SMOOTH REPRESENTATIONS OF $\mathrm{GL}_m(\mathrm{D})$ VI : SEMISIMPLE TYPES

by

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Abstract. — We give a complete description of the category of smooth complex representations of the multiplicative group of a central simple algebra over a locally compact nonarchimedean local field. More precisely, for each inertial class in the Bernstein spectrum, we construct a type and compute its Hecke algebra. The Hecke algebras that arise are all naturally isomorphic to products of affine Hecke algebras of type A. We also prove that, for cuspidal classes, the simple type is unique up to conjugacy.

Contents

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Introduction

In [**12**], Bushnell and Kutzko described a general approach to understanding the category of smooth (complex) representations of a reductive p-adic group G: the theory of *types*. This is based on the Bernstein decomposition of the category [**1**] into indecomposable full subcategories, indexed by pairs (L, π), with L a Levi subgroup of G and π an irreducible cuspidal representation of L, up to a certain equivalence relation. A *type* for such a subcategory is a pair (K, ρ), with K a compact open subgroup of G and ρ an irreducible representation of K, such that the irreducible representations in the subcategory are precisely those irreducible representations of G which contain ρ. In this case, there is an equivalence of categories between the subcategory and the category of left modules over the spherical Hecke algebra $\mathscr{H}(\mathrm{G}, \rho)$. Thus one can classify the representations of G by first constructing a type for each subcategory, and then computing the spherical Hecke algebras.

This programme has been completed for general linear groups over a p-adic field (Bushnell–Kutzko [**8, 13**]), for special linear groups over a p-adic field (Bushnell–Kutzko [**9, 10**], Goldberg–Roche [**14, 15**]) up to the computation of a two-cocycle in the description of the Hecke algebra, and for three-dimensional unitary groups in odd residual characteristic (Blasco [**2**]). In this paper, following previous work in [**19, 20, 21, 23, 6**], we complete the programme for inner forms of general linear groups. The Hecke algebras that arise are all naturally isomorphic to products of affine Hecke algebras of type A.

Let D be a division algebra over a locally compact nonarchimedean local field F, and let $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$, with m a positive integer; we will also think of G as the group of automorphisms of a right D-vector space V. Denote by $\mathfrak{R}(\mathrm{G})$ the category of smooth complex representations of G. In order to describe our results more precisely, we begin by recalling the Bernstein decomposition [**1**], in the language of [**12**]. For L a Levi subgroup of G, denote by $\mathrm{X}(\mathrm{L})$ the complex torus of *unramified characters* of L: that is, smooth homomorphisms $\mathrm{L} \to \mathbb{C}^\times$ which are trivial on all compact subgroups of L. For π an irreducible cuspidal representation of L, we write $[\mathrm{L}, \pi]_\mathrm{G}$ for the G-*inertial equivalence class* of (L, π): that is, the set of pairs (L', π'), consisting of a Levi subgroup L' and an irreducible cuspidal representation π' of L', such that (L, π) and $(\mathrm{L}', \pi' \otimes \chi')$ are G-conjugate, for some unramified character $\chi' \in \mathrm{X}(\mathrm{L}')$. We denote by $\mathfrak{B}(\mathrm{G})$ the set of G-inertial equivalence classes of pairs (L, π) as above.

To $\mathfrak{s} = [\mathrm{L}, \pi]_{\mathrm{G}}$ an inertial equivalence class we associate a full subcategory $\mathfrak{R}^{\mathfrak{s}}(\mathrm{G})$ of $\mathfrak{R}(\mathrm{G})$, whose objects are those representations all of whose subquotients have cuspidal support in $\mathfrak{s}$. Then the Bernstein decomposition says that

$$\mathfrak{R}(\mathrm{G}) = \prod_{\mathfrak{s}\in\mathfrak{B}(\mathrm{G})} \mathfrak{R}^{\mathfrak{s}}(\mathrm{G}).$$

Bushnell–Kutzko's theory of types [**12**] is a strategy to understand the subcategories in this decomposition. For $\mathfrak{s} \in \mathfrak{B}(\mathrm{G})$, an *$\mathfrak{s}$-type* is a pair (K, ρ), with K a compact open subgroup of G and $(\rho, \mathscr{W})$ an irreducible smooth representation of K, such that the irreducible representations in $\mathfrak{R}^{\mathfrak{s}}(\mathrm{G})$ are precisely those irreducible representations of G which contain ρ. In that case, there is an equivalence of categories

$$\begin{aligned} \mathfrak{R}^{\mathfrak{s}}(\mathrm{G}) &\rightarrow \mathscr{H}(\mathrm{G}, \rho)\text{-Mod} \\ \mathscr{V} &\mapsto \mathrm{Hom}_{\mathrm{K}}(\mathscr{W}, \mathscr{V}), \end{aligned}$$

where $\mathscr{H}(\mathrm{G}, \rho)$ is the convolution algebra of compactly supported $\mathrm{End}_{\mathbb{C}}(\check{\mathscr{W}})$-valued function f of G which satisfy $f(hgk) = \check{\rho}(h)f(g)\check{\rho}(k)$, for $h, k \in \mathrm{K}$, $g \in \mathrm{G}$ (the spherical Hecke algebra).

Our main result is the following:

Main Theorem. — *Let $\mathfrak{s} \in \mathfrak{B}(\mathrm{G})$. There exists an $\mathfrak{s}$-type (K, ρ) in* G, *such that*

$$\mathscr{H}(\mathrm{G}, \rho) \cong \bigotimes_{i=1}^{l} \mathscr{H}(r_i, q_{\mathrm{F}}^{f_i}),$$

for some positive integers l and r_i, f_i, for $1 \le i \le l$.

Here q_{F} denotes the cardinality of the residue field of F, and $\mathscr{H}(n, q)$ is the affine Hecke algebra of type $\tilde{A}_{n-1}$ with parameter q. In particular, the category of modules of such algebras is completely understood, through the work of Kazhdan–Lusztig [**18**]. The values of l, r_i, f_i can be described as follows.

To π an irreducible cuspidal representation of $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$, we may associate two invariants. First there is the *torsion number* $n(\pi)$, the (finite) number of unramified characters χ of G such that $\pi \simeq \pi\chi$. Second, we have the *reducibility number* $s(\pi)$: writing $\tilde{\mathrm{G}} = \mathrm{GL}_{2m}(\mathrm{D})$ and $\tilde{\mathrm{P}}$ for the standard parabolic subgroup of $\tilde{\mathrm{G}}$ with Levi subgroup $\mathrm{G}\times\mathrm{G}$, it is the unique positive real number such that the induced representation $\mathrm{Ind}_{\tilde{\mathrm{P}}}^{\tilde{\mathrm{G}}} \pi \otimes \pi\nu^{s(\pi)}$ (with respect to normalized parabolic induction) is reducible, where $\nu(g) = |\mathrm{Nrd}(g)|_{\mathrm{F}}$, Nrd denotes the reduced norm of A over F, and $|\cdot|_{\mathrm{F}}$ is the normalized absolute value on F (see [**22**, Theorem 4.6]).

Now suppose $\mathfrak{s} = [\mathrm{L}, \pi]_{\mathrm{G}}$ is an inertial equivalence class. The Levi subgroup L is the stabilizer of some decomposition $\mathrm{V} = \bigoplus_{j=1}^{r} \mathrm{V}^j$ into subspaces, which gives an identification of L with $\prod_{j=1}^{r} \mathrm{GL}_{m_j}(\mathrm{D})$, where $m_j = \dim_{\mathrm{D}} \mathrm{V}^j$. We can then write $\pi = \bigotimes_{j=1}^{r} \pi_j$, for π_j an irreducible cuspidal representation of $\mathrm{G}_j = \mathrm{GL}_{m_j}(\mathrm{D})$. We define an equivalence relation on $\{1, \ldots, r\}$ by

$$(*) \qquad j \sim k \iff m_j = m_k \text{ and } [\mathrm{G}_j, \pi_j]_{\mathrm{G}_j} = [\mathrm{G}_k, \pi_k]_{\mathrm{G}_k},$$

where we have identified G_j with G_k whenever $m_j = m_k$. We may, and do, assume that $\pi_j = \pi_k$ whenever $j \sim k$, since this does not change the inertial class $\mathfrak{s}$. Denote by $S_1, \ldots, S_l$ the equivalence classes. For $i = 1, \ldots, l$, we set

$$r_i = \# S_i, \qquad f_i = n(\pi_j)s(\pi_j), \text{ for any } j \in S_i.$$

These are then the parameters appearing in the Hecke algebras of the Main Theorem.

We now describe in more detail the construction of the types. The starting point is the construction of the irreducible cuspidal representations of G, which was achieved in [**21, 23**]. In [**21**], generalizing the work of Bushnell–Kutzko for $\mathrm{GL}_n(\mathrm{F})$ [**8**], the first author constructed a set of pairs (J, λ) called *simple types*. Amongst these are the *maximal simple types*, which give rise to cuspidal representations: if (J, λ) is a maximal simple type then λ extends to a representation $\tilde{\lambda}$ of its normalizer $\tilde{\mathrm{J}}$ and the compactly-induced representation $\text{c-Ind}_{\tilde{\mathrm{J}}}^{\mathrm{G}} \tilde{\lambda}$ is irreducible and cuspidal. The main result of [**23**] is that all irreducible cuspidal representations of G arise in this way. Here we prove more:

Theorem A. — *Let π be an irreducible cuspidal representation of* G *and* $\mathfrak{s} = [\mathrm{G}, \pi]_{\mathrm{G}}$. *There is a maximal simple type* (J, λ) *contained in* π, *and any such is an* $\mathfrak{s}$*-type. Moreover, it is unique up to* G*-conjugacy: that is, if* $(\mathrm{J}_i, \lambda_i)$, *for* $i = 1, 2$, *are maximal simple types contained in* π *then there exists* $g \in \mathrm{G}$ *such that* ${}^g\mathrm{J}_1 = \mathrm{J}_2$ *and* ${}^g\lambda_1 = \lambda_2$.

The uniqueness (up to conjugacy) is proved in Corollary 6.2. We remark that, in the case of $\mathrm{GL}_n(\mathrm{F})$, the uniqueness here follows from an "intertwining implies conjugacy" result for simple types. In the case of G there is no such result for two reasons: Firstly there is an extra invariant of a simple type, the *embedding type* (see paragraph 2.1) and simple types with different embedding types may intertwine but be non-conjugate. Secondly there is an action of a galois group on simple types, and any two types in the same orbit will intertwine but may be non-conjugate; this phenomenon arises already for level zero representations in [**17**]. Nonetheless, the embedding type of a *sound simple type* (see §6) is determined by any irreducible representation containing it. Moreover, in the case of maximal simple types (which are always sound), the galois action can be realized by conjugation by an element of the normalizer of J, and the uniqueness follows from this.

We turn to the case of a general G-inertial equivalence class $\mathfrak{s} = [\mathrm{L}, \pi]_{\mathrm{G}}$, for which we have the corresponding cuspidal L-inertial equivalence class $\mathfrak{s}_{\mathrm{L}} = [\mathrm{L}, \pi]_{\mathrm{L}}$. In [**12**], Bushnell and Kutzko present a general framework for constructing an $\mathfrak{s}$-type from an $\mathfrak{s}_{\mathrm{L}}$-type $(\mathrm{J}_{\mathrm{L}}, \lambda_{\mathrm{L}})$: the theory of *covers*. We do not recall precisely the definition of a cover here, only that it should be a pair $(\mathrm{J}_{\mathrm{G}}, \lambda_{\mathrm{G}})$ which has an Iwahori decomposition with respect to any parabolic subgroup with Levi component L, such that the Hecke algebra $\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{G}})$ contains a suitable invertible element. If one has such a cover $(\mathrm{J}_{\mathrm{G}}, \lambda_{\mathrm{G}})$ then it is an $\mathfrak{s}$-type.

The normalizer $\mathrm{N}_{\mathrm{G}}(\mathrm{L})$ acts on $\mathfrak{B}(\mathrm{L})$ by conjugation and there is a Levi subgroup M of G which is minimal for the property of containing the $\mathrm{N}_{\mathrm{G}}(\mathrm{L})$-stabilizer of $\mathfrak{s}_{\mathrm{L}}$. Then we also have an M-inertial equivalence class $\mathfrak{s}_{\mathrm{M}} = [\mathrm{L}, \pi]_{\mathrm{M}}$. The strategy now is first to construct a cover $(\mathrm{J}_{\mathrm{M}}, \lambda_{\mathrm{M}})$ of $(\mathrm{J}_{\mathrm{L}}, \lambda_{\mathrm{L}})$, and then a cover $(\mathrm{J}_{\mathrm{G}}, \lambda_{\mathrm{G}})$ of $(\mathrm{J}_{\mathrm{M}}, \lambda_{\mathrm{M}})$ – by transitivity of covers, this will give the required $\mathfrak{s}$-type.

In our situation, we do indeed have an $\mathfrak{s}_{\mathrm{L}}$-type: Writing $\mathrm{L} = \prod_{j=1}^{r} \mathrm{GL}_{m_j}(\mathrm{D})$ and $\pi = \bigotimes_{j=1}^{r} \pi_j$ as above, for π_j an irreducible cuspidal representation of $\mathrm{G}_j = \mathrm{GL}_{m_j}(\mathrm{D})$, there is a maximal simple type $(\mathrm{J}_j, \lambda_j)$ which is a $[\mathrm{G}_j, \pi_j]_{\mathrm{G}_j}$-type, by Theorem A. Then, putting

$$\mathrm{J}_{\mathrm{L}} = \prod_{j=1}^{r} \mathrm{J}_j, \qquad \lambda_{\mathrm{L}} = \bigotimes_{j=1}^{r} \lambda_j,$$

it is clear that $(\mathrm{J}_{\mathrm{L}}, \lambda_{\mathrm{L}})$ is an $\mathfrak{s}_{\mathrm{L}}$-type. The Levi subgroup M is then that defined by the equivalence relation $(*)$: it is the stabilizer of the decomposition $\mathrm{V} = \bigoplus_{i=1}^{l} \mathrm{Y}^i$, where $\mathrm{Y}^i = \bigoplus_{j \in S_i} \mathrm{V}^j$.

The first case to consider is when $\mathrm{M} = \mathrm{G}$, that is, the case $l = 1$ of the Main Theorem. In this situation we have the following result, which summarizes [**21**, Proposition 5.5, Théorème 4.6] and [**23**, Théorème 5.23]:

Theorem B. — *Let* π_0 *be an irreducible cuspidal representation of* $\mathrm{G}_0 = \mathrm{GL}_{m_0}(\mathrm{D})$*, with* $m = rm_0$*, and let* $(\mathrm{J}_0, \lambda_0)$ *be a maximal simple type which is a* $[\mathrm{G}_0, \pi_0]_{\mathrm{G}_0}$*-type. Let* $\mathrm{L} = {\mathrm{G}_0}^r$ *be a Levi subgroup of* G *with irreducible cuspidal representation* $\pi = {\pi_0}^{\otimes r}$*, and put* $\mathfrak{s} = [\mathrm{L}, \pi]_{\mathrm{G}}$*. Put* $\mathrm{J_L} = {\mathrm{J}_0}^r$ *and* $\lambda_{\mathrm{L}} = {\lambda_0}^{\otimes r}$*. Then there is an* $\mathfrak{s}$*-type* $(\mathrm{J_G}, \lambda_{\mathrm{G}})$ *which is a cover of* $(\mathrm{J_L}, \lambda_{\mathrm{L}})$*, and*

$$\mathscr{H}(\mathrm{G}, \lambda_{\mathrm{G}}) \simeq \mathscr{H}(r, q_{\mathrm{F}}^f),$$

where $f = n(\pi_0)s(\pi_0)$*. Moreover, there is a simple type* (J, λ) *in* G *such that* $\lambda = \mathrm{Ind}_{\mathrm{J_G}}^{\mathrm{J}} \lambda_{\mathrm{G}}$*.*

Now we turn to the general case of arbitrary M. In order to describe the covering process, we need to recall some detail of the structure of simple types.

Let (J, λ) be a simple type contained in an irreducible representation π of G. There is a particular filtration of pro-p subgroups $\{\mathrm{H}^{t+1} : t \geq 0\}$ of J such that λ restricts to a multiple of a character $\theta^{(t)}$ on H^{t+1}, and $\theta^{(0)} \mid \mathrm{H}^{t+1} = \theta^{(t)}$. These characters are called *simple characters of level* t. Simple characters were the main object of study of [**19, 6**] and they exhibit remarkable functorial properties, as in the case D = F [**8, 7**]. In particular, it is possible to transfer them to the multiplicative group of other central simple F-algebras. A convenient and powerful way to express this is in terms of *endo-classes* (see [**7, 6**] and §4): the simple character $\theta^{(t)}$ determines and endo-class $\Theta^{(t)}$, which depends only on the representation π.

Now let $\pi = \bigotimes_{j=1}^r \pi_j$ be a cuspidal representation of L as above and denote by $\Theta_j^{(t)}$ the endo-class of level t determined by π_j. (We are assuming a normalization of the index in the filtrations.) For each integer $t \geq 0$, we define an equivalence relation on $\{1, \dots, r\}$ by

$$j \sim_t k \Longleftrightarrow \Theta_j^{(t)} \text{ is endo-equivalent to } \Theta_k^{(t)}.$$

As for the equivalence relation $(*)$, this determines a Levi subgroup M_t. Note that

$$\mathrm{M} \subseteq \mathrm{M}_0; \qquad \mathrm{M}_t \subseteq \mathrm{M}_{t'}, \text{ for } t \geq t'; \qquad \text{and } \mathrm{M}_t = \mathrm{G}, \text{ for sufficiently large } t.$$

It is useful to extend the notation and put $\mathrm{M}_t = \mathrm{M}$ for $t < 0$. Of course, although the Levi subgroups are indexed by an integer t, there are only finitely many in $\{\mathrm{M}_t : t \in \mathbb{Z}\}$.

Theorem B provides a cover $(\mathrm{J_M}, \lambda_{\mathrm{M}})$ of $(\mathrm{J_L}, \lambda_{\mathrm{L}})$ in M and the Main Theorem now follows from the transitivity of covers and:

Theorem C. — *For* $t \geq t'$*, there are a cover* $(\mathrm{J}_{\mathrm{M}_{t'}}, \lambda_{\mathrm{M}_{t'}})$ *of* $(\mathrm{J}_{\mathrm{M}_t}, \lambda_{\mathrm{M}_t})$ *in* G *and a support-preserving isomorphism of Hecke algebras* $\mathscr{H}(\mathrm{M}_t, \lambda_{\mathrm{M}_t}) \simeq \mathscr{H}(\mathrm{M}_{t'}, \lambda_{\mathrm{M}_{t'}})$*.*

As always in the theory of covers, the difficulty is in defining the groups $\mathrm{J}_{\mathrm{M}_t}$. In fact, many covers were already constructed in [**23**, §4] and we must show that we can put ourselves in the situation of *loc. cit.*. For this, we need to use the notion of a *common approximation* of simple characters from [**13**], which is essentially a reinterpretation of the notion of endo-class.

We end the introduction with a summary of the contents of each section. Section 1 consists of basic definitions, as well as recalling a very useful technique from [**6**] for reducing proofs to easier situations. Section 2 concerns simple strata and pairs, while section 3 concerns simple characters; these are the technical heart of the paper, in particular the translation principle Theorem 3.3 which is needed to cope with the fact that a simple character may be defined relative to several inequivalent simple strata. Along the way, we prove a generalization of a conjecture in [**6**] on the embedding type of a simple character. Section 4 concerns the relationship between endo-classes and common approximations. Section 5 recalls basic results about simple types but in the more general situation of lattice sequences which is needed later, and we prove the uniqueness results in section 6. Finally, the general construction of a cover is given in section 7.

Much of the material here is necessarily technical. A reader who is already familiar with the situation (and common notations) for the case $\mathrm{D} = \mathrm{F}$ and is interested only in seeing the main results could probably get by reading only sections 6 and 7.

1. Notation and preliminaries

Let F be a nonarchimedean locally compact field. For K a finite extension of F, or more generally a division algebra over a finite extension of F, we denote by $\mathcal{O}_\mathrm{K}$ its ring of integers, by $\mathfrak{p}_\mathrm{K}$ the maximal ideal of $\mathcal{O}_\mathrm{K}$ and by k_K its residue field.

For u a real number, we denote by $\lceil u \rceil$ the smallest integer which is greater than or equal to u, and by $\lfloor u \rfloor$ the greatest integer which is smaller than or equal to u, that is, its integer part.

All representations considered are smooth and complex.

1.1. Let A be a simple central F-algebra, and let V be a simple left A-module. The algebra $\mathrm{End}_\mathrm{A}(\mathrm{V})$ is an F-division algebra, the opposite of which we denote by D. Considering

V as a right D-vector space, we have a canonical isomorphism of F-algebras between A and $\mathrm{End}_{\mathrm{D}}(\mathrm{V})$.

Definition 1.1. — An $\mathcal{O}_{\mathrm{D}}$-*lattice sequence* on V is a map

$$\Lambda : \mathbb{Z} \to \{\mathcal{O}_{\mathrm{D}}\text{-lattices of V}\}$$

which is decreasing (that is, $\Lambda(k) \supseteq \Lambda(k+1)$ for all $k \in \mathbb{Z}$) and such that there exists a positive integer $e = e(\Lambda|\mathcal{O}_{\mathrm{D}})$ satisfying $\Lambda(k+e) = \Lambda(k)\mathfrak{p}_{\mathrm{D}}$, for all $k \in \mathbb{Z}$. This integer is called the $\mathcal{O}_{\mathrm{D}}$-*period* of Λ over $\mathcal{O}_{\mathrm{D}}$.

If $\Lambda(k) \supsetneq \Lambda(k+1)$ for all $k \in \mathbb{Z}$, then the lattice sequence Λ is said to be *strict*.

Associated with an $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ on V, we have an $\mathcal{O}_{\mathrm{F}}$-lattice sequence on A defined by:

$$\mathfrak{P}_k(\Lambda) = \{a \in \mathrm{A} \mid a\Lambda_i \subseteq \Lambda_{i+k},\ i \in \mathbb{Z}\}, \quad k \in \mathbb{Z}.$$

The lattice $\mathfrak{A}(\Lambda) = \mathfrak{P}_0(\Lambda)$ is a hereditary $\mathcal{O}_{\mathrm{F}}$-order in A, and $\mathfrak{P}(\Lambda) = \mathfrak{P}_1(\Lambda)$ is its Jacobson radical. They depend only on the set $\{\Lambda(k) \mid k \in \mathbb{Z}\}$.

We denote by $\mathfrak{K}(\Lambda)$ the $\mathrm{A}^\times$-*normalizer* of Λ, that is the subgroup of $\mathrm{A}^\times$ made of all elements $g \in \mathrm{A}^\times$ for which there is an integer $n \in \mathbb{Z}$ such that $g(\Lambda(k)) = \Lambda(k+n)$ for all $k \in \mathbb{Z}$. Given $g \in \mathfrak{K}(\Lambda)$, such an integer is unique: it is denoted $\upsilon_\Lambda(g)$ and called the Λ-*valuation* of g. This defines a group homomorphism υ_Λ from $\mathfrak{K}(\Lambda)$ to $\mathbb{Z}$. Its kernel, denoted $\mathrm{U}(\Lambda)$, is the group of invertible elements of $\mathfrak{A}(\Lambda)$. We set $\mathrm{U}_0(\Lambda) = \mathrm{U}(\Lambda)$ and, for $k \geqslant 1$, we set $\mathrm{U}_k(\Lambda) = 1 + \mathfrak{P}_k(\Lambda)$.

1.2. Let E be a finite extension of F contained in A. We denote by $e(\mathrm{E}/\mathrm{F})$ and $f(\mathrm{E}/\mathrm{F})$ the ramification index and residue class degree respectively.

An $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ on V is said to be E-*pure* if it is normalized by $\mathrm{E}^\times$. The centralizer of E in A, denoted B, is a simple central E-algebra. We fix a simple left B-module V_{E} and write D_{E} for the algebra opposite to $\mathrm{End}_{\mathrm{B}}(\mathrm{V}_{\mathrm{E}})$. By [**23**, Théorème 1.4] (see also [**3**, Theorem 1.3]), given an E-pure $\mathcal{O}_{\mathrm{D}}$-lattice sequence on V, there is an $\mathcal{O}_{\mathrm{D}_{\mathrm{E}}}$-lattice sequence Γ on V_{E} such that:

$$\mathfrak{P}_k(\Lambda) \cap \mathrm{B} = \mathfrak{P}_k(\Gamma), \quad k \in \mathbb{Z}.$$

It is unique up to translation of indices, and its $\mathrm{B}^\times$-normalizer is $\mathfrak{K}(\Lambda) \cap \mathrm{B}^\times$.

Definition 1.2. — A *stratum* in A is a quadruple $[\Lambda, n, m, \beta]$ made of an $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ on V, two integers m, n such that $0 \leqslant m \leqslant n-1$ and an element $\beta \in \mathfrak{P}_{-n}(\Lambda)$.

For $i = 1, 2$, let $[\Lambda, n, m, \beta_i]$ be a stratum in A. We say these two strata are *equivalent* if $\beta_2 - \beta_1 \in \mathfrak{P}_{-m}(\Lambda)$.

1.3. Given a stratum $[\Lambda, n, m, \beta]$ in A, we denote by E the F-algebra generated by β. This stratum is said to be *pure* if E is a field, if Λ is E-pure and if $\upsilon_{\Lambda}(\beta) = -n$. Given a pure stratum $[\Lambda, n, m, \beta]$, we denote by B the centralizer of E in A. For $k \in \mathbb{Z}$, we set:

$$\mathfrak{n}_k(\beta, \Lambda) = \{x \in \mathfrak{A}(\Lambda) \mid \beta x - x\beta \in \mathfrak{P}_k(\Lambda)\}.$$

The smallest integer $k \geqslant \upsilon_{\Lambda}(\beta)$ such that $\mathfrak{n}_{k+1}(\beta, \Lambda)$ is contained in $\mathfrak{A}(\Lambda) \cap \mathrm{B} + \mathfrak{P}(\Lambda)$ is called the *critical exponent* of the stratum $[\Lambda, n, m, \beta]$, denoted $k_0(\beta, \Lambda)$.

Definition 1.3. — The stratum $[\Lambda, n, m, \beta]$ is said to be *simple* if it is pure and if we have $m \leqslant -k_0(\beta, \Lambda) - 1$.

Given $n \geqslant 0$ and Λ an $\mathcal{O}_{\mathrm{D}}$-lattice sequence, there is another stratum which plays a very similar role to simple strata, namely the *null stratum* $[\Lambda, n, n-1, 0]$. In particular, a simple stratum $[\Lambda, n, n-1, \beta]$ is equivalent to a null stratum if and only if $\beta \in \mathfrak{P}_{1-n}(\Lambda)$.

1.4. Let $[\Lambda, n, m, \beta]$ be a simple stratum in A. The *affine class* of Λ is the set of all $\mathcal{O}_{\mathrm{D}}$-lattice sequences on V of the form:

$$a\Lambda + b : k \mapsto \Lambda(\lceil (k-b)/a \rceil),$$

with $a, b \in \mathbb{Z}$ and $a \geqslant 1$. The period of $a\Lambda + b$ is a times the period $e(\Lambda|\mathcal{O}_{\mathrm{D}})$ of Λ. The *affine class* of the stratum $[\Lambda, n, m, \beta]$ is the set of all (simple) strata of the form

$$[\Lambda', n', m', \beta],$$

where $\Lambda' = a\Lambda + b$ is in the affine class of Λ, $n' = an$ and m' is any integer such that $\lfloor m'/a \rfloor = m$.

In the course of the paper, there will be several objects associated to a simple stratum $[\Lambda, n, m, \beta]$, in particular simple characters (see §3). By a straightforward induction (see [**6**, Lemma 2.2]), these objects depend only on the affine class of the stratum.

1.5. This article makes use of a number of results of Grabitz [**16**] which are based on the following definition.

Definition 1.4. — A simple stratum $[\Lambda, n, m, \beta]$ in A is *sound* if Λ is strict, $\mathfrak{A} \cap \mathrm{B}$ is principal and $\mathfrak{K}(\mathfrak{A}) \cap \mathrm{B}^\times = \mathfrak{K}(\mathfrak{A} \cap \mathrm{B})$, where $\mathfrak{A} = \mathfrak{P}_0(\Lambda)$ is the hereditary $\mathcal{O}_\mathrm{F}$-order defined by Λ.

The condition on $\mathfrak{A} \cap \mathrm{B}$ forces $\mathfrak{A}$ to be a principal $\mathcal{O}_\mathrm{F}$-order. In the split case, a simple stratum $[\Lambda, n, m, \beta]$ is sound if and only if Λ is strict and $\mathfrak{A}$ is principal.

1.6. In [**6**, §2.7], we developed a process to reduce many proofs to the case of sound strata, which we recall briefly here: Let $[\Lambda, n, m, \beta]$ be a simple stratum in A and let e denote the period of Λ over $\mathcal{O}_\mathrm{D}$. Write B for the centralizer of the field $\mathrm{E} = \mathrm{F}(\beta)$ in A, fix a simple left B-module V_E and write D_E for the E-algebra opposite to the algebra of B-endomorphisms of V_E. Let Γ denote an $\mathcal{O}_{\mathrm{D}_\mathrm{E}}$-lattice sequence on V_E such that $\mathfrak{P}_k(\Lambda) \cap \mathrm{B} = \mathfrak{P}_k(\Gamma)$ for $k \in \mathbb{Z}$, and let e' denote its period over $\mathcal{O}_{\mathrm{D}_\mathrm{E}}$. We fix an integer l which is a multiple of e and e' and set:

$$\Lambda^\ddagger : k \mapsto \Lambda(k) \oplus \Lambda(k+1) \oplus \cdots \oplus \Lambda(k+l-1),$$

which is a strict $\mathcal{O}_\mathrm{D}$-lattice sequence on $\mathrm{V}^\ddagger = \mathrm{V} \oplus \cdots \oplus \mathrm{V}$ (l times). Thus we can form the simple stratum $[\Lambda^\ddagger, n, m, \beta]$ in $\mathrm{A}^\ddagger = \mathrm{End}_\mathrm{D}(\mathrm{V}^\ddagger)$, where β is the block diagonal element $\mathrm{diag}(\beta, \dots, \beta) \in \mathrm{A}^l \subseteq \mathrm{A}^\ddagger$. By [**6**, Lemma 2.17], the stratum $[\Lambda^\ddagger, n, m, \beta]$ is sound.

As we have mentioned, there will be several objects associated to a simple stratum $[\Lambda, n, m, \beta]$ through the course of the paper. If one identifies A with the "$(1,1)$-block" of $\mathrm{A}^\ddagger$ and intersect (or restrict) these objects for $[\Lambda^\ddagger, n, m, \beta]$ to A one recovers the corresponding objects for $[\Lambda, n, m, \beta]$ (see, for example, [**23**, Théorème 2.17]). Using this, in several proofs we write something like: "by passing to $\Lambda^\ddagger$ we may assume we are in the sound case" (Lemma 2.11, Proposition 2.14, Lemma 3.5)). By this we mean that we may prove the result for the sound stratum $[\Lambda^\ddagger, n, m, \beta]$ and then deduce the result for $[\Lambda, n, m, \beta]$ by intersecting with A. In general, it is safe to do this provided there is no conjugation involved in the statement. An example of this is given already in [**6**, Theorem 4.16].

2. Simple strata and simple pairs

2.1. We begin by recalling the definition of a *type* of embedding, from [**5, 6**].

We fix a simple central F-algebra A and a simple left A-module V, and denote by D the opposite algebra of $\mathrm{End}_{\mathrm{A}}(\mathrm{V})$. An *embedding* in A is a pair (E,Λ) consisting of a finite field extension E of F contained in A and an E-pure $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ on V. Given such a pair, we denote by $\mathrm{E}^{\diamond}$ the maximal finite unramified extension of F which is contained in E and whose degree divides the reduced degree of D over F.

Two embeddings (E_i,Λ_i) are *equivalent* if there is an element $g\in\mathrm{G}$ such that Λ_1 is in the translation class of $g\Lambda_2$ and $\mathrm{E}_1^{\diamond}=g\mathrm{E}_2^{\diamond}g^{-1}$. An equivalence class for this relation is called an *embedding type* in A.

Lemma 2.1. — *Let* (E,Λ) *be an embedding and put* $e=e(\mathrm{E}/F)$, $f=f(\mathrm{E}/\mathrm{F})$. *Let* E' *be a finite field extension of* F *such that* $e(\mathrm{E}'/F)=e$ *and* $f(\mathrm{E}'/F)=f$. *Then there is an embedding* $\iota:\mathrm{E}'\hookrightarrow\mathrm{A}$ *such that* $(\iota(\mathrm{E}'),\Lambda)$ *is an embedding with the same embedding type as* (E,Λ).

Proof. — When Λ is strict, this is [**5**, Corollary 3.16(ii)]. For the general case, we fix a simple right $\mathrm{E}\otimes_{\mathrm{F}}\mathrm{D}$-module S and put $\mathrm{A}(\mathrm{S})=\mathrm{End}_{\mathrm{D}}(\mathrm{S})$. Let B be the commutant of E in A, and let D_{E} be the commutant of E in A(S). We also fix a decomposition $\mathrm{V}=\mathrm{V}^1\oplus\cdots\oplus\mathrm{V}^l$ into simple right $\mathrm{E}\otimes_{\mathrm{F}}\mathrm{D}$-modules (which are all copies of S) such that the lattice sequence Λ decomposes into the direct sum of the $\Lambda^j=\Lambda\cap\mathrm{V}^j$, for $j\in\{1,\dots,l\}$. From [**21**, §1.3], after choosing identifications $\mathrm{V}^j\simeq\mathrm{S}$ for each j, we have an F-algebra embedding $\iota:\mathrm{A}(\mathrm{S})\to\mathrm{A}$. Denote by $\mathfrak{S}$ the unique (up to translation) E-pure strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence on S. By the strict case, there is an embedding $\rho:\mathrm{E}'\hookrightarrow\mathrm{A}(\mathrm{S})$ such that $(\rho(\mathrm{E}'),\mathfrak{S})$ is an embedding with the same embedding type as $(\mathrm{E},\mathfrak{S})$. (Indeed, by [**6**, Remark 2.12], any embedding $(\rho(\mathrm{E}'),\mathfrak{S})$ has the same embedding type as $(\mathrm{E},\mathfrak{S})$.) By conjugating the embedding, we may assume $\rho(\mathrm{E}')^{\diamond}=\mathrm{E}^{\diamond}$. Then the embedding $\iota\circ\rho$ has the required property. □

2.2. We recall the definitions of *simple pair* and *endo-equivalence* from [**6**, Definitions 1.4, 1.7] (see also [**7**, Definition 1.5]):

Definition 2.2. — A *simple pair* over F is a pair (k,β) consisting of a non-zero element β of some finite extension of F and an integer $0\leqslant k\leqslant -k_{\mathrm{F}}(\beta)-1$.

Let A be a simple central F-algebra and V be a simple left A-module. A *realization* of a simple pair (k,β) in A is a stratum in A of the form $[\Lambda,n,m,\varphi(\beta)]$ made of:

(i) a homomorphism φ of F-algebra from $\mathrm{F}(\beta)$ to A;
(ii) an $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ on V normalized by the image of $\mathrm{F}(\beta)^{\times}$ under φ;
(iii) an integer m such that $\lfloor m/e_{\varphi(\beta)}(\Lambda)\rfloor = k$.

The integer $-n$ is then the Λ-valuation of $\varphi(\beta)$. By [**19**, Proposition 2.25] we have:

$$k_0(\varphi(\beta),\Lambda) = e_{\varphi(\beta)}(\Lambda)k_{\mathrm{F}}(\beta),$$

which implies that any realization of a simple pair is a simple stratum.

Definition 2.3. — (i) For $i = 1, 2$, let (k_i, β_i) be a simple pair over F. We say that these pairs are *endo-equivalent*, denoted:

$$(k_1,\beta_1) \approx (k_2,\beta_2),$$

if $k_1 = k_2$ and $[\mathrm{F}(\beta_1) : \mathrm{F}] = [\mathrm{F}(\beta_2) : \mathrm{F}]$, and if there exists a simple central F-algebra A together with realizations $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ of (k_i, β_i) in A, with $i = 1, 2$, which intertwine in A.

This defines an equivalence relation on simple pairs, from the following Proposition:

Proposition 2.4 **(**[**6**, Propositions 1.7, 1.9]**)**. — *For $i = 1, 2$, let (k, β_i) be a simple pair over* F*, and suppose these pairs are endo-equivalent. Let* A *be a simple central* F*-algebra and let $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ be a realization of (k, β_i) in* A*, for $i = 1, 2$. These strata then intertwine in* A.

Moreover, if $n_1 = n_2$, $m_1 = m_2$, and $(F[\varphi_i(\beta_i)], \Lambda)$ have the same embedding type then these strata are conjugate in $\mathfrak{K}(\Lambda)$.

2.3. Let $[\Lambda, n, m, b]$ be a stratum in a simple central F-algebra $\mathrm{A} = \mathrm{End}_{\mathrm{D}}(\mathrm{V})$ and let $[\widetilde{\Lambda}, n, m, b]$ be the induced stratum in the split central simple F-algebra $\widetilde{\mathrm{A}} = \mathrm{End}_{\mathrm{F}}(\mathrm{V})$, where $\widetilde{\Lambda}$ denotes the $\mathcal{O}_{\mathrm{F}}$-lattice sequence defined by Λ. By [**19**, Théorème 2.23], this latter stratum is simple if and only the first one is, and in this case they are realizations of the same simple pair over F.

We fix a uniformizer ϖ_{F} of F and set

$$\mathscr{y} = \mathscr{y}(b,\Lambda) = \varpi_{\mathrm{F}}^{n/g} b^{e/g},$$

where $e = e(\Lambda|\mathcal{O}_{\mathrm{F}})$ and $g = \gcd(n, e)$. Set

$$\overline{\mathscr{y}} = \overline{\mathscr{y}}(b,\Lambda) = \mathscr{y} + \mathfrak{P}_1(\widetilde{\Lambda}),$$

considered as an element of $\mathfrak{P}_0(\widetilde{\Lambda})/\mathfrak{P}_1(\widetilde{\Lambda})$.

Definition 2.5. — With notation as above:

(i) the characteristic polynomial of $\overline{\mathscr{Y}}$ (in $k_{\mathrm{F}}[X]$) is called the *characteristic polynomial of the stratum* $[\Lambda, n, m, b]$;

(ii) the minimum polynomial of $\overline{\mathscr{Y}}$ (in $k_{\mathrm{F}}[X]$) is called the *minimum polynomial of the stratum* $[\Lambda, n, m, b]$.

Remark 2.6. — (i) Since $\overline{\mathscr{Y}}$ depends only on the equivalence class of the stratum $[\Lambda, n, n-1, b]$ (and the choice of the uniformizer ϖ_{F}), the same is true of the minimum and characteristic polynomials.

(ii) If b normalizes Λ then, by [**4**, Lemma 2.1.9], the element $\mathscr{Y}$ depends only on the strict lattice sequence whose image is the image of Λ; hence the same is true of the minimum and characteristic polynomials.

(iii) The characteristic polynomial of a stratum may also be computed as the reduction modulo $\mathfrak{p}_{\mathrm{F}}$ of the characteristic polynomial of $\mathscr{Y}$ in $\widetilde{\mathrm{A}}$; of course, the same is *not* true for the minimum polynomial.

Proposition 2.7. — *Let $[\Lambda, n, n-1, b]$ be a stratum in* A. *Then $[\Lambda, n, n-1, b]$ is equivalent to a simple stratum if and only if its minimum polynomial is irreducible and not X. Moreover, if $[\Lambda, n, n-1, \beta]$ is a simple stratum equivalent to $[\Lambda, n, n-1, b]$ then, writing* $\mathrm{E} = \mathrm{F}[\beta]$, *we have*

$$\mathfrak{O}_{\mathrm{E}} + \mathfrak{P}_1(\Lambda) = \mathfrak{O}_{\mathrm{F}}[\mathscr{Y}(b, \Lambda)] + \mathfrak{P}_1(\Lambda).$$

Proof. — Note first that both conditions imply that the b normalizes Λ: if the minimum polynomial of $[\Lambda, n, n-1, b]$ is irreducible and not X then $\overline{\mathscr{Y}}(b, \Lambda)$ is invertible so $\mathscr{Y}(b, \Lambda)$ normalizes Λ, whence so does b. Hence, using Remark 2.6(ii), we may (and do) assume in the proof that Λ is strict. Also, the final assertion is clear since, by the minimality of β, the element $\mathscr{Y}(\beta, \Lambda) + \mathfrak{p}_{\mathrm{E}}$ generates the extension $k_{\mathrm{E}}/k_{\mathrm{F}}$, and $\mathscr{Y}(\beta, \Lambda) + \mathfrak{P}_1(\Lambda) = \mathscr{Y}(b, \Lambda) + \mathfrak{P}_1(\Lambda)$.

Suppose $[\Lambda, n, n-1, b]$ is equivalent to a simple stratum $[\Lambda, n, n-1, \beta]$ and put $\mathrm{E} = \mathrm{F}[\beta]$. Then $[\widetilde{\Lambda}, n, n-1, \beta]$ is also simple; in particular, $\overline{\mathscr{Y}}(b, \Lambda) = \overline{\mathscr{Y}}(\beta, \Lambda)$ is a non-zero element of k_{E} in $\mathfrak{P}_0(\widetilde{\Lambda})/\mathfrak{P}_1(\widetilde{\Lambda})$ so has irreducible minimum polynomial not X.

For the converse, suppose $[\Lambda, n, n-1, b]$ has irreducible minimum polynomial $\overline{f}(X) \in k_{\mathrm{F}}[X]$ and put $\delta = \deg(\overline{f}(X))$. Since $[\Lambda, n, n-1, b]$ has characteristic polynomial which is a power of $\overline{f}(X)$, it is non-split fundamental. Now the proof follows that of [**4**, Theorem 3.2.1] and we only sketch the difference so this proof should be read alongside *loc. cit.* – in particular, we will refer to notations used in the proofs there.

Following the ideas of [**4**, §3], we treat first the simpler case when $\overline{f}(X)$ is also irreducible as an element of $k_{\mathrm{D}}[X]$ – that is, when d is coprime to δ. We fix L/F a maximal unramified subfield of D, so that $k_{\mathrm{L}} = k_{\mathrm{D}}$.

Let $f(X) \in \mathcal{O}_{\mathrm{F}}[X]$ be any monic polynomial which reduces modulo $\mathfrak{p}_{\mathrm{F}}$ to give $\overline{f}(X)$. We choose a matrix $\overline{\gamma} \in \mathrm{M}_{\delta}(k_{\mathrm{F}})$ with minimum polynomial $\overline{f}(X)$. In [**4**, Definition 3.2.4], Broussous defines the notion of $\overline{\gamma}$-*standard form*, which we will use here. Since $[\Lambda, n, n-1, b]$ has characteristic polynomial which is a power of $\overline{f}(X)$, it is non-split fundamental and, by [**4**, Proposition 3.2.5] there is a $u \in \mathrm{U}(\Lambda)$ such that $[\Lambda, n, n-1, ubu^{-1}]$ is equivalent to a stratum in $\overline{\gamma}$-standard form. Since the property of being equivalent to a simple stratum is unchanged by conjugation in $\mathrm{U}(\Lambda)$, we may as well assume $[\Lambda, n, n-1, b]$ is itself in $\overline{\gamma}$-standard form.

Now we follow the proof of [**4**, Theorem 3.2.1] in this case. In *op. cit.* p.221, an element β is defined and the proof of *op. cit.* Proposition 3.2.8 shows that there is $u \in \mathrm{U}(\Lambda)$ such that $[\Lambda, n, n-1, \beta]$ is equivalent to $[\Lambda, n, n-1, ubu^{-1}]$. (More precisely, [**4**, Proposition 3.2.9] is applied to the $\mathcal{O}_{\mathrm{L}}$-order $\mathrm{M}_{n_i}(\mathcal{O}_{\mathrm{L}})$, in the notation there.) Moreover, β is minimal over F by [**4**, Lemma 3.2.10] so $[\Lambda, n, n-1, b]$ is equivalent to the simple stratum $[\Lambda, n, n-1, u^{-1}\beta u]$.

Finally suppose we are in the general case where $\overline{f}(X)$ is not irreducible in $k_{\mathrm{L}}[X]$ and we decompose $\overline{f}(X) = \overline{p}_0(X)\cdots\overline{p}_{s-1}(X)$ into irreducibles. Now we follow [**4**, §3.3] to reduce to previous case. The proof is essentially identical (but easier) so we will not repeat it – the only point is that, in [**4**, Proposition 3.3.2], the orders $\mathfrak{A}_0$ and $\mathfrak{B}_0$ can be taken to be equal, by the case where $\overline{f}(X)$ is irreducible, and then the lattice sequences denoted $\mathscr{M}^1$ and $\mathscr{L}$ are equal, which implies that $\mathscr{M}$ is the lattice sequence here denoted Λ and, in the notation of [**4**, Lemma 3.3.10], we have $\mathfrak{A} = \mathfrak{A}'$. □

For $j = 1, \ldots, r$, let $[\Lambda^j, n, n-1, \beta_j]$ be a simple stratum in $\mathrm{A}^j = \mathrm{End}_{\mathrm{D}}(\mathrm{V}^j)$ with $e(\Lambda^j|\mathcal{O}_{\mathrm{F}}) = e$. Put $\mathrm{V} = \mathrm{V}^1 \oplus \cdots \oplus \mathrm{V}^r$, and set $\Lambda = \Lambda^1 \oplus \cdots \oplus \Lambda^r$, a lattice sequence in V of $\mathcal{O}_{\mathrm{F}}$-period e. Write $\mathrm{A} = \mathrm{End}_{\mathrm{D}}(\mathrm{V})$ and denote by e^j the idempotents in $\mathfrak{P}_0(\Lambda)$ corresponding to the decomposition of V. We put $\beta = \sum_{j=1}^{r} \beta_j$. Then $[\Lambda, n, n-1, \beta]$ is a stratum in A.

Corollary 2.8. — *With notation as above, suppose the strata* $[\Lambda^j, n, n-1, \beta_j]$ *are all equivalent to simple (or null) strata. Then they have the same minimum polynomial if and only if* $[\Lambda, n, n-1, \beta]$ *is equivalent to a simple (or null) stratum.*

Proof. — Writing $\mathrm{A} = \bigoplus_{i,j} \mathrm{End}_{\mathrm{D}}(\mathrm{V}^j, \mathrm{V}^i)$, it is clear that $\mathscr{y}(\beta, \Lambda)$ is block diagonal of the form $\mathrm{diag}(\mathscr{y}(\beta_1, \Lambda^1), \ldots, \mathscr{y}(\beta_r, \Lambda^r))$. In particular, the minimum polynomial of $\overline{\mathscr{y}}(\beta, \Lambda)$ is the gcd of the minimum polynomials of $\overline{\mathscr{y}}(\beta_j, \Lambda^j)$. The result is now immediate from Proposition 2.7, with the case of null strata coming from the case where the minimum polynomial is X. □

Corollary 2.9. — *Let* (k, β) *be a simple pair over* F *and let* $[\Lambda^1, n_1, m_1, \varphi_1(\beta)]$ *be a realization in some simple central* F*-algebra* A^1. *The minimum polynomial of* $[\Lambda^1, n_1, m_1, \varphi_1(\beta)]$ *depends only on the endo-equivalence class of the pair* $(-k_{\mathrm{F}}(\beta) - 1, \beta)$.

Proof. — First note that Corollary 2.8 essentially says that the minimum polynomial is independent of the realization. For suppose $[\Lambda^2, n_2, m_2, \varphi_2(\beta)]$ is another realization in a simple central F-algebra A^2. Since the minimum polynomial depends only on the induced strata in $\widetilde{\mathrm{A}}^1$ and $\widetilde{\mathrm{A}}^2$, we may as well suppose that both algebras are split – that is, $\mathrm{A}^j = \mathrm{End}_{\mathrm{F}}(\mathrm{V}^j)$, for some F-vector space V^j, $j = 1, 2$. Moreover, by scaling we may assume that Λ^1 and Λ^2 have the same period so that $n_1 = n_2 = n$. Put $m = \max\{m_1, m_2\}$.

Now set $\mathrm{V} = \mathrm{V}^1 \oplus \mathrm{V}^2$ and use the notation of Corollary 2.8; also let $\varphi = \varphi_1 + \varphi_2$ denote the diagonal embedding of $\mathrm{F}[\beta]$ in A. The stratum $[\Lambda, n, m, \varphi(\beta)]$ is then a realization of the simple pair (k, β) so it is simple and $[\Lambda, n, n-1, \varphi(\beta)]$, being pure, is equivalent to a simple stratum. Hence, by Corollary 2.8, the strata $[\Lambda^j, n, n-1, \varphi_j(\beta)]$ have the same minimum polynomial.

Finally, suppose (k, γ) is a simple pair endo-equivalent to (k, β). Let $[\Lambda, n, m, \varphi(\beta)]$ and $[\Lambda, n, m, \rho(\gamma)]$ be realizations in some split simple central F-algebra $\mathrm{A} = \mathrm{End}_{\mathrm{F}}(\mathrm{V})$. By Proposition 2.4, these strata are conjugate by some $u \in \mathfrak{K}(\Lambda)$ so, by conjugating the embedding ρ, we may assume that $[\Lambda, n, n-1, \varphi(\beta)]$ is equivalent to $[\Lambda, n, n-1, \rho(\gamma)]$. In particular, they have the same minimum polynomial. □

2.4. The following is a generalization of [**8**, Lemma 2.4.11], [**16**, Lemma 1.9]:

Lemma 2.10. — *Let* $[\Lambda, n, n-1, b]$ *be a stratum in* A. *It is intertwined by every element of* G *if and only if* $(b + \mathfrak{P}_{1-n}(\Lambda)) \cap \mathrm{F} \neq \emptyset$.

Proof. — The proof follows the same scheme as that of [**8**, Lemma 2.4.11] (see *op. cit.* pp.77–78) so we only sketch the argument. Suppose $[\Lambda, n, n-1, b]$ is intertwined by every element of G. If $b \in \mathfrak{P}_{1-n}(\Lambda)$ then $0 \in (b + \mathfrak{P}_{1-n}(\Lambda)) \cap \mathrm{F}$ so we assume $b \notin \mathfrak{P}_{1-n}(\Lambda)$. Then b defines a non-zero map $\overline{\beta}$ in

$$\mathfrak{P}_{-n}(\Lambda)/\mathfrak{P}_{1-n}(\Lambda) = \bigoplus_{i=0}^{e-1} \mathrm{Hom}_{k_{\mathrm{D}}}(\Lambda(i)/\Lambda(i+1), \Lambda(i-n)/\Lambda(i-n+1)),$$

where $e = e(\Lambda|\mathcal{O}_{\mathrm{D}})$ is the $\mathcal{O}_{\mathrm{D}}$-period of Λ. (Note that there is, in general, redundancy in this sum: the spaces $\mathrm{Hom}_{k_{\mathrm{D}}}(\Lambda(i)/\Lambda(i+1), \Lambda(i-n)/\Lambda(i-n+1))$ may be 0.)

Since $\overline{\beta}$ is non-zero, by moving Λ in its translation class we can suppose that it defines a non-zero map in $\mathrm{Hom}_{k_{\mathrm{D}}}(\Lambda(0)/\Lambda(1), \Lambda(-n)/\Lambda(1-n))$. If $e \nmid n$ then we can find $g \in \mathrm{G} \cap \mathfrak{P}_0(\Lambda)$ such that g induces the identity map on $\Lambda(0)/\Lambda(1)$ but the zero map on $\Lambda(i)/\Lambda(i+1)$, for $1 \leq i \leq e-1$. But then $\overline{gb}$ is zero on $\Lambda(0)/\Lambda(1)$, while $\overline{bg}$ coincides with $\overline{b}$ on $\Lambda(0)/\Lambda(1)$, so is non-zero, which contradicts the assumption that g intertwines $[\Lambda, n, n-1, b]$.

Thus e divides n and we put $t = -n/e$. Fix L/F a maximal unramified subfield of D, and ϖ_{D} a uniformizer of D which normalizes L (so acts via conjugation as a generator of Gal(L/F)) and such that $\varpi_{\mathrm{D}}^d = \mathrm{W}_{\mathrm{F}}$. Then the coset $b\varpi_{\mathrm{D}}^{-t} + \mathfrak{P}_1(\Lambda)$ is intertwined by every $g \in \mathrm{G}$ which commutes with ϖ_{D}. In particular, since elementary matrices (with respect to a suitable basis) commute with ϖ_{D}, we find that $(b\varpi_{\mathrm{D}}^{-t} + \mathfrak{P}_1) \cap \mathcal{O}_{\mathrm{D}} \neq \emptyset$ and, since conjugation by ϖ_{D} acts by Frobenius on k_{D}, the fact that ϖ_{D} intertwines implies that $(b\varpi_{\mathrm{D}}^{-t} + \mathfrak{P}_1) \cap \mathcal{O}_{\mathrm{F}} \neq \emptyset$. Thus $b \equiv \varpi_{\mathrm{D}}^t\lambda \pmod{\mathfrak{P}_{1-n}(\Lambda)}$, for some $\lambda \in \mathcal{O}_{\mathrm{F}}^\times$. Finally, the fact that every element of $\mathcal{O}_{\mathrm{L}}^\times$ intertwines the stratum implies that conjugation by ϖ_{D}^t acts trivially on k_{D}, so d divides t and $\varpi_{\mathrm{D}}^t\lambda \in \mathrm{F}$, as required.

The converse is trivial. □

2.5. Let A be a simple central F-algebra and V be a simple left A-module. Let $[\Lambda, n, m, \beta]$ be a simple stratum, set $\mathrm{E} = \mathrm{E}_\beta = \mathrm{F}[\beta]$, and let $\mathrm{B} = \mathrm{B}_\beta$ denote the A-centralizer of E. We identify $\widetilde{\mathrm{A}} = \mathrm{End}_{\mathrm{F}}(\mathrm{V})$ with $\mathrm{A} \otimes_{\mathrm{F}} \mathrm{End}_{\mathrm{A}}(\mathrm{V})$. From [**23**, Définition 2.25] (see also [**4**, §4.2]) a *tame corestriction relative to* E/F on A is a (B, B)-bimodule homomorphism $s = s_\beta : \mathrm{A} \to \mathrm{B}$ such that $\widetilde{s} = s \otimes \mathrm{id}_{\mathrm{End}_{\mathrm{A}}(\mathrm{V})}$ is a tame corestriction relative to E/F on $\widetilde{\mathrm{A}}$, in the sense of [**8**, Definition 1.3.3].

***Lemma 2.11* (cf. [8, Lemma 2.4.12]).** — *Let* $[\Lambda, n, m, \beta_i]$ *be equivalent simple strata. Then, putting* $\mathrm{E}_i = \mathrm{F}[\beta_i]$, *we have:*

(i) $\mathcal{O}_{\mathrm{E}_1} + \mathfrak{P}_1(\Lambda) = \mathcal{O}_{\mathrm{E}_2} + \mathfrak{P}_1(\Lambda)$;

(ii) *there are tame corestrictions* s_i *on* A *relative to* E_i/F *such that, for all* $k \in \mathbb{Z}$ *and* $x \in \mathfrak{P}_k(\Lambda)$,

$$s_1(x) \equiv s_2(x) \pmod{\mathfrak{P}_{k+1}(\Lambda)};$$

(iii) *there are prime elements* ϖ_i *of* E_i *such that* $\varpi_1 \mathrm{U}^1(\Lambda) = \varpi_2 \mathrm{U}^1(\Lambda)$;

(iv) *the pairs* (E_i, Λ) *have the same embedding type.*

Proof. — We begin by proving (i)-(iii), for which we may assume that Λ is strict by passing first to $\Lambda^{\ddagger}$ (cf. paragraph 1.6). To return to Λ, notice that the condition in (iii) is equivalent to $\varpi_1 \equiv \varpi_2 \pmod{\mathfrak{P}_{e+1}(\Lambda)}$, with $e = e(\Lambda|\mathcal{O}_{\mathrm{E}_i})$, so for (i) and (iii) we can simply intersect with A. The same is true for (ii) since tame corestrictions also decompose by blocks, by [**23**, Proposition 2.26].

The simple strata $[\widetilde{\Lambda}, n, m, \beta_i]$ in $\widetilde{\mathrm{A}}$ are equivalent so, by [**8**, Lemma 2.4.12], we have the results corresponding to (i) and (ii) in $\widetilde{\mathrm{A}}$, while (iii) is a by-product of the proof (see also [**11**, Lemma 5.2] and its proof). Intersecting with A gives the result here.

In the case of strict sequences, (iv) is given by [**5**, Lemma 5.2]. Moreover, writing $\mathscr{L}$ for the strict lattice sequence with the same image as Λ, since $\mathfrak{P}_0(\Lambda) = \mathfrak{P}_0(\mathscr{L})$ and $\mathfrak{P}_1(\Lambda) = \mathfrak{P}_1(\mathscr{L})$, the same proof (using *op. cit.* Lemma 2.3.6) works in the general case to show that the maximal unramified subextensions of E_i are conjugate in $\mathrm{U}^1(\mathscr{L}) = \mathrm{U}^1(\Lambda)$. □

2.6. Now let V_E be a simple left B-module, let D_E be the algebra opposite to $\mathrm{End}_\mathrm{B}(\mathrm{V}_\mathrm{E})$, and let $\Gamma = \Gamma_\beta$ be the unique (up to translation) $\mathcal{O}_{\mathrm{D}_\mathrm{E}}$-lattice sequence V_E such that $\mathfrak{P}_k(\Lambda) \cap \mathrm{B} = \mathfrak{P}_k(\Gamma)$, for all $k \in \mathbb{Z}$.

***Definition 2.12* ([23, Définition 3.21]).** — A *derived stratum* of $[\Lambda, n, m, \beta]$ is a stratum of the form $[\Gamma, m, m-1, s(b)]$, for some $b \in \mathfrak{P}_{-m}(\Lambda)$ and some tame corestriction s on A relative to E/F.

The following result is a slight strengthening of [**23**, Proposition 3.30]:

***Proposition 2.13*.** — *Let* $[\Lambda, n, m, \beta]$ *be a simple stratum and let* $b \in \mathfrak{P}_{-m}(\Lambda)$ *be such that the derived stratum* $[\Gamma, m, m-1, s(b)]$ *is equivalent to a simple (or null) stratum* $[\Gamma, m, m-1, c]$. *Then there is a simple stratum* $[\Lambda, n, m-1, \beta']$ *equivalent to* $[\Lambda, n, m-$

$1, \beta + b]$ *and, moreover, for any such stratum, writing* $\mathrm{E}' = \mathrm{F}[\beta']$ *and* $\mathrm{E}_1 = \mathrm{F}[\beta, c]$*, we have:*

(i) $e(\mathrm{E}'/\mathrm{F}) = e(\mathrm{E}_1/\mathrm{F})$*,* $f(\mathrm{E}'/\mathrm{F}) = f(\mathrm{E}_1/\mathrm{F})$ *and* $k_0(\beta', \Lambda) = \begin{cases} k_0(\beta, \Lambda) & \textit{if } c \in \mathrm{E}, \\ -m & \textit{otherwise;} \end{cases}$

(ii) $\mathcal{O}_{\mathrm{E}'} + \mathfrak{P}_1(\Lambda) = \mathcal{O}_{\mathrm{E}_1} + \mathfrak{P}_1(\Lambda) = \mathcal{O}_{\mathrm{E}}[\mathscr{y}(s(b))] + \mathfrak{P}_1(\Lambda)$.

Proof. — The first assertion is proved in [**23**, Proposition 3.30], under the hypothesis that the derived stratum $[\Gamma, m, m-1, s(b)]$ is itself simple. If it is only equivalent to the simple stratum $[\Gamma, m, m-1, c]$ then $c - s(b) \in \mathfrak{P}_{1-m}(\Gamma)$ so, by [**23**, Proposition 2.29], there is $d \in \mathfrak{P}_{1-m}(\Lambda)$ such that $s(d) = c - s(b)$. Replacing b by $b + d$ we reduce to the case that the derived stratum is simple and the result here follows.

For (i), we may pass first to $\Lambda^{\ddagger}$, where the result follows from [**16**, Proposition 9.5].

For (ii), the second equality follows from Proposition 2.7, while the independence of $\mathcal{O}_{\mathrm{E}'} + \mathfrak{P}_1(\Lambda)$ on the choice of β' comes from Lemma 2.11. In particular, we need only find a single β' for which the first equality holds.

We fix a simple right $\mathrm{E}_1 \otimes_{\mathrm{F}} \mathrm{D}$-module S and put $\mathrm{A}(\mathrm{S}) = \mathrm{End}_{\mathrm{D}}(\mathrm{S})$. Let C be the commutant of E_1 in A, and let D_1 be the commutant of E_1 in A(S). We also fix a decomposition $\mathrm{V} = \mathrm{V}^1 \oplus \cdots \oplus \mathrm{V}^l$ into simple right $\mathrm{E}_1 \otimes_{\mathrm{F}} \mathrm{D}$-modules (which are all copies of S) such that the lattice sequence Λ decomposes into the direct sum of the $\Lambda^j = \Lambda \cap \mathrm{V}^j$, for $j \in \{1, \ldots, l\}$. From [**21**, §1.3], after choosing identifications $\mathrm{V}^i \simeq \mathrm{S}$, we have an F-algebra embedding $\iota : \mathrm{A}(\mathrm{S}) \to \mathrm{A}$ and an isomorphism of $(\mathrm{A}(\mathrm{S}), \mathrm{C})$-bimodules

$$\mathrm{A}(\mathrm{S}) \otimes_{\mathrm{D}_1} \mathrm{C} \to \mathrm{A}.$$

We denote by B(S) the commutant of E in A(S) and let S_{E} be a simple left B(S)-module. By [**23**, Lemme 3.31], the tame corestriction s on A relative to E/F takes the form $s_1 \otimes \mathrm{id}_{\mathrm{C}}$, for s_1 a tame corestriction on A(S) relative to E/F.

Denote by $\mathfrak{S}$ the unique (up to translation) E_1-pure strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence on S, and by $\mathfrak{S}_\beta$ the unique (up to translation) $\mathcal{O}_{\mathrm{D}_{\mathrm{E}}}$-lattice sequence on S_{E} compatible with the filtration from $\mathfrak{S}$. Set $n_0 = -v_{\mathfrak{S}}(\beta)$ and $m_0 = -v_{\mathfrak{S}}(c)$ and pick $b_0 \in \mathfrak{P}_{-m_0}(\mathfrak{S})$ such that $s_1(b_0) = c$. By [**23**, Lemme 3.32], the stratum $[\mathfrak{S}, n_0, m_0 - 1, \beta + b_0]$ is pure, so equivalent to a simple stratum $[\mathfrak{S}, n_0, m_0 - 1, \beta + b']$, with $s_1(b') \in c + \mathfrak{P}_{1-m_0}(\mathfrak{S}_\beta)$. We put $\mathrm{E}' = \mathrm{F}[\beta + b']$.

Suppose first that $c \in \mathrm{E}$. Then, by (i), we have $k_0(\beta + b', \mathfrak{S}) = k_0(\beta, \mathfrak{S})$ so that $[\mathfrak{S}, n_0, m_0, \beta]$ and $[\mathfrak{S}, n_0, m_0, \beta + b']$ are equivalent simple strata. Now Lemma 2.11(i) implies

$$(\dagger) \qquad \mathcal{O}_{\mathrm{E}'} + \mathfrak{P}_1(\mathfrak{S}) = \mathcal{O}_{\mathrm{E}_1} + \mathfrak{P}_1(\mathfrak{S}).$$

Now suppose $c \notin \mathrm{E}$. Let $x \in \mathcal{O}_{\mathrm{E}'}$ and put $r = -k_0(\beta, \mathfrak{S})$, which is strictly greater than m_0. Then certainly $a_\beta(x) \in \mathfrak{P}_{-m_0}(\mathfrak{S})$ so, by [**23**, Proposition 2.29], we can write $x = \gamma + y$, with $\gamma \in \mathfrak{P}_0(\mathfrak{S})$ and $y \in \mathfrak{P}_{r-m_0}(\mathfrak{S}) \subseteq \mathfrak{P}_1(\mathfrak{S})$. Thus

$$0 \;=\; (\beta + b')(\gamma + y) - (\gamma + y)(\beta + b') \;\equiv\; a_\beta(y) + b'\gamma - \gamma b' \pmod{\mathfrak{P}_{1-m_0}(\mathfrak{S})}.$$

Applying s_1 and using $s_1(b') \in c + \mathfrak{P}_{1-m_0}(\mathfrak{S})$, we see that $a_c(\gamma) \in \mathfrak{P}_{1-m_0}(\mathfrak{S}_\beta)$. Now $k_0(c, \mathfrak{S}_\beta) = -m_0$ so we deduce that $\gamma \in \mathcal{O}_{\mathrm{D}_1} + \mathfrak{P}_1(\mathfrak{S}_\beta)$, since D_1 is the commutant of $\mathrm{E}_1 = \mathrm{E}[c]$ in $\mathrm{B}(\mathrm{S})$. In particular, we see that $\mathcal{O}_{\mathrm{E}'} \subseteq \mathcal{O}_{\mathrm{D}_1} + \mathfrak{P}_1(\mathfrak{S})$ so that the image of the residue field $k_{\mathrm{E}'}$ in $\mathfrak{P}_0(\mathfrak{S})/\mathfrak{P}_1(\mathfrak{S})$ is contained in the image of k_{D_1}. Since (the images of) $k_{\mathrm{E}'}$ and k_{E_1} are then subfields of k_{D_1} of the same cardinality (by (i)), they must coincide and we deduce again that

$$(\dagger) \qquad \mathcal{O}_{\mathrm{E}'} + \mathfrak{P}_1(\mathfrak{S}) = \mathcal{O}_{\mathrm{E}_1} + \mathfrak{P}_1(\mathfrak{S}).$$

Finally, we must translate this back to A, using the embedding $\iota : \mathrm{A}(\mathrm{S}) \to \mathrm{A}$; we will identify β and c with their images under ι. The image of the simple stratum $[\mathfrak{S}, n_0, m_0 - 1, \beta + b']$ under ι is a simple stratum $[\Lambda, n, m-1, \beta + \iota(b')]$, and we have

$$s(\iota(b')) = \iota(s_1(b')) \;\equiv c \equiv s(b) \pmod{\mathfrak{P}_{1-m}(\Gamma)},$$

since $\iota(\mathfrak{P}_{1-m_0}(\mathfrak{S})) \subseteq \mathfrak{P}_{1-m}(\Lambda)$. Thus, as in the end of the proof of [**23**, Proposition 3.30], there exists $h \in \mathrm{U}_1(\Lambda)$ such that, putting $\beta' = h^{-1}(\beta + \iota(b'))h$, we get a simple stratum $[\Lambda, n, m-1, \beta']$ equivalent to $[\Lambda, n, m-1, \beta + b]$. Then $\mathrm{F}[\beta'] = h^{-1}\iota(\mathrm{E}')h$ and

$$\mathcal{O}_{\mathrm{F}[\beta']} = h^{-1}\iota(\mathcal{O}_{\mathrm{E}'})h \equiv \iota(\mathcal{O}_{\mathrm{E}'}) \pmod{\mathfrak{P}_1(\Lambda)}.$$

Finally, by ($\dagger$), we have $\iota(\mathcal{O}_{\mathrm{E}'}) \equiv \iota(\mathcal{O}_{\mathrm{E}_1}) \pmod{\mathfrak{P}_1(\Lambda)}$ and, since we have identified E_1 with $\iota(\mathrm{E}_1)$, we deduce

$$\mathcal{O}_{\mathrm{F}[\beta']} + \mathfrak{P}_1(\Lambda) = \mathcal{O}_{\mathrm{E}_1} + \mathfrak{P}_1(\Lambda).$$

This completes the proof of Proposition 2.13. □

2.7. We also have a converse to Proposition 2.13:

***Proposition 2.14* (cf. [8, Theorem 2.4.1], [16, Proposition 9.3])**
Let $[\Lambda, n, m, \beta]$ be a pure stratum equivalent to the simple stratum $[\Lambda, n, m, \gamma_1]$. Let s_1 be a tame corestriction on A *relative to* $\mathrm{F}[\gamma_1]/\mathrm{F}$. *Then the derived stratum $[\Gamma_1, m, m-1, s_1(\beta-\gamma_1)]$ is equivalent to a simple (or null) stratum.*

We will need the following Lemma, which is in fact a special case of the Proposition.

***Lemma 2.15* (cf. [8, (2.4.10)]).** — *Let $[\Lambda, n, m, \beta_i]$ be equivalent simple strata and let s_1 be a tame corestriction on* A *relative to* $\mathrm{F}[\beta_1]/\mathrm{F}$. *Then there exists $\delta \in \mathrm{F}[\beta_1]$ such that $s_1(\beta_1 - \beta_2) \equiv \delta \pmod{\mathfrak{P}_{1-m}(\Lambda)}$.*

Proof. — By passing to $\Lambda^{\ddagger}$, we may assume we are in the strict sound case. The proof is then identical to that of [8, (2.4.10)], replacing [8, Proposition 1.4.6] by [4, Proposition 4.3.3] and [8, Theorem 1.5.8] by [4, Proposition 4.1.1]. □

Proof of Proposition 2.14. — By passing to an equivalent stratum, we may assume that the stratum $[\Lambda, n, m-1, \beta]$ is simple. If $[\Lambda, n, m, \beta]$ is also simple then the result follows from Lemma 2.15; thus we may assume $k_0(\beta, \Lambda) = -m$. By passing to $\Lambda^{\ddagger}$, we may assume we are in the strict sound case. We write $\mathrm{E}_1 = \mathrm{F}[\gamma_1]$ and B_1 for the A-centralizer of E_1.

By [16, Propositions 3.8, 9.3], there exists a simple stratum $[\Lambda, n, m, \gamma_2]$ equivalent to $[\Lambda, n, m, \beta]$ such that the derived stratum $[\Gamma_2, m, m-1, s_2(\beta-\gamma_2)]$ is equivalent to a simple (or null) stratum, where s_2 is a tame corestriction on A relative to $\mathrm{F}[\gamma_2]/\mathrm{F}$. Moreover, this derived stratum is non-scalar, by Proposition 2.13, since $k_0(\beta, \Lambda) > k_0(\gamma_2, \Lambda)$, and thus, by Proposition 2.7, it has irreducible minimum polynomial.

We write $\mathrm{E}_2 = \mathrm{F}[\gamma_2]$ and B_2 for the A-centralizer of E_2. By Lemma 2.11, we may assume that the tame corestriction s_2 is chosen such that, for all $k \in \mathbb{Z}$ and $x \in \mathfrak{P}_k(\Lambda)$,

$$s_1(x) \equiv s_2(x) \pmod{\mathfrak{P}_{k+1}(\Lambda)}.$$

We also use Lemma 2.11 to choose uniformizers ϖ_i for E_i such that $\varpi_1 \mathrm{U}^1(\Lambda) = \varpi_2 \mathrm{U}^1(\Lambda)$. Again by Lemma 2.11, the residue fields k_{E_i} have a common image in $\mathfrak{P}_0(\Lambda)/\mathfrak{P}_1(\Lambda)$ so that we may identify them. Moreover, $\mathfrak{P}_0(\Gamma_i)/\mathfrak{P}_1(\Gamma_i)$ have a common image in $\mathfrak{P}_0(\Lambda)/\mathfrak{P}_1(\Lambda)$: by [23, Proposition 2.29], the maps $s_i : \mathfrak{P}_k(\Lambda) \to \mathfrak{P}_k(\Gamma_i)$ are surjective, for all $k \in \mathbb{Z}$, so $\mathfrak{P}_0(\Gamma_i)/\mathfrak{P}_1(\Gamma_i)$ are the common image of s_i in $\mathfrak{P}_0(\Lambda)/\mathfrak{P}_1(\Lambda)$.

Put $b_i = \beta - \gamma_i$. By the choices of ϖ_i and s_i, we have

$$\mathscr{y}(s_1(b_2), \Gamma_1) \equiv \mathscr{y}(s_2(b_2), \Gamma_2) \pmod{\mathfrak{P}_1(\Lambda)}.$$

In particular (given the identifications we have made), we see that the two strata $[\Gamma_i, m, m-1, s_i(b_2)]$ have the same minimum and characteristic polynomials. In particular, $[\Gamma_1, m, m-1, s_1(b_2)]$ has irreducible minimum polynomial so, by Proposition 2.7, it is equivalent to a simple stratum $[\Gamma_1, m, m-1, c]$.

Finally, by Lemma 2.15, the stratum $[\Gamma_1, m, m-1, s_1(\gamma_2 - \gamma_1)]$ is equivalent to a scalar stratum $[\Gamma_1, m, m-1, \delta]$, whence $[\Gamma_1, m, m-1, s_1(b_1)]$ is equivalent to the simple stratum $[\Gamma_1, m, m-1, c+\delta]$, as required. □

3. Simple characters and refinement

3.1. Let A be a central simple F-algebra and let $[\Lambda, n, 0, \beta]$ be a simple stratum in A. To this simple stratum, in [**23**, §2.4], one attaches compact open subgroups $\mathrm{H}(\beta, \Lambda) \subseteq \mathrm{J}(\beta, \Lambda)$ of $\mathrm{A}^\times$, together with filtrations

$$\mathrm{H}^{m+1}(\beta, \Lambda) = \mathrm{H}(\beta, \Lambda) \cap \mathrm{U}^{m+1}(\Lambda), \quad \mathrm{J}^{m+1}(\beta, \Lambda) = \mathrm{J}(\beta, \Lambda) \cap \mathrm{U}^{m+1}(\Lambda), \qquad m \geq 0,$$

and a finite set $\mathcal{C}(\Lambda, 0, \beta)$ of characters of $\mathrm{H}^1(\beta, \Lambda)$, called *simple characters of level* 0, depending on the choice of an additive character

$$\psi_\mathrm{F} : \mathrm{F} \to \mathbb{C}^\times$$

which is trivial on $\mathfrak{p}_\mathrm{F}$ but not on $\mathcal{O}_\mathrm{F}$, and which will now be fixed once and for all.

By restriction to $\mathrm{H}^{m+1}(\beta, \Lambda)$, we get also a set $\mathcal{C}(\Lambda, m, \beta)$ of *simple characters of level* m. If $\lfloor n/2 \rfloor \leqslant m$, then $\mathrm{H}^{m+1}(\beta, \Lambda) = \mathrm{U}^{m+1}(\Lambda)$, and the set $\mathcal{C}(\Lambda, m, \beta)$ reduces to the single character ψ_β of $\mathrm{U}^{m+1}(\Lambda)$ defined by

$$\psi_\beta : x \mapsto \psi_\mathrm{F} \circ \mathrm{tr}_{\mathrm{A/F}}(\beta(x-1)),$$

where $\mathrm{tr}_{\mathrm{A/F}}$ denotes the reduced trace of A over F, which depends only on the equivalence class of $[\Lambda, n, m, \beta]$. More generally, for any m, the subgroup $\mathrm{H}^{m+1}(\beta, \Lambda)$ and the set $\mathcal{C}(\Lambda, m, \beta)$ depend only on the equivalence class of $[\Lambda, n, m, \beta]$.

Note that we will use the following common convention: the trivial character of the group $\mathrm{U}^{t+1}(\Lambda)$ will be considered as a simple character for the trivial stratum $[\Lambda, t, t, 0]$.

3.2. Various properties of simple characters can be found in [**23, 6**]. For now we recall two of them, the first of which is a special case of the intertwining formula [**23**, Théorème 2.24]:

Proposition 3.1. — *Let* A *be a central simple* F*-algebra, let* $[\Lambda, n, 0, \beta]$ *be a simple stratum in* A *and let* $\theta \in \mathcal{C}(\Lambda, 0, \beta)$*. Then, writing* B *for the* A*-centralizer of* β *as usual, we have*

$$\mathrm{I_G}(\theta) = \mathrm{J}^1(\beta, \Lambda)\mathrm{B}^\times \mathrm{J}^1(\beta, \Lambda).$$

The following fundamental result is one of the main results of [**6**]:

Proposition 3.2 **([6**, Theorem 1.12**])**. — *Let* A *be a simple central* F*-algebra. For* $i = 1, 2$*, let* $[\Lambda, n, m, \beta_i]$ *be a simple stratum in* A*, and let* $\theta_i \in \mathcal{C}(\Lambda, m, \beta_i)$ *be a simple character. Write* K_i *for the maximal unramified extension of* F *contained in* $\mathrm{F}(\beta_i)$*. Assume that* θ_1 *and* θ_2 *intertwine in* $\mathrm{A}^\times$ *and that the* $(\mathrm{F}[\beta_i], \Lambda)$ *have the same embedding type. Then there is an element* $u \in \mathfrak{K}(\Lambda)$ *such that:*

(i) $\mathrm{K}_1 = u\mathrm{K}_2u^{-1}$*;*

(ii) $\mathcal{C}(\Lambda, m, \beta_1) = \mathcal{C}(\Lambda, m, u\beta_2u^{-1})$*;*

(iii) $\theta_2(x) = \theta_1(uxu^{-1})$*, for all* $x \in \mathrm{H}^{m+1}(\beta_2, \Lambda)$*.*

3.3. One of the technical difficulties with simple characters is that they do not determine the simple stratum used to define them: that is, we may have $\mathcal{C}(\Lambda, m, \beta_1) \cap \mathcal{C}(\Lambda, m, \beta_2) \neq \emptyset$, for inequivalent strata $[\Lambda, n, m, \beta_i]$ (though certain invariants of the strata are equal – see later). In order to cope with this, we need the following translation principle, which is the main result of this section.

Theorem 3.3 **(cf. [11**, Translation Principle 2.11**])**. — *Let* $[\Lambda, n, m, \gamma_i]$ *be simple strata with* $\mathcal{C}(\Lambda, m, \gamma_1) \cap \mathcal{C}(\Lambda, m, \gamma_2) \neq \emptyset$*. Let* $[\Lambda, n, m-1, \beta_1]$ *be a simple stratum such that* $[\Lambda, n, m, \beta_1]$ *is equivalent to* $[\Lambda, n, m, \gamma_1]$*. Then there is a simple stratum* $[\Lambda, n, m-1, \beta_2]$ *such that* $[\Lambda, n, m, \beta_2]$ *is equivalent to* $[\Lambda, n, m, \gamma_2]$ *and* $\mathcal{C}(\Lambda, m-1, \beta_1) = \mathcal{C}(\Lambda, m-1, \beta_2)$*.*

The proof of this translation principle, which will take up most of the remainder of this section, begins with the following special case, in which $\beta_1 = \gamma_1$:

Lemma 3.4 **(cf. [8**, Theorem 3.5.9**], [16**, Proposition 9.10**])**. — *Let* $[\Lambda, n, m, \gamma_i]$ *be simple strata with* $\mathcal{C}(\Lambda, m, \gamma_1) \cap \mathcal{C}(\Lambda, m, \gamma_2) \neq \emptyset$*. Then* $\mathrm{H}^m(\gamma_1, \Lambda) = \mathrm{H}^m(\gamma_2, \Lambda)$ *and there is a simple stratum* $[\Lambda, n, m, \beta_2]$ *equivalent to* $[\Lambda, n, m, \gamma_2]$ *such that* $\mathcal{C}(\Lambda, m-1, \beta_2) = \mathcal{C}(\Lambda, m-1, \gamma_1)$*.*

Proof. — From [**6**, Lemma 4.12], we have already that $\mathrm{H}^m(\gamma_1,\Lambda) = \mathrm{H}^m(\gamma_2,\Lambda)$. The remainder of the proof is *mutatis mutandis* that of [**8**, Theorem 3.5.9]: we replace [**8**, Theorem 3.3.2] by [**23**, Théorème 2.23], [**8**, Lemma 2.4.11] by Lemma 2.10, [**8**, Theorem 2.2.8] by Proposition 2.13, [**8**, 3.3.20] by [**23**, Proposition 2.15], and [**8**, 3.5.8] by [**6**, Theorem 4.16]; for the proof of [**8**, Lemma 3.5.10] we replace [**8**, Corollary 3.3.17] by [**23**, Proposition 2.24] and [**8**, Proposition 3.3.9] by [**23**, Lemme 2.30(2)]. □

3.4. The technical crux of the translation principle is contained in the following lemma:

Lemma 3.5 **(cf.** [**11**, Lemma 5.2]**).** — *Let* $[\Lambda,n,m-1,\beta_i]$ *be simple strata with* $\mathcal{C}(\Lambda,m-1,\beta_1)\cap\mathcal{C}(\Lambda,m-1,\beta_2)\neq\emptyset$*. Then, putting* $\mathrm{E}_i=\mathrm{F}[\beta_i]$*, we have:*

(i) $\mathcal{O}_{\mathrm{E}_1}+\mathfrak{P}_1(\Lambda)=\mathcal{O}_{\mathrm{E}_2}+\mathfrak{P}_1(\Lambda)$*;*

(ii) *the pairs* (E_i,Λ) *have the same embedding type;*

(iii) *there are prime elements* ϖ_i *of* E_i *such that* $\varpi_1\mathrm{U}^1(\Lambda)=\varpi_2\mathrm{U}^1(\Lambda)$*;*

(iv) *there are tame corestrictions* s_i *on* A *relative to* E_i/F *such that, for all* $k\in\mathbb{Z}$ *and* $x\in\mathfrak{P}_k(\Lambda)$*,*

$$s_1(x)\equiv s_2(x)\pmod{\mathfrak{P}_{k+1}(\Lambda)}.$$

Note that (ii) in this lemma answers Conjecture 4.17 of [**6**] – indeed, it is a generalization of that conjecture since here we do not assume that the sequence Λ is strict. Also, the hypothesis $\mathcal{C}(\Lambda,m-1,\beta_1)\cap\mathcal{C}(\Lambda,m-1,\beta_2)\neq\emptyset$ is equivalent to $\mathcal{C}(\Lambda,m-1,\beta_1)=\mathcal{C}(\Lambda,m-1,\beta_2)$, by [**6**, Theorem 4.16].

Proof. — In the split case when Λ is strict, this is [**11**, Lemma 5.2], while the case of arbitrary Λ follows by passing to $\Lambda^\ddagger$.

We proceed by induction on m. When $m=n$ the result is immediate from Lemma 2.11. Note that $k_0(\beta_i,\Lambda)$ is independent of i, by [**6**, Lemma 4.7]. If $k_0(\beta_i,\Lambda)<-m$ then again the result is clear from the induction hypothesis, since the conclusions (i)–(iv) are independent of m, so we assume $k_0(\beta_i,\Lambda)=-m>-n$. By replacing $[\Lambda,n,m-1,\beta_1]$ by an equivalent stratum, Lemma 3.4 says we may (and do) assume $\mathcal{C}(\Lambda,0,\beta_1)=\mathcal{C}(\Lambda,0,\beta_2)$ without affecting the conclusion of the lemma, by Lemma 2.11.

For $i=1,2$, let $[\Lambda,n,m,\gamma_i]$ be a simple stratum equivalent to $[\Lambda,n,m,\beta_i]$. Then

$$\mathcal{C}(\Lambda,m,\gamma_1)=\mathcal{C}(\Lambda,m,\beta_1)=\mathcal{C}(\Lambda,m,\beta_2)=\mathcal{C}(\Lambda,m,\gamma_2)$$

so, by induction applied to the simple strata $[\Lambda, n, (m+1)-1, \gamma_i]$, (i)–(iv) are satisfied with $E_{\gamma_i} = F[\gamma_i]$ in place of E_i, and we pick uniformizers ϖ_{γ_i} and tame corestrictions s_{γ_i} satisfying (iii), (iv). Moreover, by replacing $[\Lambda, n, m, \gamma_1]$ by an equivalent stratum, Lemma 3.4 says we may assume $\mathcal{C}(\Lambda, 0, \gamma_1) = \mathcal{C}(\Lambda, 0, \gamma_2)$.

Put $c_i = \beta_i - \gamma_i$ and let $\theta \in \mathcal{C}(\Lambda, m-1, \beta_1) = \mathcal{C}(\Lambda, m-1, \beta_2)$. By [**23**, Proposition 2.15], we have $\theta = \vartheta_i \psi_{c_i}$, for some $\vartheta_i \in \mathcal{C}(\Lambda, m-1, \gamma_i)$. Hence

$$\vartheta_1 = \vartheta_2 \psi_{c_2 - c_1}$$

and $\vartheta_1, \vartheta_2 \in \mathcal{C}(\Lambda, m-1, \gamma_1)$ both restrict to the same character $\vartheta \in \mathcal{C}(\Lambda, m, \gamma_1)$. Since ϑ_1, ϑ_2 are both intertwined by $B_{\gamma_1}^{\times}$, the same is true of $\psi_{c_2-c_1}$. In particular, restricting to $H^m(\gamma_1, \Lambda) \cap B_{\gamma_1}^{\times} = U^m(\Gamma_{\gamma_1})$, we see that $\psi_{s_{\gamma_1}(c_2-c_1)} | U^m(\Gamma_{\gamma_1})$ is intertwined by all of $B_{\gamma_1}^{\times}$ and, by Lemma 2.10, $(s_{\gamma_1}(c_2 - c_1) + \mathfrak{P}_{1-m}(\Gamma_{\gamma_1})) \cap E_{\gamma_1} \neq \emptyset$. In particular, the stratum $[\Gamma_{\gamma_1}, m, m-1, s_{\gamma_1}(c_2 - c_1)]$ is equivalent to a simple (or null) scalar stratum.

By Proposition 2.13, there is a simple stratum $[\Lambda, n, m-1, \gamma_1']$ equivalent to the stratum $[\Lambda, n, m-1, \gamma_1 + (c_2 - c_1)]$. Since $\vartheta_2 \in \mathcal{C}(\Lambda, m-1, \gamma_1)$, by [**23**, Proposition 2.15] we have $\vartheta_1 = \vartheta_2 \psi_{c_2-c_1} = \vartheta_2 \psi_{\gamma_1' - \gamma_1} \in \mathcal{C}(\Lambda, m-1, \gamma_1')$. Hence $\mathcal{C}(\Lambda, m-1, \gamma_1') = \mathcal{C}(\Lambda, m-1, \gamma_1)$. Moreover, putting $c_1' = \beta_1 - \gamma_1'$ we see that $c_2 - c_1' \in \mathfrak{P}_{1-m}(\Lambda)$; in particular, for any tame corestriction $s_{\gamma_1'}$ on A relative to $E_{\gamma_1'}/F$, we have $s_{\gamma_1'}(c_2 - c_1') \in \mathfrak{P}_{1-m}(\Lambda)$.

Thus, replacing γ_1 by γ_1' we may assume that $s_{\gamma_1}(c_2 - c_1) \in \mathfrak{P}_{1-m}(\Lambda)$. By (iv), we also have $s_{\gamma_2}(c_2 - c_1) \in \mathfrak{P}_{1-m}(\Lambda)$. In particular, looking at the derived strata $[\Gamma_{\gamma_i}, m, m-1, s_{\gamma_i}(c_j)]$, with $i, j \in \{1, 2\}$, and using the inductive hypotheses (iii), (iv), we get

$$\mathscr{y}(s_{\gamma_1}(c_1)) \equiv \mathscr{y}(s_{\gamma_1}(c_2)) \equiv \mathscr{y}(s_{\gamma_2}(c_2)) \pmod{\mathfrak{P}_1(\Lambda)}.$$

(The elements $\mathscr{y}$ here are computed with respect to the uniformizers ϖ_{γ_i} satisfying (iii).) By Proposition 2.14 the derived strata $[\Gamma_{\gamma_i}, m, m-1, s_{\gamma_i}(c_i)]$ are equivalent to simple or null strata so, by Proposition 2.13 (applied to the strata $[\Lambda, n, m, \gamma_i]$ and $\beta' = \beta_i$) and the inductive hypothesis (i), we have

$$\mathcal{O}_{E_1} + \mathfrak{P}_1(\Lambda) = \mathcal{O}_{E_{\gamma_1}}[\mathscr{y}(s_{\gamma_1}(c_1))] + \mathfrak{P}_1(\Lambda) = \mathcal{O}_{E_{\gamma_2}}[\mathscr{y}(s_{\gamma_2}(c_2))] + \mathfrak{P}_1(\Lambda) = \mathcal{O}_{E_2} + \mathfrak{P}_1(\Lambda)$$

and we have proved (i). Now (ii) follows exactly as in the proof of Lemma 2.11 (see also [**5**, Lemma 5.2]); indeed the proof gives the existence of $u \in U^1(\Lambda)$ such that $u^{-1}K_1 u = K_2$, where K_i is the maximal unramified subextension of E_i/F.

For the remainder, we may pass to $\Lambda^\ddagger$ and assume we have soundly embedded strict lattice sequences with $\mathfrak{P}_0(\Lambda)$ principal, as in the proof of Lemma 2.11. (By [**6**, Proposition 4.11], we have $\mathcal{C}(\Lambda^\ddagger, m-1, \beta_1) = \mathcal{C}(\Lambda^\ddagger, m-1, \beta_2)$; cf. the proof of *op. cit.* Theorem 4.16.) Recall that we have $\theta \in \mathcal{C}(\Lambda, m-1, \beta_1) = \mathcal{C}(\Lambda, m-1, \beta_2)$, which we extend to a common simple character in $\mathcal{C}(\Lambda, 0, \beta_1) = \mathcal{C}(\Lambda, 0, \beta_2)$. Then $\theta^u \in \mathcal{C}(\Lambda, 0, u^{-1}\beta_1 u)$ surely intertwines $\theta \in \mathcal{C}(\Lambda, 0, \beta_2)$ so, by Proposition 3.2, there is $g \in \mathfrak{K}(\Lambda)$ such that $\theta^{ug} = \theta$ and $(ug)^{-1}\mathrm{K}_1(ug) = \mathrm{K}_2$. Since ug then normalizes θ, we have $ug \in \mathfrak{K}(\Gamma_{\beta_2})\mathrm{J}^1(\beta_b, \Lambda)$.

In particular, there is $x \in \mathfrak{K}(\Gamma_{\beta_2})$ such that $h = ugx \in \mathrm{J}^1(\beta_2, \Lambda)$, $h^{-1}\mathrm{K}_1 h = \mathrm{K}_2$ and $\theta^h = \theta$. Thus, replacing β_1 by $h^{-1}\beta_1 h$, we may assume that $\mathrm{K}_1 = \mathrm{K}_2 = \mathrm{K}$, without affecting the conclusion of the Lemma (since $h \in \mathrm{U}^1(\Lambda)$).

Now we will utilise the *interior lifting* and *base change* processes of [**6**] to reduce to the split case.

We suppose first that we are in the special case $\mathrm{K} = \mathrm{F}$, that is E_i/F is totally ramified. Fix an unramified extension L/F which splits A, so that $\mathrm{L}_i = \mathrm{E}_i \otimes_\mathrm{F} \mathrm{L}$ is a field, for $i = 1, 2$. The algebra $\overline{\mathrm{A}} = \mathrm{A} \otimes_\mathrm{F} \mathrm{L}$ is then a split simple L-central algebra and we choose a simple left $\overline{\mathrm{A}}$-module $\overline{\mathrm{V}}$. There is a unique (up to translation) strict $\mathcal{O}_\mathrm{L}$-lattice sequence $\overline{\Lambda}$ on $\overline{\mathrm{A}}$ such that $\mathfrak{P}_k(\overline{\Lambda}) = \mathfrak{P}_k(\Lambda) \otimes_{\mathcal{O}_\mathrm{F}} \mathcal{O}_\mathrm{L}$, for all $k \in \mathbb{Z}$ (see [**19**, §2.2]). Identifying A with $\mathrm{A} \otimes_\mathrm{F} 1 \subseteq \overline{\mathrm{A}}$, we get strata $[\overline{\Lambda}, n, m-1, \beta_i]$, which are simple.

Denote by $\mathcal{C}(\overline{\Lambda}, m-1, \beta_i)$ the set of simple characters with respect to the character $\psi_\mathrm{F} \circ \mathrm{tr}_{\mathrm{L}/\mathrm{F}}$. The base change process from [**6**, §7.2] gives rise to injective $\mathfrak{K}(\Lambda)$-equivariant maps

$$\boldsymbol{b}^i = \boldsymbol{b}^i_{\mathrm{L}/\mathrm{F}} : \mathcal{C}(\Lambda, m-1, \beta_i) \to \mathcal{C}(\overline{\Lambda}, m-1, \beta_i).$$

Moreover, by [**6**, Proposition 7.6], we have $\boldsymbol{b}^1(\theta) = \boldsymbol{b}^2(\theta)$ so $\mathcal{C}(\overline{\Lambda}, m-1, \beta_1) \cap \mathcal{C}(\overline{\Lambda}, m-1, \beta_2) \neq \emptyset$. In particular, by the split case we get uniformizers ϖ_i^L of L_i such that

$$(\ddagger) \qquad \varpi_1^\mathrm{L} + \mathfrak{P}_{e+1}(\overline{\Lambda}) = \varpi_2^\mathrm{L} + \mathfrak{P}_{e+1}(\overline{\Lambda}),$$

with $e = e(\overline{\Lambda}|\mathcal{O}_\mathrm{L}) = e(\Lambda|\mathcal{O}_\mathrm{F})$, and tame corestrictions s_i^L on $\overline{\mathrm{A}}$ relative to L_i/L for the character $\psi_\mathrm{F} \circ \mathrm{tr}_{\mathrm{L}/\mathrm{F}}$ such that, for all $k \in \mathbb{Z}$ and $x \in \mathfrak{P}_k(\overline{\Lambda})$,

$$s_1^\mathrm{L}(x) \equiv s_2^\mathrm{L}(x) \pmod{\mathfrak{P}_{k+1}(\overline{\Lambda})}.$$

Multiplying through ($\ddagger$) by a unit, we see that we may assume $\varpi_1^\mathrm{L} = \varpi_1$ is a uniformizer of E_1 and $\varpi_2^\mathrm{L} = \varpi_2\zeta$, for some uniformizer ϖ_2 of E_2 and $\zeta \in \mathcal{O}_\mathrm{L}^\times$ a root of unity of order coprime to p. Thus we have

$$\varpi_1\varpi_2^{-1} \;\equiv\; \zeta \pmod{\mathfrak{P}_1(\overline{\Lambda})}.$$

Now the Galois group Gal(L/F) acts on $\overline{A}$, fixing $\varpi_1\varpi_2^{-1}$, so the image of ζ in k_{L} is fixed by $\mathrm{Gal}(k_{\mathrm{L}}/k_{\mathrm{F}})$. In particular, $\zeta \in \mathcal{O}_{\mathrm{F}}^{\times}$ so, replacing ϖ_2 by $\varpi_2\zeta$, we get

$$\varpi_1 + \mathfrak{P}_{e+1}(\overline{\Lambda}) = \varpi_2 + \mathfrak{P}_{e+1}(\overline{\Lambda}),$$

and intersecting with A completes the proof of (iii).

The argument for the tame corestrictions is similar: We check that, if s_i is an arbitrary tame corestriction on A relative to E_i/F, then $s_i \otimes 1$ is a tame corestriction on $\overline{\mathrm{A}}$ relative to L_i/L. By [**23**, Proposition 2.26], s_i^{L} and $s_i \otimes 1$ differ by a unit u_i in $\mathcal{O}_{\mathrm{L}_i}$ and, changing by a root of unity, we can assume $u_1 \in 1 + \mathfrak{p}_{\mathrm{L}_1}$. We have $u_2 \equiv \zeta \pmod{\mathfrak{p}_{\mathrm{L}_i}}$, for some root of unity $\zeta \in \mathcal{O}_{\mathrm{L}}^{\times}$. Then, for all $k \in \mathbb{Z}$ and $a \in \mathfrak{P}_k(\Lambda)$

$$s_1(a) \otimes 1 \equiv s_1^{\mathrm{L}}(a \otimes 1) \equiv s_2^{\mathrm{L}}(a \otimes 1) \equiv s_2(a) \otimes \zeta \pmod{\mathfrak{P}_{k+1}(\overline{\Lambda})}.$$

Again, the Galois group Gal(L/F) acts on $\overline{A}$, fixing $s_1(a) \otimes 1$, so

$$s_2(a) \otimes \zeta \equiv s_2(a) \otimes \zeta^{\sigma} \pmod{\mathfrak{P}_1(\overline{\Lambda})}, \qquad \text{for all } a \in \mathfrak{P}_0(\Lambda),\ \sigma \in \mathrm{Gal}(\mathrm{L}/\mathrm{F}).$$

By [**23**, Proposition 2.29], the map $s_2 : \mathfrak{P}_0(\overline{\Lambda}) \to \mathfrak{P}_0(\overline{\Gamma}_2)$ is surjective so, as above, we deduce that $\zeta \in \mathcal{O}_{\mathrm{F}}^{\times}$ and, after replacing s_2 by ζs_2, we may assume $\zeta = 1$. Finally, intersecting with A completes with proof of (iv).

Finally we consider the case where $\mathrm{K} \neq \mathrm{F}$. Denote by C the A-centralizer of K, fix a simple left C-module W, and let D_{K} be the algebra opposite to $\mathrm{End}_{\mathrm{C}}(\mathrm{W})$. Let Γ_{K} be the unique (up to translation) $\mathcal{O}_{\mathrm{D}_{\mathrm{K}}}$-lattice sequence on W such that

$$\mathfrak{P}_k(\Lambda) \cap \mathrm{C} = \mathfrak{P}_k(\Gamma_{\mathrm{K}}), \qquad k \in \mathbb{Z}.$$

Then $[\Gamma_{\mathrm{K}}, n, m-1, \beta_i]$ is a simple stratum in C, for $i = 1, 2$, by [**6**, Proposition 5.2]. From [**6**, Theorem 5.8, Proposition 6.12], we get *interior lifting* maps

$$\boldsymbol{l}^i = \boldsymbol{l}^i_{\mathrm{K}/\mathrm{F}} : \mathcal{C}(\Lambda, m-1, \beta_i) \to \mathcal{C}(\Gamma_{\mathrm{K}}, m-1, \beta_i),$$

which are injective and $\mathfrak{K}(\Gamma_{\mathrm{K}})$-equivariant. Moreover, by [**6**, Proposition 6.13], we have $\boldsymbol{l}^1(\theta) = \boldsymbol{l}^2(\theta)$ so that $\mathcal{C}(\Gamma_{\mathrm{K}}, m-1, \beta_1) \cap \mathcal{C}(\Gamma_{\mathrm{K}}, m-1, \beta_2) \neq \emptyset$. From the case K = F above, we find uniformizers ϖ_i of E_i with $\varpi_1\mathrm{U}^1(\Gamma_{\mathrm{K}}) = \varpi_2\mathrm{U}^1(\Gamma_{\mathrm{K}})$; in particular, $\varpi_1\mathrm{U}^1(\Lambda) =$

$\varpi_2 \mathrm{U}^1(\Lambda)$ which proves (iii). We also get tame corestrictions s_i^{K} on C relative to E_i/K satisfying (iv): for all $k \in \mathbb{Z}$ and $x \in \mathfrak{P}_k(\Gamma_{\mathrm{K}})$,

$$s_1^{\mathrm{K}}(x) \equiv s_2^{\mathrm{K}}(x) \pmod{\mathfrak{P}_{k+1}(\Gamma_{\mathrm{K}})}.$$

Finally, if s_{K} is any tame corestriction on A relative to K/F then $s_i = s_i^{\mathrm{K}} \circ s_{\mathrm{K}}$ are tame corestrictions on A relative to E_i/F, which satisfy (iv) since $\mathfrak{P}_{k+1}(\Gamma_{\mathrm{K}}) \subseteq \mathfrak{P}_{k+1}(\Lambda)$. □

3.5. Now we are ready to complete the proof of the translation principle.

Proof of Theorem 3.3. — For $i = 1, 2$, we have simple strata $[\Lambda, n, m, \gamma_i]$ such that $\mathcal{C}(\Lambda, m, \gamma_1) \cap \mathcal{C}(\Lambda, m, \gamma_2) \neq \emptyset$; these sets of simple characters are then equal, by [**6**, Theorem 4.16]. Moreover, by Lemma 3.4, we may replace $[\Lambda, n, m, \gamma_1]$ by an equivalent stratum so that $\mathcal{C}(\Lambda, m-1, \gamma_1) = \mathcal{C}(\Lambda, m-1, \gamma_2)$.

Let B_i denote the A-centralizer of $\mathrm{F}[\gamma_i]$, let V_i be a simple left B_i-module, and let D_i be the algebra opposite to $\mathrm{End}_{\mathrm{B}_i}(\mathrm{V}_{\mathrm{F}[\gamma_i]})$. Denote by Γ_i the unique (up to translation) $\mathcal{O}_{\mathrm{D}_i}$-lattice sequence in V_i such that

$$\mathfrak{P}_k(\Lambda) \cap \mathrm{B}_i = \mathfrak{P}_k(\Gamma_i), \quad k \in \mathbb{Z}.$$

We use Lemma 3.5 to choose uniformizers ϖ_i for $\mathrm{F}[\gamma_i]$ such that $\varpi_1 \mathrm{U}^1(\Lambda) = \varpi_2 \mathrm{U}^1(\Lambda)$, and tame corestrictions s_i on A relative to $\mathrm{F}[\gamma_i]/\mathrm{F}$ such that, for all $k \in \mathbb{Z}$ and $x \in \mathfrak{P}_k(\Lambda)$,

$$s_1(x) \equiv s_2(x) \pmod{\mathfrak{P}_{k+1}(\Lambda)}.$$

Again by Lemma 3.5, the residue fields $k_{\mathrm{F}[\gamma_i]}$ have a common image in $\mathfrak{P}_0(\Lambda)/\mathfrak{P}_1(\Lambda)$ so that we may identify them. Moreover, $\mathfrak{P}_0(\Gamma_i)/\mathfrak{P}_1(\Gamma_i)$ have a common image in $\mathfrak{P}_0(\Lambda)/\mathfrak{P}_1(\Lambda)$, as in the proof of Proposition 2.14.

Now let $[\Lambda, n, m-1, \beta_1]$ be a simple stratum such that $[\Lambda, n, m, \beta_1]$ is equivalent to $[\Lambda, n, m, \gamma_1]$. If the stratum $[\Lambda, n, m, \beta_1]$ is itself simple then the result follows from Lemma 3.4 (applied with β_1 in place of γ_1), so we assume this is not the case. We write $\beta_1 = \gamma_1 + b$, with $b \in \mathfrak{P}_{-m}(\Lambda)$, and pick a simple character $\vartheta \in \mathcal{C}(\Lambda, m-1, \gamma_1)$, so that $\vartheta\psi_b$ is a simple character in $\mathcal{C}(\Lambda, m-1, \beta_1)$.

By Proposition 2.14, the derived stratum $[\Gamma_1, m, m-1, s_1(b)]$ is equivalent to a simple stratum, which is non-scalar by Proposition 2.13, since $k_0(\beta_1, \Lambda) > k_0(\gamma_1, \Lambda)$. Thus, by

Proposition 2.7, $[\Gamma_1, m, m-1, s_1(b)]$ has irreducible minimum polynomial. However, by the choices of ϖ_i and s_i, we have

$$\mathscr{Y}(s_1(b), \Gamma_1) \equiv \mathscr{Y}(s_2(b), \Gamma_2) \pmod{\mathfrak{P}_1(\Lambda)}.$$

In particular (given the identifications we have made), we see that the strata $[\Gamma_i, m, m-1, s_i(b)]$ have the same minimum and characteristic polynomials. In particular, $[\Gamma_2, m, m-1, s_2(b)]$ has irreducible minimum polynomial so, by Proposition 2.7, it is equivalent to a simple stratum.

Finally, by Proposition 2.13, there is a simple stratum $[\Lambda, n, m-1, \beta_2]$ equivalent to $[\Lambda, n, m-1, \gamma_2 + b]$ and, by [**23**, Proposition 2.15], we have $\vartheta\psi_b \in \mathcal{C}(\Lambda, m-1, \beta_2)$. In particular, $\mathcal{C}(\Lambda, m-1, \beta_1) \cap \mathcal{C}(\Lambda, m-1, \beta_2) \neq \emptyset$ so, by [**6**, Theorem 4.16], we have $\mathcal{C}(\Lambda, m-1, \beta_1) = \mathcal{C}(\Lambda, m-1, \beta_2)$ as required. □

3.6. We will need one corollary of the translation principle, which is in fact a generalization of Proposition 2.14:

Corollary 3.6. — *Let $[\Lambda, n, m, \gamma]$ be a simple stratum and let $[\Lambda, n, 0, \beta]$ be a simple stratum such that $\mathcal{C}(\Lambda, m, \beta) \cap \mathcal{C}(\Lambda, m, \gamma) \neq \emptyset$. Suppose $\vartheta \in \mathcal{C}(\Lambda, m-1, \gamma)$ and $\theta \in \mathcal{C}(\Lambda, m-1, \beta)$ coincide on $\mathrm{H}^{m+1}(\gamma, \Lambda) = \mathrm{H}^{m+1}(\beta, \Lambda)$. Then there is $c \in \mathfrak{P}_{-m}(\Lambda)$ such that $\theta = \vartheta\psi_c$ and, for any such c, the derived stratum $[\Gamma_\gamma, m, m-1, s_\gamma(c)]$ is equivalent to a simple (or null) stratum.*

Proof. — The entire statement depends only on the equivalence class of the stratum $[\Lambda, n, m-1, \beta]$ so, by replacing with an equivalent stratum, we may assume this stratum is simple. Moreover, that such $c \in \mathfrak{P}_{-m}(\Lambda)$ exists is clear so we need only prove that the derived stratum is simple.

By the translation principle (Theorem 3.3), there is a simple stratum $[\Lambda, n, m-1, \beta']$ such that $[\Lambda, n, m, \beta']$ is equivalent to $[\Lambda, n, m, \gamma]$ and $\mathcal{C}(\Lambda, m-1, \beta') = \mathcal{C}(\Lambda, m-1, \beta)$. Then, by [**23**, Proposition 2.15], we have $\theta|\mathrm{H}^m(\beta', \Lambda) = \vartheta'\psi_{c'}$, for some $\vartheta' \in \mathcal{C}(\Lambda, m-1, \gamma)$ and $c' = \beta' - \gamma$. Moreover, the derived stratum $[\Gamma_\gamma, m, m-1, s_\gamma(c')]$ is simple (or null), by Proposition 2.14.

Now we have two simple characters, $\vartheta', \vartheta|\mathrm{H}^m(\gamma, \Lambda)$ in $\mathcal{C}(\Lambda, m-1, \gamma)$, which differ by the character $\psi_{c-c'}$. Since the simple characters are intertwined by $\mathrm{B}_\gamma^\times$, so is $\psi_{c-c'}$ and, in particular, its restriction to $\mathrm{U}^m(\Gamma_\gamma)$. Then, by Lemma 2.10, there is a $\delta \in \mathrm{F}[\gamma]$

such that $s_\gamma(c-c')-\delta\in\mathfrak{P}_{1-m}(\Gamma)$. In particular, $[\Gamma_\gamma,m,m-1,s_\gamma(c)]$ is equivalent to $[\Gamma_\gamma,m,m-1,s_\gamma(c')+\delta]$, which is simple (or null). □

4. Endo-classes and common approximations

In this section, we collect together some results concerning endo-classes of ps-characters and their relationship with common approximations (see [**13**, §8]). Much of this is implicit in [**13**] in the split case.

4.1. Let (k,β) be a simple pair and, for $i=1,2$, let $[\Lambda_i,n_i,m_i,\varphi_i(\beta)]$ be a realization in a simple central F-algebra A_i. According to [**19**, §3.3], there is a canonical *transfer* map

$$\tau:\mathcal{C}(\Lambda_1,m_1,\varphi_1(\beta))\to\mathcal{C}(\Lambda_2,m_2,\varphi_2(\beta)).$$

Denote by $\mathfrak{C}_{(k,\beta)}$ the set of pairs $([\Lambda,n,m,\varphi(\beta)],\theta)$ made of a realization $[\Lambda,n,m,\varphi(\beta)]$ of (k,β) in a simple central F-algebra and a simple character $\theta\in\mathcal{C}(\Lambda,m,\varphi(\beta))$. Then the transfer maps τ induce an equivalence relation on $\mathfrak{C}_{(k,\beta)}$.

Definition 4.1. — A *potential simple character* over F (or *ps-character* for short) is a triple (Θ,k,β) made of a simple pair (k,β) over F and an equivalence class Θ in $\mathfrak{C}_{(k,\beta)}$.

When the context is clear, we will often denote by Θ the ps-character (Θ,k,β). Given a realization $[\Lambda,n,m,\varphi(\beta)]$ of (k,β), we will denote by $\Theta(\Lambda,m,\varphi)$ the simple character θ such that the pair $([\Lambda,n,m,\varphi(\beta)],\theta)$ belongs to Θ.

Definition 4.2. — For $i=1,2$, let (Θ_i,k_i,β_i) be a ps-character over F. We say that these ps-characters are *endo-equivalent*, denoted:

$$\Theta_1\approx\Theta_2,$$

if $k_1=k_2$ and $[\mathrm{F}(\beta_1):\mathrm{F}]=[\mathrm{F}(\beta_2):\mathrm{F}]$, and if there exist a simple central F-algebra A and realizations $[\Lambda,n_i,m_i,\varphi_i(\beta_i)]$ of (k_i,β_i) in A, for $i=1,2$, such that the simple characters $\Theta_1(\Lambda,m_1,\varphi_1)$ and $\Theta_2(\Lambda,m_2,\varphi_2)$ intertwine in $\mathrm{A}^\times$.

That this defines an equivalence relation on ps-characters follows from a major result in [**6**]:

Proposition 4.3 (**[6**, Theorem 1.11]**)**. — *For $i = 1, 2$, let (Θ_i, k_i, β_i) be a ps-character over* F*, and suppose that $\Theta_1 \approx \Theta_2$. Let* A *be a simple central* F*-algebra and let $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ be realizations of (k_i, β_i) in* A*, for $i = 1, 2$. Then $\theta_1 = \Theta_1(\Lambda, m_1, \varphi_1)$ and $\theta_2 = \Theta_2(\Lambda, m_2, \varphi_2)$ intertwine in $\mathrm{A}^\times$.*

In the situation of Proposition 4.3, if $(F[\varphi_i(\beta_i)], \Lambda)$ have the same embedding type then we can apply Proposition 3.2 to conclude that the realizations θ_1, θ_2 are actually conjugate.

We will use the common convention that, for each $t \geq 0$, there is the *trivial ps-character* $\Theta_0^{(t)}$, whose realization on any lattice sequence Λ is the trivial character of the group $\mathrm{U}^{t+1}(\Lambda)$; then $\{\Theta_0^{(t)}\}$ forms a singleton equivalence class under endo-equivalence.

4.2. Let $[\Lambda, n, 0, \beta]$ be a simple stratum in $\mathrm{A} = \mathrm{End}_{\mathrm{D}}(\mathrm{V})$. Let $\theta \in \mathcal{C}(\Lambda, 0, \beta)$ be a simple stratum and denote by $(\Theta, 0, \beta)$ the ps-character that it determines – that is, θ is a realization of Θ.

For $t \geq 0$, let $[\Lambda, n, t, \beta^{(t)}]$ be a simple stratum equivalent to the pure stratum $[\Lambda, n, t, \beta]$ and write $\mathrm{E}^{(t)} = \mathrm{F}[\beta^{(t)}]$. Then the restriction $\theta|\mathrm{H}^{t+1}(\beta, \Lambda)$ is a simple character in $\mathcal{C}(\Lambda, t, \beta^{(t)})$ and we denote by $(\Theta^{(t)}, k^{(t)}, \beta^{(t)})$ the ps-character determined by this restriction, with $k^{(t)} = \lfloor t/e(\Lambda|\mathcal{O}_{\mathrm{E}^{(t)}}) \rfloor$.

Remark. We allow the case $t \geq n$, in which case we interpret the stratum $[\Lambda, n, t, \beta]$ to be equivalent to a null stratum $[\Lambda, t, t, 0]$ and $\Theta^{(t)}$ is the trivial ps-character.

Lemma 4.4. — *With notation as above let $(\Theta_\gamma, k, \gamma)$ be a ps-character which is endo-equivalent to $\Theta^{(t)}$. Then there is an embedding $\iota_\gamma : \mathrm{F}[\gamma] \hookrightarrow \mathrm{A}$ such that*

$$\theta|\mathrm{H}^{t+1}(\beta, \Lambda) = \Theta_\gamma(\Lambda, t, \iota_\gamma).$$

Proof. — Put $\mathrm{E}_\gamma = \mathrm{F}[\gamma]$. Since the ps-characters Θ_γ, $\Theta^{(t)}$ are endo-equivalent, the fields E_γ, $\mathrm{E}^{(t)}$ have the same invariants by [**6**, Lemma 4.8]:

$$e(\mathrm{E}_\gamma/\mathrm{F}) = e(\mathrm{E}^{(t)}/\mathrm{F}), \quad f(\mathrm{E}_\gamma/\mathrm{F}) = f(\mathrm{E}^{(t)}/\mathrm{F}), \quad \text{and} \quad k_{\mathrm{F}}(\gamma) = k_{\mathrm{F}}(\beta^{(t)}).$$

Then, by Lemma 2.1, there is an embedding $\iota_\gamma : \mathrm{K}_\gamma \hookrightarrow \mathrm{A}$ such that $[\Lambda, n, t, \iota_\gamma(\gamma)]$ is a pure stratum with the same embedding type as $[\Lambda, n, t, \beta^{(t)}]$, which is simple since $k_{\mathrm{F}}(\gamma) = k_{\mathrm{F}}(\beta^{(t)})$ and $[\Lambda, n, t, \beta^{(t)}]$ is simple.

Finally, since the ps-characters are endo-equivalent, by Propositions 4.3 and 3.2, the realization $\Theta_\gamma(\Lambda, t, \iota_\gamma)$ is conjugate to $\theta|\mathrm{H}^{t+1}(\beta,\Lambda)$ by some $u \in \mathfrak{K}(\Lambda)$. Conjugating our embedding ι_γ by u gives the desired embedding. □

For $j = 1, \dots, r$, let $\mathrm{A}^j = \mathrm{End}_\mathrm{D}(\mathrm{V}^j)$, let $[\Lambda^j, n_j, 0, \beta_j]$ be a simple stratum in A^j, write $\mathrm{E}_j = \mathrm{F}[\beta_j]$, and let $\theta_j \in \mathcal{C}(\Lambda^j, 0, \beta_j)$. We normalize so that the lattice sequences Λ^j have the same $\mathcal{O}_\mathrm{F}$-period e.

As above, for $t \geq 0$, let $[\Lambda^j, n_j, t, \beta_j^{(t)}]$ be a simple stratum equivalent to the pure stratum $[\Lambda^j, n_j, t, \beta_j]$, and write $\mathrm{E}_j^{(t)} = \mathrm{F}[\beta_j^{(t)}]$. The restriction $\theta_j|\mathrm{H}^{t+1}(\beta_j, \Lambda^j)$ is a simple character in $\mathcal{C}(\Lambda^j, t, \beta_j^{(t)})$ and we denote by $(\Theta_j^{(t)}, k_j^{(t)}, \beta_j^{(t)})$ the ps-character determined by this restriction, with $k_j^{(t)} = \lfloor t/e(\Lambda|\mathcal{O}_{\mathrm{E}_j^{(t)}})\rfloor$.

We put $\mathrm{V} = \mathrm{V}^1 \oplus \cdots \oplus \mathrm{V}^r$, and set $\Lambda = \Lambda^1 \oplus \cdots \oplus \Lambda^r$, a lattice sequence in V of $\mathcal{O}_\mathrm{F}$-period e. Write $\mathrm{A} = \mathrm{End}_\mathrm{D}(\mathrm{V})$ and denote by e^j the idempotents in $\mathfrak{P}_0(\Lambda)$ corresponding to the decomposition of V. We put $\beta = \sum_{j=1}^r \mathrm{e}^j \beta_j \mathrm{e}^j$. Then $[\Lambda, n, 0, \beta]$ is a stratum in A, with $n = \max_j n_j$. We write L for the stabilizer in $\mathrm{G} = \mathrm{Aut}_\mathrm{D}(\mathrm{V})$ of the decomposition $\mathrm{V} = \mathrm{V}^1 \oplus \cdots \oplus \mathrm{V}^r$, and A_L for its stabilizer in A.

Definition 4.5. — A *common approximation of* (θ_j) *of level* t *on* Λ is a pair $([\Lambda, n, t, \gamma], \vartheta)$ consisting of: a simple stratum $[\Lambda, n, t, \gamma]$ with $\gamma \in \mathrm{A}_\mathrm{L}$ and $0 \leq t \leq n$, such that

$$\mathrm{H}^{t+1}(\gamma, \Lambda) \cap \mathrm{L} = \prod_{j=1}^r \mathrm{H}^{t+1}(\beta_j, \Lambda^j);$$

and a simple character $\vartheta \in \mathcal{C}(\Lambda, 0, \gamma)$ such that

$$\vartheta|\mathrm{H}^{t+1}(\gamma, \Lambda) \cap \mathrm{L} = \theta_1 \otimes \cdots \otimes \theta_r.$$

When we have such a common approximation, we will identify γ with its images $\mathrm{e}^j\gamma\mathrm{e}^j$ in A^j.

Lemma 4.6. — *Let* $0 \leq t \leq n$. *Then the following are equivalent:*

(i) *There is a common approximation of* (θ_j) *of level* t *on* Λ.

(ii) *The ps-characters* $\Theta_j^{(t)}$ *are endo-equivalent.*

Proof. — (ii)⇒(i) Let $(\Theta_\gamma, k, \gamma)$ be a ps-character which is endo-equivalent to all $\Theta_j^{(t)}$. Then, by Lemma 4.4, for each j there is an embedding $\iota_j : \mathrm{F}[\gamma] \hookrightarrow \mathrm{A}^j$ such that $\Theta_\gamma(\Lambda^j, t, \iota_j) = \theta_j|\mathrm{H}^{t+1}(\beta_j, \Lambda^j)$. Denote by ι the diagonal embedding $\iota : \mathrm{F}[\gamma] \hookrightarrow \bigoplus_{j=1}^r \mathrm{A}^j \subseteq$

A and let ϑ be any simple character in $\mathcal{C}(\Lambda, 0, \iota(\gamma))$ which restricts to $\Theta_\gamma(\Lambda, t, \iota)$ on $\mathrm{H}^{t+1}(\gamma, \Lambda)$. Then $([\Lambda, n, t, \iota(\gamma)], \vartheta)$ is a common approximation as required.

(i)⇒(ii) Suppose $([\Lambda, n, t, \gamma], \vartheta)$ is a common approximation of (θ_j) of level t on Λ. Then the characters $\theta_j|\mathrm{H}^{t+1}(\beta_j, \Lambda^j)$ are simple characters in $\mathcal{C}(\Lambda^j, t, \gamma)$ and, by [**23**, Théorème 2.17], these characters are all transfers of each other relative to γ; hence the corresponding ps-characters (which are supported by the simple pair (k, γ), with $k = \lfloor t/e(\Lambda|\mathcal{O}_{\mathrm{F}[\gamma]})\rfloor$) are endo-equivalent. □

We suppose now that there is a common approximation $([\Lambda, n, t, \gamma], \vartheta)$ of (θ_j). Denote by B_γ the A-centralizer of $\mathrm{E}_\gamma = \mathrm{F}[\gamma]$, by V_γ a simple left B_γ-module, by D_γ the opposite algebra to $\mathrm{End}_{\mathrm{B}_\gamma}(\mathrm{V}_\gamma)$, and by s_γ a tame corestriction on A. Note that, since $\gamma \in \mathrm{A}_\mathrm{L}$, the restriction of s_γ to A^j is also a tame corestriction (see [**23**, Proposition 2.26]). Also, the idempotents e^j lie in B_γ so correspond to a decomposition $\mathrm{V}_\gamma = \mathrm{V}_\gamma^1 \oplus \cdots \oplus \mathrm{V}_\gamma^r$. Let Γ_γ be an $\mathcal{O}_{\mathrm{D}_\gamma}$-lattice sequence in V_γ such that $\mathfrak{P}_n(\Lambda) \cap \mathrm{B}_\gamma = \mathfrak{P}_n(\Gamma_\gamma)$, for all $n \in \mathrm{Z}$, and put $\Gamma_\gamma^j = \Gamma_\gamma \cap \mathrm{V}_\gamma^j$, for $1 \leq j \leq r$.

Since θ_j and ϑ coincide on $\mathrm{H}^{t+1}(\gamma, \Lambda)$, Corollary 3.6 says that there is $c_j \in \mathfrak{P}_{-t}(\Lambda^j)$ such that $\theta_j|\mathrm{H}^t(\beta_j, \Lambda^j) = \vartheta\psi_{c_j}$, and that the derived stratum $[\Gamma_\gamma^j, t, t-1, s_\gamma(c_j)]$ is equivalent to a simple (or null) stratum. The following result is a generalization of Corollary 2.9.

Corollary 4.7. — *In the situation above, the derived strata* $[\Gamma_\gamma^j, t, t-1, s_\gamma(c_j)]$ *have the same minimum polynomial if and only if the ps-characters* $\Theta_j^{(t-1)}$ *are endo-equivalent.*

Proof. — Suppose first that the minimum polynomials of the derived strata $[\Gamma_\gamma^j, t, t-1, s_\gamma(c_j)]$ are all the same. Note that they are irreducible since these strata are equivalent to simple strata. Then the derived stratum $[\Gamma_\gamma, t, t-1, s_\gamma(c)]$ is equivalent to a simple stratum by Corollary 2.8 and, by Proposition 2.13, there is a simple stratum $[\Lambda, n, t-1, \gamma']$ equivalent to $[\Lambda, n, t-1, \gamma+c]$, so that $\vartheta\psi_c \in \mathcal{C}(\Lambda, t-1, \gamma')$. Then, for any $\vartheta' \in \mathcal{C}(\Lambda, 0, \gamma')$ extending $\vartheta\psi_c$, the pair $([\Lambda, n, t-1, \gamma'], \vartheta')$ is a common approximation of level $t-1$. Hence, by Lemma 4.6, the ps-characters $\Theta_j^{(t-1)}$ are endo-equivalent.

Conversely, suppose the ps-characters $\Theta_j^{(t-1)}$ are endo-equivalent so, by Lemma 4.6, there is a common approximation $([\Lambda, n, t-1, \gamma'], \vartheta')$ of level $t-1$. We then have $\vartheta' = \vartheta\psi_c$ and, by Corollary 3.6, the derived stratum $[\Gamma_\gamma, t, t-1, s_\gamma(c)]$ is equivalent to a simple (or null) stratum. Hence, by Corollary 2.8, the derived strata $[\Gamma_\gamma^j, t, t-1, s_\gamma(c_j)]$ have the same minimum polynomial. □

5. Simple types

In this section we recall some results from [**21**] concerning simple types. In later sections we will need these in slightly more generality than in *op. cit.* – in particular, in the case where we have a non-strict lattice sequence. Already in the case of $\mathrm{GL}(n, \mathrm{F})$, simple types on non-strict lattice sequence are required in [**13**], although this is not immediately apparent. The proofs are mostly identical to those in [**20, 21**].

5.1. Let $[\Lambda, n, 0, \beta]$ be a simple stratum in $\mathrm{A} = \mathrm{End}_{\mathrm{D}}(\mathrm{V})$, and use all the usual notation from the previous sections. Since β is fixed, we will omit it from the notations; when Λ is fixed, we will omit that also.

Lemma 5.1. — *Let $\theta \in \mathcal{C}(\Lambda, 0, \beta)$ be a simple character. Then there is a unique irreducible representation η of J^1 which contains θ; moreover, $\eta|\mathrm{H}^1$ is a multiple of θ, the dimension $\dim(\eta) = (\mathrm{J}^1 : \mathrm{H}^1)^{1/2}$ and*

$$\dim \mathrm{I}_g(\eta) = \begin{cases} 1 & \textit{if } g \in \mathrm{J}^1\mathrm{B}^\times\mathrm{J}^1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. — The proof of all but the final assertion is identical to that of [**8**, Proposition 5.1.1)], replacing [**8**, Theorem 3.4.1)] by [**23**, Proposition 2.31]. The proof of the final assertion is identical to that of [**8**, Proposition 5.1.8)], replacing the exact sequences there by those of [**23**, Proposition 2.27]. □

Lemma 5.2. — *For $i = 1, 2$, let $[\Lambda^i, n_i, 0, \beta]$ be a simple stratum in* A, *let $\theta_i \in \mathcal{C}(\Lambda^i, 0, \beta)$, and let η_i be the unique irreducible representation of $\mathrm{J}^1(\beta, \Lambda^i)$ which contains θ_i. Then*

$$\frac{\dim(\eta_1)}{\dim(\eta_2)} = \frac{(\mathrm{J}^1(\beta, \Lambda^1) : \mathrm{J}^1(\beta, \Lambda^2))}{(\mathrm{U}^1(\Lambda^1) \cap \mathrm{B} : \mathrm{U}^1(\Lambda^2) \cap \mathrm{B})}.$$

Proof. — Again, the proof is identical to that of [**8**, Proposition 5.1.2], replacing the exact sequence there with [**23**, Proposition 2.27]. □

5.2. Recall that a *β-extension of θ* is a representation κ of J which extends the representation η given by Lemma 5.1 and such that $I_G(\kappa) \supset \mathrm{B}^\times$. In the case that Λ is strict, the existence of β-extensions is given by [**20**, Théorème 2.28]. Using this, we proceed here via a simplified version of the compatibility argument used there.

Definition 5.3. — Let $[\Lambda, n, 0, \beta]$ be a simple stratum in A and let Λ' be an E-pure lattice sequence in V such that $\mathfrak{P}_0(\Lambda) = \mathfrak{P}_0(\Lambda')$. Let θ, θ' be simple characters which are realizations of the same ps-character on Λ, Λ' respectively, and let κ, κ' be extensions of the representations η, η' given by Lemma 5.1 respectively. We say that κ, κ' are *compatible* (or *mutually coherent*) if

$$\mathrm{Ind}_{\mathrm{J}}^{(\mathrm{U}(\Lambda)\cap \mathrm{B})\mathrm{U}^1(\Lambda)} \kappa \simeq \mathrm{Ind}_{\mathrm{J}(\beta,\Lambda')}^{(\mathrm{U}(\Lambda)\cap \mathrm{B})\mathrm{U}^1(\Lambda)} \kappa'.$$

Proposition 5.4. — *With the notations of Definition 5.3, the notion of compatibility induces a bijection*

$$\{\beta\text{-}extensions\ of\ \theta\} \longleftrightarrow \{\beta\text{-}extensions\ of\ \theta'\}.$$

In particular, there is a β-extension κ of θ, and then the set of β-extension of θ is given by

$$\left\{\kappa \otimes \left(\chi \circ \mathrm{N}_{\mathrm{B}/\mathrm{E}}\right) : \chi \in \widehat{\mathrm{U}_{\mathrm{E}}/\mathrm{U}_{\mathrm{E}}^1}\right\}.$$

Proof. — The first assertion follows as in [**20**, Lemmes 2.23,2.24] (cf. also [**8**, Proposition 5.25]). Now taking Λ' to be the strict lattice sequence in V with the same image as Λ, the final assertion follows from [**20**, Théorème 2.28]. □

5.3. We continue with a simple stratum $[\Lambda, n, 0, \beta]$ and a simple character $\theta \in \mathcal{C}(\Lambda, 0, \beta)$, together with the unique irreducible representation η of J^1 containing θ. Let V_{E} be a simple left B-module, let D_{E} be the opposite algebra to $\mathrm{End}_{\mathrm{B}}(\mathrm{V}_{\mathrm{E}})$, and set $m_{\mathrm{E}} = \dim_{\mathrm{D}_{\mathrm{E}}} \mathrm{V}_{\mathrm{E}}$. We write Γ for the unique (up to translation) $\mathcal{O}_{\mathrm{D}_{\mathrm{E}}}$-lattice sequence on V_{E} such that $\mathfrak{P}_k(\Lambda) \cap \mathrm{B} = \mathfrak{P}_k(\Gamma)$, for all $k \in \mathbb{Z}$.

We suppose given a decomposition $\mathrm{V} = \mathrm{V}^1 \oplus \cdots \oplus \mathrm{V}^r$ which is *subordinate to* $\mathfrak{P}_0(\Gamma)$ in the sense of [**23**, Définition 5.1]: that is, it is a decomposition of $\mathrm{E}\otimes\mathrm{D}$-bimodules and, writing e^j for the idempotents of $\mathfrak{P}_0(\Gamma)$ defined by the decomposition and $m_j = \dim_{\mathrm{D}_{\mathrm{E}}} \mathrm{e}^j \mathrm{V}_{\mathrm{E}}$, there is an isomorphism of E-algebras $\Psi : \mathrm{B} \to \mathrm{M}_{m_{\mathrm{E}}}(\mathrm{D}_{\mathrm{E}})$ such that:

(i) for $1 \leq j \leq r$, the idempotent $\Psi(\mathrm{e}^j)$ is $\mathrm{I}^j = \mathrm{diag}(0, \ldots, \mathrm{Id}_{m_j}, \ldots, 0)$:

(ii) The hereditary order $\Psi(\mathfrak{P}_0(\Gamma))$ is the $\mathcal{O}_{\mathrm{E}}$-subalgebra of $\mathrm{M}_{m_{\mathrm{E}}}(\mathcal{O}_{\mathrm{D}_{\mathrm{E}}})$ consisting of matrices whose reduction modulo $\mathfrak{p}_{\mathrm{D}_{\mathrm{E}}}$ is upper triangular by blocks of size $(m_1, \ldots, m_r)$.

Note then that $\Lambda^j = \Lambda \cap \mathrm{V}^j$ is in the affine class of a strict lattice sequence of $\mathcal{O}_{\mathrm{D}_{\mathrm{E}}}$-period 1 in V^j.

Let P be the parabolic subgroup of G stabilizing the flag

$$\{0\} \subset \mathrm{V}^1 \subset \mathrm{V}^1 \oplus \mathrm{V}^2 \subset \cdots \subset \mathrm{V},$$

and write $\mathrm{P} = \mathrm{LN}$, where L is the stabilizer of the decomposition $\mathrm{V} = \bigoplus_{j=1}^r \mathrm{V}^j$ and N is the unipotent radical. Write $\mathrm{P}_- = \mathrm{LN}_-$ for the opposite parabolic relative to L.

We define the groups

$$\mathrm{J_P} = \mathrm{H}^1(\mathrm{J} \cap \mathrm{P}), \quad \mathrm{J_P^1} = \mathrm{H}^1\left(\mathrm{J}^1 \cap \mathrm{P}\right), \quad \mathrm{H_P^1} = \mathrm{H}^1\left(\mathrm{J}^1 \cap \mathrm{N}\right),$$

and define the character θ_P of $\mathrm{H_P^1}$ by $\theta_\mathrm{P}(hu) = \theta(h)$, for $h \in \mathrm{H}^1$ and $u \in \mathrm{J}^1 \cap \mathrm{N}$. We also put $\mathrm{J_L} = \mathrm{J} \cap \mathrm{L}$ and $\mathrm{J_L^1} = \mathrm{J}^1 \cap \mathrm{L}$ and notice that, since the decomposition is subordinate to $\mathfrak{P}_0(\Lambda) \cap \mathrm{B}$, we have

$$\mathrm{J_P}/\mathrm{J_P^1} \simeq \mathrm{J_L}/\mathrm{J_L^1} \simeq \mathrm{U}(\Gamma)/\mathrm{U}^1(\Gamma) \simeq \mathrm{J}/\mathrm{J}^1.$$

In particular, given a representation of $\mathrm{U}(\Gamma)$ trivial on $\mathrm{U}^1(\Gamma)$, we can also regard it as a representation of $\mathrm{J_P}$ (respectively $\mathrm{J_L}$, J) trivial on $\mathrm{J_P^1}$ respectively $\mathrm{J_L^1}$, J^1).

The following Proposition summarizes the results of [**23**, Propositions 5.3–5] (see also *op. cit.* §5.8); the results there are in the case that Λ is strict but, given our preliminary results above, identical proofs apply in the general case.

Proposition 5.5. — *Let η_P denote the natural representation of $\mathrm{J_P^1}$ on on the $\mathrm{J} \cap \mathrm{N}$-invariants of η. Then η_P is the unique irreducible representation of $\mathrm{J_P^1}$ which contains θ_P. Moreover, $\mathrm{Ind}_{\mathrm{J_P}}^{\mathrm{J}^1}\eta_\mathrm{P}$ is isomorphic to η and*

$$\dim \mathrm{I}_g(\eta_\mathrm{P}) = \begin{cases} 1 & \text{if } g \in \mathrm{J_P^1}\mathrm{B}^\times\mathrm{J_P^1}, \\ 0 & \text{otherwise.} \end{cases}$$

5.4. Now let κ be a β-extension of θ and let κ_P denote the natural representation of $\mathrm{J_P}$ on on the $\mathrm{J} \cap \mathrm{N}$-invariants of κ. The proof of [**23**, Proposition 5.8] (see also *op. cit.* §5.8) again generalizes to the non-strict case and gives:

Proposition 5.6. — *κ_P is an irreducible representation of $\mathrm{J_P}$ with the following properties:*

(i) *$\kappa_\mathrm{P}|\mathrm{H}^1(\beta, \Lambda)$ is a multiple of θ;*
(ii) *κ_P is trivial on $\mathrm{J_P} \cap \mathrm{N}$ and $\mathrm{J_P} \cap \mathrm{N}_-$;*
(iii) *$\kappa_\mathrm{P}|\mathrm{J_L} \simeq \kappa_1 \otimes \cdots \otimes \kappa_r$, for some β-extensions κ_j containing θ_j;*
(iv) *$\mathrm{I_G}(\kappa_\mathrm{P}) = \mathrm{J_P}\mathrm{B}^\times\mathrm{J_P}$;*

(v) *if* $m_j = m_k$ *then* $\kappa_j \simeq \kappa_k$.

(vi) *if* ξ *is an irreducible representation of* $\mathrm{U}(\Gamma)$ *trivial on* $\mathrm{U}^1(\Gamma)$*, then*

$$\mathrm{Ind}_{\mathrm{J_P}}^{\mathrm{J}}(\kappa_{\mathrm{P}} \otimes \xi) \simeq \kappa \otimes \xi, \qquad \textit{and} \qquad \mathrm{I_G}(\kappa_{\mathrm{P}} \otimes \xi) = \mathrm{J_P I_{B^\times}}(\xi)\mathrm{J_P}.$$

Proof. — The only property not given by [**23**, Proposition 5.8 or §5.8] is (v), where imitate the proof of [**8**, Corollary 7.2.6]. There is a permutation matrix in $w \in \mathrm{M}_{m_\mathrm{E}}(\mathrm{D_E})$ (which we have identified with B via Ψ above) which swaps V^j with V^k, and leaves all other V^i fixed. In particular, it lies in B and normalizes $\mathrm{J_P} \cap \mathrm{L}$. Then w intertwines κ_P so normalizes $\kappa_\mathrm{P}|\mathrm{J_L}$ and hence induces an isomorphism between κ_j and κ_k. □

5.5. Finally, we consider the case where all m_j are equal to some integer s, so that $\mathrm{U}(\Gamma)/\mathrm{U}^1(\Gamma) \simeq \mathrm{GL}_s(k_{\mathrm{D_E}})^r$ and let ξ be the inflation to $\mathrm{U}(\Gamma)$ of the representation $\sigma^{\otimes r}$, for σ an irreducible cuspidal representation of $\mathrm{GL}_s(k_{\mathrm{D_E}})$.

We put $\lambda = \kappa \otimes \xi$, $\lambda_\mathrm{P} = \kappa_\mathrm{P} \otimes \xi$ and $\lambda_\mathrm{L} = \lambda_P|\mathrm{J_L}$.

Proposition 5.7. — *The pair* $(\mathrm{J_P}, \lambda_\mathrm{P})$ *is a cover of* $(\mathrm{J_L}, \lambda_\mathrm{L})$ *and*

$$\mathscr{H}(\mathrm{G}, \lambda_\mathrm{P}) \cong \mathscr{H}(r, q_{\mathrm{D_E}}^s).$$

We remark that the parameter $q_{\mathrm{D_E}}^s$ here is the same as that in Theorem B of the introduction, by [**22**, Theorem 4.6].

Proof. — In the case that Λ is strict, this is given by [**21**, Proposition 5.5, Théorème 4.6]. The idea of the proof is to reduce to this case, as in the proof of [**21**, Théorème 5.6]. Let $\mathscr{L}$ be the strict lattice sequence with the same image as Λ and we make the same constructions for $\mathscr{L}$, which we denote with the superscript $^{\mathscr{L}}$. In particular, we choose a β-extension $\kappa^{\mathscr{L}}$ compatible with κ. Hence (as in [**21**, Proposition 4.5]) we have a support-preserving isomorphism

$$\mathscr{H}(\mathrm{G}, \lambda) \cong \mathscr{H}(\mathrm{G}, \lambda^{\mathscr{L}}).$$

Moreover, we have $\lambda = \mathrm{Ind}_{\mathrm{J_P}}^{\mathrm{J}} \lambda_\mathrm{P}$ and $\lambda^{\mathscr{L}} = \mathrm{Ind}_{\mathrm{J_P^{\mathscr{L}}}}^{\mathrm{J^{\mathscr{L}}}} \lambda_\mathrm{P}^{\mathscr{L}}$, so we get a support-preserving isomorphism

$$\mathscr{H}(\mathrm{G}, \lambda_\mathrm{P}) \cong \mathscr{H}(\mathrm{G}, \lambda_\mathrm{P}^{\mathscr{L}}).$$

Then the assertions follow from (the proof of) [**21**, Proposition 5.5]. □

Definition 5.8. — A pair (J, λ) as in this paragraph is called a *simple type*; if $r = 1$ then it is called a *maximal simple type*.

Recall that the main result of **[23]** (Théorèmes 5.21, 5.23) is that every irreducible cuspidal representation of G contains a maximal simple type.

6. Intertwining and conjugacy

In this section we consider the unicity of the simple type contained in an irreducible representation π of $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$. That is, we suppose the inertial class $\mathfrak{s}(\pi)$ of π is homogeneous: there are a positive integer r dividing m, an irreducible cuspidal representation ρ of the group $\mathrm{G}_0 = \mathrm{GL}_{m/r}(\mathrm{D})$ and unramified characters χ_i of G_0, with $i \in \{1, \ldots, r\}$, such that π is isomorphic to a quotient of the normalized parabolically induced representation $\rho\chi_1 \times \cdots \times \rho\chi_r$. Unlike the situation for $\mathrm{D} = \mathrm{F}$, the simple type is *not* uniquely determined up to conjugacy in general, as there is a galois action we must take into account.

We consider the set $\mathfrak{S}$ of *sound simple types*

$$\mathfrak{S} = \left\{ \begin{array}{l} \text{simple types } (\mathrm{J}(\beta, \Lambda), \lambda) \text{ such that } \Lambda \text{ is strict, } \mathfrak{P}_0(\Lambda) \\ \text{is principal and } (\mathrm{F}[\beta], \Lambda) \text{ is soundly embedded} \end{array} \right\}.$$

For $(\mathrm{J}, \lambda) \in \mathfrak{S}$, we use all the associated notation of §5: that is, there are a (sound) simple stratum $[\Lambda, n, 0, \beta]$, a simple character $\theta \in \mathcal{C}(\Lambda, 0, \beta)$, a β-extension κ and a representation ξ which is the inflation to $\mathrm{U}(\Gamma)$ of the representation $\overline{\xi} = \sigma^{\otimes r}$ of

$$\mathrm{U}(\Gamma)/\mathrm{U}^1(\Gamma) \simeq \mathrm{GL}_s(k_{\mathrm{D_E}})^r,$$

for σ an irreducible cuspidal representation of $\mathrm{GL}_s(k_{\mathrm{D_E}})$. Note that, implicit in the isomorphism above are the choice of a decomposition $\mathrm{V} = \mathrm{V}^1 \oplus \cdots \oplus \mathrm{V}^r$ subordinate to $\mathfrak{P}_0(\Gamma)$ and the choice of an E-algebra isomorphism $\Psi : \mathrm{B} \to \mathrm{M}_{m_\mathrm{E}}(\mathrm{D_E})$ as in paragraph 5.3.

We fix a uniformizer ϖ of $\mathrm{D_E}$. The Galois group $\mathscr{G} = \mathrm{Gal}(k_{\mathrm{D_E}}/k_\mathrm{E})$ identifies, via reduction, with the group generated by $\mathrm{Ad}(\varpi)$, the inner automorphism given by conjugation by ϖ. The Galois group $\mathscr{G}$ acts on the representations of $\mathrm{GL}_s(k_{\mathrm{D_E}})$. Moreover, a different choice of E-algebra isomorphism Ψ could result in a change in the identification $\mathrm{U}(\Gamma)/\mathrm{U}^1(\Gamma) \simeq \mathrm{GL}_s(k_{\mathrm{D_E}})^r$ by conjugating each factor by an element of $\mathscr{G}$, rather than just by an inner automorphism. Thus we define $[\sigma]$ to be the orbit of σ under the action of $\mathscr{G}$ and set

$$[\lambda] = \left\{ \begin{array}{l} \text{equivalence classes of representations } \kappa \otimes \xi', \text{ for } \xi' \\ \text{the inflation to } \mathrm{U}(\Gamma) \text{ of } \sigma_1 \otimes \cdots \otimes \sigma_r, \text{ with } \sigma_i \in [\sigma] \end{array} \right\}.$$

We also define an equivalence relation on $\mathfrak{S}$ by: $(\mathrm{J}, \lambda) \sim (\mathrm{J}, \lambda')$ if and only if there is an irreducible representation π of G such that π contains both λ and λ'.

***Theorem 6.1* (cf. [8, Theorem 5.7.1]).** — *Let* (J,λ) *and* (J',λ') *be sound simple types. Then* $(\mathrm{J},\lambda)\sim(\mathrm{J}',\lambda')$ *if and only if there exists* $g\in\mathrm{G}$ *such that* ${}^g\mathrm{J}'=\mathrm{J}$ *and* $[{}^g\lambda']=[\lambda]$.

Proof. — The proof follows that of [**8**, Theorem 5.7.1]. Suppose first that $(\mathrm{J},\lambda)\sim(\mathrm{J}',\lambda')$ and use all notation as above, with a prime $'$ to indicate the corresponding objects for $\mathrm{J}',\lambda')$, in particular writing $\mathrm{E}'=\mathrm{F}[\beta']$. We also write Θ for the ps-character defined by θ. Then [**6**, Theorems 9.2, 9.3] imply that:

- Θ' is endo-equivalent to Θ;
- (E,Λ) and (E',Λ') have the same embedding type.

In particular, by Propositions 4.3 and 3.2, and the definition of embedding type, there is $g\in\mathrm{G}$ such that $g\Lambda'$ is in the translation class of Λ, ${}^g\mathrm{H}^1(\beta',\Lambda')=\mathrm{H}^1(\beta,\Lambda)$ and ${}^g\theta'=\theta$. Replacing λ' by ${}^g\lambda'$ we may assume that $g=1$, so that $\theta'=\theta$; moreover, since changing Λ in its translation class affects nothing, we may assume $\Lambda'=\Lambda$.

Now the $\mathrm{U}(\Lambda)$ intertwining of θ is $\mathrm{J}(\beta,\Lambda)$ so we get $\mathrm{J}'=\mathrm{J}$ and $\mathrm{J}^1(\beta',\Lambda')=\mathrm{J}^1(\beta,\Lambda)$. By unicity in Lemma 5.1, we get $\eta'=\eta$. Moreover, since the intertwining of θ is $\mathrm{J}\mathrm{B}^\times\mathrm{J}=\mathrm{J}(\mathrm{B}')^\times\mathrm{J}$, the β-extension κ is also a β'-extension and we may assume $\kappa'=\kappa$.

As in the proof of [**8**, Theorem 5.7.1], the cuspidality of $\overline{\xi}$ can be interpreted in purely group-theoretic terms. In particular, if we identify J/J^1 with $\mathrm{GL}_s(k_{\mathrm{D_E}})^r$, then $\overline{\xi'}$ decomposes as $\sigma'_1\otimes\cdots\otimes\sigma'_r$ with all σ'_i cuspidal.

Now λ, λ' are contained in some irreducible representation π of G and are therefore intertwined by some $x\in\mathrm{G}$. Since λ, λ' both restrict to a multiple of θ on H^1, we have also that x intertwines θ and thus $x\in\mathrm{J}^1\mathrm{B}^\times\mathrm{J}^1$. In particular, we may assume $x\in\mathrm{B}^\times$. Then, arguing as in [**8**, Proposition 5.3.2], we see that x intertwines ξ with ξ', when we interpret them as representations of $\mathrm{U}(\Gamma)$.

To finish, we argue again as in the proof of [**8**, Theorem 5.7.1], using results from [**17**]. In particular, we will use some notation from [**17**, §0.8], writing $\tilde{\mathrm{W}}_{\mathrm{B}}$ for the generalized affine Weyl group in $\mathrm{B}^\times$, which we have identified with $\mathrm{GL}_{m_{\mathrm{E}}}(\mathrm{D_E})$ via Ψ. By the affine Bruhat decomposition, we may assume $x\in\tilde{\mathrm{W}}_{\mathrm{B}}$.

Since $\overline{\xi}$ and $\overline{\xi'}$ are cuspidal, the same proof as that of [**17**, Proposition 1.2] shows that x normalizes $\mathrm{L}\cap\mathrm{U}(\Gamma)$, where L is the Levi subgroup of G which is the stabilizer of the decomposition $\mathrm{V}=\mathrm{V}^1\oplus\cdots\oplus\mathrm{V}^r$. Likewise, [**17**, Lemma 1.5] implies that

$$\mathrm{Hom}_{\mathrm{U}(\Gamma)\cap x^{-1}\mathrm{U}(\Gamma)x}(\xi',\xi^x)=\mathrm{Hom}_{\mathrm{U}(\Gamma)/\mathrm{U}^1(\Gamma)}(\overline{\xi'},\overline{\xi}^x).$$

Now $\overline{\xi}^x = \sigma_1 \otimes \cdots \otimes \sigma_r$, with each $\sigma_i \in [\sigma]$ and, since $\overline{\xi'}$, $\overline{\xi}^x$ are irreducible, we deduce $\sigma'_i \simeq \sigma_i \in [\sigma]$. Hence the equivalence class of λ' is in $[\lambda]$, so $[\lambda'] = [\lambda]$, as required.

The converse is given by [**23**, Proposition 5.19]. □

Although, in general, equivalent sound simple types are not conjugate, they are in the special case of cuspidal representations:

Corollary 6.2. — *Suppose* (J, λ), (J', λ') *are maximal simple types. Then* $(\mathrm{J}, \lambda) \sim (\mathrm{J}', \lambda')$ *if and only if there exists* $g \in \mathrm{G}$ *such that* ${}^g\mathrm{J}' = \mathrm{J}$ *and* ${}^g\lambda' \simeq \lambda$.

Proof. — Suppose (J, λ) and (J', λ') are equivalent maximal simple types. By Theorem 6.1, there there exists $g \in \mathrm{G}$ such that ${}^g\mathrm{J}' = \mathrm{J}$ and $[{}^g\lambda'] = [\lambda]$. That is, as in the proof of Theorem 6.1, we can write ${}^g\lambda' \simeq \kappa \otimes \xi'$, with $\overline{\xi'} \simeq \overline{\xi}^\gamma$, for some $\gamma \in \mathscr{G}$. But the action of γ can be realized as conjugation by a power of ϖ, which normalizes Γ, so there is $y \in \mathfrak{K}(\Gamma)$ such that ${}^y\xi' \simeq \xi$. Since y also normalizes κ, we deduce that ${}^{yg}\lambda' \simeq \lambda$. □

This also completes the proof of Theorem A of the introduction.

7. Semisimple types

Suppose we have cuspidal representations π_j of $\mathrm{G}^j = \mathrm{Aut}_\mathrm{D}(\mathrm{V}^j)$, for $1 \le j \le r$, and we think of $\mathrm{L} = \prod_{j=1}^r \mathrm{G}_j$ as a Levi subgroup in $\mathrm{G} = \mathrm{Aut}_\mathrm{D}(\mathrm{V})$, where $\mathrm{V} = \bigoplus_{j=1}^r \mathrm{V}^j$. The aim of this section is to prove Theorem C: a maximal simple type for $(\mathrm{L}, \pi_1 \otimes \cdots \otimes \pi_r)$ admits a cover, with an explicitly computable Hecke algebra.

For $j = 1, \dots, r$, the cuspidal representation π_j contains a (maximal) simple type $(\mathrm{J}(\beta_j, \Lambda^j), \lambda_j)$, where $[\Lambda^j, n_j, 0, \beta_j]$ is a simple stratum in $\mathrm{A}^j = \mathrm{End}_\mathrm{D}(\mathrm{V}^j)$, θ_j is a simple character of $\mathrm{H}^1(\beta_j, \Lambda^j)$, κ_j is a β_j-extension containing θ_j, σ_j is a cuspidal representation of $\mathrm{J}(\beta_j, \Lambda^j)/\mathrm{J}^1(\beta_j, \Lambda^j)$ and $\lambda_j = \kappa_j \otimes \sigma_j$. We write $(\Theta_j, 0, \beta_j)$ for the ps-character defined by θ_j. Then our type in L is $(\mathrm{J}_\mathrm{L}, \lambda_\mathrm{L})$, given by

$$\mathrm{J}_\mathrm{L} = \prod_{j=1}^r \mathrm{J}(\beta_j, \Lambda^j), \quad \lambda_\mathrm{L} = \lambda_1 \otimes \cdots \otimes \lambda_r.$$

For $j = 1, \dots, r$, we write B^j for the A_j-centralizer of β_j. Then B_j has the form $\mathrm{End}_{\mathrm{D}_{\mathrm{E}_j}}(\mathrm{W}^j)$, for some right $\mathrm{D}_{\mathrm{E}_j}$-vector space W^j. We write Γ^j for the unique strict $\mathcal{O}_{\mathrm{D}_{\mathrm{E}_j}}$-lattice sequence in W^j such that $\mathfrak{P}_0(\Gamma^j) = \mathfrak{P}_0(\Lambda^j) \cap \mathrm{B}^j$. Since π_j is cuspidal, Γ^j is a sequence of $\mathcal{O}_{\mathrm{D}_{\mathrm{E}_j}}$-period 1 and then the normalizer in $\mathrm{G}^j \cap \mathrm{B}^j$ of Γ^j is just $\mathfrak{K}(\Lambda^j) \cap \mathrm{B}^j$; moreover, Λ^j is a strict lattice sequence and $\mathfrak{P}_0(\Lambda^j)$ is a principal order in A^j.

7.1. The homogeneous case. — We suppose first that the $(\Theta_j, 0, \beta_j)$ are all endo-equivalent to some fixed ps-character $(\Theta, 0, \beta)$.

By Lemma 4.4, for each j there is a realization $\Theta(\Lambda^j, 0, \iota_j)$ equal to θ_j. Hence we may (and do) assume that all θ_j are defined relative to the same simple pair $(0, \beta)$ and are realizations of the same ps-character Θ.

We have $\mathrm{V} = \bigoplus_{j=1}^r \mathrm{V}^j$ and put $\mathrm{W} = \bigoplus_{j=1}^r \mathrm{W}^j$, so that $\mathrm{B} = \mathrm{End}_{\mathrm{D}_\mathrm{E}}(\mathrm{W})$. Write e^j for the idempotent in B with image W^j and kernel $\bigoplus_{i\neq j} \mathrm{W}^i$. As an element of A it has image V^j and kernel $\bigoplus_{i\neq j} \mathrm{V}^i$.

Let Γ be an $\mathcal{O}_{\mathrm{D}_\mathrm{E}}$-lattice sequence in W such that:

(i) $\Gamma \cap \mathrm{W}^j$ is in the affine class of Γ^j, and

(ii) Γ is subordinate to the decomposition $\mathrm{W} = \bigoplus_{j=1}^r \mathrm{W}^j$.

Note that condition (ii) is generically satisfied: that is, amongst all lattice sequences satisfying (i), those also satisfying (ii) are dense (in the building of $\mathrm{B}^\times$). A particular example of such a sequence is given by

$$\Gamma(k) \;=\; \bigoplus_{j=1}^{r} \Gamma^j\left(\left\lfloor \frac{k+j}{r} \right\rfloor\right), \qquad k \in \mathbb{Z},$$

which is strict of $\mathcal{O}_{\mathrm{D}_\mathrm{E}}$-period r but not principal in general.

We fix Γ satisfying (i),(ii) and let Λ be the corresponding $\mathcal{O}_\mathrm{D}$-lattice sequence in V given by [**23**, Théorème 1.7]. Then $\mathrm{e}^j \in \mathfrak{P}_0(\Gamma) \subseteq \mathfrak{P}_0(\Lambda)$ so Λ is decomposed by (indeed subordinate to) the decomposition $\mathrm{V} = \bigoplus_{j=1}^r \mathrm{V}^j$ and the lattice sequence $r \mapsto \Lambda(r) \cap \mathrm{V}^j$ is in the affine class of Λ^j. In fact, replacing Λ^j by this sequence changes nothing in the construction of the type $(\mathrm{J}(\beta_j, \Lambda^j), \lambda_j)$ so we may (and do) assume this done, in which case $\Lambda = \bigoplus_{j=1}^r \Lambda^j$.

Let P be the parabolic subgroup of G stabilizing the flag

$$\{0\} \subset \mathrm{V}^1 \subset \mathrm{V}^1 \oplus \mathrm{V}^2 \subset \cdots \subset \mathrm{V},$$

and write $\mathrm{P} = \mathrm{LN}$, where L is the stabilizer of the decomposition $\mathrm{V} = \bigoplus_{j=1}^r \mathrm{V}^j$ and N is the unipotent radical. Write $\mathrm{P}_- = \mathrm{LN}_-$ for the opposite parabolic relative to L.

Now $[\Lambda, n, 0, \beta]$ is a simple stratum in $\mathrm{A} = \mathrm{End}_{\mathrm{D}}(\mathrm{V})$, for a suitable integer n, and we have $\mathrm{H}^1(\beta,\Lambda)\cap\mathrm{L} \cong \prod_{j=1}^{r} \mathrm{H}^1(\beta,\Lambda^j)$, with similar decompositions for J^1 and J. Moreover, $\theta = \Theta(V, 0, \Lambda)$ is a simple character such that

$$\theta|\mathrm{H}^1(\beta,\Lambda)\cap\mathrm{L} = \theta_1 \otimes\cdots\otimes\theta_r.$$

As in §5, we define the groups

$$\mathrm{J_P} = \mathrm{H}^1(\beta,\Lambda)\left(\mathrm{J}(\beta,\Lambda)\cap\mathrm{P}\right), \qquad \mathrm{J}^1_{\mathrm{P}} = \mathrm{H}^1(\beta,\Lambda)\left(\mathrm{J}^1(\beta,\Lambda)\cap\mathrm{P}\right),$$

noting that $\mathrm{J_L} = \mathrm{J_P}\cap\mathrm{L}$.

Let κ_{P} be the irreducible representation of $\mathrm{J_P}$ given by Proposition 5.6, so that $\kappa_{\mathrm{P}}|\mathrm{J_L} \simeq \bigotimes_{j=1}^{r}\kappa'_j$, for some β-extensions κ'_j containing θ_j. Then we can choose the decompositions $\lambda_j = \kappa_j\otimes\sigma_j$ of the maximal simple types above so that $\kappa_j = \kappa'_j$, which we assume done.

We define an equivalence relation $\sim$ on $\{1,\dots,r\}$ by

$$j\sim k \iff \sigma_j \simeq \sigma_k^{\gamma},\ \text{for some } \gamma\in\mathrm{Gal}(k_{\mathrm{D_e}}/k_{\mathrm{E}}),$$

and denote by $I_1,\dots,I_l$ the equivalence classes. Put $r_i = \#I_i$ and define s_i by $\mathrm{J}(\beta,\Lambda^j)/\mathrm{J}^1(\beta,\Lambda^j) \simeq \mathrm{GL}_{s_i}(k_{\mathrm{D_E}})$, for any $j\in I_i$. Note also that, by conjugating the types λ_j by a suitable element of $\mathrm{D}_{\mathrm{E}}^{\times}$, we may (and do) assume that $\sigma_j\simeq\sigma_k$ whenever $j\sim k$. Put $\mathrm{Y}^i = \bigoplus_{j\in I_i}\mathrm{V}^j$ and denote by M the Levi subgroup which stabilizes the decomposition $\mathrm{V} = \bigoplus_{i=1}^{l}\mathrm{Y}^i$. Note that this is the Levi subgroup M of the introduction.

Now $\mathrm{J_P}/\mathrm{J}^1_{\mathrm{P}} \cong \prod_{j=1}^{r}\mathrm{J}(\beta,\Lambda^j)/\mathrm{J}^1(\beta,\Lambda^j)$ so we can define a representation σ of $\mathrm{J_P}$ inflated from $\bigotimes_{j=1}^{r}\sigma_j$. Then we put $\lambda_{\mathrm{P}} = \kappa_{\mathrm{P}}\otimes\sigma$. We put $\mathrm{J_M} = \mathrm{J_P}\cap\mathrm{M}$ and $\lambda_{\mathrm{M}} = \lambda_{\mathrm{P}}|\mathrm{J_M}$.

Proposition 7.1. — *The pair* $(\mathrm{J_P},\lambda_{\mathrm{P}})$ *is a cover of* $(\mathrm{J_M},\lambda_{\mathrm{M}})$, *which is a cover of* $(\mathrm{J_L},\lambda_{\mathrm{L}})$. *Moreover, we have a support-preserving Hecke algebra isomorphism*

$$\mathscr{H}(\mathrm{M},\lambda_{\mathrm{M}}) \cong \mathscr{H}(\mathrm{G},\lambda_{\mathrm{P}})$$

and

$$\mathscr{H}(\mathrm{M},\lambda_{\mathrm{M}}) \cong \bigotimes_{i=1}^{l}\mathscr{H}(r_i, q_{\mathrm{D_E}}^{s_i}).$$

Proof. — The proof of the first assertion is identical to that of [**23**, Proposition 5.17]; indeed the proof there shows that $\mathrm{I_G}(\lambda_{\mathrm{P}}) \subseteq \mathrm{J_P}(\mathrm{B}^{\times}\cap\mathrm{M})\mathrm{J_P}$ and then the first Hecke algebra isomorphism also follows from [**12**, 7.2].

That $(\mathrm{J_M}, \lambda_\mathrm{M})$ is a cover of $(\mathrm{J_L}, \lambda_\mathrm{L})$ follows from Proposition 5.7, as does the second Hecke algebra isomorphism. □

7.2. The general case. — We now treat the general case, where the ps-characters Θ_j are not all endo-equivalent. We define an equivalence relation $\sim$ on $\{1, \ldots, r\}$ by

$$j \sim k \iff \Theta_j, \Theta_k \text{ are endo-equivalent,}$$

and denote by $I_1, \ldots, I_l$ the equivalence classes. We put $\mathrm{Y}^{(i)} = \bigoplus_{j \in I_i} \mathrm{V}^j$. As in the homogeneous case, we assume that, for fixed i, every $j \in I_i$ has the same ps-character, defined relative to the same simple pair. We write $\Lambda^{(i)}$ for an $\mathcal{O}_\mathrm{D}$-lattice sequence in $\mathrm{Y}^{(i)}$, as in the homogeneous case. By changing in their affine class, we may (and do) assume that all the $\Lambda^{(i)}$ have the same $\mathcal{O}_\mathrm{F}$-period; as in the homogeneous case, we suppose also that we have replaced the lattice sequences Λ^j with sequences in their affine class, so that $\Lambda^{(i)} = \bigoplus_{j \in I_i} \Lambda^j$.

We write M for the Levi subgroup of G which stabilizes the decomposition $\mathrm{V} = \bigoplus_{i=1}^{l} \mathrm{Y}^{(i)}$, and $(\mathrm{J_M}, \lambda_\mathrm{M})$ for the cover of $(\mathrm{J_L}, \lambda_\mathrm{L})$ in M, given by the homogeneous case in Proposition 7.1. Note that this M is now *not* the Levi subgroup of the introduction, but one rather larger.

We put $\Lambda = \bigoplus_{i=1}^{l} \Lambda^{(i)}$, a (not necessarily strict) $\mathcal{O}_\mathrm{D}$-lattice sequence in V, and $\beta = \sum_{j=1}^{r} \beta_j \in \mathrm{A} = \mathrm{End_D(V)}$. Then $[\Lambda, n, 0, \beta]$ is a (non-simple) stratum in A, for a suitable integer n.

For $0 \leq t \leq n$, we write $\Theta_j^{(t)}$ for the ps-character defined by the character $\theta_j | \mathrm{H}^{t+1}(\beta_j, \Lambda^j)$.

Theorem 7.2 **(cf. [13**, Main Theorem, p.94**])**. — *There is a cover* (K, τ) *of the type* $(\mathrm{J_M}, \lambda_\mathrm{M})$ *with the following properties:*

(i) $\mathrm{U}^{n+1}(\Lambda) \subseteq \mathrm{K} \subseteq \mathrm{U}(\Lambda)$*;*

(ii) *if the ps-characters* $\Theta_j^{(t)}$*, for* $1 \leq j \leq r$*, are endo-equivalent, and* $([\Lambda, n, 0, \gamma], \vartheta, t)$ *is a common approximation of* $(\theta_1, \ldots, \theta_r)$*, then*

- (a) K *contains and normalizes* $\mathrm{H}^{t+1}(\gamma, \Lambda) \cdot (\mathrm{H}^t(\gamma, \Lambda) \cap \mathrm{M})$*;*
- (b) $\tau | \mathrm{H}^{t+1}(\gamma, \Lambda)$ *is a multiple of* ϑ*;*
- (c) $\tau | \mathrm{H}^t(\gamma, \Lambda) \cap \mathrm{L}$ *is a multiple of* $\theta_1 \otimes \cdots \otimes \theta_r$*;*

(iii) *there is a support-preserving isomorphism of Hecke algebras* $\mathscr{H}(\mathrm{M}, \lambda_\mathrm{M}) \simeq \mathscr{H}(\mathrm{G}, \tau)$.

Proof. — The proof is by induction on r, the case $r=1$ being empty. So let $r>1$ and suppose that $t\ge 0$ is minimal such that the ps-characters $\Theta_j^{(t)}$ are endo-equivalent. Let $([\Lambda,n,0,\gamma],\vartheta,t)$ be a common approximation of $(\theta_1,\dots,\theta_r)$. If $t=0$ then the Theorem is given by Proposition 7.1, so we assume $t>0$. We allow $t=n$ (that is, the ps-characters $\Theta_j^{(n-1)}$ are not all endo-equivalent) in which case ϑ is the trivial character of $\mathrm{U}^{n+1}(\Lambda)$. We use the notation of §4.

For $1\le j\le r$, let $c_j\in\mathfrak{P}_{-t}(\Lambda^j)$ be such that $\theta_j|\mathrm{H}^{t+1}(\beta_j,\Lambda^j)=\vartheta\psi_{c_j}$. By Corollary 3.6, the derived stratum $[\Gamma_\gamma^j,t,t-1,s_\gamma(c_j)]$ is equivalent to a simple (or null) stratum and we write $\phi_j(X)$ for the minimum polynomial of this stratum (so that the characteristic polynomial is a power of $\phi_j(X)$). We define an equivalence relation on $\{1,\dots,r\}$ by

$$j\sim_t k \iff \Theta_j^{(t-1)},\Theta_k^{(t-1)} \text{ are endo-equivalent.}$$

Note that, by Corollary 4.7, we have $j\sim_t k$ if and only if $\phi_j(X)=\phi_k(X)$. Let J denote an equivalence class for $\sim_t$ for which the minimum polynomial is not X; then J is a union of certain equivalence classes I_i but is not the whole of $\{1,\dots,r\}$, or else we would contradict the minimality of t.

Set $\mathrm{Z}=\bigoplus_{j\in J}\mathrm{V}^j$ and $\mathrm{Z}'=\bigoplus_{j\notin J}\mathrm{V}^j$; let $\bar{\mathrm{M}}$ be the Levi subgroup which stabilizes the decomposition $\mathrm{V}=\mathrm{Z}\oplus\mathrm{Z}'$ and let $\bar{\mathrm{P}}=\bar{\mathrm{M}}\bar{\mathrm{N}}$ be a parabolic subgroup with Levi component $\bar{\mathrm{M}}$ and opposite $\bar{\mathrm{P}}^-=\bar{\mathrm{M}}\bar{\mathrm{N}}^-$. By the inductive hypothesis, we have a cover $(\mathrm{K}_{\bar{\mathrm{M}}},\tau_{\bar{\mathrm{M}}})$ of $(\mathrm{K}_{\mathrm{M}},\tau_{\mathrm{M}})$ satisfying the conditions of the theorem (with $\bar{\mathrm{M}}$ in place of G). We define the group K by

$$\mathrm{K}=\mathrm{K}_{\bar{\mathrm{M}}}\mathrm{H}^t(\gamma,\Lambda)\cdot(\mathrm{U}^1(\Lambda)\cap\mathrm{B})\Omega_{q-t+1}(\gamma,\Lambda)\cap\bar{\mathrm{N}},$$

where $\Omega_{q-t+1}(\gamma,\Lambda)$ is the group defined in [**23**, §2.8]. Then [**23**, Corollaire 4.6] says that there is a unique irreducible representation τ of K such that (K,τ) is a cover of $(\mathrm{K}_{\bar{\mathrm{M}}},\tau_{\bar{\mathrm{M}}})$, which has all the required properties by transitivity of covers. [Note that, although it is assumed in [**23**, §4] that the lattice sequence Λ is strict, this extra condition is never used.] □

Now Theorem C of the introduction follows from Theorem 7.2 and Proposition 7.1, whence the Main Theorem.

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