Mémoire

M2, Mathématiques Fondamentales

Silting theory and s-torsion pairs

Author
Esha Gupta

Supervisor
Pierre-Guy Plamondon

Université Paris Cité

Université de Versailles Saint-Quentin

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Chapter 1

Introduction

Throughout these notes, k will denote an algebraically closed field and Λ a finite-dimensional algebra over k. We will denote by $\operatorname{mod}\Lambda$ the category of finite-dimensional right Λ-modules and by $\operatorname{proj}\Lambda$ the category of finitely generated projective Λ-modules. We will use $\mathcal{D}(\operatorname{mod}\Lambda)$ (resp. $\mathcal{D}^b(\operatorname{mod}\Lambda)$) to denote the derived category (resp. bounded derived category) of $\operatorname{mod}\Lambda$. Both of these are triangulated categories with the suspension functor given by the shift functor. One of the classical problems in representation theory is to determine when two rings have the same representation theory. This was answered by Morita by proving that two rings have equivalent module categories if and only if one arises as the endomorphism ring of a special module, called a **progenerator**, over the other. This was taken forward by Rickard to the derived setting, where he proved that two rings have equivalent derived module categories if and only if one arises as the endomorphism ring of a special complex, called a **tilting complex**, over the other.

Tilting complexes can be equipped with a notion of mutation which allows one to produce a new tilting complex from a given one by replacing a summand. However, the problem is that the mutations of tilting complexes are not always possible. Silting theory, thus, can be viewed as a completion of tilting theory under the operation of mutation, so that it is always possible to mutate a silting object at an indecomposable summand to get a new silting object.

Our starting point in this thesis was the famous theorem by Adachi, Iyama, and Reiten ([2, Theorem 2.7, Theorem 3.2]) which proves that there is an isomorphism of posets between the following:

1. basic 2-term silting complexes in $K^b(\operatorname{proj}\Lambda)$,
2. basic τ-tilting modules in $\operatorname{mod}\Lambda$,
3. functorially finite torsion pairs in $\operatorname{mod}\Lambda$.

Moreover, we know by a result of Demonet, Iyama, and Jasso, that whenever this poset is finite, it is a lattice. The proof of this result relies on the fact that, in this case, all torsion pairs in $\operatorname{mod}\Lambda$ are functorially finite, and that the poset of all torsion pairs is known to always be a lattice. The latter result is an easy consequence of the following equivalent characterization of torsion classes: A full subcategory $\mathcal{T} \subseteq \operatorname{mod}\Lambda$ is a torsion class if and only if it is closed under quotients and extensions. In particular, for the path algebra of a quiver of type A_n, this poset is the famous **Tamari lattice** of order $n+1$. There are several equivalent ways of describing Tamari lattices–they are the posets of binary trees with n leaves, ordered by tree rotation operations, they are the posets of triangulations of a convex n-gon, ordered by flip operations, and many more. Moreover, the number of elements in a Tamari lattice of order n is given by the nth Catalan number C_n. They can also be embedded as skeletons of some nice polytopes called the **Stasheff polytopes** or **associahedrons** (Figure 1.1). An easy proof for the enumeration of 2-term silting objects in type A_n by Catalan numbers can be given using the geometric model for τ-tilting complexes for gentle algebras introduced in [16].

In this thesis, our main goal is to generalize the above results to the poset of d-term silting objects to as much extent as possible. This is done by following the same steps as for the $d = 2$ case and generalizing the relevant notions at each step. The module category is replaced by an appropriate extriangulated subcategory

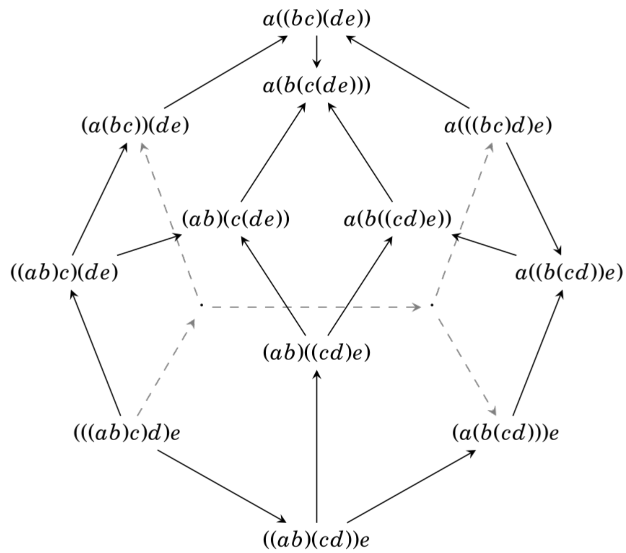


Figure 1.1: Stasheff polytope K_5 [20]

of $\mathcal{D}^b(\operatorname{mod}\Lambda)$ and the notion of torsion pairs is replaced by s-torsion pairs in these categories. The two main results are as follows:

Theorem 1.0.1. *Let Λ be a hereditary algebra. There exists an injective poset homomorphism*

$$\phi : d\text{-}\operatorname{silt}\Lambda \to \operatorname{stors}\mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$$

given by $M \mapsto (\{N' \in \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda) \mid \operatorname{Hom}(M, \Sigma^m N') = 0,\ \forall\ m > 0\}, \{N' \in \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda) \mid \operatorname{Hom}(N, \Sigma^{m+1} N') = 0,\ \forall\ m < 0\})$.

Theorem 1.0.2. *Let Λ be a hereditary algebra of finite representation type and set $\mathcal{C} := D^{[-(d-2),0]}(\operatorname{mod}\Lambda)$. Then the poset $\operatorname{stors}\mathcal{C}$ is a lattice.*

We also generalize the model for 2-term silting complexes in kA_n to a model for the entire bounded derived category $\mathcal{D}^b(\operatorname{mod} kA_n)$, which we use to calculate explicitly the number of 3-term silting complexes in kA_n. This turns out to be the Fuss-Catalan number $A_{n+1}(3,1)$, a well-studied generalization of Catalan numbers [18, 21]. This observation leads us to believe that the number of d-term silting objects in kA_n is given by the Fuss-Catalan number $A_{n+1}(d,1)$.

The thesis is organized as follows: § 2.1 is devoted to the definitions and terminology related to extriangulated categories. In § 2.2, we recall Adachi, Enomoto, and Tsukamoto's definitions of extriangulated categories with negative extensions and s-torsion pairs on them. § 2.3 is devoted to the definitions of silting objects and their mutations along with their relation to t-structures. It also deals with the special case of 2-term silting complexes and torsion pairs. § 2.4 is used to introduce some background results on the derived categories of hereditary algebras which we will extensively use in Chapter 3. Finally, § 2.5 recalls the geometric model for the derived categories of gentle algebras introduced in [5, 15, 16] and the explicit calculation for 2-term silting complexes. Sections 3.1 and 3.2 deal with the proofs of Theorem 1.0.1 and Theorem 1.0.2 respectively. § 3.3 is used to introduce the geometric model for the derived category of kA_n and to count the number of 3-term silting objects for these algebras.

Before going forward, we set up some notation that we will use throughout this work. For an algebra Λ, an object $P \in \mathcal{D}(\operatorname{mod}\Lambda)$ is said to be a **perfect complex** if it is quasi-isomorphic to a bounded complex of finitely generated projective Λ-modules. We will denote by $\operatorname{per}\Lambda$ the full subcategory of $\mathcal{D}(\operatorname{mod}\Lambda)$ of perfect complexes. This is equivalent to the homotopy category of bounded chain complexes of finitely generated projective Λ-modules, which will be denoted by $K^b(\operatorname{proj}\Lambda)$. For a subcategory or a set of objects $\mathcal{S}$ of a triangulated category $\mathcal{C}$, $\operatorname{thick}(\mathcal{S})$ will denote the **thick subcategory of $\mathcal{C}$ generated by $\mathcal{S}$**, i.e., the smallest triangulated subcategory of $\mathcal{C}$ containing $\mathcal{S}$ and closed under taking isomorphisms and direct summands. For an object M of an abelian category, we denote by $\operatorname{add} M$ (respectively, $\operatorname{Fac} M, \operatorname{Sub} M$) the category of all direct summands (respectively, factor modules, submodules) of finite direct sums of copies of M. For a collection $\mathcal{X}$ of objects in an a category $\mathcal{C}$, set $\mathcal{X}^{\perp} := \{C \in \mathcal{C} \mid \mathcal{C}(\mathcal{X}, C) = 0\}$ and ${}^{\perp}\mathcal{X} := \{C \in \mathcal{C} \mid \mathcal{C}(C, \mathcal{X}) = 0\}$. Finally, for $n, m \in \mathbb{Z}$, set

$$\mathcal{D}^{\leq n} := \{X \in \mathcal{D}^b(\operatorname{mod} \Lambda) \mid \mathrm{H}^{>n}(X) = 0\}, \ \mathcal{D}^{\geq m} := \{X \in \mathcal{D}^b(\operatorname{mod} \Lambda) \mid \mathrm{H}^{<m}(X) = 0\}.$$

We also set $\mathcal{D}^{[m,n]} := \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m}$.

Chapter 2

Preliminaries

2.1 Extriangulated categories

In this section, we recall the basic definitions and terminology associated with extriangulated categories. These were first introduced by Nakaoka and Palu [14, § 2] as a simultaneous generalization of exact and triangulated categories. A particularly important class of examples for us of extriangulated categories would be that of extension-closed subcategories in the derived category of an algebra.

Let $\mathcal{C}$ be an additive category and $\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Ab}$ a biadditive functor. An **$\mathbb{E}$-extension** is a triplet (A, δ, C), where $A, C \in \mathcal{C}$ and $\delta \in \mathbb{E}(C, A)$.

Definition 2.1.1. *Let (A, δ, C), (A', δ', C') be a pair of $\mathbb{E}$-extensions. A* morphism $(a, c) : (A, \delta, C) \to (A', \delta', C')$ *is a pair of morphisms $a : A \to A'$ and $c : C \to C'$ such that $\mathbb{E}(C, a)(\delta) = \mathbb{E}(c, A')(\delta')$.*

The above definition allows us to define the category $\mathbb{E}$-$\mathrm{Ext}(\mathcal{C})$ of $\mathbb{E}$-extensions, with compositions and identities induced naturally from the compositions and identities in $\mathcal{C}$.

Let (A, δ, C), (A', δ', C') be two $\mathbb{E}$-extenions. Then the biadditivity of $\mathbb{E}$ gives a natural isomorphism

$$\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A').$$

We denote by $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ the element corresponding to $(\delta, 0, 0, \delta')$ through this isomorphism.

Definition 2.1.2. *Let $A, C \in \mathcal{C}$. Two sequences of morphisms $A \xrightarrow{x} B \xrightarrow{y} C$ and $A \xrightarrow{x'} B' \xrightarrow{y'} C$ are called **equivalent** if there exists an isomorphism $b : B \to B'$ such that the following diagram commutes.*

$$\begin{tikzcd} & B \arrow[dd, "b"] \arrow[dr, "y"] & \\ A \arrow[ur, "x"] \arrow[dr, "x'"'] & & C \\ & B' \arrow[ur, "y'"'] & \end{tikzcd}$$

We will denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$. Moreover, we denote the class $[A \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} A \oplus C \xrightarrow{[0\ 1]} C]$ as 0 and the class $[A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C']$ as $[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$.

Definition 2.1.3. *A **realization** $\mathfrak{s}$ of $\mathbb{E}$ is a map that assigns an equivalence class $[A \xrightarrow{x} B \xrightarrow{y} C]$ to every $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$ such that, if (A, δ, C) and (A', δ', C') are two $\mathbb{E}$-extensions with $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ and $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$, then for any morphism $(a, c) \in \mathbb{E}$-$\mathrm{Ext}(\mathcal{C})(\delta, \delta')$, there exists $b : B \to B'$ which makes the following diagram commute.*

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] & B \arrow[r, "y"] \arrow[d, dashed, "b"] & C \arrow[d, "c"] \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd}$$

In this case, we say that the triplet (a, b, c) **realizes** (a, c). Note that the above definition is independent of the choices of the representatives of the equivalence classes.

Definition 2.1.4. *We say a realization $\mathfrak{s}$ of $\mathbb{E}$ is **additive** if*

1. *For any $A, C \in \mathcal{C}$, $\mathfrak{s}(0) = 0$.*
2. *For any pair of $\mathbb{E}$-extensions (A, δ, C) and (A', δ', C'), $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$.*

Definition 2.1.5. *An **external triangulation** of $\mathcal{C}$ is a pair $(\mathbb{E}, \mathfrak{s})$, with $\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Ab}$ a biadditive functor and $\mathfrak{s}$ an additive realization of $\mathbb{E}$, that satisfies the following conditions:*

1. *Let (A, δ, C) and (A', δ', C') be a pair of $\mathbb{E}$-extensions with $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ and $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. For any commutative square*

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "y"] \arrow[d, "b"] & C \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd}$$

   in $\mathcal{C}$, there exists a morphism $(a, c) : \delta \to \delta'$ which is realized by (a, b, c).

2. *Let (A, δ, C) and (A', δ', C') be a pair of $\mathbb{E}$-extensions with $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ and $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. For any commutative square*

$$\begin{tikzcd} A \arrow[r, "x"] & B \arrow[r, "y"] \arrow[d, "b"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "c"] \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd}$$

   in $\mathcal{C}$, there exists a morphism $(a, c) : \delta \to \delta'$ which is realized by (a, b, c).

3. *Let (A, δ, D) and (B, δ', F) be a pair of $\mathbb{E}$-extensions with $\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{f'} D]$ and $\mathfrak{s}(\delta') = [B \xrightarrow{g} C \xrightarrow{g'} F]$. Then there exists an object $E \in \mathcal{C}$, a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "f'"] \arrow[d, "g"] & D \arrow[d, "d"] \\ A \arrow[r, "h"'] & C \arrow[r, "h'"'] \arrow[d, "g'"'] & E \arrow[d, "e"] \\ & F \arrow[r, equal] & F \end{tikzcd}$$

   in $\mathcal{C}$, and an $\mathbb{E}$-extension (A, δ'', E) with $\mathfrak{s}(\delta'') = [A \xrightarrow{h} C \xrightarrow{h'} E]$, satisfying the following conditions:

   (a) $\mathfrak{s}(\mathbb{E}(F, f')(\delta')) = [D \xrightarrow{d} E \xrightarrow{e} F]$.

(b) $\mathbb{E}(d,A)(\delta'') = \delta$.

(c) $\mathbb{E}(E,f)(\delta'') = \mathbb{E}(e,B)(\delta')$.

4. *Let (A,δ,D) and (F,δ',B) be a pair of $\mathbb{E}$-extensions with $\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{f'} D]$ and $\mathfrak{s}(\delta') = [F \xrightarrow{g} C \xrightarrow{g'} B]$. Then there exists an object $E \in \mathcal{C}$, a commutative diagram*

$$\begin{tikzcd}
F \arrow[r, equal] \arrow[d, "e"'] & F \arrow[d, "g"] & \\
E \arrow[r, "h"] \arrow[d, "a"'] & C \arrow[r, "h'"] \arrow[d, "g'"] & D \arrow[d, equal] \\
A \arrow[r, "f"'] & B \arrow[r, "f'"'] & D
\end{tikzcd}$$

in $\mathcal{C}$, and an $\mathbb{E}$-extension (E,δ'',D) with $\mathfrak{s}(\delta'') = [E \xrightarrow{h} C \xrightarrow{h'} D]$, satisfying the following conditions:

(a) $\mathfrak{s}(\mathbb{E}(f,F)(\delta')) = [F \xrightarrow{e} E \xrightarrow{a} A]$.

(b) $\mathbb{E}(D,a)(\delta'') = \delta$.

(c) $\mathbb{E}(f',E)(\delta'') = \mathbb{E}(B,e)(\delta')$.

We call the triple $(\mathcal{C},\mathbb{E},\mathfrak{s})$ an **extriangulated category**. Note that in the above definition, the second condition is the dual of the first and the fourth condition is the dual of the third. It follows that if $(\mathcal{C},\mathbb{E},\mathfrak{s})$ is an extriangulated category, then so is $(\mathcal{C}^{op},\mathbb{E}^{op},\mathfrak{s}^{op})$, where $\mathbb{E}^{op}(A,C) = \mathbb{E}(C,A)$, and $\mathfrak{s}^{op}(\delta \in \mathbb{E}^{op}(C,A)) = \mathfrak{s}(\delta)$.

Following [14], we now introduce some terminology to refer to the structures in an extriangulated category.

Definition 2.1.6. 1. *A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a **conflation** if $[A \xrightarrow{x} B \xrightarrow{y} C] = \mathfrak{s}(\delta)$ for some $\delta \in \mathbb{E}(C,A)$.*

2. *A morphism $f \in \mathcal{C}(A,B)$ is called an **inflation** if it is a part of some conflation $A \xrightarrow{f} B \to C$.*

3. *A morphism $f \in \mathcal{C}(A,B)$ is called a **deflation** if it is a part of some conflation $K \to A \xrightarrow{f} B$.*

We will write the above conflation as $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$.

Examples 2.1.1. 1. *[14, Proposition 3.22] Let $\mathcal{C}$ be a triangulated category with suspension Σ. Define $\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Ab}$ as $\mathrm{Hom}_{\mathcal{C}}(-,\Sigma(-))$. For $\delta \in \mathbb{E}(C,A)$, take a distinguished triangle*

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \Sigma A$$

and define $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$. Note that this does not depend on the choice of the distinguished triangle. Then $(\mathcal{C},\mathbb{E},\mathfrak{s})$ is an extriangulated category.

2. *Let $\mathcal{E}$ be an exact category. Then setting $\mathbb{E}(C,A)$ to be the collection of equivalence classes of short exact sequences of the form $A \xrightarrow{x} B \xrightarrow{y} C$ in $\mathcal{E}$ and $\mathfrak{s}([A \xrightarrow{x} B \xrightarrow{y} C])$to be $[A \xrightarrow{x} B \xrightarrow{y} C]$, we get that $(\mathcal{E},\mathbb{E},\mathfrak{s})$ is an extriangulated category. The details of this can be found in [14, Example 2.13].*

3. *[14, Remark 2.18] Let $(\mathcal{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category. For two collections $\mathcal{X}$ and $\mathcal{Y}$ of objects in $\mathcal{C}$, let $\mathcal{X} \star \mathcal{Y}$ denote the full subcategory of $\mathcal{C}$ consisting of those $M \in \mathcal{C}$ which admit a conflation $X \to M \to Y \dashrightarrow$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.*

*Let $\mathcal{D} \subset \mathcal{C}$ be a full additive subcategory, closed under isomorphisms. Then $\mathcal{D}$ is said to be **extension-closed** if $\mathcal{D} \star \mathcal{D} \subset \mathcal{D}$. Note that if $\mathcal{D}$ is an extension-closed subcategory of $\mathcal{C}$, and we define $\mathbb{E}_{\mathcal{D}}$ to be the restriction of $\mathbb{E}$ on $\mathcal{D}^{op} \times \mathcal{D}$, and $\mathfrak{s}_{\mathcal{D}}$ to be the restriction of $\mathfrak{s}$, then $(\mathcal{D},\mathbb{E}_{\mathcal{D}},\mathfrak{s}_{\mathcal{D}})$ becomes an extriangulated category as well. In particular, any extension-closed subcategory of a triangulated category is extriangulated.*

Let $\delta \in \mathbb{E}(C,A)$. We have two natural transformations $\delta_\# : \mathcal{C}(-,C) \to \mathbb{E}(-,A)$ and $\delta^\# : \mathcal{C}(A,-) \to \mathbb{E}(C,-)$ defined as

$$(\delta_\#)_W : \mathcal{C}(W,C) \to \mathbb{E}(W,A) \quad (\phi \mapsto \mathbb{E}(\phi,A)(\delta)),$$
$$(\delta^\#)_W : \mathcal{C}(A,W) \to \mathbb{E}(C,W) \quad (\phi \mapsto \mathbb{E}(C,\phi)(\delta))$$

for $W \in \mathcal{C}$. Any conflation induces two long-exact sequences in Ab.

Proposition 2.1.1. *Let $\mathcal{C}$ be an extriangulated category. Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be a conflation. Then for each $W \in \mathcal{C}$, the following sequences are exact.*

$$\mathcal{C}(W,A) \xrightarrow{\mathcal{C}(W,f)} \mathcal{C}(W,B) \xrightarrow{\mathcal{C}(W,g)} \mathcal{C}(W,C) \xrightarrow{(\delta_\#)_W} \mathbb{E}(W,A) \xrightarrow{\mathbb{E}(W,f)} \mathbb{E}(W,B) \xrightarrow{\mathbb{E}(W,g)} \mathbb{E}(W,C)$$

$$\mathcal{C}(C,W) \xrightarrow{\mathcal{C}(g,W)} \mathcal{C}(B,W) \xrightarrow{\mathcal{C}(f,W)} \mathcal{C}(A,W) \xrightarrow{(\delta^\#)_W} \mathbb{E}(C,W) \xrightarrow{\mathbb{E}(g,W)} \mathbb{E}(B,W) \xrightarrow{\mathbb{E}(f,W)} \mathbb{E}(A,W)$$

2.2 Extriangulated categories with negative first extensions

We now define negative first extension structures on an extriangulated category, which were introduced in [1] to give a generalized framework for the study of t-structures on triangulated categories and torsion pairs in abelian categories.

Definition 2.2.1. *Let $(\mathcal{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category. A **negative first extension structure** on $\mathcal{C}$ consists of the following data:*

1. *An additive functor $\mathbb{E}^{-1} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Ab}$.*
2. *For each $\delta \in \mathbb{E}(C,A)$, two natural transformations $\delta_\#^{-1} : \mathbb{E}^{-1}(-,C) \to \mathcal{C}(-,A)$, $\delta_{-1}^\# : \mathbb{E}^{-1}(A,-) \to \mathcal{C}(C,-)$, such that for each conflation $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and each $W \in \mathcal{C}$, the sequences*

$$\mathbb{E}^{-1}(W,A) \xrightarrow{\mathbb{E}^{-1}(W,f)} \mathbb{E}^{-1}(W,B) \xrightarrow{\mathbb{E}^{-1}(W,g)} \mathbb{E}^{-1}(W,C) \xrightarrow{(\delta_\#^{-1})_W} \mathcal{C}(W,A) \xrightarrow{\mathcal{C}(W,f)} \mathcal{C}(W,B),$$

$$\mathbb{E}^{-1}(C,W) \xrightarrow{\mathbb{E}^{-1}(g,W)} \mathbb{E}^{-1}(B,W) \xrightarrow{\mathbb{E}^{-1}(f,W)} \mathbb{E}^{-1}(A,W) \xrightarrow{(\delta_{-1}^\#)_W} \mathcal{C}(C,W) \xrightarrow{\mathcal{C}(g,W)} \mathcal{C}(B,W)$$

   are exact.

*In this case, we call $\mathcal{C} = (\mathcal{C},\mathbb{E},\mathfrak{s},\mathbb{E}^{-1})$ an **extriangulated category with negative first extensions**.*

Examples 2.2.1.
1. *We saw previously that a triangulated category $(\mathcal{C},\Sigma)$ can be viewed as an extriangulated category in a canonical way. Moreover, if we define $\mathbb{E}^{-1} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Ab}$ as $\mathrm{Hom}_\mathcal{C}(-,\Sigma^{-1}(-))$, $\delta_\#^{-1}$ as $\mathrm{Hom}_\mathcal{C}(-,\Sigma^{-1}\delta)$, and $\delta_{-1}^\#$ as $\mathrm{Hom}_\mathcal{C}(\delta,-)$, for all $\delta \in \mathbb{E}(C,A)$, then $(\mathcal{C},\mathbb{E},\mathfrak{s},\mathbb{E}^{-1})$ becomes an extriangulated category with negative first extensions.*
2. *Continuing Example 2.1.1.3, any extension-closed subcategory of an extriangulated category with negative first extensions is also an extriangulated category with negative first extensions.*

2.2.1 s-torsion pairs

In this section, we study s-torsion pairs in an extriangulated category with negative first extensions, which were introduced in [1, § 3]. These are a generalization of the notion of t-structures for triangulated categories and of torsion pairs in abelian categories. In the following, let $\mathcal{C} = (\mathcal{C},\mathbb{E},\mathfrak{s},\mathbb{E}^{-1})$ be an extriangulated category with negative first extensions.

Definition 2.2.2. *A pair $(\mathcal{T},\mathcal{F})$ of full subcategories of $\mathcal{C}$ is called an s-**torsion pair** in $\mathcal{C}$ if*

1. $\mathcal{C} = \mathcal{T} \star \mathcal{F}$*,*

2. $\mathcal{C}(\mathcal{T}, \mathcal{F}) = 0$*,*

3. $\mathbb{E}^{-1}(\mathcal{T}, \mathcal{F}) = 0$*.*

In this case, $\mathcal{T}$ (respectively, $\mathcal{F}$) is called a **torsion class** (respectively, **torsion-free class**) in $\mathcal{C}$.

An important fact about s-torsion pairs is that a torsion class (respectively, torsion-free class) is uniquely determined by a torsion-free class (respectively, torsion class). We prove this in the following proposition.

Proposition 2.2.1. *[1, Proposition 3.2] Let* $(\mathcal{T}, \mathcal{F})$ *be an* s*-torsion pair in* $\mathcal{C}$*. Then*

1. ${}^{\perp}\mathcal{F} = \mathcal{T}$*.*

2. $\mathcal{T}^{\perp} = \mathcal{F}$*.*

In particular, $\mathcal{T}$ *and* $\mathcal{F}$ *are extension-closed subcategories that are closed under direct summands.*

Proof. We only prove 1. The proof of 2 is similar. By the definition of s-torsion pairs, $\mathcal{T} \subset {}^{\perp}\mathcal{F}$. Let $C \in {}^{\perp}\mathcal{F}$. Since $\mathcal{C} = \mathcal{T} \star \mathcal{F}$, there exists a conflation

$$T \xrightarrow{f} C \xrightarrow{g} F \dashrightarrow$$

such that $T \in \mathcal{T}$ and $F \in \mathcal{F}$. By the definition of negative first extension structures, we get the following exact sequence:

$$\mathbb{E}^{-1}(T, F) \to \mathcal{C}(F, F) \to \mathcal{C}(C, F).$$

where the left-hand side and the right-hand side vanish. Therefore, $\mathcal{C}(F, F) = 0$, which implies that $F = 0$. Using Proposition 2.1.1, this gives that the natural transformation $\mathcal{C}(f, -) : \mathcal{C}(C, -) \to \mathcal{C}(T, -)$ is an isomorphism, which, by Yoneda lemma, gives that f is an isomorphism. Thus, $C \cong T \in \mathcal{T}$. ∎

We denote the set of all s-torsion pairs in $\mathcal{C}$ by $\operatorname{stors}\mathcal{C}$. There is a natural poset structure $\leq$ on this set given as follows: for $(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2) \in \operatorname{stors}\mathcal{C}$, $(\mathcal{T}_1, \mathcal{F}_1) \leq (\mathcal{T}_2, \mathcal{F}_2)$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ which, using the above proposition, is equivalent to $\mathcal{F}_1 \supseteq \mathcal{F}_2$. The term "$s$" in an s-torsion pair stands for "shift-closed" which is justified by the following lemma.

Lemma 2.2.1. *[1, Lemma 3.3] Let* $(\mathcal{C}, \Sigma)$ *be a triangulated category viewed as an extriangulated category with negative first extensions. Let* $(\mathcal{T}, \mathcal{F})$ *be a pair of subcategories of* $\mathcal{C}$ *satisfying 1 and 2 in Definition 2.2.2. Then the following are equivalent.*

1. $(\mathcal{T}, \mathcal{F})$ *is an* s*-torsion pair.*

2. $\mathcal{T}$ *is closed under positive shifts, i.e.,* $\Sigma\mathcal{T} \subset \mathcal{T}$*.*

3. $\mathcal{F}$ *is closed under negative shifts, i.e.,* $\Sigma^{-1}\mathcal{F} \subset \mathcal{F}$*.*

Proof. We will only prove 1⇔3. The proof of 1⇔2 is similar. By definition, $(\mathcal{T}, \mathcal{F})$ is an s-torsion pair if and only if $\mathbb{E}^{-1}(\mathcal{T}, \mathcal{F}) = \mathcal{C}(\Sigma\mathcal{T}, \mathcal{F}) = 0$ if and only if $\mathcal{C}(\mathcal{T}, \Sigma^{-1}\mathcal{F}) = 0$ if and only if $\Sigma^{-1}F \subset \mathcal{T}^{\perp} = \mathcal{F}$ (by Lemma 2.2.1). ∎

We now show that s-torsion pairs are indeed a generalization of t-structures.

Example 2.2.1. *1. [1, Example 3.4] Let* $\mathcal{C}$ *be a triangulated category. A pair* $(\mathcal{U}, \mathcal{V})$ *of subcategories of* $\mathcal{C}$ *is called a* ***t-structure*** *on* $\mathcal{C}$ *if it satisfies the following conditions.*

- $\mathcal{C} = \mathcal{U} \star \mathcal{V}$*, i.e., for each* $C \in \mathcal{C}$*, there exists a triangle* $U \to C \to D \to \Sigma U$ *such that* $U \in \mathcal{U}$ *and* $V \in \mathcal{V}$*.*
- $\mathcal{C}(\mathcal{U}, \mathcal{V}) = 0$*.*

- $\mathcal{U}$ *is closed under positive shifts.*
- $\mathcal{V}$ *is closed under negative shifts.*

It follows from Lemma 2.2.1 that t-structures on $\mathcal{C}$ are exactly the s-torsion pairs in $\mathcal{C}$, and that the last two conditions are equivalent to each other. Thus it is enough to check either of them. Before going forward, we introduce some terminology related to t-structures which would be useful to us later.

For a t-structure $(\mathcal{U}, \mathcal{V})$, $\mathcal{U}$ is called the ***aisle*** *and $\Sigma\mathcal{V}$ the* ***co-aisle*** *of the t-structure. The category $\mathcal{U} \bigcap \Sigma\mathcal{V}$ is said to be the* ***heart*** *of the t-structure. The heart of a t-structure is always abelian [8].*

2. *Let $\mathcal{A}$ be an abelian category. A pair $(\mathcal{T}, \mathcal{F})$ of subcategories of $\mathcal{A}$ is called a* ***torsion pair*** *if*

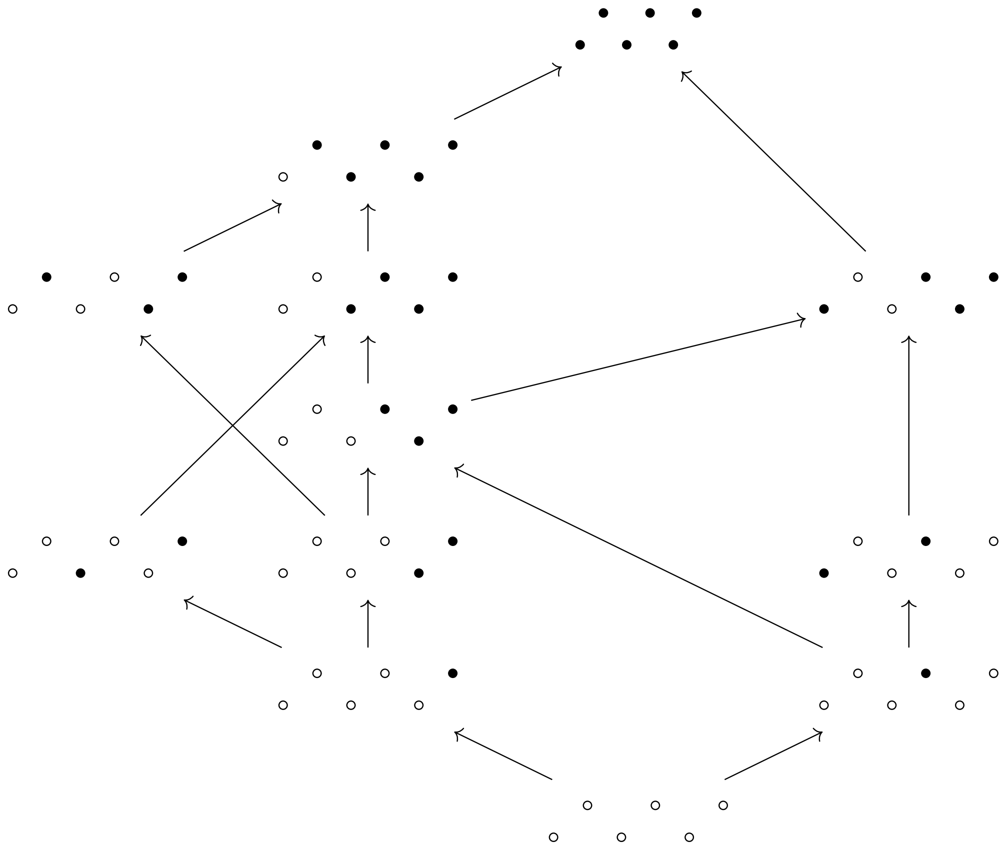
   - $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$, *and*
   - $\mathcal{A} = \mathcal{T} \star \mathcal{F}$, *i.e., for each $A \in \mathcal{A}$, there exists a short exact sequence $0 \to T \to A \to F \to 0$ in $\mathcal{A}$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.*

   We can view $\mathcal{A}$ as an extension-closed subcategory of the triangulated category $\mathcal{D}^b(\mathcal{A})$. Hence it becomes an extriangulated category with negative first extensions. Then s-torsion pairs in $\mathcal{A}$ viewed as an extriangulated category with negative first extensions are the same as torsion pairs in $\mathcal{A}$ viewed as an abelian category.

3. *Let $\Lambda = kQ$ with $Q = 1 \to 2$. Recall that the* ***Auslander-Reiten quiver*** *of a Krull-Schmidt category $\mathcal{C}$ is a quiver whose vertices are the isomorphism classes of indecomposable objects in $\mathcal{C}$ and for $X, Y \in \mathcal{C}$, there are $\dim_k(\mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y))$ many arrows from the isomorphism class of X to the isomorphism class of Y ([10, § 4.8, Chapter 1]). Using Proposition 2.4.2, we get that the AR quiver of $\mathcal{D}^{[-1,0]}(\mathrm{mod}\,\Lambda)$ is given by*

$$\begin{tikzcd}
& P_1 \arrow[dr] & & P_2[1] \arrow[dr] & & S_1[1] \\
P_2 \arrow[ur] & & S_1 \arrow[ur] & & P_1[1] \arrow[ur] &
\end{tikzcd}$$

   Using an intrinsic characterization of torsion classes in $\mathcal{D}^{[-1,0]}(\mathrm{mod}\,\Lambda)$ introduced in § 3.2, we can conclude, using a finite check, that the poset of s-torsion pairs in $\mathcal{C} := \mathcal{D}^{[-1,0]}(\mathrm{mod}\,\Lambda)$ is given as follows.



Here the vertices are the AR quiver of $\mathcal{C}$ with the arrows suppressed, and the solid dots represent the additive generators of the torsion classes.

2.3 Silting Complexes

In this section, we will look at the notions of silting and tilting objects in triangulated categories. Tilting objects play an important role in the Morita theory of derived categories and silting objects are a generalization of them introduced in [12] to study t-structures on the bounded derived category of finite-dimensional representations over a Dynkin quiver. We will also study the relationship between silting objects and t-structures.

Throughout this section, $\mathcal{C}$ will denote a Hom-finite, Krull-Schmidt triangulated category with suspension Σ.

2.3.1 Silting objects

Definition 2.3.1. *An object M of $\mathcal{C}$ is called a* ***presilting object*** *if $\mathrm{Hom}_{\mathcal{C}}(M, \Sigma^i M) = 0$ for all $i > 0$, a* ***silting object*** *if in addition $\mathcal{C} = \mathrm{thick}(M)$, and a* ***tilting object*** *if further $\mathrm{Hom}_{\mathcal{C}}(M, \Sigma^i M) = 0$ for all $i < 0$.*

Example 2.3.1. *Let $\mathcal{C} = K^b(\mathrm{proj}\,\Lambda)$, where Λ is a finite-dimensional algebra over an algebraically closed field k. Then the complex $M = \cdots \to 0 \to \Lambda \to 0 \to \cdots$ is a silting object in $\mathcal{C}$ as $\mathrm{thick}(M)$ contains all the finite-dimensional indecomposable projectives concentrated in degree 0, and every other complex can be obtained as a combination of taking cones and shifts from them.*

Two silting objects P and Q are said to be **equivalent** if $\mathrm{add}\,P = \mathrm{add}\,Q$. We denote the set of equivalence classes of silting objects in $\mathcal{C}$ as $\mathrm{silt}\,\mathcal{C}$. More generally, we say that an object M in a Krull-Schmidt category is **basic** if $M = \oplus_{i=1}^n M_i$ with M_i indecomposable and $M_i \ncong M_j$ for all $i \neq j$. Thus silt$\,\mathcal{C}$ is in bijection with the set of basic silting objects in $\mathcal{C}$.

Definition 2.3.2. *Let $d \geq 1$. An object $P^\bullet \in \operatorname{silt}(K^b(\operatorname{proj}\Lambda))$, where Λ is a finite-dimensional algebra over k, is said to be **d-term** if it is isomorphic to a complex $Q^\bullet$ such that*

$$Q^i = 0$$

for all $i \notin \{-(d-1), \cdots, -1, 0\}$, i.e., if it is only concentrated in the first d negative degrees.

For $\mathcal{C} = K^b(\operatorname{proj}\Lambda)$, the subset of silt$\,\mathcal{C}$ consisting of the equivalence classes of d-term silting objects is denoted d-silt$\,\Lambda$.

Example 2.3.2. *Let $\Lambda = kQ$, where $Q = 1 \to 2$. There are precisely five equivalence classes of 2-term silting objects in $K^b(\operatorname{proj}\Lambda)$. We can represent them using the following graph.*

$$\begin{tikzcd}
 & \Lambda \arrow[dl] \arrow[ddr] & \\
S_1 \oplus P_1 \arrow[dd] & & \\
 & & S_2 \oplus \Sigma P_1 \arrow[ddl] \\
S_1 \oplus \Sigma S_2 \arrow[dr] & & \\
 & \Sigma\Lambda &
\end{tikzcd}$$

2.3.2 Mutations of silting objects

Silting objects can be equipped with a notion of mutation which allows us to put a poset structure on the set of silting objects. For this, we need the following definition.

Definition 2.3.3. *Let $\mathcal{D} \subseteq \mathcal{C}$ be an additive subcategory and $X \in \mathcal{C}$. A left $\mathcal{D}$-approximation of X is a map $X \xrightarrow{f} D$ with $D \in \mathcal{D}$ such that the sequence*

$$\operatorname{Hom}(D, D') \xrightarrow{\operatorname{Hom}(f,D')} \operatorname{Hom}(X, D') \to 0$$

is exact for all $D' \in \mathcal{D}$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & D \arrow[dl, dashed] \\
D' &
\end{tikzcd}$$

*A left $\mathcal{D}$-approximation f of X is said to be **minimal left $\mathcal{D}$-approximation** if f is a left minimal morphism. Dually we can also define a **minimal right $\mathcal{D}$-approximation** of X.*

Definition 2.3.4. *Let $M = X \oplus N$ be a basic silting object in $\mathcal{C}$ with X indecomposable. The **left mutation** of M at X, denoted $\mu_X^+(M)$, is the object $X' \oplus N$, where X' is the cone of the minimal left $\operatorname{add}(N)$-approximation of X, i.e., the following is a triangle*

$$X \to N' \to X' \to X[1]$$

with $N' \in \operatorname{add}(N)$.

*Dually we can define the **right mutation** of M at X, denoted $\mu_X^-(M)$.*

Theorem 2.3.1. *[13, Theorem 7.1][3, Theorem 2.31 and Proposition 2.33] The objects $\mu^+_X(M)$ and $\mu^-_X(M)$ are basic silting objects. Moreover, $\mu^+_{X'} \circ \mu^-_X(M) \cong M \cong \mu^-_{X'} \circ \mu^+_X(M)$.*

We can encode the above notion of mutation in a graph as follows. We define the **silting quiver** of $\mathcal{C}$, denoted $Q(\operatorname{silt}\mathcal{C})$, to be the quiver with vertex set $\operatorname{silt}\mathcal{C}$ and arrows as: for $M, M' \in \operatorname{silt}\mathcal{C}$, there is an arrow $M \to M'$ if and only if M' is a left mutation of M. For $\mathcal{C} = K^b(\operatorname{proj}\Lambda)$, we define $Q(d\text{-}\operatorname{silt}\Lambda)$ to be the full subquiver of $Q(\operatorname{silt}\mathcal{C})$ consisting of the d-term silting objects.

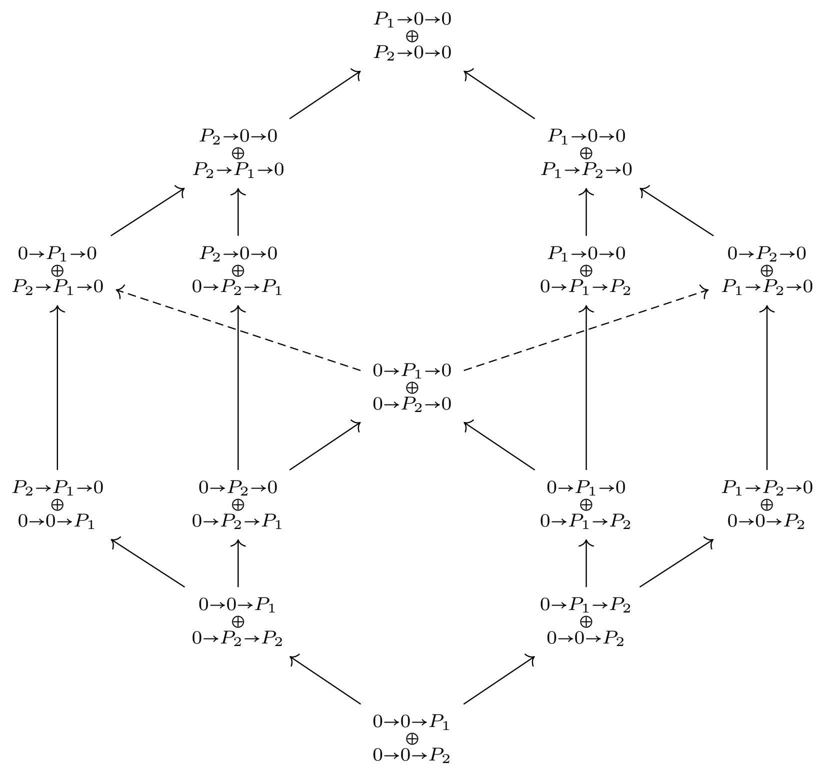
Examples 2.3.1. *1. The graph described in Example 2.3.2 is just the quiver $Q(2\text{-}\operatorname{silt} kA_2)$.*

2. The quiver $Q(3\text{-}\operatorname{silt} kA_2)$ is given as follows.

$$
\begin{tikzcd}
 & & P_1 \oplus P_2 & \\
 & S_1 \oplus P_1 \arrow[ur] & & \\
\Sigma S_1 \oplus P_1 \arrow[ur] & S_1 \oplus \Sigma P_2 \arrow[u] & & P_2 \oplus \Sigma P_1 \arrow[uul] \\
 & \Sigma P_2 \oplus \Sigma P_1 \arrow[u] \arrow[urr] & & \\
\Sigma^2 P_2 \oplus S_1 \arrow[uur] & \Sigma S_1 \oplus \Sigma P_1 \arrow[u] \arrow[uul] & & P_2 \oplus \Sigma^2 P_1 \arrow[uu] \\
 & \Sigma^2 P_2 \oplus \Sigma S_1 \arrow[ul] \arrow[u] & & \Sigma P_2 \oplus \Sigma^2 P_1 \arrow[u] \arrow[uull] \\
 & & \Sigma^2 P_2 \oplus \Sigma^2 P_1 \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Note that this is isomorphic to the Hasse quiver of the poset of s-torsion classes introduced in Example 2.2.1(3). This is a particular case of a general but weaker result (Theorem 3.1.1).

3. Let Q be the quiver $\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$ *and $I = \langle \alpha\beta, \beta\alpha \rangle$ an admissible ideal of kQ. Then the quiver $3\text{-}\operatorname{silt} kQ/I$ is as follows.*



The notion of mutations provides a way of constructing new silting objects from a known one. It is not always possible to recover all of the silting objects in this way. However, in certain cases, it is possible to do so.

Theorem 2.3.2. *[3, Theorem 3.1] Let $\mathcal{C} = K^b(\operatorname{proj}\Lambda)$, where Λ is a finite-dimensional hereditary algebra. Then the action of iterated silting mutation on $\operatorname{silt}\mathcal{C}$ is transitive.*

Let $P, P' \in \operatorname{silt}\mathcal{C}$. Define an order relation $\leq$ on $\operatorname{silt}\mathcal{C}$ by setting $P \leq P'$ if $\operatorname{Hom}(P', \Sigma^m P) = 0$ for all $m > 0$. This is a partial order on $\operatorname{silt}\mathcal{C}$ by [3, Theorem 2.11].

Theorem 2.3.3. *[13, Theorem 7.2][3, Theorem 2.35] The Hasse diagram of $(\operatorname{silt}\mathcal{C}, \leq)$ is $Q(\operatorname{silt}\mathcal{C})$.*

Thus mutations correspond to the minimal order relations between two silting objects.

Since we are working in a Krull-Schmidt triangulated category, we can decompose every basic silting object as a direct sum of indecomposables. The number of such indecomposables is an invariant of the algebra Λ as stated in the following theorem.

Theorem 2.3.4. *[3, Theorem 2.27][13, Theorem 3.1] The number of indecomposable summands of a basic silting object in $K^b(\operatorname{proj}\Lambda)$ is given by the rank of the Grothendieck group of $\operatorname{mod}\Lambda$.*

For $\Lambda = kA_n$, $n \geq 1$, we also have the following theorem which ensures that the number of indecomposable summands of a basic presilting object in $K^b(\operatorname{proj}\Lambda)$ is $\leq n$.

Theorem 2.3.5. *[4, Theorem 1.2, Theorem 2.15] Let Λ be a finite representation type algebra and P a presilting object in $K^b(\operatorname{proj}\Lambda)$. Then there exists some object $P' \in K^b(\operatorname{proj}\Lambda)$ such that $P \oplus P'$ is a silting object.*

2.3.3 Silting objects and t-structures

The study of silting objects is closely related to the study of t-structures as described in [13]. We will study this relationship here after introducing some basic definitions related to t-structures.

Definition 2.3.5. *A t-structure $(\mathcal{U},\mathcal{V})$ on $\mathcal{C}$ is said to be **bounded** if*

$$\bigcup_{n\in\mathbb{Z}} \Sigma^n\mathcal{U} = \mathcal{C} = \bigcup_{n\in\mathbb{Z}} \Sigma^n\mathcal{V}.$$

Definition 2.3.6. *Let $\mathcal{A}$ be an abelian category. Given an object $A\in\mathcal{A}$, a **Jordan-Hölder sequence** or **composition series** for A is a finite filtration, i.e., a finite sequence of subobject inclusions into X, starting with the zero objects*

$$0 = X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_{n-1} \hookrightarrow X_n = X$$

such that at each stage i, the quotient X_i/X_{i-1} (i.e., the coimage of the monomorphism $(X_{i-1}\hookrightarrow X_i)$ is a simple object of $\mathcal{A}$.

*If a Jordan-Hölder sequence for A exists, then A is said to be of **finite length**. Finally, $\mathcal{A}$ is said to be a **length category** if every object in it is of finite length.*

There is a natural poset structure on the set of t-structures on $\mathcal{C}$, defined as $(\mathcal{U},\mathcal{V}) \le (\mathcal{U}',\mathcal{V}')$ if $\mathcal{U}\subseteq\mathcal{U}'$, which is equivalent to $\mathcal{V}\supseteq\mathcal{V}'$.

Theorem 2.3.6. *[13, Theorem 6.1 and Theorem 7.13] The poset of equivalence classes of silting objects in $K^b(\operatorname{proj}\Lambda)$ is isomorphic to the poset of bounded t-structures on $\mathcal{D}^b(\operatorname{mod}\Lambda)$ with length heart.*

Although we will not look at the proof of the theorem here, we can describe the isomorphism explicitly. Let M be a silting object in $K^b(\operatorname{proj}\Lambda)$. Define the following full subcategories of $\mathcal{D}^b(\operatorname{mod}\Lambda)$.

$$\mathcal{U}_M = \{N\in\mathcal{D}^b(\operatorname{mod}\Lambda) \mid \operatorname{Hom}(M,\Sigma^m N) = 0,\ \forall\ m>0\},$$

$$\mathcal{V}_M = \{N\in\mathcal{D}^b(\operatorname{mod}\Lambda) \mid \operatorname{Hom}(M,\Sigma^m N) = 0,\ \forall\ m<0\}$$

Then $M\mapsto(\mathcal{U}_M,\Sigma^{-1}\mathcal{V}_M)$ is a poset isomorphism between the sets from Theorem 2.3.6. The inverse of this map factors through an intermediate collection of objects called the 'simple-minded collections' as described in [13].

The above map also restricts to give a bijection between some silting objects and an appropriate subset of bounded t-structures on $\mathcal{D}^b(\operatorname{mod}\Lambda)$. Let $\nu := \otimes^{\mathbf{L}}_{\Lambda} D\Lambda$ denote the Nakayama functor of $\mathcal{D}^b(\Lambda)$. For $m,n\in\mathbb{Z}$ and $m\le n$, define $\mathcal{D}^{[m,n]}_{-} = \mathcal{D}^{\le n}\cap\nu\mathcal{D}^{\ge m+1}$ and $\mathcal{D}^{[m,n]}_{+} = \mathcal{D}^{\le n}\cap\nu^{-1}\mathcal{D}^{\ge m-1}$.

Theorem 2.3.7. *[11, Corollary 3.6] There is a poset isomorphism*

$$(\operatorname{silt} K^b(\operatorname{proj}\Lambda))\cap\mathcal{D}^{[1-d,0]} \cong \{\textit{bounded } t\textit{-structures in } \mathcal{D}^b(\operatorname{mod}\Lambda) \textit{ with length hearts in } \mathcal{D}^{[-d,0]}_{-}\}$$

given by the restriction of the previous map $M\mapsto(\mathcal{U}_M,\Sigma^{-1}\mathcal{V}_M)$.

2.3.4 2-term silting objects and torsion pairs

For $d=2$, the bijection introduced above can be reformulated to give a bijection between 2-term silting objects in $K^b(\operatorname{proj}\Lambda)$ and functorially finite torsion pairs in $\operatorname{mod}\Lambda$. In this subsection, we will introduce the necessary terminology related to torsion pairs and describe an equivalent way of looking at this bijection. We first recall the definition of torsion pairs introduced in § 2.2.1.

Definition 2.3.7. *Let $\mathcal{A}$ be an abelian category. A pair $(\mathcal{T},\mathcal{F})$ of subcategories of $\mathcal{A}$ is called a **torsion pair** if*

- $\operatorname{Hom}(\mathcal{T},\mathcal{F}) = 0$, *and*
- $\mathcal{A} = \mathcal{T}\star\mathcal{F}$, *i.e., for each $A\in\mathcal{A}$, there exists a short exact sequence $0\to T\to A\to F\to 0$ in $\mathcal{A}$ with $T\in\mathcal{T}$ and $F\in\mathcal{F}$.*

In the case when $\mathcal{A} = \operatorname{mod}\Lambda$ for some finite-dimensional k-algebra Λ, the above definition can be reformulated as follows.

Definition 2.3.8. *[7, Chapter VI, Definition 1.1] Let $\mathcal{T}, \mathcal{F} \subseteq \operatorname{mod}\Lambda$ be two additive subcategories. The pair $(\mathcal{T}, \mathcal{F})$ is a **torsion pair** in $\operatorname{mod}\Lambda$ if*

1. $\operatorname{Hom}(T, F) = 0$ *for all* $T \in \mathcal{T}$ *and* $F \in \mathcal{F}$*;*
2. $\operatorname{Hom}(T, -) \mid_{\mathcal{F}} = 0$ *implies that* $T \in \mathcal{T}$*;*
3. $\operatorname{Hom}(-, F) \mid_{\mathcal{T}} = 0$ *implies that* $F \in \mathcal{F}$*.*

The notion of torsion pairs is in fact a generalization of the study of torsion groups and torsion-free groups in the category of abelian groups. Borrowing the terminology from this example, for a torsion pair $(\mathcal{T}, \mathcal{F})$, we call T a **torsion class** and $\mathcal{F}$ a **torsion-free class**.

Examples 2.3.2.
1. *An arbitrary class C of objects in $\mathcal{A}$ induces a torsion pair as follows: let $\mathcal{F} = \{N \mid \operatorname{Hom}_{\mathcal{A}}(-, N) \mid_C = 0\}$ and $T = \{M \mid \operatorname{Hom}_{\mathcal{A}}(M, -) \mid_{\mathcal{F}} = 0\}$. Then $(\mathcal{T}, \mathcal{F})$ is a torsion pair, and T is the smallest torsion class containing C. The dual construction gives the smallest torsion-free class containing C.*
2. *If $(\mathcal{T}, \mathcal{F})$ is a torsion pair in an abelian category $\mathcal{A}$, then $(\mathcal{F}^{op}, \mathcal{T}^{op})$ is a torsion pair in the abelian category $\mathcal{A}^{op}$.*

Torsion classes and torsion-free classes in module categories can be characterized intrinsically as stated in the following theorem.

Proposition 2.3.1. *[7, Chapter VI, Proposition 1.4] Let $\mathcal{T}, \mathcal{F}$ be additive subcategories of $\operatorname{mod}\Lambda$. Then $\mathcal{T}$ is the torsion class of some torsion pair $(\mathcal{T}, \mathcal{F})$ if and only if $\mathcal{T}$ is closed under quotients and extensions. Dually $\mathcal{F}$ is the torsion-free class of some torsion pair $(\mathcal{T}, \mathcal{F})$ if and only if $\mathcal{F}$ is closed under submodules and extensions.*

We will denote the set of all torsion pairs in $\operatorname{mod}\Lambda$ by $\operatorname{tors}\Lambda$. Note that the set $\operatorname{tors}\Lambda$ has a natural poset structure given by the inclusion of torsion classes, i.e., $(\mathcal{T}_1, \mathcal{F}_1) \leq (\mathcal{T}_2, \mathcal{F}_2)$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$, which is equivalent to $\mathcal{F}_1 \supseteq \mathcal{F}_2$. Moreover, using Theorem 2.3.1, we get that the intersection of torsion classes (torsion-free classes) is still a torsion class (torsion-free class) as the property of being closed under quotients (submodules) and extensions is closed under intersections. Before stating the next result, we need the following definition.

Definition 2.3.9. *A poset $(P, \leq)$ is called a **lattice** if every two-element subset $\{a, b\} \subseteq P$ has a **join**, i.e., a least upper bound, and a **meet**, i.e., a greatest lower bound. Moreover, it is called a **complete lattice** if every subset $S \subseteq P$ has a join and a meet.*

The above discussion gives us the following result immediately.

Proposition 2.3.2. *[19, Proposition 5.1] $\operatorname{tors}\Lambda$ is a complete lattice, with the meet given by the intersection of torsion classes and the join given by the intersection of torsion-free classes.*

We will now define a subset of $\operatorname{tors}\Lambda$ of 'functorially finite' torsion pairs. A subcategory $\mathcal{X}$ of an additive category $\mathcal{C}$ is said to be **contravariantly finite** in $\mathcal{C}$ if every object M of $\mathcal{C}$ admits a right $\mathcal{X}$-approximation. Dually, we say that $\mathcal{X}$ is **covariantly finite** if every object M of $\mathcal{C}$ admits a left $\mathcal{X}$-approximation. Furthermore, a subcategory $\mathcal{X}$ of $\mathcal{C}$ is said to be **functorially finite** in $\mathcal{C}$ if it is both contravariantly and covariantly finite in $\mathcal{C}$.

Theorem 2.3.8. *[17] Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\operatorname{mod}\Lambda$. Then $\mathcal{T}$ is functorially finite if and only if $\mathcal{F}$ is functorially finite.*

In this case, the pair $(\mathcal{T}, \mathcal{F})$ is called a **functorially finite torsion pair**.

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\operatorname{mod}\Lambda$. We say that $X \in \mathcal{T}$ (resp. $Y \in \mathcal{F}$) is Ext-**projective** (respectively Ext-**injective**) if $\operatorname{Ext}^1(X, \mathcal{T}) = 0$ (respectively $\operatorname{Ext}^1(\mathcal{T}, Y){=}0$). We denote by $P(\mathcal{T})$ the direct sum of one copy of each indecomposable Ext-projective object in $\mathcal{T}$ up to isomorphism. Dually we denote by $I(\mathcal{F})$ the direct sum of one copy of each indecomposable Ext-injective object in $\mathcal{F}$ up to isomorphism. We have the following result which gives a simpler characterization of functorially finite torsion pairs.

Proposition 2.3.3. *[17][2, Proposition 1.1] Let $(\mathcal{T},\mathcal{F})$ be a torsion pair in* $\operatorname{mod}\Lambda$*. Then the following are equivalent.*

1. *$(\mathcal{T},\mathcal{F})$ is functorially finite.*
2. *$\mathcal{T} = \operatorname{Fac} X$ for some X in $\operatorname{mod}\Lambda$.*
3. *$\mathcal{F} = \operatorname{Sub} Y$ for some Y in $\operatorname{mod}\Lambda$.*
4. *$\mathcal{T} = \operatorname{Fac} P(\mathcal{T})$.*
5. *$\mathcal{F} = \operatorname{Sub} I(\mathcal{F})$.*

We will denote the set of all functorially finite torsion pairs in $\operatorname{mod}\Lambda$ by $\operatorname{ftors}\Lambda$. We can now state the main theorem of this section.

Theorem 2.3.9. *[2, Theorem 2.7 and Theorem 3.2] There exists a poset isomorphism*

$$\phi : 2\text{-}\operatorname{silt}\Lambda \to \operatorname{ftors}\Lambda$$

given by $P \mapsto \operatorname{Fac}(H^0(P))$.

Since a subposet of a lattice might not be a lattice, $\operatorname{ftors}\Lambda$ is not necessarily a lattice. However, the following theorem guarantees that if this set is finite, then it is indeed a lattice.

Theorem 2.3.10. *[9, Theorem 1.2] Let Λ be a finite-dimensional algebra. Then, Λ is τ-tilting finite if and only if every torsion class (equivalently, torsion-free class) in* $\operatorname{mod}\Lambda$ *is functorially finite.*

Our main goal in this thesis is to generalize the above two results to $d\text{-}\operatorname{silt}\Lambda$ for $d \geq 2$. We will use the collection of torsion classes in an appropriate extriangulated category as a generalization of $\operatorname{tors}\Lambda$ for this purpose. We would then also like to show that the poset of torsion classes for these categories is a lattice. Before proving these results in the next chapter, we need a few results on the derived categories of hereditary algebras.

2.4 Derived categories of hereditary algebras

Definition 2.4.1. *An algebra Λ is called **hereditary** if all submodules of projective modules over Λ are again projective.*

Throughout this section, we will assume Λ to be a hereditary, basic finite-dimensional k-algebra. The following result gives a nice characterization of such algebras.

Proposition 2.4.1. *[6, Proposition I.2.28] A basic, finite-dimensional algebra Λ is hereditary if and only if $\Lambda \cong kQ$ with Q an acyclic quiver.*

We want to understand the indecomposable objects of $\mathcal{D}^b(\operatorname{mod}\Lambda)$ and the irreducible maps between these.

Definition 2.4.2. *A complex $X^\bullet = (X^i, d^i)$ is called a **stalk complex** if there exists some i_0 such that $X^{i_0} \neq 0$ and $X^i = 0$ for all $i \neq i_0$. In this case, the object X^{i_0} is called the **stalk**.*

Lemma 2.4.1. *[10, p. 49] Let $X^\bullet$ be an indecomposable object in $\mathcal{D}^b(\operatorname{mod}\Lambda)$. Then $X^\bullet$ is isomorphic to a stalk complex with an indecomposable stalk.*

Let Γ_Λ be the Auslander-Reiten (AR) quiver of Λ. Denote by Γ_i a copy of Γ_Λ for $i \in \mathbb{Z}$, by $\tilde{\Gamma}$ the quiver obtained from the disjoint union $\bigsqcup_{i\in\mathbb{Z}} \Gamma_i$ by adding an arrow from the injective module $I(a)$ in Γ_i to the projective module $P(b)$ in Γ_{i+1} for each arrow from b to a in Q. We can describe the above process informally as attaching copies of Γ_Λ one after another.

Proposition 2.4.2. *[10, p. 52] The Auslander-Reiten quiver of $\mathcal{D}^b(\operatorname{mod}\Lambda)$ is $\tilde{\Gamma}$.*

Example 2.4.1. *1. Let A_3 be the Dynkin quiver with linear orientation, i.e., $A_3 = 1 \to 2 \to 3$. Then the AR quiver of A_3 is given as*

$$\begin{tikzcd}
& & P_1 \arrow[dr] & & \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1
\end{tikzcd}$$

Hence the AR quiver of $\mathcal{D}^b(\operatorname{mod}\Lambda)$ is as follows:

$$\begin{tikzcd}
& & P_1 \arrow[dr] & & \Sigma P_3 \arrow[dr] & & \Sigma S_2 \arrow[dr] & & \Sigma I_1 \\
\cdots & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & \Sigma P_2 \arrow[ur] \arrow[dr] & & \Sigma I_2 \arrow[ur] & \cdots \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & \Sigma P_1 \arrow[ur] & &
\end{tikzcd}$$

2. The following example illustrates that for non-hereditary algebras, the AR quiver of the derived category can be very different from the above construction.

Let Q be the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ and $I = \langle \alpha\beta, \beta\alpha \rangle$ an admissible ideal of kQ. Then the AR quiver of the algebra $\Lambda = kQ/I$ is given by

$$\begin{tikzcd}
& P_2 \arrow[dr] & & P_1 \arrow[dr] & \\
S_1 \arrow[ur] & & S_2 \arrow[ur] & & S_1
\end{tikzcd}$$

where the two copies of S_1 have been identified. However the AR quiver of $\mathcal{D}^{[-1,0]}(\operatorname{mod}\Lambda)$ is given by

$$\begin{tikzcd}
& S_1 \to 0 \arrow[dr] & & P_2 \to 0 \arrow[dr] & & & & \\
0 \to S_2 \arrow[ur] \arrow[dr] & & P_2 \to P_1 \arrow[ur] \arrow[dr] & & S_2 \to 0 \arrow[dr] & & P_1 \to 0 \arrow[dr] & \\
& 0 \to P_1 \arrow[ur] & & 0 \to S_1 \arrow[ur] \arrow[dr] & & P_1 \to P_2 \arrow[ur] \arrow[dr] & & S_1 \to 0 \\
& & & & 0 \to P_2 \arrow[ur] & & 0 \to S_2 \arrow[ur] &
\end{tikzcd}$$

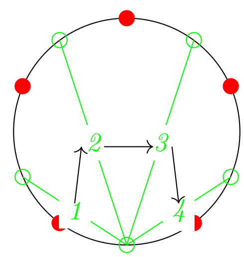
where we have identified the two copies of $0 \to S_2$ and the two copies of $S_1 \to 0$.

2.5 Geometric model for the derived categories of gentle algebras

As seen in the last example of the previous section, giving the description of the (bounded) derived category of an arbitrary finite dimensional algebra is a difficult task. We saw that for hereditary algebras this is possible using the description of the A-R quiver of the module category. There is yet another class of algebras, called **gentle algebras**, for which most of the information in the bounded derived category can be completely determined using a geometric model introduced in [15]. We present here the particular case of path algebras of linear quivers of type A_n, i.e., $1 \to 2 \to \cdots \to n$. We use a slightly different notation from [5].

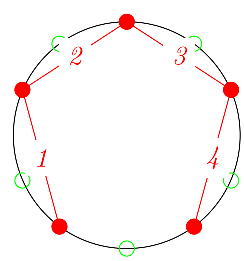
Let $n \geq 2$, $Q = 1 \to 2 \to \cdots \to n$, and $\mathbb{D}$ the 2-dimensional unit disc. Let $M = M_\circ \sqcup M_\bullet$ denote a collection of points on the boundary of $\mathbb{D}$, such that $M_\circ$ and $M_\bullet$ contain $n+1$ points each, and the points of $M_\circ$ and $M_\bullet$ alternate. The elements of $M_\circ$ and $M_\bullet$ will be represented by symbols $\circ$ and $\bullet$, respectively. Thus, $(\mathbb{D}, M, \varnothing)$ becomes a **marked surface** in the sense of [5]. We now describe a particular **admissible $\circ$-dissection**, Δ, of this marked surface, which we will use throughout this work. This is obtained by taking a fixed point in $M_\circ$ and connecting it to the other points in $M_\circ$ by arcs. The arcs are labeled from 1 to n from left to right. The angles between the arcs represent the arrows between the corresponding vertices.

Example 2.5.1. *The following figure illustrates the above dissection for $n = 4$. The black arrows represent the arrows of the quiver, viewed as angles between the corresponding arcs of the vertices.*



We also have the dual dissection $\Delta^\star$ of the above dissection, which is defined to be the unique admissible $\bullet$-dissection (up to homotopy) such that each arc of $\Delta^\star$ intersects exactly one arc of Δ ([5, Proposition 1.13]). We give each arc γ of $\Delta^\star$ the same label as the unique arc of Δ it intersects. We will denote this label by $l(\gamma)$. We also give each point p in the interior of γ the same label, i.e., $l(p) := l(\gamma)$. The angles between the arcs of $\Delta^\star$ are labeled by the arrows of the quiver in a dual way.

Example 2.5.2. *For $n = 4$, the dual dissection is as follows.*



Using the above dual dissection, we can construct all the indecomposable objects of $\mathcal{D}^b(\operatorname{mod} \Lambda)$ as described below.

Definition 2.5.1. *[5, Definition 1.8] A $\circ$-arc (or $\bullet$-arc) is a smooth map γ from the open interval $(0,1)$ to S such that its endpoints $\lim_{x\to 0}\gamma(x)$ and $\lim_{x\to 1}\gamma(x)$ are in $M_\circ$ (or in $M_\bullet$, respectively). The curve γ is required not to be contractible (at the limit) to a point in $M_\circ$ (or $M_\bullet$, respectively).*

We will only consider arcs up to homotopy. Two arcs are said to **intersect** if all choices of homotopic representatives intersect.

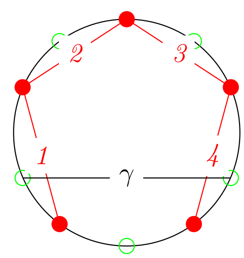
Definition 2.5.2. *[5, Definition 2.4] A* ***graded*** $\circ$*-arc* (γ, f) *is a* $\circ$*-arc* γ*, together with a function* $f : \gamma \cap \Delta^\star \to \mathbb{Z}$*, where* $\gamma \cap \Delta^\star$ *is the totally ordered set of intersection points of* γ *with* $\Delta^\star$*, where the order is induced from the direction of* γ*.*

The function f *is required to satisfy the following: if* p *and* q *are in* $\gamma \cap \Delta^\star$ *and* q *is the successor of* p*, then* γ *enters a polygon enclosed by* $\bullet$*-arcs of* $\Delta^\star$ *via* p *and leaves it via* q*. If the* $\circ$ *in this polygon is to the left of* γ*, then* $f(q) = f(p) - 1$*; otherwise,* $f(q) = f(p) + 1$*.*

In the above definition, we assume that all arcs intersect the arcs of $\Delta^\star$ minimally and transversally.

To each graded $\circ$-arc (γ, f), we can associate an indecomposable object in $\mathcal{D}^b(\operatorname{mod}\Lambda)$ as follows. For each $i \in \mathbb{Z}$, define the set $S_i := \{p \in \gamma \cap \Delta^\star \mid f(p) = i\}$. The complex $P^\bullet_{(\gamma,f)}$ is defined as follows. Set $P^i_{(\gamma,f)} := \oplus_{p \in S_i} P_{l(p)}$. As mentioned before, if p and q are in $\gamma \cap \Delta^\star$ and q is the successor of p, then γ enters a polygon enclosed by $\bullet$-arcs of $\Delta^\star$ via p and leaves it via q. If the $\circ$ in this polygon is to the left of γ, we have the map $P_{l(p)} \to P_{l(q)}$ which is a composition of the maps corresponding to the angles of this polygon. If the $\circ$ in this polygon is to the right of γ, then we have a map $P_{l(q)} \to P_{l(p)}$ which is again a composition of the maps corresponding to the angles of this polygon.

Example 2.5.3. *Continuing with the previous example for* $n = 4$*, we consider the graded* $\circ$*-arc* (γ, f) *with* $f(p) = 3, f(q) = 4$*, where* p, q *are the intersection points of* $\Delta^\star$ *and* γ*. Then* $P^\bullet_{(\gamma,f)} = \cdots \to 0 \to P_1 \to P_4 \to 0 \to \cdots$ *concentrated in degrees* $3, 4$*.*



The above construction gives all the indecomposable objects of $\mathcal{D}^b(\operatorname{mod} kA_n)$.

Theorem 2.5.1. *[15, Theorem 3.3] Graded* $\circ$*-arcs (upto homotopy) in the marked surface* $(\mathbb{D}, M, \varnothing)$ *are in bijection with indecomposable objects in* $\mathcal{D}^b(\operatorname{mod} kA_n)$ *for all* $n \geq 1$*.*

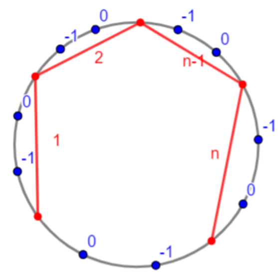
Moreover, we can understand maps between these indecomposable objects in terms of 'oriented graded intersections' of the corresponding graded $\circ$-arcs. The reader is invited to look at [15, § 3, § 4] for more details. We only present here the following lemma which is relevant to our current work.

Lemma 2.5.1. *[5, Lemma 3.5] Let* (γ, f) *and* (δ, g) *be two graded* $\circ$*-arcs, and let* $P^\bullet_{(\gamma,f)}$ *and* $P^\bullet_{(\delta,g)}$ *be the corresponding objects in* $\mathcal{D}^b(\operatorname{mod}\Lambda)$*. If* $P^\bullet_{(\gamma,f)} \oplus P^\bullet_{(\delta,g)}$ *is presilting, then* γ *and* δ *may only intersect at their endpoints.*

We can modify the above model to get a converse to the previous lemma. This, along with Theorem 2.3.4, will allow us to calculate the number of basic silting objects in type A_n by calculating the number of certain collections of arcs in the modified model. We present the case of 2-term silting objects in the following section. The general case of d-term silting objects is presented in § 3.3.

2.5.1 2-term silting complexes in type A_n

We now present a simplified model for the objects in $K^{[-1,0]}(\operatorname{proj}\Lambda)$, where $\Lambda = kA_n$ for some $n \geq 1$, which is a 'dual' version of the model introduced in [16]. We start with the admissible $\bullet$-dissection introduced above. Between any two consecutive red points, we mark two blue points, labeled -1 and 0 in the clockwise direction. We denote the label of a blue point p by $m(p)$.



Let γ be an arc connecting two blue points and intersecting the arcs of $\Delta^\star$ minimally and transversally. We define a function $f_\gamma : \gamma \cap \Delta^\star \to \mathbb{Z}$ as follows. For the first point p (in the finite total order) in $\gamma \cap \Delta^\star$, $f_\gamma(p) = m(\gamma(0))$. We then define f on all the other points via the method described in Definition 2.5.2. We say that the arc γ is a **slalom** if $f(q) = m(\gamma(0))$, where q is the last point in $\gamma \cap \Delta^\star$. We can associate an object $P^\bullet_{(\gamma, f_\gamma)}$ to the arc γ as explained before above Example 2.5.3. We then have the following immediate modification of Theorem 2.5.1.

Theorem 2.5.2. *Slaloms (up to homotopy) in the above model are in bijection with indecomposable objects in* $K^{[-1,0]}(\operatorname{proj} kA_n)$ *for all* $n \geq 1$.

Theorem 2.5.3. *Two slaloms* γ, γ' *intersect in the interior of* $\mathbb{D}$ *if and only if* $P^\bullet_{(\gamma, f_\gamma)} \oplus P^\bullet_{(\gamma', f_{\gamma'})}$ *is not a presilting object.*

Thus, using Theorem 2.3.4, the number of 2-term basic silting objects in $K^b(\operatorname{proj} kA_n)$ is the number of collections of n slaloms that do not intersect in the interior of $\mathbb{D}$. We denote this number by D_{n+1}. Note that such a collection will be maximal with respect to this property as the number of indecomposable summands of any presilting object is less than or equal to n. In the following figures, we have marked different sets of blue points with different colours for ease of reference (Figure 2.1a).

Let Γ be a collection of n slaloms that do not intersect in the interior of $\mathbb{D}$. Our first claim is that in such a collection, either of the green $0, -1$ has to be an endpoint of some slalom. This is because otherwise, the slalom in Figure 2.1b will contradict the maximality of the collection.

(a) (b)

Figure 2.1

We can now divide the problem into the following cases.

1. The green 0 is the endpoint of a slalom: Consider the leftmost slalom connected to the green 0, say γ. We have the following cases.

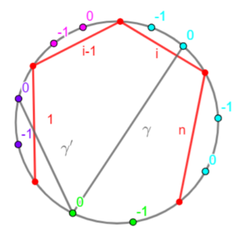
   (a) $P_\gamma = P_1$: From Figure 2.2a, we see that the arc γ divides the disc into 2 parts, the smaller one of which cannot contain any other slalom from our collection. Thus the remaining $n - 1$ slaloms of our collection lie in the other half. We note that in this case, there cannot be a slalom with an endpoint at the purple 0 as this would correspond to a complex in degrees $0, 1$, which is not allowed. Thus, removing the purple 0, we are reduced to the case of calculating 2-term silting

(a) (b)

Figure 2.2

objects in A_{n-1} as shown in Figure 2.2b. Thus the number of silting complexes, in this case, is D_n.

(b) $P_\gamma = P_i$ for some $i > 1$: We first note that, in this case, if the purple -1 is not connected to anything, we can add the following slalom γ' to our collection, contradicting its maximality.



Let γ'' be the rightmost slalom whose endpoint is the purple -1. Then $P_{\gamma'} = P_1 \to P_j$ for some $1 < j \leq i$. If $j < i$, then the pink slalom in Figure 2.3a contradicts the maximality of the collection. Thus $j = i$ (Figure 2.3b).

(a) (b)

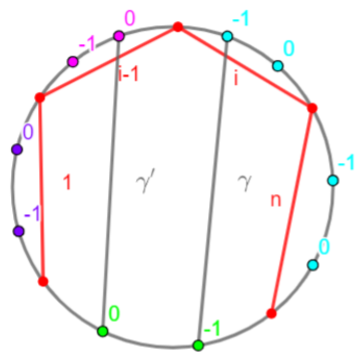
Figure 2.3

As argued in the previous case, we can relabel the numbered points in such a way that we are reduced to the problem of calculating collections of $i-2$ mutually non-intersecting slaloms in A_{i-2} (Figure 2.4a) and of $n-i$ mutually non-intersecting slaloms in A_{n-i} (Figure 2.4b). Thus the number of silting objects, in this case, is $\Sigma_{l=2}^{n} D_{l-1} D_{n-l+1}$.

2. The green 0 is not an endpoint of any slalom: Then the green -1 has to be an endpoint of some slalom. Consider the leftmost such, say, γ. Then $P_\gamma = P_i[1]$ for some $1 \leq i \leq n$. If $i > 1$, then the following slalom γ' will contradict the maximality of the collection. Thus $i = 1$, and a similar relabeling as Case 1(a) gives that the number of silting objects in this case is D_n.

(a) (b)

Figure 2.4



Thus the total number of basic 2-term silting objects in $K^b(\mathrm{proj}\, kA_n)$ is given by the recursive formula

$$D_{n+1} = \Sigma_{i=0}^{n} D_i D_{n-i},$$

with $D_0 = 1$. This is the same recurrence relation defining the **Catalan numbers**. Thus the number of basic 2-term silting objects in A_n is the Catalan number C_{n+1}. In § 3.3, we will calculate the number of basic 3-term silting objects in $K^b(\mathrm{proj}\, kA_n)$, which will turn out to be the **Fuss-Catalan numbers** (conjecturally), a well-studied generalization of Catalan numbers.

Chapter 3

Poset of d-term silting objects

As mentioned before, our main goal in this thesis is to generalize the results described in §2.3.4. In particular, we would like to show that the poset $d\text{-}\operatorname{silt}\Lambda$ is isomorphic to some appropriate subposet of the poset of torsion classes in an extriangulated category and that the poset of torsion classes, in this case, is a lattice.

For the rest of this chapter, Λ will denote a hereditary, basic, finite-dimensional k-algebra. Note that in this case, we know the structure of the derived category as described in §2.4.

3.1 d-term silting objects and s-torsion pairs

Let M be a d-term silting object in $K^b(\operatorname{proj}\Lambda)$. Recall that from Theorem 2.3.6, we have an injective poset homomorphism from the poset of equivalence classes of silting objects in $K^b(\operatorname{proj}\Lambda)$ to the poset of t-structures on $\mathcal{D}^b(\operatorname{mod}\Lambda)$. This map is given by

$$N \mapsto (\mathcal{U}_N, \Sigma^{-1}\mathcal{V}_N)$$

where

$$\mathcal{U}_N = \{N' \in \mathcal{D}^b(\operatorname{mod}\Lambda) \mid \operatorname{Hom}(N, \Sigma^m N') = 0,\ \forall\ m > 0\},$$
$$\mathcal{V}_N = \{N' \in \mathcal{D}^b(\operatorname{mod}\Lambda) \mid \operatorname{Hom}(N, \Sigma^m N') = 0,\ \forall\ m < 0\}.$$

Set $\mathcal{U}'_M := \mathcal{U}_M \cap \mathcal{D}^{[-(d-2),0]}$ and $\mathcal{V}'_M := \Sigma^{-1}\mathcal{V}_M \cap \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$.

Lemma 3.1.1. *$(\mathcal{U}'_M, \mathcal{V}'_M)$ is an s-torsion pair in the extriangulated category $\mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$.*

Proof. Let $\mathcal{C} := \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$. Since $(\mathcal{U}_M, \Sigma^{-1}\mathcal{V}_M)$ is a t-structure in $\mathcal{D}^b(\operatorname{mod}\Lambda)$, $\operatorname{Hom}(\mathcal{U}'_M, \mathcal{V}'_M) = 0$. Moreover, since $\mathcal{U}_M$ is closed under positive shifts, $\mathbb{E}^{-1}(\mathcal{U}'_M, \mathcal{V}'_M) = 0$. We want to show that $\mathcal{C} = \mathcal{U}'_M \star \mathcal{V}'_M$.

We first claim that $\mathcal{D}^{\leq(-d+1)}(\operatorname{mod}\Lambda) \subseteq \mathcal{U}_M$. To prove this, by the definition of $\mathcal{U}_M$, we need to show that $\operatorname{Hom}(\Sigma^{-m}M, \mathcal{D}^{\leq(-d+1)}(\operatorname{mod}\Lambda)) = 0$ for all $m > 0$. Let $X \in \mathcal{D}^{\leq(-d+1)}(\operatorname{mod}\Lambda)$. Without loss of generality, we can assume that X is indecomposable. Using Lemma 2.4.1, we get that $X \cong N[i]$ for some indecomposable module $N \in \operatorname{mod}\Lambda$ and $i \geq -d+1$. Since $m > 0$, $\operatorname{Hom}(\Sigma^{-m}M, X) = 0$.

Our next claim is that $\mathcal{D}^{\geq 1}(\operatorname{mod}\Lambda) \subseteq \Sigma^{-1}\mathcal{V}_M$. Note that this is equivalent to showing that $\mathcal{D}^{\geq 0}(\operatorname{mod}\Lambda) \subseteq \mathcal{V}_M$. By definition of $\mathcal{V}_M$, we need to prove that $\operatorname{Hom}(\Sigma^{-m}M, \mathcal{D}^{\geq 0}(\operatorname{mod}\Lambda)) = 0$ for all $m < 0$. Let $X \in \mathcal{D}^{\geq 0}(\operatorname{mod}\Lambda)$. Without loss of generality, we can assume that X is indecomposable. Using Lemma 2.4.1, we get that $X \cong N[i]$ for some indecomposable module $N \in \operatorname{mod}\Lambda$ and $i \leq 0$. Since $m < 0$, $\operatorname{Hom}(\Sigma^{-m}M, X) = 0$.

We are now ready to prove that $\mathcal{C} = \mathcal{U}'_M \star \mathcal{V}'_M$. Let $Z \in \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$. Then using Lemma 2.4.1, $Z \cong \oplus_{i=1}^{m} Z_i$ with $Z_i \cong T_i^{m_i}$ for some indecomposable modules T_i in $\operatorname{mod}\Lambda$ and $0 \leq m_i \leq d-2$. If $Z \in \mathcal{U}_M$, then we have a conflation

$$Z \to Z \to 0,$$

which implies that $Z \in \mathcal{U}'_M \star \mathcal{V}'_M$. Similarly, if $Z \in \Sigma^{-1}\mathcal{V}_M$, then we have a conflation

$$0 \to Z \to Z,$$

implying that $Z \in \mathcal{U}'_M \star \mathcal{V}'_M$.

Finally, suppose $Z \notin \mathcal{U}_M, \Sigma^{-1}\mathcal{V}_M$. Since $(\mathcal{U}_M, \Sigma^{-1}\mathcal{V}_M)$ is a t-structure in $\mathcal{D}^b(\operatorname{mod}\Lambda)$, there exists a conflation

$$U \xrightarrow{u} Z \xrightarrow{v} V$$

with $U \in \mathcal{U}_M$ and $V \in \Sigma^{-1}\mathcal{V}_M$. We decompose $U = \oplus_i U_i$ with U_i indecomposable objects in $\mathcal{D}^b(\operatorname{mod}\Lambda)$ and set u_i to be the composition $U_i \hookrightarrow \oplus_i U_i \xrightarrow{u} Z$. We can assume that $u_i \neq 0$ for all i. Since U_i is an indecomposable in $\mathcal{D}^b(\operatorname{mod}\Lambda)$, Lemma 2.4.1 implies that $U_i \cong \Sigma^{l_i} N_i$ with N_i an indecomposable in $\operatorname{mod}\Lambda$ and $l_i \in \mathbb{Z}$. If $l_i \geq d-1$, then $\operatorname{Hom}(\Sigma^{l_i} N_i, Z) = 0$ which implies that $u_i = 0$, a contradiction. Therefore, $l_i \leq d-2$ for all i. Now if $l_i < 0$, then $\Sigma^{l_i} N_i \in \mathcal{D}^{\geq 1}(\operatorname{mod}\Lambda) \subseteq \Sigma^{-1}\mathcal{V}_M$ by the second claim above. Since $\mathcal{U}_M$ is closed under direct summands, $U_i \in \mathcal{U}_M \cap \Sigma^{-1}\mathcal{V}_M = 0$, a contradiction. Therefore, $l_i \geq 0$ and $U_i \in \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$ for all i. Hence, $U \in \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$.

Similarly, we can decompose $V = \oplus_j V_j$ with V_j indecomposable objects in $\mathcal{D}^b(\operatorname{mod}\Lambda)$ and set v_i to be the composition $Z \xrightarrow{v} \oplus_j V_j \twoheadrightarrow V_j$. Again, we can assume that $v_j \neq 0$ for all j. By Lemma 2.4.1, $V_j \cong \Sigma^{n_j} L_j$ for some indecomposable modules L_j in $\operatorname{mod}\Lambda$ and $n_j \in \mathbb{Z}$. If $n_j \leq -1$, then $\operatorname{Hom}(Z, \Sigma^{n_j} L_j) = 0$, a contradiction to the fact that $v_j \neq 0$. Therefore, $n_j \geq 0$. If $n_j > d-2$, then $\Sigma^{n_j} L_j \in \mathcal{D}^{\leq -d+1}(\operatorname{mod}\Lambda) \subseteq \mathcal{U}_M$ using the first claim. Since $\Sigma^{-1}\mathcal{V}_M$ is closed under direct summands, $\Sigma^{n_j} L_j \in \mathcal{U}_M \cap \Sigma^{-1}\mathcal{V}_M$, a contradiction. Therefore, $n_j \leq d-2$ for all j and $V_j \in \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$. Hence $V \in \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$.

Therefore $Z \in \mathcal{U}'_M \star \mathcal{V}'_M$ and $(\mathcal{U}'_M, \mathcal{V}'_M)$ is an s-torsion pair. □

Theorem 3.1.1. *There exists an injective poset homomorphism*

$$\phi : d\text{-}\operatorname{silt}\Lambda \to \operatorname{stors}\mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$$

given by $M \mapsto (\mathcal{U}'_M, \mathcal{V}'_M)$.

Proof. Let ϕ be the map $M \mapsto (\mathcal{U}'_M, \mathcal{V}'_M)$ which is well-defined by Lemma 3.1.1. Suppose $M, N \in d\text{-}\operatorname{silt}\Lambda$ such that $M \leq N$. Since $M \mapsto (\mathcal{U}_M, \Sigma^{-1}\mathcal{V}_M)$ is a map of posets, $\mathcal{U}_M \subseteq \mathcal{U}_N$. And hence $\mathcal{U}'_M \subseteq \mathcal{U}'_N$.

We now show that ϕ is injective. Suppose M, M' are two distinct d-term silting objects. Since the map $M \mapsto (\mathcal{U}_M, \Sigma^{-1}\mathcal{V}_M)$ is injective, we get that $\mathcal{U}_M \neq \mathcal{U}_{M'}$. Without loss of generality, we can assume that there exists some $U \in \mathcal{U}_M \setminus \mathcal{U}_{M'}$. Since $\mathcal{U}_M$ and $\mathcal{U}_{M'}$ are closed under direct sums and summands, we can assume that U is indecomposable. Using Lemma 2.4.1, we get that $U \cong A[n]$ for some indecomposable A in $\operatorname{mod}\Lambda$. Since $\mathcal{D}^{\geq 1}(\operatorname{mod}\Lambda) \subseteq \Sigma^{-1}\mathcal{V}_M$ and $\Sigma^{-1}\mathcal{V}_M \cap \mathcal{U}_M = 0$, we get that $n \geq 0$. If $n \geq d-1$, then using the first claim in the previous proof, we get that $U \in \mathcal{U}_M \cap \mathcal{U}_{M'}$, a contradiction. Therefore $n \leq d-2$ and $U \in \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$. Thus $U \in \mathcal{U}'_M \setminus \mathcal{U}'_{M'}$ and ϕ is injective. □

3.2 Torsion classes in s-torsion pairs

Let Λ be a hereditary algebra. Consider $\mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$ as an extriangulated category, where $d \geq 2$. We want to give a characterization of subcategories of $\mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$ which can be obtained as torsion classes of some s-torsion pairs. Moreover, we want these conditions to be closed under the intersection of subcategories in order to prove a lattice structure on s-torsion pairs.

Theorem 3.2.1. *Let Λ be a hereditary algebra of finite representation type and $\mathcal{T} \subset \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$ an additive subcategory satisfying the following conditions:*

1. *$\mathcal{T}$ is closed under cones, i.e., if* $M' \xrightarrow{x} M \xrightarrow{y} M'' \overset{\delta}{\dashrightarrow}$ *is a conflation in $\mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$, and $M, M' \in \mathcal{T}$, then $M'' \in \mathcal{T}$;*
2. $\mathcal{T} = {}^{\perp}(\mathcal{T}^{\perp})$.

Then $(\mathcal{T}, \mathcal{T}^{\perp})$ is an s-torsion pair.

Proof. Let $\mathcal{C} := \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$. Then, by definition, $\mathcal{C}(\mathcal{T},\mathcal{T}^{\perp}) = 0$. We need to prove that for all $M \in \mathcal{T}$, $\mathbb{E}^{-1}(M,\mathcal{T}^{\perp}) = 0$, which is equivalent to proving that $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod}\Lambda)}(M[1],\mathcal{T}^{\perp}) = 0$. Without loss of generality, we can assume that M is indecomposable. This implies that $M \simeq N[t]$ for some $0 \le t \le d-2$ and N an indecomposable module in $\operatorname{mod}\Lambda$. If $0 \le t \le d-3$, then $M \to 0 \to M[1]$ is a conflation in $\mathcal{C}$. Hence, using the first hypothesis on $\mathcal{T}$, we get that $M[1] \in \mathcal{T}$, which implies that $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod}\Lambda)}(M[1],\mathcal{T}^{\perp}) = 0$. If $t = d-2$, then $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod}\Lambda)}(M[1],\mathcal{T}^{\perp}) = 0$ as $\mathcal{T}^{\perp} \subset \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$.

Now we need to show that $\mathcal{C} = \mathcal{T} \star \mathcal{T}^{\perp}$. Let $M \in \mathcal{C}$. Without loss of generality, we can again assume M to be indecomposable. This implies that $M \cong N[t]$ for some $0 \le t \le d-2$ and N an indecomposable module in $\operatorname{mod}\Lambda$. Since Λ is of finite representation type, the category $\mathcal{T}$ has only finitely many indecomposables, which implies that there exists a minimal right $\mathcal{T}$-approximation of M, say $f : T \to M$. We claim that $C(f) \in \mathcal{T}^{\perp}$.

Our first step is to prove that $C(f) \in \mathcal{C}$. Since f is a minimal morphism, $T \simeq T_1 \oplus T_2$ for some $T_1 \in \operatorname{mod}\Lambda[t]$ and $T_2 \in \operatorname{mod}\Lambda[t-1]$, where $\operatorname{mod}\Lambda[l]$ denotes the category $\mathcal{D}^{[-l,-l]}(\operatorname{mod}\Lambda)$ which is equivalent to $\operatorname{mod}\Lambda$ for any $l \ge 0$. Let $i_1 : T_1 \to T$ and $i_2 : T_2 \to T$ denote the canonical sections, and set $f_1 = f \circ i_1$ and $f_2 = f \circ i_2$. Suppose $0 \le t \le d-3$. Then we have a triangle $M \to C(f) \to T[1] \to M[1]$. Since $M, T[1] \in \mathcal{C}$, therefore $C(f) \in \mathcal{C}$. Now let $t = d-2$. We claim that $f_1 : T_1 \to M$ is a minimal right $\mathcal{T} \cap \operatorname{mod}\Lambda[d-2]$-approximation of M in $\operatorname{mod}\Lambda[d-2]$. To prove this, suppose there is a map $g : X \to M$ with $X \in \mathcal{T} \cap \operatorname{mod}\Lambda[d-2]$. Since f is a right $\mathcal{T}$-approximation of M, there exists some $h : X \to T$ such that $f \circ h = g$.

$$\begin{tikzcd}
 & X \arrow[dl, dashed, "\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}"'] \arrow[d, "g"] \\
T_1 \oplus T_2 \arrow[r, "\begin{bmatrix} f_1 & f_2 \end{bmatrix}"'] & M
\end{tikzcd}$$

Let $h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$ with $h_1 : X \to T_1$ and $h_2 : X \to T_2$. Since $X \in \operatorname{mod}\Lambda[d-2]$, $h_2 = 0$ and $g = f_1 \circ h_1$. Moreover, if $f_1 = f_1 \circ u$ for some $u : T_1 \to T_1$, then $f = f \circ \begin{bmatrix} u & 0 \\ 0 & id \end{bmatrix}$, which implies that $\begin{bmatrix} u & 0 \\ 0 & id \end{bmatrix}$ is an isomorphism, and hence, u is an isomorphism. Thus, f_1 is a minimal right $\mathcal{T} \cap \operatorname{mod}\Lambda[d-2]$-approximation of M in $\operatorname{mod}\Lambda[d-2]$.

We now claim that $\mathcal{T} \cap \operatorname{mod}\Lambda[d-2]$ is a torsion class in $\operatorname{mod}\Lambda[d-2]$. For this, it is enough to show that $\mathcal{T} \cap \operatorname{mod}\Lambda[d-2] = {}^{\perp_{\operatorname{mod}\Lambda[d-2]}}\mathcal{F}$ for some $\mathcal{F} \subset \operatorname{mod}\Lambda[d-2]$, where ${}^{\perp_{\operatorname{mod}\Lambda[d-2]}}\mathcal{F} := \{X \in \operatorname{mod}\Lambda[d-2] \mid \operatorname{Hom}(X,\mathcal{F}) = 0\}$. Define $\mathcal{F} = \mathcal{T}^{\perp} \cap \operatorname{mod}\Lambda[d-2]$. Let $Z \in {}^{\perp_{\operatorname{mod}\Lambda[d-2]}}\mathcal{F}$. Since $Z \in \operatorname{mod}\Lambda[d-2]$, $\operatorname{Hom}(Z, \operatorname{mod}\Lambda[i]) = 0$ for all $0 \le i \le d-3$. This implies that $Z \in {}^{\perp}(\mathcal{T}^{\perp}) = \mathcal{T}$. Conversely, if $Z \in \mathcal{T} \cap \operatorname{mod}\Lambda[d-2]$, then $\operatorname{Hom}(Z,\mathcal{F}) = 0$ and $Z \in {}^{\perp_{\operatorname{mod}\Lambda[d-2]}}\mathcal{F}$. Thus $\mathcal{T} \cap \operatorname{mod}\Lambda[d-2] = {}^{\perp_{\operatorname{mod}\Lambda[d-2]}}\mathcal{F}$, and $\mathcal{T}$ is a torsion class. Since the minimal right approximation of an object by a torsion class is a monomorphism, we get that $f_1 : H^{-d+2}(T_1) = T_1 \to H^{-d+2}(M) = N$ is a monomorphism.

Applying the octahedral axiom to the triangles $T_1 \to T \to T_2 \to T_1$, $T \to M \to C(f) \to T[1]$, and $T_1 \to M \to C(f_1) \to T_1[1]$, where $C(f)$ and $C(f_1)$ denote the cones of f and f_1 respectively, we get the following diagram,

$$\begin{tikzcd}
T_1 \arrow[r, equal] \arrow[d, "i_1"'] & T_1 \arrow[d, "f_1"] & & \\
T = T_1 \oplus T_2 \arrow[r, "f"] \arrow[d] & M \arrow[r] \arrow[d] & C(f) \arrow[r] \arrow[d, equal] & T[1] \arrow[d] \\
T_2 \arrow[r] \arrow[d] & C(f_1) \arrow[r] \arrow[d] & C(f) \arrow[r] & T_2[1] \\
T_1[1] \arrow[r, equal] & T_1[1] & &
\end{tikzcd}$$

and the triangle $T_2 \to C(f_1) \to C(f) \to T_2[1]$. Using the long exact sequence of cohomology for this triangle, we get

$$0 = H^{-d+1}(M) \to H^{-d+1}(C(f_1)) \to H^{-d+2}(T_1) \to H^{-d+2}(M).$$

Since the last map in the above sequence is injective, $H^{-(d+1)}(C(f_1)) = 0$. Hence $C(f_1) \in \operatorname{mod}\Lambda[d-2]$. Using the triangle $C(f_1) \to C(f) \to T_2[1] \to C(f_1)[1]$ and the fact that $\operatorname{mod}\Lambda[d-2]$ is closed under extensions, we get that $C(f) \in \operatorname{mod}\Lambda[d-2]$, and hence $C(f) \in \mathcal{C}$.

To show that $\mathcal{C} = \mathcal{T} \star \mathcal{T}^{\perp}$, the only thing that remains is to check that $C(f) \in \mathcal{T}^{\perp}$. Let $X \in \mathcal{T}$. We want to show that $\mathcal{C}(X, C(f)) = 0$. Using the conflation $T \to M \to C(f)$ in the extriangulated category $\mathcal{C}$, we get the following long exact sequence

$$\mathcal{C}(X,T) \xrightarrow{\mathcal{C}(X,f)} \mathcal{C}(X,M) \to \mathcal{C}(X,C(f)) \to \mathbb{E}(X,T) \to \mathbb{E}(X,M).$$

Since f is a right $\mathcal{T}$-approximation of M, the first map is surjective. Thus it is enough to show that $\mathbb{E}(X,f) : \mathbb{E}(X,T) \to \mathbb{E}(X,M)$ is injective to conclude that $\mathcal{C}(X,C(f)) = 0$. Let $X[-1] \xrightarrow{g} T \in \mathbb{E}(X,T)$ such that $f \circ g = 0$. We want to show that $g = 0$. Since $X[-1] \xrightarrow{g} T \xrightarrow{g'} C(g) \to X$ is a triangle, hence $T \xrightarrow{g'} C(g) \to X \to T[1]$ is also a triangle. The second condition of the hypothesis implies that $\mathcal{T}$ is closed under extensions, and since $T, X \in \mathcal{T}$, we get that $C(g) \in \mathcal{T}$. Associated to the above triangle, we get the following exact sequence

$$\operatorname{Hom}(C(g),M) \xrightarrow{\operatorname{Hom}(g',M)} \operatorname{Hom}(T,M) \xrightarrow{\operatorname{Hom}(g,M)} \operatorname{Hom}(X[-1],M) \to \mathbb{E}(C(g),M) \to \cdots$$

where $f \in \operatorname{Ker}(\operatorname{Hom}(g,M))$. Thus $f \in \operatorname{Im}(\operatorname{Hom}(g',M))$ and there exists $h : C(g) \to M$ such that $h \circ g' = f$. Moreover, since $C(g) \in \mathcal{T}$, and f is a minimal right $\mathcal{T}$-approximation of M, there exists some $h' : C(g) \to T$ such that $f \circ h' = h$.

$$\begin{tikzcd} & C(g) \arrow[dl, dashed, "h'"'] \arrow[d, "h"] \\ T \arrow[r, "f"'] & M \end{tikzcd}$$

Thus, $f \circ h' \circ g' = f$. Since f is right minimal, this implies that $h' \circ g' : T \to T$ is an isomorphism. Let u be the inverse of this morphism. Using the exact sequence

$$\operatorname{Hom}(C(g),T) \xrightarrow{k_1} \operatorname{Hom}(T,T) \xrightarrow{k_2} \operatorname{Hom}(X[-1],T),$$

we get that $Id_T = (u \circ h') \circ g' \in \operatorname{Im}(k_1) = \operatorname{Ker}(k_2)$, which implies that $g = 0$.

Thus there is a conflation $T \xrightarrow{f} M \to C(f)$ in $\mathcal{C}$ with $T \in \mathcal{T}$ and $C(f) \in \mathcal{T}^{\perp}$. □

We also have the following dual to the above theorem.

Theorem 3.2.2. *Let Λ be a hereditary algebra of finite representation type and $\mathcal{F} \subset \mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$ an additive subcategory satisfying the following conditions:*

1. *$\mathcal{F}$ is closed under cocones, i.e., if $M' \xrightarrow{x} M \xrightarrow{y} M'' \overset{\delta}{\dashrightarrow}$ is a conflation in $\mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$, and $M, M'' \in \mathcal{F}$, then $M' \in \mathcal{F}$;*
2. *$\mathcal{F} = ({}^{\perp}\mathcal{F})^{\perp}$.*

Then $({}^{\perp}\mathcal{F}, \mathcal{F})$ is an s-torsion pair.

The above conditions characterizing a torsion class or a torsion-free class are closed under intersections, and hence the set of torsion classes in $\mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda)$ forms a lattice.

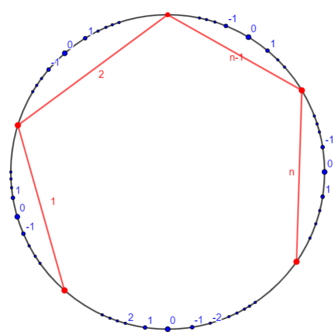
Corollary 3.2.1. *Let Λ be a hereditary algebra of finite representation type and set $\mathcal{C} := D^{[-(d-2),0]}(\operatorname{mod}\Lambda)$. Then the poset* $\operatorname{stors}\mathcal{C}$ *is a lattice.*

Proof. Let $(\mathcal{T}_1, \mathcal{F}_1)$, $(\mathcal{T}_2, \mathcal{F}_2)$ be two torsion pairs in $\operatorname{stors}\mathcal{C}$. Then $\mathcal{T}_1 \cap \mathcal{T}_2$ is an additive subcategory closed under cones as both $\mathcal{T}_1$ and $\mathcal{T}_2$ are. We claim that $\mathcal{T}_1 \cap \mathcal{T}_2 = {}^{\perp}((\mathcal{T}_1 \cap \mathcal{T}_2)^{\perp})$. By definition, $\mathcal{T}_1 \cap \mathcal{T}_2 \subseteq {}^{\perp}((\mathcal{T}_1 \cap \mathcal{T}_2)^{\perp})$. Now let $X \in {}^{\perp}((\mathcal{T}_1 \cap \mathcal{T}_2)^{\perp})$. Then $X \in {}^{\perp}(\mathcal{T}_1^{\perp}), {}^{\perp}(\mathcal{T}_2^{\perp})$ as $\mathcal{T}_1 \cap \mathcal{T}_2 \subseteq \mathcal{T}_1, \mathcal{T}_2$. However, since $\mathcal{T}_1, \mathcal{T}_2$ are torsion classes, ${}^{\perp}(\mathcal{T}_1^{\perp}) = \mathcal{T}_1$ and ${}^{\perp}(\mathcal{T}_2^{\perp}) = \mathcal{T}_2$. Therefore $X \in \mathcal{T}_1, \mathcal{T}_2 \implies X \in \mathcal{T}_1 \cap \mathcal{T}_2$. Thus $\mathcal{T}_1 \cap \mathcal{T}_2 = {}^{\perp}((\mathcal{T}_1 \cap \mathcal{T}_2)^{\perp})$ and $\mathcal{T}_1 \cap \mathcal{T}_2$ satisfies the hypotheses of Theorem 3.2.1. Hence $(\mathcal{T}_1 \cap \mathcal{T}_2, (\mathcal{T}_1 \cap \mathcal{T}_2)^{\perp})$ is an s-torsion pair which is the meet of $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$. Dually, the additive subcategory $\mathcal{F}_1 \cap \mathcal{F}_2$ satisfies the hypotheses of Theorem 3.2.2, and hence $({}^{\perp}(\mathcal{F}_1 \cap \mathcal{F}_2), \mathcal{F}_1 \cap \mathcal{F}_2)$ is an s-torsion pair in $\mathcal{C}$. Since $\mathcal{F}_1 \cap \mathcal{F}_2 \subseteq \mathcal{F}_1, \mathcal{F}_2$, therefore $\mathcal{T}_1 = {}^{\perp}\mathcal{F}_1, \mathcal{T}_2 = {}^{\perp}\mathcal{F}_2 \subseteq {}^{\perp}(\mathcal{F}_1 \cap \mathcal{F}_2)$. Moreover, if $(\mathcal{T}, \mathcal{F})$ is an s-torsion pair such that $(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2) \leq (\mathcal{T}, \mathcal{F})$, i.e., $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{T}$, then $\mathcal{F} \subseteq \mathcal{F}_1, \mathcal{F}_2 \implies \mathcal{F} \subseteq \mathcal{F}_1 \cap \mathcal{F}_2$. Thus $\mathcal{T} = {}^{\perp}\mathcal{F} \supseteq {}^{\perp}(\mathcal{F}_1 \cap \mathcal{F}_2)$, and $({}^{\perp}(\mathcal{F}_1 \cap \mathcal{F}_2), \mathcal{F}_1 \cap \mathcal{F}_2) \leq (\mathcal{T}, \mathcal{F})$. This proves that $({}^{\perp}(\mathcal{F}_1 \cap \mathcal{F}_2), \mathcal{F}_1 \cap \mathcal{F}_2)$ is the join of $(\mathcal{T}_1, \mathcal{F}_1)$, $(\mathcal{T}_2, \mathcal{F}_2)$. $\square$

3.3 d-term silting objects in type A_n

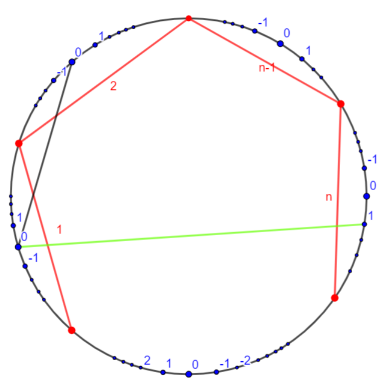
In this section, we will generalize the model for 2-term complexes in $K^b(\operatorname{proj}\Lambda)$ introduced in §2.5 to a model for the entire category $K^b(\operatorname{proj}\Lambda)$ where Λ is the path algebra of a linear quiver of type A_n $(1 \to 2 \to \cdots \to n)$. This will allow us to calculate the explicit number of basic d-term silting objects for these algebras.

Let $n \geq 2$. We start with the admissible $\bullet$-dissection, $\Delta^\star$, of the marked surface $(\mathbb{D}, M, \varnothing)$ as introduced before Example 2.5.2. Between any two adjacent red points, we mark blue points indexed by $\mathbb{Z}$ in the clockwise direction. We will denote the index of a blue point p by $m(p)$.



Let γ be an arc connecting two blue points and intersecting (non-trivially) the arcs of $\Delta^\star$ minimally and transversally. We define a function $f_\gamma : \gamma \cap \Delta^\star \to \mathbb{Z}$ as follows. For the first point p (in the finite total order) in $\gamma \cap \Delta^\star$, $f_\gamma(p) = m(\gamma(0))$. We then define f_γ on all the subsequent points via the method described in Definition 2.5.2. We say that the arc γ is a **slalom** if $f_\gamma(q) = m(\gamma(1))$, where q is the last point in $\gamma \cap \Delta^\star$.

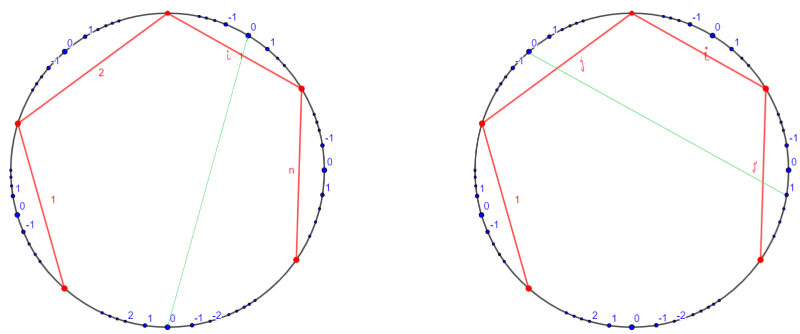
Example 3.3.1. *In the following figure, the green arc is a slalom while the black arc is not.*



We now describe a way to associate an object in $K^b(\operatorname{proj}\Lambda)$ to a slalom γ. For each $i \in \mathbb{Z}$, define the set $S_i := \{p \in \gamma \cap \Delta^\star \mid f_\gamma(p) = i\}$. The complex $P_\gamma^\bullet$ is defined as follows. Set $P_\gamma^i := \oplus_{p \in S_i} P_{l(p)}$. As mentioned before, if p and q are in $\gamma \cap \Delta^\star$ and q is the successor of p, then γ enters a polygon enclosed by $\bullet$-arcs of $\Delta^\star$ via p and leaves it via q. If the red points in this polygon are to the left of γ, we have the map $P_{l(p)} \to P_{l(q)}$ which is a composition of the maps corresponding to the angles of this polygon. If the red points in this polygon are to the right of γ, then we have a map $P_{l(q)} \to P_{l(p)}$ which is again a composition of the maps corresponding to the angles of this polygon.

Example 3.3.2. *Let γ denote the green arc in Example 3.3.1. Then $P_\gamma = \cdots \to 0 \to P_1 \to P_n \to 0 \to \cdots$ concentrated in degrees $0, 1$.*

Since we ask for the slaloms to intersect the arcs of $\Delta^\star$ minimally and transversally, it is easy to see that for each slalom γ, $\gamma \cap \Delta^\star$ contains either one or two points.



In the first case, the complex associated to γ is of the form $P_i[s]$ for some $1 \le i \le n$ and $s \in \mathbb{Z}$. In the second case, it is of the form $\cdots \to 0 \to P_j \xrightarrow{f} P_{j' \to 0 \to \cdots}$ with $1 \le j < j' \le n$, f the unique non-zero map (upto scalar multiplication) from $P_j \to P_{j'}$, and concentrated in degrees $s, s+1$ for some $s \in \mathbb{Z}$. Using Lemma 2.4.1, we know that these are all the indecomposable objects in $K^b(\operatorname{proj}\Lambda)$. Thus we have the following proposition.

Proposition 3.3.1. *There is a bijection between the indecomposable objects in $K^b(\operatorname{proj}\Lambda)$ and the slaloms (upto homotopy) in the marked surface $(\mathbb{D}, M, \varnothing)$ with the dissection $\Delta^\star$.*

We would now like to describe morphisms and extensions between the indecomposable objects in $K^b(\operatorname{proj}\Lambda)$ in terms of intersections of the corresponding slaloms.

Proposition 3.3.2. *Let γ, γ' be two slaloms with a common endpoint. Then exactly one of $\operatorname{Hom}(P_\gamma, P_{\gamma'})$ and $\operatorname{Hom}(P_{\gamma'}, P_\gamma)$ is non-zero. Moreover $\operatorname{Ext}^i(P_\gamma, P_{\gamma'}) = \operatorname{Ext}^i(P_{\gamma'}, P_\gamma) = 0$ for all $i \neq 0$.*

Proof. We have the following cases for γ, γ' to have a common endpoint.

1. $|\gamma \cap \Delta^\star| = |\gamma' \cap \Delta^\star| = 2$: In this case, $P_\gamma = P_i \to P_j$ for some $1 \le i < j \le n$ and concentrated in degrees $t, t-1$ for some $t \in \mathbb{Z}$. Similarly, $P_{\gamma'} = P_{i'} \to P_{j'}$ for some $1 \le i' < j' \le n$ and concentrated in degrees $t', t'-1$ for some $t' \in \mathbb{Z}$. Since they have a common endpoint, we have the following possibilities.

   - $t = t'$ and $j = j'$: In this case, wlog, we can assume that $i < i'$. Then we have a map from $P_i \to P_{i'}$ which makes the following diagram commute.

$$\begin{tikzcd} P_i \arrow[r] \arrow[d] & P_j \arrow[d, equal] \\ P_{i'} \arrow[r] & P_{j'} \end{tikzcd}$$

   Note that this map is not chain homotopic to zero as there is no non-zero map from $P_j \to P_{i'}$ since $j > i'$. One can similarly verify that the other parts of the result also hold.

- $t = t'$ and $i = i'$: Again, wlog, we can assume that $j < j'$. Then we have a map from $P_j \to P_{j'}$ which makes the following diagram commute.

$$\begin{tikzcd}
P_i \arrow[r] \arrow[d, equal] & P_j \arrow[d] \\
P_{i'} \arrow[r] & P_{j'}
\end{tikzcd}$$

Again this map is not chain homotopic to zero as there is no non-zero map from $P_j \to P_{i'}$ and it can be verified that the other parts of the result also hold.

- $t - 1 = t'$ and $i = j'$: In this case, $\mathrm{Hom}(P_\gamma, P_{\gamma'}) \neq 0$ because of the following map, which is non-zero because there are no non-zero maps from $P_j \to P_{j'}$ or from $P_i \to P_{i'}$.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d, equal] & P_j \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P_{i'} \arrow[r] & P_{j'} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

From the above diagram, it is easy to see that $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) \cong \mathrm{Hom}(P_\gamma, P_{\gamma'}[i]) = 0$ for all $i \neq 0$. Now suppose $f \in \mathrm{Hom}(P_{\gamma'}, P_\gamma)$. Then the commutativity of the following diagram along with the fact that g is injective implies that $f = 0$.

$$\begin{tikzcd}
\cdots \arrow[r] & P_{i'} \arrow[r, "g'"] \arrow[d] & P_{j'} \arrow[r] \arrow[d, "f^{t'}"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_i \arrow[r, "g"] & P_j \arrow[r] & \cdots
\end{tikzcd}$$

The following diagrams illustrate why $\mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = 0$ for all i.

$$\begin{tikzcd}
\cdots \arrow[r] & P_{i'} \arrow[r, "g'"] \arrow[d, "g'"'] & P_{j'} \arrow[r] \arrow[d, "g"] \arrow[dl, equal] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P_i \arrow[r, "g"'] & P_j \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_{i'} \arrow[r, "g'"] \arrow[d, "g \circ g'"] \arrow[dl, "g'"'] & P_{j'} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P_i \arrow[r, "g"'] & P_j \arrow[r, "g"'] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

- $t' - 1 = t$ and $i' = j$: This is identical to the previous case.

2. $|\gamma \cap \Delta^\star| = |\gamma' \cap \Delta^\star| = 1$: In this case, $P_\gamma = P_i[t]$ for some $1 \leq i \leq n$ and $t \in \mathbb{Z}$. Since γ, γ' have a common endpoint, $P_{\gamma'} = P_{i'}[t]$ for some $1 \leq i' \leq n$. Assume, wlog, that $i < i'$. Then we have a non-zero map from $P_i \to P_{i'}$ and it is easy to see that the result holds.

3. $|\gamma \cap \Delta^\star| = 1$, $|\gamma' \cap \Delta^\star| = 2$: In this case, $P_\gamma = P_i[-t]$ for some $1 \leq i \leq n$ and $t \in \mathbb{Z}$ and $P_{\gamma'} = P_{i'} \to P_{j'}$ for some $1 \leq i' < j' \leq n$ and concentrated in degrees $t', t' - 1$ for some $t' \in \mathbb{Z}$. We have the following two cases:

   - $t = t'$ and $i = j'$: In this case, we have the following non-zero map from $P_\gamma \to P_{\gamma'}$.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P_{i'} \arrow[r, "g'"'] & P_{j'} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

It is clear that $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = 0$ for all $i \neq 0$. The following diagrams illustrate that $\mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = 0$ for all $i \in \mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r] & P_{i'} \arrow[r] \arrow[d] & P_{j'} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_i \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_{i'} \arrow[r, "g'"] \arrow[d, "g'"'] & P_{j'} \arrow[r] \arrow[d] \arrow[dl, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_i \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

- $t = t' - 1$ and $i = i'$: In this case, we have the following non-zero map from $P_{\gamma'} \to P_\gamma$.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_{i'} \arrow[r, "g'"] \arrow[d, equal] & P_{j'} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_i \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

It is clear that $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = 0$ for all $i \neq 0$. The following diagrams illustrate that $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = 0$ for all $i \in \mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_{i'} \arrow[r, "g'"] & P_{j'} \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d, "g'"] \arrow[dl, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_{i'} \arrow[r, "g'"] & P_{j'} \arrow[r] & \cdots
\end{tikzcd}$$

4. $|\gamma \cap \Delta^\star| = 2$, $|\gamma' \cap \Delta^\star| = 1$: This is identical to the previous case.

$\square$

Theorem 3.3.1. *Let γ, γ' be two slaloms. Then γ and γ' intersect in the interior of $\mathbb{D}$ if and only if $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) \neq 0$ or $\mathrm{Ext}^i(P_{\gamma'}, P_\gamma) \neq 0$ for some $i > 0$.*

Proof. We first suppose that γ and γ' intersect in the interior of $\mathbb{D}$. We have the following cases:

1. $|\gamma \cap \Delta^\star| = |\gamma' \cap \Delta^\star| = 1$: In this case, $P_\gamma = P_i[t]$ for some $1 \leq i \leq n$ and $t \in \mathbb{Z}$ and $P_{\gamma'} = P_{i'}[t']$ for some $1 \leq i' \leq n$ and $t' \in \mathbb{Z}$. Without loss of generality, we can assume that $t < t'$ and $i' \leq i$. Then $\mathrm{Ext}^{t'-t}(P_{i'}[t'], P_i[t]) \neq 0$.

2. $|\gamma \cap \Delta^\star| = 1$, $|\gamma' \cap \Delta^\star| = 2$: In this case, $P_\gamma = P_i[-t]$ for some $1 \leq i \leq n$ and $t \in \mathbb{Z}$. We have the following possibilities for $P_{\gamma'}$:

   - $P_{\gamma'} = P_{i'} \to P_{j'}$ for some $1 \leq i' < i < j' \leq n$ and concentrated in degrees $t', t' - 1$ for some $t' \in \mathbb{Z}$: If $t' - 1 < t$, then $\mathrm{Ext}^{t-t'+1}(P_{\gamma'}, P_\gamma) \neq 0$. If $t \leq t' - 1$, then $\mathrm{Ext}^{-t+t'}(P_\gamma, P_{\gamma'}) \neq 0$.
   - $P_{\gamma'} = P_{i'} \to P_{j'}$ for some $1 \leq i' < i = j' \leq n$ and concentrated in degrees $t', t' - 1$ with $t' > t$: Then $\mathrm{Ext}^{t'-t}(P_\gamma, P_{\gamma'}) \neq 0$ as illustrated in the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & P_{i'} \arrow[r] & P_{j'} \arrow[r] & \cdots
\end{tikzcd}$$

- $P_{\gamma'} = P_{i'} \to P_{j'}$ for some $1 \le i' = i < j' \le n$ and concentrated in degrees $t', t'+1$ with $t' < t$: In this case $\mathrm{Ext}^{t-t'}(P_{\gamma'}, P_\gamma) \neq 0$.

3. $|\gamma \cap \Delta^\star| = 2$, $|\gamma' \cap \Delta^\star| = 1$: Identical to the previous case.
4. $|\gamma \cap \Delta^\star| = |\gamma' \cap \Delta^\star| = 2$: Tracing the boundary of the disc in the clockwise direction, wlog, we can assume that γ starts before γ'. Then $P_\gamma = P_i \to P_j$ for some $1 \le i < j \le n$ and concentrated in degrees $t, t+1$ for some $t \in \mathbb{Z}$, and we have the following possibilities:

   - $P_{\gamma'} = P_i \to P_{j'}$ for some $j' \ge j$ and concentrated in degrees $t', t'+1$ with $t' > t$: Then the following diagram illustrates that $\mathrm{Ext}^{t'-t}(P_\gamma, P_{\gamma'}) \neq 0$.

$$\begin{tikzcd}
\cdots \arrow[r] & P_i \arrow[r] \arrow[d, equal] & P_j \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P_i \arrow[r] & P_{j'} \arrow[r] & \cdots
\end{tikzcd}$$

   - $P_{\gamma'} = P_{i'} \to P_j$ for some $i < i' < j$ and concentrated in degrees $t', t'+1$ with $t' > t$: Then using a similar diagram as before, we get that $\mathrm{Ext}^{t'-t}(P_\gamma, P_{\gamma'}) \neq 0$.
   - $P_{\gamma'} = P_{i'} \to P_{j'}$ for some $i < i' < j < j'$ and concentrated in degrees $t', t'+1$ with for some $t' \in \mathbb{Z}$: If $t' > t$, then $\mathrm{Ext}^{t'-t}((P_\gamma, P_{\gamma'}) \neq 0$ as shown below.

$$\begin{tikzcd}
\cdots \arrow[r] & P_i \arrow[r] \arrow[d] & P_j \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P_{i'} \arrow[r] & P_{j'} \arrow[r] & \cdots
\end{tikzcd}$$

   If $t' \le t$, then $\mathrm{Ext}^{t-t'+1}(P_{\gamma'}, P_\gamma) \neq 0$ as illustrated below.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_{i'} \arrow[r] \arrow[d] & P_{j'} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P_i \arrow[r] & P_j \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

We now prove the converse that if $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) \neq 0$ or $\mathrm{Ext}^i(P_{\gamma'}, P_\gamma) \neq 0$ for some $i > 0$, then γ, γ' intersect in the interior of $\mathbb{D}$. Using Proposition 3.3.2, we know that if γ, γ' intersect on the boundary then $\mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = \mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = 0$ for all $i \neq 0$. Thus it suffices to show that if γ, γ' do not intersect, then $\mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = \mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = 0$ for all $i > 0$. We again have the following cases.

1. $|\gamma \cap \Delta^\star| = |\gamma' \cap \Delta^\star| = 1$: Suppose $P_\gamma = P_i[-t]$ for some $1 \le i \le n$ and $t \in \mathbb{Z}$ and $P_{\gamma'} = P_{i'}[-t']$ for some $1 \le i' \le n$ and $t' \in \mathbb{Z}$. Wlog we can assume $t < t'$. Then $i > i'$. Clearly $\mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = 0$ for all $i > 0$. Moreover $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = 0$ for all $i > 0$ as there are no maps from $P_i \to P_{i'}$.
2. $|\gamma \cap \Delta^\star| = 1$, $|\gamma' \cap \Delta^\star| = 2$: In this case, $P_\gamma = P_i[-t]$ for some $1 \le i \le n$ and $t \in \mathbb{Z}$. We have the following possibilities for $P_{\gamma'}$:

   - $P_{\gamma'} = P_{i'} \to P_{j'}$ for some $1 \le i' < j' < i \le n$ and concentrated in degrees $t', t'-1$ for some $t' \in \mathbb{Z}$: Clearly, $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = 0$ for all $i \in \mathbb{Z}$. The following diagrams illustrate that $\mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = 0$ for all $i \in \mathbb{Z}$.

$$
\begin{tikzcd}
\cdots \arrow[r] & P_{i'} \arrow[r] \arrow[d] & P_{j'} \arrow[r] \arrow[d] \arrow[dl] & \cdots \\
\cdots \arrow[r] & P_i \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdots \arrow[r] & P_{i'} \arrow[r] \arrow[d] & P_{j'} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_i \arrow[r] & \cdots
\end{tikzcd}
$$

- $P_{\gamma'} = P_{i'} \to P_i$ for some $1 \le i' < i \le n$ and concentrated in degrees $t', t'-1$ for some $t' < t$: Clearly $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = 0$ for all $i > 0$. Using the same diagrams as above, we see that $\mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = 0$ for all $i \in \mathbb{Z}$.
- $P_{\gamma'} = P_i \to P_{j'}$ for some $1 \le i < j' \le n$ and concentrated in degrees $t', t'+1$ for some $t' > t$: Clearly $\mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = 0$ for all $i > 0$. The following diagrams illustrate that $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = 0$ for all $i \in \mathbb{Z}$.

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d] \arrow[dl, equal] & \cdots \\
\cdots \arrow[r] & P_i \arrow[r] & P_{j'} \arrow[r] & \cdots
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdots \arrow[r] & P_i \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P_i \arrow[r] & P_{j'} \arrow[r] & \cdots
\end{tikzcd}
$$

- $P_{\gamma'} = P_{i'} \to P_{j'}$ for some $1 \le i < i' < j' \le n$ and concentrated in degrees $t', t'-1$ for some $t' \in \mathbb{Z}$: As argued in the first subcase, $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = \mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = 0$ for all $i \in \mathbb{Z}$.

3. $|\gamma \cap \Delta^\star| = 1$, $|\gamma' \cap \Delta^\star| = 2$: Identical to the previous case.

4. $|\gamma \cap \Delta^\star| = |\gamma' \cap \Delta^\star| = 2$: Tracing the boundary of the disc in the clockwise direction, wlog, we can assume that γ starts before γ'. Then $P_\gamma = P_i \to P_j$ for some $1 \le i < j \le n$ and concentrated in degrees $t, t+1$ for some $t \in \mathbb{Z}$, and we have the following possibilities:

   - $P_{\gamma'} = P_i \to P_{j'}$ with $i < j' < j$ and concentrated in degrees $t', t'+1$ with $t' \ge t$: Clearly, $\mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = 0$ for all $i > 0$. Moreover, since there are no non-zero maps from P_j to $P_{j'}$ or P_i, $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = 0$ for all $i \in \mathbb{Z}$.
   - $P_{\gamma'} = P_{i'} \to P_{j'}$ with $i < i' < j' < j$ and concentrated in degrees $t', t'+1$ for $t' \in \mathbb{Z}$: The following diagrams illustrate that $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = \mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = 0$ for all $i \in \mathbb{Z}$.

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d] \arrow[dl] & P_j \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P_{i'} \arrow[r] & P_{j'} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_{i'} \arrow[r] \arrow[d] & P_{j'} \arrow[r] \arrow[d] \arrow[dl] & \cdots \\
\cdots \arrow[r] & P_i \arrow[r] & P_j \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

   - $P_{\gamma'} = P_{i'} \to P_j$ with $i < i' < j$ and concentrated in degrees $t', t'+1$ for $t' \le t$: The The following diagrams illustrate that $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = \mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = 0$ for all $i > 0$.

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d] & P_j \arrow[r] \arrow[d] \arrow[dl, equal] & \cdots \\
\cdots \arrow[r] & P_{i'} \arrow[r] & P_j \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_{i'} \arrow[r] \arrow[d] & P_j \arrow[r] \arrow[d] \arrow[dl, equal] & \cdots \\
\cdots \arrow[r] & P_i \arrow[r] & P_j \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

   - $P_{\gamma'} = P_j \to P_{j'}$ with $i < j < j'$ and concentrated in degrees $t', t'+1$ with $t' \ge t+1$: It is clear that $\mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = 0$ for all $i > 0$. The following diagrams illustrate that $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = 0$ for all $i \in \mathbb{Z}$.

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d] & P_j \arrow[r] \arrow[d] \arrow[dl] & \cdots \\
\cdots \arrow[r] & P_j \arrow[r] & P_{j'} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdots \arrow[r] & P_i \arrow[r] \arrow[d] & P_j \arrow[r] \arrow[d] \arrow[dl, equal] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P_j \arrow[r] & P_{j'} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

   - $P_{\gamma'} = P_{i'} \to P_{j'}$ with $i < j < i' < j'$ and concentrated in degrees $t', t'+1$ for $t' \in \mathbb{Z}$: As done in the previous cases, one can easily show that $\mathrm{Ext}^i(P_\gamma, P_{\gamma'}) = \mathrm{Ext}^i(P_{\gamma'}, P_\gamma) = 0$ for all $i \in \mathbb{Z}$.

□

For ease of terminology, we say that a collection $\gamma_1, \gamma_2, \dots, \gamma_m$ of slaloms is **mutually non-intersecting** if γ_i does not intersect γ_j in the interior of $\mathbb{D}$ for all $1 \le i \ne j \le n$.

Corollary 3.3.1. *Let $\gamma_1, \gamma_2, \dots, \gamma_m$ be a collection of distinct slaloms. Then $P = \oplus_{i=1}^m P_{\gamma_i}$ is a (basic) presilting object in $K^b(\operatorname{proj}\Lambda)$ if and only if the collection is mutually non-intersecting.*

Proof. Suppose γ_i intersects γ_j in the interior of $\mathbb{D}$ for some $1 \le i, j \le m$. Then the previous theorem implies that either $\operatorname{Ext}^l(P_{\gamma_i}, P_{\gamma_j}) \ne 0$ or $\operatorname{Ext}^l(P_{\gamma_j}, P_{\gamma_i}) \ne 0$ for some $l > 0$. In either case $\operatorname{Ext}^l(P, P) \cong \oplus_{s=1}^n \oplus_{t=1}^n \operatorname{Ext}^l(P_{\gamma_s}, P_{\gamma_t}) \ne 0$, which implies that P is not a presilting object.

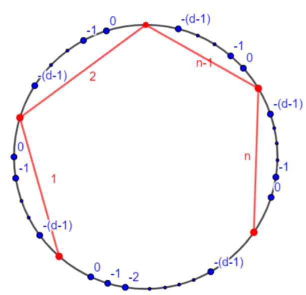
Conversely if γ_i does not intersect γ_j in the interior of $\mathbb{D}$ for all $1 \le i \ne j \le m$, then $\operatorname{Ext}^l(P_{\gamma_i}, P_{\gamma_j}) = 0$ for all $l > 0$ and $1 \le i \ne j \le m$. Moreover since P_{γ_i} is indecomposable, $\operatorname{Ext}^l(P_{\gamma_i}, P_{\gamma_i}) = 0$ for all $l > 0$ and $1 \le i \le m$. Therefore, $\operatorname{Ext}^l(P, P) = 0$ for all $l > 0$ and P is a presilting object. □

Using Theorem 2.3.5, we know that in type A_n, a basic presilting object is silting if and only if it has n indecomposable summands. Thus we immediately get the following result.

Corollary 3.3.2. *There is a bijection between* $\operatorname{silt} kA_n$ *and collections of mutually non-intersecting n slaloms in the above model.*

3.3.1 Counting the number of 3-term silting objects in type A_n

The last corollary allows us to count the number of basic d-term silting objects in type A_n. This can be done by counting the number of collections of mutually non-intersecting n slaloms in the following restricted model, where we only consider the blue points labeled $0, -1, -2, \dots, -(d-1)$.



In this section, we present the explicit calculation for $d = 3$. Since we are only interested in 3-term silting objects, i.e., objects concentrated in degrees $0, -1, -2$, we only consider the blue points labeled $0, -1, -2$ on the disc as illustrated in Figure 3.1a. For ease of reference, we have marked different sets of $0, -1, -2$ with different colours.

We denote the number of basic 3-term silting objects, or equivalently the number of collections of mutually non-intersecting n slaloms in Figure 3.1a, in type A_n, by B_n. Our goal is to calculate B_n. Let Γ be a collection of mutually non-intersecting n slaloms. Such a collection will be maximal with respect to the property of mutual non-intersection as the number of indecomposable summands of any presilting object is less than or equal to n. Our first claim is that in such a collection of n slaloms, either of the green $0, -1, -2$ has to be an endpoint of some slalom. This is because otherwise, the slalom in Figure 3.1b will contradict the maximality of the collection of mutually non-intersecting slaloms. We can now divide the problem into the following cases.

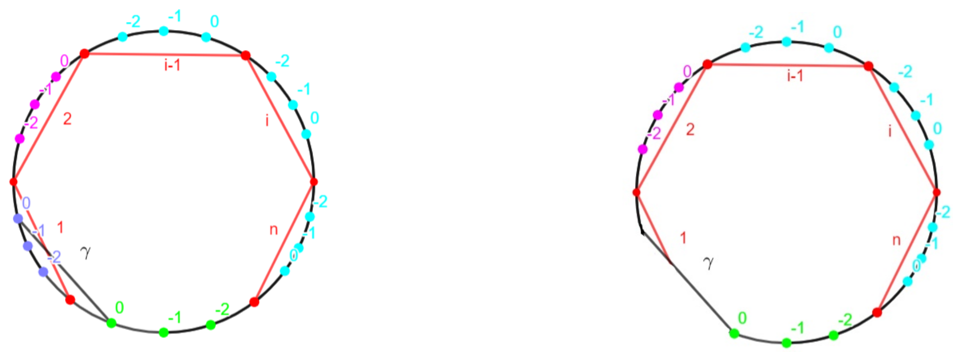
1. The green 0 is the endpoint of a slalom: Consider the leftmost slalom connected to the green 0, say γ. We have the following cases.

   (a) $P_\gamma = P_1$: From Figure 3.2a, we see that the arc γ divides the disc into 2 parts, the smaller one of which cannot contain any other slalom from our collection. Thus the remaining $n-1$ slaloms of our collection lie in the other half. We note that in this case, there cannot be a slalom with

(a) (b)

Figure 3.1

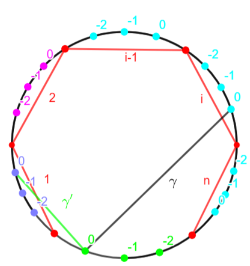
an endpoint at the purple 0 as this would correspond to a complex in degrees $0, 1$, which is not allowed. Thus, removing the purple 0, we are reduced to the case of calculating 3-term silting objects in A_{n-1} as shown in Figure 3.2b. Thus the number of silting complexes, in this case, is B_{n-1}.



(a) (b)

Figure 3.2

(b) $P_\gamma = P_i$ for some $i > 1$: We first note that, in this case, if both purple $-1, -2$ are not connected to anything, we can add the following slalom γ' to our collection, contradicting the maximality of the collection of mutually non-intersecting slaloms.



Thus we have the following subcases:

- The purple -1 is an endpoint of some slalom but the purple -2 is not: Let γ' be the rightmost slalom whose endpoint is the purple -1. Then $P_{\gamma'} = P_1 \to P_j \to 0$ for some $1 < j \leq i$. If $j < i$, then the pink slalom in Figure 3.3a contradicts the maximality of the collection of mutually non-intersecting slaloms. Thus $j = i$ (Figure 3.3b).

(a) (b)

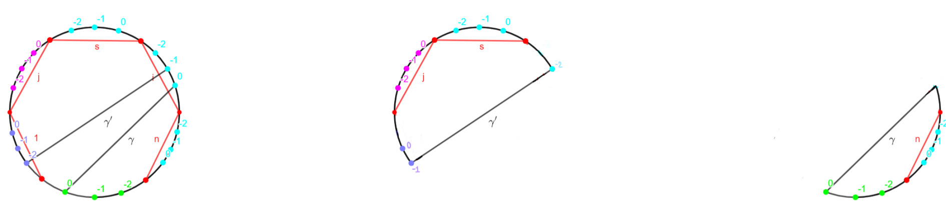
Figure 3.3

As argued in the previous case, we can relabel the numbered points in such a way that we are reduced to the problem of calculating collections of $i-2$ mutually non-intersecting slaloms in A_{i-2} (Figure 3.4a) and of $n-i$ mutually non-intersecting slaloms in A_{n-i} (Figure 3.4b). Thus the number of silting objects, in this case, is $\Sigma_{l=2}^{n} B_{l-2} B_{n-l}$.

(a) (b)

Figure 3.4

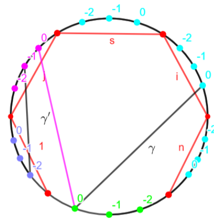
- The purple -2 is an endpoint of some slalom: Let γ' be the rightmost slalom with an endpoint at the purple -2. Then $P_{\gamma'} = P_1 \to P_j \to 0$ for some $1 < j \le i$. Suppose $j = i$ (Figure 3.5a). Then the relabeling in figures 3.5b and 3.5c gives us $\Sigma_{l=2}^{n} B_{l-2} B_{n-l}$ many silting objects in this case.



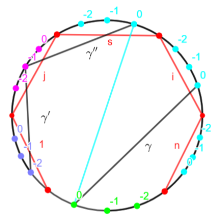
(a) (b) (c)

Figure 3.5

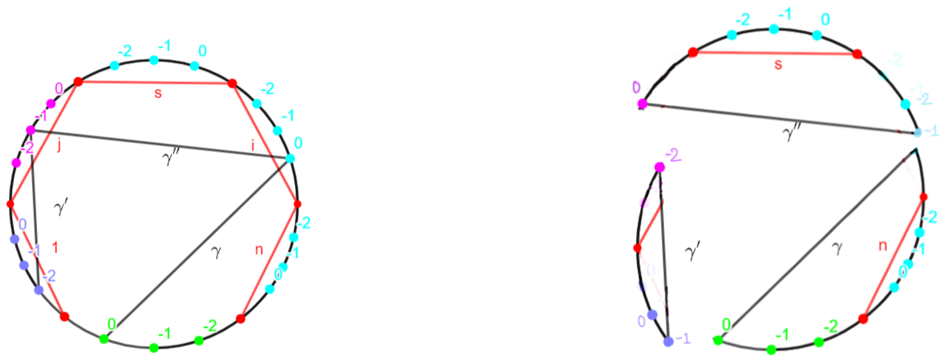
Now, suppose $j < i$. If there is no slalom corresponding to a complex of the form $0 \to P_j \to P_s$ with $s > j$, then the pink slalom in the following figure will contradict the maximality of the collection.



Thus there exists a slalom γ'' corresponding to a complex $0 \to P_j \to P_{s'}$ with $i \geq s' > j$. Let s be the maximum of such s'. If $s < i$, then the blue slalom in the following figure contradicts the maximality of the collection.



Thus $s = i$ (Figure 3.6a), and an appropriate relabeling of the numbered points reduces the problem to calculating collections of $j-2$ mutually non-intersecting slaloms in A_{j-2}, of $i-j-1$ mutually non-intersecting slaloms in A_{i-j-1} and of $n-i$ mutually non-intersecting slaloms in A_{n-i} (Figure 3.6b). Thus the total number of silting complexes, in this case, is $\Sigma_{u=2}^{n}\Sigma_{v=2}^{u-1}B_{v-2}B_{u-v-1}B_{n-u}$.

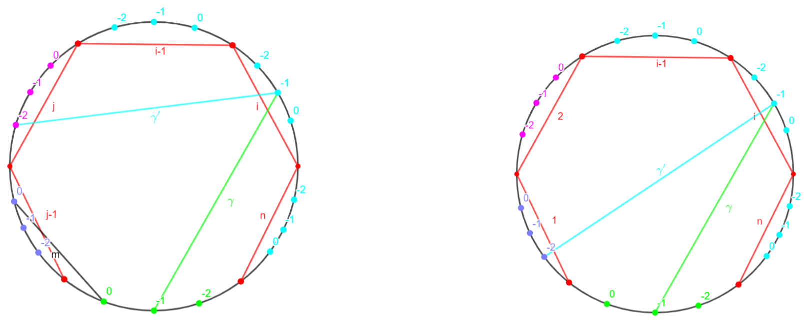


(a) (b)

Figure 3.6

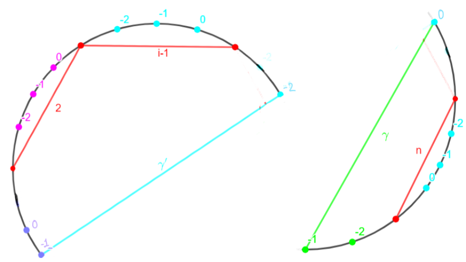
2. The green 0 is not an endpoint of any slalom: We have the following subcases:

   (a) The green -1 is an endpoint of a slalom: Consider the leftmost slalom whose endpoint is the green -1, say, γ. Then $P_\gamma = P_i[1]$ for some $1 \leq i \leq n$. Note that in this case there cannot be a slalom with an endpoint at the -2 inside the arc with the label i. Suppose $i > 1$. Then there should be another slalom ending at the -1 inside the arc with the label i as, otherwise, the slalom corresponding to P_{i-1} will contradict the maximality of the collection. Now, such a slalom will correspond to a complex of the form $P_{i'} \to P_i \to 0$ with $i' < i$. Let j be the smallest among such possible i'. If $j > 1$, then the slalom corresponding to $Pj-1$ will contradict the maximality of Γ (Figure 3.7a). Thus $j = 1$ and we have the situation in Figure 3.7b. As done before, we can argue that we can relabel the numbered points as shown in the following figure to reduce the problem to counting collections of $i-2$ mutually non-intersecting slaloms in A_{i-2} and of $n-i$ mutually non-intersecting slaloms in A_{n-i}.



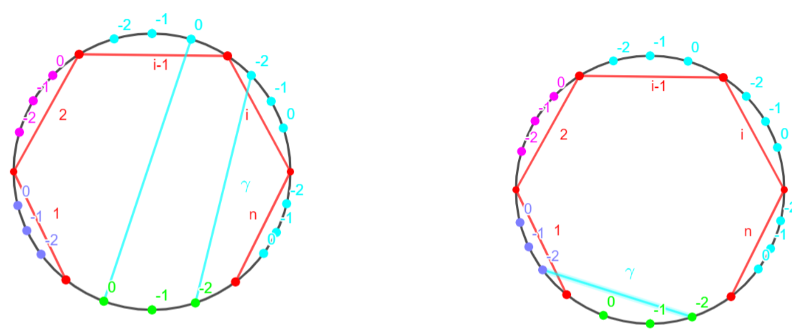
(a) (b)

Figure 3.7



If $i = 1$, we can use an argument similar to Case 1(a). Thus the total number of silting complexes, in this case, is $B_{n-1} + \Sigma_{i=2}^{n} B_{i-2} B_{n-i}$.

(b) The green -1 is not an endpoint of any slalom: Since there has to be a slalom with an endpoint at one of the green $0, -1, -2$, in this case, there should be a slalom with an endpoint at the green -2. Consider the leftmost such slalom, say γ. Then $P_\gamma = P_i[2]$ for some i. If $i > 1$, then the slalom corresponding to P_{i-1} will contradict the maximality of the collection as shown in Figure 3.8a.



(a) (b)

Figure 3.8

Thus $i = 1$ (Figure 3.8b), and using an argument similar to Case 1(a), we get that the total number of silting complexes, in this case, is B_{n-1}.

Thus the total number of basic 3-term silting objects in $K^b(\operatorname{proj} kA_n)$ is given by the recursive formula

$$B_n = 3B_{n-1} + 3\Sigma_{i=2}^{n} B_{i-2} B_{n-i} + \Sigma_{u=2}^{n} \Sigma_{v=2}^{u-1} B_{v-2} B_{u-v-1} B_{n-u}$$

with $B_0 = 1$. This is the recurrence formula satisfied by the Fuss-Catalan numbers $A_n(3,1)$, which count the number of complete ternary trees on n internal vertices.

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