AN ANALOGUE OF THE CARTAN DECOMPOSITION FOR p-ADIC SYMMETRIC SPACES OF SPLIT p-ADIC REDUCTIVE GROUPS

by

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Abstract. — Let k be a non-Archimedean locally compact field of residue characteristic p, let G be a connected reductive group defined over k, let σ be an involutive k-automorphism of G and H an open k-subgroup of the fixed points group of σ. We denote by G_k (resp. H_k) the group of k-points of G (resp. H). In this paper, we obtain an analogue of the Cartan decomposition for the reductive symmetric space $H_k\backslash G_k$ in the case where G is k-split and p is odd. More precisely, we obtain a decomposition of G_k as a union of (H_k, K)-double cosets, where K is the stabilizer of a special point in the Bruhat-Tits building of G over k. This decomposition is related to the H_k-conjugacy classes of maximal σ-anti-invariant k-split tori in G. In a more general context, Benoist and Oh obtained a polar decomposition for any p-adic reductive symmetric space. In the case where G is k-split and p is odd, our decomposition makes more precise Benoist-Oh's polar decomposition and generalizes results of Offen for GL_n.

Résumé. — Soit k un corps localement compact non archimédien de caractéristique résiduelle p, soit G un groupe réductif connexe défini sur k, soit σ un k-automorphisme involutif de G et soit H un k-sous-groupe ouvert du groupe des points de G fixes par σ. On note G_k (resp. H_k) le groupe des k-points de G (resp. H). Dans cet article, nous obtenons un analogue de la décomposition de Cartan pour l'espace symétrique réductif $H_k\backslash G_k$ lorsque G est déployé sur k et p est impair. Plus précisément, nous obtenons une décomposition de G_k sous la forme d'une réunion de doubles classes modulo (H_k, K), où K désigne le stabilisateur d'un point spécial de l'immeuble de Bruhat-Tits de G sur k. Cette décomposition est liée aux classes de H_k-conjugaison des tores k-déployés σ-anti-invariants maximaux de G. Dans un cadre plus général, Benoist et Oh ont obtenu une décomposition polaire pour les espaces symétriques réductifs p-adiques quelconques. Dans le cas où G est déployé sur k et où p est impair, notre décomposition précise la décomposition polaire de Benoist et Oh et généralise des résultats de Offen pour GL_n.

1. Introduction

Let k be a non-Archimedean locally compact field of odd residue characteristic. Let G be a connected reductive group defined over k, let σ be an involutive k-automorphism of G and let H be an open k-subgroup of the fixed points group of σ. We denote by G_k (resp.

H_k) the group of k-points of G (resp. H). Harmonic analysis on the reductive symmetric space $\mathrm{H}_k\backslash\mathrm{G}_k$ is the study of the action of G_k on the space of complex square integrable functions on $\mathrm{H}_k\backslash\mathrm{G}_k$. This study is related to the classification of H_k-distinguished representations of G_k, that is representations having a non-zero space of H_k-invariant linear forms. Offen [**19**] has investigated the harmonic analysis of spherical functions in some cases related to GL_n. Blanc and Delorme [**3**] have studied H_k-distinguishedness for families of parabolically induced representations of G_k. Lagier [**16**], and independently Kato and Takano [**15**], have introduced the notion of relative cuspidality for irreducible H_k-distinguished representations of G_k and constructed "Jacquet maps" at the level of invariant linear forms. In this paper, we investigate the geometry of the reductive symmetric space $\mathrm{H}_k\backslash\mathrm{G}_k$.

Connected reductive groups can be considered as reductive symmetric spaces. Indeed, if G' is such a group, the map:

$$\sigma : (x, y) \mapsto (y, x)$$

defines a k-involution of $\mathrm{G} = \mathrm{G}' \times \mathrm{G}'$ whose fixed points group H is the diagonal image of G' in G, and the reductive symmetric space $\mathrm{H}_k\backslash\mathrm{G}_k$ naturally identifies with G'_k via the map $(x, y) \mapsto x^{-1}y$. Moreover, if K' is a subgroup of G'_k, and if we set $\mathrm{K} = \mathrm{K}' \times \mathrm{K}'$, then this map induces a bijective correspondence:

$$\{(\mathrm{H}_k, \mathrm{K})\text{-double cosets of } \mathrm{G}_k\} \leftrightarrow \{\mathrm{K}'\text{-double cosets of } \mathrm{G}'_k\}.$$

In particular, if K' is the G'_k-stabilizer of a special point in the Bruhat-Tits building of G' over k, the decomposition of $\mathrm{H}_k\backslash\mathrm{G}_k$ into K-orbits corresponds to the Cartan decomposition of G'_k relative to K' (see [**6**, Proposition 4.4.3]).

In this paper, we obtain an analogue of the Cartan decomposition for $\mathrm{H}_k\backslash\mathrm{G}_k$ when the group G is k-split. In a more general context (k any non-Archimedean locally compact field of odd characteristic and G any connected reductive group over k), Benoist and Oh [**2**] have obtained a polar decomposition for $\mathrm{H}_k\backslash\mathrm{G}_k$. In the case where k has odd residue characteristic and G is k-split, our decomposition is a refinement of Benoist-Oh's polar decomposition (see paragraph 4.6). This decomposition can be seen as a p-adic analogue of the Cartan decomposition for real reductive symmetric spaces (see [**10**, Theorem 4.1]). It generalizes the decompositions obtained by Offen (see [**19**, Proposition 3.1]) for $\mathrm{G} = \mathrm{GL}_{2n}$ in Cases 1 and 3 (*ibid.*).

Let $\{\mathrm{A}^j \mid j \in \mathrm{J}\}$ be a set of representatives of the H_k-conjugacy classes of maximal σ-anti-invariant k-split tori of G (called maximal (σ, k)-*split* tori in [**11**], see also Definition

4.1). These tori, as well as related entities, have been studied by A. Helminck, G. Helminck and Wang [**11, 12, 13**]. In particular, the set J is finite and the A^j, $j \in \mathrm{J}$, are all conjugate under G_k. Let S be a σ-stable maximal k-split torus of G containing a maximal (σ, k)-split torus A. For each $j \in \mathrm{J}$, we choose $y_j \in \mathrm{G}_k$ such that $y_j \mathrm{A} y_j^{-1} = \mathrm{A}^j$. Our main result is the following theorem (see Theorem 4.9).

Theorem 1.1. — *Assume* G *is k-split. Let* K *be the stabilizer in* G_k *of a special point in the apartment attached to* S. *Then:*

$$\mathrm{G}_k = \bigcup_{j \in \mathrm{J}} \mathrm{H}_k y_j \mathrm{S}_k \mathrm{K}. \tag{1.1}$$

If one compares with Offen's decompositions [**19**, Proposition 3.1], one sees that in each of his Cases 1 and 3 (where $\mathrm{G} = \mathrm{GL}_{2n}$ for $n \geqslant 1$), the set J reduces to a single element and y_j can be chosen to be trivial. In general however, one cannot avoid to have several non-H_k-conjugate maximal σ-anti-invariant k-split tori of G appearing in (1.1).

To prove Theorem 1.1, we make a large use of the Bruhat-Tits theory [**6, 7**]. First, let G be any connected reductive group over k, and let $\mathscr{B}$ be its Bruhat-Tits building. It is endowed with an action of σ. Then we have (see Proposition 3.4):

Proposition 1.2. — *$\mathscr{B}$ is the union of its σ-stable apartments.*

Note that in the case where $\mathrm{G} = \mathrm{G}' \times \mathrm{G}'$ and $\sigma(x, y) = (y, x)$ as above, then the building $\mathscr{B}$ identifies with the product of two copies of the building of G' over k and Proposition 1.2 simply says that two arbitrary points in the building of G' are always contained in a common apartment.

When G is k-split, we obtain the following refinement of Proposition 1.2 (see Proposition 4.5).

Proposition 1.3. — *Assume* G *is k-split, and let x be a special point of $\mathscr{B}$. There is a σ-stable maximal k-split torus* S *of* G *such that the apartment corresponding to* S *contains x and the maximal σ-anti-invariant subtorus of* S *is a maximal (σ, k)-split torus of* G.

As we will see in paragraph 5.3, Proposition 1.3 is no longer true for non-split groups.

In section 2, we recall the main properties of the Bruhat-Tits building attached to a connected reductive group defined over k.

In section 3, we study the set of all apartments containing a given σ-stable subset of the building, and we prove Proposition 1.2.

In section 4, we prove our main theorem for G a k-split group.

In section 5, we study in more details the case of $G_k = GL_n(k)$ and $\sigma(g) =$ transpose of g^{-1}, and the case of $G_k = GL_n(k')$ with k' quadratic over k and $\mathrm{id} \neq \sigma \in \mathrm{Gal}(k'/k)$. When $n = 2$ and k' is totally ramified over k, the second case provides an example of a non-split group for which Proposition 1.3 is not satisfied.

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2. The Bruhat-Tits building

Let k be a non-Archimedean non-discrete locally compact field, and let ω be its normalized valuation. In this section, we recall the main properties of the Bruhat-Tits building attached to a connected reductive group defined over k. The reader may refer to Bruhat-Tits [**6, 7**] or to more concise presentations [**17, 21, 23**].

If G is a linear algebraic group defined over k, the group of its k-points will be denoted by G_k or $G(k)$, and its neutral component will be denoted by G°. If X is a subset of G, then $N_G(X)$ (resp. $Z_G(X)$) denotes the normalizer (resp. the centralizer) of X in G, and, given $g \in G$, we write ${}^gX$ for gXg^{-1}.

2.1. Let G be a connected reductive group defined over k, and let S be a maximal k-split torus of G. We denote by $X^*(S) = \mathrm{Hom}(S, GL_1)$ (resp. by $X_*(S) = \mathrm{Hom}(GL_1, S)$) the group of algebraic characters (resp. cocharacters) of S. We define a map:

$$X_*(S) \times X^*(S) \to \mathbf{Z} \tag{2.1}$$

as follows. If $\lambda \in X_*(S)$ and $\chi \in X^*(S)$, then $\chi \circ \lambda$ is an endomorphism of the multiplicative group GL_1, which corresponds to an endomorphism of the ring $\mathbf{Z}[t, t^{-1}]$. It is of the form $t \mapsto t^n$ for some $n \in \mathbf{Z}$. This integer n is denoted by $\langle \lambda, \chi \rangle$. The map (2.1) defines a perfect duality (see [**4**, §8.6]).

2.2. Let N (resp. Z) denote the normalizer (resp. the centralizer) of S in G. If we extend the map (2.1) by $\mathbf{R}$-linearity, there exists a unique group homomorphism:

$$\nu : Z_k \to X_*(S) \otimes_{\mathbf{Z}} \mathbf{R} \tag{2.2}$$

such that the condition:

$$\langle \nu(z), \chi \rangle = -\omega(\chi(z))$$

holds for any $z \in \mathrm{Z}_k$ and any k-rational character $\chi \in \mathrm{X}^*(\mathrm{Z})_k$ (see [**23**, §1.2]). According to [**17**, Proposition 1.2], the kernel of (2.2) is the maximal compact subgroup of Z_k.

2.3. Let C denote the connected centre of G and let $\mathrm{X}_*(\mathrm{C})$ be the group of its algebraic cocharacters. It is a subgroup of the free abelian group $\mathrm{X}_*(\mathrm{S})$. We denote by $\mathscr{A}$ the space:

$$\mathrm{V} = (\mathrm{X}_*(\mathrm{S}) \otimes_{\mathbf{Z}} \mathbf{R})/(\mathrm{X}_*(\mathrm{C}) \otimes_{\mathbf{Z}} \mathbf{R})$$

considered as an affine space on itself and by $\mathrm{Aff}(\mathscr{A})$ the group of its affine automorphisms. By making V act on $\mathscr{A}$ by translations, we can think of V as a subgroup of $\mathrm{Aff}(\mathscr{A})$. It is the kernel of the natural group homomorphism $\mathrm{Aff}(\mathscr{A}) \to \mathrm{GL}(\mathrm{V})$ which associates to any affine automorphism its linear part.

2.4. The map (2.2) induces a homomorphism:

$$\mathrm{Z}_k \to \mathrm{Aff}(\mathscr{A}) \tag{2.3}$$

which we still denote by ν. Its image is contained in V. An important property of this homomorphism is that it extends to a homomorphism $\mathrm{N}_k \to \mathrm{Aff}(\mathscr{A})$ (see [**23**, §1.2]). It does not extend in a unique way, but two homomorphisms extending (2.3) to N_k are conjugated by a *unique* element of $\mathrm{Aff}(\mathscr{A})$ (see [**17**, Proposition 1.8]).

2.5. The affine space $\mathscr{A}$ endowed with an action of N_k defined by a group homomorphism $\nu : \mathrm{N}_k \to \mathrm{Aff}(\mathscr{A})$ extending the homomorphism (2.3) is called the (reduced) *apartment* attached to S. It satisfies the conditions:

A1 $\mathscr{A}$ is an affine space on V;

A2 ν is a group homomorphism $\mathrm{N}_k \to \mathrm{Aff}(\mathscr{A})$ extending the canonical homomorphism $\mathrm{Z}_k \to \mathrm{V}$.

It has the following unicity property: if $(\mathscr{A}', \nu')$ satisfies **A1** and **A2**, then there is a unique affine and N_k-equivariant isomorphism from $\mathscr{A}'$ to $\mathscr{A}$.

Remark 2.1. — As in Tits [**23**], one obtains the *non-reduced* apartment $\mathscr{A}_{\mathrm{nr}}$ by replacing V by $\mathrm{X}_*(\mathrm{S}) \otimes_{\mathbf{Z}} \mathbf{R}$. It is not as canonical as the reduced one: two homomorphisms extending the map $\nu_{\mathrm{nr}} : \mathrm{Z}_k \to \mathrm{Aff}(\mathscr{A}_{\mathrm{nr}})$ to N_k are conjugated by an element of $\mathrm{Aff}(\mathscr{A}_{\mathrm{nr}})$ which is not necessarily unique (see [**17**, §1] and also [**23**, §1.2]).

2.6. Let $\Phi = \Phi(\mathrm{G},\mathrm{S})$ denote the set of roots of G relative to S. It is a subset of $\mathrm{X}^*(\mathrm{S})$. Therefore, any root $a \in \Phi$ can be seen as a linear form on $\mathrm{X}_*(\mathrm{S}) \otimes_{\mathbf{Z}} \mathbf{R}$ which is trivial on the subspace $\mathrm{X}_*(\mathrm{C}) \otimes_{\mathbf{Z}} \mathbf{R}$, hence as a linear form on V (see [**17**, §1]).

For $a \in \Phi$, we denote by U_a the root subgroup associated to a, which is a unipotent subgroup of G normalized by Z (see [**4**, Proposition 21.9]), and by s_a the reflection corresponding to a, considered as an element of GL(V) — or, more precisely, of the quotient of $\nu(\mathrm{N}_k)$ by $\nu(\mathrm{Z}_k)$.

2.7. Let $a \in \Phi$ and $u \in \mathrm{U}_a(k) - \{1\}$. The intersection:

$$\mathrm{U}_{-a}(k)u\mathrm{U}_{-a}(k) \cap \mathrm{N}_k \tag{2.4}$$

consists of a single element, called $m(u)$, whose image by ν is an affine reflection the linear part of which is s_a (see [**5**, §5]). The set $\mathscr{H}_{a,u}$ of fixed points of $\nu(m(u))$ is an affine hyperplane of $\mathscr{A}$, which is called a *wall* of $\mathscr{A}$.

A *chamber* of $\mathscr{A}$ is a connected component of the complementary in $\mathscr{A}$ of the union of its walls. Note that a chamber is open in $\mathscr{A}$.

A point $x \in \mathscr{A}$ is said to be *special* if, for all root $a \in \Phi$, there is a root $b \in \Phi \cap \mathbf{R}_+ a$ and an element $u \in \mathrm{U}_b(k) - \{1\}$ such that $x \in \mathscr{H}_{b,u}$ (see [**18**, §1.2.3] and also [**23**, §1.9]).

2.8. Let $\theta(a,u)$ denote the affine function $\mathscr{A} \to \mathbf{R}$ whose linear part is a and whose vanishing hyperplane is the wall $\mathscr{H}_{a,u}$ of fixed points of $\nu(m(u))$. We fix a base point in $\mathscr{A}$, so that $\mathscr{A}$ can be identified with the vector space V. For $r \in \mathbf{R}$, we set:

$$\mathrm{U}_a(k)_r = \{u \in \mathrm{U}_a(k) - \{1\} \mid \theta(a,u)(x) \geqslant a(x) + r \text{ for all } x \in \mathscr{A}\} \cup \{1\}.$$

Thus we obtain a filtration of $\mathrm{U}_a(k)$ by subgroups. If we change the base point in $\mathscr{A}$, this filtration is only modified by a translation of the indexation.

2.9. Let Ω be a non-empty subset of $\mathscr{A}$. We set:

$$\mathrm{N}_\Omega = \{n \in \mathrm{N}_k \mid \nu(n)(x) = x \text{ for all } x \in \Omega\},$$

and we denote by U_Ω the subgroup of G_k generated by all the $\mathrm{U}_a(k)_r$ such that the affine function $x \mapsto a(x) + r$ is non-negative on Ω. According to [**17**, §12], this subgroup is compact in G_k, and we have $n\mathrm{U}_\Omega n^{-1} = \mathrm{U}_{\nu(n)(\Omega)}$ for $n \in \mathrm{N}_k$. In particular, N_Ω normalizes U_Ω. The subgroup $\mathrm{P}_\Omega = \mathrm{N}_\Omega \mathrm{U}_\Omega$ is open in G_k (*loc.cit.*, Corollary 12.12).

2.10. Let $\Phi = \Phi^- \cup \Phi^+$ be a decomposition of Φ into positive and negative roots. We denote by U^+ and U^-) the subgroup of G_k generated by the U_a for all $a \in \Phi^+$ (resp. for all $a \in \Phi^-$). Then the group P_Ω has the following Iwahori decomposition:

$$\mathrm{P}_\Omega = (\mathrm{U}_\Omega \cap \mathrm{U}^-) \cdot (\mathrm{U}_\Omega \cap \mathrm{U}^+) \cdot \mathrm{N}_\Omega \tag{2.5}$$

(see [**17**, Corollary 12.6] and also [**6**, §7.1.4]).

2.11. In [**6, 7**], Bruhat and Tits associate to the apartment $(\mathscr{A}, \nu)$ a G_k-set $\mathscr{B} = \mathscr{B}(\mathrm{G}, k)$ containing $\mathscr{A}$, called the (reduced) *building* of G over k and satisfying the following conditions:

B1 The set $\mathscr{B}$ is the union of the $g \cdot \mathscr{A}$ for $g \in \mathrm{G}_k$.

B2 The subgroup N_k is the stabilizer of $\mathscr{A}$ in G_k, and $n \cdot x = \nu(n)(x)$ for all $x \in \mathscr{A}$ and $n \in \mathrm{N}_k$.

B3 For all $a \in \Phi$ and $r \in \mathbf{R}$, the subgroup $\mathrm{U}_a(k)_r$ defined in paragraph 2.8 fixes the subset $\{x \in \mathscr{A} \mid a(x) + r \geqslant 0\}$ pointwise.

The building has the following unicity property: if $\mathscr{B}'$ is a G_k-set containing $\mathscr{A}$ and satisfying **B1**, **B2** and **B3**, then there is a unique G_k-equivariant bijection from $\mathscr{B}'$ to $\mathscr{B}$ (see [**23**, §2.1] and also [**20**, §1.9]).

2.12. The subsets of $\mathscr{B}$ of the form $g \cdot \mathscr{A}$ with $g \in \mathrm{G}_k$ are called *apartments*. According to **B1**, the building is the union of its apartments. For $g \in \mathrm{G}_k$, the apartment $g \cdot \mathscr{A}$ can be naturally endowed with a structure of affine space and an action of ${}^g\mathrm{N}_k$ by affine isomorphisms. Up to unique isomorphism, it is the apartment attached to the maximal k-split torus ${}^g\mathrm{S}$ (see paragraph 2.5). This defines a unique G_k-equivariant map:

$$\mathrm{S}' \mapsto \mathscr{A}(\mathrm{S}') \subseteq \mathscr{B} \tag{2.6}$$

between maximal k-split tori of G and apartments of $\mathscr{B}$, such that S maps to $\mathscr{A}$.

Note that the building $\mathscr{B}$ does not depend on the maximal k-split torus S. Indeed, let S' be a maximal k-split torus of G, let $(\mathscr{A}', \nu')$ be the apartment attached to S' and $\mathscr{B}'$ be the building of G over k relative to this apartment (see paragraph 2.11). If we identify $\mathscr{A}'$ with the unique apartment of $\mathscr{B}$ corresponding to S' *via* (2.6), then $\mathscr{B}' = \mathscr{B}$.

2.13. The building has the following important properties (see [**6**, §7.4] and [**17**, §13]):

(1) Let Ω be a non-empty subset of $\mathscr{A}$. Then P_Ω is the subgroup of G_k made of those elements fixing Ω pointwise.

(2) Let $g \in \mathrm{G}_k$. There is $n \in \mathrm{N}_k$ such that $g \cdot x = n \cdot x$ for any $x \in \mathscr{A} \cap g^{-1} \cdot \mathscr{A}$.

In particular, Property (1) together with **B2** imply that $\mathrm{N}_\Omega = \mathrm{N}_k \cap \mathrm{P}_\Omega$.

2.14. Let σ be a k-automorphism of G. There is a unique bijective map from $\mathscr{B}$ to itself, still denoted σ, such that:

(1) the condition:

$$\sigma(g \cdot x) = \sigma(g) \cdot \sigma(x)$$

holds for any $g \in \mathrm{G}_k$ and $x \in \mathscr{B}$;

(2) the map σ permutes the apartments and, for any apartment $\mathscr{A}$, the restriction of σ to $\mathscr{A}$ is an affine isomorphism from $\mathscr{A}$ to $\sigma(\mathscr{A})$.

This makes (2.6) into a σ-equivariant map. In particular, an apartment is σ-stable if and only if its corresponding maximal k-split torus of G is σ-stable (see [**7**, §4.2.12]).

3. Existence of σ-stable apartments

From now on, k will be a non-Archimedean locally compact field of odd residue characteristic. Let G be connected reductive group defined over k and let σ be a k-involution on G. According to paragraph 2.14, the building $\mathscr{B}$ of G over k is endowed with an action of σ. In this section, we prove that, given $x \in \mathscr{B}$, there exists a σ-stable apartment containing x. We keep using notation of Section 2.

3.1. Let Ω be a non-empty σ-stable subset of $\mathscr{B}$ contained in some apartment, and let $\mathrm{Ap}(\Omega)$ be the set of all apartments of $\mathscr{B}$ containing Ω. It is a non-empty set on which the group P_Ω acts transitively (see [**17**, Corollary 13.7]). Because Ω is σ-stable, both P_Ω and $\mathrm{Ap}(\Omega)$ are σ-stable. Note that the σ-stable apartments containing Ω are exactly the σ-fixed points in $\mathrm{Ap}(\Omega)$.

3.2. Let us fix an apartment $\mathscr{A} \in \mathrm{Ap}(\Omega)$ and an element $u \in \mathrm{P}_\Omega$ such that $\sigma(\mathscr{A}) = u \cdot \mathscr{A}$. Let N denote the normalizer in G of the maximal k-split torus of G corresponding to $\mathscr{A}$. As σ is involutive, we have:

$$\sigma(u)u \in \mathrm{P}_\Omega \cap \mathrm{N}_k = \mathrm{N}_\Omega. \tag{3.1}$$

The map $\rho : g \mapsto g \cdot \mathscr{A}$ induces a P_Ω-equivariant bijection between the homogeneous spaces $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ and $\mathrm{Ap}(\Omega)$. The automorphism:

$$\theta : x \mapsto u^{-1}\sigma(x)u$$

of the group G_k stabilizes P_Ω and N_Ω. Indeed $\sigma(\mathrm{N}_k) = u\mathrm{N}_k u^{-1}$, and:

$$\theta(\mathrm{N}_\Omega) = u^{-1}\sigma(\mathrm{P}_\Omega \cap \mathrm{N}_k)u = \mathrm{P}_\Omega \cap u^{-1}\sigma(\mathrm{N}_k)u = \mathrm{N}_\Omega.$$

Note that the condition (3.1) implies that $\theta \circ \theta$ is conjugation by some element of N_Ω. As N_Ω is θ-stable, the map:

$$(\sigma, g\mathrm{N}_\Omega) \mapsto u\theta(g\mathrm{N}_\Omega), \quad g \in \mathrm{P}_\Omega,$$

defines an action of σ on $\mathrm{P}_\Omega/\mathrm{N}_\Omega$, making ρ into a σ-equivariant bijection. Note that this action differs from the natural action of σ on $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ (which obviously has fixed points).

3.3. Let Ω be a non-empty σ-stable subset of $\mathscr{B}$ contained in some apartment.

Proposition 3.1. — *Assume that Ω contains a point of a chamber of $\mathscr{B}$. Then Ω is contained in some σ-stable apartment.*

Proof. — We describe the quotient $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ as a projective limit of finite σ-sets. According to [**9**, §1.2], Example (f), the group G_k is locally compact and totally disconnected. Therefore we can choose a decreasing filtration $(\mathrm{Q}_i)_{i\geqslant 0}$ of the open subgroup P_Ω of G_k satisfying the following properties:

(A) The intersection of the Q_i is reduced to $\{1\}$.
(B) For any $i \geqslant 0$, the subgroup Q_i is compact open and normal in P_Ω.

For $i \geqslant 0$, let $\mathrm{P}_{\Omega,i}$ denote the intersection $\mathrm{N}_\Omega\mathrm{Q}_i \cap \theta(\mathrm{N}_\Omega\mathrm{Q}_i)$. These subgroups form a decreasing filtration of P_Ω, and we claim that this filtration satisfies the following properties:

(1) The intersection of the $\mathrm{P}_{\Omega,i}$ is reduced to N_Ω.
(2) For any $i \geqslant 0$, the subgroup $\mathrm{P}_{\Omega,i}$ is θ-stable and of finite index in P_Ω.

As N_Ω is θ-stable, it is contained in the intersection of the $\mathrm{P}_{\Omega,i}$. Let g be in this intersection. For any $i \geqslant 0$, there exist $n_i \in \mathrm{N}_\Omega$ and $q_i \in \mathrm{Q}_i$ such that $g = n_iq_i$. Because of Property (A) above, q_i converges to 1. Therefore n_i converges to a limit contained in the closed subgroup N_Ω, and this limit is g. This proves Property (1).

Now recall that $\theta \circ \theta$ is conjugation by some element of N_Ω. This implies that $\mathrm{P}_{\Omega,i}$ is θ-stable. As $\mathrm{P}_{\Omega,i}$ is open in P_Ω and contains N_Ω, the quotient $\mathrm{P}_\Omega/\mathrm{P}_{\Omega,i}$ can be identified with the quotient of U_Ω, which is compact, by some open subgroup. This gives us the expected result.

Because of Property (2), the map:

$$(\sigma, g\mathrm{P}_{\Omega,i}) \mapsto u\theta(g\mathrm{P}_{\Omega,i}), \quad g \in \mathrm{P}_\Omega,$$

defines an action of σ on the finite quotient $\mathrm{P}_\Omega/\mathrm{P}_{\Omega,i}$, which gives us a projective system $(\mathrm{P}_\Omega/\mathrm{P}_{\Omega,i})_{i\geqslant 0}$ of finite σ-sets. As P_Ω is complete, and thanks to Property (1), the natural σ-equivariant map from $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ to the projective limit of the $\mathrm{P}_\Omega/\mathrm{P}_{\Omega,i}$ is bijective.

Lemma 3.2. — *Let* $(\mathrm{X}_i)_{i\geqslant 0}$ *be a projective system of finite* σ*-sets. For all* $i \geqslant 0$*, assume the transition maps* $\varphi_i : \mathrm{X}_{i+1} \to \mathrm{X}_i$ *to be surjective and* X_i *to have odd cardinality. Then the projective limit* X *has a* σ*-fixed point.*

Proof. — For each $i \geqslant 0$, the set X_i^σ of σ-fixed points of X_i is non-empty, since X_i has odd cardinality. This defines a projective system $(\mathrm{X}_i^\sigma)_{i\geqslant 0}$ whose transition maps may not be surjective. For each $i \geqslant 0$, let Y_i denote the intersection in X_i of the images of the X_{i+n}^σ, for $n \geqslant 0$. Then Y_i is non-empty, and the transition maps $\varphi_i : \mathrm{Y}_{i+1} \to \mathrm{Y}_i$ are surjective. Therefore, the projective limit $\mathrm{Y} = \mathrm{X}^\sigma \subseteq \mathrm{X}$ of the system $(\mathrm{Y}_i)_{i\geqslant 0}$ is non-empty. □

Let p denote the residue characteristic of k.

Lemma 3.3. — *Let* K *be a normal subgroup of finite index in* P_Ω *containing* N_Ω*. Then the index of* K *in* P_Ω *is a power of* p*.*

Proof. — Let S be the maximal k-split torus associated to $\mathscr{A}$, let Φ be the set of roots of G relative to S and let $\Phi = \Phi^- \cup \Phi^+$ be a decomposition of Φ into positive and negative roots. According to (2.5), the group P_Ω has the following Iwahori decomposition:

$$\mathrm{P}_\Omega = (\mathrm{U}_\Omega \cap \mathrm{U}^-) \cdot (\mathrm{U}_\Omega \cap \mathrm{U}^+) \cdot \mathrm{N}_\Omega.$$

The fact that Ω contains a point of a chamber of $\mathscr{B}$ implies that the group N_Ω is reduced to $\mathrm{Ker}(\nu)$, hence normalizes the groups $\mathrm{V}^+ = \mathrm{U}_\Omega \cap \mathrm{U}^+$ and $\mathrm{V}^- = \mathrm{U}_\Omega \cap \mathrm{U}^-$. The index of K in P_Ω can be decomposed as follows:

$$(\mathrm{P}_\Omega : \mathrm{K}) = (\mathrm{P}_\Omega : \mathrm{V}^+\mathrm{K}) \cdot (\mathrm{V}^+\mathrm{K} : \mathrm{K}).$$

In a first hand, the index:

$$(\mathrm{V}^+\mathrm{K} : \mathrm{K}) = (\mathrm{V}^+ : \mathrm{V}^+ \cap \mathrm{K})$$

is a power of p, as V^+ is a pro-p-group. On the other hand, the index:

$$(\mathrm{P}_\Omega : \mathrm{V}^+\mathrm{K}) = (\mathrm{V}^- : \mathrm{V}^- \cap \mathrm{V}^+\mathrm{K})$$

is a power of p as V^- is a pro-p-group. The result follows. □

According to Lemma 3.3, the cardinality of each $\mathrm{P}_\Omega/\mathrm{P}_{\Omega,i}$, with $i \geqslant 0$, is odd (recall that p is different from 2). Proposition 3.1 now follows from Lemma 3.2. □

3.4. We now prove the first main result of this section.

Proposition 3.4. — *For any $x \in \mathscr{B}$, there exists a σ-stable apartment containing x.*

Proof. — Let x be a point in $\mathscr{B}$, and let y be a point of a chamber of $\mathscr{B}$ whose adherence contains x. The set $\Omega = \{y, \sigma(y)\}$ is a σ-stable subset of $\mathscr{B}$ satisfying the conditions of Proposition 3.1. Hence we get a σ-stable apartment of $\mathscr{B}$ containing y. Such an apartment contains the adherence of the chamber of y. In particular, it contains x. □

3.5. Let S be a σ-stable maximal k-split torus, and let N (resp. Z) denote the normalizer (resp. the centralizer) of S in G. Let $\mathrm{X} = \mathrm{X}(\mathrm{S})$ denote the set of all $g \in \mathrm{G}_k$ such that $g^{-1}\sigma(g) \in \mathrm{N}_k$, let $\mathscr{A}$ denote the σ-stable apartment corresponding to S and, given $x \in \mathscr{A}$, let P_x denote the subgroup P_Ω (see paragraph 2.10) with $\Omega = \{x\}$.

Proposition 3.5. — X *is a finite union of* $(\mathrm{H}_k, \mathrm{Z}_k)$*-double cosets and* $\mathrm{G}_k = \mathrm{XP}_x$.

Proof. — Let us fix a minimal parabolic k-subgroup P of G containing the torus S. According to [**13**, Proposition 6.8], the map $g \mapsto \mathrm{H}_k g \mathrm{P}_k$ induces a bijection between the $(\mathrm{H}_k, \mathrm{Z}_k)$-double cosets in X and the $(\mathrm{H}_k, \mathrm{P}_k)$-double cosets in G_k. The first part of the proposition then follows from [**13**, Corollary 6.16].

Note that we have $g \in \mathrm{X}$ if and only if $g \cdot \mathscr{A}$ is σ-stable. For $g \in \mathrm{G}_k$, we set $x' = g \cdot x$. According to Proposition 3.4, there is a σ-stable apartment $\mathscr{A}'$ containing x'. Let $g' \in \mathrm{X}$ be such that $\mathscr{A}' = g' \cdot \mathscr{A}$. According to Property (2) of paragraph 2.13, there is $n \in \mathrm{N}_k$ such that we have $g'^{-1}g \cdot x = n \cdot x$. Hence we get $g \in \mathrm{XN}_k\mathrm{P}_x$. As $\mathrm{XN}_k = \mathrm{X}$, we obtain the expected result. □

4. Decomposition of $\mathrm{H}_k \backslash \mathrm{G}_k$

In all this section, we assume that G is k-split. Let H be an open k-subgroup of the fixed points group G^σ. Equivalently, H is a k-subgroup of G^σ containing $(\mathrm{G}^\sigma)^\circ$ (see [**1**]).

4.1. If T is a σ-stable torus in G, we write T^+ for the neutral component of $\mathrm{T} \cap \mathrm{H}$ and T^- for the neutral component of the subgroup $\{t \in \mathrm{T} \mid \sigma(t) = t^{-1}\}$. The torus T is the almost direct product of T^+ and T^-, that is $\mathrm{T} = \mathrm{T}^+\mathrm{T}^-$ and the intersection $\mathrm{T}^+ \cap \mathrm{T}^-$ is finite (see [**4**, xi]).

***Definition 4.1* (Helminck-Wang [13], §4.4).** — A σ-stable torus T of G is said to be (σ, k)*-split* if it is k-split and if $\mathrm{T} = \mathrm{T}^-$.

By [**13**, Proposition 10.3], two arbitrary maximal (σ,k)-split tori of G are G_k-conjugated.

4.2. Let $\mathscr{D}G$ denote the derived subgroup of G, and recall that C denotes the connected centre of G. This latter subgroup is a k-split torus of G.

Lemma 4.2. — *Let* T *be a k-split torus of* G.

(1) *There is a k-subtorus* T′ *of* C *such that the groups* $T\cdot\mathscr{D}G$ *and* $T'\cdot\mathscr{D}G$ *are equal.*

(2) *If* T *is (σ,k)-split, then any* T′ *satisfying (1) is (σ,k)-split.*

(3) *Assume that* $\mathscr{D}G$ *is contained in* H *and* T *is (σ,k)-split. Then any* T′ *satisfying (1) is (σ,k)-split and has the same dimension as* T.

Proof. — We set $\tilde{G}=G/\mathscr{D}G$ and, for any k-subgroup K of G, we write $\tilde{K}$ for the image of K in $\tilde{G}$. According to [**4**, Proposition 14.2], the group G is the almost direct product of C and $\mathscr{D}G$, which means that G is equal to the product $C\cdot\mathscr{D}G$ and that the intersection $C\cap\mathscr{D}G$ is finite. This implies that $\tilde{C}=\tilde{G}$. Let f denote the k-rational map $C\to\tilde{C}$. It is surjective with finite kernel. Hence $\tilde{G}$ is a k-split torus, and we denote by $\tilde{\sigma}$ the involutive k-automorphisme of $\tilde{G}$ induced by σ. We now prove the lemma in three steps.

(1) By [**4**, Proposition 8.2(c)], the neutral component of the inverse image $f^{-1}(\tilde{T})$ is a k-split subtorus of C which we denote by T′. It has finite index in $f^{-1}(\tilde{T})$. The image $f(T')$ is then a subtorus of finite index in the connected group $\tilde{T}$, so that $\tilde{T}'=\tilde{T}$.

(2) Now assume that T is (σ,k)-split, and let T′ satisfy (1). Let us consider the map $t\mapsto t\sigma(t)$ from T′ to itself. As $\tilde{T}'=\tilde{T}$ is a $(\tilde{\sigma},k)$-split torus, the image of this map is a connected k-subgroup contained in the kernel of f, which is finite.

(3) Assume that $\mathscr{D}G$ is contained in H and T is (σ,k)-split. Then the map $T\to\tilde{T}$ has finite kernel, which implies that T and $\tilde{T}$ have the same dimension. Now let T′ satisfy (1). According to (2), such a torus is (σ,k)-split, and it has the same dimension as $\tilde{T}'=\tilde{T}$. This ends the proof of Lemma 4.2. □

4.3. Let S be a σ-stable maximal (k-split) torus of G, let $\mathscr{A}$ be the apartment corresponding to S and let Φ be the set of roots of G relative to S. Let $x\in\mathscr{A}$ be a special point (see paragraph 2.7), and write U_x for U_Ω (see paragraph 2.10) with $\Omega=\{x\}$. Let $a\in\Phi$ be a σ-invariant root, which means that $a\circ\sigma=a$.

Lemma 4.3. — *Assume that* $U_{-a}(k)$ *is contained in* $\{g\in G_k \mid \sigma(g)=g^{-1}\}$. *Then there are* $n\in N_k$ *and* $c\in U_x$ *such that* $n=c^{-1}\sigma(c)$ *and* $\nu(n)$ *is the affine reflection of* $\mathscr{A}$ *which let* x *invariant and whose linear part is* s_a.

Proof. — We fix a base point in the apartment $\mathscr{A}$, so that it can be identified with the vector space V. For any $b \in \Phi$, this defines a filtration of the group $U_b(k)$ (see paragraph 2.8). For $u \in U_b(k) - \{1\}$, we denote by $\varphi_b(u)$ the greatest real number $r \in \mathbf{R}$ such that $u \in U_b(k)_r$. Let us choose $w \in U_{-a}(k) - \{1\}$ such that x is contained in the wall $\mathscr{H}_{-a,w}$. Thus $\nu(m(w))$ is the affine reflection of $\mathscr{A}$ which fixes x and whose linear part is s_a, and we can set:

$$n = m(w) \in N_k.$$

Moreover $\theta(-a, w)$, which is the unique affine function from $\mathscr{A}$ to $\mathbf{R}$ whose linear part is $-a$ and whose vanishing hyperplane is $\mathscr{H}_{-a,w}$, vanishes on x. Therefore it is equal to:

$$y \mapsto -a(y) + a(x),$$

which implies that $\varphi_{-a}(w) = a(x)$. According to **B3** (see paragraph 2.11), it follows that w fixes x.

The group $U_{-a}(k)$ is isomorphic to the additive group of k. Thus, for $r \in \mathbf{R}$, the subgroup $U_{-a}(k)_r$ corresponds through this isomorphism to a non-trivial sub-$\mathcal{O}$-module of k, where $\mathcal{O}$ denotes the ring of integers of k (see [**17**, Proposition 7.7]). Therefore, there is a unique element $v \in U_{-a}(k)$ such that $w = v^2$ and $\varphi_{-a}(v) = \varphi_{-a}(w)$, hence $v \in U_x$.

The map $U_a(k) \times U_a(k) \to G_k$ defined by $(u, u') \mapsto uwu'$ is injective and the intersection given by (2.4) consists of a single element, which is n. If we choose $u, u' \in U_a(k)$ such that $uwu' = n$, then the element:

$$\sigma(u')^{-1} w \sigma(u)^{-1} = \sigma(n)^{-1}$$

is contained in the intersection (2.4). Hence $\sigma(n)^{-1}$ is equal to n, and the unicity property implies that $u' = \sigma(u)^{-1}$. Moreover, according to [**17**, Lemma 7.4(ii)], the real numbers $\varphi_a(u)$ and $\varphi_a(\sigma(u))$ are both equal to $-\varphi_{-a}(w)$. This implies that u and $\sigma(u)$ are contained in U_x. Since v is σ-anti-invariant and $w = v^2$, we get the expected result by choosing $c = (uv)^{-1}$. □

Remark 4.4. — Note that $\sigma(c) \in U_x$. Indeed we have $\sigma(v) = v^{-1} \in U_x$ and $\sigma(u) \in U_x$. Hence $n = c^{-1}\sigma(c) \in N_k \cap U_\Omega$, which is contained in N_Ω with $\Omega = \{x, \sigma(x)\}$.

4.4. Let $\mathscr{B}$ denote the building of G over k.

Proposition 4.5. — *Let x be a special point of $\mathscr{B}$. There is a σ-stable maximal k-split torus* S *of* G *such that the apartment corresponding to* S *contains x and such that* S^- *is a maximal (σ, k)-split torus of* G.

Remark 4.6. — In paragraph 5.3, we give an example of a *non-split* k-group G such that Proposition 4.5 does not hold.

Proof. — Let $\mathscr{A}$ be a σ-stable apartment containing x (see Proposition 3.4) and let S be the corresponding maximal k-split torus of G. Assume that $\mathscr{A}$ has been chosen such that the dimension of the (σ,k)-split torus S^- is maximal. If it is a maximal (σ,k)-split torus of G, then Proposition 4.5 is proved. Assume that this is not the case, and let A be a maximal (σ,k)-split torus of G containing S^-. The dimension of A is greater than $\dim \mathrm{S}^-$ (if not, the containment $\mathrm{S}^- \subseteq \mathrm{A}$ would imply that $\mathrm{S}^- = \mathrm{A}$). Let G' be the neutral component of the centralizer of S^- in G. It is a k-split connected reductive subgroup of G containing S and A, which is naturally endowed with a non-trivial action of σ. Let C' denote the connected centre of G'.

Lemma 4.7. — *There is* $a \in \Phi(\mathrm{G}',\mathrm{S})$ *such that the corresponding root subgroup* U'_a *is not contained in* H*, and such a root is* σ*-invariant.*

Proof. — Assume that $\mathrm{U}'_a \subseteq \mathrm{H}$ for each root $a \in \Phi(\mathrm{G}',\mathrm{S})$. Then the derived subgroup $\mathscr{D}\mathrm{G}'$, which is generated by the U'_a for $a \in \Phi(\mathrm{G}',\mathrm{S})$, is contained in H (see [**14**, Theorem 27.5(e)]). According to Lemma 4.2(iii), there exists a (σ,k)-subtorus A' of C' such that $\mathrm{A}\cdot\mathscr{D}\mathrm{G}' = \mathrm{A}'\cdot\mathscr{D}\mathrm{G}'$ and $\dim(\mathrm{A}) = \dim(\mathrm{A}')$. The subgroup generated by C' and S is a k-torus of G'. As G' is k-split, S is a maximal torus of G', hence it contains C'. Therefore S^- contains A' which has the same dimension as A, and this dimension is greater than $\dim \mathrm{S}^-$. This gives us a contradiction.

Now let a be a root in $\Phi(\mathrm{G}',\mathrm{S})$ such that U'_a is not contained in H. The root a and its conjugate $a \circ \sigma$ coincide on S^+ and are both trivial on S^-. As S is the almost direct product of S^+ and S^- (see paragraph 4.1), they are equal. Therefore a is σ-invariant. This ends the proof of Lemma 4.7. □

Let $a \in \Phi(\mathrm{G}',\mathrm{S})$ as in Lemma 4.7. If we think of a as a root in $\Phi(\mathrm{G},\mathrm{S})$, then U_a is σ-stable and is not contained in H. Moreover, we have the following result.

Lemma 4.8. — $\mathrm{U}_a(k)$ *is contained in* $\{g \in \mathrm{G}_k \mid \sigma(g) = g^{-1}\}$.

Proof. — As G is k-split, U_a is k-isomorphic to the additive group. Thus the action of σ on $\mathrm{U}_a(k)$ corresponds to an involutive automorphism of the k-algebra $k[t]$. It has the form $t \mapsto \lambda t$ for some $\lambda \in k^\times$ with $\lambda^2 = 1$. As U_a is not contained in H, we have $\lambda = -1$. This gives us the expected result. □

According to Lemma 4.3, there are $n \in \mathrm{N}_k$ and $c \in \mathrm{U}_x$ such that $n = c^{-1}\sigma(c)$ and $\nu(n)$ is the affine reflection of $\mathscr{A}$ which let x invariant and whose linear part is s_a. For any $t \in \mathrm{S}$, note that we have:

$$\begin{aligned}\sigma(ctc^{-1}) &= cn\sigma(t)n^{-1}c^{-1}\\ &= cs_a(\sigma(t))c^{-1}.\end{aligned}$$

Let $\mathscr{A}'$ denote the apartment $c \cdot \mathscr{A}$ and let $\mathrm{S}' = {}^c\mathrm{S}$ be the corresponding maximal k-split torus of G. Then $\mathscr{A}'$ contains x and is σ-stable. Moreover, as the root a is trivial on S^- and s_a fixes the kernel of a pointwise, the conjugate ${}^c(\mathrm{S}^-)$ is a (σ, k)-split subtorus of S'. Thus S'^- has dimension not smaller than $\dim \mathrm{S}^-$.

Now let S_a denote the maximal k-split torus in the set of all $t \in \mathrm{S}$ such that $s_a(t) = t^{-1}$. As a is σ-invariant, such a torus is σ-stable. Moreover, it is one-dimensional and its intersection with $\mathrm{Ker}(a)$ is finite. Therefore ${}^c\mathrm{S}_a$ is a non-trivial (σ, k)-split subtorus of S' which is not contained in ${}^c(\mathrm{S}^-)$. Thus the dimension of S'^-, which contains ${}^c(\mathrm{S}_a\mathrm{S}^-)$, is greater than $\dim \mathrm{S}^-$, which contradicts the maximality property of $\mathscr{A}$. This ends the proof of Proposition 4.5. □

4.5. Let A be a maximal (σ, k)-split torus of G, let S be a σ-stable maximal k-split torus of G containing A and let $\mathscr{A}$ denote the apartment corresponding to S. Let $\{\mathrm{A}^j \mid j \in \mathrm{J}\}$ be a set of representatives of the H_k-conjugacy classes of maximal (σ, k)-split tori in G. According to [**13**], the set J is finite. Let $x \in \mathscr{A}$ be a special point and write K for its stabilizer in G_k.

Theorem 4.9. — *For $j \in \mathrm{J}$, let $y_j \in \mathrm{G}_k$ such that ${}^{y_j}\mathrm{A} = \mathrm{A}^j$. We have:*

$$\mathrm{G}_k = \bigcup_{j \in \mathrm{J}} \mathrm{H}_k y_j \mathrm{S}_k \mathrm{K}.$$

Proof. — By Proposition 4.5, for any $g \in \mathrm{G}_k$, there is a σ-stable maximal k-split torus S' of G such that the apartment corresponding to it contains $g \cdot x$ and such that S'^- is a maximal (σ, k)-split torus of G. Let $j \in \mathrm{J}$ be such that S'^- is H_k-conjugate to A^j. According to [**12**, Lemma 2.2], there is $h \in \mathrm{H}_k$ such that $\mathrm{S}' = {}^{hy_j}\mathrm{S}$. Hence $g \cdot x$ is contained in $hy_j \cdot \mathscr{A}$. According to Property (2) of paragraph 2.13, there exists $n \in \mathrm{N}_k$ such that $g \cdot x = hy_j n \cdot x$. Therefore G_k is the union of the $\mathrm{H}_k y_j \mathrm{N}_k \mathrm{K}$ for $j \in \mathrm{J}$. As x is special, we have $\mathrm{N}_k\mathrm{K} = \mathrm{S}_k\mathrm{K}$ and we get the expected result. □

4.6. In the case where G is not necessarily k-split, we have the following result. For each j, let $W_{G_k}(A^j)$ (resp. $W_{H_k}(A^j)$) be the quotient of the normalizer of A^j in G_k (resp. in H_k) by its centralizer. According to [**13**], the group $W_{G_k}(A^j)$ is the Weyl group of a root system. For $j \in J$, let $\mathscr{N}_j \subseteq N_{G_k}(A^j)$ be a set of representatives of:

$$W_{H_k}(A^j)\backslash W_{G_k}(A^j)$$

and let $y_j \in G_k$ be such that ${}^{y_j}A = A^j$. Let P be a minimal parabolic k-subgroup of G containing S and such that $P \cap \sigma(P)$ is a Levi component of P (see [**13**, §4]). Let ϖ be a uniformizer of k, and write Λ for the lattice made of the images of ϖ by the various algebraic cocharacters of A and Λ^- for the subset of anti-dominant elements of Λ relative to P. Then one can derive from Proposition 3.5 the existence of a compact subset Q of G_k such that:

$$G_k = \bigcup_{j\in J} \bigcup_{n\in\mathscr{N}_j} H_k n y_j \Lambda^- Q. \tag{4.1}$$

Benoist and Oh [**2**] have obtained a similar decomposition of G_k, with a weaker condition on the base field k (they assume k to have odd characteristic).

Remark 4.10. — In the split case, starting from Theorem 4.9, one can obtain a sharper result than the decomposition (4.1).

Let us mention that the question of the disjointness of the various components appearing in the decomposition (4.1) has been investigated by Lagier [**16**].

5. Examples

Let k be a non-Archimedean locally compact field of odd residue characteristic. Let $\mathcal{O}$ be its ring of integers and $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}$.

5.1. In this paragraph, we consider the k-split reductive group $G = GL_n$, $n \geqslant 1$, endowed with the k-involution $\sigma : g \mapsto {}^tg^{-1}$, where ${}^tg$ denotes the transpose of g. We set $K = GL_n(\mathcal{O})$ and $H = G^\sigma$, and write S for the diagonal torus of G.

We start with the following lemma.

Lemma 5.1. — *Let* V *be a finite dimensional k-vector space and* B *a symmetric bilinear form on* V*. Then any free $\mathcal{O}$-submodule of finite rank of* V *has a basis which is orthogonal relative to* B.

Proof. — Let Λ be a free $\mathcal{O}$-submodule of finite rank of V. The proof goes by induction on the rank of Λ. If B is null, then the result is trivial. If not, we denote by B_Λ the restriction of B to $\Lambda \times \Lambda$. Its image is of the form $\mathfrak{p}^m$ for some integer $m \in \mathbf{Z}$. If ϖ is a uniformizer of k, then the form $B^0_\Lambda = \varpi^{-m} B_\Lambda$ has image $\mathcal{O}$ on $\Lambda \times \Lambda$. Therefore, it defines a non trivial bilinear form:

$$\bar{B}^0_\Lambda : \Lambda/\mathfrak{p}\Lambda \times \Lambda/\mathfrak{p}\Lambda \to \mathcal{O}/\mathfrak{p}.$$

Let $e \in \Lambda$ be a vector whose reduction modulo $\mathfrak{p}$ is not isotropic relative to $\bar{B}^0_\Lambda$, which means that $B^0_\Lambda(e,e)$ is a unit of $\mathcal{O}$. Then Λ is the direct sum of $\mathcal{O}e$ and $\Lambda \cap ke^\perp$, where $ke^\perp$ denotes the orthogonal of ke in V. Indeed, it follows from the decomposition:

$$x = \frac{B(e,x)}{B(e,e)} e + \left(x - \frac{B(e,x)}{B(e,e)} e \right)$$

for any $x \in \Lambda$. As $\Lambda \cap ke^\perp$ is a free $\mathcal{O}$-submodule of finite rank of V whose rank is smaller than the rank of Λ, we conclude by induction. □

We introduce the set Y of all $g \in G_k$ such that ${}^t gg \in S_k$. Using Lemma 5.1, we get the following decomposition of G_k.

Proposition 5.2. — *We have* $G_k = YK$.

Proof. — We make G_k act on the quotient G_k/K, which can be identified to the set of all $\mathcal{O}$-lattices (that is, cocompact free $\mathcal{O}$-submodules) of the k-vector space $V = k^n$. Let B denote the symmetric bilinear form on V making the canonical basis of V into an orthonormal basis. According to Lemma 5.1, for any $g \in G_k$, the $\mathcal{O}$-lattice Λ corresponding to the class gK has a basis which is orthogonal relative to B. This means that there exists $u \in K$ such that the element $g' = gu^{-1} \in gK$ maps the canonical basis of V to an orthogonal basis of Λ. Therefore we have $g' \in Y$, thus $g \in YK$. □

We now investigate the maximal (σ, k)-split tori of G. Note that S is a maximal (σ, k)-split torus of G.

Proposition 5.3. — *The map* $g \mapsto {}^g S$ *induces a bijection between* (H_k, N_k)*-double cosets of* Y *and* H_k*-conjugacy classes of maximal* (σ, k)*-split tori of* G.

Proof. — One immediately checks that this map is well defined and injective. For $g \in G_k$, the conjugate ${}^g S$ is a maximal (σ, k)-split torus of G if anf only if $g^{-1}\sigma(g) \in S_k$, which amounts to saying that $g \in Y$ and proves surjectivity. □

Let $\mathscr{Q}$ denote the set of all equivalence classes of non-degenerate quadratic forms on k^n. For $a = \mathrm{diag}(a_1, \dots, a_n) \in \mathrm{S}_k$ we denote by Q_a the diagonal quadratic form $a_1\mathrm{X}_1^2 + \cdots + a_n\mathrm{X}_n^2$. Note that the map $a \mapsto \mathrm{Q}_a$ induces a surjective map from S_k to $\mathscr{Q}$.

We write H^0 and H^1 for the set of σ-fixed points and the first set of nonabelian cohomology of σ, respectively.

Proposition 5.4. — (1) *The map $g \mapsto {}^tgg$ induces an injection ι from the set of $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of* Y *to* $\mathrm{H}^1(\mathrm{N}_k)$.

(2) *Given $a \in \mathrm{S}_k$, the class of a in $\mathrm{H}^1(\mathrm{N}_k)$ is in the image of ι if and only if $\mathrm{Q}_a \sim \mathrm{X}_1^2 + \cdots + \mathrm{X}_n^2$.*

Proof. — We have an exact sequence:

$$\mathrm{H}_k \to \mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k) \to \mathrm{H}^1(\mathrm{N}_k) \to \mathrm{H}^1(\mathrm{G}_k),$$

where the map from $\mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k)$ to $\mathrm{H}^1(\mathrm{N}_k)$ is induced by $g \mapsto {}^tgg$. As the set of $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of Y is a subset of $\mathrm{H}_k\backslash\mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k)$, we get the first assertion. To obtain the second one, it is enough to remark that $\mathrm{H}^1(\mathrm{G}_k)$ canonically identifies with $\mathscr{Q}$. □

Remark 5.5. — Recall (see [**22**, IV §2.3]) that for $a, b \in \mathrm{S}_k$, the quadratic forms $\mathrm{Q}_a, \mathrm{Q}_b$ are equivalent if, and only if they have the same discriminant and the same Hasse invariant.

Proposition 5.6. — *Let $\{a^j \mid j \in \mathrm{J}\} \subseteq \mathrm{S}_k$ form a set of representatives of* $\mathrm{Im}(\iota)$ *in* $\mathrm{H}^1(\mathrm{N}_k)$. *For $j \in \mathrm{J}$, we choose $y_j \in \mathrm{Y}$ such that ${}^ty_jy_j = a^j$. Then:*

$$\mathrm{G}_k = \bigcup_{j\in\mathrm{J}} \mathrm{H}_k y_j \mathrm{S}_k \mathrm{K}.$$

Proof. — Propositions 5.2 and 5.3 imply that G_k is the union of the components $\mathrm{H}_k y_j \mathrm{N}_k \mathrm{K}$ for $j \in \mathrm{J}$. As $\mathrm{N}_k\mathrm{K} = \mathrm{S}_k\mathrm{K}$, we get the expected result. □

Example 5.7. — In the case where $n = 2$, we give an explicit description of $\mathrm{Im}(\iota)$. Let ϖ denote a uniformizer of $\mathcal{O}$ and $\xi \in \mathcal{O}^\times$ a non square unit of $\mathcal{O}$, so that $\{1, \xi, \varpi, \xi\varpi\}$ is a set of representatives of $k^\times$ modulo $k^{\times 2}$. The set of elements of $k^\times$ which are represented by the quadratic form $\mathrm{Q}_1 = \mathrm{X}^2 + \mathrm{Y}^2$ depends on the image of p in $\mathbf{Z}/4\mathbf{Z}$. If $p \equiv 1$ mod. 4, all elements of $k^\times$ are represented by Q_1. If $p \equiv 3$ mod. 4, an element of $k^\times$ is represented by Q_1 if and only if its normalized valuation if even. We set:

$$\mathrm{J} = \begin{cases} \{1, \xi, \varpi, \xi\varpi\} & \text{if } p \equiv 1 \text{ mod. } 4, \\ \{1, \xi\} & \text{if } p \equiv 3 \text{ mod. } 4. \end{cases}$$

For each $j \in \mathrm{J}$, set $a^j = \mathrm{diag}(j, j)$. Then the elements a^j form a set of representatives of $\mathrm{Im}(\iota)$ in $\mathrm{H}^1(\mathrm{N}_k)$.

5.2. In this paragraph, we consider the connected reductive k-group $\mathrm{G} = \mathrm{Res}_{k'/k}\mathrm{GL}_n$, where k' is a quadratic extension of k, endowed with the involutive k-automorphism σ of G induced by the non-trivial element of $\mathrm{Gal}(k'/k)$.

We set $\mathrm{H} = \mathrm{G}^\sigma$, so that we have $\mathrm{G}_k = \mathrm{GL}_n(k')$ and $\mathrm{H}_k = \mathrm{GL}_n(k)$. We denote by S the diagonal torus of G and by K the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}')$ of G_k, where $\mathcal{O}'$ denotes the ring of integers of k'. Note that S is σ-invariant.

As usual, N (resp. Z) denotes the normalizer (resp. the centralizer) of S in G. Let $\mathfrak{S}_n$ denote the group of permutation matrices in G_k, so that N_k is the semidirect product of $\mathfrak{S}_n$ by Z_k. Note that S_k (resp. Z_k) is the subgroup of all diagonal matrices of G_k with entries in k (resp. in k').

Lemma 5.8. — $\mathrm{H}^1(\mathrm{N}_k)$ *can be identified with the set of conjugacy classes of elements of* $\mathfrak{S}_n$ *of order* 1 *or* 2.

Proof. — According to Hilbert's Theorem 90, the group $\mathrm{H}^1(\mathrm{Z}_k)$ is trivial. Therefore we have an exact sequence:

$$1 \to \mathrm{H}^1(\mathrm{N}_k) \to \mathrm{H}^1(\mathrm{N}_k/\mathrm{Z}_k). \tag{5.1}$$

As σ acts trivially on $\mathrm{N}_k/\mathrm{Z}_k \simeq \mathfrak{S}_n$, the set $\mathrm{H}^1(\mathrm{N}_k/\mathrm{Z}_k)$ can be identified to the set of $\mathfrak{S}_n$-conjugacy classes of $\mathrm{Hom}(\mathbf{Z}/2\mathbf{Z}, \mathfrak{S}_n)$, that is, to the set of conjugacy classes of elements of $\mathfrak{S}_n$ of order 1 or 2. This proves that $\mathrm{H}^1(\mathrm{N}_k)$ can be naturally embedded in the set of conjugacy classes of elements of $\mathfrak{S}_n$ of order $\leqslant 2$.

Now two elements $w, w' \in \mathfrak{S}_n$ define the same class in $\mathrm{H}^1(\mathrm{N}_k)$ if and only if they are conjugate in $\mathfrak{S}_n$, thus if and only if $w\mathrm{Z}_k$ and $w'\mathrm{Z}_k$ define the same class in $\mathrm{H}^1(\mathrm{N}_k/\mathrm{Z}_k)$. Therefore (5.1) is a bijection. □

Proposition 5.9. — (1) *The number of* H_k*-conjugacy classes of* σ*-stable maximal* k*-split tori in* G_k *is* $[n/2]+1$.

(2) *There is a unique* H_k*-conjugacy class of maximal* (σ, k)*-split tori in* G_k.

Proof. — (1) Let X denote the set of all $g \in \mathrm{G}_k$ such that $g^{-1}\sigma(g) \in \mathrm{N}_k$. Then the map $g \mapsto {}^g\mathrm{S}$ defines an injective map from the set of $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of X to $\mathrm{H}^1(\mathrm{N}_k)$. Therefore we are reduced to proving that this map is surjective, and the first assertion will follow from Lemma 5.8. For $n = 2$, let τ denote the non-trivial element of $\mathfrak{S}_2$ and choose an element $a \in k'$ which is not in k. Then the element:

$$u = \begin{pmatrix} a & \sigma(a) \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(k') \tag{5.2}$$

satisfies the relation $u^{-1}\sigma(u)=\tau$. For an arbitrary integer $n\geqslant 2$, let $w\in\mathfrak{S}_n$ have order $\leqslant 2$. Then there is an integer $0\leqslant i\leqslant [n/2]$ such that w is conjugate to the element:

$$\tau_i=\mathrm{diag}(\tau,\dots,\tau,1,\dots,1)\in \mathrm{GL}_n(k'),$$

where $\tau\in\mathrm{GL}_2(k')$ appears i times and $1\in\mathrm{GL}_1(k')$ appears $n-2i$ times. Thus:

$$u_i=\mathrm{diag}(u,\dots,u,1,\dots,1)\in \mathrm{GL}_n(k') \tag{5.3}$$

satisfies the relation $u_i^{-1}\sigma(u_i)=\tau_i$. Therefore any 1-cocycle in N_k is G_k-cohomologous to the neutral element $1\in\mathrm{G}_k$, which proves the first assertion.

(2) For any $0\leqslant i\leqslant [n/2]$, the dimension of the (σ,k)-split torus $({}^{u_i}\mathrm{S})^-$ is equal to i. According to (1), the map:

$$\mathrm{H}_k g\mathrm{N}_k\mapsto \text{class of } g^{-1}\sigma(g) \text{ in } \mathrm{H}^1(\mathrm{N}_k)$$

is a bijection from the set of $(\mathrm{H}_k,\mathrm{N}_k)$-double cosets of X to $\mathrm{H}^1(\mathrm{N}_k)$, and the elements of this latter set are the classes of the τ_i for $0\leqslant i\leqslant [n/2]$. This gives us the expected result.

This ends the proof of Proposition 5.9. □

Proposition 5.10. — *For $0\leqslant i\leqslant [n/2]$, let u_i denote the element defined by (5.2) and (5.3). Then we have:*

$$\mathrm{G}_k=\bigcup_{i=0}^{[n/2]}\mathrm{H}_k u_i \mathrm{Z}_k\mathrm{K}.$$

Proof. — According to the proof of Proposition 5.9, the set X is the union of the double cosets $\mathrm{H}_k u_i\mathrm{N}_k$ with $0\leqslant i\leqslant [n/2]$. The result then follows from Proposition 3.5 and from the fact that $\mathrm{N}_k\mathrm{K}=\mathrm{Z}_k\mathrm{K}$. □

5.3. In this paragraph, we give an example (due to Bertrand Lemaire) of a non-split k-group such that Proposition 4.5 does not hold. We set $\mathrm{G}=\mathrm{Res}_{k'/k}\mathrm{GL}_2$, where k' is now a *ramified* quadratic extension of k. The k-involution σ is still induced by the non-trivial element of $\mathrm{Gal}(k'/k)$ and we set $\mathrm{H}=\mathrm{GL}_2$. Let $\mathscr{B}'$ (resp. $\mathscr{B}$) denote the building of G (resp. H) over k.

Bruhat and Tits [**8**] give a description of the faces of $\mathscr{B}$ in terms of hereditary $\mathcal{O}$-orders of $\mathrm{M}_2(k)$. More precisely, there is a bijective correspondence:

$$\mathrm{F}\mapsto\mathscr{M}_\mathrm{F}$$

between the faces of $\mathscr{B}$ and the hereditary $\mathcal{O}$-orders of $\mathrm{M}_2(k)$, such that the stabilizer of F in $\mathrm{GL}_2(k)$ in the normalizer of $\mathscr{M}_\mathrm{F}$ in $\mathrm{GL}_2(k)$. For $x\in\mathscr{B}$, we will denote by $\mathscr{M}_x$ the hereditary order corresponding to the face of $\mathscr{B}$ which contains x. We have a similar correspondence between faces of $\mathscr{B}'$ and hereditary $\mathcal{O}'$-orders of $\mathrm{M}_2(k')$. Moreover, as k' is tamely ramified over k, there is a bijective correspondence j from the set $\mathscr{B}'^{\sigma}$ of σ-fixed points of $\mathscr{B}'$ to $\mathscr{B}$ such that, for any $x \in \mathscr{B}'^{\sigma}$, we have:

$$\mathscr{M}_{j(x)} = \mathscr{M}_x \cap \mathrm{M}_2(k).$$

Let q denote the cardinality of the residue field of k. As k' is totally ramified over k, any vertex of $\mathscr{B}$ (resp. $\mathscr{B}'$) has exactly $q+1$ neighbours in $\mathscr{B}$ (resp. in $\mathscr{B}'$). Let x be a σ-invariant point of $\mathscr{B}'$. Recall that, according to Proposition 3.4, it is contained in a σ-stable apartment.

• If $j(x)$ is in a chamber of $\mathscr{B}$, then x has $q+1$ neighbours in $\mathscr{B}'$ but only two σ-fixed ones. Thus x has non-σ-fixed neighbours.

• If $j(x)$ is a vertex of $\mathscr{B}$, then x has $q+1$ neighbours in $\mathscr{B}'$ as in $\mathscr{B}$. Therefore any neighbour of x in $\mathscr{B}'$ is σ-invariant, which implies that any σ-stable apartment containing x is σ-invariant. For instance, this is the case of the vertex x corresponding to the $\mathcal{O}'$-order $\mathrm{M}_2(\mathcal{O}')$, as its image $j(x)$ corresponds to the maximal $\mathcal{O}$-order $\mathrm{M}_2(\mathcal{O}') \cap \mathrm{M}_2(k) = \mathrm{M}_2(\mathcal{O})$. For such a special point, Proposition 4.5 does not hold.

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