# THE SIGN OF LINEAR PERIODS 

U.K. ANANDAVARDHANAN, H. LU, N. MATRINGE, V. SÉCHERRE, AND C. YANG


#### Abstract

Let $G$ be a group with subgroup $H$, and let $(\pi, V)$ be a complex representation of $G$. The natural action of the normalizer $N$ of $H$ in $G$ on the space $\operatorname{Hom}_{H}(\pi, \mathbb{C})$ of $H$-invariant linear forms on $V$, provides a representation $\chi_{\pi}$ of $N$ trivial on $H$, which is a character when $\operatorname{Hom}_{H}(\pi, \mathbb{C})$ is one dimensional. If moreover $G$ is a reductive group over a $p$-adic field, and $\pi$ is smooth irreducible, it is an interesting problem to express $\chi_{\pi}$ in terms of the possibly conjectural Langlands parameter $\phi_{\pi}$ of $\pi$. In this paper we consider the following situation: $G=\mathrm{GL}_{m}(D)$ for $D$ a central division algebra of dimension $d^{2}$ over a $p$-adic field $F, H$ is the centralizer of a non central element $\delta \in G$ such that $\delta^{2}$ is in the center of $G$, and $\pi$ has generic Jacquet-Langlands transfer to the split form $\mathrm{GL}_{m d}(F)$ of $G$. In this setting the space $\operatorname{Hom}_{H}(\pi, \mathbb{C})$ is at most one dimensional. When $\operatorname{Hom}_{H}(\pi, \mathbb{C}) \simeq \mathbb{C}$ and $H \neq N$, we prove that the value of the $\chi_{\pi}$ on the non trivial class of $\frac{N}{H}$ is $(-1)^{m} \epsilon\left(\phi_{\pi}\right)$ where $\epsilon\left(\phi_{\pi}\right)$ is the root number of $\phi_{\pi}$. Along the way we extend many useful multiplicity one results for linear and Shalika models to the case of non split $G$, and we also classify standard modules with linear periods and Shalika models, which are new results even when $D=F$.


## 1. INTRODUCTION

Let $G$ be a group and $H$ be a subgroup of $G$, abbreviated as $H \leqslant G$. Denote by $N$ the normalizer of $H$ in $G$ and let $\pi$ be a complex representation of $G$. The group $N$ acts naturally on $\operatorname{Hom}_{H}(\pi, \mathbb{C})$ by $n \cdot L=L \circ \pi(n)^{-1}$. When moreover $\operatorname{Hom}_{H}(\pi, \mathbb{C})$ is one dimensional, this action is given by a character $\chi_{\pi}$ of $N$ which factors through the quotient $\frac{N}{H}$. Furthermore, if $G$ is a reductive group over a $p$-adic field, and $\pi$ is smooth irreducible, it is an interesting problem to determine $\chi_{\pi}$ in terms of the possibly conjectural Langlands parameter $\phi_{\pi}$ of $\pi$. This paper considers this question in the following situation: $G=\mathrm{GL}_{m}(D)$ for $D$ a central division algebra of dimension $d^{2}$ over a $p$-adic field $F$, and $H$ is the centralizer of a non central element $\delta \in G$ such that $\delta^{2}$ is in the center of $G$, and $\pi$ has generic transfer to $\mathrm{GL}_{m d}(F)$. Such a situation is indeed a multiplicity one situation, i.e.
$\operatorname{Hom}_{H}(\pi, \mathbb{C})$ has dimension at most one, see Theorem 2.5 of this paper for a more general statement. Our main result is the following (see Theorem 6.1, and Section 2.3 for the unexplained terminology used in its statement):

Theorem 1.1. Suppose that $N \neq H$ so the group $\frac{N}{H}=\left\{\overline{I_{m}}, \bar{u}\right\}$ has order two. Let $\pi$ be a smooth irreducible representation of $\mathrm{GL}_{m}(D)$ with generic transfer to $\mathrm{GL}_{m d}(F)$, and such that $\operatorname{Hom}_{H}(\pi, \mathbb{C}) \neq\{0\}$. Then:
(a) The Langlands parameter $\phi_{\pi}$ of $\pi$ is symplectic.
(b) $\operatorname{Hom}_{H}(\pi, \mathbb{C}) \simeq \mathbb{C}$.
(c) The character $\chi_{\pi}$ of $\frac{N}{H}$ given by its natural action on $\operatorname{Hom}_{H}(\pi, \mathbb{C})$ is expressed by the formula

$$
\chi_{\pi}(\bar{u})=(-1)^{m} \epsilon\left(\phi_{\pi}\right)
$$

where the root number $\epsilon\left(\phi_{\pi}\right)$ is independent of the choice of any non trivial additive character of $F$ used to define it.

This in particular extends [Pra, Theorem 4] of Prasad, which is the special case where $G$ is an inner form of $\mathrm{GL}_{2}$. It also extends [LM, Theorem 3.2] which is the special case of split $G=\mathrm{GL}_{n}(F)$ and $H \simeq \mathrm{GL}_{n / 2}(F) \times$ $\mathrm{GL}_{n / 2}(F)$. Actually [Pra, Theorem 4] was used in [CCL] to study Heegner points on a general class of modular curves, and it was mentioned to us by Li Cai that our main result could have possible number theoretic applications to more general problems of this type.

The sign of all irreducible representations seems more difficult to compute, and will require a detailed study of intertwining periods as in (Mat7, if one wants to apply the techniques of the present paper. We hope to come back to it later.

When we restrict to representations with generic transfer, then there is an explicit classification of distinguished representations in terms of their Langlands parameters when $\delta^{2}$ is not a square in the center $Z$ of $G$ and the residual characteristic of $F$ is not 2: the Prasad and Takloo-Bighash conjecture for trivial twisting character is indeed proved in the papers Xue, Suz1], Sé1], and the final step [SX] which relies on the result of [Cho as a separate case. Actually in this paper, we only need to know that the Langlands parameter of such a representation is symplectic, and this follows from [Xue, Suz1, [SX] and Cho without any restriction on $p$. We also establish such a classification (and actually more) when $\delta^{2}$ is a square in $Z$, extending results from Mat3] and Mat2] for the split case, thanks in particular to results of [BPW] on discrete series representations.

The computation of the sign relies on a local/global argument for cuspidal representations, relying on some global results from XZ]. It is then extended to discrete series, and then to generic representations using some explicit linear forms on induced representations called intertwining periods. The use of such linear forms to analyze the sign of induced representations already appears in [LM, Lemma 3.5]. In fact these linear forms also appear in the local/global argument in the computation the sign of generic unramified representations when $\delta^{2}$ is not a square in $Z$, whereas the results of [FJ] are used when it is.

We now briefly describe the different parts of the paper. In Section 3 , for $H$ such that $N \neq H$, we recall the classification of $H$-distinguished standard modules in terms of distinguished discrete series representations of $G$ due to Suzuki in the setting of twisted linear periods, i.e. when $\delta^{2}$ is not a square in $Z$ (see [Suz1]), and prove it in the setting of linear periods, i.e. when $\delta^{2}$ is a square in $Z$, in Theorem 3.12 (see Section 2.4 for more on the terminology). We also obtain analogues of Theorem 3.12 in 3.21 , where linear periods are allowed to be twisted by a character, and extend the classification of generic distinguished representations obtained in [Mat3] in Theorem[3.22. Using the Friedberg-Jacquet integrals theory, which is valid in our setting, we are also able to classify standard modules with a Shalika model in Theorem 3.20, and prove multiplicity at most one for them in Corollary 3.3. Parts of the results described above require extending some results from [Mat2] to inner forms, which require in particular Appendix B.

In Section 4, we reduce the computation of the sign of a representation of $G$ with generic transfer to that of cuspidal representations, using closed and open local intertwining periods.

The sign of cuspidal representations is then computed in Section 5 by a global argument, using local results from [FJ], global results from [XZ], and an explicit unramified computation.

The main result of the paper is Theorem 6.1.
There are three appendices. The first one, Appendix A, extends a multiplicity at most one result of Chong Zhang ([Zha]) for linear periods of local inner forms of $\mathrm{GL}_{n}$ from irreducible unitary to all irreducible representations, and proves more results of this type. The second one, Appendix B, extends parts of the theory of Bernstein-Zelevinsky derivatives to representations of non-Archimedean inner forms of $\mathrm{GL}_{n}$. It is in particular used to compute the sign of Steinberg representations. It is also used in the third
appendix, namely Appendix C. to prove that generic representations cannot have a linear model with respect to $\mathrm{GL}_{p}(D) \times \mathrm{GL}_{q}(D)$ when $|p-q| \geqslant 2$, thus extending Mat2, Theorem 3.2].

Acknowledgement. This paper was written under the impulse of a question asked to the fourth named author by Erez Lapid, and we thank him for this as well as for useful comments, especially for drawing our attention to the paper [M. We also thank Dipendra Prasad. Li Cai and Miyu Suzuki for useful remarks. H. Lu and C. Yang are supported by the by the National Natural Science Foundation of China (No 12301031 and 12001191 respectively).

## 2. Notations and preliminary results

Let $F$ be a field of characteristic different from 2, and let $D$ be a division algebra with center $F$ and of dimension $d^{2}$ over $F$. For $m$ a non-negative integer, we set $G_{m}=\mathrm{GL}_{m}$. The group $G_{0}(D)$ is trivial by definition. For $m \geqslant 1$, we denote by $\delta$ a non central element in $G:=G_{m}(D)$ such that $\kappa:=\delta^{2}$ is in the center of $G$, and set $H$ to be the centralizer of $\delta$ in $G$. The group $G$ is equal to the group consisting of invertible elements of the central simple $F$-algebra $A:=\mathcal{M}_{m}(D)=: \mathcal{M}_{m}$. Then $H$ is a symmetric subgroup obtained as the fixed points of the inner automorphism

$$
\theta:=\operatorname{Ad}(\delta): g \mapsto \delta g \delta^{-1}
$$

of $G$. We identify the center $Z$ of $G$ with $F^{\times}$.
2.1. Symmetric pairs. Here we identify three different situations for the pair $(G, H)$. The assertions below follow from basic results in linear algebra, extended to vector spaces over division algebras as in Lip.

1) If $\delta^{2} \in\left(F^{\times}\right)^{2}$, then there are non negative integers $p \geqslant q$ such that $p+q=m$, and such that $\delta$ is up to scaling in $F^{\times}$conjugate of

$$
\delta_{L_{p, q}}:=\operatorname{diag}\left(I_{p},-I_{q}\right) .
$$

With such a choice of $\delta$ we identify $H$ to the maximal block diagonal Levi subgroup

$$
L_{p, q}:=\operatorname{diag}\left(G_{p}(D), G_{q}(D)\right)
$$

of $G$. When convenient, we will prefer to take

$$
\delta=\delta_{H_{p, q}}:=\operatorname{diag}\left(I_{p-q}, 1,-1, \ldots,(-1)^{q-1}\right)
$$

in the conjugacy class of $\delta$, the centralizer of which we denote by $H_{p, q}$ and can be described as in Mat1. Finally when convenient, for $p=q$, we will take

$$
\delta=\delta_{M_{p, p}}:=\left(\begin{array}{ll} 
& I_{p} \\
I_{p} &
\end{array}\right)
$$

with $M_{p, p}=H$ the centralizer of $\delta_{M_{p, p}}$.
2) If $\delta^{2} \in F^{\times}-\left(F^{\times}\right)^{2}$, then $E=F[\delta] \subseteq A$ is a quadratic extension of $F$, and $n:=m d$ is even.
(a) There exists an $F$-algebra homomorphism from $E$ to $D$ (it is unique up to $D^{\times}$-conjugacy by the Skolem-Noether theorem). When $F$ is a local field, this simply means that $d$ is even. We may assume $E \subseteq D$. Then $H$ identifies with $G_{m}\left(C_{E}\right)$, where $C_{E}$ is the central $E$-division algebra centralizing $E$ in $D$, and $\delta$ identifies with the matrix $\delta . I_{m} \in G_{m}(E)$.
(b) There is no $F$-algebra homomorphism from $E$ to $D$. When $F$ is a local field this means $d$ odd. Then we put $D_{E}:=D \otimes_{F} E$, and it is a central $E$-division algebra. We realize $D_{E}$ inside $A$ as the matrices set of matrices $\left(\begin{array}{cc}x & \kappa y \\ y & x\end{array}\right)$ with $x$ and $y$ in $D$. In this situation we make the following identifications:

$$
\delta=\operatorname{diag}\left(\left(\begin{array}{ll} 
& \kappa \\
1 &
\end{array}\right), \ldots,\left(\begin{array}{ll} 
& \kappa \\
1 &
\end{array}\right)\right) \in G
$$

and

$$
H=H_{m}:=G_{m / 2}\left(D_{E}\right) .
$$

However we will also take

$$
\delta=\left(\begin{array}{ll} 
& \kappa I_{m / 2} \\
I_{m / 2} &
\end{array}\right)
$$

when convenient, in which case

$$
H=H_{m}^{\prime}:=\left\{\left(\begin{array}{cc}
A & \kappa B \\
B & A
\end{array}\right) \in G, A, B \in \mathcal{M}_{m}(D)\right\}
$$

2.2. Normalizers. In this paragraph we compute the normalizer of $H$ in $G$ in the different cases singled out in the previous paragraph.

1) Take $H$ under the form $H=L_{p, q}$. If $p \neq q$, the normalizer $N:=N_{G}(H)$ is equal to $H$. Indeed for $n$ in $N$, the matrix $n \delta_{L_{p, q}} n^{-1}$ must still belong to the center of $H$, so it must be of the form $\operatorname{diag}\left(\lambda I_{p}, \mu I_{q}\right)$. Reasoning on the possible choices of eigenvalues and their multiplicity, we deduce
that $\lambda=1$ and $\mu=-1$. Hence $n$ centralizes $\delta_{L_{p, q}}$ and so it belongs to $H$. If $p=q$ a similar reasoning implies that

$$
N=H \sqcup u H
$$

where

$$
u=u_{m}:=\left(\begin{array}{cc}
0 & I_{m / 2} \\
-I_{m / 2} & 0
\end{array}\right)
$$

If we choose $H=H_{m / 2, m / 2}$ then we take

$$
u=u_{m}:=\delta_{M_{m / 2, m / 2}}
$$

whereas if we choose $H=M_{m / 2, m / 2}$ then we take

$$
u=u_{m}:=\operatorname{diag}\left(I_{m / 2},-I_{m / 2}\right)
$$

2) (a) If $d$ is even, then by Cho, one can write $D=C_{E}+\iota C_{E}$ where $\iota$ is such that

$$
\theta(\iota)=-\iota .
$$

In this situation conjugation by $\iota$ on the central $E$-division algebra $C_{E}$ induces the Galois involution on $E$ because otherwise $\iota$ would centralize $E$, which it does not. Now take $n \in N$. The inner automorphism $\operatorname{Ad}(n)$ of $\mathcal{M}_{m}(D)$ is an $F$-linear automorphism of $\mathcal{M}_{m}\left(C_{E}\right)$, which stabilizes its center $E$. If it induces the identity on $E$, then $n \in H$ by the definition of $H$, and if not then $\operatorname{Ad}\left(\iota^{-1} n\right)$ induces the identity of $E$, hence $\iota n \in H$. The conclusion here is that

$$
N=H \sqcup u H
$$

where $u=u_{m}:=\iota I_{m}$.
(b) If $d$ is odd, a similar reasonning, using the Skolem-Noether theorem proves that

$$
N=H \sqcup u H
$$

where this time

$$
u=u_{m}:=\operatorname{diag}\left(1,-1, \ldots,(-1)^{m-1}\right)
$$

if $H=H_{m}$ and

$$
u=u_{m}:=\operatorname{diag}\left(I_{m},-I_{m}\right)
$$

if $H=H_{m}^{\prime}$.
2.3. Local Langlands correspondence for inner forms of $\mathrm{GL}_{n}$. Here $F$ is a local field of characteristic zero. We denote by $\left.\right|_{F}$ the normalized absolute value of $F$. When $F$ is non Archimedean, we denote its residual cardinality by $q_{F}$, and by $\mathfrak{w}_{F}$ its uniformizer. We denote by Nrd the reduced
norm on $G$, and for $g \in G$ we set

$$
\nu(g)=|\operatorname{Nrd}(g)|_{F} .
$$

We only consider smooth admissible complex representations, which we call representations. When $F$ is Archimedean we moreover require them to be Casselman-Wallach representations as in [JJ for example. When $F$ is $p$-adic, we recall that thanks to [HT] and Hen, one can associate to any irreducible representation $\pi$ of $\mathrm{GL}_{n}(F)$ a semi-simple representation $\phi_{\pi}$ of dimension $n$ of the Weil-Deligne group $\mathrm{WD}_{F}$ of $F$, which we call its Langlands parameter. When $F$ is Archimedean, taking the Weil-Deligne group of $F$ to be equal to its Weil group, one can do the same association thanks to Langlands' work [Lan], and we refer to [Kna] for more details on $\mathrm{GL}_{n}$.

We denote by JL the Jacquet-Langlands correspondence, which is a bijection from the set of isomorphism classes of discrete series, also called essentially square integrable representations of $G$, to that of $\mathrm{GL}_{n}(F)$ (see [DKV] and $[\mathrm{BR}$ when $F$ is archimedean). In Bad1 when $F$ is non Archimedean and $\overline{\mathrm{BR}}$ when $F$ is Archimedean, the Jacquet-Langlands correspondence was extended to an injective map from the Grothendieck group of finite length representations of $G$ to that of $\mathrm{GL}_{n}(F)$. These references actually define a surjective morphism $\mathrm{LJ}_{n}$ in the other direction, which has the Jacquet-Langlands correspondence as a section.

An irreducible representation of $\mathrm{GL}_{n}(F)$ is called generic if it has a non degenerate Whittaker model, see [JS]. Using the standard product notation for normalized parabolic induction, it follows from from [Zel] and [Jac2] that and irreducible representation of $\mathrm{GL}_{n}(F)$ is generic if and only if it is a product of discrete series representations.

Definition 2.1. Let $\pi$ be an irreducible representation of $G$. We say that it is generic if it has generic transfer to $\mathrm{GL}_{n}(F)$, i.e., if it is of the form $\mathrm{LJ}_{n}\left(\pi_{0}\right)$ for $\pi_{0}$ a generic representation of $\mathrm{GL}_{n}(F)$, in which case we set

$$
\mathrm{JL}(\pi):=\pi_{0} .
$$

We moreover define the Langlands parameter $\phi_{\pi}$ by

$$
\phi_{\pi}:=\phi_{\pi_{0}} .
$$

Remark 2.2. If $\pi$ is generic, then it can be written as a product of discrete series representations

$$
\pi=\delta_{1} \times \cdots \times \delta_{r}
$$

and

$$
\mathrm{JL}(\pi)=\mathrm{JL}\left(\delta_{1}\right) \times \cdots \times \mathrm{JL}\left(\delta_{r}\right)
$$

Definition 2.3. We say that a generic representation $\pi$ of $G$ is symplectic if its Langlands parameter $\phi_{\pi}$ is a symplectic representation of $\mathrm{WD}_{F}$, i.e. if it preserves a non degenerate alternate form.

For $p$-adic $F$, we refer to [ Zel$]$ and $[\mathrm{Tad}]$ for the parametrization of discrete series representations of $G$ in terms of cuspidal segments, as well as for the vocabulary related to these objects. In [Tad, a positive integer $s(\rho) \mid d$ is attached to any cuspidal representation $\rho$ of $G$. It can be described as the smallest non negative real number $s$ such that $\rho \times \nu^{s(\rho)} \rho$ is reducible. For $\rho$ a cuspidal representation of $G$, we set

$$
\pi_{t}(\rho):=\nu^{(1-t) s(\rho) / 2} \rho \times \cdots \times \nu_{\rho}^{(t-1) s(\rho) / 2} \rho .
$$

Then $\pi_{t}(\rho)$ has a unique irreducible quotient, denoted by $\operatorname{St}_{t}(\rho)$.
If $\rho$ is a cuspidal representation of $G_{a}(D)$ with transfer

$$
\mathrm{JL}(\rho)=\mathrm{St}_{r}\left(\rho^{\prime}\right)
$$

to $\mathrm{GL}_{a d}(F)$ for some $r$ dividing $a d$ and $\rho^{\prime}$ a cuspidal representation of $\mathrm{GL}_{\frac{a d}{r}}(F)$, then

$$
\mathrm{JL}\left(\mathrm{St}_{t}(\rho)\right)=\mathrm{St}_{t r}\left(\rho^{\prime}\right)
$$

In particular, when $\rho=\mathbf{1}_{D^{\times}}$we have $r=d$ and $\rho^{\prime}=\mathbf{1}_{F^{\times}}$, i.e.

$$
\mathrm{JL}\left(\mathrm{St}_{t}\left(\mathbf{1}_{D^{\times}}\right)\right)=\mathrm{St}_{t d}\left(\mathbf{1}_{F^{\times}}\right)
$$

Let $F$ any local field of characteristic zero, and $\pi$ be a generic representation of $G$. For $\psi$ a non trivial character of $F$, we denote by $\epsilon\left(1 / 2, \phi_{\pi}, \psi\right)$ the Langlands root number attached to $\phi_{\pi}$ as in Tat, and we set

$$
\epsilon(\pi, \psi):=\epsilon\left(1 / 2, \phi_{\pi}, \psi\right) .
$$

It is known that this root number is the same as the one defined in [GJ] via the Godement-Jacquet functional equation. If moreover $\pi$ is symplectic, then $\epsilon\left(1 / 2, \phi_{\pi}, \psi\right)$ is independent of the choice of $\psi$ and we write:

$$
\epsilon(\pi):=\epsilon(\pi, \psi) .
$$

When $F$ is $p$-adic, we will use the following well-known formulae concerning root numbers which can be found in [GJ] and [GR].
(a) If $\rho$ is a cuspidal representation of $G_{a}$ with transfer $\mathrm{JL}(\rho)=\mathrm{St}_{r}\left(\rho^{\prime}\right)$ to $\mathrm{GL}_{a d}(F)$, then

$$
\epsilon\left(\mathrm{St}_{t}(\rho), \psi\right)=\epsilon\left(\rho^{\prime}, \psi\right)^{r t}
$$

except when $\rho$ is an unramified character $\mu \circ \mathrm{Nrd}$ of $G_{1}=D^{\times}$, in which case the formula is given by

$$
\epsilon\left(\mathrm{St}_{t}(\mu \circ \mathrm{Nrd}), \psi\right)=\left(-\mu\left(\mathfrak{w}_{F}\right)\right)^{d t+1} \epsilon(\mu, \psi)^{d t} .
$$

In particular when $\operatorname{St}_{t}(\mu)$ is self-dual, we observe that

$$
\epsilon\left(\mathrm{St}_{t}(\eta), \psi\right)=\epsilon(\eta, \psi)^{d t}
$$

when $\mu=\eta$ is the unramified quadratic character of $F^{\times}$, which is the same formula as that in Equation 2.1), whereas when $\mu=\mathbf{1}_{F^{\times}}$ we obtain

$$
\epsilon\left(\mathrm{St}_{t}\left(\mathbf{1}_{D^{\times}}\right), \psi\right)=(-1)^{d t+1} \epsilon\left(\mathbf{1}_{F^{\times}}, \psi\right)^{d t}
$$

(b) If $\pi$ is an irreducible representation of $G$ with central character $\omega_{\pi}$,

$$
\epsilon(\pi, \psi) \epsilon\left(\pi^{\vee}, \psi\right)=\omega_{\pi}(-1)
$$

(c) If $\pi_{1}$ and $\pi_{2}$ are irreducible representations of $G_{m_{1}}$ and $G_{m_{2}}$ respectively, such that $\pi_{1} \times \pi_{2}$ is irreducible, then

$$
\epsilon\left(\pi_{1} \times \pi_{2}, \psi\right)=\epsilon\left(\pi_{1}, \psi\right) \epsilon\left(\pi_{2}, \psi\right)
$$

(d) If $\pi$ is a generic unramified representation of $\mathrm{GL}_{n}(F)$ and $\psi$ is unramified as well, then

$$
\epsilon(\pi, \psi)=1
$$

2.4. Local linear periods and their sign. Here $F$ is a local field of characteristic zero.

Definition 2.4. Let $\pi$ be an irreducible representation of $G$ and $\chi$ be a character of $H$. We say that $\pi$ is $\chi$-distinguished if $\operatorname{Hom}_{H}(\pi, \chi) \neq\{0\}$. Moreover if $\pi$ is distinguished:

1) if $\delta^{2} \in\left(F^{\times}\right)^{2}$, we call an element in $\operatorname{Hom}_{H}(\pi, \chi)-\{0\}$ a $\chi$-linear period.
2) if $\delta^{2} \notin\left(F^{\times}\right)^{2}$, we call an element in $\operatorname{Hom}_{H}(\pi, \chi)-\{0\}$ a twisted $\chi$-linear period. Furthermore for such an invariant linear form $L$ :
(a) if d is odd we say that $L$ is a twisted $\chi$-linear period of odd type.
(b) if $d$ is even we say that $L$ is a twisted $\chi$-linear period of even type.

We will mostly consider the case of the trivial character $\chi=\mathbf{1}_{H}$ of $H$, in which case we will omit $\chi$ in the terminology above.

10 U.K. ANANDAVARDHANAN, H. LU, N. MATRINGE, V. SÉCHERRE, AND C. YANG
If $\pi$ is distinguished, then by (JR, GuO, BM, Appendix A and Appen$\operatorname{dix} \mathbb{C}$, we have much information on the dimension of the space $\operatorname{Hom}_{H}(\pi, \mathbb{C})$.

Theorem 2.5. Let $\pi$ be an irreducible representation and suppose that $H \not \nsim$ $H_{p, q}$ such that $|p-q| \geqslant 2$. Then

$$
\operatorname{dim}\left(\operatorname{Hom}_{H}(\pi, \mathbb{C})\right) \leqslant 1
$$

If $|p-q| \geqslant 2$ this conclusion holds for irreducible unitary representations as well as for generic representations. In fact for generic representations we have

$$
\operatorname{dim}\left(\operatorname{Hom}_{H}(\pi, \mathbb{C})\right)=0
$$

when $|p-q| \geqslant 2$.
Remark 2.6. When $G$ is split, it is proved in [JR that $\operatorname{dim}\left(\operatorname{Hom}_{H_{p, q}}(\pi, \mathbb{C})\right) \leqslant$ 1 for any irreducible $\pi$, without any restriction on $p$ and $q$.

In particular for any distinguished generic representation $\pi$ of $G$, we define the character $\chi_{\pi}$ of $\frac{N}{H}$ by the equality for all $n \in N$ :

$$
L \circ \pi(n)^{-1}=\chi_{\pi}(\bar{n}) L
$$

The problem is that of computing $\chi_{\pi}$ in terms of its Langlands parameter $\phi_{\pi}$.

We recall that $\frac{N}{H}$ is trivial in the case of linear periods when $p \neq q$, so we forget about this case unless explicitly stated. Hence our three cases are now:

1) Linear periods with respect to $H$ conjugate to $L_{m / 2, m / 2}$.
2) (a) Twisted linear periods of odd type.
(b) Twisted linear periods of even type.

Distinguished generic representations of $G$ always have a symplectic Langlands parameter as explained in Corollary 3.24. As we shall see that the solution to our problem involves the quantity $\epsilon(\pi)$, it will be convenient to define the sign of the (possibly twisted linear period on the) distinguished representation $\pi$ by:

$$
\operatorname{sgn}(\pi)=\frac{\chi_{\pi}(\bar{u})}{\epsilon(\pi)}=\chi_{\pi}(\bar{u}) \epsilon(\pi) .
$$

## 3. Results on distinguished standard modules

### 3.1. Friedberg-Jacquet integrals for inner forms and applications

 to linear periods and Shalika models. Here $G=G_{m}(D)=: G_{m}$ with$m$ even. In Section 5.2, allowing $F$ to be Archimedean, we will use the Friedberg-Jacquet integrals in the case where $G$ is $F$-split. Indeed the full theory of such integrals is developed in [FJ] over all local fields when $G$ is $F$-split. In the present paragraph we consider general $G$, but restrict to $F$ non Archimedean. We explain why the Friedberg-Jacquet integrals are well defined, and why their gcd is the Godement-Jacquet L-function for $G$ not necessarily $F$-split. The very clever computations in [FJ] easily extend to our $G$, but some proofs of [FJ] are only given for Archimedean $F$ as this case is more difficult. The details of their proofs were written in Mat2] and Mat5] in the non Archimedean case for split $G$, but the situation in Mat2 is complicated by the fact that the Shalika functionals are with respect to the Shalika model of the mirabolic subgroup. Hence some integrals of Shalika functions in Mat2] are not obviously absolutely convergent anymore, as in the case of Shalika functionals with respect to the Shalika subgroup of $G$. Here we briefly explain why [FJ] applies without modifications to our setting, giving a simplified version of the arguments in Mat2]. The reason for studying Friedberg-Jacquet integrals is that following [Mat5] for split $G$, one proves using such integrals that if $\delta$ is a discrete series representation of $G$, and if $\chi$ is a character of $H=L_{m / 2, m / 2}$ trivial on $G_{m / 2}$ diagonally embedded, then $\delta$ has a $\chi$-linear period if and only if it has a Shalika model. This result is then used in Section 3.4 for the classification of standard modules of $G_{m}$ with a linear period when $m$ is odd, though this classification is not really needed in this paper. However it is used in Appendix A to prove multiplicity one of linear periods for the pair $\left(G_{2 q+1}, G_{q+1} \times G_{q}\right)$.

First we recall the definition and properties of Godement-Jacquet integrals for induced representations of $G$. We recall that $n=m d$. Let

$$
\pi=\pi_{1} \times \cdots \times \pi_{r}
$$

be a representation of $G$ with each $\pi_{i}$ irreducible. For $\Phi$ a Schwartz function on $\mathcal{M}:=\mathcal{M}_{m}$, and $c$ a matrix coefficient of $\pi$, we set

$$
Z(s, \Phi, c):=\int_{G} \Phi(g) c(g) \nu(g)^{s+\frac{(n-1)}{2}} d g
$$

We moreover say that $\pi$ is a standard module of $G$ if each $\delta_{i}:=\pi_{i}$ is a discrete series such that $e\left(\delta_{i}\right) \geqslant e\left(\delta_{i+1}\right)$ for $i=1, \ldots, r-1$, where $e(\delta)$ is the unique real number such that $\nu^{-e(\delta)} \delta$ is unitary. We recall that the Langlands classification (see [Tad]) says that standard modules of $G$ have a unique irreducible quotient, and conversely that each irreducible representation of $G$ is the quotient of a standard module.

The first part of the following statement is proved in GJ]. The second part for Langlands quotients of standard modules is proved in Jac1, (2.3) Proposition, (3.4) Theorem] for split $G$ with a proof valid for any $G$, following GJ.

Theorem 3.1 (Jacquet). (a) For $\pi=\pi_{1} \times \cdots \times \pi_{r}$ a representation of $G$ with each $\pi_{i}$ irreducible, the integral $Z(s, \Phi, c)$ converges for $s$ of real part large enough, and extends to a rational function of $q_{F}^{-s}$. Moreover the vector space spanned by the integrals $Z(s, \Phi, c)$ is a fractional ideal of $\mathbb{C}\left[q_{F}^{ \pm s}\right]$ and the normalized $\operatorname{gcd} L(s, \pi)$ of this fractional ideal is equal to the following the product of GodementJacquet L-factors

$$
L(s, \pi)=\prod_{i=1}^{r} L\left(s, \pi_{i}\right) .
$$

(b) If moreover $\pi$ is a standard module with Langlands quotient $\mathrm{LQ}(\pi)$, then

$$
L(s, \pi)=L(s, \mathrm{LQ}(\pi))
$$

We now denote by $S_{m}$ the Shalika subgroup of $G_{m}$. By definition it is the set of matrices

$$
S_{m}:=\left\{s(g, x):=\operatorname{diag}(g, g)\left(\begin{array}{cc}
I_{m / 2} & x \\
& I_{m / 2}
\end{array}\right), g \in G_{m / 2}, x \in \mathcal{M}_{m / 2}\right\}
$$

We fix $\psi$ a non trivial character of $F$, and denote by $\Theta$ the character of $S_{m}$ defined by the formula

$$
\Theta(s(g, x))=\psi(\operatorname{Trd}(x))
$$

where $\operatorname{Trd}$ is the reduced trace of $\mathcal{M}_{m / 2}$. If $\pi$ is a smooth representation of $G$, and $L \in \operatorname{Hom}_{S_{m}}(\pi, \Theta)-\{0\}$, we call the space

$$
S_{L}(\pi, \Theta):=\left\{S_{v}: g \mapsto L(\pi(g) v), v \in \pi\right\}
$$

the Shalika model of $\pi$ attached to $L$ (and $\psi$ ). When $G$ is split it is known thanks to JR (and AGJ when $F$ is Archimedean) that the space $\operatorname{Hom}_{S_{m}}(\pi, \Theta)$ is at most one dimensional for any irreducible $\pi$. For $G$ as in this paragraph and $\pi$ a discrete series representation, the multiplicity of Shalika functionals is again known to be at most one by BPW]. We will extend their result to all irreducible representations in Corollary 3.3 and thus remove the dependence to $L$ in the notation for the Shalika model of irreducible representations. The following generalizes the $p$-adic part of [FJ, Proposition 3.1], and follows from its proof.

Proposition 3.2. Let $\pi$ be a representation of $G$ as in Theorem 3.1, (a), and suppose that

$$
\operatorname{Hom}_{S_{m}}(\pi, \Theta) \neq\{0\} .
$$

Take $L \in \operatorname{Hom}_{S_{m}}(\pi, \Theta)-\{0\}$ and $S \in S_{L}(\pi, \Theta)$, and $\alpha$ a character of $F^{\times}$. Then the integral

$$
\Psi_{\alpha}(s, S)=\int_{G_{m / 2}} S\left(\operatorname{diag}\left(g, I_{n}\right)\right) \alpha(\operatorname{Nrd}(g)) \nu(g)^{s-1 / 2} d g
$$

is absolutely convergent for s of real part large enough and extends to a rational function of $q_{F}^{-s}$. Moreover the vector space spanned by the integrals $\Psi_{\alpha}(s, S)$ is a fractional ideal of $\mathbb{C}\left[q_{F}^{ \pm}\right]$and the normalized $g c d$ of this fractional ideal is the Godement-Jacquet L-factor $L(s, \alpha \pi)$.

Proof. We can suppose $\alpha$ trivial as in the proof of [FJ, Proposition 3.1] and set $\Psi(s, S):=\Psi_{\alpha}(s, S)$. We denote by $K_{m}$ the maximal compact subgroup $\operatorname{GL}_{m}\left(O_{D}\right)$ of $G_{m}$ where $O_{D}$ is the ring of integers of $D$. For $S \in S_{L}(\pi, \Theta)$ and $\phi \in \mathcal{C}_{c}^{\infty}\left(\mathcal{M}_{m / 2} \times G_{m / 2} \times K_{m}\right)$ a smooth function with compact support in $\mathcal{M}_{m / 2} \times G_{m / 2} \times K_{m}$, we define

$$
\left.S_{\phi}(g)=\int_{u, b, k} S\left(g\left(\begin{array}{ll}
b^{-1} & \\
& 1
\end{array}\right)\left(\begin{array}{ll}
1 & u \\
& 1
\end{array}\right)\left(\begin{array}{ll}
1 & \\
& b
\end{array}\right) k\right)\right) \phi(u, b, k) \nu(b)^{n / 2} d b d u d k
$$

with variable

$$
(u, b, k) \in \mathcal{M}_{m_{2}} \times G_{m / 2} \times K_{m}
$$

Clearly $S_{\phi} \in S_{L}(\pi, \Theta)$ and one can write $S=S_{\phi}$ for some well chosen $\phi$ by Mat2, Lemma 4.2]. For such a $\phi$, we denote by $K_{\phi}$ its support in the variable $b \in G_{m / 2}$, and by $\phi^{\prime}$ the characteristic function of $K_{\phi}^{-1}$. We then attach to $\phi$ a map

$$
\Phi_{\phi}\left(\left(\begin{array}{ll}
a & x \\
& b
\end{array}\right), k\right)=\int_{u, v} \phi(u, b, k) \phi^{\prime}(v) \Theta(u a-v x) d u d v
$$

with compact support in the variable $(a, x, b, k) \in \mathcal{M}_{m / 2} \times \mathcal{M}_{m / 2} \times G_{m / 2} \times K_{m}$ by properties of the Fourier transform.

Let $\Omega$ be the open set of matrices $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ of $\mathcal{M}_{m}$ with $\left(\begin{array}{ll}c & d\end{array}\right)$ of rank $m / 2$ in $\mathcal{M}_{m / 2, m}$, and $\Omega_{0}$ its subset of matrices $\left(\begin{array}{ll}a & b \\ 0 & d\end{array}\right)$ with $d$ invertible. The map

$$
r:(p, k) \mapsto p k
$$

from $\Omega_{0} \times K_{m}$ to $\Omega$ is proper and surjective. We then define $\Phi_{\phi^{*}}$ on $\Omega$, by

$$
\Phi_{\phi} *(p k)=\int_{k^{\prime} \in K_{m} \cap P_{(m / 2, m / 2)}} \Phi_{\phi}\left(p k^{\prime-1}, k^{\prime} k\right) d k^{\prime}
$$

for $(p, k) \in \Omega_{0} \times K_{m}$. The map $\Phi_{\phi^{*}}$ is well defined and has compact support in $\Omega$ as $r$ is proper, it is moreover fixed by some open compact subgroup of $K_{m}$ which stabilizes $\Omega$ by right translation, hence it is smooth. We thus extend $\Phi_{\phi^{*}}$ by zero outside $\Omega$ to obtain a smooth function on $\mathcal{M}_{m}$. We now define the integrals

$$
I\left(S, \Phi_{\phi}, a, b\right)=\int_{x \in \mathcal{M}_{m / 2}, k \in K_{m}} S\left(\left(\begin{array}{ll}
a & x \\
& b
\end{array}\right) k\right) \Phi_{\phi}\left(\left(\begin{array}{ll}
a & x \\
& b
\end{array}\right), k\right) d x d k
$$

and

$$
\begin{gathered}
J(S, \phi, a, b)= \\
\int_{u \in \mathcal{M}_{m / 2}, k \in K_{m}} S\left(\left(\begin{array}{cc}
a & \\
& I_{m / 2}
\end{array}\right)\left(\begin{array}{cc}
I_{m / 2} & u \\
& I_{m / 2}
\end{array}\right)\left(\begin{array}{cc}
I_{m / 2} & \\
& b
\end{array}\right) k\right) \phi(u, b, k) d u d k .
\end{gathered}
$$

It is proved in Mat2, Lemma 4.3] that they both converge absolutely and are equal, and have compact support with respect to the variables $a \in \mathcal{M}_{m / 2}$, and $b \in G_{m / 2}$, and $k \in K_{m}$. Now we claim that the integral

$$
\int_{(a, b) \in G_{m / 2}^{2}} I\left(S, \Phi_{\phi}, a, b\right) \nu(a)^{s+n} \nu(b)^{s+\frac{(n-1)}{2}} d a d b
$$

converges absolutely for $s$ of real part large enough, and is of the form $P\left(q^{-s}\right) L(s, \pi)$ for $P \in \mathbb{C}\left[X^{ \pm 1}\right]$ in its realm of convergence. Indeed:

$$
\begin{gathered}
\int_{(a, b) \in G_{m / 2}^{2}}\left|I\left(S, \Phi_{\phi}, a, b\right) \nu(a)^{s+n} \nu(b)^{s+\frac{(n-1)}{2}}\right| d a d b \\
\leqslant \int_{(a, b) \in G_{m / 2}^{2}} I\left(|S|,\left|\Phi_{\phi}\right|, a, b\right) \nu(a)^{\Re(s)+n} \nu(b)^{\Re(s)+\frac{(n-1)}{2}} d a d b \\
=\int_{(a, b) \in G_{m / 2}^{2}} I\left(|S|,\left|\Phi_{\phi}\right|, a, b\right) \left\lvert\, \nu(a)^{\Re(s)+n} \nu(b)^{\mathfrak{R}(s)+\frac{(n-1)}{2}} d a d b\right. \\
=\int_{g \in G}|S(g)|\left|\Phi_{\phi *}(g)\right| \nu(g)^{\Re(s)+\frac{(n-1)}{2}} d g \\
=\int_{g \in G}\left|S^{U}(g) \Phi_{\phi *}(g)\right| \nu(g)^{\Re(s)+\frac{(n-1)}{2}} d g
\end{gathered}
$$

where the penultimate equality follows from Iwasawa decomposition, and in the last one $U \leqslant K_{m}$ is a compact open subgroup of $G$ fixing $\Phi_{\phi *}$ under right translation and $S^{U}$ is the right average of $S$ by $U$. Because $S^{U}$ is a genuine matrix coefficient of $\pi$, we recognize the absolute value of an integrand of a Godement-Jacquet integral for $\pi$ in the last equation hence the absolute
convergence. This also implies the equality

$$
\int_{(a, b) \in G_{m / 2}^{2}} I\left(S, \Phi_{\phi}, a, b\right) \nu(a)^{s+n} \nu(b)^{s+\frac{(n-1)}{2}} d a d b=Z\left(s, \Phi_{\phi *}, S^{U}\right)
$$

hence the existence of the Laurent polynomial $P$. On the other hand

$$
\begin{gathered}
\int_{(a, b) \in G_{m / 2}^{2}} I\left(S, \Phi_{\phi}, a, b\right) \nu(a)^{s+n} \nu(b)^{s+\frac{(n-1)}{2}} d a d b \\
=\int_{(a, b) \in G_{m / 2}^{2}} J(S, \phi, a, b) \nu(a)^{s+n} \nu(b)^{s+\frac{(n-1)}{2}} d a d b \\
=\Psi\left(s, S_{\phi}\right)
\end{gathered}
$$

So far we have proved every part of the statement we want to obtain, except the assertion on the normalized gcd of the integrals $\Psi(s, S)$. This assertion is easier and follows word for word from [FJ] beginning of p. 111 and Lemmas 3.2 and 3.3].
Q.E.D.

For $a$ and $b$ two non negative integers, we define the character $\chi_{\alpha}$ of $L_{a, b}$ by

$$
\chi_{\alpha}\left(\operatorname{diag}\left(g_{1}, g_{2}\right)\right)=\alpha\left(\operatorname{Nrd}\left(g_{1}\right) \operatorname{Nrd}\left(g_{2}\right)^{-1}\right) .
$$

Let $V_{\pi}$ be the space of $\pi$ as in Theorem 3.1, (a). For $L \in \operatorname{Hom}_{S_{m}}(\pi, \Theta)$, and $v \in \pi$, we define $S_{v, L} \in S_{L}(\pi, \Theta)$ by the formula

$$
S_{v, L}(g)=L(\pi(g) v)
$$

We then define the linear map

$$
\Lambda_{L, \alpha}: v \mapsto \lim _{s \rightarrow \frac{1}{2}} \frac{\Psi_{\alpha}\left(s, S_{v, L}\right)}{L(s, \alpha \pi)}
$$

which makes sense thanks to Proposition 3.2. As explained in JR, p.117], we deduce the following from Proposition 3.2 .

Corollary 3.3. Let $\pi$ be a representation of $G$ as in Theorem 3.1, (a), and suppose moreover that it has a Shalika model, then it is $\chi_{\alpha}$-distinguished. More precisely we have an exact sequence

$$
0 \rightarrow \operatorname{Hom}_{S_{m}}(\pi, \Theta) \rightarrow \operatorname{Hom}_{L_{m / 2, m / 2}}\left(\pi, \chi_{\alpha}\right) .
$$

In particular

$$
\operatorname{dim}\left(\operatorname{Hom}_{S_{m}}(\pi, \Theta)\right) \leqslant 1
$$

whenever $\pi$ is irreducible.
Proof. The linear map

$$
L \in \operatorname{Hom}_{S_{m}}(\pi, \Theta) \mapsto \Lambda_{L, \alpha^{-1}} \in \operatorname{Hom}_{L_{m / 2, m / 2}}\left(\pi, \chi_{\alpha}\right)
$$

is injective as $\Lambda_{L}$ is nonzero as soon as $L$ is thanks to Proposition 3.2. The second statement follows from Appendix A which uses the classification of standard modules with a linear period (i.e. $\alpha=\mathbf{1}_{F^{\times}}$) given in Theorem 3.12 .
Q.E.D.

Corollary 3.4. Let $\delta$ a discrete series representation of $G$. Then $\delta$ has a $\chi_{\alpha^{-}}$ linear period if and only if it has a Shalika model, and if so it is symplectic, hence selfdual, and $\operatorname{Hom}\left(\delta, \chi_{\alpha}\right)$ has dimension one.

Proof. All arguments in Mat5, Section 2], in particular Proposition 2.2 there, are valid for $G$. We mention in particular that thanks to Proposition 3.2, the statement of Mat5, Proposition 2.3] is valid for the standard module $\pi$ of $G$, hence the proof of [Mat5, Theorem 2.1] works without any modification. We highlight that the most delicate part there is the extension of linear form argument at the end of the proof of Mat5, Theorem 2.1], which works equally well here thanks to Theorem 3.1, (b). Q.E.D.

Remark 3.5. In the split case it is proved in Yan, Corollary 3.6] that Corollary 3.4 holds more generally for generic representations, using Gan's technique of relating periods via Theta correspondence. However this uses the realization of contragredient of a representation using transpose inverse, hence it seems to not extend directly to the case of inner forms. As well the unfolding principle of [SV] does not seem to extend easily to linear periods twisted by a non unitary character, as was already observed in Mat5, Remak 2.1] in the split case. In Theorem 3.21 below we extend [Yan, Corollary 3.6] to standard modules for a wide family of characters $\chi_{\alpha}$.

Remark 3.6. For $F$-split $G$, the sign of generic representations is given at once by the functional equation of the Friedberg-Jacquet integrals. Again this functional equation is established using the realization of contragredient using transpose inverse, so this is one reason why the theory above does not include this functional equation. Note that our sign computation will prove the Friedberg-Jacquet functional equation of generic unitary representations of $G$ at least at the value $s=1 / 2$.

### 3.2. Preliminaries for the geometric lemma applied to linear peri-

 ods. Here we provide the necessary material to apply the geometric lemma of [Off2, Corollary 5.2] in the context of linear periods. Let $P=P_{\left(m_{1}, \ldots, m_{r}\right)}$ be a standard upper block triangular parabolic subgroup of $G$ attached to the composition ( $m_{1}, \ldots, m_{r}$ ) of $m$, and suppose $p \geqslant q$ are two non negativeintegers such that $p+q=m$. We recall that from an immediate generalization of the arguments at the beginning of Mat3, Section 3] (see also [Yan] for a different approach), the double cosets $P \backslash G / H_{p, q}$ are parametrized by sequences of non negative integers of the form

$$
\begin{gathered}
s= \\
\left(m_{1,1}^{+}, m_{1,1}^{-}, m_{1,2}, \ldots, m_{1, r}, m_{2,1}, m_{2,2}^{+}, m_{2,2}^{-}, m_{2,3}, \ldots, m_{2, r}, \ldots, m_{r, 1}, \ldots, m_{r, r-1}, m_{r, r}^{+}, m_{r, r}^{-}\right)
\end{gathered}
$$

such that if one puts

$$
m_{i, i}:=m_{i, i}^{+}+m_{i, i}^{-},
$$

then the matrix $\left(m_{i, j}\right)_{i, j=1, \ldots, r}$ is symmetric, and for all $i$ one has

$$
\sum_{j=1}^{r} m_{i, j}=m_{i}
$$

and

$$
\sum_{i=1}^{r} m_{i, i}^{+}-\sum_{i=1}^{r} m_{i, i}^{-}=p-q .
$$

We denote by $\mathcal{I}_{P}$ this set of parametrizing sequences. Given $s \in \mathcal{I}_{P}$, we can associate to it a standard parabolic subgroup $P_{s}=M_{s} U_{s} \subset P$ with the standard Levi $M_{s}$ of type

$$
\left(m_{1,1}, \cdots, m_{1, r}, m_{2,1} \cdots, m_{r, r-1}, m_{r, r}\right)
$$

where we ignore the zero's in the above partition of $m$, as in Mat3, Section 3]. We also have an involution $\theta_{s}$ on $M_{s}$ and a positive character $\delta_{s}$ of $M_{s}^{\theta_{s}}$, that was denoted by

$$
\delta_{s}:=\delta_{P_{s}^{\theta_{s}}} \delta_{P_{s}}^{-1 / 2}
$$

and was computed explicitly in Mat3, Proposition 3.6]. For $\alpha$ a character of $F^{\times}$, and $a$ and $b$ two non negative integers, we denote by

$$
\xi_{\alpha}: H_{p, q} \rightarrow \mathbb{C}^{\times}
$$

the character of $H_{p, q}$ corresponding via the natural isomorphism between $H_{p, q}$ and $L_{p, q}$ described in [Mat2, Section 2] to the character

$$
\chi_{\alpha}: L_{p, q} \rightarrow \mathbb{C}^{\times}
$$

defined in Section 3.1. We observe that the computation in Mat3, Proposition 3.6] remains valid in our setting, under the following form:

Lemma 3.7. Set

$$
\mu:=\nu^{d},
$$

and

$$
h=\operatorname{diag}\left(h_{1,1}^{+}, h_{1,1}^{-}, h_{1,2}, \ldots, h_{r, r-1}, h_{r, r}^{+}, h_{r, r}^{-}\right) \in M_{s}^{\theta_{s}}
$$

Then:
(a)
$\delta_{s}(h)=\prod_{1 \leqslant i<j \leqslant r} \mu\left(h_{i, i}^{+}\right)^{\frac{m_{j, j}^{+}-m_{j, j}^{-}}{2}} \mu\left(h_{i, i}^{-}\right)^{\frac{m_{j, j}^{-}-m_{j, j}^{+}}{2}} \mu\left(h_{j, j}^{+}\right)^{\frac{m_{i, i}^{+}-m_{i, i}^{-}}{2}} \mu\left(h_{j, j}^{-}\right)^{\frac{m_{i, i}^{-}-m_{i, i}^{+}}{2}}$.
(b)

$$
\xi_{\alpha}(h)=\prod_{i=1}^{r} \xi_{\alpha}\left(h_{i, i}^{+}\right) \xi_{\alpha}\left(h_{i, i}^{-}\right)^{-1}
$$

3.3. Distinguished generalized Steinberg representations. We start with a result that will be used in classifying standard modules with linear periods in terms of their essentially square-integrable support.

Theorem 3.8. Let $\pi$ be a discrete series representation of $G_{m}$ for $m \geqslant 2$ and let $p$ and $q$ be two non-negative integers such that $p+q=m$. If $\pi$ is $\left(H_{p, q}, \mu\right)$-distinguished for $\mu$ a character of $H_{p, q}$, then $p=q$.

Proof. We write $\pi=\operatorname{St}_{t}(\rho)$ for $\rho$ a cuspidal representation of $G_{r}$. We can suppose that both $t$ and $r \geqslant 2$ thanks to Propositions B. 3 and B.5. Then by an explicitation of Off2, Corollary 5.2], we deduce thanks to the case $t=1$ and $r \geqslant 2$ that $m_{i, i}^{+}=m_{i, i}^{-}$whenever $m_{i, i}$ is nonzero so $p=q$. We refer to the proof of Theorem 3.12 below for more details on such type of computation in a more difficult case.
Q.E.D.

We will also use the following fact, which follows from the results in [SX] and Xue in the case of twisted linear periods, and is a special case of Corollary 3.4 for linear periods.

Proposition 3.9. If $\pi$ is a discrete series representation of $G$ which is distinguished, then $\pi$ is symplectic, and hence it is selfdual.

Remark 3.10. In the proof of [Mat2, Theorem 5.1], there is an order mistake: p. 16 the third equality should read

$$
\int_{M_{2 m} \backslash G_{2 m}}|\phi(g)|^{2} d g=\int_{\mathcal{M}_{m}}\left(|\phi|^{2}\right)^{K}(x) d x
$$

Observing that $\int_{\mathcal{M}_{m}}\left(|\phi|^{2}\right)^{K}(x) d x$ is also equal to $\int_{K}\left(\int_{\mathcal{M}_{m}}|\phi(x k)|^{2} d x\right) d k$, the Parseval identity should then be applied to each $\phi_{k}:=\phi(. k)$ restricted to $\mathcal{M}_{m}$, i.e. equality number four should start as

$$
\int_{K}\left(\int_{\mathcal{M}_{m}}|\phi(x k)|^{2} d x\right) d k=\int_{K}\left(\int_{\mathcal{M}_{m}}\left|\widehat{\phi_{k}}(x)\right|^{2} d x\right) d k=\ldots
$$

The rest of the equalities should then be corrected accordingly, and we refer to the proof given in [Duh, Lemme 5.7] for the correct full proof.

The following result is enough for our purpose, and in the case of linear periods it follows from the proof of Theorem 3.8 except when $r=1$.

Proposition 3.11. Let $\rho \neq \mathbf{1}_{D^{\times}}$be a cuspidal representation of $G_{r}$, and suppose that $\mathrm{St}_{t}(\rho)$ is $H$-distinguished. Then there is up to scaling at most one $H$-invariant linear form on $\pi_{t}(\rho)$, so that the $H$-invariant linear form on $\mathrm{St}_{t}(\rho)$ descends from that on $\pi_{t}(\rho)$. Moreover if $t$ is odd, then $\rho$ is distinguished.

Proof. In the case of twisted linear periods, this is part of the proof of BM, Proposition 5.6]. In the case of linear periods, when $r \geqslant 2$, it follows again from the geometric lemma [Off2, Corollary 5.2] and [Mat3, Section 3] that necessarily $m_{i, t+1-i}=r$ for all $i$ smaller than $\lfloor(t+1) / 2\rfloor$, and moreover that $m_{(t+1) / 2,(t+1) / 2}^{+}=m_{(t+1) / 2,(t+1) / 2}^{-}=r / 2$ when $t$ is odd. This implies our statement in the case $r \neq 1$. When $r=1$ the assertions for twisted linear periods are proved in Cho, and we explain why they also hold for linear periods. By Proposition 3.9, we can suppose that $\rho \neq \mathbf{1}_{D^{\times}}$is quadratic character $\eta$ of $D^{\times}$. If some non open orbit contributed to the distinction of $\pi_{t}(\rho)$, because $\eta$ takes negative value, applying the geometric lemma would lead to a sign issue as in [Mat4, Proof of Proposition 3.6]. Finally note that in this case the integer $t$ cannot be odd for central character reasons. Q.E.D.
3.4. Classification of distinguished standard modules. Here $F$ is a $p$-adic field. The results that we state here are known for twisted linear periods when $F$ is Archimedean by [ST], but have to be checked for linear periods (which should be possible following [ST]). In any case, we do not need such results in the Archimedean setting for our purpose.

The Langlands parameter of an irreducible representation of $G$ is understood in terms of its essentially square-integrable support once it is realized as the quotient of the unique standard module lying above it. Now, it is clear that if the Langlands quotient of a standard module is distinguished, then the standard module is itself distinguished so this observation could be a starting point for computing the sign of all distinguished irreducible representations of $G$. However in this paper we only compute the sign of generic representations. We feel that the computation of the sign for all irreducible representations would involve much finer properties of admissible intertwining periods on standard modules. Nevertheless, we
state the classification of distinguished standard modules in terms of their essentially square-integrable support, from which the classification of distinguished generic representations follows.

For twisted linear periods, this classification is due to Suzuki Suz1, Theorem 1.3], and we recall it in (a) of Theorem 3.12 below. Here we provide the proof in the case of linear periods. We also consider the case $p=q+1$ in order to obtain the full statement of Theorem 2.5 later in Appendix A. We follow the main steps of the proof of [Mat3, Theorems 3.13 and 3.14], but we in fact correct the proof in question as one part of Claim (2) in the proof of [Mat3, Theorems 3.14] is incorrect.

Theorem 3.12. Let $\pi=\delta_{1} \times \cdots \times \delta_{r}$ be a standard module of $G$ where each $\delta_{i}$ is an essentially square integrable representation of $G_{m_{i}}$, and for any $1 \leqslant k \leqslant r$ set $\pi_{k}$ to be the standard module

$$
\pi_{k}:=\delta_{1} \times \cdots \times \delta_{k-1} \times \delta_{k+1} \times \cdots \times \delta_{r} .
$$

(a) The representation $\pi$ has a twisted linear period if and only if there exists an involution $\tau \in S_{r}$ such that for all $1 \leqslant i \leqslant r$ one has $\delta_{\tau(i)}=\delta_{i}^{\vee}$, and moreover each $\delta_{i}$ has a twisted linear period when $\tau(i)=i$.
(b) Suppose that $G=G_{m}$ for $m$ even, and that $H=H_{m / 2, m / 2}$. The representation $\pi$ is distinguished if and only if there exists an involution $\tau \in S_{r}$ such that for all $1 \leqslant i \leqslant r$ one has $\delta_{\tau(i)}=\delta_{i}^{\vee}$, and moreover $m_{i}$ is even and $\delta_{i}$ is $H_{m_{i} / 2, m_{i} / 2}$-distinguished when $\tau(i)=i$.
(c) Suppose that $G=G_{m}$ for $m$ odd, and that $H=H_{(m+1) / 2,(m-1) / 2}$. Then $\pi$ is distinguished if and only if there exists an index $1 \leqslant i_{0} \leqslant r$ such that $\delta_{i_{0}}=\mathbf{1}_{D^{\times}}$, and $\pi_{i_{0}}$ is $H=H_{(m-1) / 2,(m-1) / 2}$-distinguished.

Proof. We only deal with linear periods. We refer for Section 3.2 for the definitions of $\mu$ and $\xi_{\alpha}: H_{p, q} \rightarrow \mathbb{C}^{\times}$, and we set

$$
\mu_{\alpha}=\xi_{\alpha}^{d}
$$

Since the right to left implications of (b) and (c) are easier, we start with them. When $m$ is even we can suppose that

$$
\pi=\delta_{1} \times \cdots \times \delta_{a} \times \tau_{1} \times \cdots \times \tau_{b} \times \delta_{a}^{\vee} \times \cdots \times \delta_{1}^{\vee}
$$

where the $\tau_{i}$ are distinguished discrete series. The middle product

$$
\tau_{1} \times \cdots \times \tau_{b}
$$

is distinguished by [Mat3, Proposition 3.8] so that the full product is distinguished by [Off2, Proposition 7.2]. When $m$ is odd we can suppose that

$$
\pi=\delta_{1} \times \cdots \times \delta_{a} \times \tau_{1} \times \cdots \times \tau_{b} \times \mathbf{1}_{D \times} \times \delta_{a}^{\vee} \times \cdots \times \delta_{1}^{\vee}
$$

Note that the representations $\tau_{i}$ are $\xi_{\alpha}$-distinguished for any choice of $\alpha$ thanks to Corollary 3.4 , in particular they are $\mu_{\left.\right|^{\frac{1}{2}}}$-distinguished. Then the representation $\tau_{1} \times \cdots \times \tau_{b} \times \mathbf{1}_{D \times}$ is $H_{k+1, k}$-distinguished for the relevant $k$ by [Mat3, Proposition 3.8], so that the full product is distinguished by [Off2, Proposition 7.2] again.

Now we move on to the direct implications. We set $P=M U$ to be the standard parabolic subgroup of type ( $m_{1}, m_{2}, \cdots, m_{r}$ ). Each $\delta_{i}$ is attached to a cuspidal segment $\Delta_{i}$ and we use the notation $\delta_{i}=\delta\left(\Delta_{i}\right)$. Given $s \in \mathcal{I}_{P}$, we associate to $s$ the standard parabolic subgroup $P_{s}=M_{s} U_{s} \subset P, \theta_{s}$ and $\delta_{s}$ as in Section 3.2. The geometric lemma says that if $\pi$ is distinguished, then there is an $s \in \mathcal{I}_{P}$ such that the normalized Jacquet module $r_{M_{s}, M}(\sigma)$ is $\left(M_{s}^{\theta_{s}}, \delta_{s}\right)$-distinguished. If we write for each $1 \leqslant i \leqslant r$, with obvious intuitive notations,

$$
\Delta_{i}=\left[\Delta_{i, r}, \Delta_{i, r-1}, \cdots, \Delta_{i, 1}\right]
$$

so that

$$
r_{M_{s}, M}(\sigma)=\delta\left(\Delta_{1,1}\right) \otimes \cdots \delta\left(\Delta_{1, r}\right) \otimes \delta\left(\Delta_{2,1}\right) \otimes \cdots \otimes \delta\left(\Delta_{r, r}\right)
$$

then putting $\delta_{i, j}:=\delta\left(\Delta_{i, j}\right)$ we have

$$
\left\{\begin{array}{l}
\delta_{i, j} \cong \delta_{j, i}^{\vee}, \quad i \neq j  \tag{3.1}\\
\delta_{i, i} \text { is }\left(H_{m_{i, i}^{+}, m_{i, i}^{-}}, \mu_{i}\right) \text {-distinguished },
\end{array}\right.
$$

where the character $\mu_{i}$ above could be described explicitly, but we only state their needed properties. We start with the following useful observations.
(a) All characters $\mu_{i}$ are of the form $\xi_{\alpha_{i}}$ for some character $\alpha_{i}$ of $F^{\times}$.
(b) If $m_{i, i}>1$ for some $i$, then both $m_{i, i}^{+}$and $m_{i, i}^{-}$are nonzero otherwise $\delta_{i}$ would be a character by (3.1), and $m_{i, i}^{+}=m_{i, i}^{-}$by Theorem 3.8. In particular, we don't care about the value of $\alpha_{i}$ thanks to Corollary 3.4, hence we can suppose that all $\mu_{i}$ are trivial, and moreover $\delta_{i}$ is selfdual by the same corollary.
(c) Denoting by $u_{1}<\cdots<u_{t}$ the indices such that $m_{u_{k}, u_{k}}=1$, and setting

$$
\epsilon_{k}=m_{u_{k}, u_{k}}^{+}-m_{u_{k}, u_{k}}^{-} \in\{1,-1\} .
$$

Then $t$ is even and

$$
\sum_{k=1}^{t} \epsilon_{k}=0
$$

when $m$ is even whereas $t$ is odd and

$$
\sum_{k=1}^{t} \epsilon_{k}=1
$$

when $m$ is odd. Moreover it follows from Lemma 3.7 that the characters $\mu_{u_{i}}$ are given by the formula

$$
\mu_{u_{k}}=\mu^{\frac{\epsilon_{k}\left(-\sum_{i<k} \epsilon_{i}+\sum_{j>k} \epsilon_{j}\right)}{2}},
$$

which after some rewriting gives
(a)

$$
\mu_{u_{k}}=\mu^{\frac{1}{2}+\epsilon_{k}\left(\sum_{j>k} \epsilon_{j}\right)}
$$

when $m$ is even,
(b)

$$
\mu_{u_{k}}=\mu^{\frac{1}{2}+\epsilon_{k}\left(-\frac{1}{2}+\sum_{j>k} \epsilon_{j}\right)}
$$

when $m$ is odd.
It then follows from the above formulae that the $\mu_{i}$ 's satisfy the following inductive relations which are the correct version of Mat3, Theorems 3.14, (2)]:

$$
\mu_{u_{k+1}}=\mu^{-1} \mu_{u_{k}} \quad \text { or } \quad \mu_{u_{k+1}}=\mu_{u_{k}}^{-1}
$$

depending on $\epsilon_{k+1}=\epsilon_{k}$ or not. Moreover,

- when $m$ is even: $\mu_{u_{1}}=\mu^{-1 / 2}$ and $\mu_{u_{t}}=\mu^{1 / 2}$.
- when $m$ is odd:

$$
\begin{aligned}
& -\mu_{u_{1}}=\mu^{0}=\mathbf{1} \text { if } m_{1,1}=m_{1,1}^{+} \\
& -\mu_{u_{1}}=\mu^{-1} \text { if } m_{1,1}=m_{1,1}^{-} \\
& -\mu_{u_{t}}=\mu^{0}=1 \text { if } m_{t, t}=m_{t, t}^{+} \\
& -\mu_{u_{t}}=\mu^{-1} \text { if } m_{t, t}=m_{t, t}^{-}
\end{aligned}
$$

In particular setting $\mu_{u_{i}}^{\prime}:=\mu_{u_{t+1-i}}^{-1}$, we see that the characters $\mu_{u_{i}}^{\prime}$ satisfy the same relations and have the same initial values. This implies that the multiset $\left\{\mu_{u_{1}}, \ldots, \mu_{u_{t}}\right\}$ is stable under the $\operatorname{map} \chi \mapsto \chi^{-1}$. However because it contains only positive characters, hence only the trivial character as a quadratic character, it can be partitioned by pairs $\left\{\chi, \chi^{-1}\right\}$ when $t$ is even, whereas it can be partitioned by such pairs together with one time $\mathbf{1}$ when $t$ is odd.

We move on to the application of the geometric lemma. We can and will assume that $\pi$ is right-ordered, i.e. that the right ends $r\left(\Delta_{i}\right)$ of its cuspidal segments satisfy $e\left(r\left(\Delta_{i}\right)\right) \geqslant e\left(r\left(\Delta_{i+1}\right)\right)$. We now start to prove by induction on $n$ the following statement: If $r_{M_{s}, M}(\sigma)$ is $\left(M_{s}^{\theta_{s}}, \delta_{s}\right)$-distinguished, then $s$ is a monomial matrix, which will finish our proof thanks to the above observations. We have three cases.

Case (I) If $m_{1,1}>1$, then by the observation (b) above $\delta\left(\Delta_{1,1}\right)$ is selfdual. By the right orderning of $\pi$, we have $\Delta_{1}=\Delta_{1,1}$. We then can finish the proof by induction.

Case (II) If $m_{1,1}=1$ we use observation (c) above. When $t$ is even, we have $r\left(\Delta_{1}\right)=\mu^{-1 / 2}$. This is absurd as the central character of $\pi$ will not be trivial. When $t$ is odd the only possible case is $m_{1,1}=m_{1,1}^{+}$so $r\left(\Delta_{1}\right)=\mathbf{1}$, which implies that $\Delta_{1}=\Delta_{1,1}$ thanks to the right ordering, and we conclude by induction.

Case (III) If $m_{1,1}=0$, then by the right ordering of $\pi$, we have $\Delta_{1}=\Delta_{1, l}$ for some $l>1$. We need to show that $\Delta_{l}=\Delta_{l, 1}$, and the rest follows from induction. If not so, then the only possibility is $m_{l, l}=1$ and $\Delta_{l}=\left[\Delta_{l, 1}, \Delta_{l, l}\right]$. We write $\Delta_{1}=\left[\mu^{a}, \mu^{b}\right]$ with $b>0$. So $\Delta_{l, 1}=\left[\mu^{-b}, \mu^{-a}\right]$ and $\Delta_{l, l}=\mu^{-b-1}$. By the observation (3.2) we know that there is a $u_{k}$ such that $\mu_{u_{k}}=\mu^{c}$ with $c>b+1$. This means that $\Delta_{u_{k}, u_{k}}=\mu^{c}$. This is absurd as the segments are right-ordered.
Q.E.D.
3.5. Classification of standard modules with a Shalika model and $\chi$-distinction. In this paragraph we prove Theorem 3.20 below, which is the classification of standard modules with a Shalika model. Denoting by $\mathfrak{R}(\alpha)$ the unique real number such that if $\alpha: F^{\times} \rightarrow \mathbb{C}^{\times}$is a character, then

$$
|\alpha|=| |_{F}^{\Re(\alpha)},
$$

we moreover extend the classification of distinguished standard modules to the case of $\chi$-distinguished ones, for $\chi$ of the form $\chi_{\alpha}$ with $\alpha$ such that $-\frac{d}{2} \leqslant \mathfrak{R}(\alpha) \leqslant \frac{d}{2}$.

We start by proving that certain type of induced representations admit a Shalika model with meromorphic families of Shalika functionals. Let $\pi_{i}$ be representations of $G_{m_{i}}$ for $i=1, \ldots, l$ and $\underline{s}=\left(s_{1}, \ldots, s_{l}\right) \in \mathbb{C}^{l}$. Then if $\pi=\pi_{1} \times \cdots \times \pi_{l}$ is a representation of $G$, we set

$$
\pi[\underline{s}]=\nu^{d s_{1}} \pi_{1} \times \cdots \times \nu^{d s_{l}} \pi_{l}=\mu^{s_{1}} \pi_{1} \times \cdots \times \mu^{s_{l}} \pi_{l} .
$$

Iwasawa decomposition allows one to realize all these representations on the same vector space, and we thus refer to [MO, 2.5] for the definition of a meromorphic family in the variable $\underline{s}$ of linear forms on $\pi[\underline{s}]$.

Proposition 3.13. Let $\delta$ be a discrete representations of $G_{m}$, then there is a nonzero meromorphic family $\left(\lambda_{s}\right)_{s \in \mathbb{C}}$ with

$$
\lambda_{s} \in \operatorname{Hom}_{S_{2 m}}\left(\delta[s] \times \delta^{\vee}[-s], \Theta\right)
$$

In particular $\delta[s] \times \delta^{\vee}[-s]$ admits a Shalika model for all $s \in \mathbb{C}$.
Proof. The proof is exactly the same as the one given in the proof of Mat3, Lemma 3.11] together with its correction in [Mat5, Proposition 5.3], and relies on the Bernstein principle of meromorphic continuation of equivariant linear forms in a generic multiplicity one situation, and the multiplicity at most one of Shalika models for irreducible representations proved in Corollary 3.3 .
Q.E.D.

Remark 3.14. In general, the fact that any representation of the form $\pi \times \pi^{\vee}$ has a Shalika model follows immediately from the computation of twisted Jacquet modules in [PR, Theorem 2.1].

Remark 3.15. The representation $\delta$ could be replaced by any irreducible unitary representation as follows from its proof.

Here is another result of the same flavor.
Proposition 3.16. Let $\pi_{i}$, resp. $\pi_{j}^{\prime}$, be representations of $G_{m_{i}}$, resp. $G_{m_{j}^{\prime}}$, for $i=1, \ldots, l$, resp. $j=1, \ldots, l^{\prime}$, and set

$$
\pi=\pi_{1} \times \cdots \times \pi_{l}
$$

and

$$
\pi^{\prime}=\pi_{1}^{\prime} \times \cdots \times \pi_{l^{\prime}}^{\prime}
$$

to be the corresponding induced representations of $G_{m}$ and $G_{m}^{\prime}$ respectively. Suppose that $m$ and $m^{\prime}$ are even, and that there are two nonzero meromorphic families $\left(\lambda_{\underline{s}}\right)_{\underline{s} \in \mathbb{C}^{l}}$ and $\left(\lambda_{\underline{s^{\prime}}}^{\prime}\right)_{\underline{s}^{\prime} \in \mathbb{C}^{l^{\prime}}}$ respectively in

$$
\lambda_{\underline{s}} \in \operatorname{Hom}_{S_{m}}(\pi[\underline{s}], \Theta)
$$

and

$$
\lambda_{\underline{s^{\prime}}} \in \operatorname{Hom}_{S_{m^{\prime}}}\left(\pi^{\prime}\left[\underline{s^{\prime}}\right], \Theta\right) .
$$

Then there is a nonzero meromorphic family $\Lambda_{\underline{s}, s^{\prime}}$ in

$$
\operatorname{Hom}_{S_{m+m^{\prime}}}\left(\pi[\underline{s}] \times \pi^{\prime}\left[\underline{s}^{\prime}\right], \Theta\right)
$$

Proof. The proof follows from the explicit construction of the Shalika functional on $\pi[\underline{s}] \times \pi^{\prime}\left[\underline{s}^{\prime}\right]$ given in Mat6] which is valid for inner forms, with the following observations. First there is a quantifier ordering problem in Mat6, Lemma 3.2]: the compact set $C$ depends on the function $f$ there. However in our situation, this compact set will be independent of $\left(\underline{s}, \underline{s^{\prime}}\right) \in \mathbb{C}^{l+l^{\prime}}$ for a given holomorphic section $f_{\underline{s}, \underline{s^{\prime}}}$ as follows from the proof of Propositions 4.1 and 4.2 there. In particular with the notations of Mat6, Lemma 3.2] the $\operatorname{map} \phi_{\underline{s}, \underline{s^{\prime}}}\left(f_{\underline{s}, \underline{s^{\prime}}}\right)$ is meromorphic in the variable $\left(\underline{s}, \underline{s^{\prime}}\right)$ as follows from the first equality in the proof of the lemma. More precisely it follows from the assumption on $\lambda_{\underline{s}}$ and $\lambda_{\underline{s}^{\prime}}$, there exists a nonzero rational map $R\left(\underline{s}, \underline{s}^{\prime}\right)$ in the variables $q_{F}^{-s_{i}}$ and $q_{F}^{-s_{j}^{\prime}}$ such that $R\left(\underline{s}, \underline{s}^{\prime}\right) \lambda_{\underline{s}, \underline{s}^{\prime}}$ is holomorphic and nonzero, where we set

$$
\lambda_{\underline{s,}, \underline{s}^{\prime}}:=\lambda_{\underline{s}} \otimes \lambda_{\underline{s^{\prime}}}
$$

as in the lemma. In particular according to the aforementioned equality, we deduce

$$
R\left(\underline{s}, \underline{s}^{\prime}\right) \phi_{\underline{s}, \underline{s}^{\prime}}\left(f_{\underline{s}, \underline{s}^{\prime}}\right)
$$

is holomorphic for any choice of $f$. Later in the proof of the lemma, the $\operatorname{map} R\left(\underline{s}, \underline{s}^{\prime}\right) \Phi_{\underline{s}, \underline{s}^{\prime}}\left(f_{\underline{s}, \underline{s^{\prime}}}\right)$ will thus be holomorphic as well, by compactness of the quotient $P^{\prime} \backslash G^{\prime}$ there. Finally it will be nonzero for some choice of $f$ by the end of the proof of the same lemma.
Q.E.D.

We are now in position to prove the first part of the theorem of interest to us.

Proposition 3.17. Let $\tau$ be an irreducible representation with a Shalika model, and $\delta_{1} \times \cdots \times \delta_{l}$ be discrete series representations. Then the induced representation

$$
\pi_{\underline{s}}:=\delta_{1}\left[s_{1}\right] \times \cdots \times \delta_{l}\left[s_{l}\right] \times \tau \times \delta_{l}^{\vee}\left[-s_{l}\right] \times \cdots \times \delta_{1}^{\vee}\left[-s_{1}\right]
$$

admits a Shalika model for all values of $\underline{s} \in \mathbb{C}^{l}$.
Proof. By Propositions 3.13 and 3.16 , the representation

$$
\pi_{\underline{s}}^{\prime}:=\tau \times \delta_{1}\left[s_{1}\right] \times \delta_{1}^{\vee}\left[-s_{1}\right] \times \cdots \times \delta_{l}\left[s_{l}\right] \times \delta_{l}^{\vee}\left[-s_{l}\right]
$$

admits a nonzero meromorphic family of Shalika functionals on its space. It then follows from the meromorphy and generic invertibility of the standard intertwining operator from $\pi_{\underline{s}}$ to $\pi_{\underline{s}}^{\prime}$ that so does the representation $\pi_{\underline{s}}$. The result now follows from a usual leading term argument (see Section 4.2 for more details on such arguments).
Q.E.D.

Remark 3.18. It also follows from the proof above that the order of the representations in the product of the statement of the proposition could be permuted as one likes, but the same conclusion would still hold.

Remark 3.19. One could also only suppose the representations $\delta_{i}$ to be unitary in the above statement according to Remark 3.15.

As a corollary, we obtain the following theorem:
Theorem 3.20. A standard module of $G=G_{m}$ has a Shalika model if and only if it is $H_{m / 2, m / 2^{-d i s t i n g u i s h e d ~(s e e ~ T h e o r e m ~ 3.12) . ~}}^{\text {2 }}$.

Proof. The right to left implication follows from Theorem 3.12 and Proposition 3.17. The other direction follows from Corollary 3.3.
Q.E.D.

Finally we end with a word on $\chi$-distinction for standard modules. As observed in [Suz2, Proposition 3.4], the proof of Theorem 3.12 holds with no modification in the case of twisted linear periods when one adds a twisting character $\chi$ (the reason being that the so called modulus assumption stated in MOY] holds in this situation), and we refer to it for the statement of the classification of $\chi$-distinguished standard modules in this case. Here we give similar statements in the case of linear models for $H_{m / 2, m / 2}$ and $H_{(m+1) / 2,(m-1) / 2}$.

Theorem 3.21. Let $\pi, \delta_{i}$ and $\pi_{i}$ be as in the statement of Theorem 3.12, and $\alpha$ be a character of $F^{\times}$.
(a) If $m$ is even and $\pi$ is $H_{m / 2, m / 2}$-distinguished, then it is $\left(H_{m / 2, m / 2}, \xi_{\alpha}\right)$ distinguished, and if moreover $-\frac{d}{2} \leqslant \Re(\alpha) \leqslant \frac{d}{2}$ the converse implication holds.
(b) If $m$ is odd, $-\frac{d}{2} \leqslant \mathfrak{R}(\alpha)<\frac{d}{2}$ and $\pi$ is $\left(H_{(m+1) / 2,(m-1) / 2}, \xi_{\alpha}\right)$-distinguished, then there exists an index $i_{0}$ such that $\delta_{i_{0}}=\alpha \circ \nu$, and $\pi_{i_{0}}$ is $H=H_{(m-1) / 2,(m-1) / 2^{-}}$distinguished.

Proof. The first implication of (a) follows from Theorem 3.20 . The proof of the converse implications of (a) and (a) are completely similar to the proofs for trivial $\alpha$ given in Theorem 3.12, and we briefly explain the modifications to be done. In equation (3.1) the character $\mu_{i}$ must be replaced by $\mu_{i} \chi_{\alpha}$ with the same definition of $\mu_{i}$. Let us set

$$
\nu_{\alpha}:=\alpha \circ \nu_{D^{\times}} .
$$

The proofs of Cases (I) and (III) are the same as when $\alpha=1$.

In the proof of Case (II) when $m$ is even, the character $r\left(\Delta_{1}\right)$ must be replaced by $\nu_{\alpha}^{ \pm 1} \mu^{-1 / 2}=\nu_{\alpha}^{ \pm 1} \nu^{-d / 2}$ where the sign $\pm 1$ is positive if $m_{1,1}=m_{1,1}^{+}$ and negative if $m_{1,1}=m_{1,1}^{-}$. According to the case the argument when $\mathfrak{R}(\alpha)<d / 2$ or $-\mathfrak{R}(\alpha)<d / 2$ is the same. When $m_{1,1}=m_{1,1}^{+}$and $\mathfrak{R}(\alpha)=d / 2$ or $m_{1,1}=m_{1,1}^{-}$and $\mathfrak{R}(\alpha)=-d / 2$, then the argument is different. In this situation, because the central character is trivial and the standard module is right ordered, all $m_{i}$ 's are equal to 1 , and all $\mu_{u_{i}}$ must be unitary. The $m_{k}$ 's either come in pairs $\left(m_{i, j}, m_{j, i}\right)$ for $i<j$, and for the remaining $m_{u_{k}, u_{k}}$ from $k=1, \ldots, t$, the first case of Equation (3.2) can never happen, hence the sign of $\epsilon_{k}$ changes at each step. We deduce that $\pi$ is of the form

$$
\pi=\left(\left.\nu_{\alpha \mid}\right|^{-d / 2} \times \nu_{\alpha| |^{-d / 2}}^{-1}\right)^{t / 2} \times \prod_{j=1}^{(m-t) / 2}\left(\chi_{j} \times \chi_{j}^{-1}\right)
$$

for some unitary characters $\chi_{j}$ of $D^{\times}$when $m_{1,1}=m_{1,1}^{+}$, whereas it is of the form

$$
\pi=\left(\nu_{\alpha| |^{d / 2}} \times\left.\nu_{\alpha| |}^{-1}\right|^{d / 2}\right)^{t / 2} \times \prod_{j=1}^{(m-t) / 2}\left(\chi_{j} \times \chi_{j}^{-1}\right)
$$

when $m_{1,1}=m_{1,1}^{-}$.
We now discuss the proof of Case (II) when $m$ is odd . The character $r\left(\Delta_{1}\right)$ must be replaced by $\nu_{\alpha}$ when $m_{1,1}=m_{1,1}^{+}$and by $\nu_{\alpha^{-1}| |^{-d}}$ when $m_{1,1}=m_{1,1}^{-}$, but this second case cannot happen for right-ordering and central character reasons. When $m_{1,1}=m_{1,1}^{+}$and $\mathfrak{R}(\alpha)<d / 2$ the argument is the same as in the case of trivial $\alpha$.
Q.E.D.

We take the opportunity to extend [Mat3, Theorems 3.13] in the split case to the case of inner forms. When $m$ is even we obtain the same statement as in the case (a) of standard modules in Theorem 3.21 , as a special case of it. When $m$ is odd we obtain the following statement.

Theorem 3.22. Let $\pi, \delta_{i}$ and $\pi_{i}$ be as in in the statement of Theorem 3.12, and $\alpha$ be a character of $F^{\times}$with $-\frac{d}{2} \leqslant \mathfrak{R}(\alpha)<\frac{d}{2}$. Suppose moreover that $\pi$ is generic and that $m$ is odd. Then $\pi$ is $\left(H_{(m+1) / 2,(m-1) / 2}, \chi_{\alpha}\right)$-distinguished if and only if there exists a generic $H_{(m-1) / 2,(m-1) / 2}$-distinguished generic representation $\pi^{\prime}$ of $G_{m-1}$ such thate $\pi=\pi^{\prime} \times \nu_{\alpha}$.

Proof. By Theorem 3.21, if $\pi^{\prime}$ is $H_{(m-1) / 2,(m-1) / 2^{-} \text {-distinguished, it is also }}$ $\mu^{1 / 2} \chi_{\alpha}$-distinguished thanks to Theorem 3.21, hence $\pi=\pi^{\prime} \times \nu_{\alpha}$ is $\chi_{\alpha^{-}}$ distinguished by [Mat3, Proposition 3.8]. The converse implication follows from Theorem 3.21,
Q.E.D.

Remark 3.23. When $G$ is split, one can allow $\mathfrak{R}(\alpha)=\frac{1}{2}$ in the statement of Theorem 3.22 above because $\pi$ is $\chi$-distinguished if and only if $\pi^{\vee}$ is $\chi^{-1}$-distinguished by the realization of the contragredient of an irreducible representation using transpose inverse.

Together with Proposition 3.9 we obtain:
Corollary 3.24. Suppose that $\pi$ is an irreducible representation of $G$ with either a twisted linear period, or a linear period with respect to $H_{p, q}$ for $|p-q| \leqslant 1$. Then it is selfdual. If moreover $\pi$ is generic, and $\pi$ has a twisted linear period or a linear period with respect to $H_{m, m}$, then it is symplectic. In the latter case the converse also holds.

Proof. Only the first statement requires a proof, but it is an immediate consequence of Theorem 3.12 and Proposition 3.9, together with the Langlands quotient theorem.
Q.E.D.

## 4. Sign and parabolic induction

By convention $G_{0}$ is the trivial group, and the sign of any representation of $G_{0}$ is equal to 1 . We study the sign of parabolically induced representations, using closed and open intertwining periods. This natural strategy is not new as it was already used in [LM, Lemma 3.5] in the case of linear periods for split $G$, as was pointed to us by Lapid.

### 4.1. Sign and irreducible parabolic induction.

Proposition 4.1. If $\pi_{1}, \pi_{2}$ are distinguished representations of $G_{m_{1}}$ and $G_{m_{2}}$ for positive integers $m_{i}$, and $\pi_{1} \times \pi_{2}$ is irreducible, then

$$
\operatorname{sgn}\left(\pi_{1} \times \pi_{2}\right)=\operatorname{sgn}\left(\pi_{1}\right) \operatorname{sgn}\left(\pi_{2}\right) .
$$

Proof. We set $m=m_{1}+m_{2}$, and let $L_{i}$ a nonzero $H_{i}$-invariant linear form on $\pi_{i}$, where $H_{i}$ is the relevant subgroup of $G_{i}$, i.e. either $H_{m_{i} / 2, m_{i} / 2}$ if we consider linear periods, or the centralizer of $\delta$ in $G_{m_{i}}$ with square in the non-square elements of $G_{m_{i}}$ in the case of twisted linear periods. The $H$-invariant linear form on $\pi_{1} \times \pi_{2}$ is given by the closed intertwining period

$$
f \mapsto \int_{P_{m_{1}, m_{2}} \cap H \backslash H} L_{1} \otimes L_{2}(f(h)) d h .
$$

In both the non twisted and twisted case, we set $u_{i}$ as in Section 2.2 such that the normalizer $N_{i}$ of $H_{i}$ is equal to $H_{i} \sqcup u_{i} H_{i}$. In particular $u:=\operatorname{diag}\left(u_{1}, u_{2}\right)$ is such that $N=H \sqcup u H$, and we observe that $\delta_{P_{m_{1}, m_{2}}}(u)=1$. The result now follows from the change of variable

$$
\begin{gathered}
\int_{P_{m_{1}, m_{2}} \cap H \backslash H} L_{1} \otimes L_{2}\left(f\left(h u^{-1}\right)\right) d h=\int_{P_{m_{1}, m_{2}} \cap H \backslash H} L_{1} \otimes L_{2}\left(f\left(u^{-1} h\right)\right) d h \\
=\int_{P_{m_{1}, m_{2}} \cap H \backslash H} L_{1} \otimes L_{2}\left(\pi_{1}\left(u_{1}^{-1}\right) \otimes \pi_{2}\left(u_{2}^{-1}\right) f(h)\right) d h \\
=\chi_{\pi_{1}}\left(\overline{u_{1}}\right) \chi_{\pi_{2}}\left(\overline{u_{2}}\right) \int_{P_{m_{1}, m_{2}} \cap H \backslash H} L_{1} \otimes L_{2}(f(h)) d h,
\end{gathered}
$$

and the identity

$$
\epsilon\left(\pi_{1} \times \pi_{2}\right)=\epsilon\left(\pi_{1}\right) \epsilon\left(\pi_{2}\right) .
$$

Q.E.D.
4.2. Sign and open intertwining periods. Before stating the next results, we need to extend somehow the definition of the character $\chi_{\pi}$. So far we have only considered irreducible representations for which multiplicity one of $H$-invariant linear form is guaranteed. In particular it made sense to talk about $\chi_{\pi}$ rather than $\chi_{L}$ for $L$ the (possibly twisted) linear period on it. Here, though we are only interested in irreducible representations, we need to consider some which are not, though of finite length. On such a finite length representation $\pi$ of $G$ one might loose multiplicity one, so we prefer to talk about the character of $\frac{N}{H}$ attached to an $H$-invariant linear form on $\pi$ when it exists. Hence if $L \in \operatorname{Hom}_{H}(\pi, \mathbb{C})-\{0\}$ is such that there exists $z$ in $\mathbb{C}^{\times}$satisfying $L \circ \pi\left(u^{-1}\right)=z . L$, we define $\chi_{L}$ to be the character of $\frac{N}{H}$ such that

$$
\chi_{L}\left(\bar{u}^{-1}\right)=z .
$$

In particular $\chi_{L}=\chi_{\pi}$ if $\pi$ is irreducible.
Now we recall the Blanc-Delorme theory of open intertwining periods (see $[\mathrm{BD}])$. For $\pi$ a finite length representation of $G$ and $s \in \mathbb{C}$, we recall our notation

$$
\pi[s]:=\mu^{s} \pi
$$

Let $\pi_{0}, \pi_{1}, \ldots, \pi_{r}$ be finite length representations of $G_{m_{0}}, G_{m_{1}}, \ldots, G_{m_{r}}$ respectively and suppose that $\pi_{0}$ possesses an $H$-invariant linear form $\ell_{0}$. Then the representation

$$
\pi_{\underline{s}}:=\pi_{r}\left[s_{r}\right] \times \cdots \times \pi_{1}\left[s_{1}\right] \times \pi_{0} \times \pi_{1}^{\vee}\left[-s_{1}\right] \times \cdots \times \pi_{r}^{\vee}\left[-s_{r}\right],
$$

is distinguished for any $s_{i} \in \mathbb{C}$ and there is an explicit enough description of it which we now recall.

We first make specific choices of the element $\delta$, hence of the element $u$ and the group $H$, well adapted to describe situation in a simple manner, and which we will also use in the next section:

1) In the case of linear periods we take
and

$$
u=\operatorname{diag}\left(I_{m_{r}}, \ldots, I_{m_{1}}, I_{m_{0} / 2},-I_{m_{0} / 2},-I_{m_{1}}, \ldots,-I_{m_{r}}\right)
$$

2) (a) In the case of odd twisted linear periods we take
and

$$
u=\operatorname{diag}\left(I_{m_{r}}, \ldots, I_{m_{1}}, \iota I_{m_{0}},-I_{m_{1}}, \ldots,-I_{m_{r}}\right)
$$

(b) In the case of even twisted linear periods we take

and

$$
u=\operatorname{diag}\left(I_{m_{r}}, \ldots, I_{m_{1}}, I_{m_{0} / 2},-I_{m_{0} / 2},-I_{m_{1}}, \ldots,-I_{m_{r}}\right)
$$

Now put

$$
\begin{gathered}
G:=G_{m_{0}+2 \sum_{i=1}^{r} m_{i}}, \\
P:=P_{\left(m_{r}, \ldots, m_{1}, m_{0}, m_{1}, \ldots, m_{r}\right)} \\
M:=M_{\left(m_{r}, \ldots, m_{1}, m_{0}, m_{1}, \ldots, m_{r}\right)},
\end{gathered}
$$

and observe that with the choices above, the parabolic subgroup $P$ is $\theta$-split, in particular $P^{\theta}=M^{\theta}$.

We denote by $\ell$ the linear form on

$$
\pi_{r} \otimes \cdots \otimes \pi_{1} \otimes \pi_{0} \otimes \pi_{1}^{\vee} \otimes \cdots \otimes \pi_{r}^{\vee}
$$

defined by

$$
\ell\left(v_{r} \otimes \cdots \otimes v_{1} \otimes v_{0} \otimes v_{1}^{\vee} \otimes \cdots \otimes v_{r}^{\vee}\right)=\ell_{0}\left(v_{0}\right) \prod_{i=1}^{r} v_{i}^{\vee}\left(v_{i}\right) .
$$

Then if $f_{\underline{s}}$ is a holomorphic section of $\pi_{\underline{s}}$, the integral

$$
I_{\underline{s}}\left(f_{\underline{s}}\right):=\int_{P^{\theta} \backslash H} \ell\left(f_{\underline{s}}(h)\right) d h
$$

converges for $s_{i+1}-s_{i}$ of real part large enough, and extends meromorphically. For a generic choice of nonzero vector $\underline{a}=\left(a_{1}, \ldots, a_{r}\right) \in \mathbb{C}^{r}$ and an integer $l$ such that for any $f:=f_{0} \in \pi:=\pi_{0}$, the limit

$$
L_{\underline{a}}(f):=\lim _{s \rightarrow 0} s^{l} I_{s \underline{a}}\left(f_{s \underline{a}}\right)
$$

is well defined and nonzero for some $f$. In short we will say that $L_{\underline{a}}$ is a leading term of $I_{\underline{s}}$ at $s=0$.

Proposition 4.2. Let $\pi_{0}$ be a finite length representation of $G_{m_{0}}$, where $m_{0}$ is a non negative integer, which admits a nonzero $H$-invariant linear
form $\ell_{0}$ which admits a sign $\chi_{\ell_{0}}$. Let $\pi_{i}$ be a finite length representation of $G_{m_{i}}$, for $m_{i} \geqslant 1$, with a central character for $i=1, \ldots, r$. Then the open intertwining period $L$ on

$$
\pi:=\pi_{r} \times \cdots \times \pi_{1} \times \pi_{0} \times \pi_{1}^{\vee} \times \cdots \times \pi_{r}^{\vee}
$$

associated to $\ell_{0}$ as above has a sign character $\chi_{L}$, moreover

$$
\frac{\chi_{L}}{\prod_{i=1}^{r} \omega_{\pi_{i}}(-1)}=\chi_{\ell_{0}}
$$

In particular when $\pi$ is irreducible:

$$
\operatorname{sgn}(\pi)=\operatorname{sgn}\left(\pi_{0}\right)
$$

Proof. By induction, up to the replacements

$$
\pi_{0}:=\pi_{r-1} \times \cdots \times \pi_{1} \times \pi_{0} \times \pi_{1}^{\vee} \times \cdots \times \pi_{r-1}^{\vee}
$$

and

$$
\ell_{0}\left(v_{0}\right):=\ell_{0}\left(v_{0}\right) \prod_{i=1}^{r-1} v_{i}^{\vee}\left(v_{i}\right)
$$

we are reduced to the $r=1$ case, which we assume for the rest of this proof.
Let $f_{s}$ be a holomorphic section of

$$
\pi_{s}:=\pi_{1}[s] \times \pi_{0} \times \pi_{1}^{\vee}[-s]
$$

The $H$-invariant linear form in this case is given by the leading term at $s=0$ of the following open intertwining period:

$$
I_{s}\left(f_{s}\right)=\int_{P^{\theta} \backslash H} \ell\left(f_{s}(h)\right) d h
$$

Now we observe that

$$
\ell\left(\pi_{1}\left(I_{m_{1}}\right) v \otimes \pi_{1}^{\vee}\left(\theta\left(-I_{m_{1}}\right)\right) v^{\vee}\right)=\omega_{\pi_{1}}(-1) \ell\left(v \otimes v^{\vee}\right)
$$

Hence by a change of variable in the integral defining $I_{s}$ for $\mathfrak{R}(s)$ large enough we obtain:

$$
\chi_{I_{s}}=\omega_{\pi_{1}}(-1) \chi_{\ell_{0}} .
$$

The first part of statement then follows by a meromorphic continuity argument. For the second part, if $\pi$ is irreducible then so is $\pi_{1}$. Hence

$$
\epsilon\left(\pi_{1} \times \pi_{1}^{\vee}\right)=\omega_{\pi_{1}}(-1)
$$

Q.E.D.
4.3. Sign of discrete series. In this section it turns out that for linear periods, the Steinberg representation $\operatorname{St}_{m}\left(\mathbf{1}_{D^{\times}}\right)$, where $m$ has to be even, exhibits an exceptional behaviour. First its root number is not given by the same formula as the other generalized Steinberg representations with a
linear period, and moreover its $H$-invariant linear form does not come from the open intertwining period on the induced representation lying above it. Hence we first prove our general result for discrete series representations in the special case of linear periods for the Steinberg representation.

Proposition 4.3. Let $m$ be an even integer. Then the Steinberg representation $\pi:=\operatorname{St}_{m}\left(\mathbf{1}_{D^{\times}}\right)$has no twisted linear period. On the other hand it has a linear period, and moreover its sign for linear period is

$$
\operatorname{sgn}(\pi)=1
$$

Proof. The first assertion follows from [Cho Theorem 0.1]. We prove the second in two steps. We start with $m=2$. Then the $H$-invariant linear form $\ell_{0}$ on $\pi$ comes from the descent of the difference of the two closed intertwining periods on the induced representation $\pi_{2}\left(\mathbf{1}_{D^{\times}}\right)$. Namely

$$
\ell_{0}(f)=f\left(I_{2}\right)-f(u)
$$

where

$$
u:=\left(\begin{array}{ll} 
& 1 \\
1 &
\end{array}\right) .
$$

Indeed this linear form clearly vanishes on the spherical vector. Hence $\chi_{\mathrm{St}_{2}\left(\mathbf{1}_{D^{\times}}\right)}(\bar{u})=-1$ as $u^{2}=I_{2}$. On the other hand $\epsilon\left(\operatorname{St}_{2}\left(\mathbf{1}_{D^{\times}}\right)\right)=-1$ and the result for $m=2$ follows.

So we suppose that $m \geqslant 4$. Now for $k$ between 1 and $m-1$, set

$$
\tau_{k}:=\mathbf{1}_{D^{\times}} \times^{\prime} \cdots \times^{\prime} \mathbf{1}_{D^{\times}} \times^{\prime} \underbrace{\mathbf{1}_{G_{2}}}_{\text {position } k} \times^{\prime} \mathbf{1}_{D^{\times}} \times^{\prime} \cdots \times^{\prime} \mathbf{1}_{D^{\times}}
$$

and

$$
\sigma:=\mathbf{1}_{D^{\times}} \times^{\prime} \cdots \times^{\prime} \mathbf{1}_{D^{\times}} \times^{\prime} \underbrace{\operatorname{St}_{2}\left(\mathbf{1}_{D^{\times}}\right)}_{\text {middle }} \times^{\prime} \mathbf{1}_{D^{\times}} \times^{\prime} \cdots \times^{\prime} \mathbf{1}_{D^{\times}}
$$

where $x^{\prime}$ stands for non-normalized parabolic induction. We denote by $\overline{\tau_{k}}$ the image of $\tau_{k}$ in the quotient $\sigma$, and we recall that

$$
\operatorname{St}_{m}\left(\mathbf{1}_{D^{\times}}\right) \simeq \sigma / \sum_{k \neq m / 2} \overline{\tau_{k}} .
$$

Arguing as in Mat7, Lemma 10.12], one checks that no $\tau_{k}$ for $k \neq m / 2$ is distinguished. By multiplicity one on $\mathrm{St}_{m}\left(\mathbf{1}_{D^{\times}}\right)$, this proves on the one hand that $\sigma$ has a unique $H$-invariant linear form, and on the other hand that the $H$-invariant linear form on $\operatorname{St}_{m}\left(\mathbf{1}_{D^{\times}}\right)$is its descent. Now the $H$-invariant linear form $L$ on $\sigma_{k}$ is a leading term of the open period attached to $\ell_{0}$ as
in the proof of Proposition 4.2, and it follows from this proposition that

$$
\chi_{L}=\chi_{\ell_{0}}
$$

Hence

$$
\chi_{\operatorname{St}_{m}\left(\mathbf{1}_{D^{\times}}\right)}=\chi_{\operatorname{St}_{2}\left(\mathbf{1}_{D^{\times}}\right)}
$$

On the other hand, by the formula for the root number of the Steinberg representation, we have

$$
\epsilon\left(\operatorname{St}_{m}\left(\mathbf{1}_{D^{\times}}\right)\right)=\epsilon\left(\operatorname{St}_{2}\left(\mathbf{1}_{D^{\times}}\right)\right)
$$

and the result follows.
Q.E.D.

The sign of cuspidal representations is computed by a global method in Theorem 5.7. Its proof uses the results above and a globalization argument. Propositions 4.1, 4.2 and Theorem 4.3 allow to reduce the proof for discrete series to the cuspidal case, and we explain this now.

Theorem 4.4. Let $\pi$ be a distinguished discrete series representation of $\mathrm{GL}_{m}(D)$. Then

$$
\operatorname{sgn}(\pi)=(-1)^{m}
$$

Proof. Thanks to Proposition 4.3, we can assume that $\pi=\operatorname{St}_{t}(\rho)$ for $\rho \neq$ $\mathbf{1}_{D^{\times}}$. In particular if we write $\mathrm{JL}(\rho)=\operatorname{St}_{r}\left(\rho^{\prime}\right)$, we have

$$
\epsilon(\pi)=\epsilon\left(\rho^{\prime}\right)^{t r}=\epsilon(\rho)^{t}
$$

On the other hand the $H$-invariant linear form on $\pi_{t}(\rho)$ is the descent of the unique open period on $\pi_{t}(\rho)$ according to Proposition 3.11 (in fact this open intertwining period has to be holomorphic at the parameter giving $\pi_{t}(\rho)$ by a reasoning similar to those done in Mat7]). By Proposition 4.2, when $t$ is odd we have

$$
\pi=\operatorname{St}_{t}(\rho) \Rightarrow \chi_{\mathrm{St}_{t}(\rho)}=\epsilon(\rho)^{(t-1)} \chi_{\rho}
$$

and the result follows from Theorem5.7. When $t$ is even, fix any non-trivial character $\psi$ of $F$. By Proposition 4.2 we obtain

$$
\chi_{\mathrm{St}_{t}(\rho)}(\bar{u})=\omega_{\rho}(-1)^{t / 2}=[\epsilon(\rho, \psi) \epsilon(\rho, \psi)]^{t / 2}=\epsilon(\rho, \psi)^{t}=\epsilon\left(S t_{t}(\rho)\right)
$$

and the sign is 1 as expected in this case.
Q.E.D.

## 5. The sign of cuspidal Representations

5.1. The sign of distinguished generic unramified representations for split $G$. Here we assume that $G=\operatorname{GL}_{n}(F)$ is $F$-split (hence $n$ is even) and we set $K:=\operatorname{GL}_{n}\left(O_{F}\right)$ where $O_{F}$ is the ring of integers of $F$. In the case of linear models, the result below follows from [Mat6, Theorem 5.1] and
[FJ, Proposition 3.2] and its proof, hence we only prove it for twisted linear periods.

Proposition 5.1. Suppose that $\pi$ is generic unramified and distinguished representation of $G$. Let $v_{0}$ be a $K$-spherical vector in $\pi$, and $\Lambda \in \operatorname{Hom}_{H}(\pi, \mathbb{C})-$ $\{0\}$. Then $\Lambda\left(v_{0}\right) \neq 0$, hence $\operatorname{sgn}(\pi)=1$.

Proof. By Theorem 3.12, and because $\pi$ is irreducible, the representation can be written under the form $\pi=\chi_{1} \times \cdots \times \chi_{2 k}$ with $n=2 k$ for unramified characters $\chi_{i}$ of $F^{\times}$such that $\chi_{2(i+1)}=\chi_{2 i+1}^{-1}$. In this situation each representation $\pi_{i}: \chi_{2 i+1} \times \chi_{2(i+1)}$ of $\mathrm{GL}_{2}(F)$ is $E^{\times}$-distinguished with distinguishing linear form $L_{i}$. Then by the proof of Proposition 4.1, setting

$$
L:=\otimes_{i=1}^{k} L_{i}: \pi_{1} \otimes \cdots \otimes \pi_{k} \rightarrow \mathbb{C}
$$

the $H$-invariant linear form on $\pi=\prod_{i=1}^{k} \pi_{i}$ is given by

$$
\Lambda: f \mapsto \int_{B_{H} \backslash H} L(f(h)) d h \sim \int_{K_{H}} L(f(h)) d h .
$$

Here the symbol " $\sim$ " means equality up to a positive constant and $B_{H}$ is the intersection of the upper triangular Borel of $G$ with $H$, i.e. the upper triangular Borel of $H$, and $K_{H}$ is $\mathrm{GL}_{k}\left(O_{E}\right)$.

The (unique up to scalar in $\mathbb{C}^{\times}$) spherical vector $f_{0} \in \pi$ is given by

$$
f_{0}(p k)=\delta_{P_{(2, \cdots, 2)}}(p)^{1 / 2} v_{1} \otimes \cdots \otimes v_{k}
$$

where each $v_{i}$ is the spherical vector of $\pi_{i}$. Hence up to appropriate normalizations:

$$
\Lambda\left(f_{0}\right)=\prod_{i} L_{i}\left(v_{i}\right) .
$$

This reduces our problem to the $n=2$ case, and we write $\pi=\chi \times \chi^{-1}$. Now we consider separately the case where $E / F$ is unramified and the case where $E / F$ is ramified.

If $E / F$ is unramified, then one can take $\delta \in U_{E}$, so that with our choice of embedding $U_{E} \leqslant \mathrm{GL}_{2}\left(O_{F}\right)$ and the well defined $E^{\times}$-invariant linear form

$$
\Lambda: f \mapsto \int_{F^{\star} \backslash E^{\times}} f(h) d h
$$

on $\pi$ is nonzero on the spherical vector $f_{0}$ as

$$
\Lambda\left(f_{0}\right)=\int_{U_{F} \backslash U_{E}} f_{0}(h) d h=\operatorname{vol}\left(U_{F} \backslash U_{E}\right)>0 .
$$

If $E / F$ is ramified, then $\delta$ can be chosen as the uniformizer $\left(\begin{array}{ll}0 & \kappa \\ 1 & 0\end{array}\right)$ of $E$, and

$$
E^{\times}=F^{\times} U_{E} \sqcup F^{\times}\left(\begin{array}{ll}
0 & \kappa \\
1 & 0
\end{array}\right) U_{E} .
$$

Because the Iwasawa decomposition of $\left(\begin{array}{cc}0 & \kappa \\ 1 & 0\end{array}\right)$ is $\left(\begin{array}{cc}\kappa & 0 \\ 0 & 1\end{array}\right)\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$, we obtain
$\Lambda\left(f_{0}\right)=\int_{U_{F} \backslash U_{E}} f_{0}(h) d h+\int_{U_{F} \backslash U_{E}} f_{0}\left(\left(\begin{array}{cc}\kappa & 0 \\ 0 & 1\end{array}\right) h\right) d h=\left(1+q^{1 / 2} \chi(\kappa)\right) \operatorname{vol}\left(U_{F} \backslash U_{E}\right)$.
This proves the expected result, except when the unramified character $\chi$ satisfies $\chi(\kappa)=-q^{-1 / 2}$, which implies $\chi=\nu_{F} \chi^{-1}$. This would imply that $\pi$ is reducible, hence this case does not occur.

Finally, we conclude that $\operatorname{sgn}(\pi)=1$ because on one hand we just proved that $\chi_{\pi}$ is trivial, but on the other hand $\epsilon(\pi)=1$. Q.E.D.
5.2. The sign of linear periods: generic unitary representations for split $G$. Here $G$ is $F$-split (i.e. $m=n$ ) and $H=L_{n / 2, n / 2}$. Moreover $F$ is allowed to be Archimedean as well. Because of a later globalization process, we are at the moment only interested in local components of cuspidal automorphic representations with cuspidal Jacquet-Langlands transfer which admit a global twisted linear period. However such representations are known to admit a global Shalika model as we shall recall later. Hence all their local components admit a local Shalika model.

So in this paragraph, we suppose that our local representation $\pi$ of $G$ admits a Shalika model. By JR this implies that $\pi$ has a linear model, and we will recall the argument in the proof below when $\pi$ is generic unitary. In fact, when $F$ is $p$-adic, it is known by [Mat6] or Gan that a generic irreducible representation $\pi$ of $G$ has a linear model if and only if it has a Shalika model. This could certainly be proved in the Archimedean setting as well due to the advanced stage of the literature in this setting, but we don't need this result here.

In any case, under this assumption, the computation of the sign of local linear periods immediately follows from the results in [FJ].

Proposition 5.2. Let $\pi$ be an irreducible generic unitary (of CasselmanWallach type when $F$ is Archimedean) representation of $G$ with a Shalika model. Then it has a linear period and $\operatorname{sgn}(\pi)=1$.

Proof. Our result is just a re-interpretation of a result of Friedberg and Jacquet. They prove in [FJ, Proposition 3.1] that if $S$ is in the Shalika model of $\pi$, the integral

$$
\Psi(s, S)=\int_{G_{n}} S\left(\operatorname{diag}\left(g, I_{n}\right)\right)|\operatorname{det}(g)|_{F}^{s-1 / 2} d g
$$

initially defined for $s$ of real part large enough in fact extends to a meromorphic function on $\mathbb{C}$, which is a holomorphic multiple of $L(s, \pi)$, and is equal to it for some choice of $S$. They moreover prove in [FJ, Proposition 3.6] the functional equation

$$
\Psi(1-s, \widetilde{S})=\gamma(s, \pi) \Psi(s, S)
$$

where

$$
\tilde{S}(g)=S\left(u g^{-t}\right) .
$$

Under our assumptions $L(1 / 2, \pi)$ is finite (and of course nonzero) so $\Psi(1 / 2, S)$ makes sense for any $S$ and because $\pi$ is selfdual:

$$
\Psi(1 / 2, \widetilde{S})=\epsilon(1 / 2, \pi) \Psi(1 / 2, S)
$$

Now observe that

$$
L: S \mapsto \Psi(1 / 2, S)
$$

is $H$-invariant, and that it is moreover nonzero as $L(1 / 2, \pi) \neq 0$. Finally up to easy changes of variable, the left hand side of the above equation exactly expresses the action of the normalizer $N$ on the linear form $L$, and our claim follows.
Q.E.D.
5.3. The global sign of twisted linear periods. Here we fix a number field $k$ with its adele ring $\mathbb{A}$, and denote by $\mathcal{P}(k)$ the set of places of $k$. We fix a central division $k$-algebra $D$, put $D_{\mathbb{A}}=D \otimes_{k} \mathbb{A}$ and set

$$
G=G_{m}\left(D_{\mathbb{A}}\right)
$$

We denote by $l$ a quadratic extension of $k$. We suppose that there exists a $k$-embedding of $l \in \mathcal{M}_{m}(D)$, which we fix. We denote by $\mathcal{H}$ the $l$-algebra centralizing $l$ in $\mathcal{M}_{m}(D)$ and set

$$
H:=\left(\mathcal{H} \otimes_{k} \mathbb{A}\right)^{\times} \leqslant G .
$$

The groups $G$ and $H$ are $\mathbb{A}$-points of algebraic groups $\mathbf{G}$ and $\mathbf{H}$ defined over $k$ by definition. We put $H_{k}:=\mathbf{H}(k)$ and $G_{k}:=\mathbf{G}(k)$. By the analysis done

38 U.K. ANANDAVARDHANAN, H. LU, N. MATRINGE, V. SÉCHERRE, AND C. YANG
in Section 2.2, there exists an element $u \in G_{k}$ such that

$$
N_{G_{k}}\left(H_{k}\right)=H_{k} \sqcup u . H_{k} .
$$

In this paragraph we prove that the global sign of certain distinguished cuspidal automorphic representations of $G$ is equal to one. We recall that a smooth cuspidal automorphic representation $\Pi$ of $G$ (see BPCZ, Section $2.7]$ ) is called distinguished if the convergent period integral

$$
\phi \mapsto \int_{H_{k} \backslash H} \phi(h) d h
$$

does not vanish on $\Pi$. The following lemma follows from a simple change of variable in the period integral.

Lemma 5.3. If $\Pi$ is a cuspidal automorphic representation of $G$, then the $H$-period on $\Pi$ is fixed by $u$.

In order to compute the global sign, we also need to compute the root number of a distinguished cuspidal automorphic representation in the following special case.

From now on, let JL denote the global Jacquet-Langlands correspondence defined in Bad2 and BR.

Lemma 5.4. Let $\Pi$ be a distinguished cuspidal automorphic representation of $G$ with cuspidal Jacquet-Langlands transfer. We suppose that $l / k$ splits at every Archimedean place of $k$, and that there are two places $v_{1}, v_{2} \in \mathcal{P}(k)$ such that $\Pi_{v_{1}}$ is cuspidal and $\mathrm{JL}_{v_{2}}\left(\Pi_{v_{2}}\right)=\operatorname{St}_{n}\left(\mathbf{1}_{v_{2}}\right)$. Then $\mathrm{JL}(\Pi)$ has a Shalika period hence it is selfdual, and moreover the central value of the Godement-Jacquet L-function of $\mathrm{JL}(\Pi)$ is nonzero:

$$
L(1 / 2, \mathrm{JL}(\Pi)) \neq 0 .
$$

In particular its Godement-Jacquet root number is trivial:

$$
\epsilon(\mathrm{JL}(\Pi))=1 .
$$

Proof. Our local assumptions imply that the assumptions of XZ, Theorem 1.4] are satisfied, hence $\mathrm{JL}(\Pi)$ has a Shalika period so it is selfdual. Then noting that standard $L$ functions of cuspidal automorphic representations are entire by [GJ, it follows again from [XZ, Theorem 1.4] that $L(1 / 2, \mathrm{JL}(\Pi)) \neq 0$. Finally the functional equation of the standard $L$ function of $\mathrm{JL}(\Pi)$ (see [GJ]) together with the fact hat $\mathrm{JL}(\Pi)$ is selfdual imply that $\epsilon(\mathrm{JL}(\Pi))=1$.
Q.E.D.

We obtain the sign of global linear periods as a corollary.

Corollary 5.5. Let $\Pi$ be a distinguished cuspidal automorphic representation of $G$ as in Lemma 5.4. Then

$$
\prod_{v \in \mathcal{P}(k)} \operatorname{sgn}\left(\Pi_{v}\right)=1
$$

Proof. First the product is well defined thanks to Proposition 5.1. Moreover the period integral is factorizable thanks to Theorem 2.5 and Theorem A.2 for Archimedean linear periods. The result now follows from, Lemmas 5.3 and 5.4 .
Q.E.D.
5.4. Application to the sign of cuspidal representations. The determination of the sign in this case follows again from a globalization argument, and the obvious computation of the sign of the trivial representation. For this latter part, we recall that the local root number of $\mathbf{1}_{D^{\times}}$is equal to -1 if $d$ is even and 1 if $d$ is odd. This immediately implies:

Lemma 5.6. Suppose that $d$ is even. Then one has $\operatorname{sgn}\left(\mathbf{1}_{D \times}\right)=-1$.
Here we compute the sign of linear periods, using Lemma 5.4 and a globalization argument of Prasad and Schulze-Pillot.

Theorem 5.7. Let $\pi$ be a distinguished cuspidal representation of $\mathrm{GL}_{m}(D)$. Then

$$
\operatorname{sgn}(\pi)=(-1)^{m}
$$

Proof. Let $\frac{x}{d}+\mathbb{Z}$ be the Hasse invariant of $\mathcal{M}_{m}(D)$ where we take $x \in[1, d]$ and coprime to $d$, and define $a$ by the equality $\frac{a}{n}=\frac{x}{d}$.

We start with the case of twisted linear periods. By Krasner's lemma and the weak approximation lemma, one can find a number field $k$ and a quadratic extension $l / k$ such that $l_{v} / k_{v}=E / F$ for some finite place $v$ of $k, l / k$ remains unsplit at every place of a set $S_{0}$ of finite places of $k$ of size $\left|S_{0}\right|=a$ which does not contain $v$, but $l / k$ splits at every Archimedean place of $k$ (see for instance [AKM ${ }^{+}$, Section 9.6] for the details). Then by the Brauer-Hasse-Noether theorem, one can also find a division algebra $\mathcal{D}$ with center $k$ such that $\mathcal{D}_{v}=\mathcal{M}_{m}(D), \mathcal{D}_{w}$ is a central division algebra $\mathfrak{D}_{w}$ over $k_{w}$ of Hasse invariant $\frac{-1}{n}+\mathbb{Z}$ for each $w$ in $S_{0}$, and which is split at every other place of $k$. In this situation we observe that $l$ automatically embeds as a $k$-subalgebra of $\mathcal{D}$ thanks to [SYY, Theorem 1.1]. Finally fix some finite $v_{1}$ outside $\{v\} \cup S_{0} \cup S_{\infty}$ such that $l_{v_{1}} / k_{v_{1}}$ is split. Putting

$$
S:=\{v\} \cup\left\{v_{1}\right\} \cup S_{0} \cup S_{\infty}
$$

with $S_{\infty}$ the set of Archimedean places of $k$, by [PSP, Theorem 4.1], there exists a cuspidal automorphic representation $\Pi$ of $\left(\mathcal{D} \otimes_{k} \mathbb{A}\right)^{\times}$which is unramified outside $S$, such that $\Pi_{v}=\pi$, and $\Pi_{w}=\mathbf{1}_{\mathfrak{D}_{w}^{\times}}$for $w$ in $S_{0}$, and $\Pi_{v_{1}}$ is cuspidal (and distinguished). Note that $\mathrm{JL}_{w}\left(\pi_{w}\right)=\mathrm{St}_{n}\left(\mathbf{1}_{w}\right)$ for all $w \in S$ hence Lemma 5.4 applies. Note also that $\mathrm{JL}_{v_{1}}\left(\Pi_{v_{1}}\right)=\Pi_{v_{1}}$ because $\mathcal{D}_{v_{1}}$ is split. Now the combination of Proposition 5.1, Proposition 5.2, Lemma 5.6 and Corollary 5.5, we obtain the formula:

$$
\operatorname{sgn}\left(\Pi_{v}\right)=(-1)^{a}
$$

The result now follows from the fact that if $m$ is even, then $a=m x$ as well, whereas if $m$ is odd then $d$ must be even, $a=m x$ is also odd because $x$ and $d$ are coprime.

For linear periods, we use again Krasner's lemma and the weak approximation lemma, but this time to find a quadratic extension $l / k$ such that $l_{v} / k_{v} \simeq(F \times F) / F$ for some finite place $v$ of $k, l / k$ remains unsplit at every place of a set $S_{0}$ of finite places of $k$ of size $\left|S_{0}\right|=a$ which does not contain $v$, and again $l / k$ is split at every Archimedean place. The rest of the proof is the same (we can however observe that in this situation $m$ is automatically even, so [SYY, Theorem 1.1] applies again).
Q.E.D.

Remark 5.8. When $G$ is split (and probably also when $G$ is split over a quadratic extension), one can give a local proof of the above result, extending that of Prasad for inner forms of $\mathrm{GL}_{2}$, and which applies directly to the class of $H$-cuspidal representations of $G$ (see [KT]). By [KT] a cuspidal representation of $G$ is automatically $H$-cuspidal. We very briefly sketch the idea, and mention that the proof uses the realization of the contragredient of an irreducible representation using transpose inverse, hence the restrictive hypothesis on $G$. One just needs to modify an argument from the thesis Ok of Youngbin Ok based on the Poisson summation formula, which itself extends to $\mathrm{GL}_{n}$ the quick proof by Deligne [Del] of a result of Fröhlich and Queyrut (FQ for $\mathrm{GL}_{1}$. Ok uses the the Poisson summation formula together with an extension of the Godement-Jacquet functional equation for relative matrix coefficients to prove the triviality of local root numbers of cuspidal representations distinguished by a Galois involution. The method of Ok in fact proves the result for relatively cuspidal representations as has been observed by Offen in [Off1]. In fact [Off1, Sections 2.2 and 2.3] apply with minor modifications to our setting, using the same results from [GJ]. The only difference is that in the Poisson summation formula [Off1, Lemma 2.2], the orthogonal of the Lie algebra $\mathcal{H}$ of $H$ is u. $\mathcal{H}$ instead of $\mathcal{H}$ itself. Following
the proof of Off1, Proposition 2.1], this will claim that the gamma factor of an $H$-cuspidal $\pi$ at $s=1 / 2$ is equal to $(-1)^{m(d-1)} \chi_{\pi}(u)=(-1)^{m} \chi_{\pi}(u)$, but this gamma factor is equal to the root number as $\pi$ is self dual and because $L(1 / 2, \pi) \neq 0$ because $\pi$ is unitary.

Remark 5.9. Using Remark 5.8, one can remove the assumption that $l_{v_{1}} / k_{v_{1}}$ is split in the proof of Theorem 5.7 .

Remark 5.10. When $F$ is a finite field, the sign problem makes sense as well. Multiplicity at most one is known for linear periods, at least on cuspidal representations ([Sé2]), and can certainly be checked for twisted linear periods following Gu0. In this setting the computation alluded to in 5.8 works equally well and relates the sign to the Godement-Jacquet gamma factors defined in [Rod], with the extra complication that $G$ has not total measure in its Lie algebra anymore. Hence one has to use cuspidality to prove vanishing of some terms, in order to use the Poisson summation formula within the Godement-Jacquet functional equation. The details will appear elsewhere, and will be consistent with our result for level zero cuspidal representations.

## 6. The sign of generic representations

As an immediate corollary of Theorem 3.12, Proposition 4.1. Proposition 4.2, and Theorem 4.4 we achieve the goal of this paper:

Theorem 6.1. Let $\pi$ be a distinguished generic representation of $\mathrm{GL}_{m}(D)$. Then

$$
\operatorname{sgn}(\pi)=(-1)^{m}
$$

## Appendix A. MUltiplicity one for linear periods of inner forms

In this appendix $F$ is allowed to be any local field of characteristic 0 . We recall that $D$ is a central $F$-division. Let $p, q$ and $m$ be positive integers such that $p+q=m$. We prove here that the symmetric pair

$$
\left(G:=\operatorname{GL}_{m}(D), H:=\operatorname{GL}_{p}(D) \times \operatorname{GL}_{q}(D)\right)
$$

is a Gelfand pair when $|p-q| \leqslant 1$, i.e., that if $\pi$ is an irreducible representation of $\mathrm{GL}_{m}(D)$ (of Casselman-Wallach type when $F$ is Archimedean), then

$$
\operatorname{dim}\left(\operatorname{Hom}_{H}(\pi, \mathbb{C})\right) \leqslant 1
$$

When $D=F$ is non Archimedean, this result is due to Jacquet and Rallis ( JR$]$ ). Later Aizenbud and Gourevitch gave another proof, inspired by that
of Jacquet and Rallis, which applies to all local fields of characteristic zero in [AG, Theorem 8.2.4]. This proof was extended when $p=q$ to unitary representations of inner forms by Chong Zhang, who obtained the following statement as a key step in the process (see Zha, Proposition 4.3 and Remark 4.7]), but we claim that this step is valid for any $p$ and $q$ :

Proposition A.1. Let $\pi$ be an irreducible representation of $G$. Then

$$
\operatorname{dim}\left(\operatorname{Hom}_{H}(\pi, \mathbb{C})\right) \operatorname{dim}\left(\operatorname{Hom}_{H}\left(\pi^{\vee}, \mathbb{C}\right)\right) \leqslant 1
$$

Proof. Following the proof of [Zha, Proposition 4.1] and using the terminology introduced in BM , one sees that the descendants of the pair $(G, H)$ can only be of diagonal type, of Galois type or of the form

$$
\left(R_{L / F} \mathrm{GL}_{a+b}\left(D^{\prime}\right), R_{L / F} \mathrm{GL}_{a}\left(D^{\prime}\right) \times R_{L / F} \mathrm{GL}_{b}\left(D^{\prime}\right)\right)
$$

for $D^{\prime}$ an $L$-division algebra, where $L$ is a finite extension of $F$ and $R_{L / F}$ is Weil restriction of scalars. This is enough to conclude according to the discussion before [Zha, Proposition 4.3] and the references to AG] given there.
Q.E.D.

In order to prove that $(G, H)$ is a Gelfand pair when $|p-q| \leqslant 1$, it is enough to prove that $H$-distinguished irreducible representations of $G$ are selfdual. This has been proved in Corollary 3.24 in the $p$-adic case, and follows from a verbatim adaptation of the proof of [BM, Theorem 6.7] in the Archimedean case. Hence we obtain:

Theorem A.2. Let $\pi$ be an irreducible representation of $G$, and suppose that $|p-q| \leqslant 1$. Then

$$
\operatorname{dim}\left(\operatorname{Hom}_{H}(\pi, \mathbb{C})\right) \leqslant 1
$$

When $\pi$ is unitary, we can remove the assumption on $p$ and $q$ by the observation in the proof of [Zha, Corollary 4.4].

## Appendix B. Bernstein-Zelevinsky theory for the mirabolic SUbGROUP OF INNER FORMS AND LINEAR PERIODS

Here we only make observations without proofs, about some results which remain valid from the mirabolic restriction theory of Bernstein-Zelevinsky ([BZ1] and [BZ2]) for inner forms of the generalized linear group. We define the mirabolic subgroup $P_{m}$ of $G_{m}:=\mathrm{GL}_{m}(D)$ as the group of matrices in $G_{m}$ with bottom row equal to $(0, \ldots, 0,1)$. We denote by $U_{m}$ its unipotent
radical, so that the mirabolic subgroup is the semi-direct product $P_{m}=$ $G_{m-1} \cdot U_{m}$, where we identify $G_{m-1}$ with a subgroup of $G_{m}$ by the embedding

$$
g \mapsto \operatorname{diag}(g, 1)
$$

The group $G_{m-1}$ thus acts naturally on the group $\widehat{U_{m}}$ of characters of $U_{m}$ by

$$
(g, \psi) \mapsto \psi\left(g^{-1} \cdot g\right),
$$

with exactly two orbits: the trivial character and the others. Moreover all non degenerate characters (see $[\mathrm{BH}]$ ) of maximal unipotent upper triangular subgroup of $G_{m}$ are conjugate under the action of the diagonal matrices in $G_{m}$. This makes some part of the machinery of Bernstein-Zelevinsky work the same and we simply observe that the following results hold with exactly the same proofs. First the functors $\Phi^{ \pm}$and $\Psi^{ \pm}$(and $\hat{\Phi}^{+}$defined as in [BZ2, Section 3], satisfy the statements of [BZ2, Propositions 3.2 and 3.4] with the same proofs (relying on [BZ1, Section 5.11]). Following the authors of [BZ2], for $\tau$ a smooth representation of $P_{m}$ and $k$ between 1 and $n$ we put

$$
\tau^{(k)}=\Psi^{-}\left(\Phi^{-}\right)^{k-1}(\tau)
$$

and

$$
\tau^{(0)}=\tau
$$

Then the Bernstein-Zelevinsky filtration still exists for smooth representations of $P_{m}$ :
Theorem B.1. [BZ2, Corollary 3.4] Let $\tau$ be a smooth representation of $P_{m}$ with $m \geqslant 2$. Then $\tau$ admits a filtration

$$
\{0\} \subseteq \tau_{m} \subseteq \cdots \subseteq \tau_{1}=\tau
$$

with

$$
\tau_{k} / \tau_{k+1} \simeq\left(\Phi^{+}\right)^{k-1} \Psi^{+}\left(\tau^{(k)}\right)
$$

Following again Bernstein and Zelevinsky, we define the derivatives of a representation of $G_{m}$ to be that of its restriction to $G_{m}$. Now let $\pi$ be a cuspidal representation of $G_{m}$. Because of the filtration above, and because $\pi^{(k)}=\{0\}$ for all $k=1, \ldots, m-1$ (this is by the definition of the derivatives, which factor through some Jacquet module), we obtain the following corollary of the Bernstein-Zelevinsky filtration above:

Corollary B.2. Let $\pi$ be a cuspidal representation of $G_{m}$. Then $\pi=$ $\left(\Phi^{+}\right)^{m-1}\left(\Psi^{+}\right)\left(\pi^{(n)}\right)$ as a representation of $P_{m}$, where $\pi^{(n)}$ is a finite dimensional D-vector space, the dimension of which is that of the generalized Whittaker model of $\pi$ in the sense of BH .

Proof. The only assertion remaining is that on the dimension of $\pi^{(n)}$. By definition its dual is the space of generalized Whittaker functionals on $\pi$, and it is finite dimensional by (MW].
Q.E.D.

In turn, this is sufficient for the following result concerning cuspidal representations of $G_{m}$ to hold with the same proof as in Mat1] (see also [CM] for the version with the twisting character).

Proposition B.3. Let $p$ and $q$ be two non negative integers such that $p+q=$ $m \geqslant 2$. Let $\mu$ be a character of $H_{p, q}$ and $\pi$ be a cuspidal representation of $G_{m}$. If $\pi$ is $\left(H_{p, q}, \mu\right)$-distinguished, then $p=q$.

We extend this result to discrete series representations in Section 3.3 . Here we treat separately the case of Steinberg representations, as its proof is different and uses the theory of derivatives.

More precisely, a careful analysis of [BZ2] and [Zel] shows that the full theory developed in these papers hold for representations with cuspidal support consisting of characters of $G_{1}=D^{\times}$. Let us explain the crucial point. The result of [BZ2, Theorem 4.2] for a representation $\pi:=\chi_{1} \times \cdots \times \chi_{r}$ induced by characters of $D^{\times}$becomes in our setting that $\pi$ is irreducible as soon as $\chi_{i} \chi_{j}^{-1} \neq \mu$ for any $i \neq j$. But following [BZ2], with notations and terminology identical to that of [BZ2, Section 4.8], the natural 1-pairing between $\pi$ and $\tilde{\pi}$ induces a $\mu$-pairing between $\pi$ and $\bar{\pi}:=\mu \pi$, which induces by restriction to the mirabolic subgroup $P_{r}$ a $\Delta$-pairing between $\pi_{\mid P_{r}}$ and $\bar{\pi}_{\mid P_{r}}$. This latter assertion becomes false if one replaces $\mu$ by $\mu^{s}$ for any other real number $s$, but for $s=1$ it implies that the proof of $[\overline{\mathrm{BZ} 2}$, Lemma 4.7] remains valid when the cuspidal representations $\rho_{i}$ there are replaced here by characters $\chi_{i}$ of $D^{\times}$. The claim one has to check in turn is that this lemma is the key to have a complete theory of derivatives. In particular the following proposition holds, as one can check from its proof in [Zel, Proposition 9.5] and the formula for Jacquet modules of generalized Steinberg representations in [Tad, Proposition 3.1].

Proposition B.4. Let $m \geqslant 1$ and $k$ be between 0 and $n$, and let $\chi$ be $a$ character of $G_{1}$. Then

$$
\operatorname{St}_{n}(\chi)^{(k)}=\nu^{k / 2} \operatorname{St}_{n-k}(\chi)
$$

The the following follows immediately as in Mat2, Theorem 3.1]
Proposition B.5. Suppose that the representation $\operatorname{St}_{n}(\chi)$ above is $\left(H_{p, q}, \mu\right)$ distinguished for some character $\mu$ of $H_{p, q}$. Then $p=q$.

## Appendix C. A multiplicity zero Result for generic REPRESENTATIONS

Here $F$ is $p$-adic.
Lemma C.1. Let $p \geqslant q$ be two non negative integers, $m_{0}$ and $m_{1}$ be two positive integers such that $m_{0}+m_{1}=p+q:=m$. Set $G=G_{m}$, and set $H=H_{p, q}$. Let $\pi_{0}$ be a generic representation of $G_{m_{0}}$ with cuspidal support containing no character of $G_{1}$, let $\pi_{1}$ a generic representation of $G_{m_{1}}$ with cuspidal support consisting of characters of $G_{1}$ only, and let $\chi$ be a character of $H$. In such a situation, if $\pi_{0} \times \pi_{1}$ is $\chi$-distinguished, then $m_{0}$ is even, $q \geqslant m_{0} / 2$, and there exist characters $\chi_{0}$ of $H_{m_{0} / 2, m_{0} / 2}$ and $\chi_{1}$ of $H_{p-m_{0} / 2, q-m_{0} / 2}$ such that $\pi_{0}$ is $\chi_{0}$-distinguished and $\pi_{1}$ is $\chi_{1}$-distinguished.

Proof. Once again this follows from the geometric lemma as in Off2. Set $P=M U$ be the standard parabolic subgroup of $G$ of type ( $m_{0}, m_{1}$ ). Reasonning as in the beginning of the proof of Theorem 3.12, and by our assumption, no duality relation can occur between the subsegments of the product $\pi_{0}$ and those of the product $\pi_{1}$. This already implies that the double cosets $P x H$ possibly contributing to distinction are such that the representative $x$ belongs to the standard Levi subgroup $M$. Moreover if one writes $x=\operatorname{diag}\left(x_{0}, x_{1}\right)$, the element $x_{0}$ can only have even $m_{i, i}=2 m_{i, i}^{+}$in its associated subpartiation by Theorem [3.8. The result now follows from the second equality in the proof of [Off2, Proposition 3.1]. Q.E.D.

As a consequence of the discussion in Appendix B about the theory of derivatives for representations the cuspidal support of which consist of characters of $G_{1}$, we also obtain the following multiplicity zero result. Its proof is the same as that of the erratum to [Mat2, Theorem 3.2], which is available on the author's webpage but not published. We thus reproduce the argument here, and recall that if $\Delta$ is a cuspidal segment, then $r(\Delta)$ denotes the cuspidal representation forming its right end.

Lemma C.2. Let $\Delta_{1}, \ldots, \Delta_{r}$ be cuspidal segments with supports consisting of characters of $G_{1}$, and suppose that they are right anti-ordered, i.e. $e\left(r\left(\Delta_{i+1}\right)\right) \geqslant e\left(r\left(\Delta_{i}\right)\right)$ for $i=1, \ldots, r-1$. Then the representation $\pi=$ $\delta\left(\Delta_{1}\right) \times \cdots \times \delta\left(\Delta_{r}\right)$ of $G$ cannot by $H_{p, q}$ distinguished if $|p-q| \geqslant 2$.

Proof. We observe that for any non negative integer $n_{i}$, the segments

$$
\Delta_{1}^{\left(n_{1}\right)}, \ldots, \Delta_{r}^{\left(n_{1}\right)}
$$

are still right anti-ordered. Then one replaces the class of representations to which the induction is applied in the proof of Mat2, Theorem 3.2] (defined by a non preceding condition which is not preserved by taking derivatives of each segments as claimed in the proof of [Mat2, Theorem 3.2]) by that of products as in the statement of the lemma. Then the proof of Mat2, Theorem 3.2] applies without any modification, except the initial steps of the induction which are for $n=2$ and 3 . For these cases we see that for $G_{2}$ and $G_{3}$, the only product of discrete series representations having a character as a quotient are respectively of the form $\xi \times \xi \nu^{-1}$ and $\xi \times \xi \nu^{-1} \times \xi \nu^{-2}$, and the corresponding segments are visibly not right anti-ordered. Q.E.D.

As an immediate consequence of Lemmas C.1 and C.2, we obtain:
Theorem C.3. Let $\pi$ be a generic representation of $G, H=H_{p, q}$ with $|p-q| \geqslant 2$, and $\chi$ be a character of $H$. Then $\pi$ cannot be $\chi$-distinguished.

## References

[AG] A. Aizenbud and D. Gourevitch. Generalized Harish-Chandra descent, Gelfand pairs, and an Archimedean analog of Jacquet-Rallis's theorem. Duke Math. J., 149(3), 2009.
[AGJ] Avraham Aizenbud, Dmitry Gourevitch, and Hervé Jacquet. Uniqueness of Shalika functionals: the Archimedean case. Pacific J. Math., 243(2):201-212, 2009.
$\left[\mathrm{AKM}^{+}\right]$U. K. Anandavardhanan, R. Kurinczuk, N. Matringe, V. Sécherre, and S. Stevens. Galois self-dual cuspidal types and Asai local factors. J. Eur. Math. Soc. (JEMS), 23(9):3129-3191, 2021.
[Bad1] Alexandru Ioan Badulescu. Jacquet-Langlands et unitarisabilité. J. Inst. Math. Jussieu, 6(3):349-379, 2007.
[Bad2] Alexandru Ioan Badulescu. Global jacquet-langlands correspondence, multiplicity one and classification of automorphic representations. Inventiones mathematicae, 172:383-438, 2008.
[BD] Philippe Blanc and Patrick Delorme. Vecteurs distributions H-invariants de représentations induites, pour un espace symétrique réductif $p$-adique $\mathrm{G} / \mathrm{H}$. In Annales de l'institut Fourier, volume 58, pages 213-261, 2008.
[BH] Colin J. Bushnell and Guy Henniart. Generalized Whittaker models and the Bernstein center. Amer. J. Math., 125(3):513-547, 2003.
[BM] P. Broussous and N. Matringe. Multiplicity one for pairs of Prasad-TaklooBighash type. Int. Math. Res. Not. IMRN, 2021(21):16423-16447, 2021.
[BPCZ] Raphaël Beuzart-Plessis, Pierre-Henri Chaudouard, and Michał Zydor. The global Gan-Gross-Prasad conjecture for unitary groups: the endoscopic case. Publ. Math. Inst. Hautes Études Sci., 135:183-336, 2022.
[BPW] Raphaël Beuzart-Plessis and Chen Wan. A local trace formula for the generalized Shalika model. Duke Math. J., 168(7):1303-1385, 2019.
[BR] A. I. Badulescu and D. Renard. Unitary dual of GL( $n$ ) at Archimedean places and global Jacquet-Langlands correspondence. Compos. Math., 146(5):11151164, 2010.
[BZ1] J. Bernstein and A. V. Zelevinskii. Representations of the group GL(n,F) where F is a non-archimedean local field. Uspekhi Matematicheskikh Nauk, 31(3):5-70, 1976.
[BZ2] J. Bernstein and A. V. Zelevinsky. Induced representations of reductive $\mathfrak{p}$-adic groups. I. In Annales scientifiques de l'École normale supérieure, volume 10, pages 441-472, 1977.
[CCL] Li Cai, Yihua Chen, and Yu Liu. Heegner points on modular curves. Trans. Amer. Math. Soc., 370(5):3721-3743, 2018.
[Cho] Marion Chommaux. Distinction of the Steinberg representation and a conjecture of Prasad and Takloo-Bighash. J. Number Theory, 202:200-219, 2019.
[CM] James W. Cogdell and Nadir Matringe. The functional equation of the JacquetShalika integral representation of the local exterior-square $L$-function. Math. Res. Lett., 22(3):697-717, 2015.
[Del] Pierre Deligne. Les constantes locales de l'équation fonctionnelle de la fonction $L$ d'Artin d'une représentation orthogonale. Invent. Math., 35:299-316, 1976.
[DKV] Pierre Deligne, David Kazhdan, and Marie-France Vignéras. Représentations des algèbres centrales simples p-adiques. Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 33:117, 1984.
[Duh] N. Duhamel. Formule de plancherel sur $\mathrm{GL}_{\mathrm{n}} \times \mathrm{GL}_{\mathrm{n}} \backslash \mathrm{GL}_{2 \mathrm{n}} . \operatorname{arXiv:1912.08497,}$ 2019.
[FJ] Solomon Friedberg and Hervé Jacquet. Linear periods. J. reine angew. Math, 443(91):139, 1993.
[FQ] A. Fröhlich and J. Queyrut. On the functional equation of the Artin $L$-function for characters of real representations. Invent. Math., 20:125-138, 1973.
[Gan] Wee Teck Gan. Periods and theta correspondence. In Representations of reductive groups, volume 101 of Proc. Sympos. Pure Math., pages 113-132. Amer. Math. Soc., Providence, RI, 2019.
[GJ] R. Godement and H. Jacquet. Zeta functions of simple algebras, volume 260. Springer, 2006.
[GR] Benedict H. Gross and Mark Reeder. Arithmetic invariants of discrete Langlands parameters. Duke Math. J., 154(3):431-508, 2010.
[Guo] Jiandong Guo. Uniqueness of generalized Waldspurger model for GL(2n). Pacific J. Math., 180(2):273-289, 1997.
[Hen] Guy Henniart. Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique. Inventiones mathematicae, 139(2):439-455, 2000.
[HT] Michael Harris and Richard Taylor. The Geometry and Cohomology of Some Simple Shimura Varieties.(AM-151), volume 151. Princeton university press, 2001.
[Jac1] Hervé Jacquet. Principal L-functions of the linear group. In Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ.,

Corvallis, Ore., 1977), Part 2, volume XXXIII of Proc. Sympos. Pure Math., pages 63-86. Amer. Math. Soc., Providence, RI, 1979.
[Jac2] Hervé Jacquet. Archimedean Rankin-Selberg integrals. In Automorphic forms and L-functions II. Local aspects, volume 489 of Contemp. Math., pages 57-172. Amer. Math. Soc., Providence, RI, 2009.
[JR] Hervé Jacquet and Stephen Rallis. Uniqueness of linear periods. Compositio Mathematica, 102(1):65-123, 1996.
[JS] H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic representations. I. Amer. J. Math., 103(3):499-558, 1981.
[Kna] A. W. Knapp. Local Langlands correspondence: the Archimedean case. In Motives (Seattle, WA, 1991), volume 55, Part 2 of Proc. Sympos. Pure Math., pages 393-410. Amer. Math. Soc., Providence, RI, 1994.
[KT] Shin-ichi Kato and Keiji Takano. Subrepresentation theorem for $p$-adic symmetric spaces. Int. Math. Res. Not. IMRN, (11):Art. ID rnn028, 40, 2008.
[Lan] R. P. Langlands. On the classification of irreducible representations of real algebraic groups. In Representation theory and harmonic analysis on semisimple Lie groups, volume 31 of Math. Surveys Monogr., pages 101-170. Amer. Math. Soc., Providence, RI, 1989.
[Lip] Huang Liping. The minimal polynomial of a matrix over a central simple algebra. Linear and Multilinear Algebra, 45(2-3):99-107, 1998.
[LM] Erez Lapid and Zhengyu Mao. Whittaker-Fourier coefficients of cusp forms on $\widetilde{\mathrm{Sp}_{n}}$ : reduction to a local statement. Amer. J. Math., 139(1):1-55, 2017.
[Mat1] Nadir Matringe. Cuspidal representations of GL (n, F) distinguished by a maximal levi subgroup, with F a non-archimedean local field. Comptes Rendus Mathematiques, 350(17-18):797-800, 2012.
[Mat2] Nadir Matringe. Linear and Shalika local periods for the mirabolic group, and some consequences. Journal of Number Theory, 138:1-19, 2014.
[Mat3] Nadir Matringe. On the local Bump-Friedberg L-function. Journal für die reine und angewandte Mathematik (Crelles Journal), 2015(709):119-170, 2015.
[Mat4] Nadir Matringe. Distinction of the Steinberg representation for inner forms of GL(n). Mathematische Zeitschrift, 287:881-895, 2017.
[Mat5] Nadir Matringe. On the local Bump-Friedberg L-function II. Manuscripta Math., 152(1-2):223-240, 2017.
[Mat6] Nadir Matringe. Shalika periods and parabolic induction for $G L(n)$ over a nonarchimedean local field. Bull. Lond. Math. Soc., 49(3):417-427, 2017.
[Mat7] Nadir Matringe. Gamma factors of intertwining periods and distinction for inner forms of GL (n). Journal of Functional Analysis, 281(10):109223, 2021.
[MO] Nadir Matringe and Omer Offen. Intertwining periods and distinction for padic galois symmetric pairs. Proceedings of the London Mathematical Society, 125(5):1179-1252, 2022.
[MOY] Nadir Matringe, Omer Offen, and Chang Yang. On local intertwining periods. J. Funct. Anal., 286(4):Paper No. 110293, 2024.
[MW] C. Mœglin and J.-L. Waldspurger. Modèles de Whittaker dégénérés pour des groupes p-adiques. Math. Z., 196(3):427-452, 1987.
[Off1] Omer Offen. On local root numbers and distinction. J. Reine Angew. Math., 652:165-205, 2011.
[Off2] Omer Offen. On parabolic induction associated with a $p$-adic symmetric space. J. Number Theory, 170:211-227, 2017.
[Ok] Youngbin Ok. Distinction and gamma factors at 1/2: Supercuspidal case. ProQuest LLC, Ann Arbor, MI, 1997. Thesis (Ph.D.)-Columbia University.
$[\mathrm{PR}] \quad$ Dipendra Prasad and A. Raghuram. Kirillov theory for $\mathrm{GL}_{2}(D)$ where $D$ is a division algebra over a non-Archimedean local field. Duke Math. J., 104(1):19-44, 2000.
[Pra] Dipendra Prasad. Some applications of seesaw duality to branching laws. Math. Ann., 304(1):1-20, 1996.
[PSP] Dipendra Prasad and Rainer Schulze-Pillot. Generalised form of a conjecture of Jacquet and a local consequence. J. Reine Angew. Math., 616:219-236, 2008.
[Rod] E.-A. Roditty. On gamma factors and bessel functions for representations of general linear groups over finite fields. Master's thesis , Tel Aviv University, 2010.
[ST] Miyu Suzuki and Hiroyoshi Tamori. Epsilon Dichotomy for Linear Models: The Archimedean Case. Int. Math. Res. Not. IMRN, (20):17853-17891, 2023.
[Suz1] Miyu Suzuki. Classification of standard modules with linear periods. J. Number Theory, 218:302-310, 2021.
[Suz2] Miyu Suzuki. A reformulation of the conjecture of Prasad and Takloo-Bighash. arXiv:2312.16393, 2023.
[SV] Yiannis Sakellaridis and Akshay Venkatesh. Periods and harmonic analysis on spherical varieties. Astérisque, (396):viii+360, 2017.
[SX] Miyu Suzuki and Hang Xue. Linear intertwining periods and epsilon dichotomy for linear models. to appear in Math. Ann., 2020.
[SYY] Sheng-Chi Shih, Tse-Chung Yang, and Chia-Fu Yu. Embeddings of fields into simple algebras over global fields. Asian J. Math., 18(2):365-386, 2014.
[Sé1] V. Sécherre. Représentations cuspidales de gl(r,d) distinguées par une involution intérieure. preprint, Ann. Scient. Éc. Norm. Sup. arxiv.org/abs/2005.05615, 2016.
[Sé2] Vincent Sécherre. Supercuspidal representations of $\mathrm{GL}_{n}(\mathrm{~F})$ distinguished by a Galois involution. Algebra Number Theory, 13(7):1677-1733, 2019.
[Tad] Marko Tadic. Induced representations of GL(n, A) for p-adic division algebras A. J. reine angew. Math, 405:48-77, 1990.
[Tat] J. Tate. Number theoretic background. In Automorphic forms, representations and L-functions, Part 2, Proc. Sympos. Pure Math., XXXIII, pages 3-26. Amer. Math. Soc., Providence, R.I., 1979.
[Xue] Hang Xue. Epsilon dichotomy for linear models. Algebra Number Theory, 15(1):173-215, 2021.
[XZ] H. Xue and W. Zhang. Twisted linear periods and a new relative trace formula. preprint, to appear in Peking Math J. arXiv:2209.08366, 2016.
[Yan] Chang Yang. Linear periods for unitary representations. Math. Z., 302(4):22532284, 2022.
[Zel] A. V. Zelevinsky. Induced representations of reductive p-adic groups. II. on irreducible representations of GL( $n$ ). Annales Scientifiques de l'École Normale Supérieure, 13(2):165-210, 1980.
[Zha] Chong Zhang. On linear periods. Math. Z., 279(1-2):61-84, 2015.
U.K. Anandavardhanan. Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai, 400076, India

Email address: anand@math.iitb.ac.in

Hengfei Lu. School of Mathematical Sciences, Beihang University, 9 Nanshan Street, Shahe Higher Education Park, Changping, Beijing, 102206, China

Email address: luhengfei@buaa.edu.cn
Nadir Matringe. Institute of Mathematical Sciences, NYU Shanghai, 3663 Zhongshan Road North Shanghai, 200062, China and Institut de Mathématiques de Jussieu-Paris Rive Gauche, Université Paris Cité, 75205, Paris, France

Email address: nrm6864@nyu.edu and matringe@img-prg.fr
Vincent Sécherre. Laboratoire de Mathématiques de Versailles, UVSQ, CNRS, Université Paris-Saclay, 78035, Versailles, France

Email address: vincent.secherre@uvsq.fr
Chang Yang. Key Laboratory of High Performance Computing and Stochastic Information Processing (HPC- SIP), School of Mathematics and Statistics, Hunan Normal University, Changsha, 410081, China

Email address: cyang@hunnu.edu.cn

