

Du logarithme à E8, en passant par les 27 droites d'une surface cubique

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Hyperlogarithmic functional equations on del Pezzo surfaces

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ABSTRACT

For any $d \in \{1, \dots, 6\}$, we prove that the web of conics on a del Pezzo surface of degree d carries a functional identity whose components are antisymmetric hyperlogarithms of weight $7 - d$. Our approach is uniform with respect to d and relies on classical results about the action of the Weyl group on the set of lines on the del Pezzo surface. These hyperlogarithmic functional identities are natural

Plan

Le logarithme

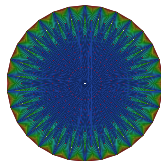
$$\text{Log}(x) + \text{Log}(y) - \text{Log}(xy) = 0$$



Hyperlogarithme

$$\sum_i \text{HLog}(U_i(x, y)) = 0$$

E_8



Le logarithme

- Formule intégrale : $\mathbf{Log}(z) = \int_1^z \frac{du}{u} \quad (z \in \mathbb{C})$

- Monodromie : $\mathcal{M}_{\textcircled{0}}(\mathbf{Log}) = \mathbf{Log} + 2i\pi$

- Développement en série : $\mathbf{Li}_1(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} \quad |z| < 1$

$$\mathbf{Li}_1(z) = -\mathbf{Log}(1 - z)$$

- Identité fonctionnelle :

$$\mathbf{Log}(x) + \mathbf{Log}(y) - \mathbf{Log}(xy) = 0$$

Le dilogarithme Li_2

- $\text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \quad (|z| < 1)$
- Formule Intégrale : $\text{Li}_2(z) = \mathbf{L}_{01}(z) = -\int^z \log(1-u) \frac{du}{u-0}$
- Monodromie : $\mathcal{M}_{(\mathbf{1})}(\text{Li}_2) = \text{Li}_2 - 2i\pi \mathbf{Log}$
- Identité fonctionnelle d'Abel ($\mathcal{A}b$) [Spence 1809]

$$\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$$

$$\mathbf{R}(z) = \frac{1}{2} \left(\mathbf{L}_{01}(z) - \mathbf{L}_{10}(z) \right)$$

Le n -ième polylogarithme Li_n pour $n \geq 1$

- $\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad (|z| < 1)$

- Formule intégrale :** $\text{Li}_n(z) = \int^z \text{Li}_{n-1}(u) \frac{du}{u}$

$$\text{Li}_n'(z) = \text{Li}_{n-1}(z)/z$$

- Monodromie :** $\mathcal{M}_{(\mathbf{1})}(\text{Li}_n) = \text{Li}_n - 2i\pi \frac{(\text{Log})^{n-1}}{(n-1)!}$

- Identités fonctionnelles en plusieurs variables ($\exists ?$) :**

$$\sum_{i \in I} c_i \text{Li}_n(\mathbf{U}_i) = \text{Elem}_{<n}$$

$$(I \text{ fini}, c_i \in \mathbb{Z}, \mathbf{U}_i \in \mathbb{Q}(x_1, \dots, x_N))$$

Le n -ième polylogarithme Li_n pour $n \geq 1$

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- Identités fonctionnelles en plusieurs variables ($\exists ?$) :**

$$\sum_{i \in I} c_i \text{Li}_n(\mathbf{U}_i) = \text{Elem}_{<n} \iff \sum_{i \in I} c_i \mathcal{L}_n(\mathbf{U}_i) = \mathbf{0}$$

$$(I \text{ fini, } c_i \in \mathbb{Z}, \mathbf{U}_i \in \mathbb{Q}(x_1, \dots, x_N))$$

Exemple : le trilogarithme

- **Identité de Spence-Kummer (1809-1840) :**

$$\begin{aligned} & 2\operatorname{Li}_3(x) + 2\operatorname{Li}_3(y) - \operatorname{Li}_3\left(\frac{x}{y}\right) + 2\operatorname{Li}_3\left(\frac{1-x}{1-y}\right) + 2\operatorname{Li}_3\left(\frac{x(1-y)}{y(1-x)}\right) - \operatorname{Li}_3(xy) \\ & + 2\operatorname{Li}_3\left(-\frac{x(1-y)}{(1-x)}\right) + 2\operatorname{Li}_3\left(-\frac{(1-y)}{y(1-x)}\right) - \operatorname{Li}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right) \\ & = 2\operatorname{Li}_3(1) - \operatorname{Log}(y)^2 \operatorname{Log}\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3} \operatorname{Log}(y) + \frac{1}{3} \operatorname{Log}(y)^3 \end{aligned}$$

$$\begin{aligned} & 2\mathcal{L}_3(x) + 2\mathcal{L}_3(y) - \mathcal{L}_3\left(\frac{x}{y}\right) + 2\mathcal{L}_3\left(\frac{1-x}{1-y}\right) + 2\mathcal{L}_3\left(\frac{x(1-y)}{y(1-x)}\right) - \mathcal{L}_3(xy) \\ & + 2\mathcal{L}_3\left(-\frac{x(1-y)}{(1-x)}\right) + 2\mathcal{L}_3\left(-\frac{(1-y)}{y(1-x)}\right) - \mathcal{L}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right) = 0 \end{aligned}$$

- **Abel 1881 (Spence 1809, Hill 1829, Rogers 1907)**

$$R(x) - R(y) - R\left(\frac{x}{y}\right) - R\left(\frac{1-y}{1-x}\right) + R\left(\frac{x(1-y)}{y(1-x)}\right) = 0 \quad (n = 2)$$

- **Spence-Kummer :** $\sum_{i=1}^9 c_i \mathcal{L}_3(U_i(x, y)) = 0 \quad (n = 3)$

- **Kummer 1840 :** $\sum_i c_i \mathcal{L}_n(U_i(x, y)) = 0 \quad (n \leq 5)$

...

- **Goncharov 1995 :** $\sum_{i=1}^{22} c_i \mathcal{L}_3(U_i(a, b, c)) = 0 \quad (n = 3)$

- **Gangl 2003 :** $\sum_i c_i \mathcal{L}_n(U_i(x, y)) = 0 \quad (n \leq 7)$

- **Charlton, Gangl, Radchenko, Rudenko, Goncharov-Rudenko, ...**

- **Identités fonctionnelles (IF) des polylogarithmes Li_n :**
 - ▶ Géométrie hyperbolique (volumes des polytopes)
 - ▶ Géométrie des tissus (tissus exceptionnels)
 - ▶ K-théorie des corps de nombres (Conjecture de Zagier)
 - ▶ Théorie des périodes (valeurs multizetas)
 - ▶ Physique des particules ('*Scattering amplitudes*')
 - ▶ Physique mathématique ('*Y-systèmes*')
 - ▶ Algèbres amassées (cluster period)
 - ▶ Symétrie miroir ('*Scattering diagrams*')

- **Problèmes :**
 - trouver des **IF** pour \mathcal{L}_n (e.g. $\exists n \geq 8 ?$)
 - mieux comprendre les **IF** polylogarithmiques

[Griffiths 2002] *The legacy of Abel in algebraic geometry*

Existe-t-il une suite d'identités fonctionnelles $(\mathcal{IF}_n)_{n \geq 1}$ tq

- $\mathcal{IF}_1 =$ équation du log $\mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}(x/y) = 0$;
- $\mathcal{IF}_2 =$ équation d'Abel (**Ab**) du dilog ;
- $\mathcal{IF}_n =$ équation *fondamentale* pour \mathbf{Li}_n , $\forall n \geq 1$?

[Goncharov-Rudenko 2018] *Motivic correlators, cluster varieties*

Conclusion Si $n > 3$, le problème de trouver des IF pour \mathbf{Li}_n n'est peut-être pas le "bon". Il semble que lorsque n augmente, celles-ci sont si compliquées qu'il devient impossible de les écrire sur une feuille de papier.

- En considérant des **hyperlogarithmes** AH^\bullet

Théorème [Castravet-Pirio]

Pour $w = 1, \dots, 6$, on a :

$$\left(\mathbf{HLog}^w \right) \quad \sum_{i=1}^{\kappa} \mathbf{AH}_{\Sigma_i}^w(\phi_i) = 0$$

→ Famille uniforme d'identités hyperlogarithmiques

$$\mathbf{HLog}^1 \iff \mathbf{Log}(x) + \mathbf{Log}(y) - \mathbf{Log}(xy) = 0$$

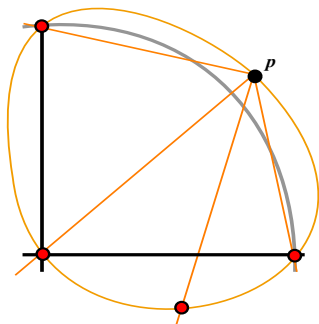
$$\mathbf{HLog}^2 \iff \mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$$

⋮

$$\mathbf{HLog}^6 \iff \sum_{i=1}^{2160} \mathbf{AH}_{\Sigma_i}^6(\phi_i(x, y)) = 0$$

Une vue géométrique de l'identité d'Abel

• (Ab)
$$\underset{\parallel U_1}{R(x)} - \underset{\parallel U_2}{R(y)} - \underset{\parallel U_3}{R\left(\frac{x}{y}\right)} - \underset{\parallel U_4}{R\left(\frac{1-y}{1-x}\right)} + \underset{\parallel U_5}{R\left(\frac{x(1-y)}{y(1-x)}\right)} = 0$$



Points base des U_i 's :

– $p_1 = [1, 0, 0]$

– $p_2 = [0, 1, 0]$

– $p_3 = [0, 0, 1]$

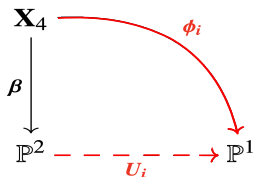
– $p_4 = [1, 1, 1]$

↪ Éclatement $\beta : X_4 = \mathbf{Bl}_{p_1, \dots, p_4}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$

Une vue géométrique de l'identité d'Abel

- $(\mathcal{A}b)$
$$\underbrace{R(x)}_{U_1} - \underbrace{R(y)}_{U_2} - \underbrace{R\left(\frac{x}{y}\right)}_{U_3} - \underbrace{R\left(\frac{1-y}{1-x}\right)}_{U_4} + \underbrace{R\left(\frac{x(1-y)}{y(1-x)}\right)}_{U_5} = 0$$

- Éclatement $\beta : X_4 = \text{Bl}_{p_1, \dots, p_4}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$



Les $\phi_1, \dots, \phi_5 : X_4 \longrightarrow \mathbb{P}^1$ sont les cinq fibrations en coniques sur la surface de del Pezzo X_4

- $(\mathcal{A}b) \iff \left(\begin{array}{l} \exists (\epsilon_i)_{i=1}^5 \in \{\pm 1\}^5 \\ \exists ! \text{ au signe près} \end{array} \text{ tq } \right) \sum_{i=1}^5 \epsilon_i R(\phi_i) = 0$

Généralisation aux surfaces de del Pezzo

- $p_1, \dots, p_r \in \mathbb{P}^2$ points en "position générale" ($r \in \{3, \dots, 8\}$)
- **Éclatement** $\beta_r : X_r = \mathbf{Bl}_{p_1, \dots, p_r}(\mathbb{P}^2) \rightarrow \mathbb{P}^2$ ($X_r = dP_{9-r}$)
 $\rightarrow X_r =$ surface de del Pezzo ($-K_{X_r}$ ample) de degré $d = 9 - r$

Prop : 1. Il y a un nombre fini κ_r de fibrations en coniques $\phi_1, \dots, \phi_{\kappa} : X_r \rightarrow \mathbb{P}^1$

2. Pour tout $i : \Sigma_i = \mathbf{Spectre}(\phi_i) \subset \mathbb{P}^1$ a $r - 1$ éléments

Def^o : L' hyperlogarithme complet antisymétrique de poids $r - 2$: $\mathbf{AH}_{\Sigma_i}^{r-2} : \widehat{\mathbb{P}^1 \setminus \Sigma_i} \rightarrow \mathbb{C}$

Théorème [Castravet-Pirio 2022]

$\exists (\epsilon_i)_{i=1}^{\kappa} \in \{\pm 1\}^{\kappa}$ unique au signe près tel que

$$\left(\mathbf{HLog}^{r-2}\right) \quad \sum_{i=1}^{\kappa} \epsilon_i \mathbf{AH}_{\Sigma_i}^{r-2}(\phi_i) = 0$$

→ Surfaces de Del Pezzo

→ Hyperlogarithmes (aka “Intégrales itérées sur \mathbb{P}^1 ”)

Hyperlogarithmes

- **Poincaré (1884), Lappo-Danilevski (1928), Chen (1973)**

- $\Sigma = \{ \sigma_1, \dots, \sigma_{m+1} = \infty \} \subset \mathbb{P}^1$

- $\left\{ \begin{array}{l} \omega_k = dz/(z - \sigma_k) \\ k = 1, \dots, m \end{array} \right\}$: base de $\mathcal{H}_\Sigma = \mathbf{H}^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\text{Log } \Sigma))$

- **Def^o** : l'hyperlogarithme $L_{\omega_{s_1} \dots \omega_{s_w}}$ pour $(s_k)_{k=1}^w \in \{1, \dots, m\}^w$ est la fonction multivaluée sur \mathbb{P}^1 définie inductivement par

$$L_{\omega_{s_1}}(z) = \int^z \omega_{s_1} = \mathbf{Log}(z - \sigma_{s_1}) \quad \text{et} \quad L_{\omega_{s_1} \omega_{s_2} \dots \omega_{s_w}}(z) = \int \frac{du}{u - \sigma_{s_1}} L_{\omega_{s_2} \dots \omega_{s_w}}(u)$$

- **Ex** : $\mathcal{H}_{\{0,1,\infty\}} = \left\langle \begin{array}{l} \eta_0 = dz/(z-0) \\ \eta_1 = dz/(z-1) \end{array} \right\rangle \quad \mathbf{Li}_n = L_{\eta_0^{\otimes(n-1)} \eta_1} \quad \forall n \geq 1$

- **Def^o** : le **symbole** de $L_{\omega_{s_1} \dots \omega_{s_w}}$ est $\mathcal{S}(L_{\omega_{s_1} \dots \omega_{s_w}}) = \omega_{s_1} \otimes \dots \otimes \omega_{s_w}$

L'Hyperlogarithme Antisymétrique complet AH

- $\Sigma = \{\sigma_1, \dots, \sigma_{m+1} = \infty\} \subset \mathbb{P}^1$
- $\left\{ \begin{array}{l} \omega_k = dz/(z - \sigma_k) \\ k = 1, \dots, m \end{array} \right\}$: base de $\mathcal{H}_\Sigma = \mathbf{H}^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\text{Log } \Sigma))$
- $\otimes_{k=1}^m \omega_k = \left(\frac{dz}{z - \sigma_1}\right) \otimes \dots \otimes \left(\frac{dz}{z - \sigma_m}\right) \in (\mathcal{H}_\Sigma)^{\otimes m}$
- $L_{\omega_1 \omega_2 \dots \omega_m}$: dépend de l'ordre des σ_i
- **Def**^o : $AH_\Sigma^m = \frac{1}{m!} \sum_{\nu \in \mathfrak{S}_m} (-1)^\nu L_{\omega_{\nu(1)} \dots \omega_{\nu(m)}} = \mathbf{Alt}(L_{\omega_1 \omega_2 \dots \omega_m})$
 - $\mathcal{S}(AH_\Sigma^m) = \mathbf{Alt}\left(\otimes_{k=1}^m \omega_k\right) = \omega_1 \wedge \dots \wedge \omega_m \in \wedge^m \mathcal{H}_\Sigma$
 - AH_Σ^m est canonique (au signe près)

$$\left(\mathbf{HLog}^{r-2}\right) \quad \sum_{i=1}^{\kappa} \epsilon_i \mathbf{AH}_{\Sigma_i}^{r-2}(\phi_i) = 0$$

- Une identité \mathbf{HLog}^{r-2} pour chaque del Pezzo $\mathbf{dP}_d = \mathbf{X}_r$ ($d = 9 - r$)

[d = 6] \mathbf{dP}_6 unique, $\mathbf{AH}_{\Sigma_i}^1 = \mathbf{Log} \quad (\forall i)$

$$\left(\mathbf{HLog}^1\right) \quad \mathbf{Log}(\mathbf{x}) - \mathbf{Log}(\mathbf{y}) - \mathbf{Log}(\mathbf{x}/\mathbf{y}) = 0$$

[d = 5] \mathbf{dP}_5 unique, $\mathbf{AH}_{\Sigma_i}^2 = \frac{1}{2}(\mathbf{L}_{01} - \mathbf{L}_{10}) = \mathbf{R} \quad (\forall i)$

$$\left(\mathbf{HLog}^2\right) \quad \sum_{i=1}^5 \epsilon_i \mathbf{R}(\phi_i) = 0 \quad (\mathcal{A}b)$$

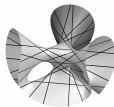
$$\left(\mathbf{HLog}^{r-2}\right) \quad \sum_{i=1}^{\kappa} \epsilon_i \mathbf{AH}_{\Sigma_i}^{r-2}(\phi_i) = 0$$

[d = 4] \mathbf{dP}_4 ∞^2 moduli \rightsquigarrow ∞^2 identités \mathbf{HLog}^3

$$\begin{aligned} \mathbf{AH}_1^3(x) + \mathbf{AH}_2^3\left(\frac{1}{y}\right) + \mathbf{AH}_3^3\left(\frac{y}{x}\right) + \dots \\ \dots + \mathbf{AH}_9^3\left(\frac{y(x-b)}{x(y-a)}\right) + \mathbf{AH}_{10}^3\left(\frac{a(b-x)}{by-ax}\right) = 0 \end{aligned}$$

[d = 3] \mathbf{dP}_3 = surface cubique dans \mathbb{P}^3 \rightsquigarrow ∞^4 identités \mathbf{HLog}^4


$$\sum_{i=1}^{27} \mathbf{AH}_i^4(\phi_i) = 0$$

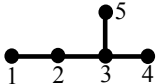


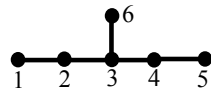
Surfaces de del Pezzo : groupe de Weyl

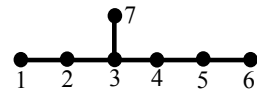
- $dP_d = X_r = \mathbf{Bl}_{\rho_1, \dots, \rho_r}(\mathbb{P}^2)$
- $\text{Pic}(dP_d) = \mathbb{Z} \mathbf{h} \oplus \left(\bigoplus_{i=1}^r \mathbb{Z} \ell_i \right) \simeq \mathbb{Z}^{r+1}$
 \cup
 $\mathbf{K}^\perp = \langle \rho_1, \dots, \rho_r \rangle$ $\rho_i = \ell_i - \ell_{i+1} \quad (i = 1, \dots, r-1)$
 $\rho_r = \mathbf{h} - \ell_1 - \ell_2 - \ell_3$
- $-(\cdot, \cdot) + \{ \rho_i \}_{i=1}^r \rightsquigarrow$ Système de racines $E_r \subset R_r = \mathbf{K}^\perp \otimes \mathbb{R}$
- Pour toute racine ρ : $s_\rho : R_r \longrightarrow R_r$ (reflexion orthog.)
 $\delta \longmapsto \delta + (\delta, \rho) \rho$
- $W_r = \langle s_{\rho_1}, \dots, s_{\rho_r} \rangle \subset O(R_r)$: groupe de Weyl de type E_r

Surfaces de Del Pezzo : diagrammes de Dynkin

$$E_4 = A_4$$


$$E_5 = D_5$$


$$E_6$$


$$E_7$$


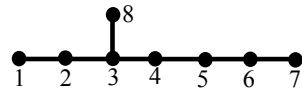
$$E_8$$


Figure – Dynkin diagram E_r (k stands for ρ_k for any $k = 1, \dots, r$)

Identité $\sum_i \mathbf{A}H_{\Sigma_i}^{r-2}(\phi_i) = 0$

- Droites dans X_r : $\mathcal{L}_r \simeq \left\{ \ell \in \mathbf{Pic}(X_r) \mid (\ell \cdot \mathbf{K}) = \ell^2 = -1 \right\} = \mathbf{W}_r \cdot \ell_1$
- “Diviseur des droites” : $L_r = \cup_{\ell \in \mathcal{L}_r} \ell \subset X_r$
- Fib^o en coniques $\rightsquigarrow \Sigma_i = \{ \sigma_i^k \}_{k=1}^{r-2} = \{ \sigma \mid \phi_i^{-1}(\sigma) = \ell_i + \ell'_i \}$
 $\phi_i : X_r \rightarrow \mathbb{P}^1$
- $\phi_i^* \left(\frac{dz}{z - \sigma_i^k} \right) = \frac{d\phi_i}{\phi_i - \sigma_i^k} \in \mathbf{H}^0(X_r, \Omega_{X_r}^1(\text{Log } L_r)) = \mathbf{H}_{X_r}$
- $\mathcal{S}(\mathbf{A}H_{\Sigma_i}^{r-2}(\phi_i)) = \phi_i^* \mathcal{S}(\mathbf{A}H_{\Sigma_i}^{r-2}) = \wedge_{k=1}^{r-2} \left(\frac{d\phi_i}{\phi_i - \sigma_i^k} \right) \in \wedge^{r-2} \mathbf{H}_{X_r}$
- $\sum_i \mathbf{A}H_{\Sigma_i}^{r-2}(\phi_i) = 0 \iff \sum_i \wedge_{k=1}^{r-2} \left(\frac{d\phi_i}{\phi_i - \sigma_i^k} \right) = 0 \quad \text{ds} \quad \wedge^{r-2} \mathbf{H}_{X_r}$

$$\bullet \quad \sum_i \mathbf{A} \mathbf{H}_{\Sigma_i}^{r-2}(\phi_i) = 0 \iff \sum_i \wedge_{k=1}^{r-2} \left(\frac{d\phi_i}{\phi_i - \sigma_i^k} \right) = 0 \quad \text{ds} \quad \wedge^{r-2} \mathbf{H}_{X_r}$$

$$\bullet \quad \mathbf{H}_{X_r} = \mathbf{H}^0 \left(\Omega_{X_r}^1(\text{Log } L_r) \right) \xrightarrow{\oplus \ell \text{Res}_\ell} \mathbb{C} \mathcal{L}_r \quad \text{linéaire injective}$$

$$\bullet \quad \rightsquigarrow \wedge^{r-2} \mathbf{H}_{X_r} \hookrightarrow \wedge^{r-2} \mathbb{C} \mathcal{L}_r \longleftarrow \mathbf{W}(E_r)\text{-représentation}$$

$$\bullet \quad \text{Vu dans } \wedge^{r-2} \mathbb{C} \mathcal{L}_r, \text{ l'élément } \sum_i \wedge_{k=1}^{r-2} \left(\frac{d\phi_i}{\phi_i - \sigma_i^k} \right)$$

- est $\mathbf{W}(E_r)$ -invariant
- se transforme comme la signature
- est nul !

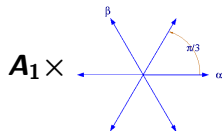
$$\implies \text{ On a donc } \sum_i \mathbf{A} \mathbf{H}_{\Sigma_i}^{r-2}(\phi_i) = 0$$

□

Logarithme

$$\text{Log}(x) + \text{Log}(y) - \text{Log}(xy) = 0$$

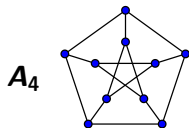
$$dP_6 = \text{BI}_3(\mathbb{P}^2) \subset \mathbb{P}^6$$



Dilogarithme

$$R(x) - R(y) \cdots + R\left(\frac{x(1-y)}{y(1-x)}\right) = 0$$

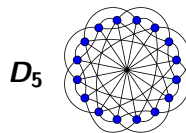
$$dP_5 = \text{BI}_4(\mathbb{P}^2) \subset \mathbb{P}^5$$



Hyperlog de poids 3

$$\sum_{i=1}^{10} \text{AH}_i^3(U_i(x, y)) = 0$$

$$dP_4 = Q_1 \cap Q_2 \subset \mathbb{P}^4$$



Hyperlog de poids 4

$$\sum_{i=1}^{27} \text{AH}_i^4(U_i(x, y)) = 0$$

$$dP_3 =$$

