then [**37**] Théorème 6.14 tells us that the unique generic irreducible subquotient π of the normalized parabolically induced representation:

$$\tau \times \tau\nu \times \cdots \times \tau\nu^{k-1}$$

is cuspidal, and that it is σ-selfdual since τ is. If $k > 1$ and $m(\pi) = km(\tau)$ is odd, then π is a σ-selfdual cuspidal representation which is not distinguished nor ω-distinguished. (For instance, this is the case when τ is the trivial character of $\mathrm{F}^\times$ and $k = n = \ell$ where $\ell \neq 2$ divides $q-1$, which gives $m(\tau) = e(\tau) = 1$).

7.2. Existence of σ-selfdual extensions of the Heisenberg representation

We now go back to our usual notation. Let $[\mathfrak{a}, \beta]$ be a maximal simple stratum in $\mathrm{M}_n(\mathrm{F})$ such that $\mathfrak{a}$ is σ-stable and $\sigma(\beta) = -\beta$. Write E for the extension $\mathrm{F}[\beta]$, and suppose that it is ramified over the field E_0 of σ-fixed points in E. Let d be the degree $[\mathrm{E} : \mathrm{F}]$ and write $n = md$.

Let $\boldsymbol{l}$ denote the residue field of E. Let us notice once and for all that, since $p \neq 2$, any character of $\mathrm{GL}_m(\boldsymbol{l})$ is of the form $\alpha \circ \det$, for some character α of $\boldsymbol{l}^\times$.

The following lemma generalizes [**11**] Lemme 3.4.6 (which is concerned with complex representations and χ trivial only).

Lemma 7.6. — *Let χ be a character of $(\mathrm{GL}_i \times \mathrm{GL}_{m-i})(\boldsymbol{l})$ for some $i \in \{0, \cdots, \lfloor m/2 \rfloor\}$. Suppose there is a χ-distinguished cuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$. Then either $m = 1$ or $m = 2i$.*

Proof. — If $m \geqslant 2$, the result follows from Proposition 2.14. Note that, if $m = 1$, then χ is the unique χ-distinguished irreducible representation of $\mathrm{GL}_1(\boldsymbol{l})$. □

Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a maximal simple character such that $\mathrm{H}^1(\mathfrak{a}, \beta)$ is σ-stable and $\theta \circ \sigma = \theta^{-1}$, and let $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ be its normalizer in G. Let η be the Heisenberg representation of $\mathrm{J}^1 = \mathrm{J}^1(\mathfrak{a}, \beta)$ containing θ, and write $\mathrm{J} = \mathrm{J}(\mathfrak{a}, \beta)$.

Lemma 7.7. — *There is a σ-selfdual representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η.*

Proof. — Conjugating by a suitable element in G, we may assume that the stratum $[\mathfrak{a}, \beta]$ satisfies the conditions of Remark 5.11. Indeed, if it doesn't, there is a $g \in \mathrm{G}$ such that θ^g is σ-selfdual and $[\mathfrak{a}^g, \beta^g]$ satisfies these conditions. This implies that $\gamma = \sigma(g)g^{-1}$ normalizes θ, that is $\gamma \in \mathbf{J}$. Now, assuming the lemma to be true for $[\mathfrak{a}^g, \beta^g]$, there exists a σ-selfdual representation $\boldsymbol{\kappa}'$ of $\mathbf{J}^g$ extending η^g. Define a representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ by $\boldsymbol{\kappa}^g = \boldsymbol{\kappa}'$. Then $\boldsymbol{\kappa}$ extends η, and it is σ-selfdual since $\gamma \in \mathbf{J}$. From now on, we will assume that $[\mathfrak{a}, \beta]$ satisfies the conditions of Remark 5.11. We will identify J/J^1 with $\mathrm{GL}_m(\boldsymbol{l})$, on which σ acts trivially.

Suppose first that R has characteristic 0. Let $\boldsymbol{\kappa}$ be a representation of $\mathbf{J}$ extending η, let μ be the character of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}\mu$ given by Lemma 5.21 and χ be the character of $\mathbf{J} \cap \mathrm{G}^\sigma$ associated with $\boldsymbol{\kappa}$ by Lemma 6.20. We claim that there is a character ν of $\mathbf{J}$ trivial on J^1 such that $(\nu \circ \sigma)\nu = \mu$. Indeed, $\boldsymbol{\kappa}\nu$ will then extend η and be σ-selfdual. We have:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{J}^1 \cap \mathrm{G}^\sigma}(\eta, 1) &= \mathrm{Hom}_{\mathbf{J} \cap \mathrm{G}^\sigma}(\boldsymbol{\kappa}, \chi^{-1}) \\ &\simeq \mathrm{Hom}_{\mathbf{J} \cap \mathrm{G}^\sigma}(\chi, \boldsymbol{\kappa}^{\sigma\vee}) \\ &\simeq \mathrm{Hom}_{\mathbf{J} \cap \mathrm{G}^\sigma}(\chi\mu^{-1}, \boldsymbol{\kappa}) \end{aligned}$$

where the isomorphism in the middle follows from the fact that σ acts trivially on $\mathbf{J} \cap \mathrm{G}^\sigma$ and by duality. Since R has characteristic 0 and $\mathbf{J} \cap \mathrm{G}^\sigma$ is compact, the latter space is isomorphic to $\mathrm{Hom}_{\mathbf{J} \cap \mathrm{G}^\sigma}(\boldsymbol{\kappa}, \chi\mu^{-1})$. By uniqueness of χ, it follows that the restriction of μ to $\mathbf{J} \cap \mathrm{G}^\sigma$ is χ^2. Restricting to $\mathrm{J} \cap \mathrm{G}^\sigma$ and writing $\mu = \varphi \circ \det$ and $\chi = \alpha \circ \det$ as characters of $\mathrm{GL}_m(\boldsymbol{l})$ for suitable characters φ, α of $\boldsymbol{l}^\times$, we get $\varphi = \alpha^2$. Let ν be the unique character of J which is trivial on J^1

and equal to $\alpha \circ \det$ as a character of $\mathrm{GL}_m(\boldsymbol{l})$. Since $\mathbf{J}$ is generated by t and J, it remains to extend ν to $\mathbf{J}$ by fixing a scalar $\nu(t) \in \mathrm{R}^\times$ such that $\nu(t)^2 = \nu(-1)\mu(t)$.

Suppose now that R is equal to $\overline{\mathbf{F}}_\ell$. As in the proof of Lemma 2.5, we use a lifting and reduction argument. Note that reducing finite-dimensional smooth $\overline{\mathbf{Q}}_\ell$-representations of profinite groups is the same as for finite groups (for which we referred to [**45**] §15). The simple character θ lifts to a simple character $\widetilde{\theta}$ with values in $\overline{\mathbf{Z}}_\ell$, defined with respect to the same simple stratum as θ, and such that $\widetilde{\theta} \circ \sigma = \widetilde{\theta}^{-1}$. By the characteristic 0 case, there is a σ-selfdual $\overline{\mathbf{Q}}_\ell$-representation $\widetilde{\boldsymbol{\kappa}}$ of $\mathbf{J}$ extending the irreducible $\overline{\mathbf{Q}}_\ell$-representation $\widetilde{\eta}$ of J^1 associated with $\widetilde{\theta}$. The reduction mod ℓ of $\widetilde{\eta}$ is a representation of J^1 containing θ, of the same dimension as η: it is thus isomorphic to η itself. Let $\widetilde{\kappa}$ denote the restriction of $\widetilde{\boldsymbol{\kappa}}$ to J. Its reduction mod ℓ, denoted κ, is a σ-selfdual representation of J extending η, and which extends to some representation $\boldsymbol{\kappa}$ of $\mathbf{J}$. Since κ is σ-selfdual, the representation $\boldsymbol{\kappa}^{\sigma\vee}$ is isomorphic to $\boldsymbol{\kappa}\mu$ for some character μ of $\mathbf{J}$ trivial on J. Since $\mathbf{J}$ is generated by J and t, there is a character ν of $\mathbf{J}$ trivial on J such that $(\nu \circ \sigma)\nu = \mu$, thus $\boldsymbol{\kappa}\nu$ is σ-selfdual.

Finally, suppose that R has characteristic $\ell > 0$, and fix an embedding $\iota : \overline{\mathbf{F}}_\ell \to \mathrm{R}$. Since θ has finite image, there is a simple $\overline{\mathbf{F}}_\ell$-character θ_0 defined with respect to the same simple stratum as θ such that $\theta_0 \circ \sigma = \theta_0^{-1}$ and $\theta = \iota \circ \theta_0$. Let $\boldsymbol{\kappa}_0$ be a σ-selfdual $\overline{\mathbf{F}}_\ell$-representation of $\mathbf{J}$ extending the irreducible $\overline{\mathbf{F}}_\ell$-representation η_0 of J^1 associated with θ_0. The irreducible representations η and $\eta_0 \otimes \mathrm{R}$ both contain θ. By uniqueness of the Heisenberg representation, they are isomorphic. It follows that $\boldsymbol{\kappa} = \boldsymbol{\kappa}_0 \otimes \mathrm{R}$ is a σ-selfdual R-representation of J extending η. □

7.3. Proof of Proposition 7.1

Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type, with associated simple character the character θ of §7.2.

Lemma 7.8. — *If $(\mathbf{J}, \boldsymbol{\lambda})$ is distinguished, then:*

(1) *either $m = 1$,*

(2) *or $m = 2r$ for some $r \geqslant 1$, and $(\mathbf{J}, \boldsymbol{\lambda})$ has index r.*

Proof. — Let $\boldsymbol{\kappa}$ be a σ-selfdual representation of $\mathbf{J}$ extending η provided by Lemma 7.7. Let $\boldsymbol{\rho}$ be the unique irreducible representation of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\lambda} \simeq \boldsymbol{\kappa} \otimes \boldsymbol{\rho}$ and i be the index of $(\mathbf{J}, \boldsymbol{\lambda})$. Lemma 6.20 tells us that $\boldsymbol{\rho}$ is χ-distinguished for some character χ of $\mathbf{J} \cap \mathrm{G}^\sigma$ trivial on $\mathrm{J}^1 \cap \mathrm{G}^\sigma$. Restricting $\boldsymbol{\rho}$ to J and identifying J/J^1 with $\mathrm{GL}_m(\boldsymbol{l})$, we get a cuspidal representation ρ of $\mathrm{GL}_m(\boldsymbol{l})$ and a character χ of $(\mathrm{GL}_i \times \mathrm{GL}_{m-i})(\boldsymbol{l})$ such that ρ is χ-distinguished. The result follows from Lemma 7.6. □

Let π be a σ-selfdual cuspidal representation of G, and suppose that the quadratic extension T/T_0 associated with it by Proposition 5.14 is ramified. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type contained in π. By Remark 5.12, we may assume that it is defined with respect to a σ-selfdual simple stratum. By Remark 5.16, E is ramified over E_0. We can thus apply the results of Paragraph 7.2 and Lemma 7.8. Proposition 7.1 now follows from Theorem 6.1 together with Lemma 7.8.

7.4. Existence of distinguished extensions of the Heisenberg representation

The second main result of this section is the following proposition.

Proposition 7.9. — *Let π be a σ-selfdual cuspidal representation of* G *with ramified quadratic extension* T/T_0. *Assume that $m=1$ or m is even, and let $(\mathbf{J},\boldsymbol{\lambda})$ be a σ-selfdual type in π of index $\lfloor m/2\rfloor$. Let J^1 be the maximal normal pro-p-subgroup of $\mathbf{J}$ and η be an irreducible component of the restriction of $\boldsymbol{\lambda}$ to J^1.*

(1) *There is a distinguished representation of $\mathbf{J}$ extending η, and any such representation of $\mathbf{J}$ is σ-selfdual.*

(2) *Let $\boldsymbol{\kappa}$ be a distinguished representation of $\mathbf{J}$ extending η, and let $\boldsymbol{\rho}$ be the unique representation of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\lambda}\simeq\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$. Then π is distinguished if and only if $\boldsymbol{\rho}$ is distinguished.*

We start with the following lemma, which slightly refines part (1) of the proposition.

Lemma 7.10. — *Let $(\mathbf{J},\boldsymbol{\lambda})$ be as in* Proposition 7.9.

(1) *There is a distinguished representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η.*

(2) *If $\ell=2$ or if m is even, such a distinguished representation $\boldsymbol{\kappa}$ is unique.*

(3) *Any distinguished representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η is σ-selfdual.*

Remark 7.11. — If $\ell\neq 2$ and $m=1$, there are exactly two distinguished representations of $\mathbf{J}$ extending η, twisted of each other by the unique non-trivial character of $\mathbf{J}$ trivial on $(\mathbf{J}\cap\mathrm{G}^\sigma)\mathrm{J}^1$. (See the proof below, which shows that $(\mathbf{J}\cap\mathrm{G}^\sigma)\mathrm{J}^1$ has index 2 in $\mathbf{J}$.)

Proof. — Let J be the maximal compact subgroup of $\mathbf{J}$, and J^1 be its maximal normal pro-p-subgroup. As usual, we fix a maximal simple stratum $[\mathfrak{a},\beta]$ defining $(\mathbf{J},\boldsymbol{\lambda})$ such that $\mathfrak{a}$ is σ-stable and $\sigma(\beta)=-\beta$, and write $\mathrm{E}=\mathrm{F}[\beta]$ and $\boldsymbol{l}$ for its residue field. We will identify J/J^1 with $\mathrm{GL}_m(\boldsymbol{l})$ equipped with an involution whose fixed points is $(\mathrm{GL}_i\times\mathrm{GL}_{m-i})(\boldsymbol{l})$ where $i=\lfloor m/2\rfloor$.

Let $\boldsymbol{\kappa}$ be an irreducible representation of $\mathbf{J}$ extending η. By Lemma 6.20, there is a character χ of $\mathbf{J}\cap\mathrm{G}^\sigma$ trivial on $\mathrm{J}^1\cap\mathrm{G}^\sigma$ associated to $\boldsymbol{\kappa}$. We claim that χ extends to a character ϕ of $\mathbf{J}$ trivial on J^1. It will then follow that $\boldsymbol{\kappa}\phi$ is distinguished and extends η.

Suppose first that $m=1$. We then have canonical group isomorphisms:

$$(\mathrm{J}\cap\mathrm{G}^\sigma)/(\mathrm{J}^1\cap\mathrm{G}^\sigma)\simeq\mathrm{J}/\mathrm{J}^1\simeq\boldsymbol{l}^\times. \tag{7.1}$$

Thus there is a unique character ϕ of J trivial on J^1 which coincides with χ on $\mathrm{J}\cap\mathrm{G}^\sigma$. Since $\mathbf{J}$ is generated by t and J, and since t normalizes ϕ, this character extends to a character of $\mathbf{J}$ trivial on J^1.

Lemma 7.12. — *Suppose that $m=1$. Then $\mathbf{J}\cap\mathrm{G}^\sigma$ is generated by $\mathrm{J}\cap\mathrm{G}^\sigma$ and t^2.*

Proof. — Since we have $\mathbf{J}=\mathrm{E}^\times\mathrm{J}^1$ when $m=1$, we may consider the exact sequence of σ-groups:

$$1\to\mathrm{U}^1_\mathrm{E}\to\mathrm{E}^\times\times\mathrm{J}^1\to\mathbf{J}\to 1.$$

Taking σ-invariants and since the first cohomology group $\mathsf{H}^1(\sigma,\mathrm{U}^1_\mathrm{E})$ is trivial, $\mathbf{J}\cap\mathrm{G}^\sigma$ is generated by $\mathrm{E}_0^\times$ and $\mathrm{J}^1\cap\mathrm{G}^\sigma$. The result follows from (7.1) and the fact that t^2 is a uniformizer of E_0. □

It follows from Lemma 7.12 that χ can be extended to a character ϕ of J trivial on J^1. Since we must have $\phi(t)^2=\chi(t^2)$ in the field R of characteristic ℓ, there are at most two such characters, with uniqueness if and only if $\ell=2$.

Suppose now that $m = 2r$ for some $r \geqslant 1$, and consider the element:

$$w = \begin{pmatrix} & \mathrm{id}_r \\ \mathrm{id}_r & \end{pmatrix} \in \mathfrak{b}^\times \subseteq \mathrm{GL}_m(\mathrm{E})$$

where id_r is the identity matrix in $\mathrm{GL}_r(\mathrm{E})$.

Lemma 7.13. — *Suppose that $m = 2r$. The group $\mathbf{J} \cap \mathrm{G}^\sigma$ is generated by $\mathrm{J} \cap \mathrm{G}^\sigma$ and tw.*

Proof. — First, notice that $t' = tw$ is σ-invariant. Any $x \in \mathbf{J}$ can be written $x = t'^k y$ for unique $k \in \mathbb{Z}$ and $y \in \mathrm{J}$. We thus have $x \in \mathbf{J} \cap \mathrm{G}^\sigma$ if and only if $y \in \mathrm{J} \cap \mathrm{G}^\sigma$. □

Since $\boldsymbol{\kappa}$ and $\mathbf{J} \cap \mathrm{G}^\sigma$ are normalized by w, we have $\mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\kappa}, \chi^{-1}) = \mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\kappa}, (\chi^w)^{-1})$, and the uniqueness of χ implies that $\chi^w = \chi$. First, consider the character of:

$$(\mathrm{J} \cap \mathrm{G}^\sigma)/(\mathrm{J}^1 \cap \mathrm{G}^\sigma) \simeq (\mathrm{GL}_r \times \mathrm{GL}_r)(\boldsymbol{l})$$

defined by χ and write it $(\alpha_1 \circ \det) \otimes (\alpha_2 \circ \det)$ for some characters α_1, α_2 of $\boldsymbol{l}^\times$. The identity $\chi^w = \chi$ implies that $\alpha_1 = \alpha_2$, thus there is a unique character ϕ of J trivial on J^1 which coincides with χ on $\mathrm{J} \cap \mathrm{G}^\sigma$. By Lemma 7.13, there is a unique character ϕ of $\mathbf{J}$ trivial on J^1 extending χ. This proves (1) and (2).

Now let $\boldsymbol{\kappa}$ be a distinguished representation of $\mathbf{J}$ extending η. It satisfies $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}\mu$ for some character μ of $\mathbf{J}$ trivial on J^1 such that $\mu \circ \sigma = \mu$ (see Lemma 5.21). Since $\boldsymbol{\kappa}$ is distinguished, μ is trivial on $\mathbf{J} \cap \mathrm{G}^\sigma$. We will prove that $\boldsymbol{\kappa}$ is σ-selfdual, that is, that the character μ is trivial.

Suppose first that $m = 1$. Thus $(\mathbf{J}, \boldsymbol{\kappa})$ is a distinguished type in G. Let π denote the cuspidal irreducible representation of G compactly induced from $\boldsymbol{\kappa}$. It is distinguished, thus σ-selfdual by Theorem 4.1. It follows that $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}\mu$ are both contained in π, thus μ is trivial.

Suppose now that $m = 2r$. Since μ is trivial on $(\mathrm{GL}_r \times \mathrm{GL}_r)(\boldsymbol{l})$, it must be trivial on $\mathrm{GL}_m(\boldsymbol{l})$. Since tw is σ-invariant, we have $\mu(tw) = 1$. Thus μ is trivial. This proves (3). □

For part (2) of Proposition 7.9, it suffices to fix a distinguished representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η and to consider the canonical isomorphism:

$$\mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\kappa}, 1) \otimes \mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\rho}, 1) \to \mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}, 1)$$

(compare with Lemma 6.20).

Proposition 7.9 reduces the problem of the distinction of π to that of $\boldsymbol{\rho}$. In the next section, we investigate the distinction of $\boldsymbol{\rho}$ *in the case where π is supercuspidal.*

8. The supercuspidal ramified case

In this section, we investigate the distinction of σ-selfdual *supercuspidal* representations of G in the case where T/T_0 ramified.

8.1. The relative degree

Let π be a σ-selfdual cuspidal representation of G such that T/T_0 is ramified. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type contained in π and let $\boldsymbol{\kappa}$ be a σ-selfdual representation of $\mathbf{J}$ extending η given by Lemma 7.7. This defines a σ-selfdual irreducible representation $\boldsymbol{\rho}$ of $\mathbf{J}$ trivial on J^1. Let J denote the maximal compact subgroup of $\mathbf{J}$ and ρ denote the cuspidal representation of $\mathrm{J}/\mathrm{J}^1 \simeq \mathrm{GL}_m(\boldsymbol{l})$ induced by $\boldsymbol{\rho}$.

Since $\boldsymbol{\rho}$ is σ-selfdual, the representation ρ is selfdual. Applying Fact 5.5 together with Lemma 2.17, we get the following lemma mentioned in Remark 7.4.

Proposition 8.1. — *Let π be a σ-selfdual supercuspidal representation of* G *such that* T/T_0 *is ramified. Then its relative degree m is either even or equal to* 1.

8.2. Distinction criterion in the ramified case

Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type of index $\lfloor m/2 \rfloor$ contained in π. We fix a distinguished representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η given by Proposition 7.9. It is σ-selfdual, thus the representation $\boldsymbol{\rho}$ of $\mathbf{J}$ trivial on J^1 which correspond to this choice is σ-selfdual. By Proposition 7.9 again, π is distinguished if and only if $\boldsymbol{\rho}$ is distinguished. We now investigate the distinction of $\boldsymbol{\rho}$. For this, we will use the admissible pairs of level zero introduced in Paragraphs 5.3 and 5.5.

Let us fix a σ-selfdual maximal simple stratum $[\mathfrak{a}, \beta]$ such that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$. Write $\mathrm{E} = \mathrm{F}[\beta]$. Let $(\mathrm{K}/\mathrm{E}, \xi)$ be an admissible pair of level zero attached to $\boldsymbol{\rho}$ in the sense of Definition 5.7. Since $\boldsymbol{\rho}$ is σ-selfdual, Proposition 5.19 tells us that there is a unique involutive E_0-automorphism of K, which we denote by σ, which is non-trivial on E and satisfies $\xi \circ \sigma = \xi^{-1}$. Let K_0 be the σ-fixed points of K and $\mathrm{E}_0 = \mathrm{K}_0 \cap \mathrm{E}$.

Lemma 8.2. — *The representation $\boldsymbol{\rho}$ is distinguished if and only if at least one of the following conditions is fulfilled:*

(1) $\ell = 2$,
(2) $m = 1$ *and $\boldsymbol{\rho}$ is trivial on* $\mathrm{E}_0^\times$,
(3) m *is even and ξ is non-trivial on* $\mathrm{K}_0^\times$.

Remark 8.3. — Note that Case 3 cannot happen when $\ell = 2$.

Proof. — The case $m = 1$ is clear. Let us suppose that $m = 2r$ for some $r \geqslant 1$. The case where the characteristic of R is 0 is given by [**24**] Proposition 6.3. Suppose R has characteristic $\ell > 0$, and fix an embedding $\iota : \overline{\mathbf{F}}_\ell \to \mathrm{R}$. Since $\xi \circ \sigma = \xi^{-1}$, the image of ξ is finite, thus contained in the image of $\overline{\mathbf{F}}_\ell$ in R. Indeed, the restriction of ξ to the units of $\mathrm{K}^\times$ has finite image, and $\xi(t)$ has order at most 4 since $\xi(\sigma(t)) = \xi(t)^{-1}$ and $\sigma(t) \in \{-t, t\}$. There is thus a $\overline{\mathbf{F}}_\ell$-character ξ_0 of $\mathrm{K}^\times$ such that $\xi_0 \circ \sigma = \xi_0^{-1}$ and $\xi = \iota \circ \xi_0$. In particular, $(\mathrm{K}/\mathrm{E}, \xi_0)$ is an admissible pair of level zero. Let $\boldsymbol{\rho}_0$ be the σ-selfdual $\overline{\mathbf{F}}_\ell$-representation attached to it. By Remark 5.9, the representation $\boldsymbol{\rho}$ is isomorphic to $\boldsymbol{\rho}_0 \otimes \mathrm{R}$. It thus suffices to prove the lemma when R is equal to $\overline{\mathbf{F}}_\ell$, which we assume now.

We consider the canonical $\overline{\mathbf{Q}}_\ell$-lift $\widetilde{\xi}$ of ξ, which has the same finite order as ξ. It satisfies the identity $\widetilde{\xi} \circ \sigma = \widetilde{\xi}^{-1}$, and the pair $(\mathrm{K}/\mathrm{E}, \widetilde{\xi})$ is admissible of level zero. Attached to it, there is thus a σ-selfdual $\overline{\mathbf{Q}}_\ell$-representation $\widetilde{\boldsymbol{\rho}}$ of $\mathbf{J}$ trivial on J^1. Note that the kernel of $\widetilde{\boldsymbol{\rho}}$ has finite index, since it contains J^1 and t^4, thus $\widetilde{\boldsymbol{\rho}}$ can be considered as a representation of a finite group. From Proposition 2.1, one checks easily that its reduction mod ℓ is $\boldsymbol{\rho}$. Note that the restriction of ξ to $\mathrm{K}_0^\times$ is either trivial or (if $\ell \neq 2$) equal to $\omega_{\mathrm{K}/\mathrm{K}_0}$.

Suppose ξ is non-trivial on $\mathrm{K}_0^\times$. Then the same holds for $\widetilde{\xi}$, and the characteristic 0 case tells us that $\widetilde{\boldsymbol{\rho}}$ is distinguished. As in the proof of Lemma 2.5, by applying Lemma 2.6, reducing mod ℓ a non-zero invariant form on $\widetilde{\boldsymbol{\rho}}$ gives us a non-zero invariant form on $\boldsymbol{\rho}$, which is thus distinguished.

Suppose now that ξ is trivial on $\mathrm{K}_0^\times$. Then the same holds for $\widetilde{\xi}$. Let $\widetilde{\alpha}$ denote the unramified ℓ-adic character of $\mathrm{K}^\times$ of order 2. Then $(\mathrm{K}/\mathrm{E}, \widetilde{\xi}\widetilde{\alpha})$ is an admissible pair of level zero. It is attached to $\widetilde{\boldsymbol{\rho}}\widetilde{\varphi}$ where $\widetilde{\varphi}$ is the unramified ℓ-adic character of $\mathbf{J}$ of order 2. Since $\widetilde{\xi}\widetilde{\alpha}$ is non-trivial on $\mathrm{K}_0^\times$, the representation $\widetilde{\boldsymbol{\rho}}\widetilde{\varphi}$ is distinguished. Thus $\boldsymbol{\rho}$ is φ-distinguished, where φ is the reduction mod ℓ of $\widetilde{\varphi}$.

If $\ell = 2$, then $\boldsymbol{\rho}$ is distinguished. Suppose now that $\ell \neq 2$. If $\boldsymbol{\rho}$ were both φ-distinguished and distinguished, one would have two linearly independent linear forms in $\mathrm{Hom}_{\mathrm{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\rho}, 1)$, and this would contradict Lemma 2.19. The result follows. □

The field extension E of F is not uniquely determined by π, unlike its maximal tamely ramified extension T. To remedy this, let D be the maximal tamely ramified sub-extension of K/F. Write $\mathrm{D}_0 = \mathrm{D} \cap \mathrm{K}_0$, and let δ_0 be the restriction of ξ to $\mathrm{D}_0^\times$.

Since $\xi \circ \sigma = \xi^{-1}$ the character δ_0 is quadratic, either trivial or (if $\ell \neq 2$) equal to $\omega_{\mathrm{D}/\mathrm{D}_0}$. We will see in Proposition 10.5 that, up to F_0-equivalence, D/D_0 and δ_0 are determined by π.

Theorem 8.4. — *Let π be a σ-selfdual supercuspidal representation of* G. *Suppose that* T/T_0 *is ramified. Let m be its relative degree and δ_0 be the quadratic character of* $\mathrm{D}_0^\times$ *associated to it.*

(1) *The representation π is distinguished if and only if at least one of the following conditions is fulfilled:*

- (a) $\ell = 2$,
- (b) $m = 1$ *and* δ_0 *is trivial,*
- (c) m *is even and* δ_0 *is non-trivial.*

(2) *Suppose that* $\ell \neq 2$. *Then π is ω-distinguished if and only if:*

- (a) *either* $m = 1$ *and* δ_0 *is non-trivial,*
- (b) *or* m *is even and* δ_0 *is trivial.*

Remark 8.5. — If R has characteristic 2, then π is always distinguished. If R has characteristic not 2, then π is either distinguished or ω-distinguished, but not both.

Proof. — By Proposition 7.9 and Lemma 8.2, it suffices to compare the restriction of ξ to $\mathrm{K}_0^\times$ with δ_0 when $\ell \neq 2$.

Suppose first that $m = 1$ and δ_0 is trivial. Since the restriction of $\boldsymbol{\rho}$ to $\mathrm{E}_0^\times$ is equal to either 1 or $\omega_{\mathrm{E}/\mathrm{E}_0}$, its restriction to $\mathrm{T}_0^\times$ is either 1 or $\omega_{\mathrm{T}/\mathrm{T}_0}$, respectively. Since δ_0 is trivial, we are in the first case, that is, the restriction of $\boldsymbol{\rho}$ to $\mathrm{E}_0^\times$ is trivial.

Suppose now that $m \neq 1$ and ξ is non-trivial on $\mathrm{K}_0^\times$. We want to prove that δ_0 is non-trivial. The restriction of ξ to $\mathrm{K}_0^\times$ is equal to $\omega_{\mathrm{K}/\mathrm{K}_0}$. Thus δ_0 is equal to $\omega_{\mathrm{D}/\mathrm{D}_0}$.

Now suppose that R has characteristic different from 2, and let χ be an unramified character of $\mathrm{F}^\times$ extending ω. Note that the twisted representation $\pi' = \pi(\chi^{-1} \circ \det)$ is supercuspidal and σ-selfdual, and that the character associated with π' is $\delta_0' = \delta_0(\chi^{-1} \circ \mathrm{N}_{\mathrm{K}/\mathrm{F}})|_{\mathrm{D}_0^\times}$, where $\mathrm{N}_{\mathrm{K}/\mathrm{F}}$ is the norm map from K to F. Suppose first that $m = 1$. Then:

$$\begin{aligned} \pi \text{ is } \omega\text{-distinguished} &\Leftrightarrow \pi' \text{ is distinguished} \\ &\Leftrightarrow \text{the character } \delta_0' \text{ is trivial} \\ &\Leftrightarrow \text{the character } \delta_0 \text{ coincides with } \chi \circ \mathrm{N}_{\mathrm{E}/\mathrm{F}} \text{ on } \mathrm{T}_0^\times. \end{aligned}$$

Suppose now that $m \neq 1$. Then:

$$\begin{aligned} \pi \text{ is } \omega\text{-distinguished} &\Leftrightarrow \pi' \text{ is distinguished} \\ &\Leftrightarrow \text{the character } \delta_0' \text{ is non-trivial} \\ &\Leftrightarrow \text{the character } \delta_0 \text{ coincides with } (\chi \circ \mathrm{N}_{\mathrm{K}/\mathrm{F}})\omega_{\mathrm{D}/\mathrm{D}_0} \text{ on } \mathrm{D}_0^\times. \end{aligned}$$

The restriction of $\chi \circ \mathrm{N}_{\mathrm{K}/\mathrm{F}}$ to $\mathrm{D}_0^\times$ is $\omega \circ \mathrm{N}_{\mathrm{D}_0/\mathrm{F}_0} = \omega_{\mathrm{D}/\mathrm{D}_0}$ to the power of $[\mathrm{K}_0 : \mathrm{D}_0]$, which is a p-power with p odd. This gives us the expected result. □

Remark 8.6. — If π is as in Theorem 8.4 and $m > 1$, its central character ω_π is always trivial on $\mathrm{F}_0^\times$. Indeed, since π and $\boldsymbol{\lambda}$ have the same central character, we can express ω_π as the product $\omega_{\boldsymbol{\kappa}}\omega_{\boldsymbol{\rho}}$, where $\omega_{\boldsymbol{\kappa}}$ and $\omega_{\boldsymbol{\rho}}$ are the central characters of $\boldsymbol{\kappa}$ and $\boldsymbol{\rho}$ on $\mathrm{F}^\times$, respectively. Since $\boldsymbol{\kappa}$ is distinguished, its central character is trivial on $\mathrm{F}_0^\times$, thus ω_π and δ_0 coincide on $\mathrm{F}_0^\times$. If δ_0 is trivial then ω_π is trivial on $\mathrm{F}_0^\times$. Now assume that δ_0 is equal to $\omega_{\mathrm{D}/\mathrm{D}_0}$. Since D/D_0 is unramified by Lemma 5.20, its restriction to $\mathrm{F}_0^\times$ is trivial if and only if $e(\mathrm{D}_0/\mathrm{F}_0)$ is even, which is the case since $e(\mathrm{D}_0/\mathrm{T}_0) = 2$ when m is even.

Remark 8.7. — On the other hand, since T/T_0 is ramified, the restriction of $\omega_{\mathrm{T}/\mathrm{T}_0}$ to $\mathrm{F}_0^\times$ is trivial if and only if $\boldsymbol{k}_0^\times$ is contained in the subgroup $\boldsymbol{l}^{\times 2}$ of squares in $\boldsymbol{l}^\times$, that is, if and only if $f(\mathrm{T}_0/\mathrm{F}_0)$ is even. It follows that, if $m = 1$ and $f(\mathrm{T}_0/\mathrm{F}_0)$ is odd (which is equivalent to n being odd by Lemma 5.15 and since m is either 1 or even) then π is distinguished if and only if ω_π is trivial on $\mathrm{F}_0^\times$.

9. The supercuspidal unramified case

Let π be a σ-selfdual cuspidal representation of $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ for $n \geqslant 1$. In this section, we investigate the case where the quadratic extension T/T_0 is unramified. By Corollary 6.24, the representation π is distinguished if and only if any of the σ-selfdual types contained in π is distinguished.

9.1. Existence of σ-selfdual extensions of the Heisenberg representation

Let $[\mathfrak{a}, \beta]$ be a maximal simple stratum as in Remark 5.11. Write $\mathrm{E} = \mathrm{F}[\beta]$ and suppose that it is unramified over $\mathrm{E}_0 = \mathrm{E}^\sigma$. Let us write $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$, $\mathrm{J} = \mathrm{J}(\mathfrak{a}, \beta)$ and $\mathrm{J}^1 = \mathrm{J}^1(\mathfrak{a}, \beta)$.

We may and will identify J/J^1 with the group $\mathrm{GL}_m(\boldsymbol{l})$, denoted $\mathcal{G}$, equipped with the residual involution $\sigma \in \mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$, where $\boldsymbol{l}$ and $\boldsymbol{l}_0$ are the residue fields of E and E_0, respectively.

Lemma 9.1. — *The group* $\mathbf{J} \cap \mathrm{G}^\sigma$ *is generated by* t *and* $\mathrm{J} \cap \mathrm{G}^\sigma$.

Proof. — Any $x \in \mathbf{J}$ can be written $x = t^m y$ for unique $m \in \mathbb{Z}$ and $y \in \mathrm{J}$. Since t is σ-invariant, we have $x \in \mathbf{J} \cap \mathrm{G}^\sigma$ if and only if $y \in \mathrm{J} \cap \mathrm{G}^\sigma$. □

Lemma 9.2. — *Any character of* $\mathbf{J} \cap \mathrm{G}^\sigma$ *trivial on* $\mathrm{J}^1 \cap \mathrm{G}^\sigma$ *extends to a character of* $\mathbf{J}$ *trivial on* J^1.

Proof. — Let χ be a character of $\mathbf{J} \cap \mathrm{G}^\sigma$ trivial on $\mathrm{J}^1 \cap \mathrm{G}^\sigma$. Since J^1 is a pro-p-group, the first cohomology group of σ in J^1 is trivial. The subgroup $\mathcal{G}^\sigma$ thus identifies with $(\mathrm{J} \cap \mathrm{G}^\sigma)/(\mathrm{J}^1 \cap \mathrm{G}^\sigma)$. The restriction of χ to $\mathrm{J} \cap \mathrm{G}^\sigma$ thus induces a character of $\mathcal{G}^\sigma$, which can be written $\alpha_0 \circ \det$ for some character α_0 of $\boldsymbol{l}_0^\times$. Let α be a character of $\boldsymbol{l}^\times$ extending α_0, and let ϕ be the character of J trivial on J^1 inducing the character $\alpha \circ \det$ of $\mathcal{G}$. Since $\mathrm{J} = \mathfrak{b}^\times \mathrm{J}^1$, the element t acts trivially on J/J^1 by conjugacy, thus normalizes ϕ. We thus may extend ϕ to $\mathbf{J}$ by setting $\phi(t) = \chi(t)$. Lemma 9.1 implies that ϕ extends χ and it is trivial on J^1. □

Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a maximal simple character such that $\mathrm{H}^1(\mathfrak{a}, \beta)$ is σ-stable and $\theta \circ \sigma = \theta^{-1}$. Let η denote the Heisenberg representation of θ on the group J^1.

Lemma 9.3. — *There is a* σ*-selfdual representation* $\boldsymbol{\kappa}$ *of* $\mathbf{J}$ *extending* η.

Proof. — Let $\boldsymbol{\kappa}$ be an irreducible representation of $\mathbf{J}$ extending η. By Lemma 5.21, there is a character μ of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}\mu$ and $\mu \circ \sigma = \mu$. We claim that there is a character ν of $\mathbf{J}$ trivial on J^1 such that $(\nu \circ \sigma)\nu = \mu$. Indeed, if this is the case, the representation $\boldsymbol{\kappa}\nu$ extends η and is σ-selfdual.

Consider first μ as a character of $\mathcal{G}$ and write $\mu = \varphi \circ \det$ for some character φ of $\boldsymbol{l}^\times$. Then we have $\varphi \circ \sigma = \varphi$, thus there is a character α of $\boldsymbol{l}^\times$ such that $(\alpha \circ \sigma)\alpha = \varphi$. Choosing such a α, there exists a unique character ν of J inducing $\alpha \circ \det$ on $\mathcal{G}$. Since $\mathbf{J}$ is generated by t and J, it remains to extend ν to $\mathbf{J}$ by choosing a scalar $\nu(t) \in \mathrm{R}^\times$ such that $\nu(t)^2 = \mu(t)$. □

9.2. Existence of distinguished extensions of the Heisenberg representation

Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type, with associated simple character the character θ of §9.1.

In this paragraph, we suppose that m is odd.

Proposition 9.4. — *Suppose that* m *is odd. There is a* σ*-selfdual distinguished representation* $\boldsymbol{\kappa}$ *of* $\mathbf{J}$ *extending* η.

Proof. — We first assume that R has characteristic 0. By Lemma 9.3, there is a σ-selfdual representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η. Let χ denote the character of $\mathbf{J} \cap \mathrm{G}^\sigma$ trivial on $\mathrm{J}^1 \cap \mathrm{G}^\sigma$ associated to $\boldsymbol{\kappa}$ by Lemma 6.20. Since m is odd, Lemma 2.3 implies that $\mathcal{G}$ possesses a σ-selfdual supercuspidal representation ρ. Let $\boldsymbol{\rho}$ be the unique representation of $\mathbf{J}$ trivial on J^1 such that $t \in \mathrm{Ker}(\boldsymbol{\rho})$ and whose restriction to J is the inflation of ρ. This representation $\boldsymbol{\rho}$ is σ-selfdual. By Lemma 2.5, it is also distinguished. Now let $\boldsymbol{\lambda}$ be the σ-selfdual type $\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$ on $\mathbf{J}$. The natural isomorphism:

$$\mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\kappa},\chi)\otimes\mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\rho},1)\to\mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda},\chi)$$

thus shows that $\boldsymbol{\lambda}$ is χ-distinguished.

By Lemma 9.2, there exists a character ϕ of $\mathbf{J}$ trivial on J^1 extending χ. The representation $\boldsymbol{\lambda}'=\boldsymbol{\lambda}\phi^{-1}$ is thus a distinguished type. Let π' be the cuspidal representation of G compactly induced from $(\mathbf{J},\boldsymbol{\lambda}')$. It is distinguished, thus σ-selfdual by Theorem 4.1. Since $\boldsymbol{\lambda}'$ and $\boldsymbol{\lambda}'^{\sigma\vee}\simeq\boldsymbol{\lambda}'\phi(\phi\circ\sigma)$ are both contained in π', it follows that $\phi(\phi\circ\sigma)$ is trivial. This implies that $\boldsymbol{\kappa}'=\boldsymbol{\kappa}\phi^{-1}$ is both σ-selfdual and distinguished.

Now assume that R has characteristic $\ell>0$. We then reduce to the characteristic 0 case as in the proof of Lemma 7.7. □

Remark 9.5. — I don't know whether Proposition 9.4 holds when m is even.

Corollary 9.6. — (1) *Any distinguished representation of* $\mathbf{J}$ *extending* η *is* σ*-selfdual.*

(2) *If* $\ell=2$*, any* σ*-selfdual representation of* $\mathbf{J}$ *extending* η *is distinguished.*

Proof. — Let us fix a distinguished σ-selfdual representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η given by Proposition 9.4. Let $\boldsymbol{\kappa}'$ be a distinguished representation of $\mathbf{J}$ extending η. Then $\boldsymbol{\kappa}'=\boldsymbol{\kappa}\phi$ for some character ϕ of $\mathbf{J}$ trivial on $(\mathbf{J}\cap\mathrm{G}^\sigma)\mathrm{J}^1$. Thus $\phi(t)=1$ and ϕ induces the character $\alpha\circ\det$ on $\mathcal{G}$, where α is a character of $\boldsymbol{l}^\times$ trivial on $\boldsymbol{l}_0^\times$, or equivalently $\alpha^{q_0+1}=1$. Thus we have $\phi(\phi\circ\sigma)=1$. This implies that $\boldsymbol{\kappa}'$ is σ-selfdual, which proves the first assertion.

Now suppose that $\boldsymbol{\kappa}'$ is a σ-selfdual representation of $\mathbf{J}$ extending η. Then $\boldsymbol{\kappa}'=\boldsymbol{\kappa}\xi$ for some character ξ of $\mathbf{J}$ such that $\xi(\xi\circ\sigma)$ is trivial. Thus $\xi(t)\in\{-1,1\}$ and there is a character ν of $\boldsymbol{l}^\times$ such that ξ induces $\nu\circ\det$ on $\mathcal{G}$ and $\nu^{q_0+1}=1$. It follows that ξ is trivial on $(\mathrm{J}\cap\mathrm{G}^\sigma)\mathrm{J}^1$. Thus, if $\ell=2$, the representation $\boldsymbol{\kappa}'$ is distinguished. □

Remark 9.7. — Let $\boldsymbol{\kappa}$ be a σ-selfdual representation of $\mathbf{J}$ extending η. Then the character χ of $\mathbf{J}\cap\mathrm{G}^\sigma$ associated to $\boldsymbol{\kappa}$ by Lemma 6.20 is quadratic and unramified.

9.3. Distinction criterion in the unramified case

Let π be a σ-selfdual supercuspidal representation of G. Associated to it by Proposition 5.14, there is a quadratic extension T/T_0. We assume that T is unramified over T_0.

Recall that, by Theorem 6.1 and Proposition 5.17, the representation π is distinguished if and only any of its σ-selfdual types is distinguished. The following result is the analogue of Proposition 7.9.

Proposition 9.8. — *Let* π *be a* σ*-selfdual supercuspidal representation of* G*, with unramified quadratic extension* T/T_0 *and relative degree* m*. Let* $(\mathbf{J},\boldsymbol{\lambda})$ *be a* σ*-selfdual type in* π*. Let* J^1 *be the maximal normal pro-p-subgroup of* $\mathbf{J}$ *and* η *be an irreducible component of the restriction of* $\boldsymbol{\lambda}$ *to* J^1*.*

(1) *The integer* m *is odd.*

(2) *There is a distinguished representation of* $\mathbf{J}$ *extending* η*, and any such extension of* η *is* σ*-selfdual.*

(3) *Let* $\boldsymbol{\kappa}$ *be a distinguished representation of* $\mathbf{J}$ *extending* η*, and let* $\boldsymbol{\rho}$ *be the unique representation of* $\mathbf{J}$ *trivial on* J^1 *such that* $\boldsymbol{\lambda} \simeq \boldsymbol{\kappa} \otimes \boldsymbol{\rho}$*. Then* π *is distinguished if and only if* $\boldsymbol{\rho}$ *is distinguished.*

Proof. — We may and will assume that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ for some maximal simple stratum $[\mathfrak{a}, \beta]$ as in Remark 5.11. Following Remark 5.16, the extension E is unramified over E_0. We thus may apply the results of Paragraph 9.1.

Let $\boldsymbol{\kappa}$ be a σ-selfdual representation of $\mathbf{J}$ extending η, the existence of which is given by Lemma 9.3, and let $\boldsymbol{\rho}$ be the irreducible representation of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\lambda}$ is isomorphic to $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$. Since $\boldsymbol{\lambda}$ and $\boldsymbol{\kappa}$ are σ-selfdual, $\boldsymbol{\rho}$ is σ-selfdual. Its restriction to J induces a cuspidal irreducible representation of $\mathrm{GL}_m(\boldsymbol{l})$, denoted ρ. Since π is supercuspidal, ρ is also supercuspidal by Fact 5.5. Lemma 2.3 implies that m is odd. We thus apply Proposition 9.4, which gives us a σ-selfdual distinguished representation extending η.

Part (2) of the proposition is given by Corollary 9.6. For (3), it suffices to fix a distinguished representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η and to consider the canonical isomorphism:

$$\mathrm{Hom}_{\mathbf{J} \cap \mathrm{G}^\sigma}(\boldsymbol{\kappa}, 1) \otimes \mathrm{Hom}_{\mathbf{J} \cap \mathrm{G}^\sigma}(\boldsymbol{\rho}, 1) \to \mathrm{Hom}_{\mathbf{J} \cap \mathrm{G}^\sigma}(\boldsymbol{\lambda}, 1)$$

as in the ramified case. □

Remark 9.9. — If one relaxes the supercuspidality assumption on π (that is, we only assume π to be σ-selfdual cuspidal with T/T_0 unramified), then its relative degree m need not be odd, in which case our proof of Proposition 9.8(2) doesn't apply (see Remarks 2.4 and 9.5). Unlike the ramified case, I thus don't know whether there is a distinguished and σ-selfdual extension $\boldsymbol{\kappa}$ of η when π is not supercuspidal.

Remark 9.10. — In both ramified and unramified cases, the distinguished representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η is not unique in general, so neither is $\boldsymbol{\rho}$. If $\boldsymbol{\kappa}$ is a distinguished representation of $\mathbf{J}$ extending η, the other ones are exactly the $\boldsymbol{\kappa}\phi$ where ϕ ranges over the set of characters of $\mathbf{J}$ trivial on $(\mathbf{J} \cap \mathrm{G}^\sigma)\mathrm{J}^1$.

From now, we will thus assume that $\boldsymbol{\kappa}$ is a distinguished σ-selfdual representation of $\mathbf{J}$ extending η. Proposition 9.8 reduces the problem of the distinction of π to that of $\boldsymbol{\rho}$. We now investigate the distinction of $\boldsymbol{\rho}$.

Let ρ be the representation of $\mathrm{GL}_m(\boldsymbol{l})$ defined by restricting $\boldsymbol{\rho}$ to J. It is σ-selfdual. By Fact 5.5, it is also supercuspidal.

Let $(\mathrm{K}/\mathrm{E}, \xi)$ be an admissible pair of level 0 attached to $\boldsymbol{\rho}$ in the sense of Definition 5.7. Since $\boldsymbol{\rho}$ is σ-selfdual, Proposition 5.19 tells us that there is a unique involutive E_0-automorphism of K, which we denote by σ, which is non-trivial on E and satisfies $\xi \circ \sigma = \xi^{-1}$. Let K_0 be the σ-fixed points of K and $\mathrm{E}_0 = \mathrm{K}_0 \cap \mathrm{E}$.

Lemma 9.11. — *The representation* $\boldsymbol{\rho}$ *is distinguished if and only if it is trivial on* $\mathrm{E}_0^\times$*.*

Proof. — Note that $\boldsymbol{\rho}(x) = \xi(x) \cdot \mathrm{id}$ for all $x \in \mathrm{E}^\times$, thus $\boldsymbol{\rho}$ is trivial on $\mathrm{E}_0^\times$ if and only if ξ is. The representation ρ is σ-selfdual, thus distinguished (see Lemma 2.5). We thus have:

$$\mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\rho}, 1) \subseteq \mathrm{Hom}_{\mathrm{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\rho}, 1) \simeq \mathrm{Hom}_{\mathrm{GL}_m(\boldsymbol{l}_0)}(\rho, 1)$$

where the space on the right hand side is non-zero (and has dimension 1). Since $\mathbf{J} \cap \mathrm{G}^\sigma$ is generated by $\mathrm{J} \cap \mathrm{G}^\sigma$ and t, we deduce that $\boldsymbol{\rho}$ is distinguished if and only if t acts trivially on the space $\mathrm{Hom}_{\mathrm{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\rho}, 1)$, that is, if and only if $\xi(t)$ is trivial. The result follows from the fact that, since ξ is σ-selfdual, it is trivial on the E/E_0-norms in $\mathrm{E}_0^\times$, thus on the units of E_0. □

Let ε_0 denote the restriction of the character ξ to $\mathrm{T}_0^\times$.

Lemma 9.12. — *The character ε_0 is quadratic and unramified.*

Proof. — As has been said in the proof of Lemma 9.11, the character ξ is trivial on the subgroup of E/E_0-norms in $\mathrm{E}_0^\times$, since $\boldsymbol{\rho}$ is σ-selfdual. Thus the restriction of ξ to $\mathrm{E}_0^\times$ is either trivial or (if $\ell \neq 2$) equal to $\omega_{\mathrm{E}/\mathrm{E}_0}$. We get the expected result by restricting to $\mathrm{T}_0^\times$, since E is unramified over E_0 and $e(\mathrm{E}_0/\mathrm{T}_0)$ is a p-power with p odd. □

We will see below (Remark 10.7) that the character ε_0 is uniquely determined by π.

Theorem 9.13. — *Let π be a σ-selfdual supercuspidal representation of* G. *Suppose that* T *is unramified over* T_0.

(1) *The representation π is distinguished if and only if ε_0 is trivial.*

(2) *Suppose that the characteristic of* R *is not* 2. *Then π is ω-distinguished if and only if ε_0 is non-trivial.*

Remark 9.14. — If R has characteristic 2, then π is always distinguished. If R has characteristic not 2, then π is either distinguished or ω-distinguished, but not both.

Proof. — By Proposition 9.8, the representation π is distinguished if and only if $\boldsymbol{\rho}$ is distinguished. Lemma 9.11 tells us that it is distinguished if and only if $\xi(t) = 1$. The restriction of ξ to $\mathrm{E}_0^\times$ is a quadratic unramified character. Since the ramification index of E_0 over T_0 is odd (for it is a p-power), ξ is trivial on $\mathrm{E}_0^\times$ if and only if it is trivial on $\mathrm{T}_0^\times$. The first assertion is proven.

Now suppose that R has characteristic different from 2, and let χ be an unramified character of $\mathrm{F}^\times$ extending ω. Note that the twisted representation $\pi' = \pi(\chi^{-1} \circ \det)$ is supercuspidal and σ-selfdual, and that the character associated with π' is $\varepsilon_0' = \varepsilon_0(\chi^{-m} \circ \mathrm{N}_{\mathrm{E}/\mathrm{F}})|_{\mathrm{T}_0^\times}$ where $\mathrm{N}_{\mathrm{E}/\mathrm{F}}$ is the norm map from E to F. Thus:

$$\begin{aligned} \pi \text{ is } \omega\text{-distinguished} &\Leftrightarrow \pi' \text{ is distinguished} \\ &\Leftrightarrow \text{the character } \varepsilon_0' \text{ is trivial on } \mathrm{T}_0^\times \\ &\Leftrightarrow \text{the character } \varepsilon_0 \text{ coincides with } \chi^m \circ \mathrm{N}_{\mathrm{E}/\mathrm{F}} \text{ on } \mathrm{T}_0^\times. \end{aligned}$$

The restriction of $\chi \circ \mathrm{N}_{\mathrm{E}/\mathrm{F}}$ to $\mathrm{T}_0^\times$ is equal to $\omega \circ \mathrm{N}_{\mathrm{T}_0/\mathrm{F}_0} = \omega_{\mathrm{T}/\mathrm{T}_0}$ to the power of $[\mathrm{E}_0 : \mathrm{T}_0]$, which is a p-power. The second assertion then follows from the fact that p and m are odd. □

Corollary 9.15. — *Let π be a supercuspidal representation of* G*. Suppose that* T/T_0 *is unramified, and that the ramification index of* T/F *is odd. Then π is distinguished if and only if it is σ-selfdual and its central character is trivial on* $F_0^\times$*.*

Proof. — Suppose that π is σ-selfdual and that its central character ω_π is trivial on $F_0^\times$. By using a σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$ contained in π as above, we can express ω_π as the product $\omega_{\boldsymbol{\kappa}}\omega_{\boldsymbol{\rho}}$, where $\omega_{\boldsymbol{\kappa}}$ and $\omega_{\boldsymbol{\rho}}$ are the central characters of $\boldsymbol{\kappa}$ and $\boldsymbol{\rho}$ on $F^\times$, respectively. Since $\boldsymbol{\kappa}$ is distinguished, its central character is trivial on $F_0^\times$, thus ω_π and ε_0 coincide on $F_0^\times$. It remains to prove that ε_0 is trivial if and only if it is trivial on $F_0^\times$.

Suppose ε_0 is trivial on $F_0^\times$. By Lemma 9.12, it is unramified, thus $\varepsilon_0^{e(T_0/F_0)}$ is trivial on $T_0^\times$. Since $e(T/F)$ is odd, $e(T_0/F_0)$ is odd too, and the expected result follows from the fact that ε_0 is quadratic. □

Remark 9.16. — In particular, when n is odd and F is unramified over F_0, a supercuspidal representation of G is distinguished if and only if it is σ-selfdual and its central character is trivial on $F_0^\times$. This has been proved by Prasad [**42**] when R has characteristic 0. Note that, since m and p are odd here, n is odd if and only if $[T : F]$ is odd.

Remark 9.17. — Note that, in the proof of Prasad [**42**] Theorem 4, the isomorphism of π with $\pi^{\sigma\vee}$ gives an element $g \in G$ which has the property that $g\sigma(g) \in J = J(\mathfrak{a}, \beta)$, but g has *a priori* no reason to normalize J. Anyway, g can be chosen in the maximal compact open subgroup $\mathfrak{a}^\times$ which contains J (which derives from [**10**] Theorem 3.5.11), thus the group generated by g and J will indeed be compact mod centre.

10. Statement of the final results

In this section we put together the main results of Sections 7 to 9. Let π be a σ-selfdual supercuspidal representation of G. Associated to it, there are its relative degree m and the quadratic extension T/T_0. It is convenient to introduce the following definition, which comes from [**3**].

Definition 10.1. — A σ-selfdual type in π is said to be *generic* if either T/T_0 is unramified, or T/T_0 is ramified and this type has index $\lfloor m/2 \rfloor$.

Remark 10.2. — It is proved in [**3**] Proposition 5.5 that a σ-selfdual type is generic in the sense of Definition 10.1 if and only if there are a σ-stable maximal unipotent subgroup N of G and a σ-selfdual non-degenerate character ψ_N of N such that $\mathrm{Hom}_{\mathbf{J}\cap N}(\boldsymbol{\lambda}, \psi_N)$ is non-zero.

Definition 10.1 is convenient to us because of the following result, which subsumes Propositions 7.1 and 8.1 (compare with Theorem 6.1).

Theorem 10.3. — *A σ-selfdual cuspidal representation of* G *is distinguished if and only if any of its generic σ-selfdual types is distinguished.*

Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a *generic* σ-selfdual type contained in π. Let $[\mathfrak{a}, \beta]$ be a σ-selfdual simple stratum such that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$. The restriction of $\boldsymbol{\lambda}$ to the maximal normal pro-p-subgroup J^1 is made of copies of a single irreducible representation η. We fix a distinguished σ-selfdual representation $\boldsymbol{\kappa}$

of $\mathbf{J}$ extending η, the existence of which is given by Propositions 7.9 and 9.8. Let $\boldsymbol{\rho}$ be the representation of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\lambda}$ is isomorphic to $\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$. Let $(\mathrm{K}/\mathrm{E},\xi)$ be a admissible pair of level 0 attached to $\boldsymbol{\rho}$ and σ be the involution of K given by Proposition 5.19. Let K_0 be the field of σ-fixed points of K. We thus have $\mathrm{K}\simeq\mathrm{K}_0\otimes_{\mathrm{F}_0}\mathrm{F}$.

Definition 10.4. — Let D be the maximal tamely ramified sub-extension of K/F. Write $\mathrm{D}_0=\mathrm{D}\cap\mathrm{K}_0$, and let δ_0 be the restriction of ξ to $\mathrm{D}_0^\times$.

It follows immediately from the definition that $\mathrm{D}_0/\mathrm{F}_0$ is tamely ramified and the character δ_0 is quadratic, either trivial or (if $\ell\neq2$) equal to $\omega_{\mathrm{D}/\mathrm{D}_0}$.

Proposition 10.5. — *The quadratic extension* D/D_0 *and the character* δ_0 *are uniquely determined by* π *up to* F_0*-equivalence. That is, if* $\mathrm{D}'/\mathrm{D}'_0$ *and* δ'_0 *are another quadratic extension and character associated to* π*, then there is an* F_0*-isomorphism* $\varphi:\mathrm{D}\to\mathrm{D}'$ *such that* $\varphi(\mathrm{D}_0)=\mathrm{D}'_0$ *and* $\delta_0=\delta'_0\circ\varphi$.

Proof. — Start with a generic σ-selfdual type contained in π. Since it is unique up to G^σ-conjugacy, we may assume this is $(\mathbf{J},\boldsymbol{\lambda})$. Fix a σ-selfdual stratum $[\mathfrak{a}',\beta']$ such that $\mathbf{J}=\mathbf{J}(\mathfrak{a}',\beta')$. By [**3**] Lemma 4.29, we may assume that the maximal tamely ramified sub-extension of $\mathrm{E}'=\mathrm{F}[\beta']$ over F is equal to T. Fix a distinguished σ-selfdual representation $\boldsymbol{\kappa}'$ of $\mathbf{J}$ extending η, let $\boldsymbol{\rho}'$ be the representation of $\mathbf{J}$ trivial on J^1 corresponding to this choice and $(\mathrm{K}'/\mathrm{E}',\xi')$ be an admissible pair of level 0 attached to $\boldsymbol{\rho}'$. This gives us a quadratic extension $\mathrm{D}'/\mathrm{D}'_0$ and a character δ'_0 of $\mathrm{D}'^\times_0$.

First, suppose that $[\mathfrak{a}',\beta']=[\mathfrak{a},\beta]$ and $\mathrm{K}'=\mathrm{K}$. We have $\boldsymbol{\kappa}'=\boldsymbol{\kappa}\phi$ for some character ϕ of $\mathbf{J}$ trivial on $(\mathbf{J}\cap\mathrm{G}^\sigma)\mathrm{J}^1$, thus $\boldsymbol{\rho}'$ is isomorphic to $\boldsymbol{\rho}\phi^{-1}$. Thus ξ' is E-isomorphic to $\xi\alpha^{-1}$ for some tamely ramified character α of $\mathrm{K}^\times$ trivial on $\mathrm{K}_0^\times$. Restricting to D_0, we get $\delta'_0=\delta_0$.

We now go back to the general case. By the previous argument, we may assume that $\boldsymbol{\kappa}'=\boldsymbol{\kappa}$, thus $\boldsymbol{\rho}'=\boldsymbol{\rho}$. Since D and D′ are both unramified of same degree m over T, they are T-isomorphic. Let us fix a T-isomorphism $f:\mathrm{D}\to\mathrm{D}'$. Write σ' for the involutive automorphism of K' given by Proposition 5.19.

Lemma 10.6. — *We have* $\sigma'\circ f=f\circ\sigma$.

Proof. — Let us identify the residual fields of K and D, denoted $\boldsymbol{t}$, and those of E and T, denoted $\boldsymbol{l}$. Note that, if φ is any T-automorphism of D, then it commutes with σ since φ and $\sigma\circ\varphi\circ\sigma^{-1}$ are both in Gal(D/T) and have the same image in $\mathrm{Gal}(\boldsymbol{t}/\boldsymbol{l})$.

We now consider the pair $(\mathrm{D}/\mathrm{T},\xi|_{\mathrm{D}^\times})$. Since D/T is unramified of degree m, it is admissible of level 0. Moreover, the $\boldsymbol{l}$-regular character of $\boldsymbol{t}^\times$ it induces is $\mathrm{Gal}(\boldsymbol{t}/\boldsymbol{l})$-conjugate to the one induced by $(\mathrm{K}/\mathrm{E},\xi)$, which doesn't depend on the identifications of residual fields we have made. We have a similar result for $(\mathrm{D}/\mathrm{T},\xi'\circ f)$ and $(\mathrm{K}'/\mathrm{E}',\xi')$. Since $\boldsymbol{\rho}'=\boldsymbol{\rho}$ we deduce that $\xi'\circ f=\xi\circ\varphi$ for some $\varphi\in\mathrm{Gal}(\mathrm{D}/\mathrm{T})$. Let α be the T-automorphism $\sigma'\circ f\circ\sigma^{-1}\circ f^{-1}$ of D′. We have:

$$\xi'\circ\alpha=\xi'^{-1}\circ f\circ\sigma^{-1}\circ f^{-1}=\xi^{-1}\circ\varphi\circ\sigma^{-1}\circ f^{-1}=\xi\circ\varphi\circ f^{-1}=\xi'.$$

It follows from admissibility of ξ' that α is trivial, as expected. □

Lemma 10.6 implies that D_0 and D'_0 are T_0-isomorphic. We thus now may assume that $D = D'$ and $D_0 = D'_0$, thus K, K′ have the same maximal unramified sub-extension D over T and there is an automorphism $\varphi \in \mathrm{Gal}(D/T)$ such that $\xi'(x) = \xi \circ \varphi(x)$ for all $x \in D^\times$. Restricting to $D_0^\times$, we deduce that $\delta'_0 = \delta_0$. □

Remark 10.7. — In particular, the character ε_0 of Paragraph 9.3, which is the restriction of δ_0 to $T_0^\times$, is uniquely determined by π.

We state the dichotomy and disjunction theorem.

Theorem 10.8. — *Let π be a σ-selfdual supercuspidal representation of* G*. Let ℓ be the characteristic of* R*.*

(1) *If $\ell \neq 2$, then π is either distinguished or ω-distinguished, but not both.*
(2) *If $\ell = 2$, then π is always distinguished.*

Proof. — See Remarks 8.5 and 9.14. □

We now state the distinction criterion theorem.

Theorem 10.9. — *Let π be a σ-selfdual supercuspidal representation of* G*. Attached to it, there are the quadratic extensions T/T_0 and D/D_0 and the character δ_0.*

(1) *Suppose that n is odd. Then π is distinguished if and only if its central character is trivial on $F_0^\times$.*
(2) *If $\ell \neq 2$, T/T_0 is ramified and D/D_0 is unramified, then π is distinguished if and only if the character δ_0 is non-trivial.*
(3) *Otherwise, π is distinguished if and only if δ_0 is trivial.*

Proof. — Item 1 is an immediate consequence of Theorem 10.8 as explained in §1.4. If $\ell \neq 2$, a σ-selfdual supercuspidal representation π is either distinguished or ω-distinguished. In the latter case, the restriction of its central character to $F_0^\times$ is ω^n, which is trivial if and only if n is even. See also Remarks 8.7 and 9.16.

For the remaining items, see Theorems 8.4 and 9.13: it suffices to check that, if T/T_0 is unramified, then δ_0 is trivial if and only if its restriction ε_0 to $T_0^\times$ is trivial, which follows from the fact that m is odd in that case. □

Remark 10.10. — The following conditions are equivalent:

(1) D/D_0 is ramified;
(2) T/T_0 is ramified and $m = 1$.

Indeed this follows from Remark 5.16, Lemma 5.20 and Proposition 8.1. The following conditions are thus also equivalent:

(1) T/T_0 is ramified and D/D_0 is unramified;
(2) F/F_0 is ramified, T_0/F_0 has odd ramification order and D/D_0 is unramified;
(3) F/F_0 is ramified, T_0/F_0 has odd ramification order and $m \neq 1$.

We now state the distinguished lift theorem. For the notion of the reduction mod ℓ of an integral irreducible $\overline{\mathbf{Q}}_\ell$-representation of G, we refer to [**47, 50**]. If π is an irreducible $\overline{\mathbf{F}}_\ell$-representation of G, we say an integral irreducible $\overline{\mathbf{Q}}_\ell$-representation of G is a *lift* of π if its reduction mod ℓ is irreducible and isomorphic to π.

Theorem 10.11. — *Let π be a σ-selfdual supercuspidal $\overline{\mathbf{F}}_\ell$-representation of* G.

(1) *The representation π admits a σ-selfdual supercuspidal lift to $\overline{\mathbf{Q}}_\ell$.*

(2) *Let $\widetilde{\pi}$ be a σ-selfdual lift of π, and suppose that $\ell \neq 2$. Then $\widetilde{\pi}$ is distinguished if and only if π is distinguished.*

Proof. — Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type in π. Let η be the Heisenberg representation contained in the restriction of $\boldsymbol{\lambda}$ to J^1, with associated simple character θ. As in the proof of Lemma 7.7, let $\widetilde{\theta}$ be the lift of θ with values in $\overline{\mathbf{Q}}_\ell$ and $\widetilde{\eta}$ be the associated Heisenberg representation, whose reduction mod ℓ is isomorphic to η. Propositions 7.9 and 9.4 tell us that there is a distinguished σ-selfdual representation $\widetilde{\boldsymbol{\kappa}}$ of $\mathbf{J}$ which extends $\widetilde{\eta}$. Its reduction mod ℓ, denoted $\boldsymbol{\kappa}$, is a σ-selfdual representation of $\mathbf{J}$ extending η, and it is distinguished thanks to Lemma 2.6. (Note that, as in the proof of Lemma 8.2, the fact that $\boldsymbol{\kappa}$ is σ-selfdual implies that it has finite image; it thus can be considered as a representation of a finite group.) Let $\boldsymbol{\rho}$ be the irreducible representation of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\lambda}$ is isomorphic to $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$. It is σ-selfdual.

The representation $\boldsymbol{\rho}$ admits a σ-selfdual $\overline{\mathbf{Q}}_\ell$-lift $\widetilde{\boldsymbol{\rho}}$ on $\mathbf{J}$ trivial on J^1. Indeed, let $(\mathrm{K}/\mathrm{E}, \xi)$ be an admissible pair of level 0 attached to $\boldsymbol{\rho}$. Then, as in the proof of Lemma 8.2, the canonical $\overline{\mathbf{Q}}_\ell$-lift $\widetilde{\xi}$ of ξ defines a pair $(\mathrm{K}/\mathrm{E}, \widetilde{\xi})$ which is admissible of level 0, and the $\overline{\mathbf{Q}}_\ell$-representation $\widetilde{\boldsymbol{\rho}}$ of $\mathbf{J}$ trivial on J^1 which is attached to it is both σ-selfdual and a lift of $\boldsymbol{\rho}$. The representation $\widetilde{\boldsymbol{\kappa}} \otimes \widetilde{\boldsymbol{\rho}}$ is thus a σ-selfdual ℓ-adic type whose reduction mod ℓ is $\boldsymbol{\lambda}$. Inducing $\widetilde{\boldsymbol{\kappa}} \otimes \widetilde{\boldsymbol{\rho}}$ to G, we get a σ-selfdual supercuspidal lift $\widetilde{\pi}$ of π.

Suppose that $\ell \neq 2$ and let $\widetilde{\omega}$ be the canonical ℓ-adic lift of ω, that is, the $\overline{\mathbf{Q}}_\ell$-character of $\mathrm{F}_0^\times$ of kernel $\mathrm{N}_{\mathrm{F}/\mathrm{F}_0}(\mathrm{F}^\times)$. By Theorem 10.8, since the representation $\widetilde{\pi}$ is σ-selfdual, it is either distinguished or $\widetilde{\omega}$-distinguished, but not both. Using the reduction argument of invariant linear forms as in Lemma 2.6, we see that if $\widetilde{\pi}$ is distinguished (respectively, $\widetilde{\omega}$-distinguished), then π is distinguished (respectively, ω-distinguished). By Theorem 10.8 applied to π, this is an equivalence. □

We end with the following result, which is useful in [**3**].

Proposition 10.12. — *Suppose that π is a distinguished supercuspidal representation of* G *and that $\ell \neq 2$. Then π has no ω-distinguished unramified twist if and only if* D/D_0 *is ramified, that is, if and only if* T/T_0 *is ramified and $m = 1$.*

Proof. — Consider an unramified twist $\pi' = \pi(\chi \circ \det)$ of π, where χ is an unramified character of $\mathrm{F}^\times$. We are looking for a χ such that π' is ω-distinguished. First, π' is σ-selfdual if and only if $\pi\chi(\chi \circ \sigma) \simeq \pi$, that is, if and only if:

$$(\chi(\chi \circ \sigma))^{t(\pi)} = \chi^{2t(\pi)} = 1 \qquad (10.1)$$

where $t(\pi)$ denotes the torsion number of π, that is the number of unramified characters α of G such that $\pi\alpha \simeq \pi$. By [**36**] §3.4, we have $t(\pi) = f(\mathrm{K}/\mathrm{F}) = f(\mathrm{D}/\mathrm{F})$. Now the quadratic character associated with π' is $\delta'_0 = \delta_0(\chi \circ \mathrm{N}_{\mathrm{K}/\mathrm{F}})|_{\mathrm{D}_0^\times}$ and we have:

$$(\chi \circ \mathrm{N}_{\mathrm{K}/\mathrm{F}})|_{\mathrm{D}_0^\times} = (\chi \circ \mathrm{N}_{\mathrm{D}_0/\mathrm{F}_0})^{[\mathrm{K}:\mathrm{D}]}. \tag{10.2}$$

By Theorem 10.11, the representation π' is ω-distinguished if and only if the character (10.2) is equal to $\omega_{\mathrm{D}/\mathrm{D}_0}$. If D is ramified over D_0, this is not possible since χ is unramified. If D/D_0 is unramified, choosing an unramified character χ of order $f(\mathrm{D}_0/\mathrm{F}_0)$ gives us an ω-distinguished twist π'. □

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Vincent Sécherre, Laboratoire de Mathématiques de Versailles, UVSQ, CNRS, Université Paris-Saclay, 78035, Versailles, France • *E-mail :* vincent.secherre@math.uvsq.fr