The Cubic Klein-Gordon Equation with Damping (Master 2 Lecture Notes)

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The objective of this course is to give a qualitative description of the asymptotic behavior in large time of all the global solutions of the one-dimensional focusing cubic Klein-Gordon equation with damping

$$\begin{cases} \partial_t^2 u + 2\alpha \partial_t u - \partial_x^2 u + u - u^3 = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

Here, $\alpha \in (0,1)$ is a fixed damping constant and $(u_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ is the initial data. We start by a study of the local and global Cauchy problem. Then, we introduce the key notion of solitary waves for this equation, and we study their stability properties. By variational techniques, it is then proved that in large time, any global solution converges strongly, at least for a subsequence, to the zero function or to a sum of decoupled solitary waves. Lastly, we describe a more detailed convergence result, for the whole sequence of time, with a characterization of all the possible asymptotic configurations and a precise convergence rate.

These lecture notes contain no new material and are entirely inspired by the references [1, 3, 4, 7, 9, 13, 14, 16].

1 The local Cauchy problem

1.1 The linear problem

A solution u of (1) will be seen as a solution of the first order system

$$\begin{cases} \partial_t u = v \\ \partial_t v = -2\alpha v + \partial_x^2 u - u + u^3 = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{NLKG}$$

and we will use the notation $\vec{u} = (u, \partial_t u) = (u, v)$. We define the energy of $\vec{u}$ by

$$E(\vec{u}) = \int_{\mathbb{R}} \left(\tfrac{1}{2} v^2 + \tfrac{1}{2} (\partial_x u)^2 + \tfrac{1}{2} u^2 - \tfrac{1}{4} u^4 \right) \mathrm{d}x. \tag{2}$$

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We check by integration by parts that it holds formally

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\vec{u}) = -2\alpha\|v\|^2$$

and thus, for $0 \le t_1 < t_2$,

$$E(\vec{u}(t_2)) - E(\vec{u}(t_1)) = -2\alpha \int_{t_1}^{t_2} \|v\|^2 \,\mathrm{d}t. \tag{3}$$

Since $\alpha > 0$, we obtain that the energy is nonincreasing for any solution for which (3) can be justified. This important qualitative property leads us to work for finite energy solutions, that is solutions such that $\vec{u}(t) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, for which the quantity E is well-defined. The space $H^1(\mathbb{R}) \times L^2(\mathbb{R})$, denoted simply by $H^1 \times L^2$ or by X, will be called the energy space. We also denote $Y = L^2 \times H^{-1}$.

The notation $\int$ will be used for $\int_{\mathbb{R}} \mathrm{d}x$. We denote $\langle \cdot, \cdot \rangle$ the L^2 scalar product for real-valued functions u_i or vector-valued functions $\vec{u}_i = (u_i, v_i)$ $(i = 1, 2)$

$$\|u\| := \|u\|, \quad \langle u_1, u_2 \rangle := \int u_1 u_2, \quad \langle \vec{u}_1, \vec{u}_2 \rangle := \int u_1 u_2 + \int v_1 v_2,$$

and we denote

$$\|\vec{u}\|_X := \sqrt{\|u\|_{H^1}^2 + \|v\|^2}, \quad \langle \vec{u}_1, \vec{u}_2 \rangle_X := \int (\partial_x u_1)(\partial_x u_2) + \int u_1 u_2 + \int v_1 v_2,$$

$$\|\vec{u}\|_Y := \sqrt{\|u\|^2 + \|v\|_{H^{-1}}^2}.$$

Lemma 1.1 ([4, Chapter 9.5])**.** *The linear problem*

$$\begin{cases} \partial_t u = v \\ \partial_t v = \partial_x^2 u - u - 2\alpha v, \end{cases} \tag{4}$$

generates a strongly continuous semigroup of contractions $(S_\alpha(t))_{t\ge 0}$ in X satisfying, for some $C_\alpha \ge 1$, $\gamma > 0$, for all $t \ge 0$,

$$\|S_\alpha(t)\|_{\mathcal{L}(X)} \le C_\alpha e^{-\gamma t}, \tag{5}$$

Moreover, $(S_\alpha(t))_{t\ge 0}$ extends to a strongly continuous semigroup of contraction in Y satisfying, for some $C'_\alpha \ge 1$, $\gamma' > 0$, for all $t \ge 0$,

$$\|S_\alpha(t)\|_{\mathcal{L}(Y)} \le C'_\alpha e^{-\gamma' t}.$$

Proof. We define the operator A_α on $H^1 \times L^2$ by

$$\begin{cases} D(A_\alpha) = H^2 \times H^1, \\ A_\alpha \vec{u} = (v, u'' - u - 2\alpha v), \text{ for any } \vec{u} \in D(A_\alpha). \end{cases}$$

We claim that the operator A_α is maximally dissipative in the sense that

- A_α is dissipative: for all $\vec{u} \in D(A_\alpha)$ and all $\lambda > 0$, $\|\vec{u} - \lambda A_\alpha \vec{u}\|_X \geq \|\vec{u}\|_X$,
- for all $\lambda > 0$ and all $\vec{f} \in X$, there exists $\vec{u} \in D(A_\alpha)$ such that $\vec{u} - \lambda A_\alpha \vec{u} = \vec{f}$.

Indeed, we have

$$\begin{aligned}\langle A_\alpha \vec{u}, \vec{w}\rangle_X &= \int v'w' + vw + (u'' - u - 2\alpha v)z \\ &= \int (v'w' + vw - u'z' - uz) - 2\alpha \int vz.\end{aligned}$$

In particular, $\langle A_\alpha \vec{u}, \vec{u}\rangle_X = -2\alpha \int v^2 \leq 0$. Moreover, for an operator on a Hilbert space, the property $\langle A_\alpha \vec{u}, \vec{u}\rangle_X \leq 0$ is known to be equivalent to the fact that A_α is dissipative. Then, we prove the surjectivity. It is enough to prove the surjectivity for $\lambda = 1$. Let $\vec{f} \in X$. We solve

$$\begin{cases} u - v = f \\ v - u'' + u + 2\alpha v = g \end{cases} \iff \begin{cases} -u'' + 2(1+\alpha)u = g + (1+2\alpha)f \\ v = u - f \end{cases}$$

Using the Fourier transform, or the convolution product, or the Lax-Milgram theorem, it is easy to find $u \in H^2$, solution of $-u'' + 2(1+\alpha)u = g + (1+2\alpha)f$. Then, we set $v = u - f \in H^1$. Moreover, it is clear that the domain $D(A_\alpha)$ is dense in X. Therefore, by the Hille-Yosida-Phillips theorem, A_α generates a semigroup of contraction $(S_\alpha(t))_{t\geq 0}$ on X.

For a solution of (4), we set

$$N(t) = \int (v^2 + (\partial_x u)^2 + u^2 + 2\alpha uv)$$

and we compute $\frac{d}{dt}N = -2\alpha N$. Thus, $N(t) = N(0)e^{-2\alpha t}$ and since

$$(1-\alpha)\int (v^2 + (\partial_x u)^2 + u^2) \leq N(t) \leq (1+\alpha)\int (v^2 + (\partial_x u)^2 + u^2)$$

we obtain the result for the bound in $\mathcal{L}(X)$.

The theory in $Y = L^2 \times H^{-1}$ is done similarly. □

Remark 1.2 (The User Guide)**.** For $\vec{u}_0 \in D(A_\alpha)$, the function $\vec{u}(t) = S_\alpha(t)\vec{u}_0$ is the unique solution of the linear problem

$$\begin{cases} \vec{u} \in C([0,+\infty), D(A_\alpha)) \cap C^1([0,+\infty), X) \\ \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = A_\alpha \vec{u} \\ \vec{u}(0) = \vec{u}_0 \end{cases}$$

For $\vec{u}_0 \in X$, the function $\vec{u}(t) = S_\alpha(t)\vec{u}_0$ is unique solution of the linear problem

$$\begin{cases} \vec{u} \in C([0,+\infty), X) \cap C^1([0,+\infty), Y) \\ \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = A_\alpha \vec{u} \\ \vec{u}(0) = \vec{u}_0 \end{cases}$$

1.2 The nonlinear problem

The standard theory of semilinear evolution equations (see for instance [4, Chapter 4.3] or [19]) yields the following result.

Proposition 1.3. *For any initial data $\vec{u}_0 \in X$, there exists a unique maximal solution*

$$\vec{u} = (u, \partial_t u) \in C([0, T_{\max}), X) \cap C^1([0, T_{\max}), Y)$$

of (NLKG) *satisfying $\vec{u}(0) = \vec{u}_0$.*

If the maximal time of existence $T_{\max}$ is finite, then $\lim_{t \uparrow T_{\max}} \|\vec{u}(t)\|_X = \infty$.

If $\vec{u}_0 \in D(A_\alpha)$, then

$$\vec{u} = (u, \partial_t u) \in C([0, T_{\max}), D(A_\alpha)) \cap C^1([0, T_{\max}), X).$$

Moreover, the map $T_{\max} : \vec{u}_0 \in X \mapsto (0, \infty)$ is lower semicontinuous, and if $\lim_{n\to\infty} \vec{u}_{0,n} = \vec{u}_0$ in X then, for any $0 < T < T_{\max}$,

$$\lim_{n\to\infty} \vec{u}_n = \vec{u} \quad \text{in } C([0, T], X),$$

where $\vec{u}_n$ is the solution of (1) *corresponding to $\vec{u}_{0,n}$.*

Proof. Observe that the map $u \mapsto u^3$ is Lipschitz continuous from bounded sets of H^1 to L^2. Indeed, in dimension one, one has $\sup_{\mathbb{R}} |u| \le C\|u\|_{H^1}$ and thus

$$|u^3 - v^3| \le C(|u|^2 + |v|^2)|u - v| \quad \text{so that} \quad \|u^3 - v^3\| \le C(\|u\|_{H^1}^2 + \|v\|_{H^1}^2)\|u - v\|.$$

Let $\bar{B}_M$ denote the closed ball of X of center 0 and radius $M > 0$. It follows that there exists $C_L > 0$ such that for all $M > 0$ and for all $u, v \in \bar{B}_M$ it holds

$$\|u^3 - v^3\| \le C_L M^2 \|u - v\|. \tag{6}$$

We rewrite (NLKG) under the following equivalent Duhamel formulation

$$\vec{u}(t) = S_\alpha \vec{u}_0 + \int_0^t S_\alpha(t - s)(0, u^3(s))\, \mathrm{d}s. \tag{7}$$

Uniqueness. Let $T > 0$. Then there exists at most one solution of (7) on $[0, T]$. Indeed, let $\vec{u}_1, \vec{u}_2$ be two solutions of (7) with the same initial data. Set

$$M = \sup_{t\in[0,T]} \max\{\|\vec{u}_1(t)\|_X; \|\vec{u}_2(t)\|_X\}.$$

We have by (7) and $\|S(t)\|_{\mathcal{L}(X)} \le C$,

$$\|\vec{u}_1(t) - \vec{u}_2(t)\|_X \le C \int_0^t \|u_1^3(s) - u_2^3(s)\|\, \mathrm{d}s \le CM^2 \int_0^t \|u_1(s) - u_2(s)\|\, \mathrm{d}s.$$

It follows from the Gronwall lemma that $\|\vec{u}_1(t) - \vec{u}_2(t)\|_X = 0$, for all $t \in [0, T]$. □

Existence of a local solution by contraction. Let $M > 0$ and fix

$$T_M = \frac{1}{2CM^2} > 0. \tag{8}$$

We claim that for any $\vec{u}_0 \in X$ such that $\|\vec{u}_0\|_X \leq M/2$, there exists a solution $\vec{u}$ of (7) on $[0, T]$. Define

$$E = \{\vec{u} \in \mathcal{C}([0, T_M], X) : \|\vec{u}(t)\|_X \leq M, \text{ for all } t \in [0, T_M]\}.$$

We equip E with the distance generated by norm of $\mathcal{C}([0, T_M], X)$, *i.e.*, for any $\vec{u}_1, \vec{u}_2 \in E$,

$$d(u, v) = \sup_{t \in [0, T_M]} \|\vec{u}_1(t) - \vec{u}_2(t)\|_X.$$

Since $\mathcal{C}([0, T_M], X)$ is a Banach space and E is closed in $\mathcal{C}([0, T_M], X)$, (E, d) is a complete metric space. For all $\vec{u} \in E$, we define $\Phi(\vec{u}) \in \mathcal{C}([0, T_M], X)$ by

$$\Phi(\vec{u})(t) = S(t)\vec{u}_0 + \int_0^t S(t - s) u^3(s) \, \mathrm{d}s,$$

for all $t \in [0, T_M]$.

First, we prove that $\Phi : E \to E$. Indeed, for any $s \in [0, T_M]$, by (6)

$$\|u^3\| \leq CM^2 \|u\| \leq CM^3,$$

It follows from $\|S_\alpha(t)\| \leq C$ and the definition of T_M in (8) that for any $t \in [0, T_M]$,

$$\|\Phi(\vec{u})(t)\|_X \leq \|\vec{u}_0\|_X + \int_0^t \|u^3(s)\| \, \mathrm{d}s \leq M + C_L T_M M^3 \leq \frac{3}{2} M.$$

Second we prove that Φ is a contraction on (E, d). Indeed, for any $\vec{u}, \vec{v} \in E$, and for any $t \in [0, T_M]$,

$$\|\Phi(\vec{u})(t) - \Phi(\vec{v})(t)\| \leq \int_0^t \|u^3(s) - v^3(s)\| \, \mathrm{d}s \leq C_L T_M M^2 d(u, v) \leq \frac{1}{2} d(u, v).$$

By the Banach Fixed-Point Theorem, Φ has a unique fixed-point $\vec{u} \in E$, which is a solution of (7).

Maximal solution. We claim that there exists a function $T_{\max} : X \to (0, \infty]$ with the following properties. For any $\vec{u}_0 \in X$, there exists $u \in \mathcal{C}([0, T_{\max}(\vec{u}_0)), X)$, such that for all $T \in (0, T_{\max}(\vec{u}_0))$, u is the unique solution of (7). Moreover, the following alternative holds:

(i) Either $T_{\max}(\vec{u}_0) = \infty$;

(ii) Or $T_{\max}(\vec{u}_0) < \infty$ and then $\lim_{t \uparrow T_{\max}(\vec{u}_0)} \|\vec{u}(t)\|_X = \infty$.

When property (i) holds, one says that the solution is globally defined, or global. When property (ii) holds, one says that the solution blows up in finite time.

Proof. Let $\vec{u}_0 \in X$ and $M = 2\|\vec{u}_0\|_X$. We define

$$T_{\max}(\vec{u}_0) = \sup\{T > 0 : \text{there exists a solution } u \text{ of (7) on } [0,T]\}.$$

We have just proved that $T_{\max}$ is well-defined and $T_{\max} \geq T_M > 0$. Now, we define a function $\vec{u} \in \mathcal{C}([0, T_{\max}(\vec{u}_0)), X)$ which is solution of (7) on $[0,T]$ for any $T \in (0, T_{\max}(\vec{u}_0))$. Let $t \in [0, T_{\max}(\vec{u}_0))$. Let $T \in [t, T_{\max}(\vec{u}_0))$. By the definition of $T_{\max}(\vec{u}_0)$ as a supremum, there exists a solution $\vec{u}_T$ of (7) on $[0,T]$. Then, we set $\vec{u}(t) = \vec{u}_T(t)$ on $[0,T]$. By the uniqueness result, this definition does not depend on the choice of $T \in [t, T_{\max}(\vec{u}_0))$. Thus, it provides a function $\vec{u} \in \mathcal{C}([0, T_{\max}(\vec{u}_0)), X)$ which is indeed a solution of (7) on $[0,T]$ for any $T \in (0, T_{\max}(\vec{u}_0))$. Last, note that by the definition of $T_{\max}(\vec{u}_0)$, this solution cannot be extended beyond $T_{\max}(\vec{u}_0)$. This solution is called the *maximal solution* of (7).

Now, we prove the *blowup alternative.* Fix any $\tau \in [0, T_{\max}(\vec{u}_0))$, set $M = 2\|u(\tau)\|$ and consider $T_M > 0$ given by (8). There exists a solution $\vec{w}$ of

$$\begin{cases} \vec{w} \in \mathcal{C}([0, T_M], X), \\ \vec{w}(t) = S(t)\vec{u}(\tau) + \displaystyle\int_0^t S(t-s) w^3(s)\,\mathrm{d}s. \end{cases} \tag{9}$$

We extend the function $\vec{w} \in \mathcal{C}([0, \tau + T_M], X)$ by setting

$$\vec{w}(t) = \begin{cases} \vec{u}(t) & \text{if } t \in [0,\tau], \\ \vec{w}(t-\tau) & \text{if } t \in [\tau, \tau + T_M]. \end{cases}$$

We observe that $\vec{w}$ is now a solution of the problem (7) on the interval $[0,T]$, for $T = \tau + T_M$. By the definition of $T_{\max}(\vec{u}_0)$, this shows that

$$\tau + T_M < T_{\max}(\vec{u}_0).$$

Assume $T_{\max}(\vec{u}_0) < \infty$. By the general definition of T_M in (8) and the value of $M = 2\|u(\tau)\|$ in the present context, we obtain

$$\frac{1}{2C_L M^2} \leq T_{\max}(\vec{u}_0) - \tau.$$

This is equivalent to

$$2C_L\|u(\tau)\|^2 \geq \frac{1}{T_{\max}(\vec{u}_0) - \tau}, \tag{10}$$

which proves that if $T_{\max}(\vec{u}_0) < \infty$, then $\lim_{t \uparrow T_{\max}(\vec{u}_0)} \|\vec{u}(t)\|_X = \infty$.

Persistence of regularity. In the above framework, since $u^3 \in C([0, T_{\max}(\vec{u}_0), H^1)$, one has $(0, u^3) \in C([0, T_{\max}(\vec{u}_0), D(A_\alpha))$. Assume now in addition that $\vec{u}_0 \in D(A_\alpha)$. Using the Duhamel formulation (7) and the properties of S_α, we obtain $u \in C([0, T_{\max}), D(A_\alpha))$ and then $\partial_t u \in C([0, T_{\max}), X)$.

Continuous dependence on the initial data. Now, we claim that

(i) The function $T_{\max} : X \to (0, \infty]$ is lower semi-continuous;

(ii) If $\vec{u}_{0,n} \to \vec{u}_0$ as $n \to \infty$ in X, then for any $T \in (0, T_{\max}(\vec{u}_0))$, $\vec{u}_n \to \vec{u}$ in $\mathcal{C}([0,T], X)$ as $n \to \infty$, where $\vec{u}_n$ and $\vec{u}$ are the solutions of (7) corresponding respectively to $\vec{u}_{0,n}$ and $\vec{u}_0$.

Let $T \in (0, T_{\max}(\vec{u}_0))$. To prove (1)-(2), it suffices to show that if $\vec{u}_{0,n} \to \vec{u}_0$ then for n large enough $T_{\max}(\vec{u}_{0,n}) > T$ and $\vec{u}_n \to \vec{u}$ in $\mathcal{C}([0,T], X)$.

Set $M = 1 + 2\sup_{t\in[0,T]} \|\vec{u}(t)\|_X$ and define

$$\tau_n = \sup\{t \in [0, T_{\max}(\vec{u}_0)) : \|\vec{u}_n(s)\|_X \leq M \text{ for all } s \in [0,t]\}.$$

Since $\|\vec{u}_{0,n}\| < M/2$ for n large enough, $\tau_n > 0$ is well-defined. Moreover, by the well-posedness theory $\tau_n > T_M$. For any $t \in [0, \min(T;\tau_n)]$, we have

$$\|\vec{u}(t) - \vec{u}_n(t)\|_X \leq \|\vec{u}_0 - \vec{u}_{0,n}\|_X + C_L M^2 \int_0^t \|\vec{u}(s) - \vec{u}_n(s)\| \,\mathrm{d}s,$$

and thus by the Gronwall Lemma, for any $t \in [0, \min(T;\tau_n)]$,

$$\|\vec{u}(t) - \vec{u}_n(t)\|_X \leq \|\vec{u}_0 - \vec{u}_{0,n}\|_X \exp\left(C_L M^2 T\right). \tag{11}$$

This proves that for any $t \in [0, \min(T;\tau_n)]$,

$$\|\vec{u}_n(t)\|_X \leq \|\vec{u}(t)\|_X + \|\vec{u}(t) - \vec{u}_n(t)\|_X \leq \frac{M}{2} + \|\vec{u}_0 - \vec{u}_{0,n}\|_X \exp\left(C_L M^2 T\right) < \frac{3M}{4},$$

for n large enough. Therefore, $\tau_n > T$, which also justifies that $T_{\max}(\vec{u}_{0,n}) > T$.

Lastly, estimate (11) implies that $\vec{u}_n \to \vec{u}$ in $\mathcal{C}([0,T], X)$. □

In this course, we systematically work in the framework of such maximal finite energy solutions.

Corollary 1.4. *In the context of Proposition 1.3, the function $t \mapsto E(\vec{u}(t))$ is C^1 on $[0, T_{\max}(\vec{u}_0))$ and for all $t \in [0, T_{\max}(\vec{u}_0))$, it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\vec{u}(t)) = -2\alpha \|v(t)\|^2.$$

Proof. Let $\vec{u}_0 \in X$ and for all $n \geq 0$, let $\vec{u}_{0,n} \in D(A_\alpha)$ be such that $\vec{u}_{0,n} \to \vec{u}_0$ as $n \to \infty$ in X. It is known that for any $T \in (0, T_{\max}(\vec{u}_0))$, $\vec{u}_n \to \vec{u}$ in $\mathcal{C}([0,T], X)$ as $n \to \infty$.

For $\vec{u}(t)$, it is rigorously checked by using (NLKG) that

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\vec{u}_n(t)) = -2\alpha \|v_n(t)\|^2.$$

In particular, for all $t \in [0, T_{\max}(\vec{u}_0))$, and all n large,

$$E(\vec{u}_n(t)) - E(\vec{u}_{0,n}) = -2\alpha \int_0^t \|v_n(s)\|^2 \,\mathrm{d}s.$$

Passing to the limit $n \to +\infty$ $E(\vec{u}_n(t)) \to E(\vec{u}(t))$ and $\|v_n(s)\|^2 \to \|v(s)\|^2$. Thus, for all $t \in [0, T_{\max}(\vec{u}_0))$,

$$E(\vec{u}(t)) - E(\vec{u}_0) = -2\alpha \int_0^t \|v(s)\|^2 \,\mathrm{d}s.$$

This proves the result. □

2 The global Cauchy problem

2.1 On blowup in finite time

The negative sign in front of u^3 in equation (1) means that the equation is focusing. In particular, the sign of the quartic term in the definition of the energy prevents us to use the decay of energy to prove global wellposedness. On the contrary, we are going to prove that there exist blow up solutions for the equation.

Together with the energy functional $E(t) := E(\vec{u}(t))$ defined in (2) and satisfying (3), we will use the following quantities

$$M(t) := \frac{1}{2}\|u(t)\|^2 + \alpha \int_0^t \|u(s)\|^2 \,\mathrm{d}s,$$
$$W(t) := \frac{1}{2}\left(\|\partial_t u(t)\|^2 + \|\partial_x u(t)\|^2 + \|u(t)\|^2\right).$$

Lemma 2.1. *It holds*

$$M'(t) = \int u(t)\partial_t u(t)\,\mathrm{d}x + \alpha\|u(t)\|^2 \tag{12}$$
$$= \int u(t)\partial_t u(t)\,\mathrm{d}x + 2\alpha \int_0^t \int u(s)\partial_t u(s)\,\mathrm{d}x\,\mathrm{d}s + \alpha\|u(0)\|^2, \tag{13}$$
$$M''(t) = 3\|\partial_t u(t)\|^2 + \|\partial_x u(t)\|^2 + \|u(t)\|^2 - 4E(t), \tag{14}$$
$$W'(t) = -2\alpha\|\partial_t u(t)\|^2 + \int u^3(t)\partial_t u(t)\,\mathrm{d}x. \tag{15}$$

Proof. Direct computations using (1) and (2). Density arguments are used as in the proof of Corollary 1.4. □

Theorem 2.2. *Let* $0 < \alpha \le \frac{1}{4}$. *If* $E(0) < 0$, *then the corresponding solution of* (1) *blows up in finite time.*

Proof. Assume that $E(0) < 0$. For the sake of contradiction, assume that the solution is global. Then, by (3), $E(t) \le E(0) < 0$. In particular, by (14), we have $M''(t) \ge -4E(t) \ge -4E(0) > 0$. It follows that $\lim_{t\to+\infty} M(t) = +\infty$. Moreover, since $M''(t) \ge 3\|\partial_t u(t)\|^2 + \|u(t)\|^2$, we also have

$$M(t)M''(t) \ge \frac{1}{2}\|u(t)\|^2(3\|\partial_t u(t)\|^2 + \|u\|^2) \ge \frac{3}{2}\left(\int u\partial_t u\right)^2 + \frac{1}{2}\|u\|^4.$$

Using the inequality $(a+b)^2 \le \frac{5}{4}a^2 + 5b^2$, and then $\alpha < \frac{1}{4}$, we have

$$(M'(t))^2 \le \frac{5}{4}\left(\int u\partial_t u\right)^2 + 5\alpha^2\|u\|^4 \le \frac{5}{6}M(t)M''(t).$$

This implies that for all $t \ge 0$,

$$(M^{-\frac{1}{5}})''(t) \le 0.$$

But $M(t) > 0$ and $\lim_{t\to+\infty} M^{-\frac{1}{5}}(t) = 0$. Thus, there exists $t_1 > 0$ such that $(M^{-\frac{1}{5}})'(t_1) < 0$. Using the concavity, we obtain for $t \geq t_1$,

$$0 \leq M^{-\frac{1}{5}}(t) \leq M^{-\frac{1}{5}}(t_1) + (t - t_1)(M^{-\frac{1}{5}})'(t_1).$$

This is contradictory for t large. □

2.2 Global solutions are bounded

Using arguments of [3] and [2, Proof of Lemma 2.7] for the undamped Klein-Gordon equation, we prove a bound on global solutions of (1).

Theorem 2.3 ([2, 3]). *Any global solution of* (1) *is bounded in* X.

Proof. Let $\vec{u}$ be a global solution of (1). From (12) and the Cauchy-Schwarz inequality,

$$|M'(t)| \leq (1 + 2\alpha)W(t). \tag{16}$$

Moreover, by (3) and (14),

$$M''(t) \geq 2W(t) - 4E(0). \tag{17}$$

The proof of the global bound now proceeds in three steps.

Step 1. We prove that

$$\liminf_{t\to\infty} M'(t) < \infty. \tag{18}$$

Proof of (18). We argue by contradiction, proving that $\lim_\infty M' = \infty$ implies the following inequality, for all t large enough,

$$(1 + \epsilon)[M'(t)]^2 < M''(t)M(t) \quad \text{where } \epsilon > 0 \text{ is to be chosen.} \tag{19}$$

Then, we reach a contradiction by a standard argument. Indeed, remark that (19) implies $\frac{\mathrm{d}^2}{\mathrm{d}t^2}[M^{-\epsilon}(t)] < 0$, and $\lim_\infty M' = \infty$ also implies $\lim_\infty M^{-\epsilon} = 0$. Thus, there exists $t_1 > 0$ such that $\frac{\mathrm{d}}{\mathrm{d}t}[M^{-\epsilon}(t_1)] < 0$, and for all $t \geq t_1$,

$$0 \leq M^{-\epsilon}(t) \leq M^{-\epsilon}(t_1) + (t - t_1)\frac{\mathrm{d}}{\mathrm{d}t}[M^{-\epsilon}(t_1)],$$

which is absurd for $t \geq t_1$ large enough.

Thus, we only need to prove (19) assuming $\lim_\infty M' = \infty$. On the one hand, by (13) and the Cauchy-Schwarz inequality, it holds

$$|M'| \leq \|u\|\|\partial_t u\| + 2\alpha \left(\int_0^t \|u(s)\|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_0^t \|\partial_t u(s)\|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} + \alpha\|u(0)\|^2.$$

Let $\epsilon > 0$ to be chosen later, we estimate

$$|M'|^2 \leq (1 + \epsilon) \left[\|u\|\|\partial_t u\| + 2\alpha \left(\int_0^t \|u(s)\|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_0^t \|\partial_t u(s)\|^2 \,\mathrm{d}s\right)^{\frac{1}{2}}\right]^2 + \left(1 + \frac{1}{\epsilon}\right) \alpha^2\|u(0)\|^4.$$

Thus, using the inequality $(AB+CD)^2 \le (A^2+C^2)(B^2+D^2)$, we obtain

$$
\begin{aligned}
|M'|^2 &\le (1+\epsilon)\left[\frac{1}{2}\|u\|^2 + \alpha\int_0^t \|u(s)\|^2\,\mathrm{d}s\right]\left[2\|\partial_t u\|^2 + 4\alpha\int_0^t \|\partial_t u(s)\|^2\,\mathrm{d}s\right] \\
&\quad + \left(1+\frac{1}{\epsilon}\right)\alpha^2\|u(0)\|^4 \\
&\le (1+\epsilon)M\left[2\|\partial_t u\|^2 + 4\alpha\int_0^t \|\partial_t u(s)\|^2\,\mathrm{d}s\right] + \left(1+\frac{1}{\epsilon}\right)\alpha^2\|u(0)\|^4.
\end{aligned}
$$

On the other hand, by (3) and (14),

$$
\begin{aligned}
M'' &= 2\|\partial_t u\|^2 + 2W + 8\alpha\int_0^t \|\partial_t u(s)\|^2\,\mathrm{d}s - 4E(0) \\
&\ge (1+\epsilon)^3\left[2\|\partial_t u\|^2 + 4\alpha\int_0^t \|\partial_t u(s)\|^2\,\mathrm{d}s\right] + W - 4E(0),
\end{aligned}
$$

by fixing any ϵ such that

$$
0 < \epsilon < \left(\frac{5}{4}\right)^{\frac{1}{3}} - 1.
$$

In particular, since $\lim_\infty W = \infty$ by (16) and the assumption $\lim_\infty M' = \infty$, we have for t large enough,

$$
M'' \ge (1+\epsilon)^3\left[2\|\partial_t u\|^2 + 4\alpha\int_0^t \|\partial_t u(s)\|^2\,\mathrm{d}s\right].
$$

Thus,

$$
(1+\epsilon)^2|M'|^2 \le MM'' + \left(1+\frac{1}{\epsilon}\right)\alpha^2\|u(0)\|^4,
$$

and using again $\lim_\infty M' = \infty$ we obtain (19) for any t large enough.

Step 2. We prove that

$$
\sup_{t\in[0,\infty)} |M'(t)| < \infty. \tag{20}
$$

Proof of (20). Combining (16) and (17), we obtain

$$
M''(t) \ge \frac{2}{1+2\alpha}|M'(t)| - 4E(0).
$$

Let

$$
\begin{aligned}
H_+(t) &= \frac{2}{1+2\alpha}M'(t) - 4E(0), \\
H_-(t) &= -\frac{2}{1+2\alpha}M'(t) - 4E(0).
\end{aligned}
$$

Then, $H_+'(t) = \frac{2}{1+2\alpha} M''(t) \geq \frac{p-1}{1+2\alpha} + (t)$. If there exists $t \geq 0$ such that $H_+(t) > 0$, then $\lim_\infty H_+ = \infty$, contradicting (18). It follows that for all $t \geq 0$,

$$M'(t) \leq 2(1+2\alpha)E(0).$$

Similarly, $H_-'(t) = -\frac{2}{1+2\alpha} M''(t) \leq -\frac{2}{1+2\alpha} H_-(t)$. It follows that $H_-(t) \leq e^{-\frac{2}{1+2\alpha}t} H_-(0)$, for all $t \geq 0$. Thus,

$$M'(t) \geq -\frac{1+2\alpha}{2} \left(4E(0) + |H_-(0)|\right).$$

and (20) is proved.

Step 3. Last, we prove the global bound

$$\sup_{t \in [0,\infty)} |W(t)| < \infty. \tag{21}$$

Proof of (21). We rewrite (17) as

$$W(t) \leq \frac{1}{2} M''(t) + 2E(0).$$

Integrating on $(t, t+1)$ and using (20), we observe that

$$\sup_{t \geq 0} \int_t^{t+1} W(s) \, \mathrm{d}s < \infty. \tag{22}$$

Moreover, by (15),

$$W' \leq -2\alpha \|\partial_t u\|^2 + \int |u|^3 |\partial_t u| \leq \frac{1}{2} \|\partial_t u\|^2 + \frac{1}{2} \int |u|^6 \leq W + \frac{1}{2} \int |u|^6.$$

For $t \geq 1$ and $\tau \in (0,1)$, integrating on $(t-\tau, t)$, we find

$$\begin{aligned} W(t) &\leq W(t-\tau) + \int_{t-\tau}^t W(s) \, \mathrm{d}s + \frac{1}{2} \int_{t-\tau}^t \int |u(s)|^6 \, \mathrm{d}x \, \mathrm{d}s \\ &\leq W(t-\tau) + \int_{t-1}^t W(s) \, \mathrm{d}s + \frac{1}{2} \int_{t-1}^t \int |u(s)|^6 \, \mathrm{d}x \, \mathrm{d}s. \end{aligned}$$

Using the Sobolev inequality (in space-time) for the last term, we obtain, for some constants $C > 0$,

$$\begin{aligned} W(t) &\leq W(t-\tau) + \int_{t-1}^t W(s) \, \mathrm{d}s + C \|u\|_{H^1((t-1,t)\times\mathbb{R})}^6 \\ &\leq W(t-\tau) + \int_{t-1}^t W(s) \, \mathrm{d}s + C \left(\int_{t-1}^t W(s) \, \mathrm{d}s \right)^3. \end{aligned}$$

Integrating in $\tau \in (0,1)$ and using (22), we find (21). □

3 The solitary waves

It is also well-known that up to sign and translation, the only stationary solution of (1) is the solitary wave $(Q, 0)$, where Q is the explicit ground state

$$Q(x) = \frac{\sqrt{2}}{\cosh(x)} = \sqrt{2}\,\mathrm{sech}(x) \tag{23}$$

which solves the equation

$$Q'' - Q + Q^3 = 0 \quad \text{on } \mathbb{R}. \tag{24}$$

We see from the explicit expression of Q in (23) that, as $x \to \infty$,

$$Q(x) = c_Q e^{-x} + O(e^{-3x}), \quad Q'(x) = -c_Q e^{-x} + O(e^{-2x}) \tag{25}$$

where $c_Q = 2\sqrt{2}$. Note that by (24), it holds $\int (\partial_x Q)^2 + Q^2 - Q^4 = 0$ and so

$$E(Q, 0) = \frac{1}{4} \int Q^4 > 0. \tag{26}$$

Let

$$\mathcal{L} = -\partial_x^2 + 1 - 3Q^2 = -\partial_x^2 + 1 - 6\,\mathrm{sech}^2,$$
$$\langle \mathcal{L}\varepsilon, \varepsilon \rangle = \int \left\{ (\partial_x \varepsilon)^2 + \varepsilon^2 - 3Q^2\varepsilon^2 \right\} \mathrm{d}x.$$

We recall some standard properties of the operator $\mathcal{L}$ (see *e.g.* [9, Lemma 1]).

Lemma 3.1. *The following properties hold.*

(i) Spectral properties. *The unbounded operator $\mathcal{L}$ on L^2 with domain H^2 is self-adjoint, its continuous spectrum is $[1, \infty)$, its kernel is* $\mathrm{span}\{Q'\}$ *and -3 is its unique negative eigenvalue with corresponding smooth normalized eigenfunction $Y = \frac{\sqrt{3}}{2}\,\mathrm{sech}^2(x)$.*

(ii) Coercivity property. *There exist $c_1, c_2 > 0$ such that, for all $\varepsilon \in H^1$,*

$$\langle \mathcal{L}\varepsilon, \varepsilon \rangle \geq c_1 \|\varepsilon\|_{H^1}^2 - c_2 \left(\langle \varepsilon, Y \rangle^2 + \langle \varepsilon, Q' \rangle^2 \right).$$

Proof. The continuous spectrum of $\mathcal{L}$ is the same as the one of the operator $-\partial_x^2 + 1$, i.e. the interval $[1, +\infty)$, since the potential $-6\,\mathrm{sech}^2$ is a compact perturbation of $-\partial_x^2 + 1$.

We check by direct computations that $\mathcal{L}Y = -3Y$ and $\mathcal{L}Q' = 0$. Since $Y > 0$, it is a standard observation that -3 is the lowest eigenvalue of $\mathcal{L}$. Moreover, since Q' only has one zero, 0 is the second eigenvalue. Lastly, we check $R = 1 - \frac{3}{2}\,\mathrm{sech}^2$ satisfies $\mathcal{L}R = R$. Since $R \in L^\infty$ and $R' \in L^2$, but $R \notin L^2$ the bottom of the continuous spectrum 1 is called a resonance. Since R only vanishes twice on $\mathbb{R}$, there is no other discrete eigenvalue. In particular, by the spectral theorem, if $\langle \varepsilon, Y \rangle = 0$ and $\langle \varepsilon, Q' \rangle = 0$, then

$$\langle \mathcal{L}\varepsilon, \varepsilon \rangle \geq \|\varepsilon\|_{H^1}^2.$$

For a general $\varepsilon \in H^1$, we decompose $\varepsilon = aY + bQ' + \eta$, where $\langle \eta, Y\rangle = 0$ and $\langle \eta, Q'\rangle = 0$. In particular, $a = \langle Y, \varepsilon\rangle$ and $b\|Q'\|^2 = \langle Q', \varepsilon\rangle$. We also have $\langle \mathcal{L}\eta, \eta\rangle \geq \|\eta\|^2$. Thus,

$$\begin{aligned}
\langle \mathcal{L}\varepsilon, \varepsilon\rangle &= -3a^2 - b^2\|Q'\|^2 + \langle \mathcal{L}\eta, \eta\rangle \\
&\geq -3a^2 - b^2\|Q'\|^2 + \|\eta\|^2 \\
&\geq -4a^2 - 2b^2 + \|\varepsilon\|^2 \\
&\geq -4\langle \varepsilon, Y\rangle^2 - 2\langle \varepsilon, Q'\rangle^2 + \|\varepsilon\|^2
\end{aligned}$$

Moreover, it is easy to see from the definition of $\mathcal{L}$ that

$$\langle \mathcal{L}\varepsilon, \varepsilon\rangle \geq \|\partial_x \varepsilon\|^2 - 5\|\varepsilon\|^2.$$

By taking a linear combinaison with coefficients 6/7 and 1/7 of the above inequalities, we find

$$\langle \mathcal{L}\varepsilon, \varepsilon\rangle \geq \frac{1}{7}\left(\|\partial_x \varepsilon\|^2 + \|\varepsilon\|^2\right) - \frac{24}{7}\langle \varepsilon, Y\rangle^2 - \frac{12}{7}\langle \varepsilon, Q'\rangle^2.$$

□

The unique negative eigenvalue of $\mathcal{L}$ is related to an instability of the solitary wave for the equation (1), described by the following functions:

$$\nu^\pm = -\alpha \pm \sqrt{\alpha^2 + 3}, \quad \vec{Y}^\pm = \begin{pmatrix} Y \\ \nu^\pm Y \end{pmatrix}, \tag{27}$$

$$\zeta^\pm = \alpha \pm \sqrt{\alpha^2 + 3}, \quad \vec{Z}^\pm = \begin{pmatrix} \zeta^\pm Y \\ Y \end{pmatrix}. \tag{28}$$

Indeed, it follows from explicit computations that the function

$$\vec{\varepsilon}^{\,\pm}(t, x) = \exp(\nu^\pm t)\vec{Y}^\pm(x)$$

is solution of the linearized problem

$$\begin{cases} \partial_t \varepsilon = \eta \\ \partial_t \eta = -\mathcal{L}\varepsilon - 2\alpha\eta. \end{cases} \tag{29}$$

Since $\nu^+ > 0$, the solution $\vec{\varepsilon}^{\,+}$ illustrates the exponential instability of the solitary wave in positive time. This means that the presence of the damping $\alpha > 0$ does not remove the exponential instability of the Klein-Gordon solitary wave. An equivalent formulation of instability is obtained by saying that the functions $\vec{Z}^\pm$ are the eigenfunctions of the adjoint linearized operator in (29):

$$\begin{pmatrix} 0 & -\mathcal{L} \\ 1 & -2\alpha \end{pmatrix} \vec{Z}^\pm = \nu^\pm \vec{Z}^\pm,$$

and as a consequence, for any solution $\vec{\varepsilon}$ of (29),

$$a^\pm = \langle \vec{\varepsilon}, \vec{Z}^\pm\rangle \quad \text{satisfies} \quad \frac{\mathrm{d}a^\pm}{\mathrm{d}t} = \nu^\pm a^\pm. \tag{30}$$

Remark 3.2. The existence of the solutions ε^+ is called linear exponential instability. More arguments are needed to prove that the solitary wave solution $(Q, 0)$ is actually nonlinearly unstable, in the following sense

$$\exists \delta_0 > 0, \forall \sigma > 0, \exists \vec{u}_0 \in X, \|\vec{u}_0 - (Q,0)\|_X \leq \sigma, \exists T > 0 : \inf_{a \in \mathbb{R}} \|\vec{u}(T) - (Q(\cdot - a), 0)\|_X \geq \delta_0$$

where $\vec{u}$ is the solution of (NLKG) with initial data $\vec{u}_0$. We will not address this question here, but it is an interesting exercise to prove this statement.

4 First decomposition result of any global solution

4.1 The Brezis-Lieb Lemma

The following is a particular case of the Brezis-Lieb lemma.

Lemma 4.1. *Let (f_n) be a sequence of functions in L^4 that converges a.e. to a function f and such that $\sup_n \|f_n\|_{L^4} < +\infty$. Then*

$$\lim_{n\to\infty} \int \left| f_n^4 - f^4 - (f - f_n)^4 \right| = 0.$$

In particular,

$$\lim_{n\to\infty} \left(\int f_n^4 - \int (f - f_n)^4 \right) = \int f^4.$$

Proof. Let $r_n = \left| f_n^4 - f^4 - (f - f_n)^4 \right|$. We have

$$\begin{aligned} r_n &= \left| (f_n - f + f)^4 - f^4 - (f - f_n)^4 \right| \\ &= \left| 4(f_n - f)^3 f + 6(f_n - f)^2 f^2 + 4(f_n - f) f^3 \right| \\ &\leq \epsilon (f_n - f)^4 + C_\epsilon f^4. \end{aligned}$$

for any $\epsilon > 0$. Thus, the nonnegative function[1] $s_{n,\epsilon} = (r_n - \epsilon(f_n - f)^4)_+$ converges a.e. to zero and is dominated by the integrable function $C_\epsilon f^4$. By the dominated convergence theorem, this proves that $\lim_{n\to+\infty} \int s_{n,\epsilon} = 0$. Now, $0 \leq r_n \leq s_{n,\epsilon} + \epsilon(f_n - f)^4$ and so

$$\limsup_{n\to+\infty} \int r_n \leq \epsilon \limsup_{n\to+\infty} \int (f_n - f)^4.$$

In particular,

$$\limsup_{n\to+\infty} \int r_n \leq C\epsilon \limsup_{n\to+\infty} \int (f_n^4 + f^4) \leq C\epsilon,$$

and ϵ being arbitrary, we obtain $\lim_{n\to+\infty} \int r_n = 0$. □

[1] The notation x_+ means $x_+ = \max(x, 0)$

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