MAT554 – Evolution equations

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Introduction

The purpose of this course is to present the one-parameter semigroup theory in relation with the resolution of several standard linear partial differential equations, like the heat equation and the Klein-Gordon equation. The last three Chapters present applications of this theory to nonlinear variants of these equations together with some qualitative results on the long time behavior of the solutions.

The present text is largely inspired by the books [6, 12] and by the first Chapters of the book [7], with several simplifications and omissions to make the content more adequate to our goals and to the expected knowledge of the students. The author has also used [5, 8, 10] to prepare this course.

Several appendices, listed below, give a *summary* on some prerequisites to follow the course. This may be useful to refresh memory on specific issues, but it is strongly advised to study other references, for instance the ones mentionned below.

A. Elements of topology and functional analysis. See [3, 4, 14, 15].

B. Lebesgue integral and L^p spaces. See [1, 2].

C. Sobolev spaces. See [1, 8, 9, 11, 13].

D. Ordinary differential equations. See [15].

Contents

Chapter 1

The finite-dimensional case

A large part of this course is devoted to the study of applications $S(t)$ for $t \geq 0$ satisfying the equation

$$\begin{cases} S(t+s) = S(t)S(s) & \text{for all } t, s \geq 0, \\ S(0) = 1. \end{cases} \tag{1.1}$$

In this Chapter, we concentrate on the scalar case $S : [0, \infty) \to \mathbf{C}$ and the matrix case $S : [0, \infty) \to \mathrm{M}_n(\mathbf{C})$.

1.1 Scalar case

It is obvious that for any $a \in \mathbf{C}$, the exponential function $t \mapsto \mathrm{e}^{ta}$ satisfies (1.1). Conversely, the following result states that exponential functions are the only continuous maps $[0, \infty) \to \mathbf{C}$ satisfying (1.1).

Theorem 1.1. *Let $S : [0, \infty) \to \mathbf{C}$ be a* continuous function *satisfying* (1.1). *Then there exists a unique $a \in \mathbf{C}$ such that $S(t) = \mathrm{e}^{ta}$ for all $t \geq 0$.*

Remark 1.2. Recall that exponential functions are differentiable and satisfy the differential equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} S(t) = aS(t) & \text{for } t \geq 0, \\ S(0) = 1. \end{cases} \tag{1.2}$$

The fact that a continuous function satisfying (1.1) has to be differentiable and to satisfy (1.2) is actually a part of the proof of the Theorem.

Proof. Reduction to (1.2). Let S satisfy the assumptions of the Theorem. The function V defined by

$$V(t) = \int_0^t S(s)\,\mathrm{d}s, \quad t \geq 0$$

is differentiable and satisfies $\frac{\mathrm{d}}{\mathrm{d}t} V(t) = S(t)$. This implies

$$\lim_{t \downarrow 0} \frac{V(t)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} V(0) = S(0) = 1.$$

Therefore, for some $t_0 > 0$, $V(t_0) \neq 0$. We use the assumption (1.1) to write, for any $t \geq 0$,

$$\begin{aligned} S(t) &= V(t_0)^{-1}V(t_0)S(t) = V(t_0)^{-1}\int_0^{t_0} S(t+s)\,\mathrm{d}s \\ &= V(t_0)^{-1}\int_t^{t+t_0} S(s)\,\mathrm{d}s = V(t_0)^{-1}(V(t+t_0)-V(t)). \end{aligned}$$

Hence, S is differentiable, with derivative

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}S(t) &= \lim_{h\downarrow 0}\frac{S(t+h)-S(t)}{h} \\ &= \lim_{h\downarrow 0}\frac{S(h)-S(0)}{h}S(t) = \frac{\mathrm{d}}{\mathrm{d}t}S(0)S(t). \end{aligned}$$

This shows that S satisfies (1.2) with $a = \frac{\mathrm{d}}{\mathrm{d}t}S(0)$.

Uniqueness of the solution of (1.2). Define $T : [0,t] \mapsto \mathbf{C}$ by

$$T(s) = \mathrm{e}^{sa}S(t-s) \quad \text{for } 0 \leq s \leq t,$$

for some fixed $t > 0$. Then, T is differentiable and satisfies $\frac{\mathrm{d}}{\mathrm{d}s}T(s) = 0$. This implies that $T(t) = T(0)$ and thus $\mathrm{e}^{ta} = S(t)$. □

1.2 Matrix semigroup

1.2.1 Matrix-valued exponential

We denote by $\mathrm{M}_n(\mathbf{C})$ the space of complex $n \times n$ matrices with complex entries. A *linear dynamical system on* $\mathrm{M}_n(\mathbf{C})$, or *matrix semigroup* is a function

$$S : [0,\infty) \to \mathrm{M}_n(\mathbf{C})$$

satisfying the equation

$$\begin{cases} S(t+s) = S(t)S(s) & \text{for all } t,s \geq 0, \\ S(0) = I. \end{cases} \tag{1.3}$$

Given any initial state $g \in \mathrm{M}_n(\mathbf{C})$, the *evolution* of this state through this system is given by $u(t) = S(t)g$ and $\{S(t)g : t \geq 0\}$ is called the *orbit* of g under S. As a key consequence of (1.3) on the evolution, we observe that an initial state g arrives after time $t+s$ at the same state as the initial state $h = S(s)g$ after time t.

We wish to extend Theorem 1.1 to this context. First, we recall the notion of matrix-valued exponential function.

Definition 1.3. For any $A \in \mathrm{M}_n(\mathbf{C})$ and $t \in \mathbf{R}$, the *matrix exponential* e^{tA} is defined by

$$\mathrm{e}^{tA} = \sum_{k=0}^{\infty}\frac{t^kA^k}{k!}. \tag{1.4}$$

As a finite dimensional vector space, all norms on $\mathrm{M}_n(\mathbf{C})$ are equivalent, and we may equip it with any norm $\|\cdot\|$. It is also clear that the partial sums of the series in (1.4) form a Cauchy sequence for any $t \geq 0$, and $\mathrm{M}_n(\mathbf{C})$ being complete, the series converges and satisfies the inequality, for all $t \geq 0$,

$$\left\|\mathrm{e}^{tA}\right\| \leq \mathrm{e}^{t\|A\|}.$$

Here, $\|\cdot\|$ denotes the norm for matrices corresponding to the norm $\|\cdot\|$:

$$\|A\| = \sup_{g\in\mathbf{C}^n, \|g\|=1} \|Ag\|.$$

Moreover, we have the following properties.

Proposition 1.4. *For any $A \in \mathrm{M}_n(\mathbf{C})$, the map*

$$t \in [0,\infty) \mapsto \mathrm{e}^{tA} \in \mathrm{M}_n(\mathbf{C})$$

is continuous and satisfies

$$\begin{cases} \mathrm{e}^{(t+s)A} = \mathrm{e}^{tA}\mathrm{e}^{sA} & \textit{for all } t, s \geq 0, \\ \mathrm{e}^{0A} = I. \end{cases} \tag{1.5}$$

Proof. By the convergence of the series $\sum_{k=0}^{\infty} \frac{1}{k!} t^k \|A\|^k$, one justifies the following computation (using the binomial expansion)

$$\left(\sum_{j=0}^{\infty} \frac{t^j A^j}{j!}\right)\left(\sum_{l=0}^{\infty} \frac{s^l A^l}{l!}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{t^{n-k}s^k A^n}{(n-k)!k!} = \sum_{n=0}^{\infty} \frac{(t+s)^n A^n}{n!} = \mathrm{e}^{(t+s)A}.$$

This proves (1.5).

Next, we see that for any $t, h \in \mathbf{R}$,

$$\mathrm{e}^{(t+h)A} - \mathrm{e}^{tA} = \mathrm{e}^{tA}\left(\mathrm{e}^{hA} - I\right)$$

and thus

$$\|\mathrm{e}^{(t+h)A} - \mathrm{e}^{tA}\| \leq \|\mathrm{e}^{tA}\|\|\mathrm{e}^{hA} - I\|.$$

To prove the continuity, we only have to prove that $\lim_{h\to 0} \|\mathrm{e}^{hA} - I\| = 0$. This follows from the bound

$$\|\mathrm{e}^{hA} - I\| = \left\|\sum_{k=1}^{\infty} \frac{h^k A^k}{k!}\right\| \leq \sum_{k=1}^{\infty} \frac{|h|^k \|A\|^k}{k!} = \mathrm{e}^{|h|\|A\|} - 1$$

and the continuity of $x \mapsto \mathrm{e}^x$. $\square$

Remark 1.5. More generally, if A and B commute, then $\mathrm{e}^{A+B} = \mathrm{e}^A\mathrm{e}^B$.

We can rephrase these properties as follows: *the map $t \in [0,\infty) \mapsto \mathrm{e}^{tA} \in \mathrm{M}_n(\mathbf{C})$ is a homomorphism from the additive semigroup* $([0,\infty), +)$ *into the multiplicative semigroup* $(\mathrm{M}_n(\mathbf{C}), \cdot)$. In particular, we introduce the following terminology.

Definition 1.6. We call $(\mathrm{e}^{tA})_{t\geq 0}$ the (one-parameter) *semigroup* generated by the matrix $A \in \mathrm{M}_n(\mathbf{C})$.

Remark 1.7. In this definition, there is no need to restrict the time to $t \in [0,\infty)$. We call the $(\mathrm{e}^{tA})_{t\in\mathbf{R}}$ the one-parameter *group* generated by the matrix $A \in \mathrm{M}_n(\mathbf{C})$.

1.2.2 Examples

1. The semigroup generated by the diagonal matrix $A = \mathrm{diag}(a_1, \dots, a_n)$ is given by
$$\mathrm{e}^{tA} = \mathrm{diag}\left(\mathrm{e}^{ta_1}, \dots, \mathrm{e}^{ta_2}\right).$$

2. Define the $k \times k$ *Jordan block*
$$J_{k,\lambda} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix}$$
with eigenvalue $\lambda \in \mathbf{C}$. Decompose A into the sum $A = D + N$, where $D = \lambda I$. Note that $N^k = 0$, and
$$\mathrm{e}^{tN} = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \dots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \dots & \frac{t^{k-2}}{(k-2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & t \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$
Since D and N commute, we obtain $\mathrm{e}^{tA} = \mathrm{e}^{t\lambda}\mathrm{e}^{tN}$.

3. Recall that any matrix $A \in \mathrm{M}_n(\mathbf{C})$ is similar to a block diagonal matrix
$$J = \begin{pmatrix} J_{k_1,\lambda_1} & 0 & \dots & 0 \\ 0 & J_{k_2,\lambda_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_{k_p,\lambda_p} \end{pmatrix}$$
called the *Jordan normal form* of A. In the formula above, several λ_j can have the same value.

4. The Jordan normal form and the next lemma relating the exponentials of similar matrices allow to reduce to cases (1) and (2) starting with any matrix.

   Lemma 1.8. *Let $A \in \mathrm{M}_n(\mathbf{C})$ and take an invertible matrix $P \in \mathrm{M}_n(\mathbf{C})$. Then, the semigroup generated by the matrix $B = P^{-1}AP$ is given by*
$$\mathrm{e}^{tB} = P^{-1}\mathrm{e}^{tA}P.$$

   Proof. Observe that for any k, $B^k = P^{-1}A^kP$ and thus
$$\begin{aligned} \mathrm{e}^{tB} &= \sum_{k=0}^{\infty} \frac{t^k B^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k P^{-1} A^k P}{k!} \\ &= P^{-1}\left(\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}\right) P = P^{-1}\mathrm{e}^{tA}P. \end{aligned}$$
□

1.2.3 Differential equation

Proposition 1.9. *Let $S(t) = \mathrm{e}^{tA}$ for some $A \in \mathrm{M}_n(\mathbf{C})$. Then the function $S : [0,\infty) \to \mathrm{M}_n(\mathbf{C})$ is differentiable and satisfies the differential equation*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} S(t) = AS(t) & \text{for } t \geq 0, \\ S(0) = I. \end{cases} \tag{1.6}$$

Conversely, every differentiable function $S : [0,\infty) \to \mathrm{M}_n(\mathbf{C})$ satisfying (1.6) *is of the form $S(t) = \mathrm{e}^{tA}$ for some $A \in \mathrm{M}_n(\mathbf{C})$. Finally, it holds $\frac{\mathrm{d}}{\mathrm{d}t} S(0) = A$.*

Proof. We start by showing that $S(t) = \mathrm{e}^{tA}$ satisfies (1.6). By (1.5), it holds, for any $t, \delta \in \mathbf{R}$,

$$\frac{S(t+\delta) - S(t)}{\delta} = \frac{S(t) - I}{\delta} S(t).$$

Thus, it is sufficient to prove that $\lim_{\delta \to 0} \frac{S(t)-I}{\delta} = A$. This follows from the estimate

$$\begin{aligned} \left\| \frac{S(t) - I}{\delta} - A \right\| = \left\| \sum_{k=2}^{\infty} \frac{\delta^{k-1} A^k}{k!} \right\| &\leq \sum_{k=2}^{\infty} \frac{|\delta|^{k-1} \|A\|^k}{k!} \\ &\leq \frac{\mathrm{e}^{|\delta| \|A\|} - 1}{|\delta|} - \|A\| \to 0 \quad \text{as } \delta \to 0. \end{aligned}$$

The remaining assertions are proved as in Theorem 1.1, replacing the complex number a by the matrix A. ☐

1.2.4 Characterization of continuous semigroups on $\mathbf{C}^n$

Theorem 1.10. *Let $S : [0,\infty) \to \mathrm{M}_n(\mathbf{C})$ be a continuous function satisfying* (1.3)*. Then, there exists $A \in \mathrm{M}_n(\mathbf{C})$ such that*

$$S(t) = \mathrm{e}^{tA} \quad \text{for all } t \geq 0.$$

Proof. Since S is continuous and $S(0) = I$ is invertible, the matrices

$$V(t_0) = \int_0^{t_0} S(s)\,\mathrm{d}s$$

are invertible for sufficiently small $t_0 > 0$. Now, we follow the computations of the proof of Theorem 1.1. ☐

1.2.5 Asymptotic and spectral properties

Definition 1.11. We say that a continuous one-parameter semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ is *bounded* if there exists $M \geq 1$ such that for all $t \geq 0$, $\|\mathrm{e}^{tA}\| \leq M$.

We say that a continuous one-parameter semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ is *stable* if

$$\lim_{t\to\infty} \|\mathrm{e}^{tA}\| = 0.$$

The stability property is directly related to the spectral properties of A.

Theorem 1.12. *The following assertions are equivalent.*

1. *The semigroup* $(\mathrm{e}^{tA})_{t\geq 0}$ *is stable.*
2. *All eigenvalues of* A *have negative real part.*

We also have the following related result.

Theorem 1.13. *The following assertions are equivalent.*

1. *The semigroup* $(\mathrm{e}^{tA})_{t\geq 0}$ *is bounded.*
2. *All eigenvalues of* A *have nonpositive real part and all eigenvalues with zero real part have trivial Jordan blocks* ($k_j = 1$)*.*

1.3 Exercises for Chapter 1

Exercise 1.1. Compute the exponential of the following matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Exercise 1.2. Prove Theorem 1.12.

Exercise 1.3. Prove Theorem 1.13.

Exercise 1.4. A semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ for $A \in \mathrm{M}_n(\mathbf{C})$ is said to be *hyperbolic* if there exists a direct decomposition $\mathbf{C}^n = X_s \oplus X_u$ where X_s and X_u are vector subspaces invariant by A, and constants $M \geq 1$ and $\gamma > 0$ such that

$$\begin{aligned} &\forall x \in X_s,\ t \geq 0, \quad \|\mathrm{e}^{tA}x\| \leq M\mathrm{e}^{-\gamma t}\|x\|, \\ &\forall y \in X_u,\ t \geq 0, \quad \|\mathrm{e}^{tA}y\| \geq M\mathrm{e}^{\gamma t}\|y\|. \end{aligned}$$

Prove the equivalence of the following assertions:

1. The semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ is hyperbolic.
2. Any eigenvalue of A has a non zero real part.

Exercise 1.5. For $t \geq 0$, we consider the *time-dependent* matrix $A(t)$ defined by

$$A(t) = \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix}$$

1. Compute explicitly $\exp\left(\int_0^t A(s)\,\mathrm{d}s\right)$.
2. Compute explicitly the solution $U(t)$ of

$$\begin{cases} U'(t) = A(t)U(t) & t \geq 0, \\ U(0) = I. \end{cases}$$

3. Compare.

Chapter 2

Uniformly continuous *versus* strongly continuous semigroups

Let $(X, \|\cdot\|)$ be a Banach space over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$.

2.1 Uniformly continuous operator semigroups

2.1.1 Definitions

Definition 2.1. A family $(S(t))_{t\geq 0}$ of bounded linear operators on a Banach space X is called a (one-parameter) *semigroup on* X if it satisfies

$$\begin{cases} S(t+s) = S(t)S(s) & \text{for all } t, s \geq 0, \\ S(0) = I. \end{cases} \tag{2.1}$$

If (2.1) holds for all $t, s \in \mathbf{R}$, $(S(t))_{t\in\mathbf{R}}$ is called a (one-parameter) *group on* X.

The interpretation of (2.1) as the property of a dynamical system is the same as in §1.2.1.

As in the previous chapter, the typical examples of (one-parameter) semigroups of operators are operator-valued exponential functions, as defined in the following proposition.

Proposition 2.2. *For any $A \in \mathcal{L}(X)$ the series $\sum_{k\geq 0} \frac{1}{k!} A^k$ defines a bounded linear operator, denoted by e^A, which satisfies $\|\mathrm{e}^A\| \leq \mathrm{e}^{\|A\|}$. Moreover,*

1. *If A, $B \in \mathcal{L}(X)$ commute, then $\mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B$.*
2. *For fixed $A \in \mathcal{L}(X)$, the function $t \mapsto \mathrm{e}^{tA}$ belongs to $\mathcal{C}^1(\mathbf{R}, \mathcal{L}(X))$ and we have*
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = \mathrm{e}^{tA}A = A\mathrm{e}^{tA}.$$
for all $t \in \mathbf{R}$.

3. *For any $T > 0$ and any $g \in X$, $u(t) = \mathrm{e}^{tA}g$ is the unique solution in $\mathcal{C}^1([0,T],X)$ of the following problem:*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t), & \textit{for all } t \in [0,T], \\ u(0) = g. \end{cases}$$

Proof. The proofs of (1) and (2) are the same as the ones of Proposition 1.9. We prove (3). We know by (2) that $u(t) = \mathrm{e}^{tA}g$ is a solution. Let v be another solution on some time interval $[0,T]$ where $T > 0$, and set $z(t) = \mathrm{e}^{-tA}v(t)$. Then, we have

$$z'(t) = \mathrm{e}^{-tA}(Av(t)) - A(\mathrm{e}^{-tA}v(t)) = 0,$$

which implies $z(t) = z(0) = g$, and so $v(t) = \mathrm{e}^{tA}g$. □

Now, we ask whether the analogue of the characterization Theorem 1.10 holds for *continuous* semigroups of bounded linear operators. The answer depends on the notion of continuity.

Definition 2.3. A one-parameter semigroup $(S(t))_{t\geq 0}$ on a Banach space X is called *uniformly continuous* if

$$t \in [0,\infty) \mapsto S(t) \in \mathcal{L}(X)$$

is continuous *with respect to the uniform operator topology on $\mathcal{L}(X)$.*

With this rather restrictive notion of continuity, we have the following analogue of Theorem 1.10.

Theorem 2.4. *Every uniformly continuous semigroup $(S(t))_{t\geq 0}$ on a Banach space X is of the form $S(t) = \mathrm{e}^{tA}$, $t \geq 0$, for some bounded operator $A \in \mathcal{L}(X)$, called* the generator *of S.*

Proof. The proof follows the same arguments as in the scalar and matrix-valued cases. Define for $t \geq 0$,

$$V(t) = \int_0^t S(s)\,\mathrm{d}s.$$

Since

$$\frac{1}{t}V(t) - S(0) = \frac{1}{t}\int_0^t (S(s) - S(0))\,\mathrm{d}s,$$

we see that $\frac{1}{t}V(t)$ converges in norm to $S(0) = I$ as $t \downarrow 0$. In particular, using Lemma A.39, the operator $V(t)$ is invertible for $t > 0$ small enough. As in the proof of Theorem 1.1, we compute

$$S(t) = V(t_0)^{-1}(V(t+t_0) - V(t)).$$

In particular, $t \mapsto S(t)$ is differentiable and using

$$\frac{S(t+h) - S(t)}{h} = \frac{S(h) - S(0)}{h}S(t) \underset{h\downarrow 0}{\longrightarrow} \left(\frac{\mathrm{d}}{\mathrm{d}t}S(0)\right)S(t)$$

it satisfies $\frac{\mathrm{d}}{\mathrm{d}t}S(t) = AS(t)$, where $A = \frac{\mathrm{d}}{\mathrm{d}t}S(0)$. By Proposition 2.2, it follows that $S(t) = \mathrm{e}^{tA}$. □

2.1.2 Asymptotic behavior

Definition 2.5. A semigroup $(S(t))_{t\geq 0}$ on a Banach space X is called *uniformly exponentially stable* if there exist constants $\gamma > 0$ and $M \geq 1$ such that $\|S(t)\| \leq M\mathrm{e}^{-\gamma t}$, for all $t \geq 0$.

Proposition 2.6. *For uniformly continuous semigroup $(S(t))_{t\geq 0}$ the following assertions are equivalent.*

1. *$(S(t))_{t\geq 0}$ is uniformly exponentially stable.*
2. *$\lim_{t\to\infty} \|S(t)\| = 0$.*
3. *There exists $t_0 > 0$ such that $\|S(t_0)\| < 1$.*
4. *There exists $t_1 > 0$ such that $\mathrm{r}(S(t_1)) < 1$, where $\mathrm{r}(S(t))$ is the spectral radius of $S(t)$ (see (A.2)).*

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are obvious. Moreover, $\mathrm{r}(S(t_0)) \leq \|S(t_0)\|$ says that (3) $\Longrightarrow$ (4).

Since $\mathrm{r}(S(t_1)) = \lim_{n\to\infty} \|S(nt_1)\|^{\frac{1}{n}}$ (see Proposition A.41), we see that (4) implies (3).

Now, we prove (3) $\Longrightarrow$ (1). Let $\alpha = \|S(t_0)\| < 1$ and $M = \sup_{0\leq s\leq t_0} \|S(s)\|$. For any $t \geq 0$, decompose $t = kt_0 + s$ with $s \in [0, t_0]$. Then,

$$\begin{aligned}\|S(t)\| &\leq \|S(s)\| \cdot \|S(kt_0)\| \leq M\|S(t_0)\|^k \\ &\leq M\alpha^k \leq M\mathrm{e}^{\frac{t-t_0}{t_0}\log\alpha} = \frac{M}{\alpha}\mathrm{e}^{t\frac{\log\alpha}{t_0}},\end{aligned}$$

where we have used $k \geq \frac{t-t_0}{t_0}$. □

For the reader information, we state without proof the *spectral mapping theorem.*

Theorem 2.7. *For every uniformly continuous semigroup $(S(t))_{t\geq 0}$, with generator $A \in \mathcal{L}(X)$, it holds, for all $t \geq 0$,*

$$\sigma(S(t)) = \mathrm{e}^{t\sigma(A)} = \big\{\mathrm{e}^{t\lambda} : \lambda \in \sigma(A)\big\}.$$

In particular, this implies the following result.

Theorem 2.8. *For the uniformly continuous semigroup $(S(t))_{t\geq 0}$ with generator A, assertions (1)-(4) of Proposition 2.6 are equivalent to $\Re\lambda < 0$ for all $\lambda \in \sigma(A)$.*

2.2 Strongly continuous semigroups

2.2.1 Definition and examples

Example 2.9. Let $X = L^2(\mathbf{R})$ and for all $t \geq 0$, define the *translation semigroup* $S(t) = \tau_t$, where for all $g \in X$,

$$\tau_y g(x) = g(x-y) \quad \text{for all } x, y \in \mathbf{R}.$$

We check that S is indeed a semigroup on X in the sense of Definition 2.1.

Now, for any $t > 0$, $x \in \mathbf{R}$, let

$$g_t(x) = t^{-\frac{1}{2}} \mathbf{1}_{[t,2t]}(x)$$

so that

$$\|g_t\|_{L^2} = 1, \quad \|S(t)g_t - g_t\|_{L^2} = \sqrt{2} \quad \text{and so} \quad \|S(t) - I\|_{\mathcal{L}(L^2)} \geq \sqrt{2}.$$

This shows that the map $t \in [0, \infty) \mapsto S(t) \in (\mathcal{L}(L^2), \|\cdot\|_{\mathcal{L}(L^2)})$ is not continuous at $t = 0$. However, it is known that given $g \in L^2(\mathbf{R})$ (see Proposition B.27)

$$\lim_{t \downarrow 0} \|\tau_t g - g\|_{L^2} = 0.$$

This leads us to the following definition of *strongly continuous semigroup.*

Definition 2.10. A *strongly continuous (one-parameter) semigroup of linear operators on X* is a map

$$S : [0, \infty) \to \mathcal{L}(X)$$

satisfying the following properties

1. for all $t, s \in [0, \infty)$, $S(t + s) = S(t)S(s)$;
2. $S(0) = I$;
3. for all $g \in X$, the function $t \in [0, \infty) \mapsto S(t)g \in X$ is continuous.

Remark 2.11. Example 2.9 illustrates that property (3) of Definition 2.10 *does not imply in general* that $S : [0, \infty) \to (\mathcal{L}(X), \| \cdot \|_{\mathcal{L}(X)})$ is continuous.

2.2.2 Basic properties of strongly continuous semigroups

Proposition 2.12. *Let S be a* strongly continuous *semigroup on X.*

1. *For all $t_0 > 0$, there exists $M_0 \geq 1$ such that*

$$\sup_{t \in [0,t_0]} \|S(t)\| < M_0.$$

2. *There exist $M \geq 1$ and $\gamma \geq 0$ such that for all $t \in [0, \infty)$,*

$$\|S(t)\| \leq M \mathrm{e}^{\gamma t}.$$

Proof. (1) From (3) of Definition 2.10, for all $g \in X$, it holds

$$\sup_{t \in [0,t_0]} \|S(t)g\| < \infty.$$

Using the Uniform Boundedness Principle (Theorem A.34), there exists $M_0 \geq 1$ such that

$$\sup_{t \in [0,t_0]} \|S(t)\| \leq M_0.$$

(2) Let $M \geq 1$ be such that $\sup_{t\in[0,1]} \|S(t)\| \leq M$. Let any $t \in [0, \infty)$. Let $n = \lfloor t \rfloor$ be the integer part of t and $s = t - n \in [0, 1)$. Then, by Definition 2.1, it holds

$$S(t) = S(s)S(n) = S(s)S(1)^n.$$

In particular, using $n \leq t$,

$$\|S(t)\| \leq \|S(s)\|\|S(1)\|^n \leq M^{1+n} \leq M\mathrm{e}^{t \log M},$$

which implies the desired estimate with $\gamma = \log M$. □

Definition 2.13. A *contraction semigroup* S on X is a semigroup satisfying the additional condition: for all $t \geq 0$, $\|S(t)\| \leq 1$.

The Hille-Yosida-Phillips theory in the next Chapter is stated for contraction semigroups, which does not restrict the generality thanks to the following result.

Proposition 2.14. *Let S be a strongly continuous semigroup on X. There exist $\gamma > 0$ and a norm $\|\cdot\|_\gamma$ on X, equivalent to $\|\cdot\|$, such that $S_\gamma(t) = \mathrm{e}^{-\gamma t}S(t)$ is a contraction semigroup on $(X, \|\cdot\|_\gamma)$.*

Proof. By Proposition 2.12, there exist $M \geq 1$ and $\gamma > 0$ such that $S_\gamma(t) = \mathrm{e}^{-\gamma t}S(t)$ satisfies $\sup_{t\geq 0} \|S_\gamma(t)\| \leq M$. For all $g \in X$, set

$$\|g\|_\gamma = \sup_{t\geq 0} \|S_\gamma(t)g\|.$$

We check that $\|\cdot\|_\gamma$ is a norm on X. Moreover, $\|\cdot\|$ and $\|\cdot\|_\gamma$ are equivalent: for any $g \in X$, $\|g\|_\gamma \geq \|g\|$ is clear from $S_\gamma(0) = I$ and $\|g\|_\gamma \leq M\|g\|$ follows from $\sup_{t\geq 0} \|S_\gamma(t)\| \leq M$.

Last, for any $s \geq 0$, we see that $\|S_\gamma(s)g\|_\gamma = \sup_{t\geq 0} \|S_\gamma(s + t)g\| \leq \|g\|_\gamma$, which proves that $S_\gamma(s)$ is a contraction. □

Now, we claim that the strong continuity of a semigroup is equivalent to an apparently weaker property.

Proposition 2.15. *Let $(S(t))_{t\geq 0}$ be a semigroup on a Banach space X. Assume that the following two properties hold*

1. *There exists $t_0 > 0$ and $M \geq 1$ such that for all $t \in [0, t_0]$, $\|S(t)\| \leq M$;*
2. *There exists a dense set $D \subset X$ such that $\lim_{t\downarrow 0} S(t)g = g$ for all $g \in D$.*

Then $(S(t))_{t\geq 0}$ is a strongly continuous semigroup.

Proof. See Exercise 2.9. □

2.3 Generator of strongly continuous semigroup

Definition 2.16. Let S be a strongly continuous semigroup on X. We call *infinitesimal generator* of S (or simply *generator*) the couple $(D(A), A)$, where $D(A)$ is the set of vectors $g \in X$ such that

$$\frac{S(\delta)g - g}{\delta} \quad \text{has a limit in } X \text{ as } \delta \downarrow 0,$$

and for all $g \in D(A)$,

$$Ag = \lim_{\delta\downarrow 0} \frac{S(\delta)g - g}{\delta}.$$

Remark 2.17. The condition defining the set $D(A)$ in the above definition is exactly the *right-differentiability* of the function $t \mapsto S(t)g$ at $t = 0$, and for $g \in D(A)$,

$$Ag = \frac{\mathrm{d}}{\mathrm{d}t}(S(t)g)_{|t=0^+}.$$

Observe that $A : D(A) \mapsto X$ is a linear map. The above result motivates the introduction of the notion of *linear unbounded operator* in the next Chapter.

Example 2.18. The generator of the translation semigroup introduced in Example 2.9 is given by

$$D(A) = H^1(\mathbf{R}), \quad Ag = \frac{\mathrm{d}}{\mathrm{d}x}g$$

where $\frac{\mathrm{d}}{\mathrm{d}x}g$ is the derivative of g in the distributional sense. See Exercise 2.10.

Proposition 2.19. *Let S be a semigroup on X and $(A, D(A))$ be its generator. Then, for all $t \geq 0$, $D(A)$ is stable by $S(t)$ and for all $g \in D(A)$,*

$$AS(t)g = S(t)Ag.$$

Moreover,

1. *The* domain *$D(A)$ is dense in X;*
2. *The* graph *of A defined by*

$$G(A) = \{(g, w) \in X \times X : g \in D(A) \text{ and } w = Ag\}$$

is closed in $X \times X$.

Proof. Let $t \geq 0$ and $g \in D(A)$. For all $\delta > 0$, it holds

$$\begin{aligned}\frac{S(\delta) - I}{\delta} S(t)g &= \frac{S(t+\delta) - S(t)}{\delta} g \\ &= S(t) \frac{S(\delta) - I}{\delta} g \xrightarrow[\delta \downarrow 0]{} S(t)Ag,\end{aligned}$$

by continuity of $S(t)$. This implies that $S(t)g \in D(A)$ and $AS(t)g = S(t)Ag$.

We need the following lemma to prove the second part of Proposition 2.19.

Lemma 2.20. *For all $\varepsilon > 0$, for all $g \in X$, let*

$$J_\varepsilon g = \frac{1}{\varepsilon} \int_0^\varepsilon S(t)g \,\mathrm{d}t.$$

Then, $J_\varepsilon : X \to X$ is a bounded linear map and $\lim_{\varepsilon \downarrow 0} J_\varepsilon g = g$. Moreover, it holds

1. *For any $\delta \geq 0$, $\varepsilon > 0$,*

$$\frac{S(\delta) - I}{\delta} J_\varepsilon = \frac{S(\varepsilon) - I}{\varepsilon} J_\delta = J_\delta \frac{S(\varepsilon) - I}{\varepsilon}. \tag{2.2}$$

2. *For any $\varepsilon \geq 0$ and $g \in X$,*

$$\int_0^\varepsilon S(t)g \,\mathrm{d}t \in D(A);$$

3. For any $\varepsilon \geq 0$ and $g \in X$,

$$S(\varepsilon)g - g = A\int_0^\varepsilon S(t)g\,\mathrm{d}t;$$

4. For any $\varepsilon \geq 0$ and $g \in D(A)$,

$$S(\varepsilon)g - g = \int_0^\varepsilon S(t)Ag\,\mathrm{d}t.$$

Proof of Lemma 2.20. The fact that $\lim_{\varepsilon\downarrow 0} J_\varepsilon g = g$ follows from $S(0)g = g$ and the continuity of $t \mapsto S(t)g$ at $t = 0$. The map J_ε is clearly linear since $S(t)$ is a linear map. The inequality

$$\|J_\varepsilon g\| \leq \frac{1}{\varepsilon}\int_0^\varepsilon \|S(t)g\|\,\mathrm{d}t,$$

and Proposition 2.12 (1) implies the bound $\|J_\varepsilon g\| \leq C_\varepsilon \|g\|$.

(1) For $\delta > 0$, it holds

$$\begin{aligned}
\frac{S(\delta)-I}{\delta}J_\varepsilon g &= \frac{1}{\varepsilon\delta}\left[S(\delta)\int_0^\varepsilon S(t)g\,\mathrm{d}t - \int_0^\varepsilon S(t)g\,\mathrm{d}t\right] \\
&= \frac{1}{\varepsilon\delta}\left[\int_0^\varepsilon S(\delta+t)g\,\mathrm{d}t - \int_0^\varepsilon S(t)g\,\mathrm{d}t\right] \\
&= \frac{1}{\varepsilon\delta}\left[\int_\delta^{\delta+\varepsilon} S(t)g\,\mathrm{d}t - \int_0^\varepsilon S(t)g\,\mathrm{d}t\right] \\
&= \frac{1}{\varepsilon\delta}\left[\int_\varepsilon^{\delta+\varepsilon} S(t)g\,\mathrm{d}t - \int_0^\delta S(t)g\,\mathrm{d}t\right],
\end{aligned}$$

where the last equality is deduced from Chasles relation: $\int_0^{\delta+\varepsilon} = \int_0^\varepsilon + \int_\varepsilon^{\delta+\varepsilon} = \int_0^\delta + \int_\delta^{\delta+\varepsilon}$. This computation shows that the roles of δ and ε can be changed. We deduce from this computation the identity

$$\frac{S(\delta)-I}{\delta}J_\varepsilon g = \frac{S(\varepsilon)-I}{\varepsilon}J_\delta g.$$

The last identity in (2.2) follows from the fact that J_δ and $S(\varepsilon)$ commute.

(2)-(3). Let $g \in X$ and $\varepsilon > 0$. Passing to the limit $\delta \downarrow 0$ in the identity given by Lemma 2.2, we obtain

$$\frac{S(\delta)-I}{\delta}J_\varepsilon g = \frac{S(\varepsilon)-I}{\varepsilon}J_\delta g \underset{\delta\downarrow 0}{\longrightarrow} \frac{S(\varepsilon)-I}{\varepsilon}g.$$

It follows that $J_\varepsilon g \in D(A)$, with

$$AJ_\varepsilon g = \frac{S(\varepsilon)-I}{\varepsilon}g.$$

(4) Note that the second part of (2.2) and the continuity of J_δ then yield, for all $g \in D(A)$,

$$J_\delta Ag = AJ_\delta g = \frac{S(\delta)-I}{\delta}g.$$

This completes the proof of the lemma. $\square$

We continue the proof of Proposition 2.19. Proof of (1). Since $\lim_{\varepsilon\downarrow 0} J_\varepsilon g = g$, it follows that $D(A)$ is dense in X.

Proof of (2). We use the sequential characterization of closed sets. Let $(g_n, w_n) \in G(A)$ be such that

$$\lim_{n\to\infty} g_n = g, \quad \lim_{n\to\infty} w_n = w,$$

where $(g, w) \in X \times X$. Passing to the limit as $n \to \infty$ in the identity

$$J_\delta w_n = J_\delta A g_n = \frac{S(\delta) - I}{\delta} g_n$$

we obtain

$$J_\delta w = \frac{S(\delta) - I}{\delta} g.$$

Passing to the limit in $\delta \downarrow 0$, this shows that $g \in D(A)$ and $w = Ag$. Thus, $G(A)$ is closed in $X \times X$. $\square$

2.4 Generators of contraction semigroups

Definition 2.21. A *contraction semigroup* $(S(t))_{t\geq 0}$ on X is a strongly continuous semigroup of linear operators on X satisfying in addition

$$\|S(t)\| \leq 1 \quad \text{for all } t \geq 0.$$

Remark 2.22. Recall that by Proposition 2.14, there is no loss of generality to restrict to contraction semigroups.

Proposition 2.23. *Let S be a contraction semigroup on X and let A be its generator. Then, A satisfies the following properties*

1. *For all $g \in D(A)$ and $\lambda \geq 0$, it holds*

$$\|g - \lambda A g\| \geq \|g\|.$$

2. *For all $h \in X$, there exists $g \in D(A)$ such that*

$$g - Ag = h.$$

Remark 2.24. The above remarkable properties of the generator associated to a contraction semigroup motivates the introduction of the notion of *maximal dissipative unbounded operators* in the next Chapter.

Proof. Proof of (1). For all $g \in D(A)$, $\lambda \geq 0$ and $\delta > 0$, using $\|S(\delta)\| \leq 1$, the homogeneity property of the norm $\|\cdot\|$ and the triangle inequality, one sees that

$$\begin{aligned}\left\| g - \lambda \frac{S(\delta)g - g}{\delta} \right\| &= \left\| \left(1 + \frac{\lambda}{\delta}\right) g - \frac{\lambda}{\delta} S(\delta) g \right\| \\ &\geq \left(1 + \frac{\lambda}{\delta}\right) \|g\| - \frac{\lambda}{\delta} \|S(\delta) g\| \geq \|g\|.\end{aligned}$$

Passing to the limit as $\delta \downarrow 0$, since $\lim_{\delta\downarrow 0} \frac{S(\delta)g-g}{\delta} = Ag$ in X, for $g \in D(A)$, we obtain $\|g - \lambda Ag\| \geq \|g\|$.

Proof of (2). We define a bounded linear operator R on X by setting, for all $h \in X$,

$$Rh = \int_0^\infty \mathrm{e}^{-t} S(t)h \,\mathrm{d}t.$$

Since $\|\mathrm{e}^{-t}S(t)h\| \leq \mathrm{e}^{-t}\|h\|$, the integral above has a sense as

$$\int_0^\infty \mathrm{e}^{-t} S(t)h \,\mathrm{d}t = \lim_{T\to\infty} \int_0^T \mathrm{e}^{-t} S(t)h \,\mathrm{d}t,$$

and $\|Rh\| \leq \int_0^\infty \mathrm{e}^{-t}\|S(t)h\| \,\mathrm{d}t \leq \|h\|$. It follows that R is indeed a bounded linear operator with $\|R\| \leq 1$.

Now, we compute, for $h \in X$ and $\delta > 0$,

$$\begin{aligned}
\frac{S(\delta) - I}{\delta} Rh &= \frac{1}{\delta} \int_0^\infty \mathrm{e}^{-t} \left(S(t+\delta)h - S(t)h\right) \,\mathrm{d}t \\
&= \frac{1}{\delta} \int_\delta^\infty \mathrm{e}^{-(t-\delta)} S(t)h - \frac{1}{\delta} \int_0^\infty \mathrm{e}^{-t} S(t)h \,\mathrm{d}t \\
&= \frac{\mathrm{e}^\delta - 1}{\delta} \int_0^\infty \mathrm{e}^{-t} S(t)h \,\mathrm{d}t - \frac{\mathrm{e}^\delta}{\delta} \int_0^\delta \mathrm{e}^{-t} S(t)h \,\mathrm{d}t \\
&= \frac{\mathrm{e}^\delta - 1}{\delta} Rh - \frac{\mathrm{e}^\delta}{\delta} \int_0^\delta \mathrm{e}^{-t} S(t)h \,\mathrm{d}t.
\end{aligned}$$

We observe that $\lim_{\delta\downarrow 0} \frac{\mathrm{e}^\delta - 1}{\delta} = 1$ and

$$\lim_{\delta\downarrow 0} \frac{\mathrm{e}^\delta}{\delta} \int_0^\delta \mathrm{e}^{-t} S(t)h \,\mathrm{d}t = h.$$

Therefore, we have proved that $Rh \in D(A)$ and

$$ARh = Ah - h \implies (I - A)Rh = h,$$

which means that $I - A : D(A) \to X$ is bijective and $R = (I - A)^{-1}$. $\square$

2.5 Exercises for Chapter 2

Exercise 2.1. Let A, B be continuous operators on a Banach space X. Prove that $\sum_{k=0}^\infty \frac{1}{k!} A^k$ defines a continuous linear operator on X which satisfies $\|\mathrm{e}^A\| \leq \mathrm{e}^{\|A\|}$. Prove that if A and B commute, then $\mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B$. Moreover, for a given A, prove that the function $t \mapsto \mathrm{e}^{tA}$ belongs to $\mathcal{C}^\infty(\mathbf{R}, \mathcal{L}(X))$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{tA} = \mathrm{e}^{tA} A = A\mathrm{e}^{tA},$$

for all $t \in \mathbf{R}$.

Exercise 2.2. Let A, B be continuous operators on a Banach space X. Show that if for all $t \in \mathbf{R}$,

$$\mathrm{e}^{t(A+B)} = \mathrm{e}^{tA} \mathrm{e}^{tB}$$

then A and B commute.

Exercise 2.3. Let A be a continuous operator on a Banach space X. We suppose that there exists $t_0 > 0$ such that $\|\mathrm{e}^{t_0 A}\| < 1$. Prove that the semigroup $(\mathrm{e}^{tA})_{t \geq 0}$ is uniformly exponentially stable, *i.e.* there exists $M \geq 1$ and $\gamma > 0$ such that, for all $t \geq 0$,

$$\|\mathrm{e}^{tA}\| \leq M\mathrm{e}^{-\gamma t}.$$

Exercise 2.4. We consider $X = \ell^p(\mathbf{C})$ pour $1 \leq p \leq \infty$, and the right shift operator, represented by the infinite matrix $(a_{i,j})_{i,j \geq 0}$,

$$a_{i,j} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute the spectrum of A. Compute e^{tA} for all $t \geq 0$.

Exercise 2.5 (Uniformly continuous semigroups on a Hilbert space)**.** Let H be a Hilbert space, with associated scalar product $(\cdot \mid \cdot)$. For $A \in \mathcal{L}(H)$, we denote A^* the adjoint of A, i.e. the unique operator satisfying $(Ag|h) = (g|A^*h)$, for any $g, h \in H$.

Let $S(t) = \mathrm{e}^{tA}$ for $t \in \mathbf{R}$ be the uniformly continuous group associated to A.

1. Prove that $S(t)^*$ defines a uniformly continuous group and for all $t \in \mathbf{R}$,

   $$S(t)^* = \mathrm{e}^{tA^*}.$$

   An operator $T \in \mathcal{L}(H)$ is *unitary* if $T^{-1} = T^*$. An operator $A \in \mathcal{L}(H)$ is *skew-adjoint* if $A^* = -A$.

2. Prove that $S(t)$ is unitary for all $t \in \mathbf{R}$ if, and only if A is skew-adjoint.

Exercise 2.6. Let X be a Banach space. Let $t \in [0, \infty) \mapsto A(t) \in \mathcal{L}(X)$ be a $\mathcal{C}^1$ function with values in $\mathcal{L}(X)$. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{A(t)} = \int_0^1 \mathrm{e}^{sA(t)} \left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right) \mathrm{e}^{(1-s)A(t)}\,\mathrm{d}s.$$

Exercise 2.7 (Multiplication semigroups on $\mathcal{C}_0(\mathbf{R}^N)$)**.** Let

$$\mathcal{C}_0(\mathbf{R}^N) = \left\{ f \in \mathcal{C}(\mathbf{R}^N) : \begin{array}{l} \text{for all } \varepsilon > 0, \text{ there exists a compact } K_\varepsilon \subset \mathbf{R}^N \\ \text{such that } |f(x)| \leq \varepsilon \text{ for all } x \in \mathbf{R}^N \setminus K_\varepsilon \end{array} \right\}.$$

We equip $\mathcal{C}_0(\mathbf{R}^N)$ with the sup-norm on $\mathbf{R}^N$

$$\|f\| = \sup_{x \in \mathbf{R}^N} |f(x)|.$$

1. Justify that $(\mathcal{C}_0(\mathbf{R}^N), \|\cdot\|)$ is a Banach space.

   Let $q : \mathbf{R}^N \to \mathbf{C}$ be a continuous function and define for all $t \geq 0$, $f \in \mathcal{C}_0(\mathbf{R}^N)$,

   $S(t)f = \mathrm{e}^{tq}f$ in the sense that for all $x \in \mathbf{R}^N$, $[S(t)f](x) = \mathrm{e}^{tq(x)}f(x)$.

2. Prove that for $t > 0$, $S(t)$ is a bounded linear operator on $\mathcal{C}_0(\mathbf{R}^N)$ if and only if

$$\sup_{x\in\mathbf{R}^N} \Re q(x) < \infty. \tag{$\star$}$$

3. Prove that $(S(t))_{t\geq 0}$ defines a uniformly continuous semigroup on $\mathcal{C}_0(\mathbf{R}^N)$ if and only if q is bounded on $\mathbf{R}^N$.

4. Prove that under assumption $(\star)$, $(S(t))_{t\geq 0}$ defines a strongly continuous semigroup on $\mathcal{C}_0(\mathbf{R}^N)$.

Exercise 2.8. Let X be a Banach space. Let L be a compact set of $\mathbf{R}$ and let $F : L \to \mathcal{L}(X)$. Assume that F is uniformly bounded on L, and assume that there exists a dense set $D \subset X$ such that for any $g \in D$, $t \in L \mapsto F(t)g \in X$ is continuous.

Prove that for any compact set K of X, the map

$$(t, g) \in L \times K \mapsto F(t)g \in X$$

is uniformly continuous.

Exercise 2.9 (Application of the previous exercise)**.** Let $(S(t))_{t\geq 0}$ be a semigroup on a Banach space X. Assume that the following two properties hold

1. There exists $t_0 > 0$ and $M \geq 1$ such that for all $t \in [0, t_0]$, $\|S(t)\| \leq M$;

2. There exists a dense set $D \subset X$ such that $\lim_{t\downarrow 0} S(t)g = g$ for all $g \in D$.

Prove the following:

(a) $\lim_{t\downarrow 0} S(t)g = g$, for all $g \in X$,

(b) $(S(t))_{t\geq 0}$ is a strongly continuous semigroup.

Exercise 2.10 (Translation semigroup)**.** On $L^2(\mathbf{R})$, define the (right-) translation semigroup

$$S(t) = \tau_t, \quad \forall t \geq 0$$

where

$$\tau_y g(x) = g(x - y) \quad \text{for all } x, y \in \mathbf{R}.$$

1. Prove that $(S(t))_{t\geq 0}$ is a strongly continuous semigroup on $L^2(\mathbf{R})$.

2. Prove that the generator $(D(A), A)$ of $(S(t))_{t\geq 0}$ is

$$D(A) = H^1(\mathbf{R}), \quad Ag = g'.$$

Hint: use Exercises C.2-C.4.

Exercise 2.11. Let (D, A) be the generator of a contraction semigroup denoted by $(S(t))_{t\geq 0}$ on X. Define

$$D(A^2) = \{g \in D : Ag \in D\}.$$

1. Prove that for all $g \in D(A^2)$,

$$\|Ag\|^2 \leq 4\|A^2 g\|\|g\|.$$

(**Hint**: Justify and use the formula $S(t)g = g + tAg + \int_0^t (t-s)S(s)A^2 g \,\mathrm{d}s$.)

2. If in addition $(S(t))_{t\in\mathbf{R}}$ is a group of isometries then for all $g \in D(A^2)$,

$$\|Ag\|^2 \leq 2\|A^2 g\|\|g\|.$$

Exercise 2.12. Consider the Banach space $(\mathcal{C}_0(\mathbf{R}), \|\cdot\|_\infty)$. Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on X with generator (D, A). Prove the equivalence of the following statements.

1. For all $f, g \in X$, and all $t \geq 0$, $S(t)(f \cdot g) = S(t)f \cdot S(t)g$.
2. D is a subalgebra of X and for all $f, g \in D$,

$$A(f \cdot g) = Af \cdot g + f \cdot Ag.$$

Chapter 3

Unbounded operators

3.1 Unbounded operators in Banach spaces

Definition 3.1. A *linear unbounded operator* in X is a pair (D, A), where D is a linear subspace of X and A is a linear mapping $D \to X$.

We say that A is *bounded* if there exists $C > 0$ such that

$$\|Ag\| \leq C,$$

for all $g \in \{h \in D : \|h\| \leq 1\}$. Otherwise, A is not bounded.

Remark 3.2. The terminology *unbounded operator* is unfortunate, but it is of general use. In what follows, for simplicity, a linear unbounded operator is just called *linear operator* or *operator*. Note that when one defines an operator, it is absolutely necessary to define its domain.

An element of $\mathcal{L}(X)$ will still be called a *bounded linear operator*.

Definition 3.3. Let (D, A) be an operator. We set

$$\begin{aligned} G(A) &= \{(g, h) \in X \times X : g \in D \text{ and } h = Ag\}, \\ A[D] &= \{h \in X : \text{there exists } g \in D \text{ such that } h = Ag\}, \end{aligned}$$

respectively called the *graph* and the *range* of A.

Remark 3.4. When $D = X$, from the Closed Graph Theorem A.35, $A \in \mathcal{L}(X)$ if and only if $G(A)$ is closed in $X \times X$.

Definition 3.5. An operator (D, A) is said to be *closed* if $G(A)$ is closed in $X \times X$.

3.2 Maximal dissipative operators

Definition 3.6. Let (D, A) be an operator.

1. We say that A is *dissipative* if for all $g \in D$ and all $\lambda \geq 0$,

$$\|(I - \lambda A)g\| \geq \|g\|.$$

2. We say that A is *maximal dissipative* if it is dissipative and for all $h \in X$, there exists $g \in D$ such that $(I - A)g = h$.

Remark 3.7. If A is maximal dissipative, it is clear from the definition that, for any h, there exists a *unique* solution g of $g - Ag = h$. Indeed, if g and $\tilde{g}$ are two such solutions, then $(g - \tilde{g}) - A(g - \tilde{g}) = 0$ and by (1), $\|g - \tilde{g}\| = 0$ so that $g = \tilde{g}$. Moreover, the operator A being dissipative, it holds

$$\|h\| = \|(I - A)g\| \geq \|g\|,$$

and thus, denoting $R = (I - A)^{-1}$, we have $R \in \mathcal{L}(X)$ and $\|R\| \leq 1$.

Proposition 3.8. *Let (D, A) be a maximal dissipative operator on X. Then,*

1. *for any $\lambda \geq 0$, the operator λA is maximal dissipative;*
2. *A is closed.*

Remark 3.9. In practice, assertion (1) of Proposition 3.8 can greatly simplify the verification that a given operator is maximal dissipative.

Proof. Proof of (1). It is clear for any $\lambda > 0$ that λA is dissipative. Given $\lambda > 0$ and $h \in X$, the equation $(I - \lambda A)g = h$ is equivalent to

$$g - Ag = \frac{1}{\lambda}h + \left(1 - \frac{1}{\lambda}\right) g \iff g = F(g),$$

where we have set

$$F(g) = R\left(\frac{1}{\lambda}h + \left(1 - \frac{1}{\lambda}\right) g\right).$$

For $\lambda > \frac{1}{2}$, we have

$$|F(g_1) - F(g_2)| \leq \left|1 - \frac{1}{\lambda}\right| \|g_1 - g_2\| \leq k\|g_1 - g_2\|,$$

where $0 \leq k < 1$. Applying the Banach Fixed-Point Theorem A.26, there exists a solution $g \in X$ of $g = F(g)$. Thus, for any $\lambda > \frac{1}{2}$, the operator λA is maximal dissipative. Iterating this argument n times, the operator λA is maximal dissipative for any $\lambda > 2^{-n}$. Since n is arbitrary, the operator λA is maximal dissipative for any $\lambda > 0$.

Proof of (2). Let $\{g_n\}_{n=0}^{\infty}$ be a sequence of D such that $\lim_{n\to\infty} g_n = g$ and $\lim_{n\to\infty} Ag_n = h$. We need to check that $g \in D$ and $Ag = h$. We have $\lim_{n\to\infty} g_n - Ag_n = g - h$. Since $R \in \mathcal{L}(X)$ and $g_n = R(g_n - Ag_n)$, we obtain by passing to the limit $n \to \infty$, $g = R(g - h)$, which means that $g \in D$ and $g - Ag = g - h$ and thus $Ag = h$. □

Corollary 3.10. *Let (D, A) be a maximal dissipative operator on X. For any $u \in D$, let*

$$\|u\|_D = \|u\| + \|Au\|.$$

Then, $(D, \|\cdot\|_D)$ is a Banach space and $A \in \mathcal{L}(D, X)$.

Remark 3.11. From Proposition 3.8, if A is a maximal dissipative operator, for any $\lambda > 0$, λA is also a maximal dissipative operator. As a consequence, using Remark 3.7 on λA, one can define the operators $R_\lambda \in \mathcal{L}(X)$ and $A_\lambda \in \mathcal{L}(X)$ as in the next definition.

Definition 3.12. Let (D, A) be a maximal dissipative operator on X. For any $\lambda > 0$, we set

$$R_\lambda = (I - \lambda A)^{-1} \quad \text{and} \quad A_\lambda = \frac{1}{\lambda}(R_\lambda - I),$$

respectively called the *resolvent of* A and the *Yosida approximation* of A.

Remark 3.13. From Remarks 3.7 and 3.11, it follows that $R_\lambda \in \mathcal{L}(X)$, $A_\lambda \in \mathcal{L}(X)$ and $\|R_\lambda\| \leq 1$. Moreover, for any $\lambda, \mu > 0$, since $I - \lambda A$ and $I - \mu A$ commute, R_λ, A_λ, R_μ and A_μ all commute. We also observe that

$$A_\lambda = AR_\lambda = R_\lambda A. \tag{3.1}$$

Proposition 3.14. *Let (D, A) be a maximal dissipative operator with* dense domain *D on X. Then,*

1. *For all $g \in X$, $\lim_{\lambda \downarrow 0} \|R_\lambda g - g\| = 0$.*
2. *For all $g \in D$, $\lim_{\lambda \downarrow 0} \|A_\lambda g - Ag\| = 0$.*

Proof. Proof of (1). Let $u \in D$. We have

$$R_\lambda u - u = R_\lambda(u - (I - \lambda A)u) = \lambda R_\lambda Au.$$

Thus, $\|R_\lambda u - u\| = \lambda \|R_\lambda Au\| \leq \lambda \|Au\| \to 0$ as $\lambda \downarrow 0$. For any $u \in X$, since $\|R_\lambda - I\| \leq \|R_\lambda\| + \|I\| \leq 1$, we argue by density.

Proof of (2). Let $u \in D$. From (1), we have

$$\lim_{\lambda \downarrow 0} \|R_\lambda Au - Au\| = 0.$$

By the definition of A_λ, we have $A_\lambda = AR_\lambda = R_\lambda A$ and thus,

$$\|A_\lambda u - u\| = \|R_\lambda Au - Au\|,$$

which implies the desired result. □

3.3 Extrapolation

The next result shows that any maximal dissipative operator (D, A) with dense domain D on X can be extended to a maximal dissipative operator (D_B, B) on a larger space Y, with domain $D_B = X$. This will be useful to formulate the notion of *weak solution* in the next Chapters.

Proposition 3.15. *Let (D, A) be an maximal dissipative operator with dense domain D on X. There exists a Banach space $(Y, \|\cdot\|_Y)$ and a maximal dissipative operator (D_B, B) on Y such that*

1. *$X \subset Y$ with dense embedding;*
2. *for all $g \in X$, $\|g\|_Y = \|Rg\|$, with $R = (I - A)^{-1}$;*
3. *$D_B = X$ with equivalent norms;*
4. *for all $g \in D$, $Bg = Ag$;*

5. *if $g \in X$ satisfies $Bg \in X$, then $g \in D$ and $Bg = Ag$.*

Proof. We define a norm on X by setting, for any $g \in X$,

$$|||g||| = \|Rg\| \leq \|g\|.$$

In general, the norm $|||\cdot|||$ is not equivalent to $\|\cdot\|$. We define Y as the (Banach) completion of the normed space $(X, |||\cdot|||)$ (see §A.4). Then $(Y, |||\cdot|||)$ is a Banach space containing a dense subspace that is isometric with X, which we denote by $X \subset Y$ with dense embedding.

For all $g \in D$, by definition of R we have $RAg = Rg - g$, which implies the bound

$$|||Ag||| = \|Rg - g\| \leq |||Rg||| + \|g\| \leq 2\|u\|.$$

This means that A can be extended to a linear map $\tilde{A} \in \mathcal{L}(X, Y)$. We define the linear operator (D_B, B) on Y by

$$D_B = X, \quad Bg = \tilde{A}g \quad \text{for all } g \in D_B.$$

All the properties (1)-(4) are checked.

Now, we prove that B is maximal dissipative for $(Y, |||\cdot|||)$. Let $\lambda > 0$, $g \in D$ and $h = Rg$, so that

$$h - \lambda Ah = R(g - \lambda Ag).$$

Since A is dissipative, we have

$$|||g - \lambda Ag||| = \|h - \lambda Ah\| \geq \|h\| = |||g|||.$$

By density, and continuity of $\tilde{A}$, we obtain, for all $g \in X$, $|||g - \lambda Bg||| \geq |||g|||$, which means that B is dissipative for $(Y, |||\cdot|||)$.

Let $f \in Y$ and a sequence $\{f_n\}_{n=0}^{\infty}$ of X such that $\lim_{n\to\infty} f_n \to f$ in Y. Let $u_n = Rf_n$. Since $\{f_n\}_{n=0}^{\infty}$is a Cauchy sequence in Y, $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X, which implies the existence of a limit u in X. Since $\tilde{A} \in \mathcal{L}(X, Y)$, passing to the limit in $f_n = u_n - \tilde{A}u_n$ proves that $f = u - \tilde{A}u = u - Bu$. In particular, B is maximal dissipative.

Finally, we prove (5). Let $g \in X$ and assume $Bg \in X$ so that $f = g - Bg \in X$. Since A is maximal dissipative, there exists $h \in D$ with $h - Bh = h - Ah = f$. Thus, $(g - h) - B(g - h) = 0$ and since B is maximal dissipative, we have $g = h \in D$. □

3.4 Unbounded operators in Hilbert spaces

In this section, we work on a real Hilbert space $(H, (\cdot \mid \cdot))$.

3.4.1 Adjoint

Two (unbounded) operators (D, A), $(\tilde{D}, \tilde{A})$ on H are said to be *adjoint* if

$$\text{for all } g \in D,\, h \in \tilde{D}, \quad (Ag \mid h) = (g \mid \tilde{A}h).$$

Without any additional condition on A, it may admit several adjoint operators.

Definition 3.16. Let (D, A) be an operator on G with *dense domain.* We define a subspace of H by setting

$$D^* = \{h \in H : \text{there exists } f \in H \text{ such that for all } g \in D,\ (Ag \mid h) = (g \mid f)\}.$$

For $h \in D^*$, set $A^*h = f$ where f is given above.

Note by density of D in H that only one vector f is associated to a given h.

Proposition 3.17. *For (D, A) an operator on H with* dense domain*, the above definition defines an operator (D^*, A^*), called the* adjoint of (D, A).

Remark 3.18. From the Riesz Representation Theorem, the domain of A^* is given by

$$D^* = \{h \in H : \text{there exists } C > 0, \text{ for all } g \in D,\ |(Ag \mid h)| \leq C\|g\|\}.$$

Moreover, from the definition it holds

$$\text{for all } g \in D,\ h \in D^*, \quad (Ag \mid h) = (g \mid A^*h).$$

Note that the domain D^* of A^* is not necessarily dense. See Exercises.

We equip $H \times H$ of the natural scalar product and we define $J : H \times H \to H \times H$ by

$$J(g, h) = (-h, g).$$

Observe that J is isometric and satisfies $J^2 = -I$. In particular, we check that for all $A \subset H \times H$, it holds $J(A^\perp) = (J(A))^\perp$.

Lemma 3.19. *For (D, A) an operator on H with* dense domain*, it holds*

$$G(A^*) = J(G(A))^\perp, \quad J(G(A^*))^\perp = \overline{G(A)}.$$

In particular, the graph of A^* is closed.

Proof. We see that

$$\begin{aligned}
(g, h) \in J[G(A)]^\perp &\iff \forall f \in D,\ (g \mid -Af) + (h \mid f) = 0 \\
&\iff \forall f \in D,\ (Af \mid g) = (f \mid h) \\
&\iff g \in D^*,\ h = A^*g \iff (g, h) \in G(A^*).
\end{aligned}$$

Since $G(A)^\perp$ is a vector space and $J^2 = -I$, it follows that $J(G(A^*)) = G(A)^\perp$ and thus $J(G(A^*))^\perp = \overline{G(A)}$. □

Lemma 3.20. *For (D, A) an operator on H with* dense domain*, it holds*

$$(A[D])^\perp = \{h \in D^* : A^*h = 0\}.$$

Proof. Indeed,

$$h \in (A[D])^\perp \iff \forall g \in D,\ (Ag \mid h) = 0 \iff (h, 0) \in G(A^*).$$

The last property means $h \in D^*$ and $A^*h = 0$. □

3.4.2 Maximal dissipative operators in the Hilbert case

Proposition 3.21. *An operator (D, A) is dissipative on H if, and only if for all $g \in D$, $(Ag \mid g) \leq 0$. This property is also written $A \leq 0$.*

Proof. First, we see that if (D, A) is dissipative, then for all $\lambda > 0$, for all $g \in D$,

$$0 \leq \|g - \lambda Ag\|^2 - \|g\|^2 = \lambda^2 \|Ag\|^2 - 2\lambda(Ag \mid g).$$

Dividing by λ and passing to the limit as $\lambda \downarrow 0$, we obtain $(Ag \mid g) \leq 0$.

Second, if $\lambda > 0$, $g \in D$ and $(Ag \mid g) \leq 0$ then

$$\|g - \lambda Ag\|^2 = \|g\|^2 + \lambda^2 \|Ag\|^2 - 2\lambda(Ag \mid g) \geq \|g\|^2,$$

which proves that (D, A) is dissipative. □

Proposition 3.22. *The domain of a maximal dissipative operator (D, A) is dense in H. In particular, Proposition 3.14 applies to any maximal dissipative operator on a Hilbert space H.*

Proof. Let $h \in D^\perp$ and $g = Rh \in D$. Then

$$0 = (g \mid h) = (g \mid g - Ag) \quad \text{and thus} \quad \|g\|^2 = (Ag \mid g) \leq 0.$$

It follows that $g = 0$ and thus $h = 0$. Thus, $D^\perp = \{0\}$ and D is dense in X. □

Proposition 3.23. *Let (D, A) be a dissipative operator on H with dense domain. Then, (D, A) is maximal dissipative if, and only if (D^*, A^*) is dissipative and $G(A)$ is closed.*

Proof. First, assume that A is maximal dissipative. By Proposition 3.8 (2), A is closed. For $h \in D^*$ and $\lambda > 0$, we have

$$\begin{aligned}(A^* h \mid R_\lambda h) &= (h \mid AR_\lambda h) = (h \mid A_\lambda h) \\ &= \frac{1}{\lambda}(h \mid R_\lambda h - h) = \frac{1}{\lambda}\left\{(h|R_\lambda h) - \|h\|^2\right\} \leq 0.\end{aligned}$$

Passing to the limit $\lambda \downarrow 0$ and using Proposition 3.21, we have proved that (D^*, A^*) is dissipative.

Second, we assume that (D^*, A^*) is dissipative and $G(A)$ is closed. Then, $(I - A)[D]$ is closed in H. Indeed, consider a sequence $\{x_n - Ax_n\}_{n=0}^\infty$, where $x_n \in D$, converging to some $y \in X$. Since A is dissipative, $\|x_n - x_m\| \leq \|(x_n - x_m) - A(x_n - x_m)\|$ and thus $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence in X. Thus $x_n \to x$. Since $G(A)$ is closed, we have $x \in D$ and $Ax_n \to Ax$, so that $x_n - Ax_n \to x - Ax = y$.

By Proposition 3.20, we have

$$(I - A)[D]^\perp = \{h \in D^* : h - A^* h = 0\} = \{0\},$$

since (D^*, A^*) is dissipative. Therefore, $(I - A)[H] = H$ which means that A is maximal dissipative. □

Definition 3.24. Let (D, A) be an operator with *dense domain*.

1. We say that A is *self-adjoint* if $D^* = D$ and $A^* = A$;

2. We say that A is *skew-adjoint* if $D^* = D$ and $A^* = -A$.

Proposition 3.25. *Let (D, A) be an operator with* dense domain.

1. *If A is self-adjoint and dissipative, then A is maximal dissipative.*

2. *If A is maximal dissipative and $G(A) \subset G(A^*)$, then A is self-adjoint.*

Proof. (1) We know that $G(A^*)$ is closed by Proposition 3.19. Since $A = A^*$, $G(A)$ is closed and the conclusion is deduced from Proposition 3.23.

(2) Let $(g, h) \in G(A^*)$ and $f = g - h = g - A^*g$. Since A is maximal dissipative, $f = \tilde{g} - A\tilde{g}$, for some $\tilde{g} \in D$, and since $G(A) \subset G(A^*)$, we have $\tilde{g} \in D^*$ and $f = \tilde{g} - A^*\tilde{g}$. Since A^* is dissipative (see Proposition 3.23), it follows that $g = \tilde{g}$. Thus, $(g, h) \in G(A)$ and $A = A^*$. $\square$

Proposition 3.26. *Let (D, A) be an operator with* dense domain. *Then A and $-A$ are maximal dissipative if, and only if A is skew-adjoint.*

Proof. First, assume that A is skew-adjoint. Let $g \in D$. We have

$$(Ag \mid g) = (g \mid A^*g) = -(g \mid Ag).$$

Thus, $(Ag \mid g) = 0$ and by Proposition 3.21, both A and $-A$ are dissipative. We prove that A and $-A$ are maximal dissipative as in the proof of Proposition 3.25.

Second, assume that A and $-A$ are maximal dissipative. By Proposition 3.21 applied to A and $-A$, we have $(Ag \mid g) = 0$ for all $g \in D$. Now, for all $g, h \in D$, we compute

$$(Ag \mid h) + (Ah \mid g) = (A(g + h) \mid g + h) - (Ag \mid g) - (Ah \mid h) = 0.$$

Therefore, $G(-A) \subset G(A^*)$. It remains to show that $G(A^*) \subset G(A)$. Let $(g, f) \in G(A^*)$ and $h = g - A^*g = g - f$. Since $-A$ is maximal dissipative, there exists $\tilde{g}$ such that $h = \tilde{g} + A\tilde{g}$, and since $G(-A) \subset G(A^*)$, we have $\tilde{g} \in D^*$ and $h = \tilde{g} - A^*\tilde{g}$. Since A^* is dissipative (by Proposition 3.23), it follows that $g = \tilde{g}$. Thus, $(g, f) \in G(A^*)$ and $A = -A^*$. $\square$

3.5 Complex Hilbert spaces

In this section, we show how from a *complex* Hilbert space, we can reduce to the case of a *real* Hilbert space. Let $(H, \langle \cdot \mid \cdot \rangle)$ be a complex Hilbert space, which means that $\langle \cdot \mid \cdot \rangle$ satisfies

$$\begin{aligned} &\langle \lambda g \mid h \rangle = \bar{\lambda} \langle g \mid h \rangle, \quad \text{for all } \lambda \in \mathbf{C}, \text{ for all } g, h \in H; \\ &\langle g \mid h \rangle = \overline{\langle h \mid g \rangle}, \quad \text{for all } g, h \in H; \\ &\langle g \mid g \rangle = \|g\|^2, \quad \text{for all } g \in H. \end{aligned}$$

Define

$$(g \mid h) = \Re \langle g \mid h \rangle.$$

One can check that $(\cdot \mid \cdot)$ is a real scalar product on H and that $(H, (\cdot \mid \cdot))$ is a real Hilbert space. In this course, we will systematically work in this framework.

Let A be a linear operator on the real Hilbert space $(H, (\cdot \mid \cdot))$. If A is $\mathbf{C}$-linear, then one can define $\mathrm{i}A$ as a linear operator on the real Hilbert space $(H(\cdot \mid \cdot))$.

Proposition 3.27. *Assume that (D, A) is an operator with dense domain on $(H, (\cdot \mid \cdot))$. Assume that A is* **C***-linear. Then, A^* is also* **C***-linear and*

$$(\mathrm{i}A)^* = -\mathrm{i}A^*.$$

In particular, if A is self-adjoint, then $\mathrm{i}A$ is skew-adjoint.

Proof. Let $g \in D$, $h = A^*g$ and let $z \in \mathbf{C}$. For all $f \in D$, one has

$$(zh \mid f) = (h \mid \bar{z}f) = (g \mid A(\bar{z}f)) = (g \mid \bar{z}Af) = (zg \mid Af).$$

Thus, $zg \in D^*$ and $zh = A^*(zg)$. It follows that A^* is **C**-linear.

In addition,

$$(-\mathrm{i}h \mid f) = (g \mid A(\mathrm{i}f)) = (g \mid \mathrm{i}Af),$$

for all $(g, h) \in G(A^*)$ and all $f \in D$. Thus, $G(-\mathrm{i}A^*) \subset G((\mathrm{i}A)^*)$. Applying this result to the operator $\mathrm{i}A$, it follows that

$$G(-\mathrm{i}(\mathrm{i}A)^*) \subset G((\mathrm{i} \cdot \mathrm{i}A)^*) = G(-A^*).$$

Therefore, $G((\mathrm{i}A)^*) \subset G(-\mathrm{i}A^*)$, and so $G((\mathrm{i}A)^*) = G(-\mathrm{i}A^*)$.

If A is self-adjoint, then $(\mathrm{i}A)^* = -\mathrm{i}A^* = -\mathrm{i}A$, which means that $\mathrm{i}A$ is skew-adjoint. □

3.6 Examples in PDE theory

We focus here on second order differential operators. We refer to Exercises 3.4-3.6 for examples with first order differential operators (see also Exercise 2.10), and to Exercise 3.8 for a third order operator.

3.6.1 The Laplacian with Dirichlet condition in L^2

In this section, $H = L^2(\mathcal{U})$ equipped with the canonical scalar product. Define

$$\begin{cases} D = \{g \in H_0^1(\mathcal{U}) : \Delta g \in L^2(\mathcal{U})\} \\ Ag = \Delta g \quad \text{for all } g \in D. \end{cases}$$

Proposition 3.28. *The operator (D, A) is self-adjoint and maximal dissipative with dense domain in H.*

Proof. We recall that for all $g \in H_0^1(\mathcal{U})$ with $\Delta g \in L^2(\mathcal{U})$ and all $h \in H_0^1(\mathcal{U})$, it holds

$$\int_{\mathcal{U}} h \, \Delta g \, \mathrm{d}x = -\int_{\mathcal{U}} \nabla h \cdot \nabla g \, \mathrm{d}x. \tag{3.2}$$

(This is proved by density, see §C.3.) It is well-known that $\mathcal{C}_c^\infty(\mathcal{U})$ is dense in L^2 (see §B.9.1-B.9.3), so D is also dense in L^2. Applying (3.2) to $h = g$, we find

$$(g, \Delta g) = -(\nabla g, \nabla g) \leq 0,$$

and thus (D, A) is dissipative.

To prove that (D, A) is maximal dissipative, we apply Lax-Milgram Theorem A.58. Let

$$a(\varphi, \psi) = \int (\varphi\psi + \nabla\varphi \cdot \nabla\psi) \, \mathrm{d}x.$$

The coercivity of the bilinear mapping a for the norm of H_0^1 is clear. Given $f \in L^2(\mathcal{U})$, it is clear that $\varphi \mapsto (f \mid \varphi)$ is a linear form on H_0^1. Thus, there exists a unique $h \in H_0^1$ such that for all $\varphi \in H_0^1$,

$$\int (\varphi h + \nabla\varphi \cdot \nabla h)\,\mathrm{d}x = \int f h\,\mathrm{d}x.$$

In particular, $h - \Delta h = f$ holds in the sense of distributions. By this identity, and since $h, f \in L^2$, we deduce that $\Delta h \in L^2$, and thus $h \in D$. It follows that $h - Ah = f$ and this proves that (D, A) is maximal dissipative.

Formula (3.2) also proves that for all $g, h \in D$,

$$(Ag \mid h) = (g \mid Ah),$$

but this means that $G(A) \subset G(A^*)$, which is enough to say that A is self-adjoint using (2) of Proposition 3.25. □

3.6.2 The Laplacian in $\mathcal{C}_0$

We suppose that $\mathcal{U}$ is a bounded domain of $\mathbf{R}^N$, with $\mathcal{C}^1$ boundary (see §C.3). Let $X = \mathcal{C}_0(\mathcal{U})$ equipped with the norm $\|\cdot\|_{L^\infty}$. Define

$$\begin{cases} D = \{g \in X \cap H_0^1(\mathcal{U}) : \Delta g \in X\} \\ Ag = \Delta g \quad \text{for all } g \in D. \end{cases}$$

Proposition 3.29. *The operator (D, A) is maximal dissipative with dense domain in X.*

Remark 3.30. We will admit the following fact, for $\mathcal{U}$ with $\mathcal{C}^1$ boundary,

$$\{g \in H_0^1(\mathcal{U}) \cap L^\infty(\mathcal{U}) : \Delta g \in L^\infty(\mathcal{U})\} \subset \mathcal{C}_0(\mathcal{U}).$$

Proof. Let $\lambda > 0$, $f \in L^\infty(\mathcal{U})$ and $M = \|f\|_{L^\infty}$. Since the domain $\mathcal{U}$ is bounded, $f \in L^\infty(\mathcal{U}) \subset L^2(\mathcal{U})$. We consider the (unique) solution $g \in H_0^1(\mathcal{U})$ of

$$g - \lambda\Delta g = f$$

given by the proof of Proposition 3.28.

Now, we show that $g \in L^\infty$ and $\|g\|_{L^\infty} \le M$. We fix a function $G \in \mathcal{C}^1(\mathbf{R})$ such that

- $|G'| \le 1$ on $\mathbf{R}$;
- G is increasing on $(0, \infty)$;
- $G = 0$ on $(-\infty, 0]$.

Let $h = G(g - M)$. Then, $h \in L^2$ and $\nabla h = G'(g-M)\nabla g \in L^2$, so that $h \in H^1$. Moreover, since $g|_{\partial\mathcal{U}} = 0$ almost everywhere on $\partial\mathcal{U}$, we have $h|_{\partial\mathcal{U}} = G(-M) = 0$ almost everywhere on $\partial\mathcal{U}$, which means $h \in H_0^1(\mathcal{U})$. By the weak formulation, we have

$$\int |\nabla g|^2 G'(g-M) + \int gG(g-M) = \int fG(g-M),$$

which rewrites

$$\int |\nabla g|^2 G'(g-M) + \int (g-M)G(g-M) = \int (f-M)G(g-M),$$

Observe that $f-M \leq 0$, $G(g-M) \geq 0$ and and $G'(u-M) \geq 0$ so that it holds

$$\int (g-M)G(g-M) \leq 0.$$

The function $s \mapsto sG(s)$ being non negative, it follows that $(g-M)G(g-M) = 0$ almost everywhere on $\mathcal{U}$. Thus $g \leq M$ almost everywhere. Changing g in $-g$, one proves that $g \geq -M$. In particular, it follows that for any $g \in D$, $\|g - \lambda\Delta g\|_{L^\infty} \geq \|g\|_{L^\infty}$.

Moreover, we have just seen that any $f \in L^\infty(\mathcal{U})$, there exists a solution $g \in H_0^1(\mathcal{U}) \cap L^\infty(\mathcal{U})$ of $g - \Delta g = f$, which satisfies in addition $\Delta g = g - f \in L^\infty(\mathcal{U})$. By Remark 3.30, it follows that $g \in \mathcal{C}_0(\mathcal{U})$ and thus $\Delta g \in \mathcal{C}_0(\mathcal{U})$, which implies that $g \in D$. This proves that (D, A) is maximal dissipative.

Finally, note that the domain is dense in X since it contains $\mathcal{C}_c^\infty(\mathcal{U})$. □

3.6.3 The Schrödinger operator with Dirichlet condition in L^2

Let $H = L^2(\mathcal{U}; \mathbf{C})$, considered as a real Hilbert space. Define

$$\begin{cases} D = \{g \in H_0^1(\mathcal{U}) : \Delta g \in L^2(\mathcal{U})\} \\ Ag = \mathrm{i}\Delta g \quad \text{for all } g \in D. \end{cases}$$

Proposition 3.31. *The operator (D, A) is skew-adjoint and maximal dissipative with dense domain in H.*

Proof. Using the abstract general statement in Proposition 3.27, the result is a direct consequence of Proposition 3.28. □

3.6.4 The Klein-Gordon operator in $H_0^1 \times L^2$

In this section, $\mathcal{U}$ is any open subset of $\mathbf{R}^N$. Let $H = H_0^1(\mathcal{U}) \times L^2(\mathcal{U})$. We equip H with the scalar product

$$((g, h) \mid (\tilde{g}, \tilde{h})) = \int (\nabla g \cdot \nabla \tilde{g} + g\tilde{g} + h\tilde{h}) \, \mathrm{d}x.$$

Define

$$\begin{cases} D = \{(g, h) \in H : \Delta g \in L^2(\mathcal{U}), h \in H_0^1(\mathcal{U})\} \\ A(g, h) = (h, \Delta g - g) \quad \text{for all } (g, h) \in D. \end{cases}$$

Proposition 3.32. *The operator (D, A) is skew-adjoint and maximal dissipative with dense domain in H.*

Proof. As before, since $\mathcal{C}_c^\infty(\mathcal{U}) \times \mathcal{C}_c^\infty(\mathcal{U})$ is contained in D and dense in H, D is indeed dense in H. Next, we see that for $(g, h) \in D$, it holds

$$(A(g, h) \mid (g, h)) = \int \{\nabla h \cdot \nabla g + hg + (\Delta g - g)h\} \, \mathrm{d}x.$$

By (3.2), we find $(A(g,h) \mid (g,h)) = 0$, which proves that (D,A) is dissipative. Let $a, b \in H$. The equation $(g,h) - A(g,h) = (a,b)$ is equivalent to

$$\begin{cases} g - h = a \\ h - (\Delta g - g) = b \end{cases} \iff \begin{cases} 2g - \Delta g = a + b \\ h = g - a. \end{cases}$$

Proceeding as in the proof of Proposition 3.28, there exists a solution $g \in H_0^1(\mathcal{U})$ such that $\Delta g \in L^2$ of the first line of the above system. By the second line, we obtain $h \in H_0^1(\mathcal{U})$, which means that (D,A) is maximal dissipative. We check similarly that $(D,-A)$ is maximal dissipative. By Proposition 3.26, this proves that A is skew-adjoint. □

3.7 Exercises for Chapter 3

Exercise 3.1. Prove Corollary 3.10.

Exercise 3.2.

1. Let $A, B \in \mathcal{L}(X)$. Prove by contradiction that it holds

$$AB - BA \neq I.$$

2. Prove that the operator A on $L^2([0,1])$ defined by

$$(Ag)(x) = xg(x)$$

is a linear bounded operator on $L^2([0,1])$.

3. Prove that the operator (D,B) on $L^2([0,1])$ defined by

$$D = \mathcal{C}^1([0,1]), \quad (Bg)(x) = g'(x)$$

cannot be extended as a bounded operator on $L^2([0,1])$.

Exercise 3.3. Let f be a measurable function, bounded on $\mathbf{R}$ and $v_0 \in L^2(\mathbf{R})$, $v_0 \neq 0$.

1. Justify that we define an (unbounded) operator (D,A) on $L^2(\mathbf{R})$ by

$$D = \left\{ u \in L^2 : \int |f(x)u(x)| \,\mathrm{d}x < \infty \right\},$$
$$\text{for all } u \in D(A),\ Au = v_0 \cdot \textstyle\int f(x)u(x)\,\mathrm{d}x.$$

2. Assume that $f \in L^2(\mathbf{R})$. Prove that A is continuous and compute its adjoint.

3. Assume that $f \notin L^2(\mathbf{R})$. Prove that the domain of A is dense and compute (D^*, A^*).

Exercise 3.4. Let $X = \mathcal{C}_b(\mathbf{R})$ the space of continuous, bounded functions on $\mathbf{R}$, equipped with the sup norm. Define the operator A in X by

$$\begin{cases} D = \{g \in \mathcal{C}^1(\mathbf{R}) \cap X : g' \in X\} \\ Ag = g' \quad \text{for all } g \in D. \end{cases}$$

1. Prove that A and $-A$ are maximal monotone. **Hint**: for $h \in X$, define $Lh(x) = \frac{1}{\lambda} \int_x^\infty e^{\frac{x-s}{\lambda}} h(s)\,\mathrm{d}s$ and prove $\|Lh\|_{L^\infty} \le \|h\|_{L^\infty}$.
2. Is the domain of A dense? **Hint**: use the function g defined on $\mathbf{R}$ by $g(x) = \sin(x^2)$.

Exercise 3.5. Denote by $\mathcal{C}_0(\mathbf{R})$ the closure of $\mathcal{D}(\mathbf{R})$ in $L^\infty(\mathbf{R})$. Answer the same questions as in the previous exercise for $Y = \mathcal{C}_0(\mathbf{R})$ and

$$\begin{cases} D = \{g \in \mathcal{C}^1(\mathbf{R}) \cap Y : g' \in Y\} \\ Bg = g' \quad \text{for all } g \in D. \end{cases}$$

Exercise 3.6. Consider $X = \{g \in \mathcal{C}([0,1]) : u(0) = 0)$ equipped with the sup norm. Define the operator A in X by

$$\begin{cases} D = \{g \in \mathcal{C}^1([0,1]) : g(0) = g'(0) = 0\} \\ Ag = g' \quad \text{for all } g \in D. \end{cases}$$

Prove that A is maximal monotone with dense domain.

Exercise 3.7. Let $\mathcal{U}$ be an open subset of $\mathbf{R}^N$. Let $H = H_0^1(\mathcal{U}) \times L^2(\mathcal{U})$. Let

$$\lambda_0 = \inf\{\|\nabla g\|_{L^2}^2 : g \in H_0^1(\mathcal{U}), \|g\|_{L^2} = 1\} \ge 0,$$

and let $m > -\lambda_0$. We equip H with the scalar product

$$((g,h) \mid (\tilde{g},\tilde{h})) = \int (\nabla g \cdot \nabla \tilde{g} + mg\tilde{g} + h\tilde{h})\,\mathrm{d}x.$$

Define

$$\begin{cases} D = \{(g,h) \in H : \Delta g \in L^2(\mathcal{U}), h \in H_0^1(\mathcal{U})\} \\ A(g,h) = (h, \Delta g - mg) \quad \text{for all } (g,h) \in D. \end{cases}$$

Prove that (D,A) is skew-adjoint and maximal dissipative with dense domain in H.

Exercise 3.8. Let $X = L^2(\mathbf{R})$. Prove that the operator A defined by

$$\begin{cases} D = H^3(\mathbf{R}) \\ Au = u''' \quad \text{for all } u \in D \end{cases}$$

is skew-adjoint. In particular, both A and $-A$ are maximal monotone with dense domain. **Hint**: use the Fourier transform.

Exercise 3.9. Let $N \ge 2$ and consider the real Hilbert space

$$H = (L^2(\mathbf{R}^N))^N.$$

A vector of H is denoted by $\boldsymbol{u} = (u_1, \dots, u_N)$. We denote

$$\nabla \cdot \boldsymbol{u} = \operatorname{div} \boldsymbol{u} = \sum_{j=1}^N \frac{\partial u_j}{\partial x_j}, \quad \Delta \boldsymbol{u} = (\Delta u_1, \dots, \Delta u_N).$$

Let

$$X = \{\boldsymbol{u} \in H : \nabla \cdot \boldsymbol{u} = 0\},$$

where the condition $\nabla \cdot \boldsymbol{u} = 0$ is understood in the sense of distributions.

1. Prove that X is a Hilbert space with the scalar product of E:

$$(\boldsymbol{u} \mid \boldsymbol{v}) = \sum_{j=1}^{N} \int_{\mathbf{R}^N} u_j v_j \, \mathrm{d}x.$$

2. Prove that the *Stokes operator* A defined by

$$\begin{cases} D = \{\boldsymbol{u} \in (H^2(\mathbf{R}^N))^N \cap X : \Delta \boldsymbol{u} \in X\} \\ A\boldsymbol{u} = \Delta \boldsymbol{u} \quad \text{for all } \boldsymbol{u} \in D, \end{cases}$$

is self-adjoint and maximal monotone with dense domain.

Chapter 4

The Hille-Yosida-Phillips Theorem

4.1 Construction of the semigroup generated by a maximal dissipative operator

Let (D, A) be a maximal dissipative operator on X, with dense domain. For $\lambda > 0$, we consider R_λ and A_λ as defined in Definition 3.12. We set

$$S_\lambda(t) = \mathrm{e}^{tA_\lambda}, \quad \text{for all } t \geq 0,$$

so that $(S_\lambda(t))_{t\geq 0}$ is the uniformly continuous semigroup generated by A_λ.

In the next result, we construct a semigroup generated by A by a compactness argument using the sequence of approximate uniformly continuous semigroups $\{(S_\lambda(t))_{t\geq 0}\}_{\lambda>0}$.

Theorem 4.1. *For all $g \in X$, the sequence $\{S_\lambda(t)g\}_{\lambda>0}$ converges as $\lambda \downarrow 0$ uniformly on any bounded interval $[0, T]$ to a function $u \in \mathcal{C}([0, \infty), X)$. Setting $S(t)g = u(t)$, $(S(t))_{t\geq 0}$ is a contraction semigroup on X.*

Moreover, if $g \in D$ then u is the unique solution of the problem

$$\begin{cases} u \in \mathcal{C}([0, \infty), D) \cap \mathcal{C}^1([0, \infty), X), \\ \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) = Au(t), \quad \textit{for all } t \geq 0, \\ u(0) = g. \end{cases} \tag{4.1}$$

Last, for all $g \in D$ and $t \geq 0$, it holds

$$S(t)Ag = AS(t)g.$$

Proof. First, by definition, for all $\lambda > 0$ and all $t \geq 0$, one has

$$S_\lambda(t) = \mathrm{e}^{\frac{t}{\lambda}(R_\lambda - I)} = \mathrm{e}^{-\frac{t}{\lambda}}\mathrm{e}^{\frac{t}{\lambda}R_\lambda}$$

and thus using $\|R_\lambda\| \leq 1$,

$$\|S_\lambda(t)\| \leq \mathrm{e}^{-\frac{t}{\lambda}}\mathrm{e}^{\frac{t}{\lambda}\|R_\lambda\|} \leq 1.$$

In particular, for any $g \in X$, $u_\lambda(t) = S_\lambda(t)g$ satisfies

$$\|u_\lambda(t)\| \leq \|g\| \quad \text{for all } \lambda > 0, \text{ and all } t \geq 0. \tag{4.2}$$

Second, consider any $g \in D$ and any $\lambda, \mu > 0$. Recall from Remark 3.13 that A_λ and A_μ commute. In particular, for any $t \geq 0$, we have

$$\begin{aligned} S_\lambda(t) - S_\mu(t) &= \mathrm{e}^{tA_\mu}\left(\mathrm{e}^{t(A_\lambda - A_\mu)} - I\right) \\ &= \mathrm{e}^{tA_\mu}\int_0^t \mathrm{e}^{s(A_\lambda - A_\mu)}(A_\lambda - A_\mu)\,\mathrm{d}s \\ &= \int_0^t S_\lambda(s)S_\mu(t-s)(A_\lambda - A_\mu)\,\mathrm{d}s. \end{aligned}$$

Using $\|S_\lambda\| \leq 1$, we obtain the inequality, for any $g \in D$,

$$\|u_\lambda(t) - u_\mu(t)\| \leq \int_0^t \|S_\lambda(s)\|\|S_\mu(t-s)\|\|A_\lambda g - A_\mu g\|\,\mathrm{d}s \leq t\|A_\lambda g - A_\mu g\|.$$

It follows from (2) of Proposition 3.14 that $\{u_\lambda\}_{\lambda>0}$ is a Cauchy sequence in $\mathcal{C}([0,T],X)$, for any $T > 0$. Denote by $u \in \mathcal{C}([0,\infty),X)$ its limit.

Setting $u(t) = S(t)g$, by passing to the limit $\lambda \downarrow 0$ in the inequality (4.2), we have $\|S(t)g\| \leq 1$ for all $t \geq 0$ and all $g \in D$. It follows that $S(t)$ can be extended to a unique linear continuous operator $S(t) \in \mathcal{L}(X)$ satisfying $\|S(t)\| \leq 1$ for all $t \geq 0$. To prove the properties of $S(t)$, we use a density argument. Let $g \in X$ and consider $\{g_n\}_{n=0}^\infty$ a sequence of D converging to g. We have

$$\begin{aligned} \|S_\lambda(t)g - S(t)g\| &\leq \|S_\lambda(t)g - S_\lambda(t)g_n\| + \|S(t)g_n - S(t)g_n\| \\ &\quad + \|S_\lambda(t)g_n - S(t)g\| \\ &\leq 2\|g_n - g\| + \|S_\lambda(t)g_n - S(t)g_n\|, \end{aligned}$$

and so $S_\lambda(t)g$ converges to $S(t)g$ as $\lambda \downarrow 0$ uniformly on $[0,T]$, for all $T > 0$. In particular, $S(0) = I$. Now, we check that $S(t)S(s) = S(t+s)$ using the same property on S_λ:

$$\begin{aligned} \|S(t)S(s)g - S(t+s)g\| &\leq \|S(t)S(s)g - S(t)S_\lambda(s)g\| \\ &\quad + \|S(t)S_\lambda(s)g - S_\lambda(t)S_\lambda(s)g\| \\ &\quad + \|S_\lambda(t+s)g - S(t+s)g\|. \end{aligned}$$

and passing to the limit as $\lambda \downarrow 0$.

Now, we prove that $u(t) = S(t)g$ satisfies (4.1) in the case where $g \in D$. Set

$$v_\lambda(t) = A_\lambda S_\lambda(t)g = S_\lambda A_\lambda(t)g = \frac{\mathrm{d}}{\mathrm{d}t}u_\lambda(t).$$

We have by the triangle inequality and $\|S(t)\| \leq 1$,

$$\|v_\lambda(t) - S(t)Ag\| \leq \|(S_\lambda(t) - S(t))A_\lambda g\| + \|A_\lambda g - Ag\|.$$

Thus, $\lim_{\lambda\downarrow 0} v_\lambda(t) = S(t)Ag$, uniformly on $[0,T]$, for all $T > 0$. Passing to the limit $\lambda \downarrow 0$ in the identity

$$u_\lambda(t) = g + \int_0^t v_\lambda(s)\,\mathrm{d}s$$

we obtain

$$u(t) = g + \int_0^t S(t)Ag \,\mathrm{d}s. \tag{4.3}$$

This means that $u \in \mathcal{C}^1([0,\infty), X)$ and, for all $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = S(t)Ag. \tag{4.4}$$

By (3.1), we have $v_\lambda(t) = AR_\lambda S_\lambda(t)g$. Moreover, by $\|R_\lambda\| \leq 1$,

$$\|R_\lambda S_\lambda(t)g - S(t)g\| \leq \|S_\lambda(t)g - S(t)g\| + \|R_\lambda S(t)g - S(t)g\|.$$

Therefore

$$\lim_{\lambda \downarrow 0} (R_\lambda S_\lambda(t)g, A(R_\lambda S_\lambda(t)g) = (S(t)g, S(t)Ag)$$

in $X \times X$. Since the graph of A is closed, $S(t)g \in D$ for all $t \geq 0$ and $AS(t)g = S(t)Ag$. In particular $u \in \mathcal{C}([0,\infty), D)$ (see Corollary 3.10) and by (4.4), u satisfies (4.1).

Finally, we justify the uniqueness statement. Let v be a solution of (4.1) and let $s > 0$. Set

$$w(t) = S(s-t)v(t),$$

for any $t \in [0,s]$. We have $w \in \mathcal{C}([0,\infty), D) \cap \mathcal{C}^1([0,\infty), X)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) = -AS(s-t)v(t) + S(s-t)\frac{\mathrm{d}}{\mathrm{d}t} v(t) = S(s-t)\left[\frac{\mathrm{d}}{\mathrm{d}t} v(t) - Av(t)\right] = 0,$$

for all $t \in [0,s]$. It follows that $w(s) = w(0)$ and so $v(s) = S(s)g$. Since $s \geq 0$ is arbitrary, the uniqueness statement is proved. $\square$

4.2 Hille-Yosida-Phillips theorem

We have already proved the main parts of the following theorem.

Theorem 4.2. *A linear operator (D, A) on X is the generator of a contraction semigroup on X if and only if A is maximal dissipative with dense domain.*

Proof. On the one hand, if (D, A) is the generator of a contraction semigroup in X then Proposition 2.23 shows that A is maximal dissipative with dense domain.

On the other hand, let A be a maximal dissipative operator with dense domain and let $(S(t))_{t\geq 0}$ be the contraction semigroup associated to A given by Theorem 4.1. Denote by $(D(B), B)$ the generator of $(S(t))_{t\geq 0}$ as given by Definition 2.16. We only have to prove that $(D(B), B) = (D, A)$.

Recall from (4.3) that for $g \in D$ and $\delta > 0$, we have

$$S(\delta)g = g + \int_0^\delta S(t)Ag \,\mathrm{d}t,$$

and thus $g \in D(B)$ with $Bg = Ag$. Thus, $G(A) \subset G(B)$. Conversely, let $h \in D(B)$. Since A is maximal dissipative, there exists $g \in D$ such that

$$g - Ag = h - Bh.$$

Since $G(A) \subset G(B)$, we have $(g-h) - B(g-h) = 0$ and B being dissipative, we conclude $g = h$, which implies $G(B) \subset G(A)$. $\square$

Finally, for completeness, we prove the uniqueness of the semigroup generated by a given maximal dissipative operator with dense domain.

Proposition 4.3. *Let (D, A) be a maximal dissipative operator on X with dense domain. Assume that (D, A) is the generator of a contraction semigroup $(T(t))_{t\geq 0}$. Then, $(T(t))_{t\geq 0}$ is the semigroup corresponding to (D, A) constructed in Theorem 4.1.*

Proof. Let $(S(t))_{t\geq 0}$ be the semigroup corresponding to (D, A) constructed in Theorem 4.1. Let $g \in D$ and $u(t) = T(t)g$. For all $t \geq 0$ and $\delta > 0$, we have

$$\frac{u(t+\delta) - u(t)}{\delta} = \frac{T(\delta) - I}{\delta} u(t) = T(t)\frac{T(\delta)g - g}{\delta} \underset{\delta\downarrow 0}{\longrightarrow} T(t)Ag.$$

Thus, $T(t)g \in D$, for all $t \geq 0$, and for all $t \geq 0$,

$$AT(t)g = T(t)Ag = \frac{\mathrm{d}^+}{\mathrm{d}t} u(t),$$

It follows that $u \in \mathcal{C}([0, \infty), D) \cap \mathcal{C}^1([0, \infty), X)$ and $\frac{\mathrm{d}}{\mathrm{d}t} u = Au$, for all $t \geq 0$. Therefore, by Theorem 4.1, we obtain $T(t)g = S(t)g$. By density of D in X, the operators $T(t)$ and $S(t)$ coincide on X. □

4.3 Isometry group of operators

Definition 4.4. A *isometry group of linear operators on* X is a map

$$S : \mathbf{R} \to \mathcal{L}(X)$$

satisfying the following properties.

1. For all $g \in X$ and all $t \in \mathbf{R}$, $\|S(t)g\| = \|g\|$;
2. For all $t, s \in \mathbf{R}$, $S(t + s) = S(t)S(s)$;
3. $S(0) = I$;
4. For all $g \in X$, the function $t \in \mathbf{R} \mapsto S(t)g \in X$ is continuous.

Theorem 4.5. *Let (D, A) be a maximal dissipative operator with dense domain and let $(S(t))_{t\geq 0}$ be the contraction semigroup generated by A. Then $(S(t))_{t\geq 0}$ is the restriction to $[0, \infty)$ of an isometry group $(S(t))_{t\in\mathbf{R}}$ if and only if $(D, -A)$ is maximal dissipative. In this case, for all $g \in D$, the function $u(t) = S(t)g$ satisfies $u \in \mathcal{C}(\mathbf{R}, D) \cap \mathcal{C}^1(\mathbf{R}, X)$ and the equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = Au(t),$$

for all $t \in \mathbf{R}$.

Proof. First, we check that the condition (D, A) and $(D, -A)$ maximal dissipative is sufficient to construct an isometry group. Denote by $(S(t))_{t\geq 0}$ the semigroup generated by A and $(\tilde{S}(t))_{t\geq 0}$ the semigroup generated by $-A$. For $g \in D$, $u(t) = S(t)g$ is solution of $\frac{\mathrm{d}}{\mathrm{d}t} u = Au$. Let $\tilde{u}(t) = \tilde{S}(t)g$ be solution of

$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{u} = -A\tilde{u}$, $\tilde{u}(0) = g$. It is thus natural to extend the definition of $S(t)$ by setting

$$S(t) = \begin{cases} S(t) & \text{if } t \geq 0; \\ \tilde{S}(-t) & \text{if } t \leq 0. \end{cases}$$

We check that $(S(t))_{t\in\mathbf{R}}$ is indeed a strongly continuous group of linear operators. Continuity is clear by the properties of $\tilde{S}$ and S. Now, we observe that, for all $t \geq 0$,

$$S(t)S(-t) = S(t)\tilde{S}(t) = S(-t)S(t) = \tilde{S}(t)S(t) = I.$$

Indeed, set $v(t) = S(t)\tilde{S}(t)g$, for some $g \in D$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = AS(t)\tilde{S}(t)g - S(t)A\tilde{S}(t)g = 0.$$

In particular, $S(t)$ is invertible for any $t \in \mathbf{R}$ and $S(t) = (S(-t))^{-1}$. It is now easy to check that for any $t, s \in \mathbf{R}$, $S(t+s) = S(t)S(s)$. Indeed, for example, if $s < 0 < t$ are such that $s + t \geq 0$, then

$$S(t) = S(t+s)S(-s) = S(t+s)(S(s))^{-1}.$$

We also see that $\|S(t)g\| \leq \|g\|$ and $\|g\| = \|S(t)S(-t)g\| \leq \|S(t)g\|$ implies $\|S(t)g\| = \|g\|$. Concerning the $\mathcal{C}^1$ regularity of $u(t) = S(t)g$, we observe that setting $\tilde{u}(t) = u(-t)$,

$$\frac{\mathrm{d}^+}{\mathrm{d}t}u(0) = Ag, \quad \frac{\mathrm{d}^+}{\mathrm{d}t}\tilde{u}(0) = -Ag$$

and thus $\frac{\mathrm{d}^-}{\mathrm{d}t}u(0) = -\frac{\mathrm{d}^+}{\mathrm{d}t}\tilde{u}(0) = \frac{\mathrm{d}^+}{\mathrm{d}t}u(0)$.

Second, we assume that $(S(t))_{t\geq 0}$ is the restriction to $[0,\infty)$ of an isometry group $(S(t))_{t\in\mathbf{R}}$, and we set $T(t) = S(-t)$, for $t \geq 0$. Then $T(t)$ is a contraction semigroup. We denote by $(D(B), B)$ its generator. For all $\delta > 0$ and $g \in X$, we have

$$\frac{T(\delta) - I}{\delta}g = \frac{S(-\delta) - I}{\delta}g = -T(\delta)\frac{S(\delta) - I}{\delta}g.$$

Thus, passing to the limit $\delta \downarrow 0$, we obtain $B = -A$, which proves the result. $\square$

4.4 Hilbert case

We consider in this section a real Hilbert space $(H, (\cdot \mid \cdot))$.

4.4.1 Skew-adjoint case

Let (D, A) be a skew-adjoint operator with dense domain. By Proposition 3.26, the operators (D, A) and $(D, -A)$ are maximal dissipative. In particular, (D, A) generates an isometry group, see Definition 4.4 and Theorem 4.5.

4.4.2 Self-adjoint case

Theorem 4.6. *Assume that (D, A) is a self-adjoint non positive operator. Let $(S(t))_{t\geq 0}$ be the semigroup generated by (D, A). Let $g \in H$ and $u(t) = S(t)g$, for any $t \geq 0$. Then, u is the unique solution of the following problem*

$$\begin{cases} u \in \mathcal{C}([0,\infty), H) \cap \mathcal{C}((0,\infty), D) \cap \mathcal{C}^1((0,\infty), H), \\ \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) = Au(t), \quad \textit{for all } t > 0, \\ u(0) = g. \end{cases} \tag{4.5}$$

Moreover, it holds, for all $t > 0$,

$$\begin{aligned} \|Au(t)\| &\leq \frac{1}{t\sqrt{2}}\|g\|, \\ -(Au(t) \mid u(t)) &\leq \frac{1}{2t}\|g\|^2. \end{aligned}$$

Finally, if $g \in D$, it holds, for all $t > 0$,

$$\|Au(t)\|^2 \leq -\frac{1}{2t}(Ag \mid g).$$

Remark 4.7. This result means that $S(t)$ has a *smoothing effect*: the solution at time $t > 0$ belongs to D even if $g \notin D$. This effect is antagonist with any reversibility of the equation, in contrast with the case of isometry group generated by skew-adjoint operators.

Proof. First, we check that for all $\lambda > 0$, the bounded operator A_λ introduced in Definition 3.12 is self-adjoint and satisfies $A_\lambda \leq 0$. Indeed, it is easy to see that if A is self-adjoint, then the bounded operator R_λ is also self-adjoint and thus $A_\lambda = \frac{1}{\lambda}(R_\lambda - I)$ is self-adjoint. Moreover, since $A_\lambda = R_\lambda A$, $g = R_\lambda g - \lambda A R_\lambda g$ and $A \leq 0$, we have, for any $g \in D$,

$$\begin{aligned} (A_\lambda g \mid g) = (AR_\lambda g \mid g) &= (AR_\lambda g \mid R_\lambda g - \lambda AR_\lambda g) \\ &= (AR_\lambda g \mid R_\lambda g) - \lambda \|AR_\lambda g\|^2 \leq 0. \end{aligned}$$

Let $\lambda > 0$, $g \in X$ and $u_\lambda(t) = S_\lambda(t)$, where $(S_\lambda(t))_{t\geq 0}$ is the semigroup associated to A_λ. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\lambda(t)\|^2 = (A_\lambda u_\lambda(t) \mid u_\lambda(t)) + (u_\lambda(t) \mid A_\lambda u_\lambda(t)) = 2(A_\lambda u_\lambda(t) \mid u_\lambda(t)) \leq 0,$$

which means that $t \mapsto \|u_\lambda(t)\|^2$ is non increasing. We check similarly that $t \mapsto \|\frac{\mathrm{d}}{\mathrm{d}t}u_\lambda(t)\|^2$ is non increasing. Using that A_λ is self-adjoint, we also compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(A_\lambda u_\lambda(t) \mid u_\lambda(t)) &= (A_\lambda \tfrac{\mathrm{d}}{\mathrm{d}t}u_\lambda(t) \mid u_\lambda(t)) + (A_\lambda u_\lambda(t) \mid \tfrac{\mathrm{d}}{\mathrm{d}t}u_\lambda(t)) \\ &= 2\|\tfrac{\mathrm{d}}{\mathrm{d}t}u_\lambda(t)\|^2, \end{aligned}$$

which implies that $t \mapsto (A_\lambda u_\lambda(t) \mid u_\lambda(t))$ is non decreasing.

By integration on $(0, t)$, we obtain

$$\|g\|^2 = \|u_\lambda(t)\|^2 - 2\int_0^t (A_\lambda u_\lambda(s) \mid u_\lambda(s))\,\mathrm{d}s \geq -2t(A_\lambda u_\lambda(t) \mid u_\lambda(t)). \tag{4.6}$$

We also obtain by the previous observations and integration on $(0,t)$

$$-(A_\lambda g \mid g) \geq -(A_\lambda u_\lambda(t) \mid u_\lambda(t)) + 2\int_0^t \|\tfrac{\mathrm{d}}{\mathrm{d}t}u_\lambda(s)\|^2\,\mathrm{d}s \geq 2t\|\tfrac{\mathrm{d}}{\mathrm{d}t}u_\lambda(t)\|^2, \quad (4.7)$$

and by integration by parts

$$\begin{aligned}
\|g\|^2 &= \|u_\lambda(t)\|^2 - 2\int_0^t (A_\lambda u_\lambda(s) \mid u_\lambda(s))\,\mathrm{d}s \\
&= \|u_\lambda(t)\|^2 - 2t\,(A_\lambda u_\lambda(t) \mid u_\lambda(t)) + 2\int_0^t s\frac{\mathrm{d}}{\mathrm{d}s}(A_\lambda u_\lambda(s) \mid u_\lambda(s))\,\mathrm{d}s \\
&= \|u_\lambda(t)\|^2 - 2t\,(A_\lambda u_\lambda(t) \mid u_\lambda(t)) + 4\int_0^t s\,\|\tfrac{\mathrm{d}}{\mathrm{d}t}u_\lambda(s)\|^2\,\mathrm{d}s,
\end{aligned}$$

which proves the estimate

$$\|g\|^2 \geq 2t^2\|\tfrac{\mathrm{d}}{\mathrm{d}t}u_\lambda(t)\|^2. \quad (4.8)$$

By (4.8), the equation $\frac{\mathrm{d}}{\mathrm{d}t}u_\lambda = A_\lambda u_\lambda = AR_\lambda u_\lambda$ (see (3.1)), it follows that for any $t > 0$, $\|AR_\lambda u_\lambda(t)\|$ is bounded as $\lambda \downarrow 0$. Moreover, $R_\lambda u_\lambda(t) \to u(t)$ in H as $\lambda \downarrow 0$. Since $G(A)$ is closed in $H \times H$, it is also *weakly closed* in $H \times H$. This implies that $u(t) \in D$ and $Au(t)$ is the weak limit of $(AR_\lambda u_\lambda(t))_\lambda$ as $\lambda \downarrow 0$. Now, to prove the estimates stated in the proposition, we only have to pass to the limit as $\lambda \downarrow 0$ in (4.6), (4.7) (for $g \in D$) and (4.8).

Finally, we prove the uniqueness of u as solution of $\frac{\mathrm{d}u}{\mathrm{d}t} = Au$ in the class specified in (4.1). Let u be as in (4.1). Let $t > 0$ and $0 < \delta < t$. By the uniqueness statement in Theorem 4.1, we know that $u(t) = S(t-\delta)u(\delta)$. Thus,

$$\begin{aligned}
\|u(t) - S(t)g\| &\leq \|S(t-\delta)(u(\delta) - g)\| + \|S(t-\delta)(S(\delta)g - g)\| \\
&\leq \|u(\delta) - g\| + \|S(\delta)g - g\|.
\end{aligned}$$

Passing to the limit $\delta \downarrow 0$, it follows that $u(t) = S(t)g$. □

4.5 Extrapolation and weak solutions

From Theorem 4.1, we know that if $g \in D$, then $S(t)g$ is solution of (4.1). If $g \in X$, then $S(t)g$ is still well-defined in X but in general it is not differentiable in X. However, the extrapolation Proposition 3.15 allows us to identify $S(t)g$ as a solution of the same equation involving the extended operator (X, B) in the space $(Y, |||\cdot|||)$. We follow the notation of Proposition 3.15. Let $(T(t))_{t\geq 0}$ be the semigroup generated by the (maximal dissipative) operator (X, B) with dense domain in $(Y, |||\cdot|||)$.

Proposition 4.8. *Let $g \in X$. Then $u(t) = S(t)g$ is the unique solution of the problem*

$$\begin{cases}
u \in \mathcal{C}([0,\infty), X) \cap \mathcal{C}^1([0,\infty), Y), \\
\dfrac{\mathrm{d}u}{\mathrm{d}t}(t) = Bu(t), \quad \text{for all } t \geq 0, \\
u(0) = g.
\end{cases} \quad (4.9)$$

Proof. It follows from Theorem 4.1 that $T(t)g$ is the unique solution of (4.9). Now, since for $g \in D$, $T(t)g = S(t)g$, it follows by density and continuity that $T(t)g = S(t)g$ for all $g \in X$. □

4.6 Exercises for Chapter 4

Exercise 4.1. Let (D, A) be a maximal dissipative operator with dense domain in a Banach space $(X, \|\cdot\|)$.

1. Prove that the following defines a norm on D, for any $g \in D$,
$$\|g\|_1 = \sqrt{\|g\|^2 + \|Ag\|^2}.$$
2. Prove that $(D, \|\cdot\|_1)$ is a Banach space.
3. More generally, we define by induction on $k \geq 2$, the space $D(A^k)$ by
$$D(A^k) = \{g \in D(A^{k-1}) : Ag \in D(A^{k-1})\}.$$
Justify that for $k \geq 2$,
$$\|g\|_k = \sqrt{\|g\|^2 + \sum_{j=1}^{k} \|A^j g\|^2}$$
defines a norm on $D(A^k)$ and that $(D(A^k), \|\cdot\|_k)$ is a Banach space.
In what follows, we denote
$$D(A^1) = D \quad \text{and} \quad D(A^0) = X.$$
4. Set
$$D_2 = D(A^2), \quad A_2 g = Ag \text{ for } g \in D_2.$$
Prove that (D_2, A_2) is a maximal dissipative operator with dense domain in $(D, \|\cdot\|_1)$.
5. We denote by $(S(t))_{t\geq 0}$ the contraction semigroupe generated by (D, A). Prove that if $g \in D(A^k)$, for a certain $k \geq 2$, then u defined by $u(t) = S(t)g$ for all $t \geq 0$, satisfies
$$u \in \mathcal{C}^{k-j}([0, \infty), D(A^j))$$
for all $j = 0, \dots, k$.
6. For $T > 0$ and $k \geq 2$, we consider a function $b \in \mathcal{C}^{k-j}([0, T], D(A^j))$ for all $j = 1, \dots, k$. We define
$$v(t) = \int_0^t S(t-s) b(s) ds.$$
Prove that
$$v \in \mathcal{C}^{k-j}([0, T], D(A^j))$$
for all $j = 0, \dots, k$.

Exercise 4.2. Product Semigroup.
Let $(X, \|\cdot\|)$ be a Banach space on $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ be two strongly continuous contraction semigroups on X. Assume that for all $t \geq 0$, it holds
$$S(t)T(t) = T(t)S(t).$$

1. Prove that for all $s, t \geq 0$, $S(s)T(t) = T(t)S(s)$.

2. For any $t \geq 0$, let $U(t) = S(t)T(t)$. Prove that $(U(t))_{t\geq 0}$ is a strongly continuous contraction semigroup on X.

   We denote by $(D(A), A)$ and $(D(B), B)$, respectively, the generators of the two semigroups $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$. We denote by $(D(C), C)$, the generator of the semigroup $(U(t))_{t\geq 0}$.

3. Prove that $D(A)$ is invariant by the semigroup $(T(t))_{t\geq 0}$ *i.e.* for all $t \geq 0$, if $g \in D(A)$, then $T(t)g \in D(A)$. Prove also that all $t \geq 0$ and $g \in D(A)$, $AT(t)g = T(t)Ag$.

4. Prove that $D(A) \cap D(B)$ is invariant by $U(t)$.

5. Prove that $D(A) \cap D(B) \subset D(C)$ and that the restriction of C to $D(A) \cap D(B)$ is $A + B$.

6. For any $\lambda \geq 0$, we denote $R_\lambda = (I - \lambda A)^{-1}$ on X. Recall why this operator is well-defined and satisfies $\|R_\lambda\| \leq 1$. Prove that if $h \in D(B)$ and $\lambda > 0$, then $R_\lambda h \in D(A) \cap D(B)$.

7. Prove that $D(A) \cap D(B)$ is dense in X.

Exercise 4.3. Apply Theorem 4.1 or Theorem 4.6 to the examples given in §3.6 and Exercises 3.4-3.8.

Chapter 5

Inhomogeneous equations

In this Chapter, (D, A) is a maximal dissipative operator with dense domain on X and $(S(t))_{t\geq 0}$ is the semigroup generated by A.

Let $T > 0$. Let $g \in D$ and $b \in \mathcal{C}([0,T], X)$. The objective is to solve the inhomogeneous problem

$$\begin{cases} u \in \mathcal{C}([0,T], D) \cap \mathcal{C}^1([0,T], X), \\ \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) = Au(t) + b(t), \quad \text{for all } t \in [0,T], \\ u(0) = g. \end{cases} \tag{5.1}$$

5.1 Necessary condition

Proposition 5.1. *Let $g \in D$ and $b \in \mathcal{C}([0,T], X)$. If u is a solution of* (5.1)*, then for all $t \in [0,T]$,*

$$u(t) = S(t)g + \int_0^t S(t-s)b(s)\,\mathrm{d}s. \tag{5.2}$$

Remark 5.2. We check that for any $b \in \mathcal{C}([0,T], X)$, the function $v : [0,T] \to X$ defined by

$$v(t) = \int_0^t S(t-s)b(s)\,\mathrm{d}s$$

belongs to $\mathcal{C}([0,T], X)$. It can be seen as a consequence of the Dominated Convergence Theorem, but in the case $b \in \mathcal{C}([0,T], X)$, there is an elementary way to justify it, saying that the map

$$(t,s) \in \{(t', s') : t' \in [0,T],\ s' \in [0,t']\} \mapsto S(t-s)b(s) \in X$$

being continuous on a compact set, is uniformly continuous.

Remark 5.3. The assumptions $g \in D$ and $b \in \mathcal{C}([0,T], X)$ may seem exactly what is needed to provide a solution u of (5.1) by the formula (5.2). However, such assumptions garantee that $u \in \mathcal{C}([0,T], X)$ but not that $u \in \mathcal{C}([0,T], D)$ in general as shown by the following example. Assume that $(S(t))_{t\in\mathbf{R}}$ is an isometry group and take $h \in X \setminus D$. In particular, we know that for all $t \in \mathbf{R}$, $S(t)h \in X \setminus D$. Take $b(t) = S(t)h$ and $g = 0$. Then, formula (5.2) defines

$$u(t) = \int_0^t S(t-s)S(s)h \,\mathrm{d}s = tS(t)h.$$

But for $t \neq 0$, we see that $u(t) \notin D$.

Proof. Let $t \in [0,T]$ and for $s \in [0,t]$ set

$$w(s) = S(t-s)u(s).$$

Now, let $s \in [0,t)$. For $\delta \in (0, t-s]$, we have

$$\frac{w(s+\delta) - w(s)}{\delta} = S(t-s-\delta)\left\{\frac{u(s+\delta) - u(s)}{\delta} - \frac{S(\delta) - I}{\delta}u(s)\right\},$$

and thus

$$\lim_{\delta \downarrow 0} \frac{w(s+\delta) - w(s)}{\delta} = S(t-s)\left\{\frac{\mathrm{d}}{\mathrm{d}t}u(s) - Au(s)\right\} = S(t-s)b(s).$$

Note that the function $s \mapsto S(t-s)b(s)$ belongs to $\mathcal{C}([0,t], X)$. We deduce that $w \in \mathcal{C}^1([0,t), X)$ with, for all $s \in [0,t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}w(s) = S(t-s)b(s).$$

Integrating in $s \in [0,\tau]$, for some $\tau \in (0,t)$, we obtain

$$S(t-\tau)u(\tau) = S(t)g + \int_0^\tau S(t-s)b(s) \,\mathrm{d}s.$$

Passing to the limit $\tau \uparrow t$, we find the desired formula for $u(t)$. □

5.2 Sufficient condition

Proposition 5.4. *Let $g \in D$ and $b \in \mathcal{C}([0,T], D)$. Then, u given by* (5.2) *is the unique solution of* (5.1).

Proof. As before

$$v(t) = \int_0^t S(t-s)b(s) \,\mathrm{d}s.$$

Since $b \in \mathcal{C}([0,T], D)$, it follows from Remark 5.2 that $v \in \mathcal{C}([0,T], D)$ and

$$Av(t) = \int_0^t AS(t-s)b(s) \,\mathrm{d}s = \int_0^t S(t-s)Ab(s) \,\mathrm{d}s.$$

Now, we prove that $v \in \mathcal{C}^1([0,T], X)$. Indeed, for $t \in [0,T)$ and $\delta \in (0, T-t)$, we have

$$\frac{v(t+\delta) - v(t)}{\delta} = \int_0^t S(t-s)\frac{S(\delta) - I}{\delta}b(s) \,\mathrm{d}s + \frac{1}{\delta}\int_t^{t+\delta} S(t+\delta-s)b(s) \,\mathrm{d}s.$$

First, we observe that

$$\int_0^t S(t-s)\frac{S(\delta)-I}{\delta}b(s)\,\mathrm{d}s = \frac{S(\delta)-I}{\delta}\int_0^t S(t-s)b(s)\,\mathrm{d}s = \frac{S(\delta)-I}{\delta}v(t).$$

Thus, passing to the limit as $\delta \to 0$,

$$\lim_{\delta\downarrow 0}\int_0^t S(t-s)\frac{S(\delta)-I}{\delta}b(s)\,\mathrm{d}s = Av(t).$$

Second, by continuity,

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_t^{t+\delta} S(t+\delta-s)b(s)\,\mathrm{d}s = b(t).$$

It follows that v is left-differentiable on $[0,T)$, and

$$\frac{\mathrm{d}^+}{\mathrm{d}t}v(t) = Av(t) + b(t).$$

We check similarly that v is right-differentiable on $(0,T]$ and so $v \in \mathcal{C}^1([0,T],X)$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = Av(t) + b(t).$$

Setting $u(t)$ as in (5.2), we have obtained $u \in \mathcal{C}^1([0,T],X) \cap \mathcal{C}([0,T],D)$ and u solves $\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + b(t)$, $u(0) = g$. □

5.3 Weak solution

Corollary 5.5. *Let $g \in X$, $b \in \mathcal{C}([0,T],X)$ and let u be given by* (5.2) *on* $[0,T]$. *Using the notation of Propositions 3.15 and 4.8, u is the unique solution of the problem*

$$\begin{cases} u \in \mathcal{C}([0,T],X) \cap \mathcal{C}^1([0,T],Y), \\ \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) = Bu(t) + b(t), \quad \textit{for all } t \in [0,T], \\ u(0) = g. \end{cases} \tag{5.3}$$

Remark 5.6. This remark allows us to see u defined by (5.2) as the unique weak solution of $\frac{\mathrm{d}u}{\mathrm{d}t}(t) = Au(t) + b(t)$ when only $g \in X$, $b \in \mathcal{C}([0,T],X)$.

The following result is deduced immediately from (5.3).

Corollary 5.7. *Let $g \in X$, $b \in \mathcal{C}([0,T],X)$ and let u be given by* (5.2) *on* $[0,T]$. *Then,*

$$u \in \mathcal{C}([0,T],D) \iff u \in \mathcal{C}^1([0,T],X).$$

Moreover, in this case, u is solution of (5.1) *on* $[0,T]$.

5.4 Exercises for Chapter 5

Exercise 5.1. Let (D, A) be a maximal dissipative operator with dense domain in a Banach space $(X, \|\cdot\|)$. For $b \in \mathcal{C}^1([0,T], X)$, we define

$$v(t) = \int_0^t S(t-s)b(s)ds.$$

Prove that v is solution of

$$\begin{cases} v \in \mathcal{C}([0,T], D) \cap \mathcal{C}^1([0,T], X), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t}(t) = Av(t) + b(t), \quad \text{for all } t \in [0,T], \\ v(0) = 0. \end{cases}$$

Exercise 5.2. Let (D, A) be a skew-adjoint operator in a Hilbert space H with scalar product $(\cdot \mid \cdot)$. Let $(S(t))_{t\in\mathbf{R}}$ be the isometry group generated by (D, A). Let $T > 0$, $g \in H$ and $b \in \mathcal{C}([0,T], H)$. Let $u \in \mathcal{C}([0,T], X)$ be defined by

$$u(t) = S(t)g + \int_0^t S(t-s)b(s)\,\mathrm{d}s.$$

Prove that the real-valued function $t \mapsto \|u(t)\|^2$ belongs to $\mathcal{C}^1([0,T]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = 2(b(t) \mid u(t)),$$

for all $t \in [0,T]$. **Hint**: use a density argument on g and b.

Exercise 5.3. Apply Proposition 5.1 and Proposition 5.4 to the examples given in §3.6 and Exercises 3.4-3.8.

Chapter 6

Nonlinear evolution equations

6.1 Assumption on the nonlinearity

Let $\bar{B}_M$ denote the closed ball of X of center 0 and radius $M > 0$.

Definition 6.1. A function $f : X \to X$ is *Lipschitz continuous on bounded subsets of* X if for all $M > 0$, there exists a constant $L_M > 0$ such that, for all $u, v \in \bar{B}_M$ it holds

$$\|f(u) - f(v)\| \leq L_M \|u - v\|. \tag{6.1}$$

Remark 6.2. Let a function $f : X \to X$ be Lipschitz continuous on bounded subsets of X, we set

$$L(M) = \inf\{L > 0 : (6.1) \text{ holds for all } u, v \in \bar{B}_M\},$$

In particular, $M \mapsto L(M)$ is a non-decreasing function of M.

Remark 6.3. Let $\mathcal{U}$ be an open subset of $\mathbf{R}^N$ with $\mathcal{C}^1$ boundary. We consider a continuous function $f : \mathbf{R} \to \mathbf{R}$ such that there exists $\alpha \geq 0$ and $C > 0$ such that

$$f(0) = 0, \quad |f(u) - f(v)| \leq C(|v|^\alpha + |v|^\alpha)|u - v|,$$

for all $u, v \in \mathbf{R}$.

1. If $N \geq 3$, assume that
$$0 \leq \alpha \leq \frac{4}{N-2}$$
(no assumption on α if $N = 1$ or 2). Then f is Lipschitz continuous from bounded subsets of $H_0^1(\mathcal{U})$ to $H^{-1}(\mathcal{U})$. A common abuse of notation consists in using the same notation for the real-valued function $f : \mathbf{R} \to \mathbf{R}$ chosen as above and the corresponding map $f : H_0^1(\mathcal{U}) \to H^{-1}(\mathcal{U})$.

2. If $N \geq 3$, assume that
$$0 \leq \alpha \leq \frac{2}{N-2}$$
(no assumption on α if $N = 1$ or 2). Then, f is Lipschitz continuous from bounded subsets of $H_0^1(\mathcal{U})$ to $L^2(\mathcal{U})$.

See Exercice 6.1.

6.2 Weak formulation of the nonlinear problem

In this Chapter, (D, A) is a maximal dissipative operator with dense domain on X and $(S(t))_{t\geq 0}$ is the semigroup generated by A. The function $f : X \to X$ is Lipschitz continuous on bounded subsets of X. We denote $L(M) > 0$ the corresponding constant in (6.1).

Let $T > 0$. For $g \in D$, a natural objective is to solve the nonlinear problem

$$\begin{cases} u \in \mathcal{C}([0,T], D) \cap \mathcal{C}^1([0,T], X), \\ \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) = Au(t) + f(u(t)), \quad \text{for all } t \in [0,T], \\ u(0) = g. \end{cases} \tag{6.2}$$

Actually, we will rather look for a weaker form of the above problem. From Proposition 5.1, a solution of (6.2) is also a solution of

$$\begin{cases} u \in \mathcal{C}([0,T], X), \\ u(t) = S(t)g + \displaystyle\int_0^t S(t-s) f(u(s))\,\mathrm{d}s. \end{cases} \tag{6.3}$$

For $g \in X$, using Corollary 5.5 and the notation of Propositions 3.15 and 4.8, we know that (6.3) is equivalent to solving the problem

$$\begin{cases} u \in \mathcal{C}([0,T], X) \cap \mathcal{C}^1([0,T], Y), \\ \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) = Bu(t) + f(u(t)), \quad \text{for all } t \in [0,T], \\ u(0) = g. \end{cases} \tag{6.4}$$

In this chapter, we will focus on the resolution of (6.3).

6.3 Local existence and uniqueness

6.3.1 Uniqueness

We start with a general uniqueness result for (6.3).

Proposition 6.4. *Let $T > 0$ and $g \in X$. Then there exists at most one solution of* (6.3).

Proof. Let u, v be two solutions of (6.3). Set

$$M = \sup_{t\in[0,T]} \max\{\|u(t)\|; \|v(t)\|\}.$$

We have by (6.3) and $\|S(t)\| \leq 1$,

$$\|u(t) - v(t)\| \leq \int_0^t \|f(u(s)) - f(v(s))\|\,\mathrm{d}s \leq L(M) \int_0^t \|u(s) - v(s)\|\,\mathrm{d}s.$$

It follows from the Gronwall Lemma D.4 (with $C_1 = 0$, $C_2 = L(M)$ and $a \equiv 1$) that $\|u(t) - v(t)\| = 0$, for all $t \in [0,T]$. □

6.3.2 Existence of a local solution by contraction

Here we prove a version of the Cauchy-Lipschitz theorem.

Theorem 6.5. *Let $M > 0$ and fix*

$$T_M = \frac{1}{2 + 2L(2M + \|f(0)\|)} > 0. \tag{6.5}$$

For any $g \in X$ such that $\|g\| \leq M$, there exists a unique solution u of (6.2) *on* $[0, T]$.

Proof. The uniqueness statement is proved by Proposition 6.4. Let $M > 0$ and let $g \in X$ be such that $\|g\| \leq M$. Fix $K = 2M + \|f(0)\|$. We introduce

$$E = \{u \in \mathcal{C}([0, T_M], X) : \|u(t)\| \leq K, \text{ for all } t \in [0, T_M]\}.$$

We equip E with the distance generated by norm of $\mathcal{C}([0, T_M], X)$, *i.e.*, for any $u, v \in E$,

$$d(u, v) = \sup_{t \in [0, T_M]} \|u(t) - v(t)\|.$$

Since $\mathcal{C}([0, T_M], X)$ is a Banach space and E is closed in $\mathcal{C}([0, T_M], X)$, (E, d) is a complete metric space. For all $u \in E$, we define $\Phi(u) \in \mathcal{C}([0, T_M], X)$ by

$$\Phi(u)(t) = S(t)g + \int_0^t S(t - s) f(u(s)) \, \mathrm{d}s,$$

for all $t \in [0, T_M]$.

First, we prove that $\Phi : E \to E$. Indeed, for any $s \in [0, T_M]$, one has $f(u(s)) = f(0) + f(u(s)) - f(0)$, and thus by the triangle inequality and the assumption on f,

$$\begin{aligned}\|f(u(s))\| &\leq \|f(0)\| + \|f(u(s)) - f(0)\| \\ &\leq \|f(0)\| + L(K)\|u(s)\| \leq \|f(0)\| + L(K)K.\end{aligned}$$

It follows from $\|S(t)\| \leq 1$ and the definition of T_M in (6.5) that for any $t \in [0, T_M]$,

$$\begin{aligned}\|\Phi(u)(t)\| &\leq \|g\| + \int_0^t \|F(u(s))\| \, \mathrm{d}s \\ &\leq M + T_M(\|f(0)\| + L(K)K) \\ &\leq M + \frac{\|f(0)\| + L(K)K}{2 + 2L(K)} \leq M + \frac{1}{2}\|f(0)\| + \frac{1}{2}K < K.\end{aligned}$$

Second we prove that Φ is a contraction on (E, d). Indeed, for any $u, v \in E$, and for any $t \in [0, T_M]$,

$$\begin{aligned}\|\Phi(u)(t) - \Phi(v)(t)\| &\leq \int_0^t \|f(u(t)) - f(v(t))\| \, \mathrm{d}s \\ &\leq T_M L(K) d(u, v) \leq \frac{1}{2} d(u, v).\end{aligned}$$

By the Banach Fixed-Point Theorem A.26, Φ has a (unique) fixed-point $u \in E$, which is a solution of (6.3). $\square$

6.3.3 Maximal solution

Theorem 6.6. *There exists a function $T_{\max} : X \to (0, \infty]$ with the following properties. For any $g \in X$, there exists $u \in \mathcal{C}([0, T_{\max}(g)), X)$, such that for all $T \in (0, T_{\max}(g))$, u is the unique solution of* (6.3)*. Moreover, the following alternative holds:*

1. *Either $T_{\max}(g) = \infty$;*
2. *Or $T_{\max}(g) < \infty$ and then $\lim_{t \uparrow T_{\max}(g)} \|u(t)\| = \infty$.*

Remark 6.7. When property (1) holds, one says that the solution is globally defined, or global. When property (2) holds, one says that the solution blows up in finite time.

To prove that a solution is global, it is enough to prove the existence of *a priori* bounds on the solution, preventing $\|u(t)\|$ to become arbitrarily large in finite time.

The fact that when the solution ceases to exist in finite time $T_{\max}(g)$, the norm of u becomes infinite as $t \uparrow T_{\max}(g)$ is related to the assumptions made on the nonlinearity. It may fail in some other contexts, notably when the notion of critical nonlinearity arises.

Proof. Let $g \in X$ and $M = \|g\|$. We define

$$T_{\max}(g) = \sup\{T > 0 : \text{there exists a solution } u \text{ of } (6.3) \text{ on } [0, T]\}.$$

By Theorem 6.5, we know that $T_{\max}$ is well-defined and $T_{\max} \geq T_M > 0$. Now, we define a function $u \in \mathcal{C}([0, T_{\max}(g)), X)$ which is solution of (6.3) on $[0, T]$ for any $T \in (0, T_{\max}(g))$. Let $t \in [0, T_{\max}(g))$. Let $T \in [t, T_{\max}(g))$. By the definition of $T_{\max}(g)$ as a supremum, there exists a solution u_T of (6.3) on $[0, T]$. Then, we set $u(t) = u_T(t)$ on $[0, T]$. By the general uniqueness statement Proposition 6.4, this definition does not depend on the choice of $T \in [t, T_{\max}(g))$. Thus, it provides a function $u \in \mathcal{C}([0, T_{\max}(g)), X)$ which is indeed a solution of (6.3) on $[0, T]$ for any $T \in (0, T_{\max}(g))$. Last, note that by the definition of $T_{\max}$, this solution cannot be extended beyond $T_{\max}$. This solution is called the *maximal solution* of (6.3).

Now, we prove the second statement of the Theorem, called the *blowup alternative.* Fix $\tau \in [0, T_{\max}(g))$, set $M = \|u(\tau)\|$ and consider $T_M > 0$ given by (6.5). By Theorem 6.5, there exists a solution v of

$$\begin{cases} v \in \mathcal{C}([0, T_M], X), \\ v(t) = S(t)u(\tau) + \displaystyle\int_0^t S(t-s) f(v(s))\, \mathrm{d}s. \end{cases} \tag{6.6}$$

We define a function $w \in \mathcal{C}([0, \tau + T_M], X)$ by setting

$$w(t) = \begin{cases} u(t) & \text{if } t \in [0, \tau], \\ v(t - \tau) & \text{if } t \in [\tau, \tau + T_M]. \end{cases}$$

We see that w is a solution of the problem (6.3) on the interval $[0, T]$, for $T = \tau + T_M$. By the definition of $T_{\max}(g)$, this shows that

$$\tau + T_M < T_{\max}(g).$$

Assume $T_{\max}(g) < \infty$. By the general definition of T_M in (6.5) and the value of $M = \|u(\tau)\|$ in the present context, we obtain

$$\frac{1}{2 + 2L(2\|u(\tau)\| + \|f(0)\|)} \leq T_{\max}(g) - \tau.$$

This is equivalent to

$$L(2\|u(\tau)\| + \|f(0)\|) \geq \frac{1}{2(T_{\max}(g) - \tau)} - 1, \tag{6.7}$$

which proves that in the case $T_{\max}(g) < \infty$, $\lim_{t\uparrow T_{\max}(g)} \|u(t)\| = \infty$. □

Remark 6.8. The proof of Theorem 6.6 actually provides a qualitative information (6.7) on the size of $\|u(t)\|$ as $t \uparrow T_{\max}$ in the case where $T_{\max} < \infty$.

In particular, we see that if the function f is (globally) Lipschitz on X (i.e. there exists $C > 0$ such that $L(M) \leq C$ for all $M > 0$), then necessarily $T_{\max}(g) = \infty$.

6.4 Continuous dependence on the initial data

Proposition 6.9. *In the context of Theorem 6.6, the following properties hold.*

1. *The function $T_{\max} : X \to (0, \infty]$ is lower semi-continuous;*
2. *If $g_n \to g$ as $n \to \infty$ in X, then for any $T \in (0, T_{\max}(g))$, $u_n \to u$ in $\mathcal{C}([0,T], X)$ as $n \to \infty$, where u_n and u are the solutions of (6.3) corresponding respectively to g_n and g.*

Proof. Let $T \in (0, T_{\max}(g))$. To prove (1)-(2), it suffices to show that if $g_n \to g$ then for n large enough $T_{\max}(g_n) > T$ and $u_n \to u$ in $\mathcal{C}([0,T], X)$.

Set $M = 1 + 2\sup_{t\in[0,T]} \|u(t)\|$ and define

$$\tau_n = \sup\{t \in [0, T_{\max}(g)) : \|u_n(s)\| \leq 2M \text{ for all } s \in [0,t]\}.$$

Since $\|g_n\| < M$ for n large enough, $\tau_n > 0$ is well-defined. Moreover, by Theorem 6.5, $\tau_n > T_M$. For any $t \in [0, \min(T;\tau_n)]$, we have

$$\|u(t) - u_n(t)\| \leq \|g - g_n\| + L(2M) \int_0^t \|u(s) - u_n(s)\| \,\mathrm{d}s,$$

and thus by the Gronwall Lemma D.4, for any $t \in [0, \min(T;\tau_n)]$,

$$\|u(t) - u_n(t)\| \leq \|g - g_n\| \exp\left(L(2M)T\right). \tag{6.8}$$

This proves that for any $t \in [0, \min(T;\tau_n)]$,

$$\|u_n(t)\| \leq \|u(t)\| + \|u(t) - u_n(t)\| \leq \frac{M}{2} + \|g - g_n\| \exp\left(L(2M)T\right) < 2M,$$

for n large enough. Therefore, $\tau_n > T$, which also justifies that $T_{\max}(g_n) > T$.

Last, we see that estimate (6.8) proves $u_n \to u$ in $\mathcal{C}([0,T], X)$. □

Remark 6.10. Even in the case where $T_{\max}(g) = \infty$, the uniform convergence $u_n \to u$ is obtained only on compact sets of time $[0,T]$.

6.5 Exercises for Chapter 6

Exercise 6.1. Let $\mathcal{U}$ be an open subset of $\mathbf{R}^N$ with $\mathcal{C}^1$ boundary. We recall that if $p \geq 2$ satisfies $(N-2)p \leq 2N$ then it holds for a constant $C > 0$,

$$\|\cdot\|_{L^p(\mathcal{U})} \leq C\|\cdot\|_{H^1(\mathcal{U})}, \quad \|\cdot\|_{H^{-1}(\mathcal{U})} \leq C\|\cdot\|_{L^{p'}(\mathcal{U})}, \tag{6.9}$$

where $1/p + 1/p' = 1$.

We consider a continuous function $f : \mathbf{R} \to \mathbf{R}$ such that there exists $\alpha \geq 0$ and $C > 0$ such that

$$f(0) = 0, \quad |f(u) - f(v)| \leq C(|u|^\alpha + |v|^\alpha)|u - v|,$$

for all $u, v \in \mathbf{R}$.

1. Let $p \in [\alpha + 1, \infty]$. Prove that for some constant $C > 0$,

   $$\|f(u) - f(v)\|_{L^{\frac{p}{\alpha+1}}} \leq C\left(\|u\|_{L^p}^\alpha + \|v\|_{L^p}^\alpha\right)\|u - v\|_{L^p}.$$

   for all $u, v \in L^p(\mathcal{U})$.

2. We assume in this question that if $N \geq 3$,

   $$0 \leq \alpha \leq \frac{4}{N-2}$$

   (no assumption on α if $N = 1$ or 2). Prove that f is Lipschitz continuous from bounded subsets of $H_0^1(\mathcal{U})$ to $H^{-1}(\mathcal{U})$.

3. We assume in this question that if $N \geq 3$,

   $$0 \leq \alpha \leq \frac{2}{N-2}$$

   (no assumption on α if $N = 1$ or 2). Prove that f is Lipschitz continuous from bounded subsets of $H_0^1(\mathcal{U})$ to $L^2(\mathcal{U})$.

Exercise 6.2. Consider the equation

$$\begin{cases} \partial_t^2 u - \Delta u = f(u, \partial_t u, \nabla u) & (t, x) \in \mathbf{R} \times \mathbf{R}^N, \\ u(0, x) = g(x), \quad \partial_t u(0, x) = h(x) & x \in \mathbf{R}^N. \end{cases}$$

Assume that the function $f : \mathbf{R} \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ is globally Lipschitz and satisfies $f(0) = 0$. Write a global existence result for any initial data $(g, h) \in H^2(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$. **Hint**: use the space $X = H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$ and a first order formulation of the equation involving the function

$$F(g, h) = (0, f(g, h, \nabla g) + g).$$

Chapter 7

The nonlinear Klein-Gordon equation

In this Chapter, we are interested in solving the local Cauchy problem and establishing global results for the nonlinear Klein-Gordon equation

$$\partial_t^2 u - \Delta u + u - \epsilon |u|^\alpha u = 0 \quad (t, x) \in \mathbf{R} \times \mathcal{U}, \tag{7.1}$$

where $\mathcal{U}$ is a domain of $\mathbf{R}^N$ with $\mathcal{C}^1$ boundary, $\epsilon = \pm 1$, and $\alpha > 0$.

We rewrite the nonlinear Klein-Gordon equation (7.1) as a system in the unknown $\boldsymbol{u} = (u, v)$,

$$\begin{cases} \partial_t u = v \\ \partial_t v = \Delta u - u + f(u), \end{cases}$$

where $f(u) = |u|^\alpha u$. This system also writes

$$\partial_t \boldsymbol{u} = A\boldsymbol{u} + \boldsymbol{f}(\boldsymbol{u}) \tag{7.2}$$

where

$$A\boldsymbol{u} = (v, \Delta u - u) \quad \text{and} \quad \boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} 0 \\ \epsilon f(u) \end{pmatrix}. \tag{7.3}$$

7.1 The linear Klein-Gordon equation

Let $\boldsymbol{H} = H_0^1(\mathcal{U}) \times L^2(\mathcal{U})$. We denote by $\boldsymbol{g} = (g, h)$ an element of $\boldsymbol{H}$. We equip $\boldsymbol{H}$ with the scalar product

$$(\boldsymbol{g} \mid \tilde{\boldsymbol{g}}) = \int (\nabla g \cdot \nabla \tilde{g} + g\tilde{g} + h\tilde{h}) \, \mathrm{d}x.$$

We also define by $\| \cdot \|$ the corresponding norm. Define

$$\begin{cases} D = \{\boldsymbol{g} = (g, h) \in H : \Delta g \in L^2(\mathcal{U}), h \in H_0^1(\mathcal{U})\} \\ A\boldsymbol{g} = (h, \Delta g - g) \quad \text{for all } \boldsymbol{g} \in D. \end{cases}$$

Recall from Proposition 3.32 that the operator (D, A) is skew-adjoint and maximal dissipative with dense domain in H. We denote by $(S(t))_{t \in \mathbf{R}}$ the corresponding group of isometries given by Theorem 4.5. It follows from Theorem 4.1

and Theorem 4.5 that for any $\boldsymbol{g} \in D$, $\boldsymbol{u}(t) = S(t)\boldsymbol{g} \in \mathcal{C}(\mathbf{R}, D) \cap \mathcal{C}^1(\mathbf{R}, \boldsymbol{H})$ is the solution of the linear problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = A\boldsymbol{u}. \tag{7.4}$$

Moreover, for any $\boldsymbol{g} \in \boldsymbol{H}$, $\boldsymbol{u}(t) = S(t)\boldsymbol{g} \in \mathcal{C}(\mathbf{R}, \boldsymbol{H})$ can be seen as the weak solution of the linear problem (7.4) (see Proposition 4.8).

The identity $\|S(t)\boldsymbol{g}\| = \|\boldsymbol{g}\|$ is interpreted as the conservation of energy for the solution $\boldsymbol{u}(t) = S(t)\boldsymbol{g}$ of the linear problem (7.4)

$$\int \left\{|v(t)|^2 + |\nabla u(t)|^2 + |u(t)|^2\right\} \mathrm{d}x = \int \left\{|h|^2 + |\nabla g|^2 + |g|^2\right\} \mathrm{d}x.$$

7.2 Local Cauchy theory for NLKG

If $N \geq 3$, assume that

$$0 \leq \alpha \leq \frac{2}{N-2}, \tag{7.5}$$

(no assumption is to be made on α if $N = 1$ or 2). Then, it follows from Remark 6.3 that $u \in H_0^1 \mapsto f(u) \in L^2$ is well-defined and Lipschitz continuous on bounded sets of H_0^1, as a function from H_0^1 to L^2. As a consequence, the map $\boldsymbol{u} \in \boldsymbol{H} \mapsto \boldsymbol{f}(\boldsymbol{u}) \in \boldsymbol{H}$ defined in (7.3) is well-defined and Lipschitz continuous on the bounded sets of $\boldsymbol{H}$ to $\boldsymbol{H}$, as introduced in Definition 6.1.

This is precisely the condition of applicability of the local nonlinear Cauchy theory developed in Chapter 6. We write the weak formulation of the problem (7.2) as follows: for $\boldsymbol{g} \in \boldsymbol{H}$ and $T > 0$,

$$\begin{cases} \boldsymbol{u} \in \mathcal{C}([0,T], \boldsymbol{H}), \\ \boldsymbol{u}(t) = S(t)\boldsymbol{g} + \displaystyle\int_0^t S(t-s)\boldsymbol{f}(\boldsymbol{u}(s))\,\mathrm{d}s. \end{cases} \tag{7.6}$$

In particular, the following result is a direct application of Theorems 6.5 and 6.6.

Theorem 7.1. *There exists a function $T_{\max} : \boldsymbol{H} \to (0, \infty]$ with the following properties. For any $\boldsymbol{g} \in \boldsymbol{H}$, there exists $\boldsymbol{u} \in \mathcal{C}([0, T_{\max}(\boldsymbol{g})), \boldsymbol{H})$ such that for all $T \in (0, T_{\max}(\boldsymbol{g}))$, $\boldsymbol{u}$ is the unique solution of (7.6). Moreover, for any $\boldsymbol{g} \in \boldsymbol{H}$, the following alternative holds:*

1. *Either $T_{\max}(\boldsymbol{g}) = \infty$;*
2. *Or $T_{\max}(\boldsymbol{g}) < \infty$ and then $\lim_{t \uparrow T_{\max}(\boldsymbol{g})} \|\boldsymbol{g}(t)\| = \infty$.*

In the next sections, we identify conditions on α, the sign ϵ in the definition of the nonlinearity $\boldsymbol{f}$ in (7.3) and possibly on the initial data $\boldsymbol{g}$ such that the solution $\boldsymbol{u}$ given the above theorem is global, or on the contrary has a finite time of existence and thus blows up in finite time.

7.3 Conservation of energy and other identities

Let $\boldsymbol{g} = (g, h) \in \boldsymbol{H}$ and let $\boldsymbol{u} = (u, v) \in \mathcal{C}([0, T_{\max}(\boldsymbol{g}), \boldsymbol{H})$ be the corresponding maximal solution of (7.6) provided by Theorem 7.1. We define the following quantities

$$
\begin{aligned}
W(t) &= \frac{1}{2}(\boldsymbol{u}(t) \mid \boldsymbol{u}(t)) = \frac{1}{2}\int \left\{|v(t)|^2 + |\nabla u(t)|^2 + |u(t)|^2\right\} \mathrm{d}x, \\
V(t) &= \frac{\epsilon}{\alpha+2}\int |u(t)|^{\alpha+2}\,\mathrm{d}x, \\
E(t) &= W(t) - V(t), \\
M(t) &= \int u^2(t)\,\mathrm{d}x.
\end{aligned}
$$

Proposition 7.2. *Let any $T \in [0, T_{\max}(g))$. The functions $t \mapsto W(t)$, $t \mapsto V(t)$, $t \mapsto E(t)$ are in $\mathcal{C}^1([0,T])$. The function $t \mapsto M(t)$ is in $\mathcal{C}^2([0,T])$. Moreover, for all $t \in [0, T_{\max}(\boldsymbol{g}))$, it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = \epsilon \int f(u(t))v(t)\,\mathrm{d}x, \tag{7.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = \epsilon \int f(u(t))v(t)\,\mathrm{d}x, \tag{7.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0, \tag{7.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}M(t) = 2\int v(t)u(t)\,\mathrm{d}x, \tag{7.10}$$

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}M(t) &= -2(\alpha+2)E(t) \\
&\quad + \alpha\int \left\{|\nabla u(t)|^2 + u^2(t)\right\}\mathrm{d}x + (\alpha+4)\int v^2(t)\,\mathrm{d}x.
\end{aligned} \tag{7.11}
$$

Proof. Let $T \in [0, T_{\max}(g))$. All the quantities appearing in these identities are well-defined (for example, $f(u(t)) \in L^2$ and $v(t) \in L^2$ for all $t \in [0,T]$, so that $\int f(u(t))v(t)\,\mathrm{d}x$ is well-defined). However, we need to use density arguments to justify them rigorously.

We introduce a sequence $\{\boldsymbol{g}_n\}_{n=0}^{\infty}$ of functions of D such that $\boldsymbol{g}_n \to \boldsymbol{g}$ in $\boldsymbol{H}$ as $n \to \infty$, and a sequence $\{\boldsymbol{b}_n\}_{n=0}^{\infty}$ of functions of $\mathcal{C}([0,T], D)$ such that $\boldsymbol{b}_n \to \boldsymbol{f}(\boldsymbol{u})$ in $C([0,T], \boldsymbol{H})$. The existence of such sequences follows directly from the density of D in $\boldsymbol{H}$. Moreover, we define

$$\boldsymbol{u}_n(t) = S(t)\boldsymbol{g}_n + \int_0^t S(t-s)\boldsymbol{b}_n(s)\,\mathrm{d}s.$$

Using $\|S(t)\| \le 1$, it is elementary to check that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\| = 0. \tag{7.12}$$

By Proposition 5.4, $\boldsymbol{u}_n$ is the unique solution of

$$
\begin{cases}
\boldsymbol{u}_n \in \mathcal{C}([0,T], D) \cap \mathcal{C}^1([0,T], \boldsymbol{H}), \\
\dfrac{\mathrm{d}\boldsymbol{u}_n}{\mathrm{d}t}(t) = A\boldsymbol{u}_n(t) + \boldsymbol{b}_n(t), \quad \text{for all } t \in [0,T], \\
\boldsymbol{u}_n(0) = \boldsymbol{g}_n.
\end{cases} \tag{7.13}
$$

We denote by W_n, V_n and M_n the analogue for each $\boldsymbol{u}_n$ of the quantities introduced above for $\boldsymbol{u}$.

It follows from the regularity of $\boldsymbol{u}_n$ that $t \mapsto W_n(t)$ is of class $\mathcal{C}^1$ and, using the fact that the operator A is skew-adjoint,

$$\frac{\mathrm{d}}{\mathrm{d}t} W_n = \left(\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{u}_n \mid \boldsymbol{u}_n \right) = (A\boldsymbol{u}_n \mid \boldsymbol{u}_n) + (\boldsymbol{b}_n \mid \boldsymbol{u}_n) = (\boldsymbol{b}_n \mid \boldsymbol{u}_n).$$

Hence,

$$W_n(t) = W_n(0) + \int_0^t (\boldsymbol{b}_n(s) \mid \boldsymbol{u}_n(s))\, \mathrm{d}s.$$

Passing to the limit $n \to \infty$, using (7.12), we find

$$W(t) = W(0) + \int_0^t (\boldsymbol{f}(\boldsymbol{u}(s)) \mid \boldsymbol{u}(s))\, \mathrm{d}s.$$

This proves that W is $\mathcal{C}^1$ on $[0,T]$ and that (7.7) holds.

To prove the identity concerning V, we first observe that for any $w, z \in \mathbf{R}$,

$$\frac{1}{\alpha+2}\left(|w+z|^{\alpha+2} - |w|^{\alpha+2}\right) - zf(w) = \int_0^z (f(\sigma) - f(w))\, \mathrm{d}\sigma,$$

and thus

$$\left| \frac{1}{\alpha+2}\left(|w+z|^{\alpha+2} - |w|^{\alpha+2}\right) - zf(w) \right| \leq C|z|^2 \left(|w|^\alpha + |w+z|^\alpha\right).$$

From this, it follows using the Hölder inequality that

$$\begin{aligned} \left| \frac{1}{\alpha+2} \int \left(|w+z|^{\alpha+2} - |w|^{\alpha+2}\right) \mathrm{d}x - \int zf(w)\, \mathrm{d}x \right| & \\ \leq \|z\|^2_{L^{\alpha+2}} \left(\|w\|^\alpha_{L^{\alpha+2}} + \|w+z\|^\alpha_{L^{\alpha+2}}\right). & \end{aligned}$$

Applying this inequality to $w = u_n(t)$ and $z = u_n(t+\delta) - u_n(t)$, one gets

$$\begin{aligned} \left| V_n(t+\delta) - V_n(t) - \epsilon \int (u_n(t+\delta) - u_n(t))\, f(u_n(t))\, \mathrm{d}x \right| & \\ \leq C\|u_n(t+\delta) - u_n(t)\|^2_{L^{\alpha+2}} \left(|V_n(t+\delta)|^{\frac{\alpha}{\alpha+2}} + |V_n(t)|^{\frac{\alpha}{\alpha+2}}\right). & \end{aligned}$$

This shows that V_n is of class $\mathcal{C}^1$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} V_n(t) = \epsilon \int \frac{\mathrm{d}}{\mathrm{d}t} u_n(t) f(u_n(t))\, \mathrm{d}x.$$

By the first line of the equation of $\boldsymbol{u}_n$, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t} V_n(t) = \epsilon \int f(u_n(t)) v_n(t)\, \mathrm{d}x.$$

We proceed as for W_n to obtain the identity (7.8) for $\boldsymbol{u}$.

The identity (7.9) is obtained directly on $\boldsymbol{u}$ since $u \in \mathcal{C}^1([0,T],L^2)$. To prove (7.10), we first work on the sequence $\{\boldsymbol{u}_n\}_{n=0}^{\infty}$. Since $\boldsymbol{u}_n \in \mathcal{C}^1([0,T],\boldsymbol{H}$, we know that $v_n \in \mathcal{C}^1([0,T],L^2)$ and using the equation and next (3.2),

$$\begin{aligned}
M_n'' &= 2\int \left(\frac{\mathrm{d}}{\mathrm{d}t}v_n\right) u_n \,\mathrm{d}x + 2\int v_n^2 \,\mathrm{d}x \\
&= 2\int (\Delta u_n - u_n + b_n)\, u_n \,\mathrm{d}x + 2\int v_n^2 \,\mathrm{d}x \\
&= -2\int \left(|\nabla u_n|^2 + u_n^2 + b_n u_n\right) \mathrm{d}x + 2\int v_n^2 \,\mathrm{d}x.
\end{aligned}$$

Integrating this identity on $[0,t]$, and passing to the limit as $n \to \infty$ in the integral form, we find

$$M'(t) = M'(0) - 2\int_0^t \int \left\{|\nabla u|^2 + u^2 - \epsilon f(u)u - v^2\right\} \mathrm{d}x\,\mathrm{d}s.$$

This means that M is of class $\mathcal{C}^2$ on $[0,T]$ and

$$M''(t) = -2\int \left\{|\nabla u|^2 + u^2 - \epsilon f(u)u - v^2\right\} \mathrm{d}x.$$

Using $f(u)u = |u|^{\alpha+2}$ and the expression of the energy, we find

$$M''(t) = -2(\alpha+2)E(t) + (\alpha-2)\int \left\{|\nabla u|^2 + u^2\right\} \mathrm{d}x + (\alpha+4)\int v^2 \,\mathrm{d}x,$$

which is (7.10). □

Corollary 7.3. *For any $t \in [0, T_{\max}(\boldsymbol{g}))$, it holds*

$$E(t) = E(0).$$

7.4 Global existence in the defocusing case

When $\epsilon = -1$, we say that the nonlinearity is *defocusing*. It follows directly from the energy conservation that the following hold.

Proposition 7.4. *Let $\epsilon = -1$. Then, for all $\boldsymbol{g} \in \boldsymbol{H}$, the maximal solution $\boldsymbol{u}$ of* (7.6) *given by Theorem 7.1 is global,* i.e. $T_{\max}(\boldsymbol{g}) = \infty$.

Proof. When $\epsilon = -1$, it is clear that $V(t) \le 0$. Thus, by the conservation of energy, for all $t \in [0, T_{\max}(\boldsymbol{g}))$,

$$\frac{1}{2}\|\boldsymbol{u}(t)\|^2 = W(t) \le E(t) = E(0).$$

This a priori bound prevent the norm of $\boldsymbol{u}(t)$ to become large and thus the solution is global. □

7.5 Global existence for small data

In this section, we consider the *focusing* case $\epsilon = 1$.

Theorem 7.5. *There exists $\mu > 0$ such that for any $\boldsymbol{g} \in \boldsymbol{H}$ with $\|g\| \le \mu$, the solution $\boldsymbol{u}$ of* (7.6) *given by given by Theorem 7.1 is global,* i.e. $T_{\max}(\boldsymbol{g}) = \infty$.

Proof. From (6.9) and the assumption on α (in the case $N \ge 3$), it follows that

$$\begin{aligned} V(t) &\le C \int |u|^{\alpha+2} \le \frac{1}{4}\|u(t)\|_{L^2}^2 + C \int |u(t)|^{2\alpha+2} \\ &\le \frac{1}{2}\|u(t)\|_{L^2}^2 + C\|u(t)\|_{H^1}^{2\alpha+2} \le \frac{1}{2}W(t) + C[W(t)]^{\alpha+1}. \end{aligned}$$

Thus, using the conservation of energy (Corollary 7.3)

$$\begin{aligned} W(t) = E(t) + V(t) &\le E(0) + \frac{1}{2}W(t) + C[W(t)]^{\alpha+1} \\ &\le W(0) + \frac{1}{2}W(t) + C[W(t)]^{\alpha+1}. \end{aligned}$$

Thus, $W(t)$ satisfies

$$\theta(W(t)) \ge -W(0) \ge -\frac{\mu^2}{2} \quad \text{where} \quad \theta(w) = Cw^{\alpha+1} - \frac{w}{2}.$$

We denote $m > 0$ and $x_m > 0$ such that

$$\begin{aligned} x_m &= \{2C(\alpha+1)\}^{-\frac{1}{\alpha}}, \quad \theta'(x_m) = 0 \\ m &= -\inf_{[0,\infty)} \theta = -\theta(m) = \frac{\alpha}{2(\alpha+1)} \{2C(\alpha+1)\}^{-\frac{1}{\alpha}}. \end{aligned}$$

If we assume that $W(0) = \frac{1}{2}\mu^2 < \min(x_m; m)$, then for all $t \in [0, T_{\max}(\boldsymbol{g}))$, $\theta(W(t)) > -m$. By continuity of $t \mapsto W(t)$, it follows that $W(t) < x_m$, for all $t \in [0, T_{\max}(\boldsymbol{g}))$, and thus $W(t)$ is uniformly bounded, which proves that the solution $\boldsymbol{u}$ is global. □

7.6 Blowup in finite time

Theorem 7.6. *If $\boldsymbol{g} \in \boldsymbol{H}$ is such that*

$$E(0) < 0$$

then the solution $\boldsymbol{u}$ of (7.6) *given by given by Theorem 7.1 blows up in finite time,* i.e. $T_{\max}(\boldsymbol{g}) < \infty$.

Remark 7.7. We note that it is easy to generate initial data $\boldsymbol{g}$ such that $E(0) < 0$. Indeed, we check that for any $g \ne 0$, there exists $\lambda > 0$ large enough such that the energy of λg is negative.

Proof. The proof is by contradiction. We assume that the solution is global $T_{\max}(\boldsymbol{g}) = \infty$, and we perform estimates for all $t \ge 0$. We know that $E(t) = E(0)$, and thus from (7.11), we have

$$M''(t) \ge -2(\alpha+2)E(0) > 0.$$

In particular, $\lim_\infty M' = \infty$ and $\lim_\infty M = \infty$. Estimate (7.11) also implies that

$$M''(t) \geq (\alpha + 4)\|v(t)\|_{L^2}^2;$$

Moreover, from (7.10), we have

$$[M'(t)]^2 \leq 4M(t)\|v(t\|_{L^2}^2$$

Therefore, we obtain for all $t \geq 0$,

$$M''(t)M(t) \geq \left(1 + \frac{\alpha}{4}\right)[M'(t)]^2.$$

This implies directly that for all $t \geq 0$,

$$[M^{-\frac{\alpha}{4}}]''(t) \leq 0.$$

Since $\lim_\infty M' = \infty$, one has $\lim_\infty M^{-\frac{\alpha}{4}} = 0$. Thus, there exists $t_1 > 0$ such that $\frac{\mathrm{d}}{\mathrm{d}t}[M^{-\frac{\alpha}{4}}(t_1)] < 0$, and for all $t \geq t_1$,

$$0 \leq M^{-\frac{\alpha}{4}}(t) \leq M^{-\frac{\alpha}{4}}(t_1) + (t - t_1)[M^{-\frac{\alpha}{4}}]'(t_1),$$

which is absurd for $t \geq t_1$ large enough. This contradiction implies that $T_{\max}(\boldsymbol{g}) < \infty$. $\square$

7.7 Exercises for Chapter 7

Exercise 7.1. The damped linear Klein-Gordon equation.

Let $\boldsymbol{H} = H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$ for $N \geq 1$. We denote by $\boldsymbol{g} = (g, h)$ an element of $\boldsymbol{H}$. We equip $\boldsymbol{H}$ with the scalar product (all functions are real-valued)

$$(\boldsymbol{g} \mid \tilde{\boldsymbol{g}}) = \int_{\mathbf{R}^N} (\nabla g \cdot \nabla \tilde{g} + g\tilde{g} + h\tilde{h})\,\mathrm{d}x.$$

We denote by $\|\cdot\|$ the corresponding norm on $\boldsymbol{H}$. Let $\alpha > 0$. Define

$$\begin{cases} D = \{\boldsymbol{g} = (g, h) \in \boldsymbol{H} : \Delta g \in L^2(\mathbf{R}^N), h \in H^1(\mathbf{R}^N)\} \\ A\boldsymbol{g} = (h, \Delta g - g - 2\alpha h) \quad \text{for all } \boldsymbol{g} = (g, h) \in D. \end{cases}$$

1. Prove that the operator (D, A) is maximal dissipative with dense domain in $\boldsymbol{H}$. **Hint**: we recall the identity, for $g, \tilde{g} \in H^1(\mathbf{R}^N)$ with $\Delta g \in L^2$, $\int_{\mathbf{R}^N} \Delta g\, \tilde{g}\ \mathrm{d}x = -\int_{\mathbf{R}^N} \nabla g \cdot \nabla \tilde{g}\ \mathrm{d}x$.

2. Is the operator (D, A) skew-adjoint ?

3. Denote by $(S(t))_{t\geq 0}$ the semigroup of contractions on $\boldsymbol{H}$ generated by (D, A). For $\boldsymbol{g} \in D$, what is the equation satisfied by $\boldsymbol{u}(t) = (u(t), v(t)) = S(t)\boldsymbol{g}$? What is the corresponding second-order equation for $u(t)$, the first component of $\boldsymbol{u}(t)$?

4. For any $\boldsymbol{g} \in \boldsymbol{H}$, let $\boldsymbol{u}(t) = (u(t), v(t)) = S(t)\boldsymbol{g}$ and

$$E(t) = \int_{\mathbf{R}^N} \left\{|v(t)|^2 + |\nabla u(t)|^2 + |u(t)|^2\right\}\,\mathrm{d}x = (\boldsymbol{u}(t), \boldsymbol{u}(t)).$$

Prove that the map $t \in [0, \infty) \mapsto E(t)$ is of class $\mathcal{C}^1$ and compute $\frac{dE(t)}{dt}$.

Exercise 7.2. The damped cubic Klein-Gordon equation in one space dimension.

We follow the framework and notation of the previous Exercise in space dimension $N = 1$. Consider the following nonlinear equation in space dimension one, for $\alpha > 0$,

$$\partial_t^2 u = \partial_x^2 u - u - 2\alpha \partial_t u + u^3. \tag{DNLKG}$$

1. For any initial data $(u(0), \partial_t u(0)) = \boldsymbol{g} \in \boldsymbol{H}$, suggest a weak formulation of the equation (DNLKG). State and justify a local well-posedness result for this weak formulation.

   For a maximal solution $\boldsymbol{u} = (u, \partial_t u)$ of (DNLKG), we introduce

   $$F(t) = \int_{\mathbf{R}} \left\{ |\partial_t u(t)|^2 + |\nabla u(t)|^2 + |u(t)|^2 - \frac{1}{2}|u(t)|^4 \right\} \mathrm{d}x,$$
   $$M(t) = \frac{1}{2}\|u(t)\|_{L^2}^2 + \alpha \int_0^t \|u(s)\|_{L^2}^2 \,\mathrm{d}s.$$

   The goal of the remaining questions is to prove that if $F(0) < 0$ then the solution $\vec{u}$ blows up in finite time. From now on, we assume $F(0) < 0$ and, for the sake of contradiction, we assume that the solution $\vec{u}$ is global for $t \geq 0$.

2. Check the following relations, for $t \geq 0$,

   $$F'(t) = -4\alpha \int_{\mathbf{R}} |\partial_t u(t)|^2 \,\mathrm{d}x,$$
   $$M'(t) = \int_{\mathbf{R}} u(t)\partial_t u(t) \,\mathrm{d}x + 2\alpha \int_0^t \int_{\mathbf{R}} u(s)\partial_t u(s) \,\mathrm{d}x \,\mathrm{d}s + \alpha\|u(0)\|_{L^2}^2,$$
   $$M''(t) = 3\|\partial_t u(t)\|_{L^2}^2 + \|\partial_x u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 - 2F(t).$$

3. Prove that $\lim_{t\to\infty} M'(t) = \infty$ and $\lim_{t\to\infty} M(t) = \infty$.

4. Prove that

   $$|M'| \leq \|u\|_{L^2}\|\partial_t u\|_{L^2} + 2\alpha \left(\int_0^t \|u(s)\|_{L^2}^2 \,\mathrm{d}s \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_t u(s)\|_{L^2}^2 \,\mathrm{d}s \right)^{\frac{1}{2}} + \alpha\|u(0)\|_{L^2}^2.$$

   Deduce the following estimate

   $$|M'|^2 \leq \frac{4}{3} M \left[2\|\partial_t u\|_{L^2}^2 + 4\alpha \int_0^t \|\partial_t u(s)\|_{L^2}^2 \,\mathrm{d}s \right] + 4\alpha^2 \|u(0)\|_{L^2}^4.$$

5. Prove that for all t large enough, $\frac{10}{9}[M'(t)]^2 < M''(t)M(t)$ and conclude.

Exercise 7.3. In the framework of the previous exercise, prove that for an initial data $\boldsymbol{g} \in \boldsymbol{H}$ sufficiently small in the norm $\|\cdot\|$, the solution of (DNLKG) is global and converges to 0 in norm $\|\cdot\|$ when $t \to \infty$. **Hint**: for $\mu > 0$ small and $\rho = 2\alpha - \mu$, use the functional

$$G(t) = \int_{\mathbf{R}} \left\{ |\partial_t u(t) + \mu u(t)|^2 + |\partial_x u(t)|^2 + (1 - \rho\mu)|u(t)|^2 - \frac{1}{2}|u(t)|^4 \right\} \mathrm{d}x.$$

Chapter 8

The nonlinear heat equation

In this Chapter, we study the local and global Cauchy problem for the nonlinear heat equation

$$\partial_t u - \Delta u - \epsilon |u|^\alpha u = 0 \quad (t,x) \in \mathbf{R} \times \mathcal{U}, \tag{8.1}$$

where $\mathcal{U}$ is a bounded domain of $\mathbf{R}^N$ with $\mathcal{C}^1$ boundary, $\epsilon = \pm 1$, and $\alpha > 0$. We consider the case of homogeneous Dirichlet boundary condition on $\partial\mathcal{U}$.

8.1 The linear heat equation

8.1.1 General setting

We follow the setting of Section 3.6.2. Let $X = \mathcal{C}_0(\mathcal{U})$ be equipped with the norm $\|\cdot\|_{L^\infty}$. Define

$$\begin{cases} D = \{g \in X \cap H_0^1(\mathcal{U}) : \Delta g \in X\} \\ Ag = \Delta g \quad \text{for all } g \in D. \end{cases}$$

From Proposition 3.29, the operator (D, A) is maximal dissipative with dense domain in X. We denote by $(S(t))_{t \geq 0}$ the semigroup generated by (D, A) given Theorem 4.1. For any $g \in D$, $u(t) = S(t)g \in \mathcal{C}([0,\infty), D) \cap \mathcal{C}^1([0,\infty), X)$ is the solution of the linear problem

$$\frac{\mathrm{d}}{\mathrm{d}t} u = Au. \tag{8.2}$$

Moreover, for any $g \in X$, $u(t) = S(t)g \in \mathcal{C}(\mathbf{R}, X)$ is a weak solution of the linear problem (8.2). This framework is suitable to study both the linear and nonlinear heat equations.

However, recall that Section 3.6.1 provides another framework to study the linear heat equation in the Hilbert space $L^2(\mathcal{U})$. Since $\mathcal{U}$ is bounded, $\mathcal{C}_0(\mathcal{U}) \subset L^2(\mathcal{U})$ and $\|\cdot\|_{L^2} \leq \sqrt{|\mathcal{U}|}\,\|\cdot\|_{L^\infty}$, where $|\mathcal{U}|$ is the Lebesgue measure of $\mathcal{U}$. In particular, we can see the semigroup $(S(t))_{t\geq 0}$ as a restriction to L^∞ of the semigroup $(\tilde{S}(t))_{t\geq 0}$ on $L^2(\mathcal{U})$ given by the setting of Section 3.6.1. In this setting, we can enjoy the *regularizing effect* (see Theorem 4.6) due to the fact that the Laplacian is a self-adjoint operator operator on the Hilbert space $L^2(\mathcal{U})$. This framework will be used in Section 8.2.2 to study the regularizing effects of the nonlinear heat equation.

8.1.2 Exponential estimates in large time

Let $\lambda_0 > 0$ be defined by

$$\lambda_0 = \inf \left\{ \|\nabla u\|_{L^2}^2;\ u \in H_0^1(\mathcal{U}) \text{ with } \|u\|_{L^2} = 1 \right\}.$$

It is known (Poincaré inequality, Theorem C.9) that $\lambda_0 > 0$ for a bounded domain.

Lemma 8.1. *For all $g \in L^2(\mathcal{U})$,*

$$\|\tilde{S}(t)g\|_{L^2} \leq \mathrm{e}^{-\lambda_0 t} \|g\|_{L^2}.$$

Proof. By density, it suffices to consider the case $g \in H_0^1(\mathcal{U}) \cap H^2(\mathcal{U})$. Then,

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{S}(t)g\|_{L^2}^2 &= \int (\tilde{S}(t)g) \Delta(\tilde{S}(t)g)\, \mathrm{d}x = -\int |\nabla(\tilde{S}(t)g)|^2\, \mathrm{d}x \\ &\leq -\lambda_0 \|\tilde{S}(t)g\|_{L^2}^2. \end{aligned}$$

We obtain the result by integrating this differential inequality on $[0, t]$. □

Lemma 8.2. *There exists $M > 1$ such that, for all $g \in \mathcal{C}_0(\mathcal{U})$, for all $t \geq 0$,*

$$\|S(t)g\|_{L^\infty} \leq M \mathrm{e}^{-\lambda_0 t} \|g\|_{L^\infty}.$$

Proof. First, we prove that for some constant $C > 0$, for any $g \in \mathcal{C}_0(\mathcal{U})$, it holds, for all $t > 0$,

$$\|S(t)g\|_{L^\infty} \leq C t^{-\frac{N}{4}} \|g\|_{L^2}. \tag{8.3}$$

Proof of (8.3). When $\mathcal{U} = \mathbf{R}^N$, this inequality is deduced from the explicit expression of the heat kernel

$$K_t(x) = (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right), \quad \|K_t(x)\|_{L^2} = C t^{-\frac{N}{4}},$$

and the representation of the solution v of $\partial_t v = \Delta v$ with initial $v(0) = h$ as a convolution product

$$v(t, x) = (K_t \star h)(x).$$

Indeed, the Young inequality implies for $t > 0$ that

$$\|v(t)\|_{L^\infty} \leq \|K_t\|_{L^2} \|h\|_{L^2} \leq C t^{-\frac{N}{4}} \|h\|_{L^2}.$$

Now, in the case of a bounded domain $\mathcal{U}$ of $\mathbf{R}^N$, we use a comparison argument. (See the same technique in the proof of Proposition 3.29.) Let $g \in \mathcal{C}_0(\mathcal{U})$ and define $h \in \mathcal{C}_c(\mathbf{R}^N)$ by $h = |g|$ on $\mathcal{U}$ and $h = 0$ on $\mathbf{R}^N \setminus \mathcal{U}$. Denote $v(t) = K_t \star h \geq 0$ as before. Set $z(t) = v(t)|_{\mathcal{U}} - S(t)g$. It is clear that $z|_{\partial\mathcal{U}} \geq 0$ and $z(0) = 0$ on $\mathcal{U}$. We introduce a function $G \in \mathcal{C}^1(\mathbf{R})$ such that

- $G' \leq 1$ on $\mathbf{R}$;
- G is increasing on $(0, \infty)$;
- $G = 0$ on $(-\infty, 0]$;

and we set $H(s) = \int_0^s G(\sigma)\,\mathrm{d}\sigma$. Define $m(t) = \int H(-z(t))\,\mathrm{d}x$. Then,

$$m' = -\int G(-z)(\partial_t z)\,\mathrm{d}x = -\int G(-z)(\Delta z)\,\mathrm{d}x = -\int G'(-z)|\nabla z|^2\,\mathrm{d}x \leq 0.$$

Since $m(0) = 0$ and $m \geq 0$, we obtain $m(t) = 0$, for all $t \geq 0$. This justifies that $z(t) \geq 0$ holds for all $t \geq 0$. Thus, $S(t)g \leq v(t)$. Proceeding similarly with $-g$, it holds $S(t)g \geq -v(t)$. Thus, on $\mathcal{U}$, for any $t \geq 0$, it holds

$$|S(t)g| \leq v(t).$$

The estimate (8.3) on $\mathcal{U}$ thus follows.

Second, by the contraction property, $\|S(t)g\|_{L^\infty} \leq \|g\|_{L^\infty} \leq M_0 \mathrm{e}^{-\lambda_0 t}\|g\|_{L^\infty}$, for $t \in (0,1]$, with $M_0 = \mathrm{e}^{\lambda_0}$. For $t \in (1,\infty)$, we write $S(t)g = S(1)S(t-1)g$, and we use (8.3) for $t = 1$ and next (8.1) for $t-1$, which provides the estimate

$$\begin{aligned}\|S(t)g\|_{L^\infty} \leq C\|S(t-1)g\|_{L^2} &\leq C\mathrm{e}^{-\lambda_0(t-1)}\|g\|_{L^2} \\ &\leq C\mathrm{e}^{\lambda_0}\mathrm{e}^{-\lambda_0 t}\|g\|_{L^2} \leq C\mathrm{e}^{\lambda_0}\sqrt{|\mathcal{U}|}\mathrm{e}^{-\lambda_0 t}\|g\|_{L^\infty}.\end{aligned}$$

Therefore, estimate (8.1) is proved for all $t \geq 0$. □

8.2 Local Cauchy theory for the nonlinear heat equation

8.2.1 General local existence result

We denote $f(u) = \epsilon|u|^\alpha u$. The $\mathcal{C}_0(\mathcal{U})$ framework is quite favorable regarding the Lipschitz continuity condition to be checked on the nonlinearity. Indeed, for any $\alpha > 0$ and for any space dimension $N \geq 1$,

$$|f(u) - f(v)| \leq C|u-v|\left(|u|^\alpha + |v|^\alpha\right),$$

and thus

$$\|f(u) - f(v)\|_{L^\infty} \leq C\|u-v\|_{L^\infty}\left(\|u\|_{L^\infty}^\alpha + \|v\|_{L^\infty}^\alpha\right).$$

Thus, $f : u \in X \mapsto f(u) \in X$ is Lipschitz continuous on bounded sets of X. The weak formulation of the problem (8.1) is the following: for $g \in X$ and $T > 0$, solve

$$\begin{cases} u \in \mathcal{C}([0,T],X), \\ u(t) = S(t)g + \displaystyle\int_0^t S(t-s)f(u(s))\,\mathrm{d}s. \end{cases} \tag{8.4}$$

The following result is a direct application of Theorems 6.5 and 6.6.

Theorem 8.3. *There exists a function $T_{\max} : X \to (0,\infty]$ with the following properties. For any $g \in X$, there exists $u \in \mathcal{C}([0,T_{\max}(g)),X)$ such that for all $T \in (0,T_{\max}(g))$, u is the unique solution of (8.4). Moreover, for any $g \in X$, the following alternative holds:*

1. *Either $T_{\max}(g) = \infty$;*
2. *Or $T_{\max}(g) < \infty$ and then $\lim_{t\uparrow T_{\max}(g)} \|g(t)\|_{L^\infty} = \infty$.*

8.2.2 Regularizing effect

Theorem 8.4. *Let $g \in X$. In the framework of Theorem 8.3, for any $T \in (0, T_{\max}(g))$, it holds*

$$u \in \mathcal{C}((0,T], H_0^1(\mathcal{U})) \cap \mathcal{C}^1((0,T], L^2(\mathcal{U})) \quad \textit{and} \quad \Delta u \in \mathcal{C}((0,T], L^2(\mathcal{U})).$$

Moreover, u solves (8.1) *on* $(0,T]$.

Proof. Using Theorem 4.6, for any $g \in L^2(\mathcal{U})$, $w(t) = \tilde{S}(t)g$ is the unique solution of the following problem

$$\begin{cases} w \in \mathcal{C}([0,\infty), L^2(\mathcal{U})) \cap \mathcal{C}((0,\infty), H_0^1(\mathcal{U})) \cap \mathcal{C}^1((0,\infty), L^2(\mathcal{U})), \\ \Delta w \in \mathcal{C}((0,\infty), L^2(\mathcal{U})), \\ \partial_t w(t) = \Delta w(t), \quad \text{for all } t > 0, \\ w(0) = g, \end{cases} \tag{8.5}$$

and it holds, for $t > 0$,

$$\|w(t)\|_{L^2} \le \|g\|_{L^2}, \quad \|\nabla w(t)\|_{L^2} \le \frac{1}{\sqrt{2t}}\|g\|_{L^2}, \quad \|\Delta w(t)\|_{L^2} \le \frac{1}{t\sqrt{2}}\|g\|_{L^2}.$$

For $g \in \mathcal{C}_0(\mathcal{U})$, we recall that $\|g\|_{L^2} \le \sqrt{|\mathcal{U}|}\,\|g\|_{L^\infty}$, where $|\mathcal{U}|$ is the Lebesgue measure of $\mathcal{U}$, and the desired regularizing properties are obtained on the linear part $w(t) = S(t)g = \tilde{S}(t)g$ of $u(t)$ as defined in (8.4). For the nonlinear part, we set

$$v(t) = \int_0^t S(t-s) f(u(s))\,\mathrm{d}s.$$

Using Theorem 4.6, we know that $s \in [0,t) \mapsto S(t-s)f(u(s))$ belongs to $\mathcal{C}([0,t), H_0^1(\mathcal{U}))$, with the following estimate

$$\|S(t-s)f(u(s))\|_{H^1} \le \left(1 + \frac{1}{\sqrt{2(t-s)}}\right) \|f(u(s))\|_{L^2}.$$

By these properties, for any $n \ge 1$, the formula

$$v_n(t) = \int_0^{\frac{n-1}{n}t} S(t-s) f(u(s))\,\mathrm{d}s$$

defines a function in $\mathcal{C}([0,T], H_0^1(\mathcal{U}))$. Moreover, for any $n < m$, it holds for any $t \in [0,T]$,

$$\begin{aligned} \|v_m(t) - v_n(t)\|_{H^1} &\le \int_{\frac{n-1}{n}t}^{\frac{m-1}{m}t} \|S(t-s)f(u(s))\|_{H_0^1}\,\mathrm{d}s \\ &\le \int_{\frac{n-1}{n}t}^{\frac{m-1}{m}t} \left(1 + \frac{1}{\sqrt{2(t-s)}}\right) \|f(u(s))\|_{L^2}\,\mathrm{d}s \\ &\le C\sqrt{\frac{T}{n}} \sup_{[0,T]} \|f(u)\|_{L^\infty}. \end{aligned}$$

Therefore, $\{v_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathcal{C}([0,T],H_0^1(\mathcal{U}))$. Since its limit in $\mathcal{C}([0,T],L^2(\mathcal{U}))$ is exactly v, we obtain that in addition $v\in\mathcal{C}([0,T],H_0^1(\mathcal{U}))$. Moreover, for $t\in[0,T]$, we deduce from the previous estimates that

$$\|u(t)\|_{H^1}\le C\left(1+\frac{1}{\sqrt{t}}\right),\quad \|f(u(t))\|_{H^1}\le C\left(1+\frac{1}{\sqrt{t}}\right).$$

Using again the regularizing effect, we deduce that

$$\|\Delta S(t-s)f(u(s))\|_{L^2}\le C\left(1+\frac{1}{\sqrt{s}}\right)\left(1+\frac{1}{\sqrt{t-s}}\right).$$

Proceeding as before, we obtain that $\Delta v\in\mathcal{C}([0,T],L^2(\mathcal{U}))$, and thus $\Delta u\in\mathcal{C}((0,T],L^2(\mathcal{U}))$.

For any $\delta\in(0,T)$ and $t\in(0,T-\delta)$, let $\tilde{u}(t)=u(t+\delta)$. Then, $\tilde{u}$ writes

$$\tilde{u}(t)=S(t)u(\delta)+\int_0^t S(t-s)f(\tilde{u}(s))\,\mathrm{d}s.$$

Using Corollary 5.7, we obtain $\tilde{u}\in\mathcal{C}^1([0,T-\delta],L^2)$ and $\tilde{u}$ satisfies the equation (8.1). Thus, also u satisfies (8.1) on $(0,T)$. □

8.3 The maximum principle for the nonlinear heat equation

Theorem 8.5. *In the context of Theorems 8.3 and 8.4, if $g\ge 0$ then for all $t\in[0,T_{\max}(g))$, it holds $u(t)\ge 0$.*

Proof. We fix a function $G\in\mathcal{C}^1(\mathbf{R})$ such that

- $G'\le 1$ on $\mathbf{R}$;
- G is increasing on $(0,\infty)$;
- $G(s)=\frac{1}{2}s^2$ on $[0,1]$;
- $G=0$ on $(-\infty,0]$;

and we set $H(s)=\int_0^s G(\sigma)\,\mathrm{d}\sigma$. We set $m(t)=\int H(-u(t))\,\mathrm{d}x$. Then,

$$\begin{aligned}m'&=-\int G(-u)(\partial_t u)\,\mathrm{d}x=-\int G(-u)(\Delta u+f(u))\,\mathrm{d}x\\&=-\int G'(-u)|\nabla u|^2\,\mathrm{d}x-\int G(-u)f(u)\,\mathrm{d}x\\&\le C\int G(-u)|u|^{\alpha+1}\le C\int G(-u)|u|\le Cm.\end{aligned}$$

We have used that $sG(s)\le CH(s)$ for $s\ge 0$ (the proof is easier if $\epsilon=1$).

Note that the above computation makes sense for $t\in(0,T_{\max}(g))$. Using the Gronwall lemma D.4, it follows that for $0<s<t<T_{\max}(g)$,

$$m(t)\le m(s)\mathrm{e}^{C(t-s)}.$$

Passing to the limit $s\downarrow 0$, using $m(0)=0$, we find $m(t)=0$ for all $t\in[0,T_{\max}(g))$, which means that $u(t)\ge 0$. □

8.4 Global existence

Theorem 8.6. *There exists $\mu > 0$ and $C \geq 1$ such that if $\|g\|_{L^\infty} \leq \mu$, then the corresponding maximal solution of* (8.4) *given by Theorem 8.3 is global and satisfies, for all $t \geq 0$,*

$$\|u(t)\|_{L^\infty} \leq C\|g\|_{L^\infty} \mathrm{e}^{-\lambda_0 t}.$$

Proof. Let u be a maximal solution of (8.4) with initial data $g \in X$ as given by Theorem 8.3. Define, for any $t \in [0, T_{\max}(g))$,

$$N(t) = \sup_{s\in[0,t]} \left\{ \mathrm{e}^{\lambda_0 s}\|u(s)\|_{L^\infty} \right\}.$$

Multiplying (8.4) by $\mathrm{e}^{\lambda_0 t}$, taking the L^∞ norm, and then using Lemma 8.2, we obtain

$$\begin{aligned}
\mathrm{e}^{\lambda_0 t}\|u(t)\|_{L^\infty} &\leq \mathrm{e}^{\lambda_0 t}\|S(t)g\|_{L^\infty} + \int_0^t \mathrm{e}^{\lambda_0 t}\|S(t-s)f(u(s))\|_{L^\infty}\,\mathrm{d}s \\
&\leq M\|g\|_{L^\infty} + M\int_0^t \mathrm{e}^{\lambda_0 t}\mathrm{e}^{-\lambda_0(t-s)}\|u(s)\|_{L^\infty}^{1+\alpha}\,\mathrm{d}s \\
&\leq M\|g\|_{L^\infty} + M\int_0^t \mathrm{e}^{\lambda_0 t}\mathrm{e}^{-\lambda_0(t-s)}\mathrm{e}^{-(1+\alpha)\lambda_0 s}[N(s)]^{1+\alpha}\,\mathrm{d}s \\
&\leq M\|g\|_{L^\infty} + \frac{M}{\alpha\lambda_0}[N(t)]^{1+\alpha}.
\end{aligned}$$

Thus, for all $t \in [0, T_{\max}(g))$, $N(t)$ satisfies the inequality

$$N(t) \leq M\|g\|_{L^\infty} + \frac{M}{\alpha\lambda_0}[N(t)]^{1+\alpha}.$$

Setting

$$\theta(w) = \frac{M}{\alpha\lambda_0}w^{1+\alpha} - w,$$

we conclude as in the proof of Theorem 7.5. Let

$$m = -\min\nolimits_{[0,\infty)} \theta \text{ and } x_m > m \text{ be such that } \theta(x_m) = m.$$

If $M\|g\|_{L^\infty} < m < x_m$, then $N(0) < x_m$, and for all $t \in [0, T_{\max}(g))$, it holds $\theta(N(t)) \geq -M\|g\|_{L^\infty} > -m$, and so by continuity, $N(t) < x_m$. In particular, the solution is global in time.

Now, we prove the exponential bound. For any $a \in (0, m)$, there exists $a \leq x_a \leq x_m$ such that $\theta(x_a) = -a$, with $x_a \leq Ca$ for some constant $C > 1$. Since $\theta(N(t)) \geq -M\|g\|_{L^\infty}$, we have for all $t \geq 0$, $N(t) \leq x_{M\|g\|_{L^\infty}} \leq CM\|g\|_{L^\infty}$, which completes the proof. □

8.5 Blowup in finite time

We consider the (normalized) eigenfunction $\Psi_0 \in H^2(\mathcal{U}) \cap H_0^1(\mathcal{U})$ of the Laplacian on the domain $\mathcal{U}$, *i.e.* the unique positive solution of

$$\Delta\Psi_0 + \lambda_0\Psi_0 = 0, \quad \Psi_0 > 0 \text{ on } \mathcal{U}, \quad \int_{\mathcal{U}} \Psi_0\,\mathrm{d}x = 1.$$

Theorem 8.7. *Let $g \in X$ be such that $g \geq 0$ on $\mathcal{U}$. Assume that*

$$\int_{\mathcal{U}} g(x)\Psi_0(x)\,\mathrm{d}x > \lambda_0^{\frac{1}{\alpha}}.$$

Then, the corresponding maximal solution of (8.4) *given by Theorem 8.3 blows up in finite time.*

Proof. We consider a non negative initial data, and thus by Theorem 8.5, the solution itself is non negative for all $t \in [0, T_{\max}(g))$. We define

$$I(t) = \int_{\mathcal{U}} u(t,x)\Psi_0(x)\,\mathrm{d}x.$$

Then, using the equation of u for $t > 0$, and next the Jensen inequality, we compute and estimate

$$\begin{aligned} I' &= \int_{\mathcal{U}} \partial_t u \Psi_0\,\mathrm{d}x = \int_{\mathcal{U}} (\Delta u + f(u))\Psi_0\,\mathrm{d}x \\ &= \int_{\mathcal{U}} u\Delta\Psi_0\,\mathrm{d}x + \int_{\mathcal{U}} u^{\alpha+1}\Psi_0\,\mathrm{d}x \\ &\geq -\lambda_0 \int_{\mathcal{U}} u\Psi_0\,\mathrm{d}x + \left(\int_{\mathcal{U}} u\Psi_0\,\mathrm{d}x\right)^{\alpha+1}. \end{aligned}$$

This means that I satisfies

$$I' \geq -\lambda_0 I + I^{\alpha+1}.$$

By a standard argument, we prove that $I' \geq 0$ on $[0, T_{\max}(g))$. Let $\delta > 0$ be such that

$$(1-\delta)[I(0)]^{\alpha} \geq \lambda_0.$$

Then, for all $t \in [0, T_{\max}(g))$, $(1-\delta)[I(t)]^{\alpha} \geq (1-\delta)[I(0)]^{\alpha} \geq \lambda_0$.

$$I' \geq \delta I^{\alpha+1}.$$

Integrating this on $[0, t]$, we find

$$[I(0)]^{-\alpha} - [I(t)]^{-\alpha} \geq \delta\alpha t,$$

which proves that $T_{\max}(g) \leq (\delta\alpha)^{-1}[I(0)]^{-\alpha}$. □

8.6 Exercises for Chapter 8

Exercise 8.1. Let $\mathcal{U}$ be a bounded smooth domain of $\mathbf{R}^N$. Let $p > 1$. Let $\varphi \in C_0(\mathcal{U}) \cap H_0^1(\mathcal{U})$ and let $u \in C([0, T_m), C_0(\mathcal{U}) \cap H_0^1(\mathcal{U})) \cap C((0, T_m), H^2(\mathcal{U})) \cap C^1((0, T_m), L^2(\mathcal{U}))$ be the maximal solution of

$$u_t - \Delta u = u^p \text{ in } (0, T_m) \times \mathcal{U}, \quad u(0) = \varphi \text{ in } \mathcal{U}.$$

For all $t \in [0, T_m)$, define

$$E(u(t)) = \frac{1}{2}\int_{\mathcal{U}} |\nabla u(t,x)|^2\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathcal{U}} u^{p+1}(t,x)\,\mathrm{d}x.$$

1. Prove that $E(u(t)) \in C([0,T_m)) \cap C^1((0,T_m))$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u(t)) = -\int_{\mathcal{U}} u_t^2(t,x)\,\mathrm{d}x.$$

(**Hint**: for $0 < s < t < T_m$, compute $\frac{1}{t-s}[E(u(t)) - E(u(s))]$ and pass to the limit as $s \to t$.)

2. What can be said about $E(u(t))$ in the case where only $\varphi \in C_0(\mathcal{U})$?

Exercise 8.2. (A simple blow up criterium)
We consider the following problem

$$\left.\begin{array}{rl} u_t - \Delta u = u + u^p, & x \in \mathcal{U},\ t > 0, \\ u = 0, & x \in \partial\mathcal{U},\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathcal{U}, \end{array}\right\} \quad (P)$$

for $p > 1$ and $\mathcal{U}$ a bounded domain of $\mathbf{R}^N$ with $\mathcal{C}^1$ boundary.

1. Let $u_0,\ v_0 \in C_0(\mathcal{U})$ and let $u,\ v$ be the corresponding solutions of (P) on their maximal interval of existence $[0, T_{\max}(u_0)$, $[0, T_{\max}(v_0))$. Assume that $u_0 \geq \alpha v_0 \geq 0$ for some $\alpha > 1$. Prove that $T_{\max}(u_0) \leq T_{\max}(v_0)$ and that for all $t \in [0, T_{\max}(u_0))$, $u(t) \geq \alpha v(t)$.

2. Assume that problem (P) has a stationary solution $w > 0$, i.e, a positive classical solution of

$$\left.\begin{array}{rl} -\Delta w = w + w^p, & x \in \mathcal{U}, \\ w = 0, & x \in \partial\mathcal{U}. \end{array}\right\}$$

Prove that if $u_0 \geq \alpha w$ on $\mathcal{U}$, for some $\alpha > 1$, then $T_{\max}(u_0) < +\infty$.

Exercise 8.3. (Unbounded solution) Find explicitly the solution $u(t,x)$ of

$$u_t - \Delta u = 2u \text{ in } (0,\infty) \times (0,\pi), \quad u(0) = \sin(x) \text{ in } (0,\pi).$$

Exercise 8.4. (Interior regularity) Let $Q = (-1,0) \times B$, where B is the unit ball of $\mathbf{R}^N$. Let $u \in C(\overline{Q}) \cap C^\infty(Q)$ and assume $u_t - \Delta u = 0$. The goal of this exercise is to prove that, for all $\alpha \in \mathbf{R}^N$, $\forall p \geq 0$,

$$|\partial_t^p \partial_x^\alpha u(0,0)| \leq C \|u\|_{L^\infty(Q)}.$$

1. Let $\varphi \in C_0^\infty(\mathbf{R}^{d+1})$ be such that $\varphi(0,0) = 1$, $\operatorname{supp}\varphi \subset (-1,1) \times B$. Set

$$w = \varphi^2 |\nabla_x u|^2 + K|u|^2, \quad \text{for some } K > 0.$$

Prove that for $K > 0$ large enough, $w_t - \Delta w \geq 0$ on Q. The constant K is now fixed to such value.

2. Deduce from the previous question the following estimate

$$|\nabla_x u(0,0)|^2 \leq K \|u\|^2_{L^\infty(Q)}.$$

3. Conclude.

Exercise 8.5. (Blow up rate) Let $u \in C([0,T_m), C_0(\mathcal{U})) \cap C((0,T_m), H^2(\mathcal{U}) \cap H_0^1(\mathcal{U})) \cap C^1((0,T_m), L^2(\mathcal{U}))$ be the maximal solution of

$$u_t - \Delta u = u^p \text{ in } (0,T_m) \times \mathcal{U}, \quad u(0) = \varphi \geq 0 \text{ in } \mathcal{U}.$$

Assume that $T_m < +\infty$ (blow up in finite time). Prove that

$$\forall t \in [0,T_m), \quad \|u(t)\|_{L^\infty} \geq k(T_m - t)^{-\frac{1}{p-1}} \quad \text{where} \quad k = (p-1)^{-\frac{1}{p-1}}.$$

Exercise 8.6. In this exercise, B denotes the open unit ball of $\mathbf{R}^3$, and S denotes the boundary of B, i.e. the unit sphere of $\mathbf{R}^3$. For $x = (x_1, x_2, x_3)$, we set as usual $|x| = \left(\sum_{j=1}^3 x_j^2\right)^{\frac{1}{2}}$. For notational reasons, we recall the integration-by-parts formula in the particular case of B (note that $\nu = x$ is the outward pointing unit normal vector field at the boundary S of B): for $u, v \in C^1(\bar{B})$, $j = 1,2,3$,

$$\int_B (\partial_{x_j} u) v \,\mathrm{d}x = -\int_B u(\partial_{x_j} v) \,\mathrm{d}x + \int_S u v x_j d\sigma(x),$$

where $\sigma(x)$ denotes the Lebesgue measure on S.

We denote by λ_1 and φ_1 the first eigenvalue and normalized eigenfunction of $-\Delta$ in B with zero boundary condition : $\varphi_1 > 0$ on B and $\int_B \varphi_1 = 1$.

Part I. Non existence results for a nonlinear elliptic equation

1. The goal of this question is to prove that there is no nonzero classical solution of the nonlinear elliptic problem

$$\begin{cases} \Delta w + w^7 = 0, & x \in B, \\ w = 0, & x \in S. \end{cases} \tag{E}$$

Let $w \in C^2(\overline{B})$ satisfy (E).

(a) Multiply (E) by w and integrate by parts to find a relation between $\int_B |\nabla w|^2$ and $\int_B w^8$.

(b) Multiply (E) by $x \cdot \nabla w$ and integrate by parts to find a relation between $\int_B |\nabla w|^2 \,\mathrm{d}x$, $\int_B w^8 \,\mathrm{d}x$ and $\int_S |\nabla w \cdot x|^2 d\sigma(x)$.

(c) Conclude that $w = 0$ on B.

2. The goal of this question is to prove that there exists no radially symmetric positive solution of

$$\begin{cases} \Delta w + w^7 = 0, & x \in B \setminus \{0\}, \\ w = 0, & x \in S. \end{cases} \tag{SE}$$

Note that we allow a possible singularity at 0. Let $w(x) = W(|x|)$ satisfy (E) and $w > 0$ on $B \setminus \{0\}$. We assume that $W \in C^2((0,1])$.

(a) Derive the equation satisfied by W on $(0,1]$.

(b) For all $s \geq 1$, we consider $Z(s)$ defined by

$$Z(s) = s^{-\frac{1}{2}} W(s^{-1}).$$

Show that Z satisfies the following equation

$$\begin{cases} (sZ')' + Z^7 - \dfrac{Z}{4s} = 0, & s > 1, \\ Z(s) > 0, & s > 1, \\ Z(1) = 0. \end{cases} \tag{ODE}$$

(c) Prove that $Z'(1) > 0$.

(d) Prove that $s \mapsto Z(s)$ cannot be monotone nondecreasing on $[1, +\infty)$. **Hint**: argue by contradiction: assume that Z is monotone nondecreasing and show that there exists $s_1 > 1$ such that for any $s \geq s_1$, $(sZ')'(s) \leq -\frac{1}{2} Z^7(s_1)$. Conclude.

We denote by $s_0 > 1$, the first zero of Z' on $[1, +\infty)$.

(e) Let any $1 \leq s_2 < s_3$; multiplying the equation (ODE) by sZ', integrating on $[s_2, s_3]$, prove the identity

$$\int_{s_2}^{s_3} Z^8(s)ds = s_2 Z^8(s_2) - s_1 Z^8(s_1) - \left[Z^2(s_2) - Z^2(s_1)\right] + 4\left[(s_2 Z'(s_2))^2 - (s_1 Z'(s_1))^2\right]. \tag{$\star$}$$

(f) Prove that Z is monotone nonincreasing on $[s_0, +\infty)$. **Hint**: argue by contradiction, using the identity ($\star$) for suitable $1 \leq s_2 < s_3$.

(g) Show that there exists $s_n \to \infty$ such that $\lim_{n \to \infty} s_n Z'(s_n) = 0$.

(h) Using again the identity ($\star$), reach a contradiction and conclude.

Part II. Global radial solutions of a nonlinear heat equation

$$\begin{cases} \partial_t u - \Delta u - u^7 = 0, & t > 0,\ x \in B, \\ u = 0, & x \in S, \\ u(0, x) = u_0(x), & x \in B. \end{cases} \tag{P}$$

We assume that $u_0 \in C^2(\bar{B})$, $u_0 \geq 0$ on B and that u_0 is a radially symmetric decreasing function on B, i.e.

$$u_0(x) = U_0(|x|) \quad \text{where} \quad U_0'(r) \leq 0 \quad \text{for } r \in [0, 1]. \tag{H}$$

We also assume that the corresponding classical solution $u \in C([0, +\infty), C^2(\bar{B}))$ of (P) is global in time. The goal of this exercice is to prove that

$$\lim_{t \to +\infty} u(t, x) = 0 \quad \text{for all } x \in B \setminus \{0\}, \tag{C1}$$

and

$$\lim_{t \to +\infty} \|u(t)\|_{L^1(B)} = 0. \tag{C2}$$

1. Prove that for all $t \geq 0$, the function $x \in B \mapsto u(t,x)$ is also a radially symmetric decreasing function, i.e., for all $x \in B$, $u(t,x) = U(t,|x|)$ and for all $r \in [0,1]$, $\partial_r U(t,r) \leq 0$.

2. We define the energy of $u(t)$ as follows

$$E(t) = \frac{1}{2}\int_B |\nabla u(t,x)|^2 \,\mathrm{d}x - \frac{1}{8}\int_B u^8(t,x)\,\mathrm{d}x.$$

   Prove that for all $t \geq 0$, $0 \leq E(t) \leq C_0$, for some constant $C_0 > 0$. Deduce that, for some $C_0' > 0$,

$$\int_0^{+\infty}\int_B |\partial_t u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C_0'.$$

3. Let $I(t) = \int_B u(t,x)\varphi_1(x)\,\mathrm{d}x$. Prove that there exists a constant $C_1 > 0$ such that

$$\forall t \geq 0, \quad I(t) \leq C_1.$$

4. Deduce from the previous question that there exists a constant $C_2 > 0$ such that

$$\forall t \geq 0, \quad \int_B u(t,x)\,\mathrm{d}x \leq C_2.$$

5. Let $R \in (0,1)$. Let $\mathcal{U}_R = \{x \in \mathbf{R}^3,\ R < |x| < 1\}$. Prove that there exists $C_3(R) > 0$ such that

$$\forall t \geq 0, \quad \|u(t)\|_{L^\infty(\mathcal{U}_R)} \leq C_3(R).$$

   In view of the next questions, we recall the following parabolic regularity result in L^p. *Let $0 < R < \frac{1}{2}$, $1 < p < \infty$ and let $f \in L^p((0,1)\times\mathcal{U}_R)$. Let $h = h(t,x)$ be solution of*

$$\begin{cases} \partial_t h - \Delta h = f, & t \in (0,1),\ x \in \mathcal{U}_R, \\ h = 0, & x \in S. \end{cases}$$

   There exists a constant $C(p,R) > 0$ such that, for all $j,k = 1,2,3$,

$$\|\partial_t h\|_{L^p([\frac{1}{2},1]\times\mathcal{U}_{2R})} + \|\partial_{x_j} h\|_{L^p([\frac{1}{2},1]\times\mathcal{U}_{2R})} + \|\partial_{x_j x_k} h\|_{L^p([\frac{1}{2},1]\times\mathcal{U}_{2R})} \leq C(p,R)\|f\|_{L^p([0,1]\times\mathcal{U}_R)}.$$

6. Prove that there exists $C_4(R) > 0$ such that, for any $1 < p < \infty$, for any $T \geq 1$,

$$\|\partial_t^2 u\|_{L^p([T,T+1]\times\mathcal{U}_R)} \leq C_4(R).$$

7. Show that for any $R \in (0,1)$,

$$\lim_{t\to+\infty}\int_{\mathcal{U}_R} |\partial_t u(t)|^2 = 0.$$

8. Let $t_n \to +\infty$. Prove that there exists a subsequence of $\{t_n\}_{n=0}^{\infty}$, denoted by $\{t_{n_k}\}_{k=0}^{\infty}$, and $w \in C^2(\bar{B} \setminus \{0\})$, w solution of (SE) on $B \setminus \{0\}$ such that
$$\lim_{k\to+\infty} u(t_{n_k}) \to w \quad \text{in } L^\infty(\mathcal{U}_R) \text{ for all } R \in (0,1).$$

9. Prove (C1).

10. Using again the functional $I(t)$, prove (C2).

Exercise 8.7 (Continuity of the blow up time for a nonlinear heat equation)**.** In this text, $C(K)$ denotes a constant depending on K, which may change from one line to another. Possible dependency on the domain $\mathcal{U}$ is not specified.

1. Preliminary. Let $\delta_0 > 0$ and $c_0 > 0$. Prove that if a positive function $g \in C^1([0,\delta_0])$ satisfies $g' \geq c_0 g^2$ on $[0,\delta_0]$, then $g(0) < \frac{1}{\delta_0 c_0}$.

   Let $\delta_1 > 0$ and $c_1, c_2 > 0$. Prove that if a positive function $g \in C^1([0,\delta_1])$ satisfies $g' \geq -c_2 + c_1 g^2$ on $[0,\delta_1]$, then $g(0) < \max\left(\sqrt{\frac{2c_2}{c_1}}, \frac{2}{\delta_1 c_1}\right)$.

   Let $\mathcal{U}$ be a smooth bounded domain of $\mathbf{R}^2$. We consider the following nonlinear heat equation
$$\begin{cases} u_t - \Delta u = u^3 & (t,x) \in (0,T) \times \mathcal{U}, \\ u(t)|_{\partial\mathcal{U}} = 0 & t \in (0,T), \\ u(0,x) = \varphi(x) & x \in \mathcal{U}. \end{cases} \tag{8.6}$$

   Recall that for any $\varphi \in C_0(\mathcal{U})$, there exist $0 < T_{\max}(\varphi) \leq +\infty$ and a unique maximal solution u of (8.6), satisfying $u \in C([0,T]), C_0(\mathcal{U})) \cap C((0,T], H^2(\mathcal{U}) \cap H_0^1(\mathcal{U})) \cap C^1((0,T], L^2(\mathcal{U}))$ for any $0 < T < T_{\max}(\varphi)$.

   Let $K > 0$. In questions 2–10, we consider $\varphi \in C_0(\mathcal{U})$ such that $\|\varphi\|_{L^\infty} < K$. Recall that in this case $T_{\max}(\varphi) > T_K$ where $T_K > 0$ depends only on K. In questions 2–10, we also assume that u blows up in finite time, i.e.
$$T_{\max}(\varphi) < +\infty.$$

   For all $t \in [0, T_{\max}(\varphi))$, define
$$M(t) = \int |u(t,x)|^2 \,\mathrm{d}x$$
   and for all $t \in (0, T_{\max}(\varphi))$, define
$$E(t) = \frac{1}{2}\int_{\mathcal{U}} |\nabla u(t,x)|^2 \,\mathrm{d}x - \frac{1}{4}\int u^4(t,x)\,\mathrm{d}x.$$

2. Recall why $E(t) \in C^1((0, T_{\max}(\varphi)))$, and check that
$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\int_{\mathcal{U}} u_t^2(t,x)\,\mathrm{d}x.$$

3. Comparing u with the solution of the equation $h' = h^3$ with $h(0) = K$, justify the existence of $0 < \tau_K < \frac{T_K}{4}$, depending only on K, such that $\sup_{t\in[0,\tau_K]} \|u(t)\|_{L^\infty} \leq 2K$. Such τ_K is now fixed.

4. Prove that there exists $C(K) > 0$ such that

$$\forall t \in [\tau_K, T_{\max}(\varphi)), \quad E(t) \leq C(K).$$

Let $0 < \delta < \frac{T_K}{4}$ and define $T_\delta = T_{\max}(\varphi) - \delta$.

5. Prove that there exists $C(K, \delta) > 0$ such that

$$\forall t \in [\tau_K, T_\delta], \quad M(t) \leq C(K, \delta).$$

Hint: study the evolution of the quantity $M(t)$ and use question 1.

6. Prove that there exists $C(K, \delta) > 0$ such that

$$\forall t \in [\tau_K, T_\delta], \quad E(t) \geq -C(K, \delta).$$

7. Prove that there exists $C(K, \delta) > 0$ such that

$$\int_{\tau_K}^{T_\delta} \int_{\mathcal{U}} u_t^2 \leq C(K, \delta).$$

8. Prove that there exists $C(K, \delta) > 0$ such that

$$\|u(t)\|_{L^8(I, L^4(\mathcal{U}))} \leq C(K, \delta),$$

for any interval $I \subset (\tau_K, T_\delta)$ such that $|I| \leq 1$.
Hint: use the expression of $\frac{\mathrm{d}}{\mathrm{d}t} \int u^2(t)$ in terms of $E(t)$ and $\int u^4(t)$.

9. Deduce that

$$\forall t \in [\tau_K, T_\delta], \quad \|u(t)\|_{L^{\frac{19}{6}}(\mathcal{U})} \leq C(K, \delta).$$

Hint: use the inequality (we admit it)

$$\begin{aligned}\|f\|_{L^\infty((0,T), L^{\frac{19}{6}}(\mathcal{U}))} \leq C(T)\big(&\|f_t\|_{L^2((0,T)\times\mathcal{U})} + \|f\|_{L^2((0,T)\times\mathcal{U})} \\ &+ \|f\|_{L^8((0,T), L^4(\mathcal{U}))}\big)\end{aligned}$$

10. Deduce from the previous question that

$$\forall t \in [0, T_\delta], \quad \|u(t)\|_{L^\infty(\mathcal{U})} \leq C(K, \delta).$$

Hint: use the Duhamel formula for $u(t)$ and estimates for the heat semigroup.

11. Conclusion. Prove that the function

$$\varphi \in C_0(\mathcal{U}) \mapsto T_{\max}(\varphi) \in (0, \infty]$$

is continuous for the $\|.\|_{L^\infty}$ topology.

Exercise 8.8 (Construction of positive solutions to a linear heat equation with singular potential)**.** Let $\mathcal{U}$ be a bounded smooth convex domain of $\mathbf{R}^N$, where $N \geq 3$. We denote by f the function distance to the boundary, i.e.

$$\forall x \in \mathcal{U}, \quad f(x) = \mathrm{dist}(x, \partial\mathcal{U}).$$

Recall the so-called Hardy inequality

$$\forall \varphi \in H_0^1(\mathcal{U}), \quad \frac{1}{4}\int_{\mathcal{U}} \frac{\varphi^2(x)}{f^2(x)}\,\mathrm{d}x \leq \int_{\mathcal{U}} |\nabla\varphi|^2. \tag{8.7}$$

We study the existence problem for the following linear heat equation

$$\begin{cases} v_t - \Delta v = \dfrac{\alpha}{f^2(x)} v & (t,x) \in (0,\infty) \times \mathcal{U}, \\ v(t)|_{\partial\mathcal{U}} = 0 & t \in (0,\infty), \\ v(0,x) = \varphi(x) & x \in \mathcal{U}, \end{cases} \tag{8.8}$$

for $\varphi \in C_0(\mathcal{U}) \cap H_0^1(\mathcal{U})$, $\varphi \geq 0$ and $0 < \alpha < \frac{1}{4}$.

For $n \geq 1$, let

$$a_n(x) = \min\left(\frac{\alpha}{f^2(x)}, n\right).$$

1. For any $n \geq 1$, prove the existence of a global solution

$$v_n \in C([0,\infty), C_0(\mathcal{U})) \cap C([0,\infty), H_0^1(\mathcal{U}))$$

of the problem

$$\begin{cases} (v_n)_t - \Delta v_n = a_n(x) v_n & (t,x) \in (0,\infty) \times \mathcal{U}, \\ v_n(t)|_{\partial\mathcal{U}} = 0 & t \in (0,T), \\ v_n(0,x) = \varphi(x) & x \in \mathcal{U}. \end{cases} \tag{8.9}$$

Hint: use the Duhamel formulation of the equation and a fixed point argument in a suitable space on a short interval of time. Then iterate the argument to obtain a global solution.

2. For all $n \geq 1$, prove that $v_n(t) \geq 0$, for all $t \geq 0$.

3. For all $n \geq 1$, prove that $v_{n+1}(t) \geq v_n(t)$, for all $t \geq 0$.

For $t \geq 0$, define

$$E_n(t) = \frac{1}{2}\int_{\mathcal{U}} |\nabla v_n(t,x)|^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathcal{U}} a_n(x) v_n^2(t,x)\,\mathrm{d}x.$$

4. Prove that the function E_n is C^1 on $[0,+\infty)$ and compute $\frac{\mathrm{d}}{\mathrm{d}t}E_n(t)$.

5. Deduce from the previous question and (8.7) the following uniform bound : there exists $C(\alpha) > 0$ such that

$$\forall n \geq 1,\ \forall t \geq 0, \quad \int |\nabla v_n(t,x)|^2\,\mathrm{d}x \leq C(\alpha)\int |\nabla\varphi|^2.$$

6. Construct a global weak solution v of (8.8) in a sense to specify.

Hint: obtain the solution v as the limit of the sequence $\{v_n\}_{n=1}^{\infty}$.

Appendix A

Elements of topology and functional analysis

A.1 Metric spaces

Let E be a non empty set.

Definition A.1. A *distance* on E is a mapping $d : E \times E \to [0, \infty)$ satisfying, for all $x, y, z \in E$:

1. Separation: $d(x, y) = 0 \iff x = y$;
2. Symmetry: $d(x, y) = d(y, x)$;
3. Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$.

A set E equipped with a distance d is called a *metric space*, denoted by (E, d).

Let A be a subset of E. Equipped with the (restriction of the) distance d, (A, d) is a metric subspace of E.

Let $(E_1, d_1), \ldots, (E_N, d_N)$ be N metric spaces and let E be the product set

$$E = \prod_{j=1}^{N} E_j.$$

Then,

$$d(X, Y) = \sqrt{\sum_{j=1}^{N} d_j^2(x_j, y_j)},$$

defined for any $X = (x_1, \ldots, x_N)$, $Y = (y_1, \ldots, y_N)$ in E, is a distance on E. One can also consider the distances

$$d_\infty(X, Y) = \sup_{j=1,\ldots,N} d_j(x_j, y_j), \quad d_1(X, Y) = \sum_{j=1}^{N} d_j(x_j, y_j).$$

Unless otherwise indicated, for $N \geq 1$, $\mathbf{R}^N$ is equipped of the Euclidian distance

$$d(X, Y) = \sqrt{\sum_{j=1}^{N} (x_j - y_j)^2}$$

A.2 Normed vector spaces

Let X denote a $\mathbf{K}$-linear space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$.

Definition A.2. A mapping $\|\cdot\| : X \to [0, \infty)$ is called *a norm* if

1. for all $g, h \in X$, $\|g + h\| \leq \|g\| + \|h\|$ (triangle inequality);
2. for all $g \in X$, $a \in \mathbf{K}$, $\|ag\| = |a|\|g\|$;
3. $\|g\| = 0$ if and only if $g = 0$.

A normed space $(X, \|\cdot\|)$ has a metric structure for the distance d associated to $\|\cdot\|$, defined by $d(g, h) = \|g - h\|$, for any two elements $g, h \in X$.

Two norms on X, denoted by $\|\cdot\|_1$ and $\|\cdot\|_2$ are say to be *equivalent* if there exists a constant $C > 0$ such that

$$\forall g \in X, \quad \|g\|_1 \leq C\|g\|_2 \quad \text{and} \quad \|g\|_2 \leq C\|g\|_1.$$

We recall that on a finite dimensional space, all norms are equivalent.

A.3 Topology of metric spaces

A.3.1 Open sets

Let (E, d) be a metric space.

Definition A.3. We call *open ball* of center $g \in E$ and radius $r > 0$ the set

$$\mathcal{B}(g, r) = \{h \in E : d(g, h) < r\}.$$

The *closed ball* of center $g \in E$ and radius $r > 0$ is the set

$$\mathcal{B}'(g, r) = \{h \in E : d(g, h) \leq r\}.$$

Definition A.4. We say that a subset $\mathcal{U}$ of E is *open* if for any point $g \in \mathcal{U}$ there exists an open ball centered at g and included in $\mathcal{U}$. A subset of E is said to be *closed* if its complement in E is open.

By convention, the empty set is open. Any union of open subsets of E is open. Any *finite* intersection of open subsets of E is open. Any intersection of closed subsets of E is closed. Any *finite* union of closed subsets of E is closed.

For any $g \in E$, we call *neighborhood of g* a subset of E containing an open subset containing g. A subset $\mathcal{U}$ of E is open if, and only if it is a neighborhood of each of its points.

If D is a subset of E, the union of all open subsets included in D is called the *interior* of D, denoted by $\mathring{D}$. Note that $g \in \mathring{D}$ if and only if D is a neighborhood of g. The intersection of all closed sets containing D is called the *closure* of D, denoted by $\bar{D}$. Note that $g \in \bar{D}$ if and only if any neighborhood of D has a non empty intersection with D. The complement of $\bar{D}$ is the interior of the complement of D. The set $\bar{D} \setminus \mathring{D}$ is called the *boundary* of D denoted by ∂D.

Definition A.5. We say that a sequence $\{g_k\}_{k=1}^{\infty}$ of (E, d) *converges to* $g \in E$, written $\lim_k g_k = g$, if $\lim_{k\to\infty} d(g_k, g) = 0$.

Remark A.6. Let $\{g_\lambda\}_{\lambda>0}$ be a family of elements in E, defined for all $\lambda > 0$ small. We will say that $\{g_\lambda\}_{\lambda>0}$ converges to g as $\lambda \to 0$, written $\lim_{\lambda\downarrow 0} g_\lambda = g$ if for any sequence $\{\lambda_k\}_{k=1}^\infty$ of positive numbers converging to 0, the sequence $\{g_{\lambda_k}\}_{k=1}^\infty$ converges to g.

Definition A.7. A *subsequence* $\{g_{\varphi(k)}\}_{k=1}^\infty$ of a sequence $\{g_k\}_{k=1}^\infty$ is characterized by a strictly increasing mapping $\varphi : \mathbf{N} \to \mathbf{N}$.

Definition A.8. We say that g is an *accumulation point* of a sequence $\{g_k\}_{k=1}^\infty$ if there exists a subsequence of $\{g_k\}_{k=1}^\infty$ converging to g.

If a sequence $\{g_k\}_{k=1}^\infty$ converges to g, then g is the unique accumulation point of this sequence. In particular, the limit of a sequence, if it exists, is unique.

Definition A.9. Let (E, d) be a metric space and A be a subset of E. We say that A is *dense* in E if its closure $\bar{A}$ is E. Equivalently, for all $f \in E$, there exists a sequence $\{a_j\}_{j=1}^\infty$ of elements of A such that $\lim_{j\to\infty} a_j = f$.

A.3.2 Continuity

Let (E_1, d_1) and (E_2, d_2) be two metric spaces. Let A be a subset of E_1 and $F : A \to E_2$ be a function. Let $g_0 \in \bar{A}$ and $h \in E_2$. We say that

$$\lim_{g\to g_0; g\in A} F(g) = h$$

if for all neighborhood V_2 of h in E_2, there exists a neighborhood V_1 of g_0 in E_1 such that $F(V_1 \cap A) \subset V_2$. The limit, if it exists, is unique.

Definition A.10. We say that $F : A \to E_2$ is *continuous* at $g_0 \in A$ if

$$\lim_{g\to g_0; g\in A} F(g) = F(g_0).$$

We say that F is *continuous on* A if it is continuous at any point of A.

The composition of two continuous functions is continuous.

Theorem A.11. *The application $F : A \to E_2$ is continuous at $g \in A$ if, and only if for any sequence $\{g_k\}_{k=1}^\infty$ of A converging to g, the sequence $\{F(g_k)\}_{k=1}^\infty$ converges to $F(g)$.*

A.3.3 Compactness

Definition A.12. We say that a metric space (E, d) is *compact* if any sequence of E admits a subsequence that converges to an element of E.

A subset A of a metric space (E, d) is compact if the metric space (A, d) is compact.

Proposition A.13. *Let A be a subset of a metric space (E, d). If A is compact, then A is closed in Y. If E is compact and A is closed in E, then A is compact.*

Theorem A.14. *Let X be a finite dimensional vector space. Then, $A \subset X$ is compact if and only if A is closed and bounded.*

In a vector space with infinite dimension, any compact set is closed and bounded, but the converse is false in general.

Theorem A.15. *Let (E_1, d_2) and (E_2, d_2) be two metric spaces. Let $F : E_1 \to E_2$ be continuous. If E_1 is compact then $F(E_1)$, the image of E_1 by F, is a compact set of E_2.*

Definition A.16. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. Let $D \subset X$ and $F : D \to Y$. We say that F is *uniformly continuous* if the function $\omega : [0, \infty) \to [0, \infty)$ defined by

$$\omega(\delta) = \sup_{\substack{g,h \in D \\ \|g-h\| \leq \delta}} \|F(g) - F(h)\|_Y \tag{A.1}$$

converges to 0 as δ converges to 0.

Theorem A.17. *Let (E_1, d_1) and (E_2, d_2) be two metric spaces. Let $F : E_1 \to E_2$ be continuous. If E_1 is compact then F is uniformly continuous.*

A.4 Complete metric spaces and Banach spaces

Let (E, d) be a metric space.

Definition A.18. We call *Cauchy sequence on E* a sequence $\{g_k\}_{k=1}^{\infty}$ such that

$$\lim_{j,k \to \infty} d(g_j, g_k) = 0.$$

In any metric space, a converging sequence is a Cauchy sequence.

Definition A.19. We say that a (E, d) is *complete* if any Cauchy sequence is convergent.

Definition A.20. We say that a normed vector space $(X, \|\cdot\|)$ is a *Banach space* if any Cauchy sequence is convergent.

Theorem A.21. *Any finite dimensional normed vector space is a Banach space.*

Theorem A.22. *The metric space (E, d) is complete if and only if any sequence $\{g_k\}_{k=0}^{\infty}$ satisfying $\sum_{k=0}^{\infty} d(g_{k+1}, g_k) < \infty$ has a limit.*

Corollary A.23. *Let $\sum_{k=0}^{\infty} g_k$ be a series of elements g_k of a Banach space $(X, \|\cdot\|)$. Suppose that $\sum_{k=0}^{\infty} \|g_k\| < \infty$. Then the series $\sum_{k=0}^{\infty} g_k$ is convergent in X.*

Proposition A.24. *Let A be a subset of a metric space (E, d). If A is complete, then A is closed in E. If E is complete and A is closed in E, then A is complete.*

Proposition A.25. *A compact metric space is complete.*

Theorem A.26 (The Banach Fixed-Point Theorem)**.** *Let (E, d) be a complete metric space. Let $\Phi : E \to E$ be a mapping such that there exists $k \in [0, 1)$ satisfying*

$$\text{for all } g, h \in E, \quad d(\Phi(g), \Phi(h)) \leq k\, d(g, h).$$

Then, there exists a unique fixed point $g_0 \in E$ of Φ, i.e. *satisfying $\Phi(g_0) = g_0$.*

Definition A.27. A map $f : (E_1, d_1) \to (E_2, d_2)$ between two metric spaces is an *isometry* if for all $g, h \in E_1$, $d_2(f(g), f(h)) = d_1(g, h)$.

Theorem A.28 (Completion of a metric space)**.** *Let (E, d) be a metric space. There exists a unique (up to isometries)* complete *metric space $(\tilde{E}, \tilde{d})$, containing E as a dense subset and such that the restriction of $\tilde{d}$ to E is d.*

Any uniformly continuous application $f : E \to Y$, where (Y, d_Y) is a complete metric space, extends uniquely as a continuous application $\tilde{f} : \tilde{E} \to Y$.

The complete metric space $(\tilde{E}, \tilde{d})$ is called the *completion* of (E, d). The completion of a complete metric space is itself. Moreover, if A is a dense subset of a complete metric space (E, d), then E is the completion of A.

A.5 Space of continuous functions

Let (E_1, d_1) and (E_2, d_2) be two metric spaces. An application $F : E_1 \to E_2$ is *bounded* if there exists $C > 0$ such that $d_2(F(g), F(h)) \leq C$ for all $g, h \in E_1$.

We denote by $\mathcal{C}(E_1, E_2)$ the set of continuous mappings from E_1 to E_2 and by $\mathcal{C}_b(E_1, E_2)$ the subset of *bounded* continuous mappings from E_1 to E_2.

If E_1 is compact, then $\mathcal{C}(E_1, E_2) = \mathcal{C}_b(E_1, E_2)$.

Definition A.29. The formula

$$D(F, G) = \sup_{g \in E_1} d_2(F(g), G(g)),$$

for $F, G \in \mathcal{C}_b(E_1, E_2)$ defines a distance on $\mathcal{C}_b(E_1, E_2)$, called the *uniform distance.*

Proposition A.30. *Suppose that (E_2, d) is complete. Then, $(\mathcal{C}_b(E_1, E_2), D)$ is a complete metric space.*

In particular, for any metric space (E, d), the normed space $(\mathcal{C}_b(E, \mathbf{R}), \| \cdot \|)$, where the norm $\| \cdot \|$ is defined by

$$\|F\| = \sup_{g \in X} |F(g)|,$$

is a Banach space.

A.6 Bounded linear operators

Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be two normed vector spaces on $\mathbf{R}$.

Definition A.31. A map $A : X \to Y$ is called a *linear operator* if for all $g, h \in X$, $\alpha, \beta \in \mathbf{R}$,

$$A(\alpha g + \beta h) = \alpha A g + \beta A h.$$

The *range* of A is $R(A) = \{v \in Y : v = Ag \text{ for some } g \in X\}$. The *null space* of A is $N(A) = \{g \in X : Ag = 0\}$. The *graph* of A is the set

$$G(A) = \{(g, v) \in X \times Y : v = Ag\}.$$

Theorem A.32. *Let $A : X \to Y$ be a linear operator. The following three properties are equivalent.*

1. *A is continuous at 0;*
2. *A is continuous on X;*
3. *there exists a constant $C \geq 0$ such that*

$$\text{for all } g \in X, \quad \|Ag\|_Y \leq C\|g\|_X.$$

We denote by $\mathcal{L}(X, Y)$ the vector space of linear continuous operators from X to Y equipped with the norm

$$\|A\|_{\mathcal{L}(X,Y)} = \sup \{\|Ag\|_Y \, ; \, \|g\|_X = 1\} .$$

Theorem A.33. *Let $(X, \|\cdot\|)$ be a normed vector space, D be a dense subspace of X, and Y a Banach space. Any linear continuous linear map from D to Y can be uniquely extended to a continuous linear map from X to Y.*

We recall the *uniform boundedness principle* (or Banach-Steinhaus theorem).

Theorem A.34. *Let X be a Banach space, Y be a normed vector space, and $A_j{}_{j \in \mathcal{J}}$ be a family of linear operators from X to Y satisfying, for all $g \in X$,*

$$\sup_{j \in J} \|A_j g\|_Y < \infty.$$

Then, the bound is uniform on the unit ball of X, i.e.

$$\sup_{j \in J} \|A_j\|_{\mathcal{L}(X,Y)} < \infty.$$

A linear operator $A : X \to Y$ is called *closed* if its graph is closed, which means that for any sequence $\{g_k\}_{k=0}^{\infty}$ of X such that $\lim_\infty g_k = g$ in X and $\lim_\infty Ag_k = v$ in Y, one has $Ag = v$.

Theorem A.35 (The Closed Graph Theorem)**.** *Let X and Y be Banach spaces and let $A : X \to Y$ be a linear mapping. Then, $A \in \mathcal{L}(X, Y)$ if and only if the graph of A is a closed subspace of $X \times Y$.*

Corollary A.36. *Let X be a Banach space and Y be a normed vector space. Then, the pointwise limit of a sequence in $\mathcal{L}(X, Y)$ belongs to $\mathcal{L}(X, Y)$.*

When $Y = X$, we denote $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$ the vector space of the bounded linear operators on X. Equipped with the composition product of applications $A \circ B$, denoted simply by AB, $\mathcal{L}(X)$ is a unitary algebra, with identity element I. The norm on $\mathcal{L}(X)$, defined by

$$\|A\|_{\mathcal{L}(X)} = \sup_{\|g\| \leq 1} \|Ag\|$$

will also be denoted by $\|\cdot\|$ when there is no risk of confusion. It is easily checked that for all $A, B \in \mathcal{L}(X)$,

$$\|AB\| \leq \|A\| \, \|B\|.$$

An element A of $\mathcal{L}(X)$ is said to be *invertible* if it admits an inverse in $\mathcal{L}(X)$, *i.e.* if there exists $B \in \mathcal{L}(X)$ such that $AB = BA = I$. We recall the following consequence of the *open mapping theorem.*

Theorem A.37. *Let X be a Banach space. Let $A \in \mathcal{L}(X)$ be bijective. Then the inverse of A, denoted by A^{-1}, belongs to $\mathcal{L}(X)$.*

Recall that if A and B are invertible, then AB is also invertible and it holds $(AB)^{-1} = B^{-1}A^{-1}$. We will use the convention $A^0 = I$.

Definition A.38. Let $A \in \mathcal{L}(X)$.

1. The *resolvent set* of A is

$$\rho(A) = \{\lambda \in \mathbf{R} : (A - \lambda I) \text{ is invertible with continuous inverse}\}.$$

2. The *spectrum* of A is $\sigma(A) = \mathbf{R} \setminus \rho(A)$.

3. We say that $\lambda \in \sigma(A)$ is an *eigenvalue* of A if $N(A - \lambda I) \neq \{0\}$. We write $\sigma_p(A)$ the set of eigenvalues (*point spectrum*).

4. If $\lambda \in \sigma_p(A)$ and $g \in X$, $g \neq 0$ satisfies $Ag = \lambda g$, then g is an associated *eigenvector*.

Note that by Theorem A.37, when X is a Banach space, for $\lambda \in \mathbf{R}$ such that $(A - \lambda I)$ is invertible, its inverse $(A - \lambda I)^{-1}$ is automatically continuous.

Lemma A.39. *Suppose that X is a Banach space. Let $A \in \mathcal{L}(X)$ be such that $\|I - A\| < 1$. Then, A is invertible and*

$$A^{-1} = \sum_{k=0}^{\infty} (I - A)^k.$$

Proposition A.40. *Suppose that X is a Banach space. Let $A \in \mathcal{L}(X)$.*

1. *The resolvant set of A is open in* $\mathbf{R}$.

2. *The spectrum of A is a non empty compact set of* $\mathbf{R}$.

The *spectral radius* is then defined by

$$\mathrm{r}(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}. \tag{A.2}$$

Proposition A.41. *Suppose that X is a Banach space. Let $A \in \mathcal{L}(X)$. The sequence $\{\|A^n\|^{1/n}\}_{n=1}^{\infty}$ is convergent and*

$$\mathrm{r}(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}}.$$

A.7 The dual space

When $Y = \mathbf{K}$, $\mathcal{L}(X, Y)$ is denoted by X^* and called the *dual* or *topological dual* space of X; it is the space of *continuous linear forms* on X. Equipped with the norm

$$\|A\|_{\mathcal{L}(X)} = \sup_{\|g\| \leq 1} |Ag|,$$

it is a Banach space.

A.8 Continuously differentiable functions

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces and let $\mathcal{U}$ be an open set of X.

Definition A.42. For a function $f : \mathcal{U} \to \mathbf{R}$, where $0 \in \mathcal{U}$, we denote $f(h) = o(h)$, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $h \in \mathcal{U}$ with $\|h\| \leq \delta$, $|f(h)| \leq \varepsilon \|h\|$.

Definition A.43. We say that an application $F : \mathcal{U} \to Y$ is *differentiable at* $g \in X$ if there exists a continuous linear map, $\mathrm{d}F_g : X \to Y$ such that

$$\|F(g+h) - F(g) - \mathrm{d}F_g(h)\|_Y = o(h).$$

If it exists, this linear map is unique and called the *differential of* F *at* g.

We say that F is *differentiable on* $\mathcal{U}$ if it is differentiable at any point of $\mathcal{U}$.

We say that F is of *class* $\mathcal{C}^1$ *on* $\mathcal{U}$ if it is differentiable at any point of $\mathcal{U}$ and if the application

$$\mathrm{d}F : \mathcal{U} \to \mathcal{L}(X, Y), \quad g \mapsto \mathrm{d}F_g$$

is continuous.

A linear combinaison of differentiable functions is differentiable, and the differential is linear. A composition of differentiable functions is differentiable.

Remark A.44. Let I be an open interval of $\mathbf{R}$ and $(X, \|\cdot\|)$ be a normed vector space. Let $F : I \to X$ be differentiable at $t_0 \in I$, so that $\mathrm{d}F_{t_0} \in \mathcal{L}(\mathbf{R}, X)$. Since $\mathcal{L}(\mathbf{R}, X)$ is isomorphic to X, we may identify $\mathrm{d}F_{t_0}$ with a vector of X, and the real $\mathrm{d}F_{t_0}(1)$ is denoted by $F'(t_0)$ or $\frac{\mathrm{d}}{\mathrm{d}t}F(t_0)$.

Theorem A.45. *Let $a, b \in \mathbf{R}$ with $a < b$ and $(X, \|\cdot\|)$ be a normed vector space. Let $F : [a,b] \to X$ and $\varphi : [a,b] \to \mathbf{R}$ two continuous mappings on $[a,b]$, differentiable on (a,b). If for all $t \in (a,b)$, it holds $\|\frac{\mathrm{d}}{\mathrm{d}t}F(t)\| \leq \varphi'(t)$ then $\|F(b) - F(a)\| \leq \varphi(b) - \varphi(a)$.*

A.9 Integral of regulated functions with values in a Banach space

Let $(X, \|\cdot\|)$ be a Banach space and let $I = [a,b]$ be a compact interval of $\mathbf{R}$. The vector space $\mathcal{B}(I, X)$ of bounded functions from I to X, equipped with the norm

$$\|f\|_\infty = \sup_{t \in I} \|f(t)\|$$

is a Banach space.

Definition A.46. We say that a function $f : [a,b] \to X$ is a *step function* if there exists $x_0 = a < t_1 < \cdots < t_{N+1} = b$ and $c_0, c_1, \ldots, c_N \in X$ such that $f(t) = c_k$ on (t_k, t_{k+1}) for any $k = 0, \ldots, N$.

We define the integral of f by

$$\int_I f = \int_a^b f(t)\,\mathrm{d}t = \sum_{k=0}^{N} c_k.$$

We denote by $\mathcal{D}$ be the subspace of step functions in $\mathcal{B}(I, X)$

This definition yields a linear map

$$\begin{aligned} T : \mathcal{D} &\to X \\ f &\mapsto \int_a^b f(t)\,\mathrm{d}t, \end{aligned}$$

such that, for any step function $g : I \to \mathbf{R}$ and any $x \in X$

$$\int_a^b g(t)x\,\mathrm{d}t = \left(\int_a^b g(t)\,\mathrm{d}t\right)x.$$

In this framework, the following inequalities are elementary

$$\left\|\int_a^b f(t)\,\mathrm{d}t\right\| \leq \int_a^b \|f(t)\|\,\mathrm{d}t \leq (b-a)\|f\|_\infty. \tag{A.3}$$

By definition, we say that a function $f \in \mathcal{B}(I, X)$ is *regulated* if it belongs to the closure of $\mathcal{D}$ in $\mathcal{B}(I, X)$. It can be proved (Exercise) that a function $f : I \to X$ is regulated if and only if it admits left and right limits at any point of I. Using the extension theorem (see Theorem A.33), we deduce that T extends uniquely to the closure of $\mathcal{D}$ in $\mathcal{B}(I, X)$. This procedure defines, for any regulated function $f : I \to X$, its integral $\int_a^b f(t)\,\mathrm{d}t \in X$. Morever, the inequalities (A.3) extend to such functions.

In this course, we often use this notion of integral for continuous functions, which are a particular case of regulated functions.

Another possibility is to extend the notions of measurability and integrability reviewed in Appendix B to mappings $f : [a, b] \to X$ using the Bochner integral. See [8, Appendix E.5] or [6, Chapter 1].

A.10 Real and complex Hilbert spaces

Definition A.47. Let H be a linear vector space on $\mathbf{R}$. A *(real) scalar product* on H is a map $(f, g) \mapsto (f \mid g)$ from $H \times H$ to $\mathbf{R}$ satisfying the following properties

1. Bilinearity: for all $f_1, f_2, g_1, g_2 \in H$, $\lambda \in \mathbf{R}$,

$$\begin{aligned} (\lambda f_1 + f_2 \mid g_1) &= \lambda(f_1 \mid g_1) + (f_2 \mid g_1), \\ (f_1 \mid \lambda g_1 + g_2) &= \lambda(f_1 \mid g_1) + (f_1 \mid g_2); \end{aligned}$$

2. Symmetry: $(f \mid g) = (g \mid f)$, for all $f, g \in H$;

3. Positivity: for all $f \in H$, $(f \mid f) \geq 0$ and

$$(f \mid f) = 0 \iff f = 0.$$

Definition A.48. Let H be a linear vector space on $\mathbf{C}$. A *hermitian scalar product* on H is a map $(f, g) \mapsto (f \mid g)$ from $H \times H$ to $\mathbf{C}$ satisfying the following properties

1. Linearity and antilinearity: for all $f_1, f_2, g_1, g_2 \in H$, $\lambda \in \mathbf{C}$,

$$(\lambda f_1 + f_2 \mid g_1) = \bar{\lambda}(f_1 \mid g_1) + (f_2 \mid g_1),$$
$$(f_1 \mid \lambda g_1 + g_2) = \lambda(f_1 \mid g_1) + (f_1 \mid g_2);$$

2. Hermitian symmetry: $(f \mid g) = \overline{(g \mid f)}$, for all $f, g \in H$;

3. Positivity: for all $f \in H$, $(f \mid f) \geq 0$ and

$$(f \mid f) = 0 \iff f = 0.$$

A real or complex vector space equipped with a scalar product has a natural normed space structure, by setting

$$\|f\| = (f \mid f)^{1/2}.$$

Moreover, the Cauchy-Schwarz inequality holds

$$|(f \mid f)| \leq \|f\| \, \|g\|.$$

Definition A.49. We say that $(H, (\cdot \mid \cdot))$ is a *Hilbert space* if it is complete for the associated norm.

We recall the Riesz Theorem.

Theorem A.50. *Let $(H, (\cdot \mid \cdot))$ be a Hilbert space. To any element $h \in H$, we associate the* continuous linear form L_h *on H defined by, for any $f \in H$,*

$$L_h(f) = (h \mid f).$$

Conversely, for any continuous linear form L on H, there exists a unique $h \in H$ such that $L = L_h$.

Definition A.51. For any subset A of a Hilbert space H, we set

$$A^\perp = \{f \in H : \forall a \in H, \ (a \mid f) = 0\}.$$

The set $A^\perp$ is a closed subspace of H.

Definition A.52. We say that a subset A of a Hilbert space H is *total* if the subpace spanned by A, denoted by $\mathrm{span}(A)$, is dense in H. Equivalently, A is total if and only if $A^\perp = \{0\}$.

Definition A.53. We say that a Hilbert space is *separable* if there exists a finite or infinite sequence of elements of H that form a total set in H.

Definition A.54. Let H be a separable Hilbert space. We call *Hilbert basis* of H a finite or infinite sequence $\{e_j\}_{j=1,2,\ldots}$ of H that is a total set of H and satisfies the relations

$$(e_j \mid e_k) = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{for } j \neq k. \end{cases}$$

In any separable Hilbert space, there exist Hilbert basis.

Theorem A.55. *Let H be a separable Hilbert space and $\{e_j\}_{j=1,2,\dots}$ be a Hilbert basis of H.*

1. *Any element $f \in H$ decomposes uniquely as a convergent series in H*

$$f = \sum_j c_j e_j, \quad c_j \in \mathbf{C}.$$

The components c_j are given by $c_j = (e_j \mid f)$ and it holds

$$\|f\|^2 = \sum_j |c_j(f)|^2 \qquad \text{(Bessel-Parseval)}$$

2. *Conversely, being given complex numbers γ_j satisfying $\sum_j \gamma_j^2 < \infty$, the series $\sum_j \gamma_j e_j$ is convergent in H and its sum f satisfies $(f \mid e_j) = \gamma_j$, for all j.*

A.11 Weak convergence in Hilbert spaces

In Hilbert spaces of infinite dimension, bounded sets are not in general of compact closure (See Exercice A.7). Thus, it is relevant to weaken the notion of convergence in such spaces.

Definition A.56. *Let $\{f_j\}_{j=0}^{\infty}$ be a sequence of elements of a separable Hilbert space H and let f be an element of H. The sequence $\{f_j\}_{j=0}^{\infty}$ is said to* weakly converge *to f, which is denoted by $f_j \rightharpoonup f$ if*

$$\forall h \in H\,, \ \lim_{j\to\infty} (h|f_j) = (h|f).$$

It is easy to see that if the weak limit exists, then it is unique.

Proposition A.57. *Let $\{f_j\}_{j\in\mathbf{N}}$ be a sequence of elements of a separable Hilbert space H and $f \in H$. Then*

$$\lim_{j\to\infty} \|f_j - f\| = 0 \implies f_j \rightharpoonup f$$

and

$$f_j \rightharpoonup f \implies \{f_j\}_{j=0}^{\infty} \text{ is bounded and } \|f\| \le \liminf \|f_j\|. \tag{A.4}$$

The first statement is easy to prove, using the Cauchy-Schwartz inequality

$$\forall h \in H, \quad |(h|f_j - f)| \le \|h\|\|f_j - f\|.$$

The second result is a consequence of the Banach-Steinhaus Theorem A.34: for any integer j, define the map L_j on H by

$$L_j(h) = (h|f_j).$$

This map is clearly linear and continuous. Moreover, for a fixed $h \in H$, the sequence $\{L_j(h)\}_{j=0}^{\infty}$ is convergent and thus bounded. The Banach-Steinhaus Theorem implies that the sequence $\{L_j\}_{j=0}^{\infty}$ is bounded and that the limiting linear map defined by

$$L : h \mapsto (h|f)$$

is continuous. Moreover, its norm is bounded by $\liminf \|L_j\|_{\mathcal{L}(H)}$. Since it holds $\|L_j\|_{\mathcal{L}(H)} = \|f_j\|$ and $\|L\|_{\mathcal{L}(H)} = \|f\|$, the result follows.

A.12 The Lax-Milgram theorem

Let $(H, (\cdot \mid \cdot))$ be a real Hilbert space. We denote by $\langle \cdot, \cdot \rangle$ the pairing of H with its dual space H^*.

Theorem A.58 (Lax-Milgram theorem)**.** *Assume that*

$$a : H \times H \to \mathbf{R}$$

is a bilinear mapping, for which there exist positive constants α, β such that

1. Continuity: *for all $(\varphi, \psi) \in H$, $|a(\varphi, \psi)| \leq \beta \|\varphi\| \|\psi\|$;*
2. Coercivity: *for all $\varphi \in H$, $a(\varphi, \varphi) \geq \alpha \|\varphi\|^2$.*

Then, for every $\ell \in H^\star$, there exists a unique $g \in H$ such that

$$a(g, \varphi) = \langle \ell, \varphi \rangle \quad \textit{for any } \varphi \in H.$$

A.13 Exercises

Exercise A.1. Let X and Y be two normed vector spaces and $\mathcal{L}(X, Y)$ be the set of continuous linear applications from X to Y, equipped with the operator norm. Show that if Y is complete then $\mathcal{L}(X, Y)$ is complete.

Exercise A.2. Let A be a continuous operator on a Banach space X such that $\|A\| < 1$. Prove that $I - A$ is invertible with inverse $\sum_{k=0}^{\infty} A^k$.

Exercise A.3. Prove that the set of inversible operators is open in $\mathcal{L}(X)$.

Exercise A.4. Prove that spectrum $\sigma(A)$ of a continuous operator is compact and included in the disk of centre 0 and radius $\|A\|$.

Exercise A.5. Prove Theorem A.33.

Exercise A.6. Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be normed vector and D be a dense subspace of X. Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of $\mathcal{L}(X, Y)$. Suppose that there exists a constant $C > 0$ such that

$$\forall n \geq 0, \quad \|T_n\|_{\mathcal{L}(X,Y)} \leq C$$

and that for any $g \in D$, the sequence $\{A_n g\}_{n=0}^{\infty}$ has a finite limit Ag as $n \to \infty$. Prove that $A : D \to Y$ defines a linear and continuous map. Prove that if A extends to $\bar{A} \in \mathcal{L}(X, Y)$, then for any $g \in X$,

$$\lim_{n \to \infty} A_n g = \bar{A} g.$$

Exercise A.7. Let H be a separable Hilbert space of infinite dimension and let $(e_j)_{j \in \mathbf{N}}$ be a Hilbert basis of H. Assume that the sequence $\{e_j\}_{j=0}^{\infty}$ has a limiting point ℓ. Montrer que pour tout $x \in H$, la suite de terme général $(e_j | x)$ tend vers zéro. En déduire que $\ell = 0$ et conclure au fait que la boule unité d'un espace de Hilbert séparable de dimension infinie n'est pas d'adhérence compacte.

Exercise A.8. Show that weak convergence does not imply strong convergence.

Exercise A.9. Let $\{f_j\}_{j=0}^{\infty}$ be a sequence in a separable Hilbert space H which weakly converges to f, and satisfying in addition

$$\lim_{j\to\infty} \|f_j\| = \|f\|.$$

Prove that the sequence $\{f_j\}_{j=0}^{\infty}$ converges strongly to f in H.

Exercise A.10. Let $\{f_j\}_{j=0}^{\infty}$ be a bounded sequence in a separable Hilbert space H. The goal of this exercise is to prove that there exist $f \in H$ and a subsequence of $\{f_j\}_{j=0}^{\infty}$ converging weakly to f. For any Hilbert basis $(e_n)_{n\geq 1}$ of H, we will consider the sequence $\{(f_j|e_n)\}_{j=0}^{\infty}$ and apply a diagonal extraction.

1. Prove that there exists a subsequence $\{f_{\psi_1(j)}\}_j$ such that $\{f_{\psi_1(j)}|e_1)\}_j$ converges to a scalar γ_1 ($\psi_1 : \mathbf{N} \to \mathbf{N}$ strictly increasing).
2. Prove that one can extract from $\{f_{\psi_1(j)}\}_j$ a subsequence $\{f_{\psi_1\circ\psi_2(j)}\}_j$ telle que $\{f_{\psi_1\circ\psi_2(j)}|e_2)\}_j$ that converges to some γ_2. Continue and construct for any n a subsequence $\{f_{\psi_1\circ\ldots\circ\psi_n(j)}\}_j$ extracted from the previous ones and such that $\{(f_{\psi_1\circ\ldots\circ\psi_j(j)}|e_n)\}_j$ converges to a scalar γ_n.
3. We define for any $j \in \mathbf{N}$, $\varphi(j) = \psi_1 \circ \ldots \circ \psi_j(j)$. Prove that for any n,

   $$\lim_{j\to+\infty} (f_{\varphi(j)}|e_n) = \gamma_n$$

   Last, prove that $\sum_n \gamma_n e_n$ defines an element of H to which the subsequence $\{f_{\varphi(j)}\}_j$ weakly converges.

Exercise A.11.

1. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of a Hilbert space $(H, (\cdot \mid \cdot))$. Suppose that $\{g_n\}_{n=1}^{\infty}$ converges weakly to $g \in H$. Prove that there exists a subsequence of (g_n), denoted by $\{g_{\varphi(k)}\}_{k=1}^{\infty}$ such that the sequence $\{h_n\}_{n=1}^{\infty}$ defined by

   $$h_k = \frac{1}{k}(g_{\varphi(1)} + \cdots + g_{\varphi(k)})$$

   converges strongly to g.

   Hint: prove that one can reduce to the case $g = 0$ and construct the subsequence $(g_{\varphi(k)})_{k\geq 0}$ by induction.
2. Let C be a convex subset of H. Prove that C is weakly closed if and only if C is strongly closed.

Appendix B

Lebesgue integral on $\mathbf{R}^N$ and L^p spaces

B.1 Lebesgue measure

Definition B.1. We say that a collection $\mathcal{B}$ of subsets of $\mathbf{R}^N$ is a *σ-algebra* on $\mathbf{R}^N$ if

1. The empty set $\emptyset$ and $\mathbf{R}^N$ belong to $\mathcal{B}$;
2. $A \in \mathcal{B}$ implies $\mathbf{R}^N \setminus A \in \mathcal{B}$;
3. For any *sequence* $\{A_j\}_{j=1}^{\infty}$ of elements of $\mathcal{B}$, it holds

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{B} \quad \text{and} \quad \bigcap_{j=1}^{\infty} A_j \in \mathcal{B}.$$

Definition B.2. The *Borel σ-algebra* $\mathrm{Bor}(\mathbf{R}^N)$ is the σ-algebra generated by the open subsets of $\mathbf{R}^N$, *i.e.* the coarsest σ-algebra on $\mathbf{R}^N$ that contains all open subsets of $\mathbf{R}^N$.

Definition B.3. We call a *measure* on $(\mathbf{R}^N, \mathcal{B})$ a function μ on $\mathcal{B}$ such that

1. $\mu(\emptyset) = 0$;
2. For any $A, B \in \mathcal{B}$,

$$A \cap B = \emptyset \implies \mu(A \cup B) = \mu(A) + \mu(B).$$

3. For any non decreasing sequence $\{A_j\}_{j=0}^{\infty}$ of elements of $\mathcal{B}$, it holds

$$\mu\left(\bigcup_j A_j\right) = \sup_j \mu(A_j) = \lim_j \mu(A_j),$$

Now, we introduce the *N-dimensional Lebesgue measure* λ_N on $\mathbf{R}^N$.

Theorem B.4. *There exists a unique measure* λ_N *on* $\mathrm{Bor}(\mathbf{R}^N)$ *such that for all box* $B = \prod_{j=1}^N (a_j, b_j)$ *of* $\mathbf{R}^N$

$$\lambda_N(B) = \prod_{j=1}^{N} (b_j - a_j).$$

A subset A of $\mathbf{R}^N$ (belonging to $\mathrm{Bor}(\mathbf{R}^N)$ or not) is said *negligible* if there exists $B \in \mathrm{Bor}(\mathbf{R}^N)$ with $A \subset B$ and $\lambda(B) = 0$.

A property $P(x)$ depending on $x \in \mathbf{R}^N$ is satisfied *almost everywhere* if it is satisfied everywhere on $\mathbf{R}^N$ except possibly on a negligible set.

B.2 Measurable functions

Definition B.5. Let $f : \mathbf{R}^N \to \mathbf{R}$. We say that f is a *measurable* function if for any open subset $\mathcal{U}$ of $\mathbf{R}$,

$$f^{-1}(\mathcal{U}) = \{x \in \mathbf{R}^N : f(x) \in \mathcal{U}\} \in \mathrm{Bor}(\mathbf{R}^N).$$

Recall in particular that any continuous function $f : \mathbf{R}^N \to \mathbf{R}$ is measurable. The sum and the product of two measurable functions is measurable. If $\{f_k\}_{k=0}^{\infty}$ are measurable functions, then so are $\limsup f_k$ and $\liminf f_k$.

Let $\mathbf{1}_A$ the *characteristic function* of $A \in \mathrm{Bor}(\mathbf{R}^N)$ be defined by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\mathbf{1}_A : \mathbf{R}^N \to \mathbf{R}$ is measurable.

B.3 Lebesgue integral for positive functions

Using approximation of positive measurable functions $f : \mathbf{R}^N \to [0, \infty)$ by *simple* functions, one defines the *Lebesgue integral* for positive measurable functions as a map $f \mapsto \int_{\mathbf{R}^N} f(x)\,\mathrm{d}x \in [0, \infty]$ (also denoted by $\int_{\mathbf{R}^N} f$ or simply $\int f$) satisfying the following four fundamental properties.

1. Linearity: for f, g, two positive measurable functions, and $\alpha, \beta \in [0, \infty)$,

$$\int_{\mathbf{R}^N} (\alpha f + \beta g) = \alpha \int_{\mathbf{R}^N} f + \beta \int_{\mathbf{R}^N} g.$$

2. Comparison: if $f \leq g$ on $\mathbf{R}^N$ then $\int_{\mathbf{R}^N} f \leq \int_{\mathbf{R}^N} g$.

3. Normalisation: for any box $B = \prod_{j=1}^N (b_j - a_j)$, one has

$$\int_{\mathbf{R}^N} \mathbf{1}_B(x)\,\mathrm{d}x = \lambda_N(B) = \prod_{j=1}^{N} (b_j - a_j).$$

4. The Beppo-Levi monotone convergence theorem

Theorem B.6. *Let* $\{f_n\}_{n=0}^{\infty}$ *be a* non decreasing *sequence of* positive *measurable functions. Define* $f(x) = \sup_n f_n(x)$ *for all* $x \in X$*. Then* f *is measurable and*

$$\int_{\mathbf{R}^N} f \, \mathrm{d}x = \lim_{n\to\infty} \int_{\mathbf{R}^N} f_n \, \mathrm{d}x.$$

Note that more generally, for any $A \in \mathrm{Bor}(\mathbf{R}^N)$, $\int \mathbf{1}_A(x) \, \mathrm{d}x = \lambda_N(A)$.

We also recall the following result.

Proposition B.7. *Let* f *be a positive measurable function on* $\mathbf{R}^N$*. Then*

$$\int_{\mathbf{R}^N} f \, \mathrm{d}x = 0 \iff f = 0 \text{ almost everywhere on } X.$$

B.4 Integral of summable functions

Definition B.8. An application $f : \mathbf{R}^N \to \mathbf{R}$ or $\mathbf{C}$ is said to be *summable* (or *integrable*) if it is measurable and $\int_{\mathbf{R}^N} |f| \, \mathrm{d}x < \infty$.

Definition B.9. Real-valued case. For $f : \mathbf{R}^N \to \mathbf{R}$ summable, the integral of f is defined by

$$\int_{\mathbf{R}^N} f \, \mathrm{d}x = \int_{\mathbf{R}^N} f^+ \, \mathrm{d}x - \int_{\mathbf{R}^N} f^- \, \mathrm{d}x,$$

where

$$f^+ = \sup(f, 0), \; f^- = \sup(-f, 0), \; 0 \leq f^\pm \leq |f|, \; f = f^+ - f^-.$$

Complex-valued case. For $f : \mathbf{R}^N \to \mathbf{C}$ summable, the integral of f is defined by

$$\int_{\mathbf{R}^N} f \, \mathrm{d}x = \int_{\mathbf{R}^N} \Re f \, \mathrm{d}x + \mathrm{i} \int_{\mathbf{R}^N} \Im f \, \mathrm{d}x.$$

Proposition B.10.

1. *The space of summable functions is a vector space on* $\mathbf{R}$ *or* $\mathbf{C}$*, denoted by* $\mathcal{L}^1(\mathbf{R}^N, \mathrm{Bor}(\mathbf{R}^N), \lambda_N)$ *or simply* $\mathcal{L}^1(\mathbf{R}^N)$*.*
2. *The application* $f \mapsto \int_{\mathbf{R}^N} f \, \mathrm{d}x$ *is a linear form on* $\mathcal{L}^1$*. Moreover, for any* $f \in \mathcal{L}^1$*, it holds*
$$\left| \int_{\mathbf{R}^N} f \, \mathrm{d}x \right| \leq \int_{\mathbf{R}^N} |f| \, \mathrm{d}x.$$
3. *Let* f_1 *and* f_2 *be two measurable functions that are equal almost everywhere. Then,* f_1 *is summable if and only if* f_2 *is summable. In this case, they have equal integral.*

B.5 Integration on a subset of X

Let $M \in \mathrm{Bor}(\mathbf{R}^N)$ and g be a function from M to $\mathbf{C}$. We define the function g_M on $\mathbf{R}^N$, that is equal to g on M and equal to 0 on $\mathbf{R}^N \setminus M$. Then, g is summable on M if, and only if g_M is summable on $\mathbf{R}^N$ and we define

$$\int_M g \, \mathrm{d}x = \int_{\mathbf{R}^N} g_M \, \mathrm{d}x.$$

It is equivalent to define directly the integral on M, by considering the σ-algebra $\mathrm{Bor}(M)$ of elements of $\mathrm{Bor}(\mathbf{R}^N)$ that are contained in M and the restriction λ_M of the Lebesgue measure λ_N to $\mathrm{Bor}(M)$. Then, we can use the general integration theory to give a sense to $\int_M g\,\mathrm{d}x$.

B.6 The dominated convergence theorem

Theorem B.11. *Let $\{f_k\}_{k=0}^{\infty}$ be a sequence of measurable functions. We assume that*

- *The sequence $\{f_k\}_{k=0}^{\infty}$ converges almost everywhere to a measurable function f;*
- *There exists a function $g \in \mathcal{L}^1$ such that for any $k \geq 1$, $|f_k| \leq g$ almost everywhere.*

Then $f \in \mathcal{L}^1$ and

$$\lim_{k\to\infty} \int_{\mathbf{R}^N} |f_k - f|\,\mathrm{d}x = 0.$$

In particular,

$$\lim_{k\to\infty} \int_{\mathbf{R}^N} f_k\,\mathrm{d}x = \int_{\mathbf{R}^N} f\,\mathrm{d}x.$$

Now, we state applications of the dominated convergence theorem to functions defined by integrals depending on a parameter.

Theorem B.12. *Let $\mathcal{U}$ be an open set of $\mathbf{R}^N$. Let $t_0 \in \mathcal{U}$. Let $f : \mathcal{U} \times X \to \mathbf{C}$ satisfying*

1. *For all $t \in \mathcal{U}$, the function $x \mapsto f(t,x)$ is measurable.*
2. *For almost all $x \in X$, the function $t \mapsto f(t,x)$ is continuous at t_0.*
3. *There exists a non negative summable function h such that, for all $t \in \mathcal{U}$ and almost all $x \in X$, we have*

$$|f(t,x)| \leq h(x).$$

Then, the function $F : \mathcal{U} \to \mathbf{C}$

$$F(t) = \int_{\mathbf{R}^N} f(t,x)\,\mathrm{d}x.$$

is well-defined on $\mathcal{U}$ and continuous at t_0.

Theorem B.13. *Let I be an interval of $\mathbf{R}$ and let $f : I \times \mathbf{R}^N \to \mathbf{C}$. Assume that there exists a subset Z of $\mathbf{R}^N$ of measure zero such that, setting $X = \mathbf{R}^N \setminus Z$, it holds*

1. *For all $t \in I$, the function $x \mapsto f(t,x)$ is summable.*
2. *The partial derivative $\frac{\partial f}{\partial t}(t,x)$ exists at all point of $I \times X$.*

3. *There exists a non negative summable function h such that for any point $(t,x) \in I \times X$,*

$$\left|\frac{\partial f}{\partial t}(t,x)\right| \leq h(x).$$

Then, the function $F : t \mapsto \int_{\mathbf{R}^N} f(t,x)\,\mathrm{d}x$ is differentiable *on I and*

$$F'(t) = \int_{\mathbf{R}^N} \frac{\partial f}{\partial t}(t,x)\,\mathrm{d}x.$$

B.7 Lebesgue spaces

B.7.1 The Lebesgue space L^1

Let $\mathcal{U}$ be any open set of $\mathbf{R}^N$ equipped with the Borel σ-algebra and the Lebesgue measure. In the space $\mathcal{L}^1 = \mathcal{L}^1(\mathcal{U}, \mathrm{Bor}(\mathcal{U}), \lambda)$, the application $f \mapsto \|f\|_{L^1} = \int |f|\,\mathrm{d}x$ is a *semi-norm*:

- For any $f \in \mathcal{L}^1$ and $a \in \mathbf{C}$, $\|a\,f\|_{L^1} = |a|\,\|f\|_{L^1}$.
- For any $f, g \in \mathcal{L}^1$, $\|f+g\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1}$.

However,

$$\|f\|_{L^1} = 0 \iff f = 0 \quad \textit{almost everywhere.}$$

This property is weaker than the *separation property* of a norm. Thus, we are led to *modify* the space $\mathcal{L}^1$ to define a norm related to $\|\cdot\|_{L^1}$. On $\mathcal{L}^1$, we define a *equivalence relation* $\sim$ (reflexive, symmetric and transitive binary relation) by

$$f \sim g \iff f = g \text{ almost everywhere.}$$

For $f \in \mathcal{L}^1$, we denote $[f]$ the *equivalence class* of f

$$[f] := \left\{g \in \mathcal{L}^1 : f = g \text{ almost everywhere}\right\}.$$

Definition B.14. The Lebesgue space L^1 is defined by $L^1 = \left\{[f] : f \in \mathcal{L}^1\right\}$.

The Lebesgue integral of $[f]$ is defined by $[f] \mapsto \int [f]\,\mathrm{d}x = \int f\,\mathrm{d}x$, and this definition is independent of f chosen in $[f]$.

The semi-norm $\|\cdot\|_{L^1}$ on $\mathcal{L}^1$ now defines a *norm* $\|\cdot\|_{L^1}$ on the space L^1,

$$\|[f]\|_{L^1} = \int |f|\,\mathrm{d}x \quad \text{where} \quad f \in [f].$$

Indeed,

$$\|[f]\|_{L^1} = 0 \iff \int |f|\,\mathrm{d}x = 0 \iff f = 0 \text{ a.e.} \quad \iff [f] = [0].$$

In practice, we will identify $[f] \in L^1$ with any representative in $\mathcal{L}^1$ of this equivalence class, denoted by f, of $[f]$ and we denote $\|[f]\|_{L^1} = \|f\|_{L^1}$.

Theorem B.15 (Fischer-Riesz)**.** *The vector space L^1, equipped with the norm $\|\cdot\|_{L^1}$ is a Banach space.*

Corollary B.16. *Let* $\{f_k\}_{k=0}^{\infty}$ *be a sequence that converges to* f *in* L^1*. Then there exists a* subsequence *of* $\{f_k\}_{k=0}^{\infty}$ *that converges* almost everywhere *to* f*.*

For continuous functions, the notion of *support* is well-known.

Definition B.17. Let $f : \mathbf{R}^N \to \mathbf{C}$ be a continuous function. We call *support* of f the closure of the open set $\{x \in \mathbf{R}^N : f(x) \neq 0\}$. The support of f is denoted by $\operatorname{supp} f$.

We denote by $\mathcal{C}_c(\mathcal{U})$ the space of continuous functions with *compact support* in $\mathcal{U}$. Equivalently, it is the space of continuous functions $\mathcal{U} \to \mathbf{C}$ that vanish outside a compact set of $\mathbf{R}^N$ included in $\mathcal{U}$.

Theorem B.18. *The space* $\mathcal{C}_c(\mathcal{U})$ *is dense in* $L^1(\mathcal{U})$*, which means that for any function* $f \in L^1(\mathcal{U})$ *and any* $\varepsilon > 0$*, there exists* $f_\varepsilon \in \mathcal{C}_c(\mathcal{U})$ *such that*

$$\int_{\mathcal{U}} |f - f_\varepsilon| \,\mathrm{d}x \leq \varepsilon.$$

Proposition B.19 (Continuity of translations in L^1)**.** *Let* $f \in L^1(\mathbf{R}^N)$*. Let* $\tau_h : L^2(\mathbf{R}^N) \to L^2(\mathbf{R}^N)$ *be defined by*

$$\tau_h f(x) = f(x - h).$$

Then,

$$\lim_{h \to 0} \|\tau_h f - f\|_{L^1} = 0.$$

Now, we introduce the notion of *locally integrable* functions.

Definition B.20. We say that a mesurable function $f : \mathcal{U} \to \mathbf{R}$ belongs to $L^1_{\mathrm{loc}}(\mathcal{U})$ if $f \cdot \mathbf{1}_K \in L^1(\mathcal{U})$ for any compact $K \subset \mathcal{U}$.

Remark B.21. Recall that if $f \in L^1_{\mathrm{loc}}(\mathcal{U})$ and $\int_{\mathcal{U}} fu = 0$ for all $u \in \mathcal{C}_c(\mathcal{U})$, then $f = 0$ almost everywhere on $\mathcal{U}$.

B.7.2 The Lebesgue space L^2

Let $\mathcal{U}$ be any open set of $\mathbf{R}^N$.

Definition B.22. A function $f : \mathcal{U} \to \mathbf{C}$ is said to be *square integrable* if it is measurable and $\int_{\mathcal{U}} |f|^2 < \infty$.

The space $\mathcal{L}^2$ of square integrable functions is stable by multiplication by a scalar and by sum since

$$|f + g|^2 \leq 2|f|^2 + 2|g|^2.$$

It is thus a vector space. Moreover, if f and g belong to $\mathcal{L}^2$, then $\overline{f}g$ is integrable since $|fg| \leq \frac{1}{2}(|f|^2 + |g|^2)$. Thus, one can define

$$\langle f \mid g \rangle_{L^2} = \int \overline{f} g$$

(note the convention for the hermitian scalar product) and

$$\|f\|_{L^2} = \langle f \mid f \rangle_{L^2}^{1/2} = \left(\int |f|^2 \right)^{1/2}.$$

We recall the *equivalence relation*

$$f \sim g \iff f = g \text{ almost everywhere.}$$

For $f \in \mathcal{L}^2$, we denote by $[f]$ the *equivalence class* of f

$$[f] := \left\{ g \in \mathcal{L}^2 : f = g \text{ almost everywhere} \right\}.$$

Definition B.23. The Lebesgue space L^2 is defined by $L^2 = \left\{ [f] : f \in \mathcal{L}^2 \right\}$.

Theorem B.24. *The space L^2 equipped with the hermitian scalar product*

$$\langle f \mid g \rangle_{L^2} = \int \overline{f} g$$

is a Hilbert space, with corresponding norm $\|f\|_{L^2} = \langle f \mid f \rangle_{L^2}^{1/2}$.

Proposition B.25. *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of L^2 functions that converges to f in L^2. Then there exists a* subsequence *of $\{f_n\}_{n=0}^{\infty}$ that converges almost everywhere to f.*

Theorem B.26. *The space $\mathcal{C}_c(\mathcal{U})$ is dense in $L^2(\mathcal{U})$: for any function $f \in L^2(\mathcal{U})$ and any $\varepsilon > 0$, there exists $f_\varepsilon \in \mathcal{C}_c(\mathcal{U})$ such that*

$$\|f - f_\varepsilon\|_{L^2(\mathcal{U})} \leq \varepsilon.$$

Proposition B.27. *Let $f \in L^2(\mathbf{R}^N)$. Then*

$$\lim_{h \to 0} \|\tau_h f - f\|_{L^2} = 0,$$

where τ_h is defined in Proposition B.19.

B.7.3 The space of essentially bounded functions L^∞

Definition B.28. A measurable function f on $\mathbf{R}^N$ is said to be *essentially bounded* if there exists $M \in \mathbf{R}$ and a negligible subset Z of $\mathbf{R}^N$ such that $|f(x)| \leq M$, for all $x \in \mathbf{R}^N \setminus Z$.

Definition B.29. If f is essentially bounded, we define its *essential upper bound* by

$$\|f\|_{L^\infty} = \inf\{M : \text{there exists a negligible set } Z \text{ such that } |f(x)| \leq M, \text{ for all } x \in \mathbf{R}^N \setminus Z\}.$$

Remark B.30. We observe that with this definition, there exists a negligible subset Z of $\mathbf{R}^N$ so that $|f(x)| \leq \|f\|_{L^\infty}$, $\forall x \in \mathbf{R}^N \setminus Z$.

Definition B.31. The space $L^\infty(\mathbf{R}^N)$ of classes of essentially bounded functions for the equivalence relation $f = g$ almost everywhere, equipped with the norm $\|\cdot\|_{L^\infty}$ is a Banach space.

B.7.4 The Lebesgue spaces L^p

Let $1 \leq p < \infty$.

Definition B.32. The space $\mathcal{L}^p(\mathbf{R}^N, \mathrm{Bor}(\mathbf{R}^N), \lambda_N)$ (or simply $\mathcal{L}^p(\mathbf{R}^N)$) is the space of measurable functions $f : \mathbf{R}^N \to \mathbf{C}$ such that $|f|^p$ is summable. We define

$$\|f\|_{L^p} = \left(\int_{\mathbf{R}^N} |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

Proposition B.33 (Minkowski inequality)**.** *For any $f, g \in \mathcal{L}^p$, it holds*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

Definition B.34. The space $L^p(\mathbf{R}^N)$ of *classes of functions* of $\mathcal{L}^p(\mathbf{R}^N)$ for the equivalence relation $f = g$ almost everywhere, equipped with the norm $\|\cdot\|_{L^p}$, is a Banach space.

Theorem B.35. *1. Let u and v be non negative summable functions on $\mathbf{R}^N$ and let $\theta \in (0,1)$. Then, $u^\theta v^{1-\theta} \in L^1(\mathbf{R}^N)$ and*

$$\int_{\mathbf{R}^N} u^\theta v^{1-\theta} \, \mathrm{d}x \leq \left(\int_{\mathbf{R}^N} u \, \mathrm{d}x \right)^\theta \left(\int_{\mathbf{R}^N} v \, \mathrm{d}x \right)^{1-\theta}.$$

2. Let $1 \leq p \leq \infty$. Let p' be the conjugate exponent *of p defined by*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

If $f \in L^p(\mathbf{R}^N)$ and $g \in L^{p'}(\mathbf{R}^N)$, then $fg \in L^1(\mathbf{R}^N)$ and

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^{p'}}.$$

B.7.5 Support

We extend the notion of support to measurable functions and to (classes of) functions in L^1.

Definition B.36. Let $f : \mathcal{U} \to \mathbf{R}$ be a measurable function. Let $(\omega_j)_{j \in J}$ the family of all open sets of $\mathcal{U}$ such that $f = 0$ almost everywhere on ω_j. Define $\omega = \cup_{j \in J} \omega_j$. Then, $f = 0$ almost everywhere on ω and the *support* of f is defined by

$$\operatorname{supp} f = \Omega \setminus \omega.$$

Note that for two measurable functions f and g, if $f = g$ almost everywhere on $\mathcal{U}$, then $\operatorname{supp} f = \operatorname{supp} g$. Thus, the notion of support for a (class of) function in L^1, or more generally in L^p, makes sense.

B.8 Multiple integrals

B.8.1 Product σ-algebra and product measure

The product σ-algebra $\mathrm{Bor}(\mathbf{R}^p) \otimes \mathrm{Bor}(\mathbf{R}^q)$ is $\mathrm{Bor}(\mathbf{R}^{p+q})$. Let λ_p be Lebesgue measure on $\mathbf{R}^p$. Then, the product measure $\lambda_p \otimes \lambda_q$ is λ_{p+q}.

B.8.2 Fubini theorems

Theorem B.37 (Fubini theorem for borelians)**.** *Let* $S \in \mathrm{Bor}(\mathbf{R}^{p+q})$.

1. *For each* $x \in \mathbf{R}^p$*, the* vertical slice $S_x = \{y \in \mathbf{R}^q : (x, y) \in S\}$ *belongs to* $\mathrm{Bor}(\mathbf{R}^q)$.
2. *The application* $x \in \mathbf{R}^p \mapsto \lambda_q(S_x) \in [0, \infty]$ *is measurable.*
3. *It holds*
$$\lambda_{p+q}(S) = \int_{\mathbf{R}^p} \lambda_q(S_x)\,\mathrm{d}x.$$

Theorem B.38 (Non negative functions: Tonelli theorem)**.** *Let* $f : \mathbf{R}^{p+q} \to [0, \infty]$ *be measurable. Then,*

1. *For all* $x \in \mathbf{R}^p$*, the partial function* $y \in \mathbf{R}^q \mapsto f(x, y)$ *is measurable and the function from* $\mathbf{R}^p$ *to* $[0, \infty]$ *defined by* $x \mapsto \int_{\mathbf{R}^q} f(x, y)\,\mathrm{d}y$ *is mesurable;*
2. *The same property holds exchanging* x *and* y*;*
3. *The following identity holds:*
$$\int_{\mathbf{R}^{p+q}} f = \int_{\mathbf{R}^p} \left\{ \int_{\mathbf{R}^q} f(x, y)\,\mathrm{d}y \right\} \mathrm{d}x = \int_{\mathbf{R}^q} \left\{ \int_{\mathbf{R}^p} f(x, y)\,\mathrm{d}x \right\} \mathrm{d}y. \quad \text{(B.1)}$$

In particular, $f \in \mathcal{L}^1(\mathbf{R}^{p+q})$ if, and only if $y \mapsto \int_{\mathbf{R}^p} f(x, y)\,\mathrm{d}x$ is integrable on $\mathbf{R}^q$.

Theorem B.39 (Real or complex-valued functions : Fubini theorem)**.** *Let* f *be a* summable *function on* $\mathbf{R}^{p+q}$*. There exist two sets* $Z_X \in \mathbf{R}^p$ *and* $Z_Y \in \mathbf{R}^q$ *with* $\lambda_p(Z_X) = \lambda_q(Z_Y) = 0$ *such that:*

- *For all* $x \notin Z_X$*, the function* $y \mapsto f(x, y)$ *is summable on* $\mathbf{R}^q$ *and the function* $x \mapsto \int_{\mathbf{R}^q} f(x, y)\,\mathrm{d}y$ *is summable on* $\mathbf{R}^p$.
- *For all* $y \notin Z_Y$*, the function* $x \mapsto f(x, y)$ *is summable on* $\mathbf{R}^p$ *and the function* $y \mapsto \int_{\mathbf{R}^p} f(x, y)\,\mathrm{d}x$ *is summable on* $\mathbf{R}^q$.
- *The identity* (B.1) *holds.*

B.9 The Fourier transform

B.9.1 Convolution product in L^1

Theorem B.40. *Let* $f, g \in L^1(\mathbf{R}^N)$ *and* $r : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ *be defined by* $r(x, y) = f(x - y)g(y)$*. Then,*

- *For almost all* $x \in \mathbf{R}^N$*, the function* $y \mapsto r(x, y)$ *is summable on* $\mathbf{R}^N$.
- *Defining, for almost all* $x \in \mathbf{R}^N$*,*
$$(f \star g)(x) = \int_{\mathbf{R}^N} r(x, y)\ \mathrm{d}y = \int_{\mathbf{R}^N} f(x - y)g(y)\ \mathrm{d}y,$$
the function $f \star g$ *is summable on* $\mathbf{R}^N$ *and*
$$\|f \star g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1}.$$

- *For any $f, g, h \in L^1(\mathbf{R}^N)$,*

$$f \star g = g \star f \quad \text{and} \quad (f \star g) \star h = f \star (g \star h).$$

The function $f \star g$ is called convolution product *of f and g.*

Theorem B.41 (Convolution product and differentiation)**.** *Let $f \in L^1(\mathbf{R}^N)$ and let g be a bounded continuous function on $\mathbf{R}^N$. Then, the function $f \star g$ defined at any point of $\mathbf{R}^N$ by the formula*

$$(f \star g)(x) = \int_{\mathbf{R}^N} f(x-y)g(y)\ \mathrm{d}y = \int_{\mathbf{R}^N} f(y)g(x-y)\ \mathrm{d}y$$

is continuous and bounded.

If, moreover for some $p \geq 1$, the partial derivatives of g up to order p exist, are continuous and bounded, then the same is true for $f \star g$ and it holds

$$\frac{\partial (f \star g)}{\partial x_j} = f \star \frac{\partial g}{\partial x_j},\ \ldots\ , \frac{\partial^p (f \star g)}{\partial x_1^{p_1} \cdots \partial_{x_N}^{p_N}} = f \star \frac{\partial^p g}{\partial x_1^{p_1} \cdots \partial_{x_N}^{p_N}}.$$

Theorem B.42 (Convolution product in L^2)**.** *It holds*

1. *Let $f \in L^1(\mathbf{R}^N)$, $g \in L^2(\mathbf{R}^N)$. Then, $(f \star g)(x)$ is well-defined for almost all $x \in \mathbf{R}^N$. The function $f \star g$ belongs to $L^2(\mathbf{R}^N)$ and*

$$\|f \star g\|_{L^2} \leq \|f\|_{L^1} \|g\|_{L^2}.$$

2. *Let $f \in L^2(\mathbf{R}^N)$, $g \in L^2(\mathbf{R}^N)$. Then, $(f \star g)(x)$ is well-defined for all x. The function $f \star g$ is continuous, bounded, converges to 0 at infinity. Moreover,*

$$\|f \star g\|_{L^\infty} \leq \|f\|_{L^2} \|g\|_{L^2}.$$

Theorem B.43 (Approximation of the identity)**.** *Let $h \in L^1(\mathbf{R}^N)$ be such that $\int_{\mathbf{R}^N} h(x)\ \mathrm{d}x = 1$. We set*

$$h_n(x) = n^N h(nx) \quad \text{for } n \geq 1.$$

- *If $f \in L^1(\mathbf{R}^N)$ then $f \star h_n$ converges to f in $L^1(\mathbf{R}^N)$ as $n \to \infty$.*
- *If $f \in L^2(\mathbf{R}^N)$ then $f \star h_n$ converges to f in $L^2(\mathbf{R}^N)$ as $n \to \infty$.*

Remark B.44. For instance, consider on $\mathbf{R}^N$ the function $h(x) = \pi^{-\frac{N}{2}} \mathrm{e}^{-|x|^2}$ which is summable, of class $\mathcal{C}^\infty$ and of integral 1. Any function of L^2 is limit in L^2 of the sequence of functions $\{f \star h_n\}_{n=1}^{\infty}$ where each $f \star h_n$ is of class $\mathcal{C}^\infty$ by the differentiation theorem of convolution product.

B.9.2 Support and convolution

Proposition B.45. *Let $f, g : \mathbf{R}^N \to \mathbf{C}$ be two continuous functions, summable on $\mathbf{R}^N$. Then,*

$$\operatorname{supp}(f \star g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}.$$

In particular, if two continuous f, g have compact support, then $f \star g$ also has a compact support.

B.9.3 Convolution and density : continuous functions

Recall that we denote $\mathcal{C}_c(\mathbf{R}^N)$ the space of continuous functions $\mathbf{R}^N \to \mathbf{C}$ with compact support. More generally, for all $p \geq 1$, we denote $\mathcal{C}_c^p(\mathbf{R}^N)$ the space of functions of class $\mathcal{C}^p$ with compact support.

Lemma B.46. *For any $p \geq 1$, the space $\mathcal{C}_c^p(\mathbf{R}^N)$ is dense in $\mathcal{C}_c(\mathbf{R}^N)$ for the uniform norm. More precisely, for any $g \in \mathcal{C}_c(\mathbf{R}^N)$, there exists a sequence of function $g_n \in \mathcal{C}_c^p(\mathbf{R}^N)$ such that, for all $n \geq 1$,*

$$\|g - g_n\|_{L^\infty} \leq 1/n \quad \textit{and} \quad \operatorname{supp} g_n \subset \operatorname{supp} g + \bar{B}(0, 1/n).$$

We denote by $h : \mathbf{R}^N \to [0, +\infty[$ the function defined on $B(0,1)$ by

$$h(x) = \alpha_N \exp\left(-\frac{|x|^2}{1 - |x|^2}\right), \quad |x|^2 = \sum_{j=1}^N x_j^2,$$

and extended by 0 outside $B(0,1)$. We choose the constant $\alpha_N > 0$ so that $\|h\|_{L^1(\mathbf{R}^N)} = 1$. We check that the function h is of class $\mathcal{C}^\infty$. For all $n \geq 1$, we denote

$$h_n(x) = n^N h(nx).$$

The function h_n is of class $\mathcal{C}^\infty$ and $\operatorname{supp} h_n \subset \bar{B}(0, 1/n)$. Moreover, $\|h_n\|_{L^1} = \|h\|_{L^1} = 1$.

Proposition B.47. *For any $p \geq 1$, the space $\mathcal{C}_c^p(\mathbf{R}^N)$ is dense in the space $(L^2(\mathbf{R}^N); \|\cdot\|_{L^2})$.*

B.9.4 Fourier transform in L^1

Definition B.48. For any $f \in L^1(\mathbf{R}^N)$, the *Fourier transform* of f is the function denoted by $\widehat{f}$ or $\mathcal{F}(f)$ and defined for all $\xi \in \mathbf{R}^N$ by

$$\widehat{f}(\xi) := \int_{\mathbf{R}^N} \mathrm{e}^{-\mathrm{i}\,\xi \cdot x} f(x)\, \mathrm{d}x.$$

Theorem B.49. *Let $f \in L^1(\mathbf{R}^N)$ and let $\widehat{f}$ be the Fourier transform of f. Then, the function $\widehat{f}$ is continuous and satisfies $\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}$. Moreover, $\lim_{|\xi| \to \infty} \widehat{f}(\xi) = 0$.*

B.9.5 Properties of the Fourier transform

Proposition B.50.

- *If f is real-valued then $\widehat{f}$ has the hermitian symmetry: $\widehat{f}(-\xi) = \overline{\widehat{f}(\xi)}$.*

  If f has the hermitian symmetry then $\widehat{f}$ is real-valued. More generally,

$$\overline{\mathcal{F}(f(x))} = \mathcal{F}\left[\overline{f(-\xi)}\right].$$

- *If f is even (resp. odd) then $\widehat{f}$ is even (resp. odd).*

  If f is real-valued and even, then $\widehat{f}$ is real-valued and even.

- *Translation:*

$$\mathcal{F}\left[f(x-x_0)\right] = \mathrm{e}^{-\mathrm{i}x_0\cdot\xi}\widehat{f}(\xi), \quad \mathcal{F}\left[\mathrm{e}^{\mathrm{i}x\cdot\xi_0}f(x)\right] = \widehat{f}(\xi-\xi_0).$$

- *Dilation: for $\lambda > 0$,*

$$\mathcal{F}\left[f(x/\lambda)\right] = \lambda^N\,\widehat{f}(\lambda\xi).$$

Theorem B.51. *If $(1+|x|)\,f \in L^1(\mathbf{R}^N)$ then $\widehat{f} \in \mathcal{C}^1(\mathbf{R}^N)$ and for all $j = 1,\ldots,N$,*

$$\frac{\partial\widehat{f}}{\partial\xi_j}(\xi) = -\mathrm{i}\int \mathrm{e}^{-\mathrm{i}\xi\cdot x}\,x_j\,f(x)\;\mathrm{d}x = -\mathrm{i}\,\mathcal{F}(x_jf)(\xi).$$

More generally, for $p \geq 1$, if $(1+|x|^p)f \in L^1(\mathbf{R}^N)$ then $\widehat{f} \in \mathcal{C}^p(\mathbf{R}^N)$.

Theorem B.52. *If $f \in \mathcal{C}^1(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$ and for $j = 1,\ldots,N$, $\dfrac{\partial f}{\partial x_j} \in L^1(\mathbf{R}^N)$, then*

$$\mathcal{F}\left(\frac{\partial f}{\partial x_j}\right)(\xi) = \int_{\mathbf{R}^N} \mathrm{e}^{-\mathrm{i}\xi\cdot x}\,\frac{\partial f}{\partial x_j}(x)\;\mathrm{d}x = \mathrm{i}\,\xi_j\,\widehat{f}(\xi).$$

More generally, for $p \geq 1$, if $f \in \mathcal{C}^p(\mathbf{R}^N)\cap L^1(\mathbf{R}^N)$ and all the partial derivatives of f up to order p belong to $L^1(\mathbf{R}^N)$, then

$$\lim_{|\xi|\to\infty} |\xi|^p\,|\widehat{f}(\xi)| = 0.$$

Theorem B.53 (Convolution and Fourier transform in L^1)**.** *Let $f, g \in L^1(\mathbf{R}^N)$. Then,*

$$\mathcal{F}(f\star g) = \widehat{f}\,\widehat{g}$$

Theorem B.54 (Fourier inversion in L^1)**.** *Let $f \in L^1(\mathbf{R}^N)$ be such that $\widehat{f} \in L^1(\mathbf{R}^N)$. Then, for almost all $x \in \mathbf{R}^N$, it holds*

$$f(x) = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \mathrm{e}^{\mathrm{i}\,x\cdot\xi}\,\widehat{f}(\xi)\,\mathrm{d}\xi.$$

B.9.6 Fourier transform in L^2

Define the *normalized Fourier transform* of f by

$$\mathcal{G}(f)(\xi) = \frac{1}{(2\pi)^{N/2}}\int_{\mathbf{R}^N} \mathrm{e}^{-\mathrm{i}\,\xi\cdot x}\,f(x)\,\mathrm{d}x,$$

and the *inverse normalized Fourier transform* of $\widehat{f}$ by

$$\overline{\mathcal{G}}(\widehat{f})(x) = \frac{1}{(2\pi)^{N/2}}\int_{\mathbf{R}^N} \mathrm{e}^{\mathrm{i}x\cdot\xi}\,\widehat{f}(\xi)\;\mathrm{d}\xi.$$

Theorem B.55. *The maps $\mathcal{G}$ and $\overline{\mathcal{G}}$ extend to bijective isometries from $L^2(\mathbf{R}^N)$ to itself and their extensions, still denoted $\mathcal{G}$ and $\overline{\mathcal{G}}$, satisfy*

$$\mathcal{G}\circ\overline{\mathcal{G}} = \overline{\mathcal{G}}\circ\mathcal{G} = \mathrm{Id}_{L^2}.$$

B.10 Exercises

Exercise B.1. Let $\varphi \in L^\infty(\mathbf{R}^n)$. For $g \in L^2(\mathbf{R}^n)$, we define $Ag = \varphi g$. Prove that A is a continuous operator on $L^2(\mathbf{R}^n)$, with norm $\|A\| = \|\varphi\|_{L^\infty}$.

Exercise B.2. Prove Proposition B.27.

Exercise B.3. Compute the following classical Fourier transforms.

1. Indicator function of an interval $[a, b]$:

$$\mathcal{F}\left(\mathbf{1}_{[a,b]}\right) = \frac{2\sin\left(\frac{b-a}{2}\xi\right)}{\xi} \exp\left(-\mathrm{i}\frac{a+b}{2}\right).$$

2. Gaussian functions: on $\mathbf{R}^N$, for $\alpha > 0$,

$$\mathcal{F}\left(\mathrm{e}^{-\alpha|x|^2}\right) = \left(\frac{\pi}{\alpha}\right)^{\frac{N}{2}} \exp\left(-\frac{|\xi|^2}{4\alpha}\right).$$

3. Rational fractions: in $\mathbf{R}$, for $\alpha > 0$,

$$\mathcal{F}\left(\frac{2\alpha}{\alpha^2 + x^2}\right) = 2\pi\,\mathrm{e}^{-\alpha|\xi|}.$$

Appendix C

Sobolev spaces

C.1 Distribution theory

Let $\mathcal{U}$ be a nonempty open set of $\mathbf{R}^N$. We denote by $\mathcal{D}(\mathcal{U})$ the space of infinitely differentiable functions on $\mathcal{U}$ with compact support on $\mathcal{U}$. If $\varphi \in \mathcal{D}(\mathcal{U})$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{N}^N$ is a multi-index, we set

$$\partial^\alpha \varphi = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_N}\right)^{\alpha_N} \varphi = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}},$$

with

$$|\alpha| = \alpha_1 + \ldots + \alpha_N.$$

We equip the space $\mathcal{D}(\mathcal{U})$ with a *pseudo-topology*, defining the notion of convergence for sequences in $\mathcal{D}(\mathcal{U})$.

Definition C.1. We say that a sequence $\{\varphi_n\}_{n=0}^\infty$ of $\mathcal{D}(\mathcal{U})$ converges to $\varphi \in \mathcal{D}(\mathcal{U})$ if

1. The support of φ_n is included in a fixed compact set K of $\mathcal{U}$ for any $n \in \mathbf{N}$;
2. For any $\alpha \in \mathbf{N}^N$, the sequence $\{\partial^\alpha \varphi_n\}_{n=0}^\infty$ converges to $\partial^\alpha \varphi$ uniformly in $\mathbf{K}$.

Next, we introduce the space $\mathcal{D}'(\mathcal{U})$ of *distributions on* $\mathcal{U}$ as the space of linear forms on $\mathcal{D}(\mathcal{U})$ with a suitable continuity property: for T a distribution on $\mathcal{D}(\mathcal{U})$ and $\langle T, \varphi \rangle$ denoting the duality product $\mathcal{D}'(\mathcal{U})$, $\mathcal{D}(\mathcal{U})$, for any sequence $\{\varphi_n\}_{n=0}^\infty$ of $\mathcal{D}(\mathcal{U})$ converging to $\varphi \in \mathcal{D}(\mathcal{U})$, the sequence $\{\langle T, \varphi_n \rangle\}_{n=0}^\infty$ converges to $\langle T, \varphi \rangle$.

The space $\mathcal{D}'(\mathcal{U})$ itself can be given a pseudo-topology.

Theorem C.2. *If $\{T_n\}_{n=0}^\infty$ is a sequence in $\mathcal{D}'(\mathcal{U})$ such that for any $\varphi \in \mathcal{D}(\mathcal{U})$ the real sequence $\{\langle T_n, \varphi \rangle\}_{n=0}^\infty$ converges in $\mathbf{R}$ to a limit denoted by $\langle T, \varphi \rangle$, then T is a distribution and we say that the sequence $\{T_n\}_{n=0}^\infty$ converges to T in* the sense of distributions.

Example C.3. The following are standard examples of distributions.

1. Locally summable functions. To any $f \in L^1_{\text{loc}}(\mathcal{U})$, we associate a distribution, still denoted by f, defined by

$$\langle f, \varphi \rangle = \int_{\mathcal{U}} f(x)\varphi(x)\,\mathrm{d}x.$$

   The identification between f and the corresponding distribution is justified by the fact that if two locally summable functions define the same distribution, then they are equal almost everywhere by Remark B.21.

2. Point masses. The Dirac distribution at the point $a \in \mathbf{R}^N$ is defined by

$$\langle \delta_a, \varphi \rangle = \varphi(a).$$

   More generally, for any $\alpha \in \mathbf{N}^N$,

$$\langle T, \varphi \rangle = \partial^\alpha \varphi(a)$$

   defines a distribution.

Now, we introduce the differentiation in sense of distributions. If T is a distribution on $\mathcal{U}$, we define another distribution $\frac{\partial T}{\partial x_j}$ on $\mathcal{U}$ by the formula

$$\text{for any } \varphi \in \mathcal{D}(\mathcal{U}), \quad \left\langle \frac{\partial T}{\partial x_j}, \varphi \right\rangle = -\left\langle T, \frac{\partial \varphi}{\partial x_j} \right\rangle.$$

We denote by $\mathcal{D}(\overline{\mathcal{U}})$ the space of of infinitely differentiable functions on $\overline{\mathcal{U}}$ with compact support on $\overline{\mathcal{U}}$. In particular, a function φ belongs to $\mathcal{D}(\overline{\mathcal{U}})$ if there exist an open set $\mathcal{O}$ of $\mathbf{R}^N$ containing $\overline{\mathcal{U}}$ and a function $\tilde{\varphi} \in \mathcal{D}(\mathcal{O})$ such that $\varphi = \tilde{\varphi}$ on $\overline{\mathcal{U}}$.

C.2 The Sobolev space $H^1(\mathcal{U})$

A given function $g \in L^2(\mathcal{U})$ is locally summable and so can be viewed as a distribution on $\mathcal{U}$. In particular, one can define its first order partial derivatives $\frac{\partial g}{\partial x_j}$ for any $j = 1, \ldots, N$ as distributions on $\mathcal{U}$. In general, $\frac{\partial g}{\partial x_j}$ does not belong to $L^2(\mathcal{U})$, which motivates the introduction of the following function space.

Definition C.4. We call the *Sobolev space of order one* on $\mathcal{U} \subset \mathbf{R}^N$ the space

$$H^1(\mathcal{U}) = \left\{ g \in L^2(\mathcal{U}) : \frac{\partial g}{\partial x_j} \in L^2(\mathcal{U}) \text{ for any } j = 1, \ldots, N \right\}.$$

We equip $H^1(\mathcal{U})$ with the real scalar product

$$(g, h)_{H^1} = \int_{\mathcal{U}} \left(gh + \sum_{j=1}^{N} \frac{\partial g}{\partial x_j} \frac{\partial h}{\partial x_j} \right) \mathrm{d}x,$$

and we denote by $\|g\|_{H^1}$ the associated norm.

Theorem C.5. *The space $(H^1(\mathcal{U}), (\cdot,\cdot)_{H^1})$ is a separable Hilbert space.*

Recall that the space $\mathcal{D}(\mathcal{U})$ is dense in $(L^2(\mathcal{U}), \|\cdot\|_{L^2})$. In general, the space $\mathcal{D}(\mathcal{U})$ is *not* dense in $(H^1(\mathcal{U}), \|\cdot\|_{H^1})$. This justifies the following definition.

Definition C.6. We denote by $H^1_0(\mathcal{U})$ the closure of $\mathcal{D}(\mathcal{U})$ in $(H^1(\mathcal{U}), \|\cdot\|_{H^1})$.

Theorem C.7. *The space $\mathcal{D}(\mathbf{R}^N)$ is dense in $H^1(\mathbf{R}^N)$. In particular,*

$$H^1_0(\mathbf{R}^N) = H^1(\mathbf{R}^N).$$

Theorem C.8. *Let $g \in H^1_0(\mathcal{U})$. We define*

$$\tilde{g} = \begin{cases} g & \text{on } \mathcal{U}, \\ 0 & \text{on } \mathbf{R}^N \setminus \mathcal{U}. \end{cases}$$

Then, $\tilde{g}$ belongs to $H^1(\mathbf{R}^N)$.

Theorem C.9 (Poincaré inequality)**.** *If the open subset $\mathcal{U}$ of $\mathbf{R}^N$ is bounded then there exists a constant $C = C(\mathcal{U}) > 0$ such that*

$$\text{for any } g \in H^1_0(\mathcal{U}), \quad \|g\|_{L^2(\mathcal{U})} \leq C \sqrt{\sum_{j=1}^{N} \left\| \frac{\partial g}{\partial x_j} \right\|^2_{L^2(\mathcal{U})}}$$

For specific properties of H^1 function in one dimension, see Exercises C.2, C.3 and C.4.

C.3 Integration on surfaces

C.3.1 Case of a local graph

We denote by $x = (x', x_N) \in \mathbf{R}^N$ a point of $\mathbf{R}^N$, where $x' \in \mathbf{R}^{N-1}$ and $x_N \in \mathbf{R}$. We consider an open set ω of $\mathbf{R}^{N-1}$, (a, b) an open interval of $\mathbf{R}$ and a function $\Phi : \omega \to (a, b)$ of class $\mathcal{C}^1$. We set

$$\begin{aligned} \mathcal{U} &= \{x \in \omega \times (a,b) : x_N > \Phi(x')\} \\ \widetilde{\mathcal{U}} &= \{x \in \omega \times (a,b) : x_N \geq \Phi(x')\} \\ \partial\mathcal{U} &= \{x \in \omega \times (a,b) : x_N = \Phi(x')\}. \end{aligned}$$

The *outer unit normal vector* at a point $(x', \Phi(x'))$ of the surface $\partial\mathcal{U}$ is defined by

$$\nu = \frac{1}{1 + |\nabla'\Phi(x')|^2} \begin{pmatrix} \frac{\partial \Phi}{\partial x_1} \\ \vdots \\ \frac{\partial \Phi}{\partial x_{N-1}} \\ -1 \end{pmatrix} \tag{C.1}$$

where ∇' denotes the $N-1$-dimension gradient of Φ. Next, for any function f defined on $\partial\mathcal{U}$, continuous and with compact support on $\partial\mathcal{U}$, we set

$$\int_{\partial\mathcal{U}} f(x)\,\mathrm{d}\sigma(x) = \int_{\omega} f(x', \Phi(x')\sqrt{1 + |\nabla'\Phi(x')|^2}\,\mathrm{d}x', \tag{C.2}$$

Note that the above defines a positive linear form on the space of continuous, compactly supported functions on $\partial\mathcal{U}$. Actually, the above integral can be defined in the more general setting of measurable functions defined on $\partial\mathcal{U}$.

Recall that X_j of class $\mathcal{C}^1$ on $\widetilde{\mathcal{U}}$ means that X_j is continuous on $\widetilde{\mathcal{U}}$, of class $\mathcal{C}^1$ on $\mathcal{U}$, and all its first partial derivatives extend continuously to $\tilde{\mathcal{U}}$. Recall also the notation

$$\operatorname{div} X = \sum_{j=1}^{N} \frac{\partial X_j}{\partial x_j} \quad \text{and} \quad X \cdot \nu = \sum_{j=1}^{N} X_j \nu_j.$$

We recall the Stokes formula in the present context.

Theorem C.10. *Let $X = (X_1, \ldots, X_N)$ be a vector field defined on $\widetilde{\mathcal{U}}$, such that the support is a compact subset of $\widetilde{\mathcal{U}}$. Suppose that the components X_j of X are of class $\mathcal{C}^1$ on $\widetilde{\mathcal{U}}$. Then*

$$\int_{\partial\mathcal{U}} X \cdot \nu \, \mathrm{d}\sigma = \int_{\mathcal{U}} \operatorname{div} X \, \mathrm{d}x.$$

C.3.2 Definition of an open set with $\mathcal{C}^1$ boundary

We continue with the definition of a $\mathcal{C}^1$ boundary. Let $\mathcal{U}$ be any open set of $\mathbf{R}^N$. Recall that the boundary $\partial\mathcal{U}$ of $\mathcal{U}$ is defined by $\partial\mathcal{U} = \overline{\mathcal{U}} \setminus \mathcal{U}$.

Definition C.11. We say that the boundary $\mathcal{U}$ is $\mathcal{C}^1$ if for any point $m \in \mathcal{U}$, there exist

1. An orthonormal system (y) of coordinates $(y_1, \ldots, y_N)$;
2. An open set ω of $\mathbf{R}^{N-1}$ containing m' and an interval (a, b) containing m_N, where (m', m_N) are the components of m in the system (y);
3. An application $\Phi : \omega \to (a, b)$ of class $\mathcal{C}^1$ such that in the system (y)

$$\mathcal{U} \cap C = \{y \in C : y_N > \Phi(y')\},$$

   where $C = \omega \times (a, b)$.

In this framework, at a given point $m \in \partial\mathcal{U}$, we associate the outer unit normal vector defined by (C.1). Its definition is proved to be independent of the choice of coordinate system (y).

If $\mathcal{U}$ is bounded with $\mathcal{C}^1$ boundary, its boundary $\partial\mathcal{U}$ can be covered by a *finite number* of cylinders $C_k = \omega_k \times (a_k, b_k)$ $(k = 1, \ldots, K)$ such as in Definition C.11, corresponding to choices of coordinates $(y)_k$ and applications Φ_k. By the compactness of $\partial\mathcal{U}$ and partition of unity, one can find K functions $(\chi_k)_{k=1,\ldots,K}$ of class $\mathcal{C}^\infty$ such that $\sum_{k=1}^{N} \chi_k = 1$ in a neighbourhood of $\partial\mathcal{U}$ and each χ_k has compact support in C_k.

Now, let $f : \partial\mathcal{U} \to \mathbf{C}$ be a continuous function. Then, for any $k = 1, \ldots, K$, the integral $\int_{\partial\mathcal{U}} f(x)\chi_k(x) \, \mathrm{d}\sigma(x)$ being defined as in (C.2), the sum

$$\sum_{k=1}^{N} \int_{\partial\mathcal{U}} f(x)\chi_k(x) \, \mathrm{d}\sigma(x)$$

is independent of the choice of C_k, the coordinates $(y)_k$, the functions Φ_k and χ_k, for any $k = 1, \dots, K$. This sum is denoted by $\int_{\partial\mathcal{U}} f(x)\,\mathrm{d}\sigma(x)$. The Stokes formula in Theorem C.10 then extends to this case. In particular, we deduce the following calculus formulas.

Theorem C.12. *Let $\mathcal{U}$ be a bounded open set of $\mathbf{R}^N$ with $\mathcal{C}^1$ boundary.*

1. *Integration by parts. For any $f, g \in \mathcal{C}^1(\overline{\mathcal{U}})$, $j = 1, \dots, N$,*

$$\int_{\mathcal{U}} g(x) \frac{\partial f}{\partial x_j}(x)\,\mathrm{d}x = \int_{\partial\mathcal{U}} f(x)g(x)\nu_j\,\mathrm{d}\sigma(x) - \int_{\mathcal{U}} f(x)\frac{\partial g}{\partial x_j}(x)\,\mathrm{d}x,$$

   where $\nu_j = \cos(\nu, e_j)$, for $j = 1, \dots, N$, are the components of the outer unit normal vector.

2. *Green formula. For any $f, g \in \mathcal{C}^2(\overline{\mathcal{U}})$,*

$$\int_{\mathcal{U}} (f\Delta g - g\Delta f)\,\mathrm{d}x = \int_{\partial\mathcal{U}} \left(f\frac{\partial g}{\partial \nu} - g\frac{\partial f}{\partial \nu}\right)\mathrm{d}\sigma,$$

   where $\frac{\partial g}{\partial \nu} = \nabla g \cdot \nu$.

Remark C.13. Now that we have defined an integral on sufficiently regular surfaces, we allow ourselves to define the set of summable or square summable functions on $\partial\mathcal{U}$ denoted by $L^1(\partial\mathcal{U})$ and $L^2(\partial\mathcal{U})$. A rigorous definition would require the notion of measurable functions on $\partial\mathcal{U}$.

C.4 Trace theorem

The properties of H^1 functions may depend strongly on the dimension of the space. For instance, in dimension one, H^1 functions are continuous (see Exercise C.3). It is not true in dimension $N \geq 2$, for example, the function

$$v(x) = |\log|x||^{\frac{1}{4}}$$

belongs to $H^1(B_1)$, where B_1 is the unit ball of $\mathbf{R}^2$. In concrete situations, one may wish to impose some condition on functions on the boundary of an open domain $\mathcal{U}$ of $\mathbf{R}^N$. In dimension one, the continuity gives sense to the value of an H^1 function at a given point. In higher dimension, it is necessary to introduce the notion of *trace*.

Theorem C.14. *Let $N \geq 2$. Let $\mathcal{U}$ be an open set of $\mathbf{R}^N$ with $\mathcal{C}^1$ boundary. Then*

1. *The space $\mathcal{C}^\infty(\overline{\mathcal{U}})$ is dense in $H^1(\mathcal{U})$:*

2. *The map*

$$\begin{aligned} \gamma_{\mathcal{U}}\ &: \mathcal{C}^\infty(\overline{\mathcal{U}}) \to \mathcal{C}^1(\partial\mathcal{U}) \\ &v \mapsto v|_{\partial\mathcal{U}} \end{aligned}$$

   satisfies

$$\|\gamma_{\mathcal{U}} v\|_{L^2(\partial\mathcal{U})} \leq C\|v\|_{H^1(\mathcal{U})}.$$

   for some constant $C > 0$.

3. *The map $\gamma_{\mathcal{U}}$ extends by continuity as a continuous linear map from $H^1(\mathcal{U})$ to $L^2(\partial\mathcal{U})$.*

This result means that whereas an H^1 function in dimension higher than or equal to 2 is not necessarily continuous, its value on the boundary $\mathcal{U}$ makes sense as an L^2 function on the boundary. In particular, this allows to characterize functions in $H_0^1(\mathcal{U})$.

Theorem C.15. *Let $N \geq 2$ Let $\mathcal{U}$ be an open set of $\mathbf{R}^N$ with $\mathcal{C}^1$ boundary. Then,*

$$H_0^1(\mathcal{U}) = \{v \in H^1(\mathcal{U}) : \gamma_{\mathcal{U}}(v) = 0\}.$$

Theorem C.16 (The Green formula)**.** *Let $N \geq 2$. Let $\mathcal{U}$ be a bounded open subset of $\mathbf{R}^N$ with $\mathcal{C}^1$ boundary. For any $g, h \in H^1(\mathcal{U})$, it holds, for any $j = 1, \ldots, N$,*

$$\int_{\mathcal{U}} \frac{\partial g}{\partial x_j} h \,\mathrm{d}x = -\int_{\mathcal{U}} g \frac{\partial h}{\partial x_j} \,\mathrm{d}x + \int_{\partial\mathcal{U}} gh\nu_j \,\mathrm{d}\sigma.$$

C.5 Higher order Sobolev spaces

Definition C.17. For any integer $s \geq 1$, we denote

$$H^s(\mathcal{U}) = \left\{ g \in L^2 : \frac{\partial g}{\partial x_j},\ j = 1, \ldots, N \right\}.$$

We equip the space $H^s(\mathcal{U})$ with the real scalar product

$$(g, h)_{H^s} = \int_{\mathcal{U}} \left(\sum_{|\alpha| \leq m}^{N} \partial^\alpha g \, \partial^\alpha h \,\mathrm{d}x \right),$$

and we denote by $\|g\|_{H^s}$ the associated norm.

Theorem C.18. *For any integer $s \geq 1$, $(H^s(\mathcal{U}), (\cdot,\cdot)_{H^s})$ is a separable Hilbert space.*

Theorem C.19. *Let $N \geq 2$. Let $\mathcal{U}$ be an open set of $\mathbf{R}^N$ with $\mathcal{C}^1$ boundary. Then*

1. *The space $\mathcal{C}^\infty(\overline{\mathcal{U}})$ is dense in $H^2(\mathcal{U})$:*

2. *The map*

$$\begin{aligned} \gamma'_{\mathcal{U}} &: \mathcal{C}^\infty(\overline{\mathcal{U}}) \to \mathcal{C}^1(\partial\mathcal{U}) \\ & v \mapsto \frac{\partial v}{\partial \nu}|_{\partial\mathcal{U}} \end{aligned}$$

satisfies

$$\|\gamma'_{\mathcal{U}} v\|_{L^2(\partial\mathcal{U})} \leq C\|v\|_{H^2(\mathcal{U})}.$$

for some constant $C > 0$.

3. *The map $\gamma'_{\mathcal{U}}$ extends by continuity as a continuous linear map from $H^2(\mathcal{U})$ to $L^2(\partial\mathcal{U})$.*

Theorem C.20. *Let $N \geq 2$. Let $\mathcal{U}$ be an open set of $\mathbf{R}^N$ with $\mathcal{C}^1$ boundary. For any $g \in H^2(\mathcal{U})$ and $h \in H^1(\mathcal{U})$, it holds, for any $j = 1, \dots, N$,*

$$-\int_{\mathcal{U}} (\Delta g) h \, \mathrm{d}x = \int_{\mathcal{U}} (\nabla g \cdot \nabla h) \, \mathrm{d}x + \int_{\partial\mathcal{U}} \frac{\partial g}{\partial \nu} h \, \mathrm{d}\sigma.$$

In particular, if $h \in H_0^1(\mathcal{U})$, it holds

$$-\int_{\mathcal{U}} (\Delta g) h \, \mathrm{d}x = \int_{\mathcal{U}} (\nabla g \cdot \nabla h) \, \mathrm{d}x.$$

Theorem C.21. *Let $N \geq 2$. Let $\mathcal{U}$ be an open set of $\mathbf{R}^N$ with $\mathcal{C}^1$ boundary. If $s > \frac{N}{2}$, then $H^s(\mathcal{U}) \subset \mathcal{C}^0(\overline{\mathcal{U}})$.*

C.6 Sobolev spaces via the Fourier transform

Let $\mathcal{U} = \mathbf{R}^N$. For $f \in L^2(\mathbf{R}^n)$ and $s \in [0, +\infty[$, we can equivalently define

$$\|f\|_{H^s} = \left(\int_{\mathbf{R}^N} \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} \qquad \text{where} \quad \langle \xi \rangle = \left(1 + \textstyle\sum_{j=1}^N \xi_j^2\right)^{1/2}$$

and

$$H^s(\mathbf{R}^N) = \{ f \in L^2(\mathbf{R}^N) : \|f\|_{H^s} < \infty \}.$$

C.7 Exercises

Exercise C.1. We denote by $\mathcal{S}(\mathbf{R}^n)$ the spaces of functions ϕ of class C^∞ on $\mathbf{R}^n$ satisfying

$$\forall \alpha, \beta \in \mathbf{N}^n, \qquad \sup_{x \in \mathbf{R}^n} |x^\alpha \partial^\beta \phi(x)| < \infty,$$

where the following notation is used $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $\partial^\beta \phi = \partial_{x_1}^{\beta_1} \dots \partial_{x_n}^{\beta_n} \phi$.

For $f \in L^2(\mathbf{R}^n)$ and $s \in [0, +\infty[$, we define

$$\|f\|_{H^s} = \left(\int_{\mathbf{R}^n} \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} \qquad \text{where} \quad \langle \xi \rangle = \left(1 + \textstyle\sum_{j=1}^n \xi_j^2\right)^{1/2}$$

and $H^s(\mathbf{R}^n) = \{ f \in L^2(\mathbf{R}^n) \mid \|f\|_{H^s} < \infty \}$.

1. Prove that $H^s(\mathbf{R}^n)$ is a hilbert space equipped with the scalar product associated to the norm $\|\cdot\|_{H^s}$.

2. Prove that for $f \in \mathcal{S}(\mathbf{R}^n)$, the Fourier transform of f is well-defined and also belongs to $\mathcal{S}(\mathbf{R}^n)$.

3. Prove that if $f \in \mathcal{S}(\mathbf{R}^n)$ then $f \in H^s(\mathbf{R}^n)$ for all $s \geq 0$.

4. We consider the trace map τ defined (for now) from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}(\mathbf{R}^{n-1})$ by

$$\tau u(x') = u(0, x'), \qquad x' = (x_2, \dots, x_n).$$

Prove that for any $u \in \mathcal{S}(\mathbf{R}^n)$ and all $\xi' \in \mathbf{R}^{n-1}$,

$$\widehat{\tau u}(\xi') = \frac{1}{2\pi} \int_{\mathbf{R}} \widehat{u}(\xi_1, \xi') d\xi_1.$$

5. Prove that for any $s > 1/2$, there exists $C = C(s) > 0$ such that for any $u \in \mathcal{S}(\mathbf{R}^n)$,
$$\|\tau u\|_{H^{s-1/2}(\mathbf{R}^{n-1})} \leq C\|u\|_{H^s(\mathbf{R}^d)}.$$
Hint: use the estimate :
$$|\widehat{\tau u}(\xi')|^2 \leq \frac{1}{4\pi^2}\left(\int_{\mathbf{R}} |\widehat{u}(\xi)|^2\langle\xi\rangle^{2s}d\xi_1\right)\left(\int_{\mathbf{R}} \langle\xi\rangle^{-2s}d\xi_1\right)$$
and express $\int_{\mathbf{R}}\langle\xi\rangle^{-2s}d\xi_1$ in terms of $\langle\xi'\rangle$ (we denote $\xi = (\xi_1, \xi')$).

6. Let $s > 1/2$. Using that $\mathcal{S}(\mathbf{R}^n)$ is dense in $H^s(\mathbf{R}^n)$ for the norm $\|\cdot\|_{H^s}$, prove that τ can be extended uniquely to a continuous linear map from $H^s(\mathbf{R}^n)$ to $H^{s-1/2}(\mathbf{R}^{n-1})$.

7. Let $s > 1/2$ and $g \in H^{s-1/2}(\mathbf{R}^{n-1})$. Define v by
$$\widehat{v}(\xi) = \widehat{g}(\xi')\frac{\langle\xi'\rangle^{2(s-1/2)}}{\langle\xi\rangle^{2s}}.$$
Prove that $v \in H^s(\mathbf{R}^n)$ and $v(0, x') = Cg(x')$ for a constant $C \neq 0$. Deduce that the extension of the trace map defined above is surjective.

Exercise C.2. Let $I \subset \mathbf{R}$ be an open interval. Define
$$H^1(I) = \left\{g \in L^2(I) : g' \in L^2(I)\right\},$$
where g' denotes the derivative of g in the sense of distributions. Alternatively,
$$H^1(I) = \left\{g \in L^2(I) : \exists v \in L^2(I) \text{ with } \int_I g\phi' = -\int_I v\phi,\ \forall\phi \in \mathcal{C}_c^1(I)\right\}.$$

1. Prove that $H^1(I)$ equipped with
$$(g \mid h)_{H^1} = (g \mid h)_{L^2} + (g' \mid h')_{L^2}$$
is a (separable) Hilbert space.

2. Let $g \in L^2(I)$. Prove that $g \in H^1(I)$ if and only if there exists $C > 0$ such that
$$\left|\int_I g\phi'\right| \leq C\|\phi\|_{L^2}, \quad \forall\phi \in \mathcal{C}_c^\infty(I).$$

Exercise C.3. 1. Let $f \in L^1_{\text{loc}}(I)$ be such that
$$\int_a^b f\phi'\,\mathrm{d}x = 0,\ \forall\phi \in \mathcal{C}_c^\infty(I).$$
Prove that there exists $C \in \mathbf{R}$ such that $f = C$ almost everywhere on I.

Hint: Let $\chi \in \mathcal{C}_c^\infty(I)$ be such that $\int_a^b \chi(x)\,\mathrm{d}x = 1$. For $\phi \in \mathcal{C}_c^\infty(I)$, we define $\tilde{\phi}$ and Φ by
$$\tilde{\phi}(x) = \phi(x) - \chi(x)\int_a^b \phi(t)dt, \quad \Phi(x) = \int_a^x \tilde{\phi}(t)dt.$$
Prove that $\Phi \in \mathcal{C}_c^\infty(I)$.

2. Let $g \in H^1(I)$ and $g' \in L^2(I)$ be its weak derivative. Let $x_0 \in I$. Set
$$G(x) = \int_{x_0}^{x} g'(t)dt.$$
Prove that $G' = g'$.

3. If I is bounded, deduce that $H^1(I) \subset C(\bar{I})$ with compact embedding *i.e.* any bounded subset of $H^1(I)$ is relatively compact in $C(\bar{I})$.

4. Assume that $b = +\infty$. Prove that $\lim_{x\to\infty} g(x) = 0$.

   (Similarly, if $a = -\infty$, then $\lim_{x\to-\infty} g(x) = 0$.)

Exercise C.4. Let $g \in L^2(\mathbf{R})$. Prove that $g \in H^1(\mathbf{R})$ if and only if there exists a constant $C > 0$ such that for all $\delta \in \mathbf{R}$, it holds
$$\|\tau_\delta g - g\|_{L^2} \leq C|\delta|,$$
where
$$\tau_\delta g(x) = g(x - \delta) \quad \text{for all } x, \delta \in \mathbf{R}.$$
Moreover, in this case, one can choose $C = \|g'\|_{L^2}$.

Appendix D

Ordinary differential equations

D.1 Inequality

Here, $\mathbf{R}$ is equipped with the norm associated to the absolute value $|\cdot|$ and $\mathbf{R}^N$ is equipped with any norm *also denoted by* $|\cdot|$.

Let $g : [a,b] \to \mathbf{R}^N$ be a function of class $\mathcal{C}^1$. Let $g'(t) = (g_1'(t), \ldots, g_N'(t))$. Then,

$$|g(b) - g(a)| \leq \int_a^b |g'(s)| \,\mathrm{d}s \leq (b-a) \sup_{[a,b]} |g'|.$$

In the more general case where V is an open set of $\mathbf{R}^M$ and $G : V \to \mathbf{R}^N$ is a function of class $\mathcal{C}^1$, we denote by $\mathbf{J}_G$ the Jacobian matrix of G and we define

$$\|\mathbf{J}_G(X)\| = \sup_{|h|=1} |\mathbf{J}_G(X) \cdot h|.$$

Then, for any $X, Y \in V$ such that the segment joining X to Y is included in V, we have

$$|G(Y) - G(X)| \leq \sup_{0 \leq \theta \leq 1} \|\mathbf{J}_G(X + \theta(Y - X))\| \cdot |Y - X|.$$

D.2 The Cauchy-Lipschitz Theorem

Let δ be a positive real number and $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^N$. We set

$$A = \big\{(t,x) \in \mathbf{R} \times \mathbf{R}^N : |t - t_0| \leq \delta,\ |x - x_0| \leq \delta\big\}.$$

We suppose that $f : A \to \mathbf{R}^N$ is continuous and satisfies

- There exists $M > 0$ such that, for all $(t,x) \in A$,

$$|f(t,x)| \leq M;$$

- There exists $C > 0$ such that, for all $(t,x) \in A$, $(t,y) \in A$,

$$|f(t,x) - f(t,y)| \leq C|x - y|.$$

We say that f is *continuous on* (t,x) and *Lipschitz continuous with respect to its second variable* on A.

Theorem D.1. *Under the above assumptions, there exists one and only one solution* (J,x) *of the equation* $\dot{x} = f(t,x)$ *such that*

- *Time of existence:* $J = [t_0 - T, t_0 + T]$ *with*
$$T = \min\Big(\delta, \frac{\delta}{2M}, \frac{1}{2C}\Big);$$
- *Initial condition:* $x(t_0) = x_0$*;*
- *For all* $t \in J$, $(t, x(t)) \in A$.

Corollary D.2. *Suppose that the function* $f : I \times \mathcal{U} \to \mathbf{R}^N$ *is of class* $\mathcal{C}^1$*. Let* $(t_0, x_0) \in I \times \mathcal{U}$*. Then, the following hold.*

- *Local existence: there exists* $T > 0$ *and a solution* (J,x) *of* $\dot{x} = f(t,x)$ *such that* $J = [t_0 - T, t_0 + T]$ *and* $x(t_0) = x_0$.
- *Uniqueness: let* $\tilde{J} \subset J$ *be an interval containing* t_0*, let* $(\tilde{J}, \tilde{x})$ *a solution of* $\dot{x} = f(t,x)$ *with initial data* (t_0, x_0)*, then* $\tilde{x}$ *coincides with the restriction of* x *to* $\tilde{J}$.

A solution (J,x) is said to *maximal* if it does not admit any extension.

Theorem D.3. *Suppose that the function* $f : I \times \mathcal{U} \to \mathbf{R}^N$ *is of class* $\mathcal{C}^1$*. Let* $(t_0, x_0) \in I \times \mathcal{U}$*. There exists an interval* $J \subset I$ *containing* t_0 *and* open in I, *and a solution* x *of* $\dot{x} = f(t,x)$ *on* J *with initial data* (t_0, x_0) *that is* maximal, *in the sense that it cannot be extended by a solution on a larger time interval.*

D.3 Gronwall lemma

Lemma D.4 (Gronwall Lemma)**.** *Let* $T > 0$ *and* $C_1, C_2 \geq 0$*. We consider two continuous functions* $a : [0,T] \to [0,\infty)$ *and* $\varphi : [0,T] \to [0,\infty)$ *such that for all* $t \in [0,T]$,
$$\varphi(t) \leq C_1 + C_2 \int_0^t a(s)\varphi(s)\,\mathrm{d}s.$$
Then, for all $t \in [0,T]$,
$$\varphi(t) \leq C_1 \exp\left(C_2 \int_0^t a(s)\,\mathrm{d}s\right).$$

Lemma D.5. *Let* $p, q : I \to \mathbf{R}$ *be two continuous functions. Suppose that* p *has positive values. Let* $t_0 \in I$*. Consider a function* $x : I \to \mathbf{R}^N$ *of class* $\mathcal{C}^1$ *such that, for all* $t \geq t_0$, $t \in I$,
$$|\dot{x}(t)| \leq p(t)|x(t)| + q(t).$$
Let ψ *be a solution on* $I \cap [t_0, +\infty[$ *of the equation*
$$\dot{\psi} = p(t)\psi + q(t)$$
such that $\psi(t_0) \geq |x(t_0)|$*. Then, for all* $t \geq t_0$, $t \in I$,
$$|x(t)| \leq \psi(t).$$

D.4 Exercises

Exercise D.1. Prove Theorem D.1.

Exercise D.2. Prove Lemma D.4.

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