PROOF OF THE TADIĆ CONJECTURE U0 ON THE UNITARY DUAL OF $\mathrm{GL}_m(\mathrm{D})$

by

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Abstract. — Let F be a non-Archimedean local field of characteristic 0, and let D be a finite-dimensional central division algebra over F. We prove that any unitary irreducible representation of a Levi subgroup of $\mathrm{GL}_m(\mathrm{D})$, with $m \geqslant 1$, induces irreducibly to $\mathrm{GL}_m(\mathrm{D})$. This ends the classification of the unitary dual of $\mathrm{GL}_m(\mathrm{D})$ initiated by Tadić.

Introduction

Let F be a non-Archimedean locally compact non-discrete field of characteristic zero (that is, a finite extension of the field of p-adic numbers for some prime number p) and let D be a finite-dimensional central division algebra over F. In [**22**], Tadić gave a conjectural classification of the unitary dual of $\mathrm{GL}_m(\mathrm{D})$, with $m \geqslant 1$, based on five statements U0,...,U4. In the same article, he proved U3 and U4. In [**4**], Badulescu and Renard proved U1, and it is known that U0 and U1 together imply U2. In this paper, we prove the remaining conjecture U0, which asserts that any unitary irreducible representation of a Levi subgroup of $\mathrm{GL}_m(\mathrm{D})$ induces irreducibly to $\mathrm{GL}_m(\mathrm{D})$. The proof is based on Bushnell-Kutzko's theory of types (see [**12**]), and more precisely on their theory of covers, which allows one to compare parabolic induction in $\mathrm{GL}_m(\mathrm{D})$ with parabolic induction in affine Hecke algebras.

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The proof consists of reducing to the case where D is commutative, for which the result is already known (Bernstein [**8**], see Theorem 1.1 below). This can be done by using particular types of $\mathrm{GL}_m(\mathrm{D})$, the so-called Bushnell-Kutzko simple types (see [**11, 19**]). Their Hecke algebras are well known and isomorphic to affine Hecke-Iwahori algebras of type A, which allows one to transport our induction problem, *via* the Hecke algebra isomorphisms of [**19**], to a very special case, in which the conjecture is known to be true. This method has been already used in [**10, 11**].

Note that in the case where D is commutative, hence equal to its centre F, Bernstein's proof of U0 is based on the proof of Kirillov's conjecture, which asserts that if $\mathscr{P}$ denotes the subgroup of $\mathrm{GL}_m(\mathrm{D})$ made of elements with last row $(0,\dots,0,1)$, then the restriction to $\mathscr{P}$ of any unitary irreducible representation of $\mathrm{GL}_m(\mathrm{D})$ is irreducible. This fact is no more true when D is non-commutative. See also Tadić [**21**] for a classification of the unitary dual of $\mathrm{GL}_m(\mathrm{F})$.

Our proof can be decomposed into three parts. In the first part (§3.1), we reduce to the case where the unitary irreducible representation of the Levi subgroup is simple in the sense of [**12**]: the inertial class of its cuspidal support contains a cuspidal pair of the form $(\mathrm{GL}_k(\mathrm{D})^r, \rho^{\otimes r})$, with $m = kr$ and where ρ is a cuspidal irreducible representation of the group $\mathrm{GL}_k(\mathrm{D})$ (see also Definition 1.3). This special case of the conjecture is denoted by S0.

In the second part, we translate the problem in terms of induction of modules over Hecke algebras. More precisely, we reduce the proof of S0 to proving that, given an integer $r \geqslant 1$ and a Levi subgroup M of $\mathrm{GL}_r(\mathrm{F})$, any unitary irreducible module over the Hecke-Iwahori algebra of M (that is, the Hecke algebra of M relative to some Iwahori subgroup) induces irreducibly to the Hecke-Iwahori algebra of $\mathrm{GL}_r(\mathrm{F})$ (see Proposition 3.3). This step demands the existence of simple types for any irreducible simple representation of $\mathrm{GL}_m(\mathrm{D})$. Such simple types have been constructed in [**17, 18, 19, 20**].

The last part of the proof consists of proving Proposition 3.3 (see above). This step is based on a result of Barbasch-Moy (see [**5, 6**]) which asserts that the functor of Iwahori-invariant vectors induces a one-to-one correspondence between unitary irreducible representations of $\mathrm{GL}_r(\mathrm{F})$ having a non-zero vector invariant under an Iwahori subgroup, and unitary irreducible modules over the Hecke-Iwahori algebra of $\mathrm{GL}_r(\mathrm{F})$.

See also [**3, 24**] for the role played by the Tadić classification in the unitary Jacquet-Langlands correspondence. More precisely, Badulescu [**3**] proved (independently from this article) a weak form of U0 (see [**3**, Proposition 3.8]) and thus determined the image by this correspondence of the unitary dual of $\mathrm{GL}_{md}(\mathrm{F})$ (where d denotes the reduced degree of D over F) in the Grothendieck group of representations of finite length of $\mathrm{GL}_m(\mathrm{D})$.

In the last section of this article, we determine the unramified characters χ of $\mathrm{GL}_m(\mathrm{D})$ for which the parabolically induced representation $\Pi(\chi) = \rho \times \rho\chi$, where ρ is a fixed cuspidal irreducible representation of $\mathrm{GL}_m(\mathrm{D})$, is reducible. Unlike [**22**], our result does not refer to the Jacquet-Langlands correspondence. This answers a question of J. Bernstein and A. Mínguez. Here again, we reduce to the case where D is commutative, for which the reducibility points are known to be $\chi = |\det|_{\mathrm{F}}$ and $\chi = |\det|_{\mathrm{F}}^{-1}$, where $|\ |_{\mathrm{F}}$ denotes the normalized absolute value of F. In the division algebra case, the reducibility points χ depend on the cuspidal representation ρ (see Theorem 4.6).

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1. Notations and preliminaries

In this section, we fix some notations and recall some well-known facts. The reader may refer to [**22**] for more details.

1.1. Let F be a non-Archimedean locally compact non-discrete field of characteristic 0, and let D be a finite-dimensional central division algebra over F. For any integer $m \geqslant 1$, we denote by $\mathrm{M}_m(\mathrm{D})$ the F-algebra of $m \times m$ matrices with coefficients in D and by $\mathrm{G}_m = \mathrm{GL}_m(\mathrm{D})$ the group of its invertible elements. For convenience, G_0 will denote the trivial group.

Let N_m be the reduced norm of $\mathrm{M}_m(\mathrm{D})$ over F and let $|\ |_\mathrm{F}$ be the normalized absolute value of F. The map $g \mapsto |\mathrm{N}_m(g)|_\mathrm{F}$ is a continuous group homomorphism from G_m to the multiplicative group $\mathbb{C}^\times$ of the field of complex numbers, which we simply denote by ν.

If ρ is a representation and χ a character of G_m for some m, we denote by $\rho\chi$ (or equivalently by $\chi\rho$) the twisted representation $g \mapsto \chi(g)\rho(g)$.

We denote by $\mathbb{N}$ the set of non-negative integers. If S is a set, a *multiset* on S is a finitely supported function from S to $\mathbb{N}$. It can be thought as an unordered finite family of elements of S. For $n \geqslant 0$ and $x_i \in \mathrm{S}$ with $1 \leqslant i \leqslant n$, we denote by $(x_1, \dots, x_n)$ the multiset whose value on $x \in \mathrm{S}$ is the number of integers $1 \leqslant i \leqslant n$ such that $x_i = x$. The integer n is then called the *size* of this multiset. We denote by M(S) the set of all multisets on S. It is naturally endowed with a structure of commutative semigroup.

1.2. For $m \geqslant 0$, we denote by Irr_m the set of all classes of irreducible representations of G_m, by $\mathscr{R}_m$ the category of smooth complex representations of finite length of G_m and by R_m the Grothendieck group of $\mathscr{R}_m$, which is a free $\mathbb{Z}$-module with basis Irr_m. In particular, Irr_0 is reduced to a single element and R_0 is isomorphic to $\mathbb{Z}$. For $\sigma \in \mathrm{Irr}_m$, we set $\deg(\sigma) = m$, which we call the *degree* of σ. We set:

$$\mathrm{R} = \bigoplus_{m \geqslant 0} \mathrm{R}_m$$

and:

$$\mathrm{Irr} = \bigcup_{m \geqslant 0} \mathrm{Irr}_m.$$

The group R is a graded free $\mathbb{Z}$-module with basis Irr. Two equivalent irreducible representations will be considered as the same element of Irr.

Given $m, n \geqslant 0$, the (normalized) parabolic induction functor:

$$\begin{array}{rcl} \mathscr{R}_m \times \mathscr{R}_n & \to & \mathscr{R}_{m+n} \\ (\sigma, \tau) & \mapsto & \sigma \times \tau \end{array}$$

induces a map $\mathrm{R}_m \times \mathrm{R}_n \to \mathrm{R}_{m+n}$. This map extends to a $\mathbb{Z}$-bilinear map $\mathrm{R} \times \mathrm{R} \to \mathrm{R}$, which makes R into an associative and commutative graded $\mathbb{Z}$-algebra (see [**9**, §2.3] and [**22**, §1]). The image of $(\sigma, \tau) \in \mathrm{R} \times \mathrm{R}$ by this map will be still denoted by $\sigma \times \tau$.

We will make no distinction between *unitary* and *unitarizable* irreducible representations, which form a subset of Irr denoted by Irr^u (see [**13**, §2.8]). Conjecture U0 is the following statement (see [**22**, §6]):

(U0) *Let $\sigma, \tau \in \mathrm{Irr}^u$ be unitary irreducible representations. Then $\sigma \times \tau \in \mathrm{Irr}$.*

Let us recall the following result of Bernstein [**8**].

Theorem 1.1 **(Bernstein).** — *Assume that* $\mathrm{D} = \mathrm{F}$*. Then* U0 *is true.*

1.3. Let $\mathscr{C}$ be the set of all cuspidal representations in Irr. Let $\rho \in \mathscr{C}$ be a cuspidal irreducible representation, and let m denote the degree of ρ. Let d be the reduced degree of D over F, that is, the square root of the dimension of D over F. By the Jacquet-Langlands correspondence (see [**14**]) one associates to ρ an essentially square integrable representation σ of the group $\mathrm{GL}_{md}(\mathrm{F})$. The classification of the discrete series of $\mathrm{GL}_{md}(\mathrm{F})$ (see [**25**]) gives us a unique positive integer b dividing md and a unique cuspidal irreducible representation τ of $\mathrm{GL}_{md/b}(\mathrm{F})$ such that σ is a quotient of the induced representation $\tau \times \mu\tau \times \ldots \times \mu^{b-1}\tau$, where $\mu : g \mapsto |\det(g)|_{\mathrm{F}}$ denotes the analogue of ν for the group $\mathrm{GL}_{md/b}(\mathrm{F})$. We denote this integer by $b(\rho)$, and we set:

$$\nu_\rho = \nu^{b(\rho)}.$$

Let $\mathscr{D}$ be the set of all essentially square integrable representations in Irr. It is parametrized by means of cuspidal irreducible representations as follows. For any $\rho \in \mathscr{C}$ and any positive integer n, the induced representation:

$$\nu_\rho^{(n-1)/2}\rho \times \nu_\rho^{-1+(n-1)/2}\rho \times \ldots \times \nu_\rho^{-(n-1)/2}\rho$$

has a unique essentially square integrable quotient, which we denote by $\delta(\rho, n)$. The map $\mathscr{C} \times (\mathbb{N} - \{0\}) \to \mathscr{D}$ obtained this way is a bijection (see [**22, 25**]).

Let $\mathscr{C}^u$ (resp. $\mathscr{D}^u$) be the set of all unitary representations in $\mathscr{C}$ (resp. in $\mathscr{D}$). Then $\delta(\rho, n)$ is unitary if and only if ρ is. In other words, the image of $\mathscr{C}^u \times (\mathbb{N} - \{0\})$ by the map above is $\mathscr{D}^u$.

1.4. Let $\mathscr{T}$ be the set of all essentially tempered representations in Irr and let $\mathscr{T}^u$ be the set of all tempered representations in $\mathscr{T}$. Given $\tau \in \mathscr{T}$, there exists a unique real number $e(\tau) \in \mathbb{R}$, which we call the *exponent* of τ, such that $\nu^{-e(\tau)}\tau$ is tempered. The map:

$$(\delta_1, \dots, \delta_k) \mapsto \delta_1 \times \dots \times \delta_k \tag{1.1}$$

induces a bijective correspondence from $\mathrm{M}(\mathscr{D}^u)$ onto $\mathscr{T}^u$ (see [**14**, B.2.*d*]).

Given $d = (\delta_1, \dots, \delta_k) \in \mathrm{M}(\mathscr{D})$, the fibers of the map $i \mapsto e(\delta_i)$ decompose $\{1, 2, \dots, k\}$ into a finite disjoint union $\mathrm{I}_1 \cup \dots \cup \mathrm{I}_l$. For $1 \leqslant i \leqslant l$, we denote by τ_i the product of the δ_j for $j \in \mathrm{I}_i$. Each τ_i is essentially tempered. Let us choose an ordering such that:

$$e(\tau_1) \geqslant \dots \geqslant e(\tau_l).$$

Then the induced representation $\tau_1 \times \dots \times \tau_l$ has a unique irreducible quotient, which we denote by $\Lambda(d)$. This representation depends only on d and not on the ordering of the τ_i, and the map $d \mapsto \Lambda(d)$ is a bijection from $\mathrm{M}(\mathscr{D})$ to Irr.

1.5. Given $\sigma \in \mathrm{Irr}$, we denote by $\sigma^\vee$ the contragredient representation of σ and by $\overline{\sigma}$ its complex conjugate representation, that is, the representation obtained by making $\mathbb{C}$ act on the space of σ by $(\lambda, v) \mapsto \overline{\lambda}v$. The representation:

$$\sigma^+ = \overline{\sigma^\vee}$$

is called the *Hermitian contragredient* of σ, and σ is said to be *Hermitian* if it is equivalent to its Hermitian contragredient. Since this is equivalent to the existence of a non-degenerate invariant Hermitian form on the space of σ, any unitary irreducible representation is Hermitian.

Given $d \in \mathrm{M}(\mathscr{D})$, we denote by d^+ the multiset on $\mathscr{D}$ whose elements are the Hermitian contragredients of the elements of d. Then (see [**22**, §2]) we have:

$$\Lambda(d)^+ = \Lambda(d^+).$$

Thus $\Lambda(d)$ is Hermitian if and only if $d^+ = d$. Note that, for $\delta \in \mathscr{D}$, the exponent of δ^+ is $-e(\delta)$.

Lemma 1.2. — *Let $\sigma,\tau\in\mathrm{Irr}$ be Hermitian representations such that $\sigma\times\tau$ is irreducible and unitary. Then σ and τ are unitary.*

Proof. — This is a standard result. The Hermitian forms on the spaces of σ and τ induce a Hermitian form h on the space of $\sigma\times\tau$. As $\sigma\times\tau$ is irreducible, its space can be endowed with a unique, up to a non-zero real scalar, non-degenerate Hermitian form. Therefore, up to a sign, h is positive definite, and σ,τ are unitary (see [**23**, §3(a)]). □

1.6. Let m be a positive integer, and let M be a Levi subgroup of G_m. A *cuspidal pair* of M is a pair (L,ρ) where L is a Levi subgroup of M and ρ a cuspidal irreducible representation of L. We denote by $\mathscr{B}(\mathrm{M})$ the set of all M-conjugacy classes of cuspidal pairs of M (the so-called *Bernstein spectrum* of M, see [**7**]).

Given an irreducible representation σ of M, there is a cuspidal pair (L,ρ) of M, unique up to M-conjugacy, such that σ is a subquotient of $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{M}}(\rho)$, where Q denotes any parabolic subgroup of M with Levi factor L and $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{M}}$ the corresponding (normalized) parabolic induction functor. This M-conjugacy class is denoted by $\mathrm{supp}(\sigma)$, and is called the (cuspidal) *support* of σ. This defines a surjective map:

$$\mathrm{supp}:\mathrm{Irr}(\mathrm{M})\to\mathscr{B}(\mathrm{M}),$$

where $\mathrm{Irr}(\mathrm{M})$ denotes the set of all classes of irreducible representations of M.

A cuspidal pair of M is said to be *inertially equivalent* to (L,ρ) if there is an unramified character χ of L such that this pair is M-conjugate to $(\mathrm{L},\rho\otimes\chi)$. The set of all cuspidal pair of M which are inertially equivalent to (L,ρ) is denoted by $[\mathrm{L},\rho]_{\mathrm{M}}$ and called the *inertial class* of (L,ρ) in M.

1.7. As any Levi subgroup of M is also a Levi subgroup of G, any cuspidal pair of M can be considered as a cuspidal pair of G. Hence we have a natural map from $\mathscr{B}(\mathrm{M})$ to $\mathscr{B}(\mathrm{G})$, and any inertial class of M defines an inertial class of G.

Definition 1.3. — (i) An irreducible representation $\sigma\in\mathrm{Irr}$ of degree m is said to be *simple* if the inertial class of its support has the form $[\mathrm{G}_k^r,\rho^{\otimes r}]_{\mathrm{G}}$, with $m=kr$ and where ρ is a cuspidal irreducible representation of G_k.

(ii) More generally, two representations $\sigma, \tau \in \mathrm{Irr}$ of degree m, n respectively are said to be *aligned* if the inertial class of the product $\mathrm{supp}(\sigma) \times \mathrm{supp}(\tau)$, considered as an element of $\mathscr{B}(\mathrm{G}_m \times \mathrm{G}_n)$, has the form $[\mathrm{G}_k^r, \rho^{\otimes r}]_{\mathrm{G}_m \times \mathrm{G}_n}$ with $m + n = kr$ and ρ as above.

Remark 1.4. — Any essentially square integrable irreducible representation is simple. If two representations $\sigma, \tau \in \mathrm{Irr}$ are aligned, then σ and τ are simple. In particular, a representation is simple if and only if it is aligned with itself.

Proposition 2.2 together with Lemma 2.5 of [**22**] have the following consequence.

Proposition 1.5. — *Let $d = (\delta_1, \dots, \delta_k)$ and $d' = (\delta'_1, \dots, \delta'_{k'})$ be in $\mathrm{M}(\mathscr{D})$. Suppose that, for any $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant k'$, the representations δ_i and δ'_j are not aligned. Then $\Lambda(d) \times \Lambda(d')$ is irreducible and equal to $\Lambda(d + d')$.*

This leads to the following result.

Proposition 1.6. — *Let $\sigma \in \mathrm{Irr}$ be an irreducible representation.*

(i) *There is a unique subset $\{\sigma_1, \dots, \sigma_k\}$ of Irr such that $\sigma = \sigma_1 \times \ldots \times \sigma_k$, and such that σ_i, σ_j are aligned if and only if $i = j$.*

(ii) *If σ is unitary, then so are the σ_i.*

Proof. — Let $d \in \mathrm{M}(\mathscr{D})$ be such that $\sigma = \Lambda(d)$. The multiset d can be written in a unique way as a sum:

$$d = d_1 + \ldots + d_k \tag{1.2}$$

such that two elements of d are aligned if and only if they are contained in the same d_i. Thus, according to Proposition 1.5, we have:

$$\Lambda(d) = \Lambda(d_1) \times \ldots \times \Lambda(d_k).$$

The unicity property comes from the unicity of decomposition (1.2). Moreover, if $d^+ = d$, then $d_i^+ = d_i$ for each integer $1 \leqslant i \leqslant k$. Therefore, if $\Lambda(d)$ is Hermitian, then so are the $\Lambda(d_i)$. By Lemma 1.2, if $\Lambda(d)$ is unitary, then so are the $\Lambda(d_i)$. □

2. Theory of types for $\mathrm{GL}_m(\mathrm{D})$

In order to prove Conjecture U0, we need some material from Bushnell-Kutzko's theory of types, which we develop in this section.

2.1. Let m be a positive integer, and let M be a Levi subgroup of $\mathrm{G} = \mathrm{G}_m$. Let J be a compact open subgroup of M, and let τ be a smooth irreducible representation of J on a complex vector space $\mathscr{V}$. Let us choose a Haar measure on M giving measure 1 to J. The *Hecke algebra* of M relative to (J, τ), which we denote by $\mathscr{H}(\mathrm{M}, \tau)$, is the convolution algebra of locally constant and compactly supported functions $f : \mathrm{M} \to \mathrm{End}_{\mathbb{C}}(\mathscr{V})$ such that:

$$f(kgk') = \tau(k) \circ f(g) \circ \tau(k')$$

for any $k, k' \in \mathrm{J}$ and $g \in \mathrm{M}$. We have a functor:

$$\mathbf{M}_\tau : \sigma \mapsto \mathrm{Hom}_{\mathrm{J}}(\tau, \sigma) \tag{2.1}$$

from the category of smooth complex representations of M to the category of right modules over $\mathscr{H}(\mathrm{M}, \tau)$. It induces a bijection between the classes of irreducible representations of M whose restriction to J contains τ and the classes of irreducible right $\mathscr{H}(\mathrm{M}, \tau)$-modules.

2.2. According to [**11**, §4.3], the Hecke algebra $\mathscr{H}(\mathrm{M}, \tau)$ can be canonically endowed with an involution $f \mapsto f^*$. A right module V over $\mathscr{H}(\mathrm{M}, \tau)$ is said to be *unitary* if there exists a positive definite Hermitian form $(x, y) \mapsto \langle x, y \rangle$ on V such that:

$$\langle vf, w \rangle = \langle v, wf^* \rangle$$

for any $v, w \in \mathrm{V}$ and $f \in \mathscr{H}(\mathrm{M}, \tau)$.

Note that $\mathbf{M}_\tau$ preserves unitarity: if an irreducible representation of M is unitary, then the irreducible module which corresponds to it is unitary.

2.3. Let $\mathfrak{s}_{\mathrm{M}}$ be an inertial class of M.

Definition 2.1 **([12], 4.2)**. — The pair (J, τ) is said to be an $\mathfrak{s}_{\mathrm{M}}$-*type* of M if the irreducible representations of M whose restriction to J contains τ are exactly those whose cuspidal support belongs to $\mathfrak{s}_{\mathrm{M}}$.

Thus, given an $\mathfrak{s}_{\mathrm{M}}$-type (J,τ), the functor $\mathbf{M}_\tau$ induces a bijection between the classes of irreducible representations of M with cuspidal support in $\mathfrak{s}_{\mathrm{M}}$ and the classes of irreducible right $\mathscr{H}(\mathrm{M},\tau)$-modules.

2.4. Let $(\mathrm{J}_{\mathrm{M}},\tau_{\mathrm{M}})$ be an $\mathfrak{s}_{\mathrm{M}}$-type of M, let $\mathfrak{s}_{\mathrm{G}}$ denote the inertial class of G corresponding to $\mathfrak{s}_{\mathrm{M}}$ and let (J,τ) be a G-cover of $(\mathrm{J}_{\mathrm{M}},\tau_{\mathrm{M}})$. We do not give here the definition of a cover (see [**12**, 8.1]), which is quite technical. We just mention that we have $\mathrm{J}\cap\mathrm{M}=\mathrm{J}_{\mathrm{M}}$ and that the restriction of τ to M is τ_{M}. The importance of the notion of cover lies in the isomorphism (2.3) below.

Given a parabolic subgroup P of G with Levi subgroup M, we denote by:

$$t_{\mathrm{P}}:\mathscr{H}(\mathrm{M},\tau_{\mathrm{M}})\to\mathscr{H}(\mathrm{G},\tau) \tag{2.2}$$

the $\mathbb{C}$-algebra homomorphism given by [**12**, Corollary 7.12]. If we denote by $\mathscr{H}$ and $\mathscr{H}_{\mathrm{M}}$ the Hecke algebras $\mathscr{H}(\mathrm{G},\tau)$ and $\mathscr{H}(\mathrm{M},\tau_{\mathrm{M}})$, then the map t_{P} makes $\mathscr{H}$ into an $\mathscr{H}_{\mathrm{M}}$-algebra. According to [**12**] (see Theorem 8.3 and Corollary 8.4), the pair (J,τ) is an $\mathfrak{s}_{\mathrm{G}}$-type of G and, for any irreducible representation σ of M with cuspidal support in $\mathfrak{s}_{\mathrm{M}}$, we have a canonical $\mathscr{H}$-module isomorphism:

$$\mathbf{M}_\tau(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma))\simeq\mathrm{Hom}_{\mathscr{H}_{\mathrm{M}}}(\mathscr{H},\mathbf{M}_{\tau_{\mathrm{M}}}(\sigma)), \tag{2.3}$$

where $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}$ denotes the (normalized) parabolic induction functor.

2.5. In this paragraph, we discuss the question of the existence of types relative to a given inertial class. Let k be a divisor of m and let $\rho\in\mathscr{C}$ be a cuspidal irreducible representation of degree k. Let r denote the positive integer such that $m=kr$. For any Levi subgroup M of G containing:

$$\mathrm{M}_0=\mathrm{G}_k^r=\mathrm{G}_k\times\ldots\times\mathrm{G}_k, \tag{2.4}$$

we denote by $\mathfrak{s}_{\mathrm{M}}=\mathfrak{s}_{\mathrm{M}}(\rho)$ the inertial class of the cuspidal pair $(\mathrm{G}_k^r,\rho^{\otimes r})$ in M and by $\mathfrak{s}_{\mathrm{G}}=\mathfrak{s}_{\mathrm{G}}(\rho)$ the inertial class of G which corresponds to it. We have the following result:

Theorem 2.2. — *There exists an* $\mathfrak{s}_{\mathrm{G}}$*-type of* G.

This is [**15**, Theorem 5.5] if ρ is of level zero (that is, if ρ has a non-zero vector invariant under the subgroup $1+\mathrm{M}_k(\mathfrak{p}_\mathrm{D})$, where $\mathfrak{p}_\mathrm{D}$ denotes the maximal ideal of the ring of integers of D) and [**20**, Théorème 5.23] if not.

2.6. In order to prove Conjecture U0, we need $\mathfrak{s}_\mathrm{G}$-types of G whose Hecke algebras we understand precisely. This requires the notion of simple type, which first appears in [**11**] and has been generalized in [**17, 18, 19**]. For a definition of simple type, see [**19**, §4.1].

Proposition 2.3. — (i) *There is a simple type of* G_k *contained in* ρ.

(ii) *Let* (U, u) *be a simple type contained in* ρ. *There is a finite extension* K *of* F *contained in* $\mathrm{M}_k(\mathrm{D})$ *such that the normalizer of* u *in* G_k *is* $\mathrm{K}^\times\mathrm{U}$.

Proof. — Note that a type of G_k is contained in ρ if and only if it is a type relative to the inertial class of the cuspidal pair (G_k, ρ). Part (i) of the result comes from [**15**, Theorem 5.4] if ρ is of level zero and from [**20**, Théorème 5.21] if not.

In order to prove part (ii), recall that the simple type (U, u) comes with a finite extension E of F contained in $\mathrm{M}_k(\mathrm{D})$ (see [**19**, §4.1]). The centralizer of E in $\mathrm{M}_k(\mathrm{D})$ is a central simple E-algebra isomorphic to $\mathrm{M}_{k'}(\mathrm{D}')$, where k' is a positive integer and D' a finite-dimensional central division algebra over E. According to [**19**, §5.1] the normalizer of u in G_k is generated by U and an element ϖ which is a positive power of a uniformizer of D'. The E-algebra $\mathrm{K} = \mathrm{E}[\varpi]$ is a totally ramified extension of E. As an extension of F, it has the required property. □

2.7. In [**19**, §5.2] one describes a process:

$$(\mathrm{U}, u) \mapsto (\mathrm{J}, \tau) \tag{2.5}$$

which associates, to any simple type (U, u) of G_k contained in ρ, an $\mathfrak{s}_\mathrm{G}$-type (J, τ) of G with the following property.

Proposition 2.4. — *For any Levi subgroup* M *of* G *containing (2.4), the restriction of* (J, τ) *to* M *is an* $\mathfrak{s}_\mathrm{M}$*-type of* M *of which* (J, τ) *is a* G*-cover.*

Proof. — According to Proposition [**19**, 5.5], the pair (J, τ) associated to (U, u) by (2.5) is an $\mathfrak{s}_\mathrm{G}$-type of G constructed as a cover of the type $(\mathrm{U}^r, u^{\otimes r})$ of the Levi subgroup M_0. The result follows from [**12**, Proposition 8.5]. □

Remark 2.5. — The reader should pay attention to the fact that, in general, the pair (J, τ) is *not* what we call a simple type in [**19**], but is the type which we denote by $(\mathrm{J}_{\mathrm{P}}, \lambda_{\mathrm{P}})$ in [**19**, §5.2]. Nevertheless, according to [**19**, Proposition 5.4], there exists a compact open subgroup $\mathrm{J}^{\dagger}$ of G containing J such that the induced representation of τ from J to $\mathrm{J}^{\dagger}$ is a simple type.

Example 2.6. — Assume that $\mathrm{D} = \mathrm{F}$ and that ρ is the trivial character of $\mathrm{GL}_1(\mathrm{F})$. Then the trivial character $1_{\mathscr{O}_{\mathrm{F}}^{\times}}$ of the unit group of the ring of integers $\mathscr{O}_{\mathrm{F}}$ is a simple type of $\mathrm{GL}_1(\mathrm{F})$ containing ρ. The pair (J, τ) associated to it by (2.5) is the trivial character of the standard Iwahori subgroup of $\mathrm{G} = \mathrm{GL}_r(\mathrm{F})$. (By *standard* we mean that the reduction of J modulo $\mathfrak{p}_{\mathrm{D}}$ is made of *upper* triangular matrices.)

2.8. Let (U, u) be a simple type contained in ρ and let (J, τ) be the $\mathfrak{s}_{\mathrm{G}}$-type of G corresponding to it by (2.5). In this paragraph, we describe the support of the Hecke algebra $\mathscr{H}(\mathrm{G}, \tau)$. Let K/F be as in Proposition 2.3, let ϖ be a uniformizer of K, let $\mathscr{N}$ be the normalizer of the diagonal torus of $\mathrm{GL}_r(\mathrm{K})$ and let W be the subgroup of $\mathscr{N}$ made of elements whose non-zero entries are in the subgroup generated by ϖ. As K is contained in $\mathrm{M}_k(\mathrm{D})$, the group $\mathrm{GL}_r(\mathrm{K})$ can naturally be considered as a subgroup of G. Set:

$$h = \begin{pmatrix} 0 & \mathrm{Id}_{r-1} \\ \varpi & 0 \end{pmatrix} \in \mathrm{W} \subset \mathrm{G},$$

where Id_{r-1} denotes the identity matrix of $\mathrm{GL}_{r-1}(\mathrm{K})$. Note that h does not normalize J in general. According to Propositions [**19**, 4.3] and [**20**, 5.10], any element of $\mathscr{H}(\mathrm{G}, \tau)$ vanishes outside JWJ. More precisely, we have the following result.

Proposition 2.7. — *Let us fix* $w \in \mathrm{W}$.

(i) *The subspace of* $\mathscr{H}(\mathrm{G}, \tau)$ *made of functions supported on* $\mathrm{J}w\mathrm{J}$ *has dimension* 1, *and any non-zero element of this subspace is invertible.*

(ii) *Let* $\varphi \in \mathscr{H}(\mathrm{G}, \tau)$ *be a non-zero element supported on* $\mathrm{J}h\mathrm{J}$. *Then for any non-zero element* f *supported on* $\mathrm{J}w\mathrm{J}$, *the convolution product* $f * \varphi$ *(resp.* $\varphi * f$*) is supported on* $\mathrm{J}wh\mathrm{J}$ *(resp. on* $\mathrm{J}hw\mathrm{J}$*).*

Proof. — We denote by $(\mathrm{J}^{\dagger}, \tau^{\dagger})$ the simple type induced by (J, τ) (see Remark 2.5). According to [**19**] (see Propositions 4.3 and 4.16 and Lemma 4.13), the result is true if we replace $\mathscr{H}(\mathrm{G},\tau)$ by the Hecke algebra $\mathscr{H}(\mathrm{G},\tau^\dagger)$. The result for $\mathscr{H}(\mathrm{G},\tau)$ follows from [**11**, Proposition 4.1.3 and Corollary 4.1.5]. □

Example 2.8. — Assume, as in Example 2.6, that $\mathrm{D} = \mathrm{F}$ and that ρ is the trivial character of $\mathrm{GL}_1(\mathrm{F})$. Then $\mathrm{K} = \mathrm{F}$ satisfies the conditions of Proposition 2.3. The choice of a uniformizer of F defines a subgroup W of $\mathrm{G} = \mathrm{GL}_r(\mathrm{F})$, and the Hecke algebra $\mathscr{H}(\mathrm{G},\tau)$ of the trivial character of the standard Iwahori subgroup J of G is supported on $\mathrm{JWJ} = \mathrm{G}$ (the Bruhat decomposition).

2.9. In this paragraph, we investigate the structure of the Hecke algebra $\mathscr{H}(\mathrm{G},\tau)$. Let $\tilde{\mathrm{K}}$ be a finite *unramified* extension of K. According to Examples 2.6 and 2.8, the trivial character $1_{\mathscr{O}_{\tilde{\mathrm{K}}}^\times}$ of the unit group of the ring of integers $\mathscr{O}_{\tilde{\mathrm{K}}}$ is a simple type of $\mathrm{GL}_1(\tilde{\mathrm{K}})$ contained in the trivial character of $\mathrm{GL}_1(\tilde{\mathrm{K}})$. The pair associated to it by (2.5), which we denote by $(\mathscr{I}, 1_{\mathscr{I}})$, is the trivial character of the standard Iwahori subgroup of $\mathrm{GL}_r(\tilde{\mathrm{K}})$. Note that W can be considered as a subgroup of both G and $\mathrm{GL}_r(\tilde{\mathrm{K}})$. Given $f \in \mathscr{H}(\mathrm{G},\tau)$ (resp. $f \in \mathscr{H}(\mathrm{GL}_r(\tilde{\mathrm{K}}), 1_{\mathscr{I}})$), we set:

$$\mathrm{supp}(f) = \{w \in \mathrm{W} \mid f(w) \neq 0\},$$

which is the support of f *in* W. For technical reasons, this is more convenient than the support in G (resp. in $\mathrm{GL}_r(\tilde{\mathrm{K}})$).

Proposition 2.9. — *For a unique (up to isomorphism) choice of finite unramified extension* $\tilde{\mathrm{K}}$ *of* K*, there is a* $\mathbb{C}$*-algebra isomorphism:*

$$\Psi : \mathscr{H}(\mathrm{GL}_r(\tilde{\mathrm{K}}), 1_{\mathscr{I}}) \to \mathscr{H}(\mathrm{G},\tau) \tag{2.6}$$

such that for any function $f \in \mathscr{H}(\mathrm{GL}_r(\tilde{\mathrm{K}}), 1_{\mathscr{I}})$*, we have:*

$$\mathrm{supp}(\Psi f) = \mathrm{supp}(f). \tag{2.7}$$

Proof. — Theorem [**19**, 4.6] gives us the result for the Hecke algebra $\mathscr{H}(\mathrm{G},\tau^\dagger)$. The result for $\mathscr{H}(\mathrm{G},\tau)$ follows from [**11**, Proposition 4.1.3]. □

Remark 2.10. — (i) Note that (2.7) makes sense because W can be seen as a subgroup of $\mathrm{GL}_r(\tilde{\mathrm{K}})$ on the left hand side, and of G on the right hand side.

(ii) The unramified extension $\tilde{\mathrm{K}}/\mathrm{K}$ does not depend on the integer r, but only on the cuspidal representation ρ.

2.10. Let us fix an extension $\tilde{\mathrm{K}}$ of F as in Proposition 2.9. Let P be the parabolic subgroup of G of upper triangular matrices with respect to the Levi subgroup $\mathrm{M}_0 = \mathrm{G}_k^r$ (see (2.4)) and let t_{P} be the $\mathbb{C}$-algebra homomorphism:

$$t_{\mathrm{P}} : \mathscr{H}(\mathrm{G}_k^r, u^{\otimes r}) \to \mathscr{H}(\mathrm{G}, \tau)$$

corresponding to P (see (2.2)). We denote by Q the (minimal) parabolic subgroup of $\mathrm{GL}_r(\tilde{\mathrm{K}})$ of upper triangular matrices. Let t_{Q} be the $\mathbb{C}$-algebra homomorphism:

$$t_{\mathrm{Q}} : \mathscr{H}(\tilde{\mathrm{K}}^{\times r}, 1_{\mathscr{O}_{\tilde{\mathrm{K}}}^{\times}}^{\otimes r}) \to \mathscr{H}(\mathrm{GL}_r(\tilde{\mathrm{K}}), 1_{\mathscr{I}})$$

corresponding to Q. Let us choose a $\mathbb{C}$-algebra isomorphism:

$$\Psi_u : \mathscr{H}(\tilde{\mathrm{K}}^{\times}, 1_{\mathscr{O}_{\tilde{\mathrm{K}}}^{\times}}) \to \mathscr{H}(\mathrm{G}_k, u) \tag{2.8}$$

such that, for any function $f \in \mathscr{H}(\tilde{\mathrm{K}}^{\times}, 1_{\mathscr{O}_{\tilde{\mathrm{K}}}^{\times}})$, we have:

$$\mathrm{supp}(\Psi_u(f)) = \mathrm{supp}(f), \tag{2.9}$$

where supp denotes the support in the group $\langle\varpi\rangle$ generated by ϖ, considered as a subgroup of $\tilde{\mathrm{K}}^{\times}$ on the left hand side and of G_k on the right hand side. Then there is a unique W-equivariant $\mathbb{C}$-algebra isomorphism:

$$\Psi_u^r : \mathscr{H}(\tilde{\mathrm{K}}^{\times r}, 1_{\mathscr{O}_{\tilde{\mathrm{K}}}^{\times}}^{\otimes r}) \to \mathscr{H}(\mathrm{G}_k^r, u^{\otimes r})$$

which agrees with Ψ_u on the first tensor factor and such that, for any function $f \in \mathscr{H}(\tilde{\mathrm{K}}^{\times r}, 1_{\mathscr{O}_{\tilde{\mathrm{K}}}^{\times}}^{\otimes r})$, we have:

$$\mathrm{supp}(\Psi_u^r(f)) = \mathrm{supp}(f),$$

where supp denotes the support in the group $\langle\varpi\rangle^r$, considered as a subgroup of $\tilde{\mathrm{K}}^{\times r}$ on the left hand side and of G_k^r on the right hand side (compare [**11**, 7.6.19]). We are now ready to state the main result of this section.

Theorem 2.11. — *Given a $\mathbb{C}$-algebra isomorphism Ψ_u as in (2.8), there is a unique $\mathbb{C}$-algebra isomorphism:*

$$\Psi_{\mathrm{G}} : \mathscr{H}(\mathrm{GL}_r(\tilde{\mathrm{K}}), 1_{\mathscr{I}}) \to \mathscr{H}(\mathrm{G}, \tau)$$

such that the diagram:

$$\begin{array}{ccc} \mathscr{H}(\mathrm{GL}_r(\tilde{\mathrm{K}}), 1_{\mathscr{I}}) & \xrightarrow{\Psi_{\mathrm{G}}} & \mathscr{H}(\mathrm{G}, \tau) \\ \uparrow t_{\mathrm{Q}} & & \uparrow t_{\mathrm{P}} \\ \mathscr{H}(\tilde{\mathrm{K}}^{\times r}, 1^{\otimes r}_{\mathscr{O}^{\times}_{\tilde{\mathrm{K}}}}) & \xrightarrow[\Psi^r_u]{} & \mathscr{H}(\mathrm{G}^r_k, u^{\otimes r}) \end{array}$$

commutes.

Proof. — The proof goes *mutatis mutandis* as in [**11**, Theorem 7.6.20]. □

Remark 2.12. — The isomorphism Ψ_{G} preserves the canonical structure of $\mathbb{C}$-algebra with involution on the Hecke algebras (see §2.2). In other words, for any $f \in \mathscr{H}(\mathrm{GL}_r(\tilde{\mathrm{K}}), 1_{\mathscr{I}})$, we have $\Psi_{\mathrm{G}}(f^*) = \Psi_{\mathrm{G}}(f)^*$. This implies that unitary modules over $\mathscr{H}(\mathrm{GL}_r(\tilde{\mathrm{K}}), 1_{\mathscr{I}})$ correspond bijectively to unitary modules over $\mathscr{H}(\mathrm{G}, \tau)$.

3. Proof of Conjecture U0

3.1. In this paragraph, we reduce the proof of Conjecture U0 to the following special case:

(S0) *Let $\sigma, \tau \in \mathrm{Irr}^u$ be aligned unitary irreducible representations. Then $\sigma \times \tau \in \mathrm{Irr}$.*

Proposition 3.1. — *Assume that* S0 *holds. Then* U0 *is true.*

Proof. — Let $\sigma, \tau \in \mathrm{Irr}^u$ be irreducible unitary representations, and let:

$$\sigma = \sigma_1 \times \ldots \times \sigma_k \quad \text{and} \quad \tau = \tau_1 \times \ldots \times \tau_{k'}$$

be the factorizations of σ and τ given by Proposition 1.6. In particular, each σ_i, τ_j is simple for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant k'$. Moreover, we can choose the ordering such that there exists a non-negative integer r for which σ_i and τ_i are aligned if $1 \leqslant i \leqslant r$, and σ_i is not aligned with τ_j if $i, j \geqslant r+1$. As σ, τ are unitary and irreducible, and according to Proposition 1.6, each representation σ_i, τ_j is unitary. We write:

$$\begin{aligned} \sigma \times \tau = (\sigma_1 \times \tau_1) \times \ldots \\ \ldots \times (\sigma_r \times \tau_r) \times \sigma_{r+1} \times \ldots \times \sigma_k \times \tau_{r+1} \times \ldots \times \tau_{k'}. \end{aligned} \tag{3.1}$$

Assuming that S0 holds, each $\sigma_i \times \tau_i$ is irreducible for $1 \leqslant i \leqslant r$. Therefore (3.1) shows that $\sigma \times \tau$ is a product of irreducible factors, no two of them being aligned. The result now follows from Proposition 1.5. □

Remark 3.2. — Statement S0 can be rephrased as follows: any simple unitary irreducible representation of a Levi subgroup of G_m, with $m \geqslant 1$, induces irreducibly to G_m.

3.2. Let $\rho \in \mathscr{C}$ be a cuspidal irreducible representation, and set $k = \deg(\rho)$. Let m be a positive integer which is a multiple of k and let r denote the positive integer such that $m = kr$. Let M be a Levi subgroup of $\mathrm{G} = \mathrm{G}_m$ of the form:

$$\mathrm{M} = \mathrm{G}_{kr_1} \times \mathrm{G}_{kr_2}, \tag{3.2}$$

where $r_1, r_2 \geqslant 1$ are positive integers such that $r_1 + r_2 = r$. As in §2.5, we denote by $\mathfrak{s}_\mathrm{M} = \mathfrak{s}_\mathrm{M}(\rho)$ the inertial class of the cuspidal pair $(\mathrm{G}_k^r, \rho^{\otimes r})$ and by $\mathfrak{s}_\mathrm{G}$ the corresponding inertial class of G. Let (U, u) be a simple type contained in ρ, let (J, τ) be the $\mathfrak{s}_\mathrm{G}$-type of G corresponding to it by (2.5) and let $(\mathrm{J}_\mathrm{M}, \tau_\mathrm{M})$ be the $\mathfrak{s}_\mathrm{M}$-type of M of which (J, τ) is a G-cover by Proposition 2.4. Let $\mathscr{H}$ and $\mathscr{H}_\mathrm{M}$ denote the Hecke algebras $\mathscr{H}(\mathrm{G}, \tau)$ and $\mathscr{H}(\mathrm{M}, \tau_\mathrm{M})$. Let P be the parabolic subgroup of G of upper triangular matrices with respect to M and let t_P be the $\mathbb{C}$-algebra homomorphism from $\mathscr{H}_\mathrm{M}$ to $\mathscr{H}$ corresponding to P (see (2.2)).

Proposition 3.3. — *Let* V *be a unitary irreducible* $\mathscr{H}_\mathrm{M}$*-module. Then the* $\mathscr{H}$*-module* $\mathrm{Hom}_{\mathscr{H}_\mathrm{M}}(\mathscr{H}, \mathrm{V})$ *is irreducible.*

Proof. — We will first prove Proposition 3.3 in a particular case.

(1) We temporarily suppose that D = F and that ρ is the trivial character of $\mathrm{GL}_1(\mathrm{F})$ (see Example 2.6). In that case, we can choose for J the standard Iwahori subgroup of G and for τ the trivial character of J. Therefore, J_M is the standard Iwahori subgroup of M and τ_M is its trivial character. The functor $\mathbf{M}_\tau$ (resp. $\mathbf{M}_{\tau_\mathrm{M}}$) associates to a representation of G (resp. M) the space of its J-invariant (resp. $\mathrm{J} \cap \mathrm{M}$-invariant) vectors.

We now recall the following crucial result of Barbasch and Moy [**5, 6**].

***Theorem 3.4* (Barbasch-Moy)**. — *The functor* $\mathbf{M}_{\tau_{\mathrm{M}}}$ *induces a bijective correspondence between unitary irreducible representation of* M *with a non-zero space of* $\mathrm{J} \cap \mathrm{M}$*-invariant vectors and unitary irreducible right* $\mathscr{H}_{\mathrm{M}}$*-modules.*

Let σ be an irreducible representation of M with a non-zero space of $\mathrm{J} \cap \mathrm{M}$-invariant vectors such that $\mathbf{M}_{\tau_{\mathrm{M}}}(\sigma)$ is isomorphic to V. By Theorem 3.4, this representation is unitary. According to (2.3), it is enough to prove that the $\mathscr{H}$-module:

$$\mathbf{M}_{\tau}(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma)) = \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma)^{\mathrm{J}}$$

is irreducible. According to Theorem 1.1, the induced representation $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma)$ is irreducible. Because $\mathbf{M}_{\tau}$ preserves irreducibility, we are done.

(2) Now the symbols $\mathrm{D}, \rho, \mathrm{J}, \tau$... recover their general meaning. We are going to reduce the general case to our particular case 1. Let $\tilde{\mathrm{K}}$ be a finite extension of F as in Proposition 2.9. We use the notations of §§2.9–2.10. Let L denote the Levi subgroup:

$$\mathrm{L} = \mathrm{GL}_{r_1}(\tilde{\mathrm{K}}) \times \mathrm{GL}_{r_2}(\tilde{\mathrm{K}}).$$

Let Q be the parabolic subgroup of $\mathrm{GL}_r(\tilde{\mathrm{K}})$ of upper triangular matrices with respect to L and let t_{Q} be the $\mathbb{C}$-algebra homomorphism from the Hecke algebra $\mathscr{H}_{\mathrm{L}} = \mathscr{H}(\mathrm{L}, 1_{\mathscr{I} \cap \mathrm{L}})$ to $\mathscr{H}(\mathrm{GL}_r(\tilde{\mathrm{K}}), 1_{\mathscr{I}})$ corresponding to Q. Let Ψ_{G} denote the $\mathbb{C}$-algebra isomorphism of Theorem 2.11.

***Proposition 3.5*.** — *There is a* $\mathbb{C}$*-algebra isomorphism:*

$$\Psi_{\mathrm{M}} : \mathscr{H}(\mathrm{L}, 1_{\mathscr{I} \cap \mathrm{L}}) \to \mathscr{H}(\mathrm{M}, \tau_{\mathrm{M}})$$

such that the diagram:

$$\begin{array}{ccc} \mathscr{H}(\mathrm{GL}_r(\tilde{\mathrm{K}}), 1_{\mathscr{I}}) & \xrightarrow{\Psi_{\mathrm{G}}} & \mathscr{H}(\mathrm{G}, \tau) \\ {\scriptstyle t_{\mathrm{Q}}} \uparrow & & \uparrow {\scriptstyle t_{\mathrm{P}}} \\ \mathscr{H}(\mathrm{L}, 1_{\mathscr{I} \cap \mathrm{L}}) & \xrightarrow[\Psi_{\mathrm{M}}]{} & \mathscr{H}(\mathrm{M}, \tau_{\mathrm{M}}) \end{array}$$

commutes.

Proof. — According to Theorem 2.11, it suffices to choose for Ψ_{M} the W-equivariant $\mathbb{C}$-algebra isomorphism which agrees with $\Psi_{\mathrm{G}_{kr_1}}$ on the first tensor factor and such that we have:

$$\mathrm{supp}(\Psi_{\mathrm{M}}(f)) = \mathrm{supp}(f)$$

for any function $f \in \mathscr{H}(\mathrm{L}, 1_{\mathscr{I}\cap\mathrm{L}})$. □

This allows us to make V into a module over $\mathscr{H}_{\mathrm{L}}$, and thus to identify the $\mathscr{H}$-module $\mathrm{Hom}_{\mathscr{H}_{\mathrm{M}}}(\mathscr{H}, \mathrm{V})$ with the $\mathscr{H}(\mathrm{GL}_r(\tilde{\mathrm{K}}), 1_{\mathscr{I}})$-module given by:

$$\mathrm{Hom}_{\mathscr{H}_{\mathrm{L}}}(\mathscr{H}(\mathrm{GL}_r(\tilde{\mathrm{K}}), 1_{\mathscr{I}}), \mathrm{V}). \tag{3.3}$$

As Ψ_{M} preserves the canonical structure of $\mathbb{C}$-algebra with involution (see Remark 2.12), V is irreducible and unitary as a $\mathscr{H}_{\mathrm{L}}$-module. Therefore (3.3) is irreducible according to case 1.

This ends the proof of Proposition 3.3. □

3.3. In this paragraph, we prove S0. With the notations of §3.2, it suffices to prove the following result.

Proposition 3.6. — *Let σ be a simple unitary irreducible representation of* M *with cuspidal support in the inertial class $\mathfrak{s}_{\mathrm{M}}$. Then the induced representation* $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma)$ *is irreducible.*

Proof. — We apply Proposition 3.3 to the $\mathscr{H}_{\mathrm{M}}$-module $\mathrm{V} = \mathbf{M}_{\tau_{\mathrm{M}}}(\sigma)$, which is irreducible and unitary (see §2.2). The $\mathscr{H}$-module $\mathbf{M}_{\tau}(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma))$ is then irreducible, thanks to (2.3). The result now follows from the fact that $\mathbf{M}_{\tau}$ preserves reducibility. □

This ends the proof of Conjecture U0, thanks to Proposition 3.1.

Remark 3.7. — In [**22**], as in this paper, the characteristic of F is assumed to be zero. However, with the works of Badulescu [**1, 2**] and Mínguez [**16**], this assumption seems to be superfluous, and the Tadić classification of the unitary dual of $\mathrm{GL}_m(\mathrm{D})$ should be available in arbitrary characteristic. More precisely, when F is of positive characteristic:

(1) Mínguez [**16**, §2.1.14] proved that the ring R of §1.2 is commutative;

(2) Badulescu [**2**] proved that any square integrable irreducible representation of a Levi subgroup of G_m induces irreducibly to G_m (see §1.4).

It would therefore be interesting to write down a classification of the unitary dual of $\mathrm{GL}_m(\mathrm{D})$ with no assumption on the characteristic of F.

4. Reducibility points

Let $\rho \in \mathscr{C}$ be a cuspidal irreducible representation of degree k. In this section, we determine the unramified characters χ of G_k such that the representation $\rho \times \rho\chi$ is reducible. This could provide a definition of the integer $b(\rho)$ of §1.3 without refering to the Jacquet-Langlands correspondence.

4.1. Let (U, u) be a simple type contained in ρ. According to Proposition 2.3, the normalizer N of u in G_k is generated by U and a uniformizer ϖ of the extension K. Let q_{F} denote the cardinal of the residue field of F.

Proposition 4.1. — *The group of unramified characters χ of G_k such that $\rho \simeq \rho\chi$ is finite.*

Proof. — According to [**19**, §5.1], the representation u extends to an irreducible representation $\widetilde{u}$ of N such that ρ is equivalent to the representation of G_k compactly induced from $\widetilde{u}$. Moreover, there is a bijection between the set of all representations of N extending u (which is made of all twists of $\widetilde{u}$ by a character of N trivial on U) and the set of all equivalence classes of irreducible representations of G_k whose restriction to U contains u (which is made of all classes of unramified twists of ρ). Given an unramified character χ of G_k, the representation $\rho\chi$ is compactly induced from the restriction $\widetilde{u}\chi_{|\mathrm{N}}$ and is equivalent to ρ if and only if χ is trivial on N, which happens exactly when $\chi(\varpi) = 1$. Let us define a positive integer n by:

$$\nu(\varpi) = q_{\mathrm{F}}^{-n}. \tag{4.1}$$

Then the group of unramified characters χ of G_k such that $\rho \simeq \rho\chi$ is cyclic of order n. □

Definition 4.2. — The *torsion number* of ρ, which we denote by $n(\rho)$, is the cardinal of the group of unramified characters χ of G_k such that $\rho \simeq \rho\chi$.

4.2. Let φ be a non-trivial element of the Hecke algebra $\mathscr{H}(\mathrm{G}_k, u)$ supported by the double coset $\mathrm{U}\varpi\mathrm{U}$ (which actually is a single coset). According to Propositions 2.7 and 2.9, such an element is invertible and $\mathscr{H}(\mathrm{G}_k, u)$ is the commutative $\mathbb{C}$-algebra generated by φ and φ^{-1}. Therefore, the irreducible $\mathscr{H}(\mathrm{G}_k, u)$-modules are one-dimensional and characterised, up to isomorphism, by a non-zero complex number given by the eigenvalue of φ.

Definition 4.3. — If V is an irreducible $\mathscr{H}(\mathrm{G}_k, u)$-module on which φ acts by $\lambda \in \mathbb{C}^\times$ and χ an unramified character of G_k, we will denote by $\mathrm{V}\chi$ the irreducible $\mathscr{H}(\mathrm{G}_k, u)$-module (with the same underlying space as V) on which φ acts by $\chi(\varpi)\lambda$.

Let $\mathbf{M} = \mathbf{M}_u$ denote the functor defined by (2.1) relative to the pair (U, u). It induces a bijective correspondence between the inertial class of ρ and the set of all classes of irreducible $\mathscr{H}(\mathrm{G}_k, u)$-modules.

Lemma 4.4. — *For any unramified character χ of G_k, the module $\mathbf{M}(\rho\chi)$ is equal to $\mathbf{M}(\rho)\chi^{-1}$.*

Proof. — This is proved in [**12**, §2]. The reader should pay attention to the fact that in [**12**], the symbol $\mathscr{H}(\mathrm{G}_k, u)$ has a slightly different meaning. To recover our $\mathscr{H}(\mathrm{G}_k, u)$, one has to apply the isomorphism given by [**12**, (2.3)]. □

Let (J, τ) be the type of G_{2k} which corresponds to (U, u) by (2.5). This is a G_{2k}-cover of the pair $(\mathrm{U}^2, u^{\otimes 2})$ considered as a type of the Levi subgroup $\mathrm{M} = \mathrm{G}_k \times \mathrm{G}_k$, so that we have $(\mathrm{J}_\mathrm{M}, \tau_\mathrm{M}) = (\mathrm{U}^2, u^{\otimes 2})$. Let $\mathscr{H}$ and $\mathscr{H}_\mathrm{M}$ denote the Hecke algebras relative to τ and τ_M respectively. Let $\mathbf{M}_\tau$ be the functor which corresponds to τ, let P be the parabolic subgroup of G_{2k} of upper triangular matrices relative to M and let t_P be the map given by (2.2). Let $\tilde{\mathrm{K}}$ be a finite extension of F as in Proposition 3.5 and let $q_{\tilde{\mathrm{K}}}$ be the cardinal of its residue field.

Proposition 4.5. — *Let V be an irreducible $\mathscr{H}(\mathrm{G}_k, u)$-module and let χ be an unramified character of G_k. Then the $\mathscr{H}$-module:*

$$\mathrm{Hom}_{\mathscr{H}_\mathrm{M}}(\mathscr{H}, \mathrm{V} \otimes \mathrm{V}\chi^{-1}) \tag{4.2}$$

is reducible if and only if $\chi(\varpi) = q_{\tilde{\mathrm{K}}}$ or $\chi(\varpi) = q_{\tilde{\mathrm{K}}}^{-1}$.

Proof. — Let σ be the unramified twist of ρ such that $\mathbf{M}(\sigma)$ is isomorphic to V. According to (2.3) and Lemma 4.4, we have a canonical $\mathscr{H}$-module isomorphism:

$$\mathbf{M}_{\tau}(\sigma \times \sigma\chi) \simeq \mathrm{Hom}_{\mathscr{H}_{\mathrm{M}}}(\mathscr{H}, \mathrm{V} \otimes \mathrm{V}\chi^{-1}). \tag{4.3}$$

(1) We temporarily suppose that D = F and that ρ is the trivial character of $\mathrm{GL}_1(\mathrm{F})$. In that case, we can choose for U the maximal compact subgroup of $\mathrm{F}^{\times}$ and for u the trivial character of U. We have $n(\rho) = 1$ and $\tilde{\mathrm{K}} = \mathrm{F}$, and the representation $\sigma \times \sigma\chi$ is reducible if and only if $\chi = |\ |_{\mathrm{F}}$ or $\chi = |\ |_{\mathrm{F}}^{-1}$.

(2) Let $\mathscr{I}$ denote the standard Iwahori subgroup of $\mathrm{GL}_2(\tilde{\mathrm{K}})$ and $1_{\mathscr{I}}$ its trivial character, which is the $\mathrm{GL}_2(\tilde{\mathrm{K}})$-cover associated by (2.5) to the trivial character, which we denote by $1_{\mathscr{O}_{\tilde{\mathrm{K}}}^{\times}}$, of the maximal compact subgroup of $\tilde{\mathrm{K}}^{\times}$. Let L denote the Levi subgroup $\mathrm{GL}_1(\tilde{\mathrm{K}}) \times \mathrm{GL}_1(\tilde{\mathrm{K}})$, let Q be the parabolic subgroup of $\mathrm{GL}_2(\mathrm{K})$ of upper triangular matrices relative to L and let t_{Q} be the $\mathbb{C}$-algebra homomorphism from $\mathscr{H}_{\mathrm{L}} = \mathscr{H}(\mathrm{L}, 1_{\mathscr{I} \cap \mathrm{L}})$ to $\mathscr{H}(\mathrm{GL}_2(\tilde{\mathrm{K}}), 1_{\mathscr{I}})$ corresponding to Q.

We make V into a module over $\mathscr{H}(\tilde{\mathrm{K}}^{\times}, 1_{\mathscr{O}_{\tilde{\mathrm{K}}}^{\times}})$ by fixing a $\mathbb{C}$-algebra isomorphism (2.8), which allows us, according to Proposition 3.5, to identify the $\mathscr{H}$-module (4.2) with the $\mathscr{H}(\mathrm{GL}_2(\tilde{\mathrm{K}}), 1_{\mathscr{I}})$-module:

$$\mathrm{Hom}_{\mathscr{H}_{\mathrm{L}}}(\mathscr{H}(\mathrm{GL}_2(\tilde{\mathrm{K}}), 1_{\mathscr{I}}), \mathrm{V} \otimes \mathrm{V}\tilde{\chi}^{-1}), \tag{4.4}$$

where $\tilde{\chi}$ denotes the unramified character of $\tilde{\mathrm{K}}^{\times}$ which takes the same value as χ on ϖ. According to case 1, this module is reducible if and only if $\tilde{\chi} = |\ |_{\tilde{\mathrm{K}}}$ or $\tilde{\chi} = |\ |_{\tilde{\mathrm{K}}}^{-1}$, which amounts to saying that (4.4) is reducible if and only if $\chi(\varpi) = q_{\tilde{\mathrm{K}}}$ or $\chi(\varpi) = q_{\tilde{\mathrm{K}}}^{-1}$. This gives us the required result. □

Let $f(\rho)$ denote the residue degree of $\tilde{\mathrm{K}}$ over F. We state the main result of this section.

Theorem 4.6. — *Let $s \in \mathbb{C}$. Then $\rho \times \rho\nu^s$ is reducible if and only if:*

$$s = f(\rho)n(\rho)^{-1} \quad \text{or} \quad s = -f(\rho)n(\rho)^{-1}.$$

Proof. — We apply Proposition 4.5 with the unramified character $\chi = \nu^s$. The result follows from the definition of $n(\rho)$ by (4.1). □

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