Théorie d'Auslander-Reiten et modules de cordes localement de dimension finie

Auslander-Reiten theory and string modules under the locally finite-dimensional setting

par

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ABSTRACT

Given a quiver Q (possibly infinite) and a set of relations ρ on Q, we say that the algebra $\Lambda = kQ/\langle\rho\rangle$ is locally semi-perfect if it is locally finite-dimensional and $e_a\Lambda e_a$ is local for all $a \in Q_0$. In the first part of this dissertation, we prove the existence of almost split sequences in ModΛ, the category of unital modules over Λ, ending in non-projective finitely presented indecomposables, for locally semi-perfect algebras which we call 'bounded on the left'. Using a duality functor, we then prove the existence of almost split sequences in modΛ, the category of locally finite-dimensional unital modules over Λ, starting at non-injective finitely copresented indecomposables, for locally semi-perfect algebras which we call 'bounded on the right'.

In the second part, we give a combinatorial characterization of the finitely presented and finitely co-presented string modules over locally finite-dimensional string algebras. We also give an explicit description of their syzygys and cosyzygys respectively.

SOMMAIRE

Étant donné un carquois Q (peut-être infini) et un ensemble de relations ρ sur Q, nous disons que l'algèbre $\Lambda = kQ/\langle\rho\rangle$ est localement semi-parfaite si elle est localement de dimension finie et si l'algèbre $e_a\Lambda e_a$ est locale pour tout $a \in Q_0$. Dans la première partie de ce mémoire, nous prouvons l'existence de suites presque scindées dans ModΛ, la catégorie des modules unitaires sur Λ, se terminant par des indécomposables non-projectifs de présentation finie, pour les algèbres localement semi-parfaites qui s'appellent 'bornées à gauche'. En utilisant un foncteur de dualité, nous prouvons ensuite l'existence de suites presque scindées dans modΛ, la catégorie des modules unitaires localement de dimension finie sur Λ, commençant par des indécomposables non injectifs de coprésentation finie, pour des algèbres localement semi-parfaites qui s'appellent 'bornées à droite'.

Dans la deuxième partie, nous donnons une caractérisation combinatoire des modules de cordes de présentation et coprésentaton finie sur les algèbres de cordes localement de dimension finie. Nous donnons aussi une description explicite de leurs syzygies et cosyzygies respectivement.

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INTRODUCTION

An important tool in the study of the representation theory of finite dimensional algebras has been the theory of representations of finite quivers along with the Auslander-Reiten theory of irreducible morphisms and almost split sequences. However, from the perspective of covering theory, it has become important to study locally finite-dimensional algebras defined by locally finite quivers with relations. One of the natural questions is the existence of almost split sequences in the module category over such an algebra. For this purpose, we shall need some further finiteness conditions. Indeed, we shall consider locally semi-perfect algebras, that is locally finite-dimensional algebras such that the idempotents associated with the vertices of the quiver are primitive. We shall prove the existence of almost split sequences for finitely presented modules (resp. finitely co-presented modules) over locally semi-perfect algebras which are left (resp. right) locally bounded. For this purpose, we introduce a Nakayama functor by using a slightly modified standard duality, which allows us to define the Auslander-Reiten translate, prove the Auslander-Reiten formula, and derive the existence of almost split sequences for such modules. As a consequence, one obtains immediately the existence of almost split sequences in the category of finite-dimensional modules over the covering of a finite-dimensional algebra.

In the second part of the thesis, we study the representation theory of string algebras, where we drop the classical finiteness conditions imposed by Butler and Ringel [11]. In particular, we give a combinatorial characterization of the strings for which the string modules are finitely presented or finitely copresented, along with calculating their syzygies and cosyzygies. String algebras with the finiteness conditions imposed by Butler and Ringel are a class of tame algebras whose representation theory is highly combinatorial in nature, and hence easier to study. By definition, such a string algebra is the path algebra of a quiver with relations that satisfies certain conditions designed to restrict the structure of the indecomposable projective and injective modules. This definition is a specialization of the definition of a special biserial algebra, for which the restrictions on the quiver imply that all indecomposable projective modules are biserial.

In [11], Butler and Ringel classified the indecomposable finite-dimensional modules over such algebras in terms of string and band modules, where they credited their method to Gel'fand and Ponomarev [14]. They also classified all the irreducible maps between these modules and, hence, the Auslander-Reiten sequences. As mentioned before, they imposed a finiteness condition on these algebras which implied that they were finite-dimensional if the quiver had finitely many vertices. These are what we will call locally bounded string algebras in this dissertation. These conditions were dropped by William Crawley-Boevey in [12], where he classified the finitely controlled modules (A module M is finitely controlled if, for every vertex v, the set e_vM is contained in a finitely generated submodule of M) over such algebras in terms of string and band modules. The next task would be to classify the irreducible morphisms and almost split sequences for such modules.

The dissertation is organized as follows. In Chapter 1, we introduce the relevant background and terminology related to quivers and algebras associated with them. In particular, we define the notions of locally semi-perfect algebras and prove some preliminary results about them. Chapter 2 is used to introduce the basics of Auslander-Reiten theory, irreducible morphisms, almost split sequences, etc., for finite-dimensional algebras. In Chapter 3, we upgrade these basics to the case of locally semi-perfect algebras to show the existence of almost split sequences for them. Chapter 4 is devoted to the definitions of modules associated with these strings and bands, along with a description of certain projective and injective string modules. In Chapter 5, we classify the strings for which the associated modules are finitely presented, which turn out to be the same as strings for which these modules are finitely generated. In particular, we give an explicit description of the syzygies of such modules. Chapter 6 deals with the dual of these results–we classify the strings for which the associated modules are finitely copresented, which turn out to be the same as the strings for which these modules are finitely cogenerated.

CHAPTER 1

Quivers and algebras

In this chapter, we will introduce the notion of quivers and the path algebras associated to them along with some examples. We will further define certain special kinds of algebras called 'locally semi-perfect' algebras, which will be our main object of study.

Throughout this chapter, k will denote an algebraically closed field. All ideals of algebras will be two-sided unless stated otherwise.

1.1 Quivers and path algebras

In this section, we define the notion of a quiver, its path algebra, and its representations.

Definition 1.1.1. *A* quiver Q *is a quadruple* (Q_0, Q_1, s, t)*, where* Q_0, Q_1 *are sets and* $s, t : Q_1 \to Q_0$ *are functions.*

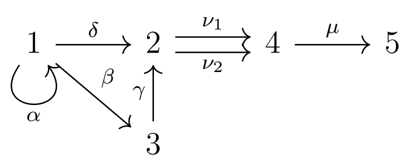
The elements of Q_0 and Q_1 are called the *vertices* and the *arrows* of the quiver respectively. For each $\alpha \in Q_1$, $s(\alpha)$ is said to be the *source* of α while $t(\alpha)$ is said to be its *target*. We denote this as $\alpha : s(\alpha) \to t(\alpha)$ or $s(\alpha) \xrightarrow{\alpha} t(\alpha)$. A vertex $a \in Q_0$ is said to be a *source vertex* if it is not the target of any arrow. Dually, a vertex $a \in Q_0$ is said to be a *sink vertex* if it is not the source of any arrow. We call the quiver Q *finite* if both Q_0 and Q_1 are finite and we call it *locally finite* if for each pair of vertices $(a, b) \in Q_0 \times Q_0$, the set of arrows starting at a and ending at b is finite. For a quiver $Q = (Q_0, Q_1, s, t)$, we define its *opposite quiver*, Q°, as the quiver (Q_0, Q_1, t, s), i.e., the quiver obtained by reversing the direction of the arrows.

Throughout this work, we will use the letter Q to denote a locally finite quiver.

Definition 1.1.2. *Let $n > 0$. A* path *of length n in Q is a sequence $\rho = \alpha_1 \cdots \alpha_n$, where $\alpha_i \in Q_1$ for all i such that $1 \le i \le n$, satisfying $s(\alpha_{i+1}) = t(\alpha_i)$ for all i such that $1 \le i < n$. In addition to this, to each $x \in Q_0$, we associate a* trivial path *ε_x of length 0.*

We will denote the length of a path ρ by $l(\rho)$.

Example 1.1.1. *The following graph is a quiver with $Q_0 = \{1, 2, 3, 4, 5\}$ and*



$Q_1 = \{\alpha, \beta, \gamma, \delta, \nu_1, \nu_2, \mu\}$. There is no source while vertex 5 is a sink. Some examples of paths are $\alpha^2\beta$, $\beta\gamma\nu_1\mu$, ε_2.

We can extend the definitions of the functions s, t to all paths as follows: for $x \in Q_0$, set $t(\varepsilon_x) = s(\varepsilon_x) := x$, and for $\rho = \alpha_1 \cdots \alpha_n$, a path of length $n > 0$, set $t(\rho) := t(\alpha_n)$ and $s(\rho) := s(\alpha_1)$. We call a path of length ≥ 1 an *oriented cycle* if its source and target coincide. We say that a quiver Q is *acyclic* if it does not contain any oriented cycles.

Definition 1.1.3. *A k-*algebra *A is a k-vector space together with a binary operation $\cdot$ such that for all $a, b, c \in A$ and $\lambda \in k$,*

1) $a \cdot (b + c) = a \cdot b + a \cdot c$.

2) $(a + b) \cdot c = a \cdot c + b \cdot c$.

3) $\lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b)$.

4) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

We define the *dimension* of a k-algebra A to be its dimension as a k-vector space.

The above definition implies that for a k-algebra A, the multiplication is determined by its values on a k-basis elements of A.

We are now ready to define the path algebra associated to a quiver. Let kQ be the k-vector space having the set of paths in Q as a basis. In order to turn kQ into a k algebra, it is enough to define the multiplication of the basis elements. Let $\rho = \alpha_1 \cdots \alpha_n$, $\rho' = \beta_1 \cdots \beta_{n'}$ be two non-trivial paths in Q. Then we define

$$\rho \cdot \rho' = \begin{cases} \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_{n'}, & \text{if } t(\rho) = s(\rho'); \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, for $x, x' \in Q_0$, we define

$$\rho \cdot \varepsilon_x = \begin{cases} \rho, & \text{if } t(\rho) = x; \\ 0, & \text{otherwise.} \end{cases}$$

$$\varepsilon_x \cdot \rho = \begin{cases} \rho, & \text{if } s(\rho) = x; \\ 0, & \text{otherwise.} \end{cases}$$

$$\varepsilon_x \cdot \varepsilon_{x'} = \begin{cases} \varepsilon_x, & \text{if } x' = x; \\ 0, & \text{otherwise.} \end{cases}$$

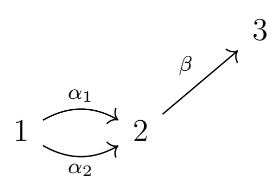
Definition 1.1.4. *Let Q be a quiver. The* path algebra *of Q, denoted by kQ, is defined to be the algebra generated as a k-vector space by the paths in Q of length ≥ 0 under the above multiplication.*

The above definition shows that kQ is infinite-dimensional if Q_0 is infinite or if Q contains cycles. Conversely, if Q is acyclic with finite Q_0, then kQ is finite-dimensional [5, Lemma 1.4]. We also have the following lemma characterizing the quivers for which kQ is unital.

Lemma 1.1.1. *The algebra kQ has an identity if and only if Q_0 is finite.*

Proof. Suppose Q_0 is finite. Let $\lambda = \sum_{a \in Q_0} \varepsilon_a$. Then for any path ρ, $\rho \cdot \lambda = \rho \cdot \varepsilon_{t(\rho)} = \rho$ and $\lambda \cdot \rho = \varepsilon_{s(\rho)} \cdot \rho = \rho$. Therefore, kQ has an identity element given by $\sum_{a \in Q_0} \varepsilon_a$. Conversely, suppose Q_0 is infinite and $1 = \sum_{i=1}^{m} \lambda_i w_i$ is the identity element, where $\lambda_i \in k$ are non-zero scalars and w_i are paths in Q. Since Q_0 is infinite, there exists $a \in Q_0$ such that $a \neq s(w_i)$ for all $1 \leq i \leq m$. Then $\varepsilon_a \cdot 1 = 0$, a contradiction. Therefore, kQ does not have an identity. □

Example 1.1.2. *Let Q be the following quiver.*



Then kQ is generated by $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \alpha_1, \alpha_2, \beta, \alpha_1\beta, \alpha_2\beta\}$ and is 8-dimensional.

For the sake of completeness, we also introduce the notion of a path of infinite length, which is a sequence either of the form $\alpha_1\alpha_2\cdots$, $\cdots\alpha_{-2}\alpha_{-1}$, or $\cdots\alpha_{-2}\alpha_{-1}\alpha_0\cdots$ of arrows of Q such that $s(\alpha_{i+1}) = t(\alpha_i)$ for all i. The first kind of paths do not have a target, the second ones do not have a source, and the last ones neither have a source nor a target.

Definition 1.1.5. *Let p be a path in Q, possibly of infinite length. In case p has a source, a path q is called an* initial subpath *of p if $p = qp'$ for some path p'. In case p has a target, a path q is called a* terminal subpath *of p if $p = p'q$ for some path p'.*

Note that in the above definition, q, p' are allowed to be trivial.

Example 1.1.3. *Continuing with Example 1.1.1, ε_1 is an initial subpath of $\alpha^2\beta$ while $\alpha\beta$ is a terminal subpath of it.*

For a quiver Q, given a vertex $a \in Q_0$ and an integer $n \geq 0$, we shall denote by $Q_n(a,-)$ the set of paths of length n starting at a, and by $Q_n(-,a)$ the set of paths of length n ending at a in Q.

1.2 Bound quivers and elementary algebras

In this section, we expand the class of algebras that can be obtained from quivers by defining a certain class of ideals of path algebras called 'admissible ideals'. We will see how this construction essentially characterizes all finite-dimensional algebras over k.

Let R_Q be the ideal of kQ generated by the arrows of Q_1.

Definition 1.2.1. *[13, § 8.3] Let $Q = (Q_0, Q_1)$ be a quiver. An ideal I of kQ is called* admissible *if*

1. *$I \subseteq R_Q^2$,*

2. *For each $a \in Q_0$, there exists $l_a \in \mathbb{N}$ such that I contains all paths of length $\geq l_a$ starting or ending at a.*

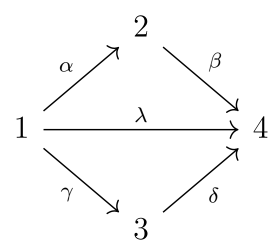
In this case, the pair (Q, I) is called a bound quiver*, and the algebra $A = kQ/I$ is called a* bound quiver algebra*.*

The reasons for choosing the above definition of admissible ideals are the following. The second condition makes sure that we do not have arbitrarily long paths in the quotient, i.e., that A is finite-dimensional for finite quivers. And the condition $I \subseteq R_Q^2$ makes sure that we do not have any redundant arrows in our quiver. Let us look at some examples.

Examples 1.2.1. *1. For any $m \geq 2$, R_Q^m is an admissible ideal.*

2. *Let Q be a finite quiver. Then the zero ideal is admissible in kQ if and only if Q is acyclic. Indeed, the zero ideal is admissible if and only if there exists $m \geq 2$ such that $R_Q^m = 0$, that is, any product of m arrows in kQ is zero. This is the case if and only if Q is acyclic.*

3. *Let Q be the following quiver.*



The ideal $I_1 = \langle \alpha\beta - \gamma\delta \rangle$ of the k-algebra kQ is admissible, but $I_2 = \langle \alpha\beta - \lambda \rangle$ is not as $\alpha\beta - \lambda \notin R_Q^2$.

It is convenient to define an admissible ideal in terms of its generators. These are called relations.

Definition 1.2.2. *A* relation *ρ on a quiver Q with coefficients in k is a non-zero k-linear combination of paths in Q of length at least two having the same source and target. Thus $\rho = \sum_{i=1}^{m} \lambda_i w_i$ such that $\lambda_i \in k^*$, $l(w_i) \geq 2$, $s(w_i) = s(w_j)$, and $t(w_i) = t(w_j)$ for all $1 \leq i, j \leq m$.*

If $m = 1$, the relation is called a *zero relation* or a *monomial relation.* If it is of the form $w_1 - w_2$ (where w_1, w_2 are two paths), it is called a *commutativity relation.*

Example 1.2.1. *For Q as in Example 1.1.1, $\alpha^2 - \alpha^3$, $\nu_1\mu$, $\delta\nu_1 - \delta\nu_2$ are some relations on Q.*

For an algebra $\Lambda = kQ/\langle\rho\rangle$, we define the *opposite algebra* of Λ, Λ°, as $kQ^\circ/\langle\rho^\circ\rangle$, where Q° is the opposite quiver of Q and $\rho^\circ = \{p^\circ \mid p \in \rho\}$.

Definition 1.2.3. *Let A be a k-algebra. The* Jacobson radical, $\mathrm{rad}A$*, of A is defined to be the two-sided ideal which is the intersection of all maximal right ideals of A.*

A finite-dimensional k-algebra A is called elementary *if $A/\mathrm{rad}A$ is a product of copies of k.*

The following theorem states that the previous construction characterizes all elementary algebras.

Theorem 1.2.1. *(Gabriel)[4, Theorem I.2.13] Let A be a finite-dimensional elementary k-algebra. Then there exists a finite quiver Q_A and an admissible ideal $I \subseteq kQ_A$ such that $A \cong kQ_A/I$.*

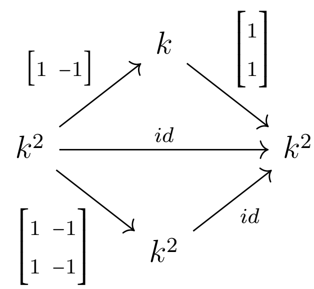
Moreover, since k is algebraically closed, every finite-dimensional k-algebra is Morita equivalent to an elementary algebra [2, Theorem 3.2].

1.3 Representations of a quiver with relations

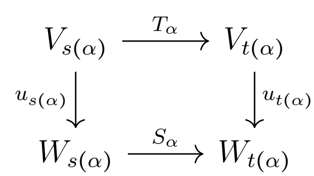
Let (Q, I) be a bound quiver such that I is generated by a set of relations ρ on Q. We will also denote this pair as (Q, ρ). Then we can talk about the representations of this bound quiver. These representations will be closely related to the modules over the algebra $kQ/\langle\rho\rangle$.

Definition 1.3.1. *A* k-representation *of the bound quiver* (Q,ρ) *is a collection of* k*-vector spaces* $(V_i)_{i\in Q_0}$ *and a family of linear maps* $T_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$*, indexed by* $\alpha \in Q_1$*, such that for each element* $\sum_{i=1}^m \lambda_i \alpha_{i,1}\cdots\alpha_{i,l_i}$ *of* ρ*,* $\sum_{i=1}^m \lambda_i T_{\alpha_{i,l_i}}\cdots T_{\alpha_{i,1}} = 0$.

Example 1.3.1. *Let* (Q,I_1) *be the bound quiver from Example 1.2.1.3. Then the following diagram gives a* k*-representation of* (Q,I_1)*.*



Definition 1.3.2. *Let* $M = ((V_i)_{i\in Q_0}, (T_\alpha)_{\alpha\in Q_1})$ *and* $N = ((W_i)_{i\in Q_0}, (S_\alpha)_{\alpha\in Q_1})$ *be two* k*-representations of a bound quiver* (Q,ρ)*. A* morphism of representations $u : M \to N$ *is a family of linear maps* $(u_i : V_i \to W_i)_{i\in Q_0}$ *such that for all* $\alpha \in Q_1$*, the following diagram commutes:*



Given k-representations L, M, N of (Q,ρ), and maps $u : L \to M$ and $v : M \to N$, we define the composition $(v \cdot u)_i := v_i \cdot u_i$ for all $i \in Q_0$. Moreover, for any k-representation $M = ((V_i)_{i\in Q_0}, (T_\alpha)_{\alpha\in Q_1})$, we have an identity morphism $id : M \to M$ given by $(id)_i = id_{M_i}$ for all $i \in Q_0$. Thus, we get a category $\mathrm{Rep}_k(Q,\rho)$ whose objects are the representations of (Q,ρ) and the morphisms are the morphisms of representations. Let $A = kQ/\langle\rho\rangle$.

Definition 1.3.3. *We say that a right* A*-module* M *is* unitary *if* $M = \sum_{x\in Q_0} M\bar{\varepsilon}_x$ *as* k*-vector spaces.*

We will denote the category of unitary right A-modules by Mod$A$. We will also identify ModΛ° with the category of unitary left Λ-modules.

Theorem 1.3.1. *[3, Theorem II.2.10] We have an equivalence of categories*

$$\mathrm{Mod}A \cong \mathrm{Rep}_k(Q,\rho)$$

given by $M \mapsto ((M\bar{\varepsilon}_i)_{i\in Q_0},(m_{\bar{\alpha}})_{\alpha\in Q_1})$*, where* $m_{\bar{\alpha}} : M\bar{\varepsilon}_{s(\alpha)} \to M\bar{\varepsilon}_{t(\alpha)}$ *denotes the map* $x \mapsto x\bar{\alpha}$*, i.e., the multiplication by* $\bar{\alpha}$*.*

Remark 1.3.1. *Although we assumed* (Q,I) *to be a bound quiver at the beginning of the section, the results mentioned here also work for all ideals* I *of* kQ *generated by relations.*

Definition 1.3.4. *Let* $M \in \mathrm{Mod}A$*. We say that* M *is* locally finite-dimensional *if* $\dim_k(M\bar{\varepsilon}_x) < \infty$ *for all* $x \in Q_0$*.*

We will denote the full subcategory of locally finite-dimensional modules of Mod$A$ by mod$A$.

1.4 Locally finite-dimensional algebras

In this section, we introduce another class of algebras obtained from quivers and relations called 'locally' finite-dimensional algebras. These will be our main object of study in this thesis.

Let ρ be a set of relations on a quiver Q and $\Lambda = kQ/\langle\rho\rangle$. Set $e_x := \bar{\varepsilon}_x$ for all $x \in Q_0$, i.e., the equivalence class of ε_x in Λ.

Definition 1.4.1. *The algebra* $\Lambda = kQ/\langle\rho\rangle$ *is said to be* locally finite-dimensional *if* $\dim(e_x\Lambda e_y) < \infty$ *for all* $x,y \in Q_0$*.*

We also define the following one-sided analogues of the above notion.

Definition 1.4.2. *1. An algebra $\Lambda = kQ/\langle\rho\rangle$ is said to be* left locally bounded *if Λe_a is finite dimensional for all a in Q_0.*

2. An algebra $\Lambda = kQ/\langle\rho\rangle$ is said to be right locally bounded *if $e_a\Lambda$ is finite dimensional for all a in Q_0.*

Note that a left or right locally bounded algebra is always locally finite-dimensional.

We also have the following slightly stronger definition, which ensures that the projective at each vertex is indecomposable.

Definition 1.4.3. *The algebra $\Lambda = kQ/\langle\rho\rangle$ is called* locally semi-perfect *if it is locally finite-dimensional, and for all $x \in Q_0$, $e_x\Lambda e_x$ is local.*

The above definitions were first introduced by Bongartz and Gabriel in [9, § 2.1], where locally semi-perfect algebras were called *locally finite-dimensional categories*, while *locally bounded categories* were defined to be left and right locally bounded locally semi-perfect algebras.

Define $P_a := e_a\Lambda$ for $a \in Q_0$.

Lemma 1.4.1. *For any Λ-module M and $a \in Q_0$, $\mathrm{Hom}_\Lambda(P_a, M) \cong Me_a$.*

Proof. Define $f : \mathrm{Hom}_\Lambda(P_a, M) \to Me_a$ as $g \mapsto g(e_a)$. Since g is a Λ-module homomorphism, $g(e_a) = g(e_a e_a) = g(e_a)e_a$, which implies that $g(e_a) \in Me_a$. f is injective because if there exists some $g \in \mathrm{Hom}_\Lambda(P_a, M)$ such that $g(e_a) = 0$, then $g(e_a \cdot \lambda) = g(e_a)\lambda = 0$ for all $\lambda \in \Lambda$, and hence $g = 0$. It is surjective because if $me_a \in Me_a$, then $g : P_a \to M$ defined by $e_a \cdot \lambda \mapsto me_a \cdot \lambda$ is a Λ-module homomorphism such that $g(e_a) = me_a$. □

Proposition 1.4.1. *P_a is a projective module for all $a \in Q_0$.*

Proof. We want to show that $\mathrm{Hom}(P_a, -)$ is an exact functor. Since it is always left exact, it is enough to show that it preserves epimorphisms. Let $f : M \to N$ be a surjective Λ-module homomorphism. Then, using Lemma 1.4.1, $\mathrm{Hom}(P_a, M) \xrightarrow{\mathrm{Hom}(P_a,f)} \mathrm{Hom}(P_a, N)$ is isomorphic to $Me_a \xrightarrow{f|_{Me_a}} Ne_a$. Let $ne_a \in Ne_a$. Since f is surjective, there exists $m \in M$ such that $f(m) = n$. Thus $f(me_a) = ne_a$ and $f|_{Me_a}$ is surjective. □

For the rest of this section, we will assume Λ to be a locally finite-dimensional algebra. Then P_a is a locally finite-dimensional unitary Λ-module for all $a \in Q_0$. Moreover, as stated in the following lemma, if Λ is locally semi-perfect, then these P_a are also indecomposable.

Lemma 1.4.2. *Let Λ be a locally semi-perfect algebra. Then for all $a \in Q_0$, P_a is indecomposable.*

Proof. Using Lemma 1.4.1, $\mathrm{End}(P_a) = \mathrm{Hom}(P_a, P_a) \cong P_a e_a = e_a \Lambda e_a$, and hence local. Since P_a has a local endomorphism algebra, it is indecomposable. □

Define $\mathrm{proj}\Lambda$ to be the full subcategory of $\mathrm{mod}\Lambda$ whose objects are finite direct sums of P_a, $a \in Q_0$. Since a finite direct sum of projective modules is projective, every module in $\mathrm{proj}\Lambda$ is projective.

Proposition 1.4.2. *Let Λ be a locally semi-perfect algebra. Then* $\mathrm{proj}\Lambda$ *is closed under direct summands.*

Proof. Let $X \in \mathrm{proj}\Lambda$ and Y a non-trivial summand of X, i.e., $X \cong Y \oplus Y'$ for non-zero $Y, Y' \in \mathrm{mod}\Lambda$. Set $A := \mathrm{End}(X)$. Let $i_Y : Y \to X$ and $\pi_Y : X \to Y$ denote the canonical injection and projection respectively. Then $f = i_Y \pi_Y$ is a non-zero idempotent in A. Since $X \in \mathrm{proj}\Lambda$, it is isomorphic to $\oplus_{i=1}^n P_{a_i}$ for some $a_i \in Q_0$. Therefore, $A \cong \oplus_{i=1}^n \oplus_{j=1}^n \mathrm{End}(P_{a_i}, P_{a_j}) \cong \oplus_{i=1}^n \oplus_{j=1}^n e_{a_j} \Lambda e_{a_i}$, and hence it is finite dimensional. In particular, it is semi-perfect. We claim that f can be written as a finite sum of primitive orthogonal idempotents in A. Suppose this is not the case. Then, in particular, f is not a primitive idempotent, which implies that $f = f_1 + f_2$, where f_1, f_2 are non-zero orthogonal idempotents in A. If both f_1, f_2 are primitive, then the claim is true, and we get a contradiction. Therefore, without loss of generality, suppose f_1 is not primitive. Then $f_1 = f_{11} + f_{12}$, where f_{11}, f_{12} are non-zero orthogonal idempotents in A. Moreover, since $f_1 f_2 = f_{11} f_2 + f_{12} f_2 = 0$, we get that $f_{11} f_1 f_2 = f_{11} f_2 = 0 = f_{12} f_1 f_2 = f_{12} f_2$. Thus, f_{11}, f_{12} are also orthogonal to f_2. Thus, repeating this process, we get that, for all $m > 0$, f can be written as a sum of m non-zero orthogonal idempotents which is a contradiction to the finite-dimensionality of A. Thus, there exists a complete set $f_1, \ldots, f_n$ of primitive orthogonal idempotents in A such that $f = f_1 + \ldots + f_r$, with $0 < r \leq n$. Let $g_j : X \to X$ denote the idempotent $\oplus_{i=1}^n P_{a_i} \xrightarrow{\pi_j} P_{a_j} \xrightarrow{\iota_j} \oplus_{i=1}^n P_{a_i}$ in A. Then $g_1, \ldots, g_n$ is another complete set of primitive orthogonal idempotents in A. Using [1, Theorem 27.10], we get that $m = n$ and there exists a permutation σ such that $Ag_i \cong Af_{\sigma(i)}$, for $i = 1, \ldots, n$. Thus $g_i = (g_i b_i f_{\sigma(i)})(f_{\sigma(i)} a_i g_i)$ and $f_{\sigma(i)} = (f_{\sigma(i)} a_i g_i)(g_i b_i f_{\sigma(i)})$, with $a_i, b_i \in A$. Setting $a = \sum_{i=1}^n (f_{\sigma(i)} a_i g_i)$, we see that $a^{-1} = \sum_{i=1}^n (g_i b_i f_{\sigma(i)})$ and $g_i = a^{-1} f_{\sigma(i)} a$. Set

$$L = P_{a_{\sigma(1)}} \oplus \ldots \oplus P_{a_{\sigma(r)}},$$

$$p = (\pi_{\sigma(1)}, \ldots, \pi_{\sigma(r)})^T a^{-1} : X \to L,$$

$$q = a(\iota_{\sigma(1)}, \ldots, \iota_{\sigma(r)}) : L \to X.$$

Then $f = qp$ and $pq = 1_L$. This yields an isomorphism $pi_Y : M \to P_{a_{\sigma(1)}} \oplus \ldots \oplus P_{a_{\sigma(r)}}$ with inverse $\pi_Y q$. □

For Λ a locally finite-dimensional algebra, we shall now define an exact contravariant functor $\mathfrak{D} : \text{Mod}\Lambda^\circ \to \text{Mod}\Lambda$ as follows. Given a module $M \in \text{Mod}\Lambda^\circ$, set $\mathfrak{D}M := \oplus_{x \in Q_0} \text{Hom}_k(e_x M, k)$. Given $f \in \text{Hom}_k(e_x M, k)$ and $u \in e_z \Lambda e_y$ with $x, y, z \in Q_0$, we define $f \cdot u \in \text{Hom}_k(e_y M, k)$ by setting $(f \cdot u)(m) = f(e_x \cdot u \cdot m)$ for $m \in e_y M$. In particular, $f \cdot e_x = f$

and $f \cdot u = 0$ in case $z \neq x$. This makes $\mathfrak{D}M$ into a unitary module in $\text{Mod}\Lambda$. Given a morphism $\psi : M \to N$ in $\text{Mod}\Lambda^{\circ}$, by restriction, we obtain k-linear maps $\psi_x : e_xM \to e_xN$ for $x \in Q_0$. This gives rise to a morphism $\mathfrak{D}\psi = \oplus_{x\in Q_0} \text{Hom}_k(\psi_x, k) : \mathfrak{D}N \to \mathfrak{D}M$ in $\text{Mod}\Lambda$. Similarly, we have an exact contravariant functor $\mathfrak{D} : \text{Mod}\Lambda \to \text{Mod}\Lambda^{\circ}$. Note that both of these functors preserve locally finite-dimensional modules. Thus we get restricted functors $\mathfrak{D} : \text{mod}\Lambda^{\circ} \to \text{mod}\Lambda$ and $\mathfrak{D} : \text{mod}\Lambda \to \text{mod}\Lambda^{\circ}$.

Proposition 1.4.3. *The functors* $\mathfrak{D} : \text{mod}\Lambda^{\circ} \to \text{mod}\Lambda$ *and* $\mathfrak{D} : \text{mod}\Lambda \to \text{mod}\Lambda^{\circ}$ *are mutually quasi-inverse dualities.*

Proof. We will show that $\mathfrak{D}^2 : \text{mod}\Lambda \to \text{mod}\Lambda$ is isomorphic to the identity functor on $\text{mod}\Lambda$. The proof for $\mathfrak{D}^2 : \text{mod}\Lambda^{\circ} \to \text{mod}\Lambda^{\circ}$ will be similar.

We first define a map $\phi_M : M \to \mathfrak{D}^2(M)$ for all $M \in \text{mod}\Lambda$. Let $m \in M$. Since M is unitary, $m = \sum_{x\in Q_0} m_x$ with $m_x \in Me_x$. By definition, $\mathfrak{D}^2(M) = \oplus_{y\in Q_0}\text{Hom}_k(e_y\mathfrak{D}M, k)$. We define $\phi(m)$ as follows. Let $f_y : e_y\mathfrak{D}M \to k$ be the map $e_y \cdot (g_x)_{x\in Q_0} \mapsto g_y(m_y)$, where $g_x \in \text{Hom}_k(Me_x, k)$. Set $\phi_M(m) := (f_y)_{y\in Q_0}$. Since $m_x = 0$ for all but finitely many $x \in Q_0$, this map is well-defined. In order to show that it is a Λ-module homomorphism, it is enough to show that $\phi_M(m \cdot u) = \phi_M(m) \cdot u$ for $u \in e_y\Lambda e_z$ with $y, z \in Q_0$. This is true because $\phi_M(m \cdot u) = (f_x)_{x\in Q_0}$, where $f_x : e_x\mathfrak{D}M \to k$ is the zero map for $x \neq z$, and $f_z : e_z\mathfrak{D}M \to k$ is the map $e_z \cdot (g_a)_{a\in Q_0} \mapsto g_z(m_y \cdot u)$. Moreover, if $\phi_M(m) = (l_x)_{x\in Q_0}$, then $\phi_M(m) \cdot u = (h_x)_{x\in Q_0}$, where $h_x : e_x\mathfrak{D}M \to k$ is the zero map for $x \neq z$ and $h_z : e_z\mathfrak{D}M \to k$ is the map $e_z \cdot (g_a)_{a\in Q_0} \mapsto l_y(u \cdot (g_a)_{a\in Q_0}) = (u \cdot (g_a)_{a\in Q_0})_y(m_y) = g_z(m_y \cdot u)$.

Now we show that ϕ_M is injective. Let $0 \neq m \in M$. Then there exists some $x \in Q_0$ such that $m_x \neq 0$. Let $\phi_M(m) = (f_y)_{y\in Q_0}$. Then $f_x((m_x)^*) = 1$, where $(m_x)^* : Me_x \to k$ denotes the linear map that takes the value 1 on m_x, and 0 on other basis elements. This gives that $\phi_M(m) \neq 0$.

Let $x \in Q_0$. The restriction of ϕ_M to Me_x gives an injective map $Me_x \to \text{Hom}_k(e_x\mathfrak{D}M, k)$.

Since M is locally finite-dimensional, both of the above modules are finite-dimensional, and hence, $Me_x \cong \mathrm{Hom}_k(e_x\mathfrak{D}M, k)$. This gives that ϕ_M is an isomorphism for all $M \in \mathrm{mod}\Lambda$. □

For $a \in Q_0$, let P_a° denote the left Λ-module Λe_a. This module is projective by an analog of Lemma 1.4.1 for left modules. Set $I_a := \mathfrak{D}(P_a^\circ)$.

Proposition 1.4.4. *I_a is an injective module for all $a \in Q_0$.*

Proof. Suppose we have two morphisms i and g with i injective.

$$\begin{array}{ccc} M & \xhookrightarrow{\ i\ } & I \\ {\scriptstyle g}\big\downarrow & & \\ N & & \end{array}$$

Since $\mathfrak{D}$ is an exact functor, we get that the map $\mathfrak{D}(i) : \mathfrak{D}(I) \to \mathfrak{D}(M)$ is surjective. Moreover, since $\mathfrak{D}$ is a duality, $\mathfrak{D}(I_a) \cong P_a^\circ$. Thus it is projective, and there exists $h : \mathfrak{D}(I_a) \to \mathfrak{D}(N)$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{D}(M) & \xleftarrow{\ \mathfrak{D}(i)\ } & \mathfrak{D}(I) \\ {\scriptstyle \mathfrak{D}(g)}\big\uparrow & \swarrow {\scriptstyle h} & \\ \mathfrak{D}(N) & & \end{array}$$

Then the map $\phi_I^{-1} \circ \mathfrak{D}(h) \circ \phi_N : N \to I$ makes the first diagram commute, where $\phi : id_{\mathrm{mod}\Lambda} \to \mathfrak{D}^2$ is the natural isomorphism obtained in the previous proposition. □

Define $\mathrm{inj}\Lambda$ to be the full subcategory of $\mathrm{mod}\Lambda$ whose objects are finite direct sums of I_a, $a \in Q_0$. Since a finite direct sum of injective modules is injective, every module in $\mathrm{inj}\Lambda$ is injective.

Definition 1.4.4. *Let M be an object of* $\mathrm{Mod}\Lambda$*. We say that M is*

1. finitely generated *if there exists an epimorphism* $f : P_0 \to M$, *for some* $P_0 \in \text{proj}\Lambda$;

2. finitely presented *if there exists an exact sequence*

$$P_1 \to P_0 \to M \to 0$$

for some $P_1, P_0 \in \text{proj}\Lambda$.

We also have the following dual notions.

Definition 1.4.5. *Let* M *be an object of* $\text{Mod}\Lambda$. *We say that* M *is*

1. finitely cogenerated *if there exists a monomorphism* $f : M \to I_0$, *for some* $I_0 \in \text{inj}\Lambda$;

2. finitely copresented *if there exists an exact sequence*

$$0 \to M \to I_0 \to I_1$$

for some $I_1, I_0 \in \text{inj}\Lambda$.

We will denote the full subcategory of finitely presented modules in $\text{mod}\Lambda$ by $\text{mod}^{+}\Lambda$ and the full subcategory of finitely co-presented modules by $\text{mod}^{-}\Lambda$. We claim that both of these categories are Krull-Schmidt categories for locally finite-dimensional algebras.

Definition 1.4.6. *Let* $\mathcal{A}$ *be an additive category. Then* $\mathcal{A}$ *is said to be* Krull-Schmidt *if every non-zero object in* $\mathcal{A}$ *decomposes into a finite direct sum of objects with local endomorphism rings.*

Theorem 1.4.1. *[19, Theorem 6.1] Let* $\mathcal{A}$ *be a* Hom*-finite additive category. Then* $\mathcal{A}$ *is Krull Schmidt if and only if* $\mathcal{A}$ *has split idempotents.*

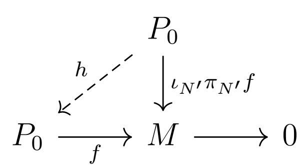
Theorem 1.4.2. *Let* $\Lambda = kQ/I$ *be a locally finite-dimensional algebra. Then,* $\text{mod}^{+}\Lambda$ *is a Krull Schmidt category.*

Proof. We first prove that $\mathrm{mod}^+\Lambda$ is Hom-finite. Let $M, N \in \mathrm{mod}^+\Lambda$ and $P_1 \to P_0 \to M \to 0$ be a projective presentation of M with $P_1, P_0 \in \mathrm{proj}\Lambda$. Applying the left exact functor $\mathrm{Hom}_\Lambda(-, N)$ to the above sequence, we get the following exact sequence

$$0 \to \mathrm{Hom}_\Lambda(M, N) \to \mathrm{Hom}_\Lambda(P_0, N) \to \mathrm{Hom}_\Lambda(P_1, N).$$

Since $P_0 \in \mathrm{proj}\Lambda$, $P_0 \cong \oplus_{i=1}^n P_{a_i}$ for $a_i \in Q_0$. Thus $\mathrm{Hom}(P_0, N) \cong \oplus_{i=1}^n \mathrm{Hom}(P_{a_i}, N) \cong \oplus_{i=1}^n N e_{a_i}$, where the last isomorphism follows from Lemma 1.4.1. Since N is locally finite-dimensional, $N e_{a_i}$ is finite-dimensional for all i, and hence $\mathrm{Hom}_\Lambda(P_0, N)$ is finite-dimensional. Thus $\mathrm{Hom}_\Lambda(M, N)$ is finite-dimensional.

We now want to show that the idempotents in $\mathrm{mod}^+\Lambda$ split. Let $M \in \mathrm{mod}^+\Lambda$ and $e : M \to M$ an idempotent. Then $M \cong e(M) \oplus (1_M - e)(M)$. Let $P_1 \xrightarrow{g} P_0 \xrightarrow{f} M \to 0$ be a projective presentation of M with $P_1, P_0 \in \mathrm{proj}\Lambda$. Set $N := e(M)$ and $N' := (1 - e)(M)$ and let $\pi_N : M \to N$, $\pi_{N'} : M \to N'$, and $\iota_N : N \to M$, $\iota_{N'} : N' \to M$ be the canonical projections and injections respectively. Since P_0 is projective, the following diagram



implies that there exists $h : P_0 \to P_0$ making the above triangle commute. We claim that the sequence $P_1 \oplus P_0 \xrightarrow{(-g,h)} P_0 \xrightarrow{\pi_N f} N \to 0$ is exact. Clearly, $\pi_N f$ is surjective being a composition of two surjective maps. Moreover,

$$(\pi_N f)(-g, h) = (-\pi_N f g, \pi_N f h) = (0, \pi_N \iota_{N'} \pi_{N'} f) = (0, 0).$$

Now suppose $p \in \mathrm{Ker}(\pi_N f)$. Then $f(p) \in N'$ and $fh(p) = \iota_{N'} \pi_{N'} f(p) = f(p)$. Thus $h(p) - p \in \mathrm{Ker}(f) = \mathrm{Im}(g)$. Thus there exists some $q \in P_1$ such that $g(q) = h(p) - p$ and $(-g, h)(q, p) = p$. This shows that $N \in \mathrm{mod}^+\Lambda$ and e splits. $\square$

CHAPTER 2

Auslander-Reiten theory

In this chapter, we will give a brief introduction to the Auslander-Reiten theory for finite-dimensional algebras over k, although everything stated here also holds for any Artin algebra over k. As we saw in the previous chapters, quiver-theoretical techniques provide a convenient way to visualise finite-dimensional algebras and certain modules over them. However, to actually compute all the finite-dimensional indecomposable modules and the homomorphisms between them, we need other tools. The notions of irreducible morphisms and almost split sequences are particularly useful for that. These were first formally introduced by Auslander [6], and Auslander and Reiten [8].

Throughout this chapter, Λ will denote a finite-dimensional algebra over k, unless stated otherwise. Moreover, all Λ-modules will be finite-dimensional.

2.1 Radical of a category

In this section, we will define the notion of the radical of a k-linear category $\mathcal{A}$, which generalizes the notion of the radical of an algebra when $\mathcal{A} = \text{mod}\Lambda$. We recall that a morphism is called a *retraction* (resp. *section*) if it has a right inverse (resp. left inverse) in $\mathcal{A}$.

Definition 2.1.1. *Let $\mathcal{A}$ be a k-linear category. An* ideal I *of $\mathcal{A}$ is given by the following data: for each pair (X, Y) of objects of $\mathcal{A}$, a k-subspace $\mathcal{I}(X, Y)$ of $\mathcal{A}(X, Y)$ such that*

1. *$f \in \mathcal{I}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ implies $g \circ f \in \mathcal{I}(X, Z)$.*

2. *$f \in \mathcal{I}(X, Y)$ and $h \in \mathcal{C}(W, X)$ implies $f \circ h \in \mathcal{I}(W, Y)$.*

Definition 2.1.2. *Let $\mathcal{I}$ be an ideal of a k-linear category $\mathcal{A}$. We define the* quotient *$\mathcal{A}/\mathcal{I}$ of $\mathcal{A}$ by $\mathcal{I}$ as follows.*

$$ob(\mathcal{A}/\mathcal{I}) := ob(\mathcal{A})$$

$$\mathcal{A}/\mathcal{I}(A, B) := \mathcal{A}(A, B)/\mathcal{I}(A, B)$$

for $A, B \in \mathcal{A}_0$. Let $f : X \to Y$, $g : Y \to Z$ be two morphisms in $\mathcal{A}$. We define their composition in $\mathcal{A}/\mathcal{I}$ as

$$(g + \mathcal{I}(Y, Z)) \circ (f + \mathcal{I}(X, Y)) = g \circ f + \mathcal{I}(X, Z).$$

Examples 2.1.1. *The following two examples will be important for us in the next chapter.*

1. *Let Λ be a k-algebra. Let $\mathcal{P}$ be the ideal of* $\mathrm{mod}\Lambda$ *formed by morphisms that factor through a module in* $\mathrm{proj}\Lambda$*. The quotient category* $\underline{\mathrm{mod}}\Lambda = \mathrm{mod}\Lambda/\mathcal{P}$ *is called the* projectively stable category.

2. *Let $\mathcal{I}$ be the ideal of* $\mathrm{mod}\Lambda$ *formed by morphisms that factor through a module in* $\mathrm{inj}\Lambda$*. The quotient category* $\overline{\mathrm{mod}}\Lambda = \mathrm{mod}\Lambda/\mathcal{I}$ *is called the* injectively stable category.

As in the case of algebras, we can also define the powers of an ideal of a category.

Definition 2.1.3. *Let $\mathcal{I}$ an ideal of $\mathcal{A}$. Set $\mathcal{I}^1 := \mathcal{I}$. For $m > 1$, we define $\mathcal{I}^m$ inductively as follows. For two objects $X, Y \in \mathcal{A}_0$,*

$$\mathcal{I}^m(X,Y) := \{g \circ f \mid g \in \mathcal{I}^{m-1}(Z,Y),\ h \in \mathcal{I}(X,Z)\}.$$

We also define $\mathcal{I}^\infty$ as

$$\mathcal{I}^\infty(X,Y) := \bigcap_{m \geq 1} \mathcal{I}^m(X,Y).$$

Definition 2.1.4. *[3, Lemma I.5.2] The* radical *of a k-category $\mathcal{A}$ is an ideal* $\mathrm{rad}_{\mathcal{A}}$*, such that for each pair (X,Y) of objects of $\mathcal{A}$,* $\mathrm{rad}_{\mathcal{A}}(X,Y)$ *is the collection of morphisms $f \in \mathcal{A}(X,Y)$ such that $1_Y - f \circ g$ is a retraction for all $g : Y \to X$. The morphisms lying in some* $\mathrm{rad}_{\mathcal{A}}(X,Y)$ *are called* radical morphisms*.*

The following theorem gives several equivalent characterizations of the radical.

Theorem 2.1.1. *[4, Theorem II.1.17] Let $\mathcal{A}$ be a k-category. A morphism $f : X \to Y$ of $\mathcal{A}$ is a radical morphism if and only if one of the following equivalent conditions holds.*

1. *$1_Y - f \circ g$ is a retraction for all $g : Y \to X$;*
2. *$1_X - g \circ f$ is a section for all $g : Y \to X$;*
3. *$f \circ g \in \mathrm{rad}(\mathrm{End}_{\mathcal{A}} Y)$ for all $g : Y \to X$;*
4. *$g \circ f \in \mathrm{rad}(\mathrm{End}_{\mathcal{A}} X)$ for all $g : Y \to X$;*
5. *$f \in F(X)$ for all maximal sub-functors F of $\mathcal{A}(-,Y)$;*
6. *$f \in F(Y)$ for all maximal sub-functors F of $\mathcal{A}(X,-)$.*

Now, we specify $\mathcal{A} = \mathrm{mod}\Lambda$. If X, Y in $\mathrm{mod}\Lambda$, we will simply write $\mathrm{rad}_{\mathrm{mod}\Lambda}(X,Y)$ as $\mathrm{rad}_\Lambda(X,Y)$. In case X or Y is indecomposable, the description of $\mathrm{rad}_\Lambda(X,Y)$ becomes simpler.

Proposition 2.1.1. *[4, Corollary II.1.10] Let $f : M \to N$ be a morphism of Λ-modules.*

1. *If M is indecomposable, then f is radical if and only if f is not a section.*

2. *If N is indecomposable, then f is radical if and only if f is not a retraction.*

2.2 Irreducible morphisms

The last proposition indicates that the essential information about $\mathrm{mod}\Lambda$ is contained in its radical, hence we want to have a method to construct all the radical morphisms of $\mathrm{mod}\Lambda$. In this section, we introduce the notion of irreducible morphisms which are an analogue of indecomposable modules for radical morphisms, i.e., the smallest building blocks for such morphisms.

Definition 2.2.1. *A homomorphism $f : X \to Y$ in* $\mathrm{mod}\Lambda$ *is said to be* irreducible *provided:*

1. *f is neither a section nor a retraction,*

2. *if $f = f_1 \circ f_2$, either f_1 is a retraction or f_2 is a section.*

The next lemma gives us a relation between irreducible morphisms and the radical of $\mathrm{mod}\Lambda$. It shows that when X, Y are indecomposable, the quotient space $\frac{\mathrm{rad}_\Lambda(X,Y)}{\mathrm{rad}^2_\Lambda(X,Y)}$ measures the irreducible morphisms from X to Y.

Lemma 2.2.1. *[5, Lemma IV.1.6] Let X, Y be indecomposable modules in* $\mathrm{mod}\Lambda$*. A morphism $f : X \to Y$ is irreducible if and only if $f \in \mathrm{rad}_\Lambda(X,Y) \setminus \mathrm{rad}^2_\Lambda(X,Y)$.*

Irreducible morphisms can also help us find some indecomposable modules over Λ.

Lemma 2.2.2. *[4, Corollary II.2.9]*

1. *The cokernel of an irreducible monomorphism is indecomposable.*
2. *The kernel of an irreducible epimorphism is indecomposable.*

The following results make it precise how irreducible morphisms 'generate' all the radical morphisms. Although the proofs use machinery from the next section, we state them here for the sake of completion.

Proposition 2.2.1. *[4, Proposition II.4.4] Let M, N be indecomposable modules and $f \in \mathrm{rad}_\Lambda^n(M,N)$ for some $n \geq 2$. Then*

1. *There exist s indecomposable modules $X_1, \dots, X_s$ and morphisms $M \xrightarrow{h_i} X_i \xrightarrow{g_i} N$ with $h_i \in \mathrm{rad}_\Lambda(M, X_i)$ and g_i a sum of compositions of $n-1$ radical morphisms between indecomposable modules such that $f = \sum_{i=1}^s g_i \circ h_i$. If, in addition, f is not in $\mathrm{rad}_\Lambda^{n+1}(M,N)$, then at least one of the h_i is irreducible and f can be written as $f = u + v$, where u is a sum of compositions of n irreducible morphisms between indecomposable modules and $v \in \mathrm{rad}_\Lambda^{n+1}(M,N)$.*
2. *There exist s indecomposable modules $X_1, \dots, X_s$ and morphisms $M \xrightarrow{h_i} X_i \xrightarrow{g_i} N$ with $g_i \in \mathrm{rad}_\Lambda(X_i, N)$ and h_i a sum of compositions of $n-1$ radical morphisms between indecomposable modules such that $f = \sum_{i=1}^s g_i \circ h_i$. If, in addition, f is not in $\mathrm{rad}_\Lambda^{n+1}(M,N)$, then at least one of the g_i is irreducible and f can be written as $f = u + v$, where u is a sum of compositions of irreducible morphisms between indecomposable modules and $v \in \mathrm{rad}_\Lambda^{n+1}(M,N)$.*

Corollary 2.2.1. *[4, Corollary II.4.5] Let M, N be indecomposable modules. Then, every radical morphism $f \in \mathrm{rad}_\Lambda(M,N)$ can be written as $f = u+v$, where u is a sum of compositions of irreducible morphisms, and $v \in \mathrm{rad}_\Lambda^\infty(M,N)$. In particular, if $\mathrm{rad}_\Lambda^\infty(M,N) = 0$, then f is a sum of compositions of radical morphisms.*

2.3 Almost split morphisms and minimal morphisms

As stated before, the consideration of irreducible morphisms came from the need to identify building blocks for radical morphisms, so that other radical morphisms could be obtained from the irreducible ones by successive compositions and linear combinations. Therefore, the next step is to study the factorisation behaviour of radical morphisms.

Definition 2.3.1. *Let L, M, N be Λ-modules.*

1. *A morphism $f : L \to M$ is called* left almost split *if it is not a section and for every morphism $u : L \to U$ that is not a section there exists $u' : M \to U$ such that $u' \circ f = u$.*

2. *A morphism $g : M \to N$ is called* right almost split *if it is not a retraction and for every morphism $v : V \to N$ that is not a retraction, there exists $v' : V \to M$ such that $g \circ v' = v$.*

We will state some families of examples of left and right almost split morphisms below. The proofs for these can be found in [5].

Examples 2.3.1. 1. *Let P be a projective indecomposable Λ-module. Then the inclusion $j : \text{rad}P \to P$ is right almost split.*

2. *Dually, if I is an injective indecomposable, then the projection $I \to I/\text{soc}I$ is left almost split.*

3. *Suppose $f : L \to M$ is left almost split and $f' : L \to M'$ is radical. Then the morphism*

$$\begin{bmatrix} f \\ f' \end{bmatrix} : L \to M \oplus M'$$

is also left almost split.

4. *Dually, if* $g : M \to N$ *is right almost split and* $g' : M' \to N$ *is radical, then*

$$\begin{bmatrix} g & g' \end{bmatrix} : M \oplus M' \to N$$

is also right almost split.

The last example suggests that the 'good' almost split morphisms should satisfy some 'minimality' condition, namely that the target of a left almost split morphism or the source of a right almost split morphism should be as small as possible.

Definition 2.3.2. 1. *A morphism* $f : L \to M$ *is called* left minimal *if for every* $h \in \mathrm{End}M$ *such that* $h \circ f = f$*,* h *is an automorphism.*

2. *A morphism* $g : M \to N$ *is called* right minimal *if for every* $h \in \mathrm{End}M$ *such that* $g \circ h = g$*,* h *is an automorphism.*

The following proposition shows that this is indeed the correct notion of minimality we wanted for almost split morphisms.

Proposition 2.3.1. *[4, Proposition II.2.19]*

1. *Let* $f : L \to M$ *be a left almost split morphism. Then,* f *is left minimal if and only if its target* M *has the least length among the targets of left almost split morphisms with source* L*. In addition, this condition uniquely determines* f *up to isomorphism.*

2. *Let* $g : M \to N$ *be a right almost split morphism. Then,* g *is right minimal if and only if its source* M *has the least length among the sources of right almost split morphisms with target* N*. In addition, this condition uniquely determines* g *up to isomorphism.*

Definition 2.3.3. 1. *A morphism is called* minimal left almost split *if it is left minimal and left almost split.*

2. *A morphism is called* minimal right almost split *if it is right minimal and right almost split.*

Example 2.3.1. *[5] For every indecomposable projective module P, the inclusion morphism* $\mathrm{rad}(P) \to P$ *is minimal right almost split. Dually, for every indecomposable injective module I, the projection $I \to I/\mathrm{soc}(I)$ is minimal left almost split.*

The following set of results gives a connection between irreducible morphisms and almost split morphisms.

Lemma 2.3.1. *Every irreducible morphism is both left and right minimal.*

Proof. We just give a proof for left minimality. The proof for right minimality is similar. Let $f : L \to M$ be irreducible and $h \in \mathrm{End}M$ be such that $h \circ f = f$. Since f is not a section, h must be a retraction, and in particular an epimorphism. But then h is an automorphism because M is finite-dimensional. □

Lemma 2.3.2. *Every nonzero minimal left or right almost split morphism is irreducible.*

Proof. We only prove the statement for minimal left almost split morphisms, the other case being dual. Let $f : L \to M$ be a minimal left almost split morphism. Then by definition, f is not a section. It is not a retraction either, because otherwise the indecomposability of L would imply that it is an isomorphism, and hence a section, which is a contradiction.

Now assume that $f = f_1 \circ f_2$ with $f_2 : L \to X$ and $f_1 : X \to M$. Suppose that f_2 is not a section. Since f is left almost split, there exists $f_2' : M \to X$ such that $f_2 = f_2' \circ f$. But then $f = f_1 \circ f_2 = f_1 \circ f_2' \circ f$ and the left minimality of f yields that $f_1 \circ f_2'$ is an automorphism. Hence, f_1 is a retraction. □

The next theorem is often called the structure theorem for irreducible morphisms. It says that irreducible morphisms with a given indecomposable source (or target) are exactly those morphisms that can be completed to a minimal almost split morphism having the same source (or target respectively).

Theorem 2.3.1. *[4, Theorem II.2.24]*

1. *Let $f : L \to M$ be minimal left almost split. Then, $f' : L \to M'$ is irreducible with $M' \neq 0$ if and only if there exists a decomposition $M = M' \oplus M''$ and a morphism $f'' : L \to M''$ such that $\begin{bmatrix} f' \\ f'' \end{bmatrix} : L \to M$ is minimal left almost split.*
2. *Let $g : M \to N$ be minimal right almost split. Then, $g' : M' \to N$ is irreducible with $M' \neq 0$ if and only if there exists a decomposition $M = M' \oplus M''$ and a morphism $g'' : M'' \to N$ such that $\begin{bmatrix} g' & g'' \end{bmatrix} : M \to N$ is minimal right almost split.*

2.4 Almost split sequences

In this section, we will present the notion of almost split sequences which are particularly important in the representation theory of algebras. We will also show that there exist sufficiently many minimal almost split morphisms inside the module category, in the sense that every indecomposable module is a source or target of a minimal almost split morphism.

Definition 2.4.1. *A short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is called an* almost split sequence *(or an* Auslander–Reiten sequence*) if f and g are irreducible morphisms.*

Remarks 2.4.1. *1. Because irreducible morphisms never split, an almost split sequence never splits.*

2. *Because of Lemma 2.2.2, f irreducible implies N indecomposable, and g irreducible implies L indecomposable: an almost split sequence always has indecomposable end terms.*

3. *Lemma 2.3.1 implies that both f and g are left and right minimal.*

Example 2.4.1. *[3, Example III.2.22] Let Λ be the algebra given by $Q = 3 \xrightarrow{a} 2 \xrightarrow{b} 1$ and $\rho = \{ab\}$. Then one can check that the sequence*

$$0 \to S_2 \to P_3 \to S_3 \to 0$$

is almost split as both of the morphisms are irreducible.

The following theorem gives several equivalent characterizations of almost split sequences.

Theorem 2.4.1. *[3, Theorem III.2.25] Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence in* $\mathrm{mod}\Lambda$*. The following are equivalent:*

1. *The given sequence is almost split.*

2. *L is indecomposable, and g is right almost split.*

3. *N is indecomposable, and f is left almost split.*

4. *The homomorphism f is minimal left almost split.*

5. *The homomorphism g is minimal right almost split.*

Corollary 2.4.1. *An almost split sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is uniquely determined by L (or by N) up to isomorphism.*

Proof. Let $0 \to L \xrightarrow{f'} M' \xrightarrow{g'} N' \to 0$ be another almost split sequence. Since f and f' are minimal left almost split, it follows from Proposition 2.3.1 that there exists an isomorphism $h : M \to M'$ such that $h \circ f = f'$. Passing to cokernels, we get an isomorphism $h' : N \to N'$ such that $h' \circ g = g' \circ h$ Thus, the sequences are isomorphic. The proof is similar when we fix N. □

We finally arrive at the theorem of the existence of almost split sequences.

Theorem 2.4.2. *[4, Theorem II.3.12] Let N be a non-projective indecomposable Λ-module, or L a non-injective indecomposable Λ-module. Then there exists an almost split sequence*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0.$$

Moreover, this sequence is uniquely determined by N, or by L, up to isomorphism.

For a non-projective indecomposable module N, the module L obtained from the above theorem is the *Auslander-Reiten translate*, denoted $\tau(N)$, of N. Dually, for a non-injective indecomposable module L, the module N obtained from the above theorem is denoted by $\tau^{-}(N)$. We will give the precise definition of τ in the next chapter.

As an easy consequence of the last theorem, we get that the module category contains enough minimal almost split morphisms.

Corollary 2.4.2.

1. *If N is an indecomposable Λ-module, then there exists a minimal right almost split morphism $g : M \to N$.*
2. *If L is an indecomposable Λ-module, then there exists a minimal left almost split morphism $f : L \to M$.*

Proof. We only prove (2); the proof of (1) is dual. If L is injective, then the projection $\pi : L \to L/\mathrm{soc}(L)$ is minimal left almost split. Otherwise, there exists an almost split sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ in which the morphism f is minimal left almost split. □

CHAPTER 3

Auslander-Reiten theory for locally semi-perfect algebras

In this chapter, our goal will be to prove the Auslander-Reiten formula for locally semi-perfect algebras defined by quivers with relations, which, in turn, would imply the existence of almost split exact sequences in the module category of such algebras. We will define appropriate generalizations of the Nakayama functor and the Auslander-Reiten translate from the finite-dimensional case.

Throughout this chapter, $\Lambda = kQ/\langle\rho\rangle$ will denote a locally finite-dimensional algebra, unless stated otherwise.

3.1 The transpose functor

Let $M \in \mathrm{Mod}\Lambda$. Then M is a right Λ-module. We want to define a left Λ-module structure on $\oplus_{x\in Q_0}\mathrm{Hom}_\Lambda(M, e_x\Lambda)$. We will denote this module by M^t. Let $x \in Q_0$. Since

$\Lambda = \oplus_{y,z\in Q_0} e_y\Lambda e_z$, it is enough to define the action of $e_y\Lambda e_z$ on $\mathrm{Hom}_\Lambda(M, e_x\Lambda)$ and then extend bilinearly. Let $f \in \mathrm{Hom}_\Lambda(M, e_x\Lambda)$ and $a \in e_y\Lambda e_z$. We define $a \cdot f \in \mathrm{Hom}_\Lambda(M, e_y\Lambda)$ such that $(a\cdot f)(m) := af(m)$ for all $m \in M$. Note that if $z \neq x$, then $a \cdot f = 0$.

We need to check that this does make M^t into a left Λ-module, i.e., we need to check that if $a \in e_y\Lambda e_z$ and $b \in e_{y'}\Lambda e_{z'}$, then $(ab)\cdot f = a\cdot(b\cdot f)$. Let $m \in M$. Then

$$((ab)\cdot f)(m) = (ab)f(m) = a(b(f(m))) = (a\cdot(b\cdot f))(m),$$

where we have used the associativity of Λ for the second equality. This shows that M^t is indeed a left Λ-module. It is also a unitary module because $e_x \cdot M^t = \mathrm{Hom}_\Lambda(M, e_x\Lambda)$. Moreover, given a morphism $\varphi : M \to N$ in $\mathrm{Mod}\Lambda$, we define $\varphi^t = \oplus_{x\in Q_0}\mathrm{Hom}_\Lambda(\varphi, e_x\Lambda)$. This yields a contravariant functor $(-)^t : \mathrm{Mod}\Lambda \to \mathrm{Mod}\Lambda^\circ$. Similarly, we have a contravariant functor $(-)^t : \mathrm{Mod}\Lambda^\circ \to \mathrm{Mod}\Lambda$.

Lemma 3.1.1. *$(-)^t$ is a left-exact functor.*

Proof. Suppose we have the following short exact sequence in $\mathrm{Mod}\Lambda$:

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0.$$

Applying $(-)^t$ to this, we get the chain complex

$$0 \to \oplus_{x\in Q_0}\mathrm{Hom}_\Lambda(N, e_x\Lambda) \xrightarrow{g'} \oplus_{x\in Q_0}\mathrm{Hom}_\Lambda(M, e_x\Lambda) \xrightarrow{f'} \oplus_{x\in Q_0}\mathrm{Hom}_\Lambda(L, e_x\Lambda),$$

where $g' = \oplus_{x\in Q_0}\mathrm{Hom}_\Lambda(g, e_x\Lambda)$ and $f' = \oplus_{x\in Q_0}\mathrm{Hom}_\Lambda(f, e_x\Lambda)$. We first show that g' is injective. Let $(n_x)_{x\in Q_0} \in \oplus_{x\in Q_0}\mathrm{Hom}_\Lambda(N, e_x\Lambda)$ such that $g'((n_x)_{x\in Q_0}) = (n_x \circ g)_{x\in Q_0} = 0$. This implies that $n_x \circ g = 0$ for all $x \in Q_0$. Since g is an epimorphism, this implies that $n_x = 0$ for all $x \in Q_0$.

Now let $(m_x)_{x\in Q_0} \in \oplus_{x\in Q_0}\mathrm{Hom}_\Lambda(M, e_x\Lambda)$ be such that $f'((m_x)_{x\in Q_0}) = (m_x \circ f)_{x\in Q_0} = 0$. Let $J = \{x \in Q_0 \mid m_x \neq 0\}$. By definition, J is a finite set. Since $m_x \circ f = 0$, m_x factors through the cokernel of f, i.e., there exists some $n_x : N \to e_x\Lambda$ such that $n_x \circ g = m_x$. Note that for $x \notin J$, we can choose $n_x = 0$. Then $(n_x)_{x\in Q_0} \in \oplus_{x\in Q_0}\mathrm{Hom}_\Lambda(N, e_x\Lambda)$ and $g'((n_x)_{x\in Q_0}) = (m_x)_{x\in Q_0}$. □

Proposition 3.1.1. *Let $\Lambda = kQ/I$ be a locally finite-dimensional algebra. Then the functor $(-)^t$ induces a duality $(-)^t : \mathrm{proj}\Lambda \to \mathrm{proj}\Lambda^\circ$.*

Proof. Let $a \in Q_0$. We claim that $(e_a\Lambda)^t \cong \Lambda e_a$ and $(\Lambda e_a)^t \cong e_a\Lambda$. We will only prove the first isomorphism. The proof of the second will be dual.

We first define a map $\phi_a : (e_a\Lambda)^t \to \Lambda e_a$. Let $(f_x)_{x\in Q_0} \in (e_a\Lambda)^t = \oplus_{x\in Q_0}\mathrm{Hom}_\Lambda(e_a\Lambda, e_x\Lambda)$. Since f_x is a Λ-module homomorphism, $f_x(e_a) = f_x(e_a \cdot e_a) = f_x(e_a) \cdot e_a$, which implies that $f_x(e_a) \in e_x\Lambda e_a$ for all $x \in Q_0$. We set $\phi((f_x)_{x\in Q_0}) := \sum_{x\in Q_0} f_x(e_a)$. Since $f_x = 0$ for all but finitely many $x \in Q_0$, this map is well-defined. Clearly, ϕ is k-linear. In order to show that it is a Λ-module homomorphism, it is enough to show that $u \cdot \phi((f_x)_{x\in Q_0}) = \phi(u \cdot (f_x)_{x\in Q_0})$ for $u \in e_y\Lambda e_z$ with $y, z \in Q_0$. This is true because

$$\phi(u \cdot (f_x)_{x\in Q_0}) = \sum_{x\in Q_0} (u \cdot (f_x))(e_a) = \sum_{x\in Q_0} u \cdot f_x(e_a) = u \cdot \phi((f_x)_{x\in Q_0}).$$

Since $\{e_x \mid x \in Q_0\}$ is a complete set of idempotents in Λ, $\Lambda e_a \cong \oplus_{x\in Q_0} e_x\Lambda e_a$ as vector spaces. Hence, if $\phi((f_x)_{x\in Q_0}) = \sum_{x\in Q_0} f_x(e_a) = 0$, then $f_x = 0$ for all $x \in Q_0$. This shows that ϕ is injective. Moreover, for $u \in \Lambda e_a$, u can be uniquely written as $u = \sum_{x\in Q_0} u_x$ with $u_x \in e_x\Lambda e_a$. Defining $f_x : e_a\Lambda \to e_x\Lambda$ as $e_a \cdot \mu \mapsto u_x \cdot \mu$ for $\mu \in \Lambda$, we get that $\phi((f_x)_{x\in Q_0}) = u$. Hence ϕ is surjective.

Dually we get an isomorphism $\psi_a : (\Lambda e_a)^t \to e_a\Lambda$. Thus, for all $a \in Q_0$, we get isomorphisms $(\phi_a)^t(\psi_a)^{-1} : e_a\Lambda \to (e_a\Lambda)^{tt}$ and $(\psi_a)^t(\phi_a)^{-1} : \Lambda e_a \to (\Lambda e_a)^{tt}$. Moreover, these isomorphisms are natural in the sense that for all morphisms $p : e_a\Lambda \to e_b\Lambda$ and $q : \Lambda e_a \to \Lambda e_b$ with $a, b \in Q_0$, the following diagrams commute:

$$\begin{array}{ccccccc} e_a\Lambda & \xrightarrow{p} & e_b\Lambda & & \Lambda e_a & \xrightarrow{q} & \Lambda e_b \\ {\scriptstyle (\phi_a)^t(\psi_a)^{-1}}\downarrow & & \downarrow{\scriptstyle (\phi_b)^t(\psi_b)^{-1}} & & {\scriptstyle (\psi_a)^t(\phi_a)^{-1}}\downarrow & & \downarrow{\scriptstyle (\psi_b)^t(\phi_b)^{-1}} \\ (e_a\Lambda)^{tt} & \xrightarrow[p^{tt}]{} & (e_b\Lambda)^{tt} & & (\Lambda e_a)^{tt} & \xrightarrow[q^{tt}]{} & (\Lambda e_b)^{tt} \end{array}$$

Since $(-)^t$ is an additive functor and $\mathrm{proj}\Lambda$ and $\mathrm{proj}\Lambda^\circ$ are additively generated by $e_a\Lambda$ and Λe_a respectively, we get that $(-)^t : \mathrm{proj}\Lambda \to \mathrm{proj}\Lambda^\circ$ is a duality. □

3.2 Nakayama functor

We now consider the composite endo-functors $\nu = \mathfrak{D} \circ (-)^t : \mathrm{Mod}\Lambda \to \mathrm{Mod}\Lambda$ and $\nu^- = (-)^t \circ \mathfrak{D} : \mathrm{Mod}\Lambda \to \mathrm{Mod}\Lambda$, and call ν the *Nakayama functor*.

Lemma 3.2.1. *Let $\Lambda = kQ/\langle\rho\rangle$ be a locally finite-dimensional algebra. Then the Nakayama functor ν restricts to a duality $\nu : \mathrm{proj}\Lambda \to \mathrm{inj}\Lambda$ with a quasi-inverse $\nu^- : \mathrm{inj}\Lambda \to \mathrm{proj}\Lambda$.*

Proof. Let $a \in Q_0$. We have a k-linear isomorphism

$$(P_a)^t = \oplus_{x\in Q_0}\mathrm{Hom}_\Lambda(e_a\Lambda, e_x\Lambda) \cong \oplus_{x\in Q_0} e_x\Lambda e_a = \Lambda e_a.$$

It is easy to verify that this is indeed a Λ-linear isomorphism. Thus, $\nu(P_a) \cong I_a$. Moreover, we have:

$$\nu^-(I_a) = \oplus_{x\in Q_0}\mathrm{Hom}_{\Lambda^\circ}(\mathfrak{D}^2(\Lambda e_a), \Lambda e_x) \cong \oplus_{x\in Q_0}\mathrm{Hom}_{\Lambda^\circ}(\Lambda e_a, \Lambda e_x) \cong \oplus_{x\in Q_0} e_a\Lambda e_x \cong e_a\Lambda.$$

This shows that ν and ν^- are quasi-inverse of each other. □

3.3 Auslander-Reiten translation

We now define the notion of transposition for a finitely-presented module. We first start by proving some preliminary results we need to do this.

Lemma 3.3.1. *Let Λ be a locally semi-perfect algebra. Then,* $\operatorname{proj}\Lambda$ *is a Krull-Schmidt category and contains all finitely generated projective modules in* $\operatorname{mod}\Lambda$*. Moreover,* $\underline{\operatorname{mod}}^{+}\Lambda$ *is also a Krull-Schmidt category.*

Proof. Since $\operatorname{proj}\Lambda$ is a full subcategory of $\operatorname{mod}^{+}\Lambda$, it is Hom-finite by the proof of Theorem 1.4.2. Moreover, by Proposition 1.4.2, it is closed under direct summands, i.e., the idempotents split. Therefore, using Theorem 1.4.1, it is a Krull-Schmidt category. This implies that every non-zero object in $\operatorname{mod}^{+}\Lambda$ decomposes into a finite direct sum of objects with local endomorphism rings. Note that if $\operatorname{End}_{\operatorname{mod}^{+}\Lambda}(M)$ is local for $M \in \operatorname{mod}^{+}\Lambda$, then $\underline{\operatorname{End}}(M) := \operatorname{End}_{\underline{\operatorname{mod}}^{+}\Lambda}(M)$ is either 0 or local. This is because by definition $\underline{\operatorname{End}}(M) = \operatorname{End}(M)/\mathcal{P}(M,M)$. If $\mathcal{P}(M,M) \neq \operatorname{End}(M)$, then it is contained in the unique maximal ideal $\operatorname{rad}(\operatorname{End}(M))$ of $\operatorname{End}(M)$. Thus, $\frac{\operatorname{rad}(\operatorname{End}(M))}{\mathcal{P}(M,M)}$ will be the unique maximal ideal of $\frac{\operatorname{End}(M)}{\mathcal{P}(M,M)}$. Thus, $\underline{\operatorname{mod}}^{+}\Lambda$ is also a Krull-Schmidt category. □

Throughout the rest of this chapter, we will assume Λ to be a locally semi-perfect algebra. Given a module $P \in \operatorname{proj}\Lambda$, we denote by $\operatorname{add}P$ the full additive subcategory of $\operatorname{proj}\Lambda$ generated by the indecomposable direct summands of P.

Lemma 3.3.2. *Let $\Sigma = \operatorname{End}_{\Lambda}(P)$, where P is a non-zero module in* $\operatorname{proj}\Lambda$*. The exact functor* $\operatorname{Hom}_{\Lambda}(P,-) : \operatorname{Mod}\Lambda \to \operatorname{Mod}\Sigma$ *restricts to an equivalence* $E_P : \operatorname{add}P \to \operatorname{proj}\Sigma$*. In particular, a morphism f in* $\operatorname{add}P$ *is a section (respectively, retraction, isomorphism) if and only if so is $E_P(f)$.*

Proof. Let $M \in \operatorname{Mod}\Lambda$, $f \in \operatorname{Hom}_{\Lambda}(P,M)$, and $\sigma \in \Sigma$. Then

$$f \cdot \sigma := f \circ \sigma$$

defines a right Σ-module structure on $\operatorname{Hom}_{\Lambda}(P,M)$. Since $P \in \operatorname{proj}\Lambda$, we can write $P = P_1 \oplus \cdots \oplus P_n$, where the P_i are indecomposable. Let $p_i : P \to P_i$ be the canonical projections and $q_i : P_i \to P$ the canonical injections. Set $e_i = q_i p_i$, for $i = 1, \dots, n$. Clearly $\{e_1, \dots, e_n\}$ is a complete set of orthogonal idempotents of Σ. We claim that each e_i is primitive. This is because if $e_i = a + b$, where a, b are orthogonal idempotents in Σ, then $P_i \cong a(P) \oplus b(P)$, and the indecomposability of P_i implies that either $a = 0$ or $b = 0$. Thus $\{e_1, \dots, e_n\}$ is a complete set of primitive orthogonal idempotents of Σ. Since Σ is a finite-dimensional algebra, therefore, $e_1\Sigma, \dots, e_n\Sigma$ are precisely the indecomposable objects in projΣ ([2, Theorem VIII.1.9]).

Note that the restriction of $E_P(p_i)$ to $e_i\Sigma$, that is $E_P(p_i) : e_i\Sigma \to \mathrm{Hom}_\Lambda(P, P_i)$, is a Σ-isomorphism with the inverse given by $E_P(q_i) : \mathrm{Hom}_\Lambda(P, P_i) \to e_i\Sigma$. Since proj$\Sigma$ is additively generated by $e_i\Sigma$ and E_P is an additive functor, we get that E_P is dense.

Given any $1 \le i, j \le n$, we claim that E_P induces a Σ-isomorphism

$$E_P : \mathrm{Hom}_\Lambda(P_i, P_j) \to \mathrm{Hom}_\Sigma(\mathrm{Hom}_\Lambda(P, P_i), \mathrm{Hom}_\Lambda(P, P_j)).$$

Define $H : \mathrm{Hom}_\Sigma(\mathrm{Hom}_\Lambda(P, P_i), \mathrm{Hom}_\Lambda(P, P_j)) \to \mathrm{Hom}_\Lambda(P_i, P_j)$ as $f \mapsto f(p_i)q_i$. Then, for $g \in \mathrm{Hom}_\Lambda(P_i, P_j)$, $f \in \mathrm{Hom}_\Sigma(\mathrm{Hom}_\Lambda(P, P_i), \mathrm{Hom}_\Lambda(P, P_j))$, and $h \in \mathrm{Hom}_\Lambda(P, P_i)$, we get that $(HE_P)(g) = E_p(g)(p_i)q_i = gp_iq_i = g$ and

$$(E_P H)(f)(h) = H(f)h = f(p_i)q_i h = f(p_i q_i h) = f(h),$$

where the second last equality holds because f is a Σ-homomorphism. Thus H is the inverse of E_P. Again, since E_P is an additive functor and add$P$ is additively generated by P_i, we get that E_P is a fully faithful functor. Thus the restriction of E_P is an equivalence. □

Definition 3.3.1. *A morphism $f : M \to N$ in* modΛ *is called* right minimal *if any map $g : M \to M$, such that $fg = f$, is an automorphism.*

Lemma 3.3.3. *Let $f : M \to N$ be a right minimal morphism in* modΛ. *If $M = M_1 \oplus M_2$ is a proper decomposition, then the restriction $f\,|_{M_i}$ is non-zero, for $i = 1, 2$.*

Proof. Let $M = M_1 \oplus M_2$ be a proper decomposition with $p_i : M \to M_i$ the canonical projection and $q_i : M_i \to M$ the canonical injection, for $i = 1, 2$. Then $1_M = q_1p_1 + q_2p_2$. Assume that $f|_{M_1} = 0$, that is $fq_1 = 0$. Then $f = f(q_1p_1 + q_2p_2) = fq_2p_2$. Since f is right minimal, q_2p_2 is an isomorphism. In particular, p_2 is a section, and hence an isomorphism. This gives that $M_1 = 0$, a contradiction. □

Definition 3.3.2. *Let $M, N \in \text{mod}\Lambda$. An epimorphism $f : M \to N$ is called* essential *if whenever fg is an epimorphism for $g : M' \to M$ with $M' \in \text{mod}\Lambda$, then g is also an epimorphism.*

Definition 3.3.3. *Let $M \in \text{mod}\Lambda$. A* projective cover *of M is an essential epimorphism $p : P \to M$ with $P \in \text{proj}\Lambda$.*

We also have the dual notions of essential monomorphisms and injective envelopes.

Definition 3.3.4. *Let $M, N \in \text{mod}\Lambda$. A monomorphism $f : M \to N$ is called* essential *if whenever gf is a monomorphism for $g : N \to N'$ with $N' \in \text{mod}\Lambda$, then g is also a monomorphism.*

Definition 3.3.5. *Let $M \in \text{mod}\Lambda$. An* injective envelope *of M is an essential monomorphism $p : M \to I$ with $I \in \text{inj}\Lambda$.*

Lemma 3.3.4. *Let $P \in \text{proj}\Lambda$ and $M \in \text{mod}\Lambda$. Then an epimorphism $p : P \to M$ is a projective cover of M if and only if p is right minimal.*

Proof. Suppose p is a projective cover of M and $f : P \to P$ such that $pf = p$. Since p is an essential epimorphism and pf is an epimorphism, f is an epimorphism. Thus there exists $f' : P \to P$ such that $ff' = 1_P$ since P is projective. Therefore $pff' = pf' = p$ and f' is also an epi. But f' is a monomorphism, so it is an isomorphism, and so is f.

Conversely, assume that p is right minimal. Let $f : N \to P$ be a morphism such that pf is an epimorphism. Then p factors through pf via a morphism $g : P \to N$ since P is projective. Since p is right minimal, the composite fg is an isomorphism, and therefore f is an epimorphism. Thus p is essential. □

Lemma 3.3.5. *Let $p : P \to M$ be an epimorphism in* modΛ *with $P \in$* projΛ*. Then there exists a decomposition $P = P_1 \oplus P_2$ such that $p\,|_{P_1}$ is right minimal and $p\,|_{P_2} = 0$.*

Proof. We may assume that P is non-zero. Since projΛ is Krull-Schmidt, P is a finite direct sum of indecomposable objects with local endomorphism rings. Hence $\Sigma := \mathrm{End}(P)$ is semi-perfect ([18, Proposition 1.1]). Since P is projective,

$$p^* = \mathrm{Hom}(P, p) : \mathrm{Hom}_\Lambda(P, P) \to \mathrm{Hom}_\Lambda(P, M)$$

is a Σ-epimorphism. Thus, there exists a Σ-projective cover $q : L \to \mathrm{Hom}_\Lambda(P, M)$ ([1, Theorem 27.6]). Since Σ is Σ-projective, there exist Σ-morphisms $\pi : \Sigma \to L$ and $\mu : L \to \Sigma$ such that $q = p^*\mu$, $p^* = q\pi$ and $\pi\mu = 1_L$.

Using Lemma 3.3.2, we know that $\mathrm{Hom}(P, -) : \mathrm{add}P \to \mathrm{proj}\Sigma$ is an equivalence. Thus there exist morphisms $\mu' : P' \to P$ and $\pi' : P \to P'$ in add$P$ such that $L = \mathrm{Hom}_\Lambda(P, P')$, $\pi = \mathrm{Hom}(P, \pi')$ and $\mu = \mathrm{Hom}(P, \mu')$. Since $\pi\mu = 1_L$, we get $\pi'\mu' = 1_{P'}$, and hence, $\mu'\pi'$ is an idempotent in Σ. Since projΛ is Krull-Schmidt, there exist morphisms $\pi'' : P \to P''$ and $\mu'' : P'' \to P$ such that $\mu''\pi'' = 1_P - \mu'\pi'$. This yields a decomposition $P \cong P' \oplus P''$. Set $p' = p\mu' : P' \to M$ and $p'' = p\mu'' : P'' \to M$. Since $q = p^*\mu = \mathrm{Hom}(P, p\mu')$ is right minimal, by Lemma 3.3.1, p' is right minimal. Moreover, $\mathrm{Hom}_\Lambda(P, \mu''\pi'') = \mathrm{Hom}_\Lambda(P, 1_P - \mu'\pi') = 1_\Sigma - \mu\pi$, and hence,

$$\mathrm{Hom}_\Lambda(P, p\mu''\epsilon'') = p^* \circ \mathrm{Hom}_\Lambda(P, \mu''\epsilon'') = p^*(1_\Sigma - \mu\pi) = p^* - q\pi = 0.$$

By Lemma 3.3.1, $p\mu''\pi'' = 0$. Since π'' is an epimorphism, $p\mu'' = 0$. That is $p'' = p\,|_{P_2} = 0$.
□

Proposition 3.3.1. *Every finitely generated module in* modΛ *admits a projective cover in* projΛ*, which is unique up to an isomorphism.*

Proof. Let M be a finitely generated module in modΛ. Then there exists an epimorphism $p : P \to M$ with $P \in \text{proj}\Lambda$. Using Lemma 3.3.5, we get that $p = (p_1\ p_2) : P_1 \oplus P_2 \to M$ such that p_1 is right minimal and $p_2 = 0$. Consequently, $p_1 : P_1 \to M$ is an epimorphism. By Lemma 3.3.4, p_1 is a projective cover of M.

Now suppose $p : P \to M$ and $p' : P' \to M$ are two projective covers of M. Then there exist epimorphisms $u : P \to P'$ and $v : P' \to P$ such that $p'u = p$ and $pv = p'$. Thus $pvu = p$ and $p'uv = p'$. Since p, p' are right minimal (Lemma 3.3.4), we get that vu and uv are isomorphisms, and hence u, v are isomorphisms. □

Definition 3.3.6. *Let $M \in \text{mod}^+\Lambda$. A projective presentation $P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0$ is called* minimal *if g is a projective cover of M and f corestricts to a projective cover $f' : P_1 \to \text{Im}(f)$.*

Definition 3.3.7. *Let $M \in \text{mod}^-\Lambda$. An injective presentation $0 \to M \xrightarrow{f} I_0 \xrightarrow{g} I_1$ is called* minimal *if f is an injective envelope of M and g restricts to an injective envelope $g' : \text{Ker}(g) \to I_1$.*

Lemma 3.3.6. *Every module $M \in \text{mod}^+\Lambda$ admits a minimal projective presentation, which is unique up to an isomorphism.*

Proof. Since $M \in \text{mod}^+\Lambda$, there exists an exact sequence $P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0$ with $P_1, P_0 \in \text{proj}\Lambda$. In particular, M is finitely generated, and Proposition 3.3.1 implies that M admits a projective cover $p_0' : P_0' \to M$. Since P_0 is projective and p_0' is essential, there exists an epimorphism $q : P_0 \to P_0'$ such that $p_0'q = g$. We claim that the image of $qf : P_1 \to P_0'$ is $\text{Ker}(p_0')$. Since $p_0'qf = gf = 0$, $\text{Im}(qf) \subseteq \text{Ker}(p_0')$. Let $a \in \text{Ker}(p_0')$. Since q is an epi, there exists $b \in P_0$ such that $q(b) = a$. Now $g(b) = p_0'(q(b)) = p_0'(a) = 0$, which implies that $b \in \text{Ker}(g)$, and hence there exists some $c \in P_1$ such that $f(c) = b$. Thus $(qf)(c) = a$ and $\text{Im}(qf) = \text{Ker}(p_0')$. This gives that $\text{Ker}(p_0')$ is finitely generated and hence, using Proposition 3.3.1, it admits a projective cover $p_1 : P_1' \to \text{Ker}(p_0')$. Setting

$p_1' = jp_1 : P_1' \to P_0'$, where $j : \mathrm{Ker}(p_0') \to P_0'$ is the inclusion map, we get that $P_1' \xrightarrow{p_1'} P_0' \xrightarrow{p_0'} M \to 0$ is a minimal projective presentation of M.

Now suppose $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ and $P_1' \xrightarrow{p_1'} P_0' \xrightarrow{p_0'} M \to 0$ are two minimal projective presentations of M. Using Proposition 3.3.1, we get that there exists an isomorphism $u : P_0 \to P_0'$ such that $p_0'u = p_0$. Thus $u : \mathrm{Ker}(p_0) \to \mathrm{Ker}(p_0')$ is an isomorphism and $up_1 : P_1 \to \mathrm{Ker}(p_1)$ is a projective cover of $\mathrm{Ker}(p_1)$. Using Proposition 3.3.1 again, this implies that there exists some isomorphism $v : P_1 \to P_1'$ such that $p_1'v = up_1$. Finally, the uniqueness of minimal projective presentation follows easily from the uniqueness of projective cover stated in Lemma 3.3.1. □

Let M be a module in $\mathrm{mod}^+\Lambda$. By Lemma 3.3.6, M admits a unique minimal projective presentation $P_1 \xrightarrow{p} P_0 \xrightarrow{q} M \to 0$ with $P_1, P_0 \in \mathrm{proj}\Lambda$. Applying the left exact contravariant functor $(-)^t$ to the above sequence, we get the following exact sequence:

$$0 \to M^t \xrightarrow{q^t} P_0^t \xrightarrow{p^t} P_1^t \xrightarrow{r} \mathrm{Coker}(p^t) \to 0.$$

Thus, $\mathrm{Coker}(p^t) \in \mathrm{mod}^+\Lambda^\circ$. We set $\mathrm{Tr}(M) := \mathrm{Coker}(p^t)$ and call it the *transpose* of M.

Lemma 3.3.7. *Let $M \in \mathrm{mod}^+\Lambda$ with $P_1 \xrightarrow{p} P_0 \xrightarrow{q} M \to 0$ a minimal projective presentation, where $P_1, P_0 \in \mathrm{proj}\Lambda$. If M has no non-zero projective summands, then the sequence $P_0^t \xrightarrow{p^t} P_1^t \xrightarrow{r} \mathrm{Tr}(M) \to 0$ is a minimal projective presentation of $\mathrm{Tr}(M)$.*

Proof. Applying the left exact functor $(-)^t$ to the sequence stated in the lemma, by Proposition 3.1.1, we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
P_1^{tt} & \xrightarrow{p^{tt}} & P_0^{tt} & \xrightarrow{q'} & M & \longrightarrow & 0 \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle 1_M} & & \\
P_1 & \xrightarrow{p} & P_0 & \xrightarrow{q} & M & \longrightarrow & 0.
\end{array}$$

On the other hand, since $\mathrm{Tr}(M)$ is finitely presented, Lemma 3.3.6 implies that $\mathrm{Tr}(M)$ admits a minimal projective presentation, say $L_0 \xrightarrow{u_0} L_1 \xrightarrow{u_1} \mathrm{Tr}(M) \to 0$. Using Lemma 3.3.5, we can decompose P_1^t as $V_2 \oplus V_1$ such that $r|_{V_2}$ is right minimal and $r|_{V_1} = 0$. Since $\mathrm{proj}\Lambda^\circ$ is closed under direct summands, $V_1, V_2 \in \mathrm{proj}\Lambda^\circ$, and hence, using Lemma 3.3.4, $r|_{V_2}$ is a projective cover of $\mathrm{Tr}(M)$. Since Proposition 3.3.1 says that projective covers are unique up to isomorphisms, we get that the map $r|_{V_2} : V_2 \to \mathrm{Tr}(M)$ is isomorphic to $L_1 \xrightarrow{u_1} \mathrm{Tr}(M)$. Thus the map $P_1^t \xrightarrow{r} \mathrm{Tr}(M) \to 0$ is isomorphic to

$$L_1 \oplus V_1 \xrightarrow{(u_1\ 0)} \mathrm{Tr}(M) \to 0.$$

Thus $\mathrm{Ker}(r) \cong \mathrm{Ker}(u_1) \oplus V_1$. Since $u_0 : L_0 \to \mathrm{Ker}(u_1)$ is the projective cover of $\mathrm{Ker}(u_1)$, we get that the map $\begin{pmatrix} u_0 & 0 \\ 0 & 1_{V_1} \end{pmatrix} : L_0 \oplus V_1 \to \mathrm{Ker}(u_1) \oplus V_1 \cong \mathrm{Ker}(r)$ is a projective cover of $\mathrm{Ker}(r)$. Since $P_0^t \in \mathrm{proj}\Lambda^\circ$, using Lemma 3.3.5, we get that there exists a decomposition $P_0^t \cong N' \oplus N$ such that $p^t|_{N'}$ is right minimal and $p^t|_N = 0$. Thus, using Lemma 3.3.4, we get that $p^t|_{N'} : N' \to \mathrm{Ker}(r)$ is a projective cover of $\mathrm{Ker}(r)$. Since Proposition 3.3.1 says that projective covers are unique up to isomorphisms, we get that the map $p^t|_{N'} : N' \to \mathrm{Ker}(r)$ is isomorphic to the map $\begin{pmatrix} u_0 & 0 \\ 0 & 1_{V_1} \end{pmatrix} : L_0 \oplus V_1 \to \mathrm{Ker}(u_1) \oplus V_1 \cong \mathrm{Ker}(r)$. Thus $p^t : P_0^t \to \mathrm{Ker}(r)$ is isomorphic to $L_0 \oplus N \oplus V_0 \xrightarrow{\begin{pmatrix} u_0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix}} \mathrm{Ker}(u_1) \oplus V_1$ for some isomorphism $v : V_0 \to V_1$. Thus, the sequence $P_0^t \xrightarrow{p^t} P_1^t \xrightarrow{r} \mathrm{Tr}(M) \to 0$ is isomorphic to the exact sequence

$$L_0 \oplus N \oplus V_0 \xrightarrow{\begin{pmatrix} u_0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix}} L_1 \oplus V_1 \xrightarrow{(u_1\ 0)} \mathrm{Tr}(M) \to 0.$$

Applying the functor $(-)^t$ to this sequence, we see that the minimal projective presentation $P_1 \xrightarrow{p} P_0 \xrightarrow{q} M \to 0$ is isomorphic to the exact sequence

$$L_1^t \oplus V_1^t \xrightarrow{\begin{pmatrix} u^t & 0 \\ 0 & 0 \\ 0 & v^t \end{pmatrix}} L_0^t \oplus N^t \oplus V_0^t \xrightarrow{(f_1\ f_2\ f_3)} M \to 0.$$

In particular, (f_1, f_2, f_3) is right minimal and $(f_1 u^t, f_3 v^t) = (0, 0)$. Since v is an isomorphism, so is v^t, and hence, $f_3 = 0$. Since (f_1, f_2, f_3) is right minimal, $V_0^t = 0$. This implies that $V_0 = 0$, and hence, $V_1 = 0$. In other words, the minimal projective presentation $P_0 \xrightarrow{p} P_0 \xrightarrow{q} M \to 0$ is isomorphic to the exact sequence

$$L_1^t \xrightarrow{\begin{pmatrix} u^t & 0 \\ 0 & 0 \end{pmatrix}} L_0^t \oplus N^t \xrightarrow{(f_1\ f_2)} M \to 0.$$

Thus $M \cong \mathrm{Coker}(p) \cong \mathrm{Coker}(u^t) \oplus N^t$, where $N^t \in \mathrm{proj}\Lambda$. Suppose that M does not have a non-zero projective summand. Then $N^t = 0$, and hence $N = 0$. Therefore, $P_0^t \xrightarrow{p^t} P_1^t \xrightarrow{r} \mathrm{Tr}(M) \to 0$ is the minimal projective presentation of $\mathrm{Tr}(M)$. □

Lemma 3.3.8. *Let M be a module in* $\mathrm{mod}^+\Lambda$. *Then* $\mathrm{Tr}(M) = 0$ *if and only if M is projective.*

Proof. Suppose M is projective. Then the minimal projective presentation of M is given by $0 \to M \to M \to 0$. Applying $(-)^t$ to this sequence, we get that

$$\mathrm{Tr}(M) = \mathrm{Coker}(M^t \to 0) = 0.$$

Conversely suppose $\mathrm{Tr}(M) = 0$. Let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ be the minimal projective presentation of M. Then p_1^t is an epi, and hence a retraction since P_1^t is projective. By Proposition 3.1.1, p_1 is a section. Thus, $P_0 = P' \oplus P''$, where $P' = \mathrm{Im}(p_1) = \mathrm{Ker}(p_0)$. In particular, $p_0 \mid_{P'} = 0$. Since p_0 is right minimal, by Lemma 3.3.3, $P' = 0$, that is, $\mathrm{Ker}(p_0) = 0$. Thus, p_0 is a monomorphism, and hence, an isomorphism. So, M is projective. □

Lemma 3.3.9. *Let M be an indecomposable non-projective module in* $\mathrm{mod}^+\Lambda$. *Then* $\mathrm{Tr}(M)$ *is indecomposable and non-projective.*

Proof. Let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ be a minimal projective presentation of M. Since M is not projective, Lemma 3.3.8 implies that $\mathrm{Tr}(M) \neq 0$.

Being indecomposable and non-projective, M has no non-zero projective summands. Thus, by Lemma 3.3.7, $P_0^t \xrightarrow{p_1^t} P_1^t \to \mathrm{Tr}(M) \to 0$ is a minimal projective presentation of $\mathrm{Tr}(M)$. Then, by Proposition 3.1.1, $\mathrm{Tr}\mathrm{Tr}(M) \cong \mathrm{Coker}(p_1^{tt}) \cong \mathrm{Coker}(p_1) \cong M$.

We now show that $\mathrm{Tr}(M)$ has no non-zero projective summand. Suppose $\mathrm{Tr}(M) \cong E \oplus P$ with $P \in \mathrm{proj}\Lambda^{\circ}$. Being finitely presented, E admits a minimal projective presentation $L_1 \xrightarrow{f} L_0 \xrightarrow{u} E \to 0$. Thus the minimal projective presentation $P_0^t \xrightarrow{p_1^t} P_1^t \to \mathrm{Tr}(M) \to 0$ is isomorphic to

$$L_1 \xrightarrow{\begin{bmatrix} f \\ 0 \end{bmatrix}} L_0 \oplus P \xrightarrow{\begin{bmatrix} u & 0 \\ 0 & 1_P \end{bmatrix}} E \oplus P \to 0.$$

Thus, we may assume that p_1 is the map $L_0^t \oplus P^t \xrightarrow{[f^t \ 0]} P_0$. In particular, $p_1 \mid_{P^t} = 0$. Since p_1 corestricts to a right minimal map $p_1' : P_1 \to \mathrm{Im}(p_1)$, it is right minimal. Hence, $P^t = 0$, and hence, $P = 0$.

Now suppose $\mathrm{Tr}(M) \cong M_1 \oplus M_2$. Then $M \cong \mathrm{Tr}\mathrm{Tr}(M) \cong \mathrm{Tr}(M_1) \oplus \mathrm{Tr}(M_2)$. Thus, $\mathrm{Tr}(M_1) = 0$ or $\mathrm{Tr}(M_2) = 0$. By Lemma 3.3.8, M_1 or M_2 is projective, and consequently, M_1 or M_2 is zero. Thus, $\mathrm{Tr}(M)$ is indecomposable. □

Given an indecomposable non-projective module M in $\mathrm{mod}^+\Lambda$, we write $\tau(M) := \mathfrak{D}\mathrm{Tr}(M)$ and call it the *Auslander-Reiten translation* of M. Lemma 3.3.9 and the fact that $\mathfrak{D}$ is a duality imply that $\tau(M)$ is a non-injective indecomposable module in $\mathrm{mod}^-\Lambda$. Dually, given an indecomposable non-injective module N in $\mathrm{mod}^-\Lambda$, $\mathfrak{D}(N)$ is a non-projective indecomposable module in $\mathrm{mod}^+\Lambda$, which implies that $\mathrm{Tr}\mathfrak{D}(N)$ is a non-projective indecomposable module as well. We denote this by $\tau^-(N) := \mathrm{Tr}\mathfrak{D}(N)$.

Lemma 3.3.10. *Let M be an indecomposable non-projective module in* $\mathrm{mod}^+\Lambda$ *with a minimal projective presentation $P_1 \xrightarrow{p} P_0 \to M \to 0$. Then $\tau M \cong \mathrm{Ker}(\nu p)$.*

Proof. By definition $\tau M = \mathfrak{D}\mathrm{Tr}(M)$. In order to calculate $\mathrm{Tr}(M)$, we apply the left exact functor $(-)^t$ to the given presentation, which gives us the sequence

$$0 \to M^t \to P_0^t \xrightarrow{p^t} P_1^t \to \mathrm{Tr}(M) \to 0.$$

Applying the exact functor $\mathfrak{D}$ to the above sequence, we get

$$0 \to \mathfrak{D}\mathrm{Tr}(M) \to \mathfrak{D}P_1^t \xrightarrow{\mathfrak{D}p^t} \mathfrak{D}P_0^t \to \mathfrak{D}M^t \to 0.$$

By definition, $\mathfrak{D}p^t = \nu p$ and $\mathfrak{D}\mathrm{Tr}M = \tau M$, and we get the required isomorphism. □

3.4 Auslander-Reiten formula

Throughout this section, we assume that $\Lambda = kQ/I$ is a locally semi-perfect algebra. We need the following technical lemma to prove our main theorem.

Lemma 3.4.1. *[3, Lemma III.3.12] Suppose we have the following commutative diagram of Λ-modules*

$$\begin{array}{ccccccc} L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ L' & \xrightarrow[f']{} & M' & \xrightarrow[g']{} & N' & & \end{array}$$

such that u, v are isomorphisms, the upper row is exact, and the lower row is a complex. Then the cohomology of the lower row at M' is isomorphic to the kernel of w.

Now, we need the usual exact contravariant functor $D = \mathrm{Hom}_k(-, k) : \mathrm{Mod}\, k \to \mathrm{Mod}\, k$. Note that if M is a finite dimensional Λ-module, then $DM \cong \mathfrak{D}(M)$ as k-vector spaces.

Lemma 3.4.2. *Let $M \in \mathrm{mod}\Lambda$ and $N \in \mathrm{Mod}\Lambda$.*

1. *There exists a bi-natural k-linear map $\phi_{M,N} : N \otimes_\Lambda M^t \to \mathrm{Hom}_\Lambda(M, N)$ with cokernel $\underline{\mathrm{Hom}}_\Lambda(M, N)$.*

2. *Suppose* Λ *is left locally bounded and* $M \in \mathrm{mod}^{+}\Lambda$. *Then there exists a bi-natural* k*-linear map* $\psi_{M,N} : D\mathrm{Hom}_{\Lambda}(M,N) \to \mathrm{Hom}_{\Lambda}(N,\nu M)$ *which is an isomorphism in case* $M \in \mathrm{proj}\Lambda$.

Proof. (1) It is easy to see that we have a k-linear map

$$\phi_{M,N} : N \otimes_{\Lambda} M^{t} \to \mathrm{Hom}_{\Lambda}(M,N) : u \otimes (f_a)_{a\in Q_0} \mapsto (v \mapsto \textstyle\sum_{a\in Q_0} u f_a(v)),$$

where $f_a \in \mathrm{Hom}_{\Lambda}(M, e_a\Lambda)$ such that $f_a = 0$ for almost all $a \in Q_0$. We claim that $\mathrm{Im}(\phi_{M,N}) = \mathcal{P}(M,N)$. First, suppose that $f \in \mathrm{Im}(\phi_{M,N})$. Then, $f = \sum_{i=1}^{n} \phi_{M,N}(u_i \otimes f_i)$, where $u_1, \ldots, u_n \in N$ and $f_1, \ldots, f_n \in M^t$. Write $f_i = (f_{i,a})_{a\in Q_0}$, where $f_{i,a} \in \mathrm{Hom}_{\Lambda}(M, e_a\Lambda)$ is such that $f_{i,a} = 0$ for almost all $a \in Q_0$. Thus, there exist $a_1, \ldots, a_m \in Q_0$ such that $f_{i,a} = 0$ for all $1 \le i \le n$ and $a \notin \{a_1, \ldots, a_m\}$. Therefore, $f(v) = \sum_{i=1}^{n} \sum_{j=1}^{m} u_i \cdot f_{i,a_j}(v)$, for $v \in M$. On the other hand, consider the morphisms

$$f_i' = \begin{pmatrix} f_{i,a_1} \\ \vdots \\ f_{i,a_m} \end{pmatrix} : M \to e_{a_1}\Lambda \oplus \cdots \oplus e_{a_m}\Lambda$$

and $g_i = (u_i\cdot, \cdots, u_i\cdot) : e_{a_1}\Lambda \oplus \cdots \oplus e_{a_m}\Lambda \to N$, the left multiplications by u_i, for $i = 1, \ldots, n$. Setting $P = e_{a_1}\Lambda \oplus \cdots \oplus e_{a_m}\Lambda$, we see that f is the composite of the following morphisms

$$M \xrightarrow{\begin{pmatrix} f_1' \\ \vdots \\ f_n' \end{pmatrix}} P \oplus \cdots \oplus P \xrightarrow{(g_1, \cdots, g_n)} N.$$

Thus $f \in \mathcal{P}(M,N)$.

Conversely, suppose that $f \in \mathcal{P}(M,N)$. Then, there exists a projective module $P \in \mathrm{proj}\Lambda$ such that $f = gh$ for some $h : M \to P$ and $g : P \to N$. Since $P \in \mathrm{proj}\Lambda$, $P \cong e_{a_1}\Lambda \oplus \cdots \oplus e_{a_n}\Lambda$ for some $a_i \in Q_0$. Write $g = (g_{a_1}, \ldots, g_{a_n})$, where $g_{a_i} : e_{a_i}\Lambda \to N$ and

$$h = \begin{pmatrix} h_{a_1} \\ \vdots \\ h_{a_n} \end{pmatrix} : M \to e_{a_1}\Lambda \oplus \cdots \oplus e_{a_n}\Lambda,$$

where $h_{a_i} \in \mathrm{Hom}_\Lambda(M, e_{a_i}\Lambda)$, for $i = 1, \ldots, n$. Consider $h_i = (h_{i,a})_{a\in Q_0} \in M^t$, where $h_{i,a} = h_{a_i}$ in case $a = a_i$ and $h_{i,a} = 0$ otherwise, for $i = 1, \ldots, n$. Write $u_i = g(e_{a_i}) \in N$, for $i = 1, \ldots, n$. Given $v \in M$, by definition, we obtain

$$\phi_{M,N}(\textstyle\sum_{i=1}^n u_i \otimes h_i)(v) = \sum_{i=1}^n \sum_{a\in Q_0} u_i h_{i,a}(v) = \sum_{i=1}^n u_i h_{a_i}(v).$$

On the other hand, since $h_{a_i}(v) \in e_{a_i}\Lambda$, we see that

$$f(v) = \textstyle\sum_{i=1}^n g_{a_i}(h_{a_i}(v)) = \sum_{i=1}^n g_{a_i}(e_{a_i} \cdot h_{a_i}(v)) = \sum_{i=1}^n u_i h_{a_i}(v).$$

That is, $f = \phi_{M,N}(\sum_{i=1}^n u_i \otimes h_i)$. This establishes our claim, and Statement (1) follows.

(2) Suppose Λ is left locally bounded. Then Λe_a is finite-dimensional for every $a \in Q_0$. Hence, every module in $\mathrm{proj}\Lambda^\circ$ is finite-dimensional. Since $M \in \mathrm{mod}^+\Lambda$, it admits an epimorphism $f : P_0 \to M$ with $P_0 \in \mathrm{proj}\Lambda$. Since $(-)^t$ is left exact, we get that $f^t : M^t \to P_0^t$ is a monomorphism, where $P_0^t \in \mathrm{proj}\Lambda^\circ$. Since M^t is a submodule of a finite-dimensional module, it is also finite-dimensional. Thus $\nu M \cong \mathfrak{D}(M^t) \cong D(M^t)$, as k-vector spaces.

We can compose the morphism $D\phi_{M,N} : D\mathrm{Hom}_\Lambda(M, N) \to D(N \otimes_\Lambda M^t)$ with the adjunction isomorphism $\eta_{M,N} : D(N \otimes_\Lambda M^t) \to \mathrm{Hom}_\Lambda(N, \mathfrak{D}(M^t))$. Hence, we get the required morphism $\psi_{M,N} = \eta_{M,N} D(\phi_{M,N}) : D\mathrm{Hom}_\Lambda(M, N) \to \mathrm{Hom}_\Lambda(N, \nu M)$. Using Lemma 3.2 from [10], we get that this map is an isomorphism if $M \in \mathrm{proj}\Lambda$. □

We are now ready to prove our main theorem which states that the *Auslander-Reiten formula* holds for certain locally finite-dimensional algebras.

Theorem 3.4.1. *Let $\Lambda = kQ/I$ be a locally semi-perfect algebra.*

1. *Suppose Λ is left locally bounded. If $N \in \mathrm{Mod}\Lambda$ and $M \in \mathrm{mod}^+\Lambda$, then there exists a binatural isomorphism*

$$\mathrm{Ext}^1_\Lambda(N, \tau M) \cong D\underline{\mathrm{Hom}}(M, N).$$

2. *Suppose* Λ *is right locally bounded. If* $M \in \mathrm{mod}^{-}\Lambda$ *and* $N \in \mathrm{mod}\Lambda$*, then there exists a binatural isomorphism*

$$\mathrm{Ext}^1_{\Lambda}(\tau^{-}M, N) \cong D\overline{\mathrm{Hom}}(N, M).$$

Proof. We will first prove Statement (1). Let $M \in \mathrm{mod}^{+}\Lambda$ with a projective presentation

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

with $P_1, P_0 \in \mathrm{proj}\Lambda$. Using Lemma 3.3.10, we get the following exact sequence:

$$0 \to \tau M \to \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \xrightarrow{\nu p_0} \nu M \to 0.$$

Applying $\mathrm{Hom}_{\Lambda}(N, -)$ to the above sequence gives the following complex

$$0 \to \mathrm{Hom}(N, \tau M) \to \mathrm{Hom}(N, \nu P_1) \xrightarrow{\mathrm{Hom}(N,\nu p_1)} \mathrm{Hom}(N, \nu P_0) \xrightarrow{\mathrm{Hom}(N,\nu p_0)} \mathrm{Hom}(N, \nu M)$$

It is well known that $\mathrm{Ext}^1_{\Lambda}(N, \tau M) \cong \frac{\mathrm{Ker}\mathrm{Hom}_{\Lambda}(N,\nu p_0)}{\mathrm{Im}\mathrm{Hom}_{\Lambda}(N,\nu p_1)}$.

On the other hand, applying the right exact functor $D\mathrm{Hom}_{\Lambda}(-, N)$ to the above projective presentation of M, we get the following exact sequence

$$D\mathrm{Hom}_{\Lambda}(P_1, N) \xrightarrow{D\mathrm{Hom}_{\Lambda}(p_1,N)} D\mathrm{Hom}_{\Lambda}(P_0, N) \xrightarrow{D\mathrm{Hom}_{\Lambda}(p_0,M)} D\mathrm{Hom}_{\Lambda}(M, N) \to 0.$$

The natural transformation ψ from Lemma 3.4.2(2) gives the following commutative diagram where the upper row is exact and the lower row is a complex.

$$\begin{array}{ccccccc}
D\mathrm{Hom}_{\Lambda}(P_1, N) & \xrightarrow{D\mathrm{Hom}_{\Lambda}(p_1,N)} & D\mathrm{Hom}_{\Lambda}(P_0, N) & \xrightarrow{D\mathrm{Hom}_{\Lambda}(p_0,N)} & D\mathrm{Hom}_{\Lambda}(M, N) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \psi_{P_1,N}} & & \downarrow{\scriptstyle \psi_{P_0,N}} & & \downarrow{\scriptstyle \psi_{M,N}} & & \\
\mathrm{Hom}_{\Lambda}(N, \nu P_1) & \xrightarrow[\mathrm{Hom}_{\Lambda}(N,\nu p_1)]{} & \mathrm{Hom}_{\Lambda}(N, \nu P_0) & \xrightarrow[\mathrm{Hom}_{\Lambda}(N,\nu p_0)]{} & \mathrm{Hom}_{\Lambda}(N, \nu M) & &
\end{array}$$

Using Lemma 3.4.2(2), $\psi_{P_1,N}$ and $\psi_{P_0,N}$ are isomorphisms. Therefore, using Lemma 3.4.1 and the fact that $\psi_{M,N} = \eta_{M,N} D(\phi_{M,N})$, we get that

$$\mathrm{Ext}^1_\Lambda(N,M) \cong \frac{\mathrm{Ker}\mathrm{Hom}_\Lambda(N,\nu p_0)}{\mathrm{Im}\mathrm{Hom}_\Lambda(N,\nu p_1)} \cong \mathrm{Ker}(\psi_{M,N}) = \mathrm{Ker}(D\phi_{M,N})$$
$$\cong D(\mathrm{Coker}\phi_{M,N}) \cong D\underline{\mathrm{Hom}}_\Lambda(M,N).$$

We now prove Statement (2). Let $M \in \mathrm{mod}^-\Lambda$ with Λ right locally bounded. Then $\mathfrak{D}(M) \in \mathrm{mod}^+\Lambda^\circ$ and Λ° is left locally bounded. Using part (1), we get that

$$\mathrm{Ext}^1_{\Lambda^\circ}(\mathfrak{D}(N),\tau\mathfrak{D}(M)) \cong D(\underline{\mathrm{Hom}}_{\Lambda^\circ}(\mathfrak{D}(M),\mathfrak{D}(N))).$$

Now $D(\underline{\mathrm{Hom}}_{\Lambda^\circ}(\mathfrak{D}(M),\mathfrak{D}(N))) \cong D(\overline{\mathrm{Hom}}_\Lambda(N,M))$ and

$$\begin{aligned}\mathrm{Ext}^1_{\Lambda^\circ}(\mathfrak{D}(N),\tau\mathfrak{D}(M)) &= \mathrm{Ext}^1_{\Lambda^\circ}(\mathfrak{D}(N),\mathfrak{D}\mathrm{Tr}\mathfrak{D}(M))\\ &= \mathrm{Ext}^1_{\Lambda^\circ}(\mathfrak{D}(N),\mathfrak{D}\tau^-(M))\\ &\cong \mathrm{Ext}^1_\Lambda(\tau^-(M),N),\end{aligned}$$

and we get the required isomorphism. □

Let $M \in \mathrm{mod}\Lambda$. We say that M is *strongly indecomposable* if $\mathrm{End}_\Lambda(M)$ is a local ring.

Theorem 3.4.2. *Let $\Lambda = kQ/I$ be a locally semi-perfect algebra.*

1. *Suppose Λ is left locally bounded. If $M \in \mathrm{mod}^+\Lambda$ is indecomposable and non-projective, then there exists an almost split sequence*

$$0 \longrightarrow \tau M \longrightarrow L \longrightarrow M \longrightarrow 0$$

in $\mathrm{Mod}\Lambda$, where τM is finite dimensional.

2. *Suppose Λ is right locally bounded. If $M \in \mathrm{mod}^-\Lambda$ is indecomposable and non-injective, then there exists an almost split sequence*

$$0 \longrightarrow M \longrightarrow L \longrightarrow \tau^- M \longrightarrow 0$$

in $\mathrm{mod}\Lambda$, where $\tau^- M$ is finite dimensional.

Proof. We first prove Statement (1). Since $M \in \text{mod}^+\Lambda$ is indecomposable and $\text{mod}^+\Lambda$ is a Krull-Schmidt category, it is strongly indecomposable. Moreover, by the paragraph following Lemma 3.3.9, $\tau(M)$ is an indecomposable module in $\text{mod}^-\Lambda$, which is a Krull-Schmidt category, and hence it is strongly indecomposable. Let

$$\phi : \text{Ext}^1(-, \tau(M)) \to D\underline{\text{Hom}}(M, -)$$

be the functorial isomorphism obtained from Theorem 3.4.1. Since ϕ is a natural tranformation, $\phi_M : \text{Ext}^1(M, \tau(M)) \to D\underline{\text{End}}(M)$ is an $\underline{\text{End}}(M)$-linear isomorphism. Since the $\underline{\text{End}}(M)$-top of $\underline{\text{End}}(M)$ is non-zero, we get that the $\underline{\text{End}}(M)$-socle of $D\underline{\text{End}}(M)$ is non-zero. As a consequence, the $\underline{\text{End}}(M)$-socle of $\text{Ext}^1(M, \tau(M))$ is non-zero. Let $\delta : 0 \to \tau(M) \to X \to M \to 0$ be a non-zero extension lying in the $\underline{\text{End}}(M)$-socle of $\text{Ext}^1(M, \tau(M))$. Hence, $\theta = \phi_M(\delta)$ is a non-zero element of the $\underline{\text{End}}(M)$-socle of $D\underline{\text{End}}(M)$. In particular, θ is annihilated by $\text{rad}(\underline{\text{End}}(M))$. Since M is not projective and $\text{End}(M)$ is local, if $\underline{f} \in \text{rad}(\underline{\text{End}}(M))$, then $f \in \text{rad}(\text{End}(M))$, and hence, $\theta(\underline{f}) = (f\theta)(\underline{1}_M) = 0$.

Let $p : L \to M$ be a morphism in $\text{Mod}\Lambda$ which is not a retraction. Since $\text{End}_\Lambda(M)$ is local, for any morphism $q : M \to L$ in $\text{Mod}\Lambda$, we have $pq \in \text{rad}(\text{End}(M))$, and hence, $\underline{pq} \in \text{rad}(\underline{\text{End}}(X))$. Thus $\theta(\underline{pq}) = 0$, that is, $(D\underline{\text{Hom}}(M, p) \circ \phi_M)(\delta) = 0$. In view of the commutative diagram

$$\begin{array}{ccc}
\text{Ext}^1(M, \tau(M)) & \xrightarrow{\text{Ext}^1(p, \tau(M))} & \text{Ext}^1(L, \tau(M)) \\
\downarrow{\scriptstyle \phi_M} & & \downarrow{\scriptstyle \phi_L} \\
D\underline{\text{Hom}}(M, M) & \xrightarrow[D\text{Hom}(M,p)]{} & D\underline{\text{Hom}}(M, L)
\end{array}$$

we see that $(\phi_L \circ \text{Ext}^1(p, \tau(M)))(\delta) = 0$. Since ϕ_L is injective, $p\delta = \text{Ext}^1(p, \tau(M))(\delta) = 0$. That is p factors through the epimorphism $X \to M$ in δ. Thus, the morphism $X \to M$ is right almost split. Since $\tau(M)$ is strongly indecomposable, δ is an almost split sequence.

Finally, since Λ is left locally bounded, all the modules in $\text{proj}\Lambda^\circ$ are finite-dimensional, and hence $\text{Tr}(M)$ is finite-dimensional, being the cokernel of a map in $\text{proj}\Lambda^\circ$. Thus, $\tau(M) = \mathfrak{D}\text{Tr}(M)$ is finite-dimensional.

We now prove Statement (2). Since Λ is right locally bounded, Λ° is left locally bounded, and $\mathfrak{D}(M) \in \text{mod}^+\Lambda^\circ$ is non-projective indecomposable. Applying Statement (1), we get that there exists an almost split sequence $0 \to \tau\mathfrak{D}(M) \to L \to \mathfrak{D}(M) \to 0$ in $\text{Mod}\Lambda^\circ$ with $\tau\mathfrak{D}(M)$ finite-dimensional. Since both $\tau\mathfrak{D}(M)$ and $\mathfrak{D}(M)$ are locally finite-dimensional, we get that L is also locally finite-dimensional. Thus $0 \to \tau\mathfrak{D}(M) \to L \to \mathfrak{D}(M) \to 0$ is an almost split sequence in $\text{mod}\Lambda^\circ$. Applying $\mathfrak{D}$, we get that $0 \to M \to \mathfrak{D}(L) \to \tau^-(M) \to 0$ is an almost split sequence in $\text{mod}\Lambda$ and $\tau^-(M)$ is finite-dimensional. □

The above result about the existence of almost split sequences was mentioned by Auslander in [7]. However, he does not provide a complete proof there.

CHAPTER 4

String algebras

In the last chapter, we proved the Auslander-Reiten formula for the class of locally semi-perfect algebras. Now we restrict our attention to a special class of algebras, called string algebras. We will start with the definition of string algebras and see how we can associate some combinatorial structures, called strings and bands, with them. Further, we will see how one can associate a Λ-module to a string or a band. We will then try to extract some properties of a given string module from the structure of the corresponding string.

4.1 String algebras

In this section, we will introduce a special class of algebras obtained from quivers called string algebras. We will also introduce the relevant combinatorial language associated with these algebras, including the notions of walks, strings, etc.

Definition 4.1.1. *Let $Q = (Q_0, Q_1)$ be a locally finite quiver and ρ a set of zero relations on Q. The algebra $\Lambda = kQ/\langle\rho\rangle$ is said to be a* string algebra *if the following conditions are satisfied:*

1. *For each* $a \in Q_0$*, there exist at most two arrows* $\alpha, \beta \in Q_1$ *such that* $s(\alpha) = s(\beta) = a$*.*

2. *For each* $a \in Q_0$*, there exist at most two arrows* $\alpha, \beta \in Q_1$ *such that* $t(\alpha) = t(\beta) = a$*.*

3. *For each* $\alpha \in Q_1$*, there exists at most one arrow* $\beta \in Q_1$ *such that* $\alpha\beta \notin \rho$

4. *For each* $\alpha \in Q_1$*, there exists at most one arrow* $\gamma \in Q_1$ *such that* $\gamma\alpha \notin \rho$*.*

In this definition, we drop the classical 'boundedness' conditions [11, § 3]:

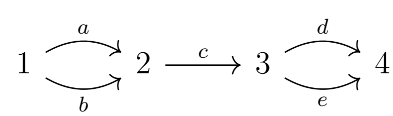
5. For each $\beta \in Q_1$, there exists some bound $n(\beta)$ such that any path $\beta_1 \cdots \beta_{n(\beta)}$ with $\beta_1 = \beta$ contains a subpath lying in ρ.

6. For each $\beta \in Q_1$, there exists some bound $n'(\beta)$ such that any path $\beta_1 \cdots \beta_{n'(\beta)}$ with $\beta_{n'(\beta)} = \beta$ contains a subpath lying in ρ.

In the sequel, we shall call the string algebras satisfying the above two conditions as *locally bounded string algebras.* The finite-dimensional indecomposable representations of locally bounded string algebras were completely classified in [11] along with the maps between these.

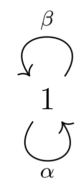
Henceforth, unless stated otherwise, we will use the letter Λ to denote a string algebra with Q the corresponding quiver and ρ the corresponding set of relations.

Examples 4.1.1. *1. Let* Q *be the following quiver.*



In order to make this a string algebra, we need some relations at vertices 2 *and* 3*. Setting* $\rho = \{bc, cd\}$ *makes* $kQ/\langle\rho\rangle$ *a string algebra, called* Λ_2*.*

2. Let Q *be the following quiver with* $\rho = \{\alpha\beta, \beta\alpha, \alpha^n, \beta^m\}$*.*



Then $kQ/\langle\rho\rangle$ is a string algebra, called $GP_{n,m}$, for all $n, m \geq 2$.

3. *The infinite linear quiver A_∞^∞*

$$\cdots \to a_{-1} \to a_0 \to a_1 \to \cdots$$

is a string algebra that is not locally bounded.

4.2 Strings and bands

In this section, we introduce the notion of strings and bands in a string algebra Λ. We will eventually look at a way of assigning Λ-modules to these combinatorial objects.

We start by introducing for each arrow $\alpha \in Q_1$, a 'formal inverse', denoted by α^{-1}, and setting $s(\alpha^{-1}) := t(\alpha)$ and $t(\alpha^{-1}) := s(\alpha)$. Let $Q_1^{-1} := \{\alpha^{-1} \mid \alpha \in Q_1\}$. We will refer to the elements of the set $Q_1 \bigsqcup Q_1^{-1}$ as letters. A letter u contained in Q_1 will be referred to as a *direct letter* and a letter u contained in Q^{-1} will be referred to as an *inverse letter*. Also if $u = \alpha^{-1}$, where α is a direct letter, then $u^{-1} := \alpha$.

Let $\Lambda = kQ/\langle\rho\rangle$ be an arbitrary string algebra, not necessarily locally bounded.

Definition 4.2.1. *A* trivial walk *in Q is a trivial path ε_x with $x \in Q_0$. A* non-trivial walk *w in Q is a formal product $\prod_{i\in S} c_i$, where S is a non-empty interval in $\mathbb{Z}$, such that c_i is either an arrow or the inverse of an arrow, and $s(c_{i+1}) = t(c_i)$ for all i such that $i, i+1 \in S$. A* reduced walk *w in Q is either a trivial walk or a non-trivial walk $\prod_{i\in S} c_i$ such that $c_{i+1} \neq c_i^{-1}$ for any i such that $i, i+1 \in S$.*

Throughout this work, the order on $\{c_i \mid i \in S\}$ in a non-trivial walk $\prod_{i\in S} c_i$ will coincide with the increasing order of S, that is,

$$\prod_{i\in S} c_i = \cdots c_i c_{i+1} \cdots.$$

Each c_i will be called an *edge* of w. When S is bounded below by an integer l, we shall call c_l the *initial edge* of w and $s(c_l)$ the *starting point* of w, denoted $s(w)$. When S is bounded above by an integer m, we shall call c_m the *terminal edge* of w and $e(c_m)$ the *ending point* of w, denoted $e(w)$. We shall call a walk $\prod_{i\in S} c_i$ a *right infinite* (respectively *left infinite, doubly infinite*) *walk* if S is bounded below and unbounded above (respectively bounded above and unbounded below, unbounded below and above). Moreover, we say two non-trivial walks $w = \prod_{i\in S} c_i$, $w' = \prod_{j\in S'} c'_j$ are *equivalent* if there exists some $m \in \mathbb{Z}$ such that $S' = \{s + m \mid s \in S\}$ and $c'_i = c_{i-m}$. Thus, up to equivalence, we can assume S to be one of $[1, n]$ for some $n > 0$, $\mathbb{N}, \mathbb{N}^-$, or $\mathbb{Z}$.

We also extend the definition of inverses to the set of all reduced walks as follows. If $w = \prod_{i\in S} c_i$ with $S = [1, n]$, $w^{-1} := \prod_{i\in S} c^{-1}_{n-i+1}$. For $w = \prod_{i\in S} c_i$, a right infinite walk (resp. left infinite walk), w^{-1} is defined to be the left infinite walk (resp. right infinite walk) $w = \prod_{i\in\mathbb{N}^-} c_i^{-1}$ (resp. $w = \prod_{i\in\mathbb{N}} c_i^{-1}$). Finally for a doubly infinite walk, $w = \prod_{i\in\mathbb{Z}} c_i$, $w^{-1} := \prod_{i\in\mathbb{Z}} c^{-1}_{-i}$, and for $w = \varepsilon_a$, $w^{-1} := w$ for all $a \in Q_0$. A reduced walk $w = c_1 \cdots c_n$ in Q for $n \geq 1$ is called a *reduced cycle* if $s(c_1) = e(c_n)$.

Let $w = \prod_{i\in S} c_i$ be a reduced walk. Let $a_i = s(c_i)$ and $a_{i+1} = e(c_i)$ for all $i \in S$. Set $\bar{S} = S$ if S has no maximal element, and otherwise, $\bar{S} = S \cup \{n + 1\}$, where n is the maximal element of S. Observe that for any $i \in S$, $i + 1 \in \bar{S}$ but $i - 1$ is not necessarily in S. We shall call $V(w) = \{a_i \mid i \in \bar{S}\}$ the *vertex set* of w and each $a \in V(w)$ a *vertex* of w. We emphasize that we think of a_i and a_j as different vertices for $i \neq j$ even if they correspond to the same vertex in Q. Note that, under this viewpoint, $V(w)$ is technically a multiset, even though, for the sake of convenience, we call it a 'set'. The vertex set of a trivial walk ε_a is defined to be the singleton set $\{a\}$. A *subwalk* of w is defined to be a reduced walk of the form $\prod_{i\in T} c_i$ for T a sub-interval of S or of the form ε_a for some $a \in V(w)$. A trivial walk ε_a has a unique *subwalk* ε_a. A subwalk w' of w is called an *initial subwalk* of w if $w = w'w''$ for a reduced walk w''. If w' is not an initial subwalk of w, then $w = uw'w''$ for a non-trivial walk u and we say that u is the *left complement* of w' in w. Dually, a subwalk w' of w is called a *terminal subwalk* of w if $w = w''w'$ for some reduced walk w''. If w' is not a terminal subwalk of w, then $w = w''w'u$ for a non-trivial walk u and we say that u is the *right complement* of w' in w. Note that in the above two definitions, w'' may be a trivial walk.

Definition 4.2.2. *Let $w = \prod_{i\in S} c_i$ be a reduced walk in (Q, ρ). We say that a path p is* contained *in w if there exists an interval $T \subseteq S$ such that $p = (\prod_{i\in T} c_i)^{\pm 1}$.*

A reduced walk w is called a string *if either it is trivial or it contains no path lying in ρ.*

Example 4.2.1. *For Λ_2, $aced^{-1}$, $ab^{-1}ab^{-1}\cdots$ are some examples of strings, while $ed^{-1}c^{-1}$, cc^{-1} are certain non-examples.*

We say that a string w is *composable* with another string w' if $e(w) = s(w')$ and the (reduced) concatenation ww' is a string, in which case the *composition* is defined to be ww'. By reduced concatenation, we mean that we remove any copy of trivial strings from the concatenation.

Definition 4.2.3. *A reduced cycle w is called a* band *if w^m is a string for all $m \geq 1$, and w is not a power of a reduced walk of smaller length.*

Example 4.2.2. *In Λ_2, ab^{-1}, cd^{-1}, ba^{-1}, and dc^{-1} are the only bands while there are infinitely many bands in $GP_{n,m}$.*

Let $w = \prod_{i\in S} c_i$ be a non-trivial string. Fix $i, j \in \bar{S}$. We shall say that j is a *successor* of i in w, or equivalently i is a *predecessor* of j in w, provided that $j = i$, or $c_i \cdots c_{j-1}$ is a path in case $i < j$, or $c_j \cdots c_{i-1}$ is the inverse of a path in case $j < i$. Moreover, we shall call i a *peak* for w provided that c_i is an arrow in case $i \in S$ and c_{i-1} is the inverse of an arrow in case $i-1 \in S$; and a *deep* for w provided that c_i is the inverse of an arrow in case $i \in S$ and c_{i-1} is an arrow in case $i-1 \in S$.

We say that a string w *starts on a peak* if it has a starting point and there is no arrow β such that βw is a string. Similarly, we say that it *starts in a deep* if it has a starting point and there is no arrow γ such that $\gamma^{-1} w$ is a string. Dually, a string w is said to *end on a peak* if it has an ending point and there is no arrow β such that $w\beta^{-1}$ is a string, and it is said to *end in a deep* if it has an ending point and there is no arrow γ such that $w\gamma$ is a string.

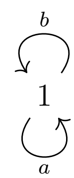
A string v is called a *substring* of a string w if it is either of the form $\prod_{i \in T} c_i$ for a subinterval T of S or ε_a for some $a \in V(w)$. A trivial string ε_a is defined to have a unique substring ε_a. We call w *zigzag finite* if it has finitely many peaks and deeps. We define the notions of *initial substrings* and *terminal substrings* similarly to the notions of initial and terminal subpaths respectively.

4.3 Definitions of string and band modules

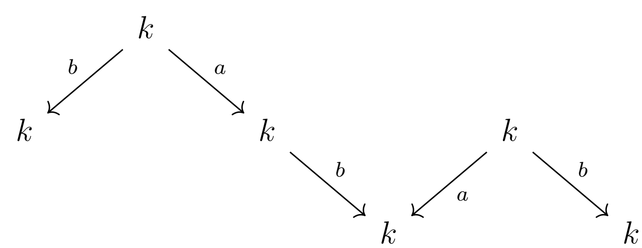
Given a string w in (Q, ρ), we can associate to it a *string module* $M(w)$ over Λ as follows. If $w = \varepsilon_a$ for a vertex $a \in Q_0$, then $M(w) := S(a)$, the simple Λ-module associated with a. Otherwise $w = \prod_{i \in S} c_i$, where $S \in \{[1,n], \mathbb{N}, \mathbb{N}^-, \mathbb{Z}\}$ and each $c_i : a_i \to a_{i+1}$, with $i \in S$ and $a_i, a_{i+1} \in Q_0$, is an edge from a_i to a_{i+1}. Now, the string module $M(w)$ has as a k-space a basis $\{v_i \mid i \in \bar{S}\}$. Its multiplication is defined as follows. Fix $i \in \bar{S}$. Given $a \in Q_0$, one defines $v_i \cdot e_a = v_i$ in case $a = a_i$, and otherwise, $v_i \cdot e_a = 0$. Given an arrow $\alpha : a \to b$ in Q, one defines $v_i \cdot \bar{\alpha} = v_{i+1}$ in case $i+1 \in \bar{S}$ and $c_i = \alpha$; and $v_i \cdot \bar{\alpha} = v_{i-1}$ in case $i-1 \in S$ and

$c_{i-1} = \alpha^{-1}$; and in all other cases, $v_i \cdot \bar{\alpha} = 0$. Since w is a reduced walk and $v_i \cdot e_{a_i} = v_i$, this definition has no ambiguity and makes $M(w)$ into a unitary right Λ-module. More explicitly, if p is a non-zero path of positive length r in Q, then $v_i \cdot \bar{p} = v_{i+r}$ in case $i+r \in \bar{S}$ and $p = c_i \cdots c_{i+r-1}$; and $v_i \cdot \bar{p} = v_{i-r}$ in case $i-r \in S$ and $p^{-1} = c_{i-r} \cdots c_{i-1}$; and in all other cases, $v_i \cdot \bar{p} = 0$. In particular, $v_i \cdot \bar{p} = 0$ in case $s(p) \neq a_i$. Henceforth, we shall call such a k-basis $\{v_i \mid i \in \bar{S}\}$ for $M(w)$ a **w**-*string basis.* For any string w there is an isomorphism $i_w : M(w) \to M(w^{-1})$ given by reversing the basis.

Example 4.3.1. *Let Q be the following quiver with $\rho = \{ab, b^2, a^2\}$.*



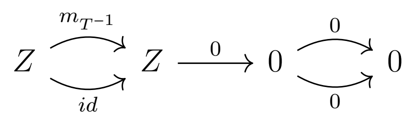
Let $u = b^{-1}aba^{-1}b$. Then $M(u)$ is 6-dimensional and we depict it as follows.



In general, we can represent any string w by a diagram of the above form by using arrows pointing towards the bottom left for inverse letters and towards the bottom right for direct letters.

Now, let $w = c_1 \cdots c_n$ be a band, where $c_i : a_i \to a_{i+1}$ are edges in Q between vertices a_i and a_{i+1} such that $a_{n+1} = a_1$. Let φ be an indecomposable automorphism of a finite-dimensional k-space V. Note that this is equivalent to giving a finite-dimensional indecomposable $k[T, T^{-1}]$ module by setting the action of T to be φ. We associate a *band module* $B(w,\varphi)$ to the pair (w,φ) as follows. First, we have an underlying k-space $B(w,\varphi) = U \otimes_k V$, where U is a k-space with a basis $\{u_1,\dots,u_n\}$. Set $u_{n+1} = u_1$ and $u_0 = u_n$. Given $\alpha \in Q_1$ and $u_i \otimes v \in B(w,\varphi)$ with $1 \le i \le n$ and $v \in V$, we define $(u_i \otimes v)\bar{\alpha} = u_{i+1} \otimes \varphi^{\delta_{in}}(v)$ if $\alpha = c_i$; and $(u_i \otimes v)\bar{\alpha} = u_{i-1} \otimes \varphi^{-\delta_{i-1,n}}(v)$ if $\alpha^{-1} = c_{i-1}$; and $(u_i \otimes v)\bar{\alpha} = 0$ in all other cases, where δ_{jn} is the Kronecker symbol. This makes $B(w,\varphi)$ into a unitary right Λ-module. Moreover, $B(w,\varphi) \cong B(w',\varphi)$ if and only if w' is a permutation of w or the inverse of w, i.e., $w' = c_{i+1}\cdots c_n c_1 \cdots c_i$ for $1 \le i \le n-1$, or $w' = c_i^{-1}\cdots c_1^{-1} c_n^{-1} \cdots c_{i+1}^{-1}$ for $1 \le i \le n-1$. We will often denote $B(w,\varphi)$ as $B(w,V)$, where V is viewed as a $k[T,T^{-1}]$ module through φ.

Example 4.3.2. *Let $w = ba^{-1}$ be a band in Λ_2 and Z an indecomposable $k[T,T^{-1}]$ module. Then $B(w,Z)$ is the following representation of (Q,ρ).*



For locally finite-dimensional string algebras, we have the following theorem about the classification of their finite-dimensional representations.

Theorem 4.3.1. *[11, § 3][14] Let Λ be a locally bounded string algebra. The modules $M(u)$, with u a finite string in Λ, and $B(w',Z)$, with w' a band in Λ, and Z a finite-dimensional, indecomposable $k[T,T^{-1}]$ module, give a complete list of finite-dimensional indecomposable Λ-modules. Moreover, $M(u) \cong M(u')$ if and only if $u = u'$ or $u^{-1} = u'$, and $M(w,Z) \cong M(w',Z')$ if and only if w is a cyclic permutation of w' or $(w')^{-1}$ and $Z \cong Z'$. Finally, no string module is isomorphic to a band module.*

Now, let Λ be an arbitrary string algebra. Suppose M is a locally finite-dimensional Λ-module. Then M is *finitely controlled* in the sense of [12], i.e., for every vertex a, Me_a is contained in a finitely generated submodule of M. In particular, the following theorem gives us a complete description of such modules.

Theorem 4.3.2. *[12, Theorem 1.2] Every finitely controlled Λ-module is isomorphic to a direct sum of copies of string modules and finite-dimensional band modules.*

4.4 Some special string modules

Let $\Lambda = kQ/\langle\rho\rangle$ be a string algebra. Since $\Lambda = \oplus_{a\in Q_0} e_a\Lambda = \oplus_{a\in Q_0}\Lambda e_a$, Λ has sufficiently many idempotents and Proposition 1.4.1 gives that $P_a = e_a\Lambda$ is a projective module. Theorem 4.4.1 shows how these modules can be obtained as string modules associated to some special strings.

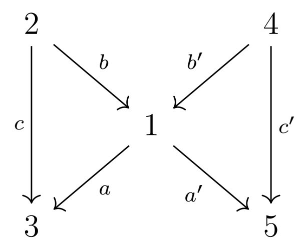
Theorem 4.4.1. *Let $a \in Q_0$. Then $P_a \cong M(p^{-1}q)$, where p, q are paths starting at a such that $p^{-1}q$ is a string starting and ending in a deep.*

Proof. We know that $P_a = e_a\Lambda$ has as basis the set of classes of non-zero paths starting at a. Let p be a maximal path (possibly trivial) in Q starting at a. If there exists at most one arrow starting at a, then let q be trivial; otherwise, let q be the maximal path starting at a and having no common initial arrow with p. Since Λ is a string algebra, any path starting at a is an initial subpath of p or q. Consider the case where p, q are non-trivial. Let $p = \prod_{i\in S}\alpha_i$ and $q = \prod_{i\in T}\beta_i$. Set $v_0 := e_a$, $v_j := \prod_{1\le i\le j}\bar{\beta}_i$ for $j \in T$, and $v_{-j} := \prod_{1\le i\le j}\bar{\alpha}_i$ for $j \in S$. Then we claim that the set $V = \{v_0\} \cup \{v_j \mid j \in T\} \cup \{v_{-j} \mid j \in S\}$ is a $p^{-1}q$ string basis for P_a.

Let u be a non-zero path of length $r \ge 1$ in Q. Then $v_0 \cdot \bar{u} \ne 0$ only if u starts at a, i.e., u is an initial subpath of p or q. In this case, $v_0 \cdot \bar{u} = v_r$ if $r \in T$ and $u = \prod_{1\le i\le r}\beta_i$; $v_0 \cdot \bar{u} = v_{-r}$ if $r \in S$ and $u = \prod_{1\le i\le r}\alpha_i$. Let $j \in T$. By the properties of string algebras, $v_j \cdot \bar{u}$ is non-zero only if $j + r \in T$ and $u = \prod_{j+1\le i\le r}\beta_i$, in which case it is equal to v_{j+r}. Finally, for $j \in S$, $v_{-j} \cdot \bar{u} = v_{-j-r}$ if $-j - r \in S$ and $u = \prod_{j+1\le i\le r}\alpha_i$; and in all other cases $v_{-j} \cdot \bar{u} = 0$. Now suppose $u = \varepsilon_b$ for some $b \in Q_0$. Then for all $v \in V$, $v \cdot \bar{u}$ is non-zero only if $b = t(v)$, in which case it equals v.

In case one of p or q is trivial, the u in the above paragraph would be an initial subpath of q or p respectively, and the proof would remain the same. In case p, q are both trivial, a is a sink and $P_a \cong S_a \cong M(\varepsilon_a)$. □

Example 4.4.1. *Let Q be the following quiver with $\rho = \{ba', b'a\}$.*



Then $P_1 = M(a^{-1}a')$, $P_2 = M(c^{-1}ba)$, $P_3 = M(e_3)$, $P_4 = M(c'^{-1}b'a')$, and $P_5 = M(e_5)$.

Let Λ be a locally finite-dimensional string algebra. The following theorem states that the injective modules $I_a, a \in Q_0$ can also be obtained as string modules.

Theorem 4.4.2. *Let $\Lambda = kQ/\langle\rho\rangle$ be a locally finite-dimensional string algebra with $a \in Q_0$. Then $I_a \cong M(pq^{-1})$ where p, q are the paths ending at a such that pq^{-1} is a string starting and ending in a deep.*

Proof. Given $a, x \in Q_0$, the set $\{\bar{p}_1, \cdots, \bar{p}_s\}$ of classes of non-zero paths from x to a is a k-basis for $P_a^{\circ}(x) = e_x\Lambda e_a$, and its dual basis $\{\bar{p}_1^*, \cdots, \bar{p}_s^*\}$ is a k-basis of $I_a(x)$. Consider the case where p, q are non-trivial. Write $p = \prod_{i\in S}\alpha_i$ and $q = \prod_{j\in T}\beta_j$, where S, T are intervals of $\mathbb{N}^-$ containing -1. Observe that any path ending at a in Q is a terminal subpath of either p or q.

Set $v_0 = e_a^*$, $v_{-i} = (\overline{\alpha_i\cdots\alpha_{-2}\alpha_{-1}})^*$ for $i \in S$, and $v_j = (\overline{\beta_j\cdots\beta_{-2}\beta_{-1}})^*$, for $j \in T$. Then we claim that the set $V = \{v_0\} \cup \{v_j \mid j \in T\} \cup \{v_{-j} \mid j \in S\}$ is a (pq^{-1})-string basis for I_a.

In order to prove the above claim, we first make the following observation. Suppose p' is a non-zero path, possibly trivial, in Q from vertex x to y and $\alpha : b \to c$ is an arrow. Then $\bar{p'}^* \cdot \bar{\alpha} = \bar{q'}^*$ if and only if $p' = \alpha q'$. Let us first suppose $p' = \alpha q'$. Let r be a path in Q starting at c. Then $(\bar{p'}^* \cdot \bar{\alpha})(\bar{r}) = \bar{p'}^*(\bar{\alpha}\bar{r})$, which is equal to 1 if $\bar{\alpha}\bar{r} = \bar{p'}$ and 0 otherwise. Therefore, $(\bar{p'}^* \cdot \bar{\alpha})(\bar{r}) = 1$ if $r = q'$ and 0 otherwise. This proves that $\bar{p'}^* \cdot \bar{\alpha} = \bar{q'}^*$. Conversely, suppose $\bar{p'}^* \cdot \bar{\alpha} = \bar{q'}^*$. Then $(\bar{p'}^* \cdot \bar{\alpha})(\bar{q'}) = 1$, which gives that $\bar{p'}^*(\bar{\alpha}\bar{q'}) = 1$, and hence $p' = \alpha q'$.

Moreover, we also note that for $a \in Q_0$, $\bar{p'}^* \cdot e_a = \bar{q'}^*$ if and only if $p' = q'$ and $a = s(p')$.

As a corollary, we conclude the following: Suppose p is a non-zero path of length $r \geq 0$ in Q. Then $v_0.\bar{p} = e_a^*.\bar{p}$ is non-zero if and only if $p = \epsilon_a$, in which case $v_0.\bar{p} = v_0$, and $v_{-i}.\bar{p} = (\alpha_i \cdots \alpha_2 \alpha_1)^*.\bar{p}$ is non-zero if and only if $p = e_{t(\alpha_i)}$ or $p = \alpha_i \cdots \alpha_{i-r+1}$ for $r \leq i$, in which case $v_{-i} = v_{-i+r}$. Similarly $v_j.\bar{p} = (\beta_j \cdots \beta_2 \beta_1)^*.\bar{p}$ is non-zero if and only if $p = e_{t(\beta_j)}$ or $p = \beta_j \cdots \beta_{i-r+1}$ for $r \leq i$, in which case $v_j = v_{j-r}$. □

Example 4.4.2. *Let Q be the quiver ${}_\infty A_\infty$ with the following orientation and without any relations*

$$\cdots \longrightarrow -2 \xrightarrow{b_2} -1 \xrightarrow{b_1} 0 \xleftarrow{a_1} 1 \xleftarrow{a_2} 2 \longleftarrow \cdots$$

Then $I_0 = M(\cdots b_2 b_1 a_1^{-1} a_2^{-1} \cdots)$, $I_k = M(a_{k+1}^{-1} \cdots)$, and $I_{-k} = M(\cdots b_{k+1})$, for $k > 0$.

In the following chapters, we will classify the finitely generated and finitely cogenerated string modules over Λ.

CHAPTER 5

Finitely-generated string modules

In this chapter, our goal would be to classify the finitely-generated string modules over a string algebra Λ. Furthermore, we will calculate their projective covers and syzygies and show that any finitely-generated string module is finitely presented as well.

Throughout this chapter, Λ will denote a locally finite-dimensional string algebra. We start by giving a combinatorial description of the top and socle of a string module.

5.1 Socle and top of string modules

Definition 5.1.1. *Let M be a Λ-module. Then the* socle *of M, denoted* $\mathrm{soc}(M)$, *is defined to be the sum of simple submodules of M.*

Lemma 5.1.1. *Let $w = \prod_{i \in S} c_i$ be a string with $i, j, l \in \bar{S}$.*

1. *If i is a successor of j in w, and j is a successor of l in w, then i is a successor of l in w.*

2. *If i is a successor of j in w, and j is a successor of i in w, then $i = j$.*

3. *If i is a successor (predecessor) of j in w and i is a peak (deep) for w, then $i = j$.*

Proof. The first statement is trivial if $i = j$ or $j = l$. Hence, we can suppose that i is a successor of j with $i \neq j$, and j is a successor of l in w with $j \neq l$. We first consider the case $j < i$. Then $c_j \cdots c_{i-1}$ is a path. In particular, c_j is an arrow. If $j < l$, then $c_j \cdots c_{l-1}$ is the inverse of a path. In particular, c_j is the inverse of an arrow, a contradiction. Thus, $l < j$ and $c_l \cdots c_{j-1}$ is a path. Therefore, $l < i$ and $c_l \cdots c_{i-1} = c_l \cdots c_{j-1} c_j \cdots c_{i-1}$ is a path. Therefore, i is a successor of l in w.

We now consider the case $i < j$. Then $c_i \cdots c_{j-1}$ is the inverse of a path. In particular, c_{j-1} is the inverse of an arrow. If $l < j$, then $c_l \cdots c_{j-1}$ is a path, and hence, c_{j-1} is an arrow, a contradiction. Thus, $j < l$ and $c_j \cdots c_{l-1}$ is the inverse of a path. Therefore, $i < l$ and $c_i \cdots c_{l-1} = c_i \cdots c_{j-1} c_j \cdots c_{l-1}$ is the inverse of a path. Therefore, by definition, i is a successor of l in w. This proves Statement (1).

Suppose i is a successor of j and j is a successor of i in w. If $j < i$, then $c_j \cdots c_{i-1}$ is a path and $c_j \cdots c_{i-1}$ is the inverse of a path, a contradiction. If $i < j$, then $c_i \cdots c_{j-1}$ is the inverse of a path and $c_i \cdots c_{j-1}$ is a path, a contradiction again. Therefore, $i = j$.

Finally, if i is a successor of j and i is a peak, then we have the following two cases: if $j < i$ then, by definition, $c_j \cdots c_{i-1}$ is a path. In particular, $i - 1 \in S$ with c_{i-1} being an arrow. If $i < j$ then, by definition, $c_i \cdots c_{j-1}$ is the inverse of a path, and hence, $i \in S$ with c_i being the inverse of an arrow. In either case, i is not a peak, a contradiction. On the other hand, if i is a predecessor of j and i is a deep, then we again have two cases: if $i > j$ then, by definition, $c_j \cdots c_{i-1}$ is the inverse of a path. In particular, $i - 1 \in S$ with c_{i-1} being the inverse of an arrow. If $j > i$ then, by definition, $c_i \cdots c_{j-1}$ is a path, and hence, $i \in S$ with c_i being an arrow. In either case, i is not a deep, a contradiction. $\square$

Recall that for a Λ-module M, and $x \in M$, $x\Lambda$ denotes the submodule of M generated by x, i.e.,

$$x\Lambda = \{x \cdot \lambda \mid \lambda \in \Lambda\}.$$

Lemma 5.1.2. *Consider the string module $M(w)$ associated to a string $w = \prod_{i \in S} c_i$, where S is an interval of $\mathbb{Z}$ and $c_i : a_i \to a_{i+1}$ are edges in Q. Let $\{v_i \mid i \in \bar{S}\}$ be a w-string basis for $M(w)$.*

1. *$\operatorname{soc}(M(w)) = \oplus_{i \in \nabla} v_i \Lambda$, where ∇ is the set of deeps for w.*

2. *$\operatorname{top}(M(w)) \cong \oplus_{i \in \Delta} S_{a_i}$, where Δ is the set of peaks for w.*

Proof. Let $i \in \nabla$. Then c_i is the inverse of an arrow in case $i \in S$ and c_{i-1} is an arrow in case $i - 1 \in S$. Given an arrow $\alpha : a \to b$ in Q, if $v_i \alpha = v_{i+1}$, then $\alpha = c_i$, a contradiction; and if $v_i \alpha = v_{i-1}$, then $\alpha^{-1} = c_{i-1}$, a contradiction as well. Thus, $v_i \alpha = 0$, for any $\alpha \in Q_1$. As a consequence, $v_i \Lambda = k v_i$, which is simple. Therefore, $\oplus_{i \in \nabla} v_i \Lambda \subseteq \operatorname{soc}(M(w))$.

Consider now $v = \sum_{i \in \Omega} \lambda_i v_i \in \operatorname{soc}(M(w))$, where $\Omega \subseteq \bar{S}$ and $\lambda_i \in k^*$. Assume that there exists some $i \in \Omega \backslash \nabla$. Then, $c_i = \alpha_i$ or $c_{i-1} = \beta_i^{-1}$, where $\alpha_i, \beta_i \in Q_1$. In the first case, $v_i \alpha_i = v_{i+1}$. This yields

$$0 = v\alpha_i = \lambda_i v_{i+1} + \sum_{j \in \Omega \setminus \{i\}} \lambda_j v_j \alpha_i.$$

Since $\lambda_i \neq 0$, we see that $v_{i+1} = v_j \alpha_i$ for some $j \in \Omega \backslash \{i\}$. If $\alpha_i = c_j$, then $v_j \alpha_i = v_{j+1} = v_{i+1}$, which is absurd since $j \neq i$. If $\alpha_i^{-1} = c_{j-1}$, then $v_j \alpha_i = v_{j-1} = v_{i+1}$. Thus, $i + 1 = j - 1$ and $c_{i+1} = c_{j-1} = \alpha_i^{-1}$, contradiction to w being a reduced walk. In other cases, $v_j \alpha_i = 0$, and hence, $v_{i+1} = 0$, a contradiction as well. In the second case, $v_i \beta_i = v_{i-1}$. This yields

$$0 = v\beta_i = \lambda_i v_{i-1} + \sum_{j \in \Omega \setminus \{i\}} \lambda_j v_j \beta_i.$$

Since $\lambda_i \neq 0$, we see that $v_{i-1} = v_j \beta_i$ for some $j \in \Omega \backslash \{i\}$. If $\beta_i^{-1} = c_{j-1}$, then $v_j \beta_i = v_{j-1} = v_{i-1}$, which is absurd because $j \neq i$. If $\beta_i = c_j$, then $v_j \beta_i = v_{j+1} = v_{i-1}$. Thus, $j + 1 = i - 1$, and hence, $c_{j+1} = c_{i-1} = \beta_i^{-1}$, contradiction to w being a reduced walk. In other cases, $v_j\beta_i = 0$, and hence, $v_{i-1} = 0$, a contradiction as well. Thus, $\mathrm{soc}(M(w)) \subseteq \oplus_{i\in\nabla} v_i\Lambda$. This shows Statement (1).

We will now prove Statement (2). By definition, $\mathrm{top}M(w) = M(w)/M(w)J$, where J is the ideal of Λ generated by the residue classes of the arrows modulo I. Given $i \in \Delta$, let $\{u_i\}$ be a k-basis of the simple module S_{a_i}. Clearly, we have a Λ-linear map

$$f : \oplus_{i\in\Delta} S_{a_i} \to M(w)/M(w)J : (\lambda_i u_i)_{i\in\Delta} \mapsto \textstyle\sum_{i\in\Delta}\lambda_i\hat{v}_i,$$

where $\lambda_i \in k$ and $\hat{v}_i = v_i + M(w)J$, for all $i \in \Delta$.

Fix $s \in \bar{S}\backslash\Delta$. As seen in the proof of Lemma 5.2.1, s is a successor in w of some $m_t \in \Delta$. Clearly, $s \neq m_t$. Thus, either $s < m_t$ with $c_s\cdots c_{m_t-1}$ being the inverse of a path or $s > m_t$ with $c_{m_t}\cdots c_{s-1}$ being a path. This implies that either $v_{m_t}c_{m_t-1}^{-1}\cdots c_s^{-1} = v_s$ or $v_{m_t}c_{m_t}\cdots c_{s-1} = v_s$. In any case, $v_s \in M(w)J$. Now, for any $v = \sum_{i\in\bar{S}}\lambda_i v_i \in M(w)$, we see that $\hat{v} = \sum_{i\in\Delta}\lambda_i\hat{v}_i$. Hence, $f((\lambda_i u_i)_{i\in\Delta}) = \hat{v}$. That is, f is surjective.

Finally, suppose that $f((\lambda_i u_i)_{i\in\Delta}) = \sum_{i\in\Delta}\lambda_i\hat{v}_i = 0$. Then, $\sum_{i\in\Delta}\lambda_i v_i \in M(w)J$. Therefore, $\sum_{i\in\Delta}\lambda_i v_i = \sum_{t\in\Omega}\mu_t v_t\bar{p}_t$, where Ω is a finite subset of $\bar{S}$, $\mu_t \in k$ and p_t is paths of length $l_t \geq 1$. Suppose that $\lambda_i \neq 0$ for some $i \in \Delta$. By definition, c_i is an arrow if $i \in S$ and c_{i-1} is the inverse of an arrow if $i-1 \in S$. Since $\lambda_i \neq 0$, we obtain $v_i = v_s\,\bar{p}_s = v_{s\pm l_s}$ for some $s \in \Omega$. That is, $i = s \pm l_s \neq s$. If $i > s$, then $p_s = c_s\cdots c_{i-1}$, and in particular, c_{i-1} is an arrow, a contradiction. If $s > i$, then $p_s^{-1} = c_i\cdots c_{s-1}$, and in particular, c_i is an inverse of an arrow, a contradiction as well. Therefore, $\lambda_i = 0$ for all $i \in \Delta$. That is, f is injective. □

5.2 Finitely generated string modules

We start by stating a few combinatorial lemmas that we will need.

Lemma 5.2.1. *Let $w = \prod_{i \in S} c_i$ be a string, where $c_i : a_i \to a_{i+1}$ are edges in Q. Then w admits at most finitely many peaks and every $i \in \bar{S}$ is a successor of some peak in w if and only if $w = p_1^{-1} q_1 \cdots p_r^{-1} q_r$, where p_i, q_i are paths in Q such that p_i is non-trivial for $1 < i \le r$ and q_i is non-trivial for $1 \le i < r$.*

Proof. Suppose w admits finitely many peaks $i_1 \le \ldots \le i_r$ in $\bar{S}$ such that every $i \in \bar{S}$ is a successor of some peak in w. We shall show that w is of the form as stated in the lemma. We start by defining p_1. If $i_1 - 1 \notin S$, then $p_1 := \varepsilon_{a_{i_1}}$. Suppose that $i_1 - 1 \in S$. Given any $i \in S$ with $i \le i_1 - 1$, i is a successor of some i_m with $1 \le m \le r$. Since $i < i_1 \le i_m$, by definition, $c_i \cdots c_{i_m - 1}$ is the inverse of a path, and in particular, c_i is the inverse of an arrow. Setting S_1 to be the set of $i \in S$ with $i < i_1$, we see that $\prod_{i \in S_1} c_i$ is the inverse of a non-trivial path. We define p_1 to be this path.

We can similarly define the path q_r. If $i_r \notin S$, then $q_r := \varepsilon_{a_{i_r}}$. Suppose that $i_r \in S$. Consider $i \in \bar{S}$ with $i_r < i$. Then i is a successor in w of some i_n with $1 \le n \le r$. Since $i_n \le i_r < i$, by definition, $c_{i_n} \cdots c_{i-1}$ is a path. In particular, c_{i-1} is an arrow. Letting T_r be the set of $i \in \bar{S}$ with $i_r < i$, we see that $\prod_{i \in T_r} c_{i-1}$ is a non-trivial path, which we define to be q_r. If $r = 1$, then $w = \prod_{i \in S_1 \cup T_1} = p_1^{-1} q_1$.

Now suppose that $r > 1$. Fix some $1 \le l < r$. Since i_l, i_{l+1} are peaks, c_{i_l} is an arrow and $c_{i_{l+1} - 1}$ is the inverse of an arrow. Therefore, we obtain a maximal j_l with $i_l < j_l < i_{l+1}$ such that $c_{i_l} \cdots c_{j_l - 1}$ is a non-trivial path, say q_l. Then, c_{j_l} is the inverse of an arrow. Consider i with $j_l < i < i_{l+1}$, which is a successor of i_t for some $1 \le t \le r$. If $t \le l$, then $i_t < i$ and we obtain a path $c_{i_t} \cdots c_{i-1} = c_{i_t} \cdots c_{j_l} \cdots c_{i-1}$, contrary to c_{j_l} being the inverse of an arrow. Thus, $l + 1 \le t$ and $i < i_t$ which gives that $c_i \cdots c_{i_t - 1}$ is the inverse of a path, and in particular, c_i is the inverse of an arrow. As a consequence, $c_{j_l} \cdots c_{i_{l+1} - 1}$ is the inverse of a non-trivial path, say p_{l+1}. It is now easy to see that $w = p_1^{-1} q_1 \cdots p_r^{-1} q_r$.

Conversely, assume that w can be written as $w = p_1^{-1} q_1 \cdots p_r^{-1} q_r$, where p_i, q_i are paths in Q

such that p_i is non-trivial for $1 < i \le r$ and q_i is non-trivial for $1 \le i < r$. For each $1 \le i \le r$, let S_i, $T_i \subseteq S$ be such that $\prod_{j \in S_i} c_j = p_i^{-1}$ and $\prod_{j \in T_i} c_j = q_i$, where $S_1 = \varnothing$ in case p_1 is trivial; and $T_r = \varnothing$ in case q_r is trivial. Let s_i be the minimal element of T_i, for $1 \le i < r$. Moreover, let s_r be the minimal element of T_r in case $T_r \ne \varnothing$, and otherwise, let s_r be the unique element of $\bar{S} \setminus S$. By definition, S_i and T_i are convex subsets of S such that

$$S = (S_1 \cup T_1) \cup (S_2 \cup T_2) \cdots \cup (S_r \cup T_r)$$

when viewed as orders. Moreover, $s_i - 1$ is the maximal element of S_i if it exists, for all $1 \le i \le r$.

We claim that $\{s_1, s_2, \ldots, s_r\}$ is the set of tops for w. Let us start with s_1. By definition, c_{s_1} is the initial arrow of q_1. If $s_1 - 1 \in S$, then $s_1 - 1 \in S_1$ with $c_{s_1 - 1}$ being the inverse of the initial arrow of p_1. So s_1 is a peak by definition. Consider now s_i with $1 < i < r$. Since s_i is the minimal element of T_i, the edge c_{s_i} is the initial arrow of q_i; and since $s_i - 1$ is the maximal element of S_i, the edge $c_{s_i - 1}$ is the inverse of the initial arrow of p_i. So s_i is a peak by definition. Let us finally consider s_r. Since $s_r - 1$ is the maximal element of S_r, the edge $c_{s_r - 1}$ is the inverse of the initial arrow of p_{r-1}. If $s_r \in S$, then s_r is the minimal element of T_r, and hence, c_{s_r} is the initial arrow of q_r. So s_r is a peak by definition.

Given $1 \le l \le r$, we claim that each $i \in S_l \cup T_l$ is a successor of s_l in w. Indeed, if $i < s_l$, since s_l is minimal in T_l, we see that $i \in S_l$ with $i \le s_l - 1$. In this case, $c_i \cdots c_{s_l - 1}$ is the inverse of an initial subpath of p_l. So i is a successor of s_l in w by definition. If $i > s_l$, since $s_l - 1$ is the maximal element of S_l, we see that $i \in T_l$. In this case, $c_{s_l} \cdots c_{i-1}$ is an initial path of q_l. So i is a successor of s_l in w.

In particular, every $i \in S$ is a successor in w of some s_l with $1 \le l \le r$. Consider now $i \in \bar{S} \setminus S$. Then $i \ge s_r$. If $i = s_r$, then it is a successor of s_r. If $i > s_r$ then, by definition, $i - 1 \in T_r$. In this case, $c_{s_r} \cdots c_{i-1} = q_r$. So i is a successor of i_r. By Lemma 5.1.1, $\{s_1, \ldots, s_r\}$ is the set of all peaks for w. $\square$

Lemma 5.2.2. *If $i, j \in \bar{S}$, then j is a successor of i in w if and only if $v_j \in v_i\Lambda$.*

Proof. Suppose j is a successor of i. If $j = i$, then $v_j = v_i e_{a_i}$. If $j < i$, then $c_j \cdots c_{i-1}$ is the inverse of a path, say p. Since p is a path of length $i-j > 0$, by definition, $v_i\bar{p} = v_{i-(i-j)} = v_j$. If $i < j$, then $c_i \cdots c_{j-1}$ is a path p. Since p is now a path of length $j - i > 0$, by definition, $v_i\bar{p} = v_{i+(j-i)} = v_j$.

Conversely, assume that $v_j \in v_i\Lambda$ with $j \neq i$. Write $v_j = \sum_{l=1}^{n} \lambda_l v_i \bar{p}_l$, where $\lambda_l \in k$ and p_l is a path of length m_l such that $\lambda_l v_i \bar{p}_l \neq 0$, for $l = 1, \ldots, n$. By definition, $v_i\bar{p}_l = v_{i \pm m_l}$. Then, $v_j = \sum_{l=1}^{n} \lambda_l v_{i \pm m_l}$, and hence, $j = i \pm m_{l_0}$, for some l_0 with $m_{l_0} > 0$. If $j = i + m_{l_0}$, then $p_{l_0} = c_i \cdots c_{i+m_{l_0}-1} = c_i \cdots c_{j-1}$. If $j = i - m_{l_0}$, then $p_{l_0}^{-1} = c_{i-m_{l_0}} \cdots c_{i-1} = c_j \cdots c_{i-1}$. In either case, j is again a successor of i. □

Having the above combinatorial results in our hands, we are now ready to prove the main theorem.

Theorem 5.2.1. *Let $w = \prod_{i \in S} c_i$ be a string, where $c_i : a_i \to a_{i+1}$ are edges in Q. The string module $M(w)$ is finitely generated if and only if w admits finitely many peaks and every $i \in \bar{S}$ is a successor of some peak in w.*

Proof. Suppose that $M(w)$ is finitely generated. By definition, $M(w)$ has a k-basis $\{v_i \mid i \in \bar{S}\}$. Then there exists a minimal subset $\{i_1, i_2, \ldots, i_r\}$ of $\bar{S}$, with $i_1 < i_2 \cdots < i_r$, such that $M(w) = \sum_{j=1}^{r} v_{i_j}\Lambda$.

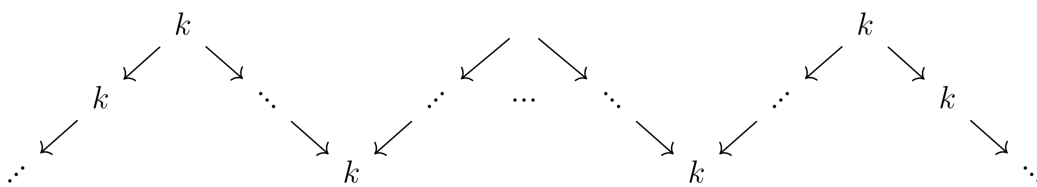
We claim that each $i \in \bar{S}$ is a successor of at least one of $i_1, i_2, \ldots, i_r$. Since $v_i \in \sum_{j=1}^{r} v_{i_j}\Lambda$, we may write $v_i = \sum_{l=1}^{t} \lambda_l v_{j_l} \bar{p}_l$, where $\lambda_l v_{j_l} \bar{p}_l \neq 0$ with $\lambda_l \in k$, $j_l \in \{i_1, \ldots, i_r\}$, and p_l a path of length m_l. Then, $v_i = \sum_{l=1}^{t} \lambda_l v_{j_l \pm m_l}$, where $v_{j_l \pm m_l} = v_{j_l}\bar{p}_l \in v_{j_l}\Lambda$. Therefore, $v_i = v_{j_{l_0} \pm m_{l_0}}$, and hence $i = j_{l_0} \pm m_{l_0}$, for some $1 \leq l_0 \leq t$. This gives that $v_i \in v_{j_{l_0}}\Lambda$. Then the previous lemma says that i is a successor of j_{l_0}.

It is now enough to show that the set $\{i_1, i_2, \ldots, i_r\}$ is the set of peaks for w. Suppose $i \in \bar{S}$ is a peak in w. Then we know that i is a successor of some i_j with $1 \leq j \leq r$. Using Lemma 5.1.1, $i = i_j$. Next, suppose that i_l is not a peak for some $1 \leq l \leq r$. Then either c_{i_l} is the inverse of an arrow or $c_{i_l - 1}$ is an arrow. In the first (resp. second) case, i_l is a successor of $i_l + 1$ (resp. $i_l - 1$). Since $i_l + 1$ (resp. $i_l - 1$) is a successor of some i_m with $1 \leq m \leq r$, Lemma 5.1.1 gives that i_l is a successor of i_m and $l \neq m$. Thus, the previous lemma gives that $v_{i_l} \in v_{i_m}\Lambda$, a contradiction to the minimality of $\{i_1, \ldots, i_r\}$.

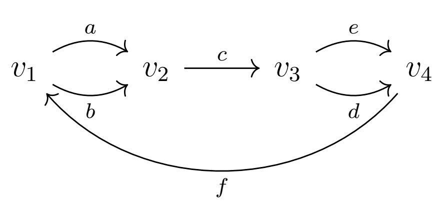
Therefore, w has finitely many peaks, and every $i \in \bar{S}$ is a successor of some peak.

Conversely, suppose w admits finitely many peaks $\{i_1, \ldots, i_r\}$ and that every $i \in \bar{S}$ is a successor of some peak. Using the previous lemma, $v_i \in v_{i_p}\Lambda$ for some $1 \leq p \leq r$. Since $\{v_i \mid i \in \bar{S}\}$ is a generating set of $M(w)$, we get that $M(w)$ is generated by $\{v_{i_1}, \ldots, v_{i_r}\}$. Therefore $M(w)$ is finitely generated. $\square$

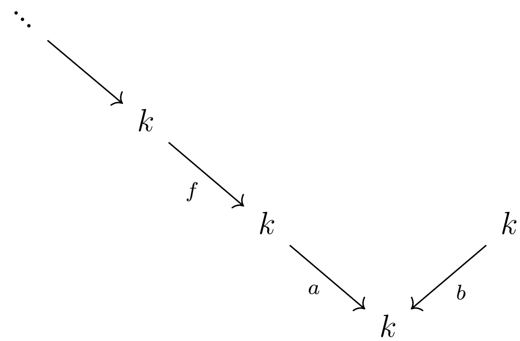
The above theorem can be informally stated as saying that a string module $M(w)$ is finitely generated if and only if the diagram representing w is of the following form. Note that the string has only finitely many peaks.



Example 5.2.1. *Let Q be the following quiver with $\rho = \{bc, cd, df, fb\}$.*



Let $w = \cdots acefacefab^{-1}$. Then the following picture demonstrates that not every $i \in \bar{S}$ is a successor of some peak, and hence $M(w)$ is not finitely generated.



5.3 Projective covers and syzygies

In this section, we will give an explicit description of the projective covers of finitely generated string modules. By Theorem 4.4.1, these will themselves be string modules. We will then use this projective cover to explicitly describe the 'syzygy' of such modules. We will use the following equivalent definition of the projective cover for our purposes.

Definition 5.3.1. *Let M be a finitely generated Λ module. The* projective cover *of M is a module $P \in \mathrm{proj}\Lambda$ with $f : P \to M$ an epimorphism such that* $\mathrm{Ker}(f) \subset \mathrm{rad}P$.

Lemma 5.3.1. *Let $w = \prod_{i \in S} c_i$ be a string, where the $c_i : a_i \to a_{i+1}$ are edges in Q. Assume that $M(w)$ is finitely generated with a w-string basis $\{v_i \mid i \in \bar{S}\}$. If Δ is the set of peaks for w, then $M(w)$ has a projective cover as follows*:

$$f_w : \oplus_{i \in \Delta} e_{a_i}\Lambda \to M(w) : (x_i)_{i \in \Delta} \mapsto \sum_{i \in \Delta} v_i x_i.$$

Proof. First, we claim that f_w is surjective. Let $j \in \bar{S}$. Then j is a successor of some $i \in \Delta$. If $j < i$, then $c_j \cdots c_{i-1} = p^{-1}$, where p is a path of length $i - j$ starting with a_i. Thus, $\bar{p} \in e_{a_i}\Lambda$ such that $f_w(\bar{p}) = v_i \bar{p} = v_{i-(i-j)} = v_j$. If $j > i$, then $c_i \cdots c_{j-1} = q$, where q is a path of length $j - i$ starting with a_i. Thus, $\bar{q} \in e_{a_i}\Lambda$ such that $f(\bar{q}) = v_i \bar{q} = v_{i+(j-i)} = v_j$. If $j = i$, then $f(e_{a_i}) = v_i e_{a_i} = v_i = v_j$. This proves our claim.

Let $x = (x_i)_{i\in\Delta} \in \mathrm{Ker}(f_w)$. For each $i \in \Delta$, we may write $x_i = \lambda_i e_{a_i} + \sum_j \lambda_{ij}\bar{p}_{ij}$, where $\lambda_i, \lambda_{ij} \in k$ and $p_{ij} \in Q_{l_{ij}}(a_i, -)$ with $l_{ij} > 0$. Then

$$f_w(x) = \sum_{i\in\Delta}\lambda_i v_i + \sum_{i\in\Delta}\sum_j \lambda_{ij} v_{i\pm l_{ij}} = 0.$$

This yields $\lambda_i = 0$ for all $i \in \Delta$. That is, $x \in \mathrm{rad}(\oplus_{i\in\Delta} e_{a_i}\Lambda)$. □

Definition 5.3.2. *Let M be a finitely generated Λ-module with $f : P \to M$ its projective cover. Then the* syzygy *of M, denoted $\Omega(M)$, is defined to be the module* $\mathrm{Ker}(f)$.

In essence, the syzygy of a module 'measures' its deviation from being a projective module.

Theorem 5.3.1. *Let $M(w)$ be a finitely-generated string module defined by a string $w = p_1^{-1}q_1 \cdots p_r^{-1}q_r$, where p_i, q_j are paths in Q such that p_i, q_j are non-trivial for $1 < i \le r$ and $1 \le j < r$ respectively. In case p_1 is finite and $p_1^{-1}q_1$ does not start in a deep, define x_1 to be the maximal path such that $x_1^{-1}p_1^{-1}q_1$ is a string. In case q_r is finite and $p_r^{-1}q_r$ does not end in a deep, define y_r to be the maximal path such that $p_r^{-1}q_r y_r$ is a string. For each $1 < i \le r$, let x_i and y_{i-1} be the maximal paths such that $p_i x_i$ and $q_{i-1}y_{i-1}$ are strings. Then*

$$\Omega(M(w)) = K_1 \oplus K_2 \oplus \cdots \oplus K_r \oplus K_{r+1},$$

where

1. *$K_1 = 0$ if x_1 is not defined; and otherwise, $K_1 = M(x)$ with x the path such that $x_1 = \alpha x$ for an arrow α;*

2. *$K_i = M(x_i^{-1}y_{i-1})$ for $2 \le i \le r$;*

3. *$K_{r+1} = 0$ if y_r is not defined; and otherwise, $K_{r+1} = M(y)$ with y the path such that $y_r = \beta y$ for an arrow β.*

Proof. We denote by l_i and d_i the length of p_i and q_i respectively, for $i = 1, \ldots, r$. Write $w = \prod_{i \in S} c_i$, where S is an interval of $\mathbb{Z}$ and $c_i : a_i \to a_{i+1}$ are edges in Q. Let $m_1, \ldots, m_r$ be the peaks for w such that $a_{m_i} = s(p_i)$. Since $s(q_{i-1}) = s(p_{i-1})$ and $e(q_{i-1}) = e(p_i)$, we see that $m_{i-1} + d_{i-1} = m_i - l_i$, which is a deep for w, for $2 \le i \le r$. Let $\{v_i \mid i \in \bar{S}\}$ be a w-string basis for $M(w)$. By Lemma 5.3.1, $M(w)$ admits a projective cover

$$f_w : P_w = \oplus_{i=1}^{r} e_{a_{m_i}} \Lambda \to M(w) : (\rho_1, \ldots, \rho_r) \mapsto \textstyle\sum_{i=1}^{r} v_{m_i} \rho_i.$$

We shall first show that $K_i \cong L_i$, a submodule of P_w, for $i = 1, \ldots, r+1$. For the sake of simplicity, we shall omit the zero components of an element $(\rho_1, \ldots, \rho_r)$ in P_w. For instance, $(e_{a_{m_1}}, 0, \ldots, 0)$ will be simply written as $(e_{a_{m_1}})$.

In case x_1 is not defined, $K_1 \cong L_1$ with $L_1 = 0$. Assume that $x_1 = \alpha x$, where $\alpha : a \to b$ is an arrow and x is a path. Put $L_1 = (\bar{p}_1 \bar{\alpha})\Lambda \neq 0$. By the maximality of x_1, we see that $\bar{p}_1 \bar{\alpha} \bar{x} \bar{\gamma} = 0$ for all $\gamma \in Q_1$. If $x = \varepsilon_b$, then $K_1 = M(x) \cong S_b$, and $L_1 = k(\bar{p}_1 \bar{\alpha}) \cong S_b$. Otherwise, $x = \alpha_1 \alpha_2 \cdots$, where $\alpha_i \in Q_1$. Then, $(\bar{p}_1 \bar{\alpha})\Lambda$ has k-basis $\{(\bar{p}_1 \bar{\alpha}), (\bar{p}_1 \bar{\alpha} \bar{\alpha}_1), (\bar{p}_1 \bar{\alpha} \bar{\alpha}_1 \bar{\alpha}_2), \ldots, \}$, which is clearly a x-string basis. Thus, $M(x) \cong L_1$. Similarly, $K_{r+1} \cong L_{r+1}$, where $L_{r+1} = 0$ in case y_r is not defined, and $L_{r+1} = (\bar{q}_r \bar{\beta})\Lambda$ in case $y_r = \beta y$ with β some arrow and y some path.

Consider $K_i = M(w_i)$, where $w_i = x_i^{-1} y_{i-1}$, for $2 \le i \le r$. By the maximality of x_i and y_{i-1}, we see that $\bar{p}_i \bar{x}_i \bar{\gamma} = 0$ and $\bar{q}_{i-1} \bar{y}_{i-1} \bar{\gamma} = 0$ for all $\gamma \in Q_1$. If x_i and y_{i-1} are trivial, then $K_i \cong S_{e(p_i)}$. Put $L_i = (\bar{q}_{i-1}, -\bar{p}_i)\Lambda$. Since $\bar{p}_i \bar{\gamma} = 0$ and $\bar{q}_{i-1} \bar{\gamma} = 0$ for all $\gamma \in Q_1$, we see that $L_i = k(\bar{q}_{i-1}, -\bar{p}_i) \cong S_{e(p_i)}$, and hence, $K_i \cong L_i$.

Assume that x_i is trivial and $y_{i-1} = \beta_{i-1,1} \beta_{i-1,2} \cdots$. Then $w_i = \beta_{i-1,1} \beta_{i-1,2} \cdots$. We set $L_i = (\bar{q}_{i-1}, -\bar{p}_i)\Lambda + (\bar{q}_{i-1} \bar{\beta}_{i-1,1})\Lambda$. Since $\bar{p}_i \bar{\gamma} = 0$ for $\gamma \in Q_1$, we see that L_i has a k-basis $\{(\bar{q}_{i-1}, -\bar{p}_i), (\bar{q}_{i-1} \bar{\beta}_{i-1,1}), (\bar{q}_{i-1} \bar{\beta}_{i-1,1} \bar{\beta}_{i-1,2}), \ldots\},$. This is a w_i-string basis for L_i because $(\bar{q}_{i-1}, -\bar{p}_i)\bar{\beta}_{i-1,1} = (\bar{q}_{i-1} \bar{\beta}_{i-1,1})$. Thus, $K_i \cong L_i$ in this case. Similarly, if y_{i-1} is trivial and $x_i = \alpha_{i1} \alpha_{i2} \cdots$, then $K_i \cong L_i$, where $L_i = (\bar{q}_{i-1}, -\bar{p}_i)\Lambda + (\bar{p}_i \bar{\alpha}_{i1})\Lambda$.

Finally, assume that $x_i = \alpha_{i1}\alpha_{i2}\cdots$ and $y_{i-1} = \beta_{i-1,1}\beta_{i-1,2}\cdots$, where $\alpha_{is}, \beta_{i-1,t}$ are arrows. Put $L_i = (-\bar{p}_i\bar{\alpha}_{i1})\Lambda + (\bar{q}_{i-1}, -\bar{p}_i)\Lambda + (\bar{q}_{i-1}\bar{\beta}_{i-1,1})\Lambda$. Since $\bar{q}_{i-1}\bar{\alpha}_{i1} = 0$ and $\bar{p}_i\bar{\beta}_{i-1,1} = 0$, we see that L_i has a k-basis

$$\{\ldots, (-\bar{p}_i\bar{\alpha}_{i1}\bar{\alpha}_{i2}), (-\bar{p}_i\bar{\alpha}_{i1}), (\bar{q}_{i-1}, -\bar{p}_i), (\bar{q}_{i-1}\bar{\beta}_{i-1,1}), (\bar{q}_{i-1}\bar{\beta}_{i-1,1}\bar{\beta}_{i-1,2}), \ldots\}.$$

Observe that $w_i = \cdots\alpha_{i2}^{-1}\alpha_{i1}^{-1}\beta_{i-1,1}\beta_{i-1,2}\cdots$. Since

$$(\bar{q}_{i-1}, -\bar{p}_i)\bar{\alpha}_{i1} = (-\bar{p}_i\bar{\alpha}_{i1}),$$

$$(\bar{q}_{i-1}, -\bar{p}_i)\bar{\beta}_{i-1,1} = (\bar{q}_{i-1}\bar{\beta}_{i-1,1}),$$

we see that the above basis is a w_i-string basis for L_i. Thus, $K_i \cong L_i$ in this case.

Now, set $L = \sum_{i=1}^{r+1} L_i$. It is not difficult to see that this is a direct sum. We first show that $L \subseteq \mathrm{Ker}(f_w)$, or equivalently, $L_i \subseteq \mathrm{Ker}(f_w)$, for $i = 1, \ldots, r+1$.

Suppose that $L_1 \neq 0$. Then, l_1 is finite and $L_1 = (\bar{p}_1\bar{\alpha})\Lambda$. Observe that $m_1 - l_1$ is a deep for w and $v_{m_1}\bar{p}_1 = v_{m_1-l_1}$. Thus, $v_{m_1-l_1}\bar{\alpha} = 0$, and consequently, $f_w((\bar{p}_1\bar{\alpha})) = v_{m_1}\bar{p}_1\bar{\alpha} = 0$. Thus, $L_1 \subseteq \mathrm{Ker}(f_w)$. Similarly, $L_{r+1} \subseteq \mathrm{Ker}(f_w)$.

Consider L_i with $2 \le i \le r$. Since $m_{i-1} + d_{i-1} = m_i - l_i$, we see that

$$f_w((\bar{q}_{i-1}, -\bar{p}_i)) = v_{m_{i-1}}\bar{q}_{i-1} - v_{m_i}\bar{p}_i = v_{m_{i-1}+d_{i-1}} - v_{m_i-l_i} = 0.$$

This shows that $(\bar{q}_{i-1}, -\bar{p}_i)\Lambda \subseteq \mathrm{Ker}(f_w)$. Let $y_{i-1} = \beta_{i-1,1}\beta_{i-1,2}\cdots$. Since $m_{i-1} + d_{i-1}$ is a deep for w, we see that $f_w((\bar{q}_{i-1}\bar{\beta}_{i-1,1})) = v_{m_{i-1}}\bar{q}_{i-1}\bar{\beta}_{i-1,1} = v_{m_{i-1}+d_{i-1}}\bar{\beta}_{i-1,1} = 0$. Thus, $(\bar{q}_{i-1}\bar{\beta}_{i-1,1})\Lambda \subseteq \mathrm{Ker}(f_w)$. Similarly, if $x_i = \alpha_{i1}\alpha_{i2}\cdots$, then $(\bar{p}_i\bar{\alpha}_{i1})\Lambda \subseteq \mathrm{Ker}(f_w)$. This implies that $L_i \subseteq \mathrm{Ker}(f_w)$ in any case.

Assume conversely that $\rho = (\rho_1, \ldots, \rho_r) \in \mathrm{Ker}(f_w)$, where $\rho_i \in e_{a_{m_i}}\Lambda$. For each $1 \le i \le r$, we may write

$$\rho_i = \textstyle\sum_{s\ge 0}\lambda_{is}\bar{p}_{is} + \sum_{t\ge 1}\mu_{it}\bar{q}_{it},$$

where $\lambda_{is}, \mu_{it} \in k$, and p_{is} is a path of length s such that p_{is} is an initial subpath of p_i for $0 \leq s \leq l_i$ and p_i is a proper initial subpath of p_{is} for $s > l_i$, while q_{it} is a path of length t such that q_{it} is an initial subpath of q_i for $1 \leq t \leq d_i$ and q_i is a proper initial subpath of p_{it} for $t > d_i$. Then

$$\begin{aligned} 0 &= f_w(\rho) = \textstyle\sum_{i=1}^r v_{m_i}\rho_i \\ &= \textstyle\sum_{i=1}^r (\sum_{s\geq 0} \lambda_{is} v_{m_i}\bar{p}_{is} + \sum_{t\geq 1} \mu_{it} v_{m_i}\bar{q}_{it}) \\ &= \textstyle\sum_{i=1}^r (\sum_{0\leq s\leq l_i} \lambda_{is} v_{m_i}\bar{p}_{is} + \sum_{1\leq t\leq d_i} \mu_{it} v_{m_i}\bar{q}_{it}) \\ &= \textstyle\sum_{0\leq s\leq l_1} \lambda_{1s} v_{m_1-s} + \sum_{i=2}^r \sum_{0\leq s<l_i} \lambda_{is} v_{m_i-s} \\ &\quad + \textstyle\sum_{i=1}^{r-1} \sum_{1\leq t<d_i} \mu_{it} v_{m_i+t} + \sum_{1\leq t\leq d_r} \mu_{rt} v_{m_r+t} \\ &\quad + \textstyle\sum_{i=2}^r (\lambda_{i,l_i} + \mu_{i-1,d_{i-1}}) v_{m_i-l_i} \end{aligned}$$

A a consequence, $\lambda_{1s} = 0$ for $0 \leq s \leq l_1$, and $\lambda_{is} = 0$ for $2 \leq i \leq r$ and $0 \leq s < l_i$, and $\mu_{it} = 0$ for $1 \leq i \leq r-1$ and $1 \leq t < d_i$, and $\mu_{rt} = 0$ for $1 \leq t \leq d_r$, and $\lambda_{i,l_i} + \mu_{i-1,d_{i-1}} = 0$ for $2 \leq i \leq r$. That is, $\rho_1 = \sum_{s>l_1} \lambda_{1s}\bar{p}_{1s} + \mu_{1,d_1}\bar{q}_1 + \sum_{t>d_1} \mu_{1t}\bar{q}_{1t}$,

$$\rho_i = -\mu_{i-1,d_{i-1}}\bar{p}_i + \mu_{i,d_i}\bar{q}_i + \textstyle\sum_{s>l_i} \lambda_{is}\bar{p}_{is} + \sum_{t>d_i} \mu_{it}\bar{q}_{it}, \text{ for } i = 2, \dots, r-1,$$

and $\rho_r = -\mu_{r-1,d_{r-1}}\bar{p}_r + \sum_{s>l_r} \lambda_{rs}\bar{p}_{rs} + \sum_{t>d_r} \mu_{rt}\bar{q}_{rt}$. This yields

$$\begin{aligned} \rho &= \textstyle\sum_{s>l_1} \lambda_{1s}(\bar{p}_{1s}) + \sum_{t>d_r} \mu_{rt}(\bar{q}_{rt}) \\ &\quad + \textstyle\sum_{i=2}^r \left(\sum_{t>d_{i-1}} \mu_{i-1,t}(\bar{q}_{i-1,t}) + \mu_{i-1,d_{i-1}}(\bar{q}_{i-1}, -\bar{p}_i) + \sum_{s>l_i} \lambda_{is}(\bar{p}_{is}) \right). \end{aligned}$$

Thus, $\rho \in L_1 + L_2 + \cdots + L_r + L_{r+1}$. Thus, $\mathrm{Ker}(f_w) = L = \oplus_{i=1}^{r+1} L_i \cong \oplus_{i=1}^{r+1} K_i$. □

Example 5.3.1. *Let Λ be the string algebra from Example 5.2.1. Let $w = ced^{-1}ef$. Then by definition, the peaks for w are 1 and 4. Therefore, the projective cover for $M(w)$ is given by $P_{v_2} \oplus P_{v_3}$, where*

$$P_{v_2} = M(cefacefa\cdots),$$

$$P_{v_3} = M(d^{-1}efacefac\cdots).$$

Moreover, using the previous theorem, we get that $K_1 = 0$, $K_2 = M(faceface\cdots)$, and $K_3 = M(cefacefa\cdots)$, hence

$$\Omega(M(w)) = M(faceface\cdots) \oplus M(cefacefa\cdots).$$

We immediately get the following theorem as a corollary of the previous theorem.

Theorem 5.3.2. *Let Λ be a string algebra. Then a string module over Λ is finitely generated if and only if it is finitely presented.*

Proof. Clearly, if a module is finitely presented, then it is finitely generated. Conversely, suppose $M(w)$ is a finitely generated module for a string w in Λ. Theorem 5.2.1 along with Lemma 5.2.1 implies that w is of the form $w = p_1^{-1}q_1 \cdots p_r^{-1}q_r$, where p_i, q_i are paths in Q such that p_i is non-trivial for $1 < i \leq r$ and q_i is non-trivial for $1 \leq i < r$. Using Theorem 5.3.1, we get that $\Omega(M(w))$ is a finite direct sum of string modules that are themselves finitely generated. Hence $\Omega(M(w))$ is finitely generated, which implies that $M(w)$ is finitely presented. □

CHAPTER 6

Finitely co-generated string modules

In this chapter, our goal would be to prove the dual results from the last chapter. We will classify the finitely co-generated string modules over Λ. Furthermore, we will calculate the injective envelopes and cosyzygies of such string modules and show that any finitely cogenerated string module is finitely copresented as well, which is again known to be the case for locally bounded string algebras. We will assume $\Lambda = kQ/\langle\rho\rangle$ to be a locally finite-dimensional string algebra for the entirety of this chapter.

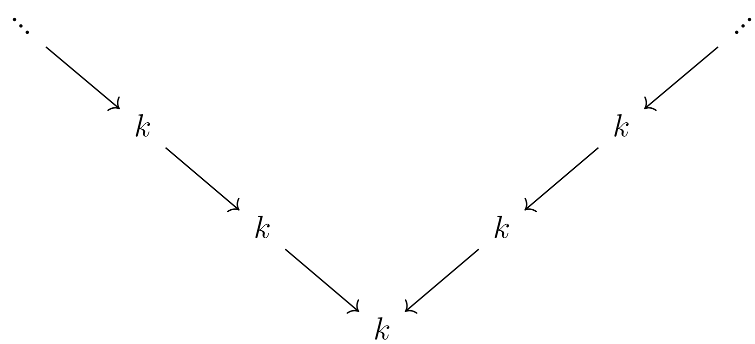
6.1 Finitely cogenerated string modules

As before, we start by stating a few combinatorial lemmas.

Lemma 6.1.1. *Let $I = \oplus_{i=1}^{r} I_{a_i}$ for some $a_i \in Q_0$ and $x \in I$ such that $x \neq 0$. Then there exists a path p in Q such that $0 \neq x \cdot \bar{p} \in \operatorname{soc}(I)$.*

Proof. Suppose $x = (x_1, \cdots, x_r)$ with $x_i \in I_{a_i}$. Let $S = \{j \mid 1 \leq j \leq r, x_j \neq 0\}$. Since $x \neq 0$, $S \neq \varnothing$. Using Theorem 4.4.2, for all $i \in S$, we can write $x_i = \sum_{j=1}^{r_i} \lambda_{i,j} v_{k_{i,j}}^{(i)}$, where $\{\ldots v_{-1}^{(i)}, v_0^{(i)}, v_1^{(i)}, \ldots\}$ is a string basis of I_{a_i} as described in Theorem 4.4.2 and $\lambda_{i,j} \neq 0$ for all $1 \leq j \leq r_i$. Let $m_i = \max_{j=1}^{r_i} |\, k_{i,j} \,|$ such that the maximum occurs at some j_i'. Let p_i be the path ending at a_i corresponding to $v_{k_{i,j_i'}}^{(i)}$, i.e., $v_{k_{i,j_i'}}^{(i)} = (\bar{p}_i)^*$. Then $x_i \cdot \bar{p}_i = \lambda_{i,j_i'}(e_{a_i})^* \in \operatorname{soc}(I_{a_i})$. Now, let $p_{i'}$ be the path with the maximum length among the p_i. Then $x_i \cdot \bar{p}_{i'}$ is either 0 or equal to $\lambda_{i,j_i'}(e_{a_i})^*$. Since $x_{i'} \cdot \bar{p}_{i'} \neq 0$, $x \cdot \bar{p}_{i'} \neq 0$ and $x \cdot \bar{p}_{i'} \in \operatorname{soc}(I)$. □

We can express the idea of the above proof as follows. Since we are starting with a direct sum of I_{a_i}, we have a direct sum of string modules of the form



Taking the maximum m_i ensures that $x_i \cdot \bar{p}_i$ is proportional to the basis vector corresponding to the deep of I_{a_i}. Further, by taking the maximum over p_i, we make sure that every other copy either becomes zero or proportional to the basis vector corresponding to the deep.

Lemma 6.1.2. *Let $w = \prod_{i \in S} c_i$ be a string, where $c_i : a_i \to a_{i+1}$ are edges in Q. Then w admits at most finitely many deeps and every $i \in \bar{S}$ is a predecessor of some deep in w if and only if $w = q_1 p_1^{-1} \cdots q_r p_r^{-1}$, where p_i, q_i are paths in Q such that p_i is non-trivial for $1 \leq i < r$ and q_i is non-trivial for $1 < i \leq r$.*

Proof. Suppose w admits finitely many deeps $d_1, \ldots, d_r$ in $\bar{S}$ such that every $i \in \bar{S}$ is a predecessor of some deep. We shall show that w is of the form as stated in the lemma.

We start with defining q_1. If $d_1 - 1 \notin S$, then $q_1 = \varepsilon_{a_{i_1}}$. Suppose that $d_1 - 1 \in S$. Given any $i \in S$ with $i \leq d_1 - 1$, i is a predecessor of some d_m with $1 \leq m \leq r$. Since $i < d_1 \leq d_m$, by definition, $c_i \cdots c_{d_m - 1}$ is a path, and in particular, c_i is an arrow. Letting S_1 be the set of $i \in S$ with $i < d_1$, we see that $\prod_{i \in S_1} c_i$ is a non-trivial path p_1.

Next, we define p_r. If $d_r \notin S$, then $p_r = \varepsilon_{a_{d_r}}$. Suppose that $d_r \in S$. Consider $i \in \bar{S}$ with $d_r < i$. Then i is a predecessor of some d_n with $1 \leq n \leq r$. Since $d_n \leq d_r < i$, by definition, $c_{d_n} \cdots c_{i-1}$ is an inverse of a path. In particular, c_{i-1} is an arrow. Let S_r be the set of $i \in \bar{S}$ with $d_r < i$, we see that $\prod_{i \in S_r} c_{i-1}$ is the inverse of a non-trivial path p_r. If $r = 1$, then $w = q_1 p_1^{-1}$.

Suppose that $r > 1$. Fix some $1 \leq l < r$. Since d_l, d_{l+1} are deeps, c_{d_l} is the inverse of an arrow and $c_{d_{l+1}-1}$ is an arrow. Therefore, we obtain a maximal j_l with $d_l < j_l < d_{l+1}$ such that $c_{d_l} \cdots c_{j_l - 1}$ is the inverse of a non-trivial path p_l. Then, c_{j_l} is an arrow. Consider i with $j_l < i < d_{l+1}$, which is a predecessor of d_t for some $1 \leq t \leq r$. If $t \leq l$, since $d_t < i$, we obtain the inverse of a path $c_{d_t} \cdots c_{i-1} = c_{d_t} \cdots c_{j_l} \cdots c_{i-1}$, contrary to c_{j_l} being an arrow. Thus, $l + 1 \leq t$, and since $i < d_t$, we see that $c_i \cdots c_{d_t - 1}$ is a path, and in particular, c_i is an arrow. As a consequence, $c_{j_l} \cdots c_{d_{l+1}-1}$ is a non-trivial path q_{l+1}. It is now easy to see that $w = q_1 p_1^{-1} \cdots q_r p_r^{-1}$.

Conversely, assume that w can be written as $w = q_1 p_1^{-1} \cdots q_r p_r^{-1}$, where p_i, q_i are paths in Q such that p_i is non-trivial for $1 \leq i < r$ and q_i is non-trivial for $1 < i \leq r$. For each $1 \leq i \leq r$, let $S_i, T_i \subseteq S$ be such that $\prod_{j \in S_i} c_j = p_i^{-1}$ and $\prod_{j \in T_i} c_j = q_i$, where $T_1 = \varnothing$ in case q_1 is trivial; and $S_r = \varnothing$ in case p_r is trivial. Let d_i be the minimal element of S_i, for $1 \leq i < r$. Moreover, let d_r be the minimal element of S_r in case $S_r \neq \varnothing$, and otherwise, let d_r be the unique element of $\bar{S} \setminus S$. By definition, S_i and T_i are convex subsets of S such that

$$S = (T_1 \cup S_1) \cup (T_2 \cup S_2) \cdots \cup (T_r \cup S_r)$$

and that $d_i - 1$ is the maximal element of T_i if it exists for all $1 \le i \le r$.

We claim that $\{d_1, d_2, \ldots, d_r\}$ is the set of deeps for w. Let us start with d_1. By definition, c_{d_1} is the inverse of the final arrow of p_1. If $d_1 - 1 \in S$, then $d_1 - 1 \in T_1$ with $c_{d_1 - 1}$ being the final arrow of q_1. So d_1 is a deep by definition. Consider now d_i with $1 < i < r$. Since d_i is the minimal element of S_i, the edge c_{d_i} is the inverse of the final arrow of p_i; and since $d_i - 1$ is the maximal element of T_i, the edge $c_{d_i - 1}$ is the final arrow of q_i. So d_i is a deep by definition. Consider finally d_r. Since $d_r - 1$ is the maximal element of T_r, the edge $c_{d_r - 1}$ is the final arrow of q_r. If $d_r \in S$, then d_r is the minimal element of S_r, and hence, c_{d_r} is the inverse of the final arrow of p_r. So d_r is a deep by definition.

Given $1 \le l \le r$, we claim that each $i \in S_l \cup T_l$ is a predecessor of d_l. Indeed, if $i < d_l$, since d_l is minimal in S_l, we see that $i \in S_l$ with $i \le d_l - 1$. In this case, $c_i \cdots c_{d_l - 1}$ is a final subpath of q_l. So i is a predecessor of d_l by definition. If $i > d_l$, since $d_l - 1$ is the maximal element of T_l, we see that $i \in T_l$. In this case, $c_{d_l} \cdots c_{i-1}$ is the inverse of a final subpath of p_l. So i is also a predecessor of d_l.

In particular, every $i \in S$ is a predecessor of some d_l with $1 \le l \le r$. Consider now $i \in \bar{S} \setminus S$. Then $i \ge d_r$. If $i = d_r$, then it is a predecessor of d_r. If $i > d_r$ then, by definition, $i - 1 \in S_r$. In this case, $c_{d_r} \cdots c_{i-1} = p_r^{-1}$. So i is also a predecessor of d_r. By Lemma 5.1.1, $\{d_1, \ldots, d_r\}$ is the set of all deeps for w. □

Theorem 6.1.1. *Let $w = \prod_{i \in S} c_i$ be a string, where the $c_i : a_i \to a_{i+1}$ are edges in Q. The string module $M(w)$ is finitely cogenerated if and only if w admits finitely many deeps and every $i \in \bar{S}$ is a predecessor of some deep in w.*

Proof. Let $\{v_i \mid i \in \bar{S}\}$ be a w-string basis of $M(w)$. We first suppose that $M(w)$ is finitely cogenerated. Then there exists an injective map $f : M(w) \to I_0$, where $I_0 \in \text{inj}\Lambda$. This means that $I_0 = \oplus_{i=1}^{r} I_{a_i}$, where $a_i \in Q_0$ for $1 \le i \le r$.

Let Ω be the set of deeps of w. We first show that Ω is finite. Using Lemma 5.1.2, we know that $\text{soc}(M(w)) = \oplus_{i\in\nabla} v_i\Lambda$, where ∇ is the set of deeps for w. Since f is an injection, $\text{soc}(M(w)) \subset \text{soc}(\oplus_{i=1}^r I_{a_i}) = \oplus_{i=1}^r \text{soc}(I_{a_i}) = \oplus_{i=1}^r k(e_{a_i})^*$. Since $\oplus_{i=1}^r k(e_{a_i})^*$ is finite-dimensional, $\text{soc}(M(w))$ is finite dimensional and ∇ is finite.

Let $i \in \bar{S}$. Then using Lemma 6.1.1, there exists some path p in Q such that

$$f(v_i \cdot \bar{p}) = f(v_i).\bar{p} = \sum_{i=1}^{r} \lambda_i (e_{a_i})^* \neq 0.$$

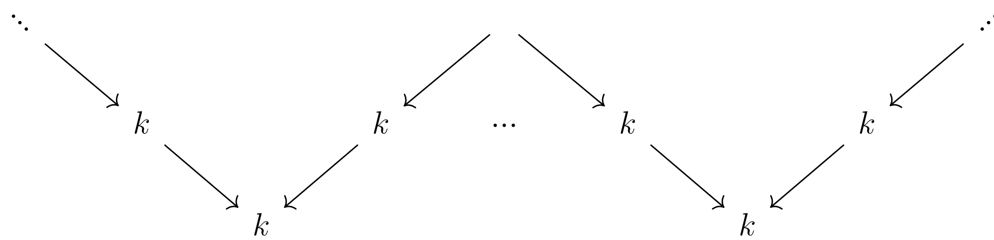
Since $v_i \cdot \bar{p} \neq 0$, there exists some $m \in \bar{S}$ such that $v_m = v_i \cdot \bar{p}$.

We claim that m is a deep. Suppose otherwise. Then either $m \in S$ and c_m is an arrow, or $m-1 \in S$ and c_{m-1} is the inverse of an arrow. In the first case, $v_m \cdot c_m = v_{m+1}$, which implies that $f(v_{m+1}) = f(v_m \cdot c_m) = f(v_m)\cdot c_m = (\sum_{i=1}^r \lambda_i (e_{a_i})^*)\cdot c_m = 0$. Since f is injective, this is a contradiction. In the second case, $v_m \cdot c_m = v_{m-1}$, which implies that $f(v_{m-1}) = f(v_m)\cdot c_m = (\sum_{i=1}^r \lambda_i (e_{a_i})^*)\cdot c_m = 0$. Since f is injective, this is again a contradiction. Hence, Lemma 5.2.2 gives that each $i \in \bar{S}$ is a predecessor of some deep.

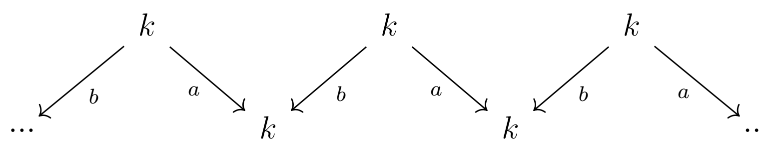
Now suppose w admits finitely many deeps $\{d_1, \cdots, d_r\}$ and every $i \in \bar{S}$ is a predecessor of some deep. Then $\text{soc}(M(w)) = \oplus_{i=1}^r v_{d_i} k$ using Lemma 5.1.2. Thus, we get a canonical inclusion $f : \text{soc}(M(w)) \to \oplus_{m=1}^r I_{a_{d_m}}$ sending v_{d_i} to the element having $(e_{a_{d_i}})^*$ as the ith coordinate and 0 as others. Let g be the canonical inclusion of $\text{soc}(M(w))$ in $M(w)$. Since $\oplus_{m=1}^r I_{a_{d_m}}$ is an injective module, we get a map $h : M(w) \to \oplus_{m=1}^r I_{a_{d_m}}$ such that $h \circ g = f$.

We claim that h is a monomorphism. For this, it is enough to show that $\text{soc}(M(w))$ is an essential submodule of $M(w)$ as that would imply that g is an essential monomorphism, which would imply that h is a monomorphism (since f is a monomorphism). Let N be a non-zero submodule of $M(w)$ such that $x = \sum_{i\in T} \lambda_i v_i$ is a non-zero element of N, where T is a finite subset of $\bar{S}$, and $\lambda_i \neq 0$ for all $i \in T$. Since every $l \in \bar{S}$ is a predecessor of some deep, there exist paths p_i in Q such that $v_i \cdot \bar{p}_i$ is a deep. Let p_n be a path with the maximum length among p_i. Then $v_i \cdot \bar{p}_n$ is either 0 or a deep. Since $v_n \cdot \bar{p}_n \neq 0$, $0 \neq x \cdot \bar{p_n} \in N \cap \operatorname{soc}(M(w))$. This shows that $\operatorname{soc}(M(w))$ is an essential submodule of $M(w)$. □

Again, the above theorem can be reformulated as saying that a string module $M(w)$ is finitely cogenerated if and only if the diagram representing w is of the following form. Note that the string has only finitely many deeps.



Example 6.1.1. *Let $w = \cdots ab^{-1}ab^{-1}\cdots$ in Λ_2. Then the diagram representing $M(w)$ is as follows.*



Since w has infinitely many deeps, $M(w)$ is not finitely cogenerated.

6.2 Injective envelopes and cosyzygies

Lemma 6.2.1. *Let $w = \prod_{i \in S} c_i$ be a string, where $c_i : a_i \to a_{i+1}$ are edges in Q, such that $M(w)$ is finitely cogenerated. If ∇ is the set of deeps for w, then $\oplus_{i \in \nabla} I_{a_i}$ is the injective envelope of $M(w)$.*

Proof. Let $i \in \nabla$. Then $I_{a_i} \cong M(p_i q_i^{-1})$, where p_i, q_i are longest paths ending at a_i such that $p_i q_i^{-1}$ is a string. We write $p_i = \cdots\alpha_{(-2,i)}\alpha_{(-1,i)}$ and $q_i = \cdots\beta_{(1,i)}\beta_{(0,i)}$. Now, we define a map $g_w : M(w) \to \oplus_{i\in\nabla} I_{a_i}$ as follows.

Let $i \in \bar{S}$. Since $M(w)$ is finitely cogenerated, i is a predecessor of some deep d_i. We note that i can be a predecessor of at most two distinct deeps. If i is a predecessor of a unique deep d, then either $c_i \cdots c_{d-1}$ is a subpath of p_d or $c_d \cdots c_{i-1}$ is the inverse of a subpath of q_d. In either case, we define $g_w(v_i)$ to be the dual of this subpath. However, if i is a predecessor of two deeps d_1 and d_2, then $c_{d_1} \cdots c_{i-1}$ is the inverse of a subpath of q_{d_1} and $c_i \cdots c_{d_2 - 1}$ is a subpath of p_{d_2} (Assuming, without loss of generality, that $d_1 \leq d_2$). In this case, set $g_w(v_i)$ to be the difference of the dual of these two paths. Clearly, g_w is an injective module homomorphism as the images of v_i are linearly independent. We will show that $\mathrm{Im}(g_w)$ is an essential submodule of $\oplus_{i\in\nabla} I_{a_i}$.

Suppose $0 \neq N \oplus_{i\in\nabla} I_{a_i}$. Let $0 \neq x \in N$. Then $x = \sum_m \lambda_m (r_m)*$, with $\lambda_m \neq 0$, where $r_m = \alpha_{(-2,n_m)} \cdots \alpha_{(-2,i)}\alpha_{(-1,i)}$ or $r_m = \beta_{(1,n_m)} \cdots \beta_{(1,i)}\beta_{(0,i)}$ for some $i \in \nabla$. Set $n := \max_m n_m$. Let $r_{m'}$ be a path with length n. Then $x \cdot r_{m'}$ is a non-zero sum of e_{a_i}. Hence we get that $x \cdot r_{m'} \in g(M(w))$. Therefore, g_w is an essential monomorphism. □

Definition 6.2.1. *Let M be a finitely cogenerated Λ-module with $i : M \to I$ its injective envelope. Then the* cosyzygy *of M, denoted by $\Omega^{-}(M)$, is defined to be the module* $\mathrm{Coker}(i)$.

In essence, the cosyzygy of a module 'measures' its deviation from being an injective module.

Theorem 6.2.1. *Let $M(w)$ be a finitely cogenerated string module defined by a string $w = q_1 p_1^{-1} \cdots q_r p_r^{-1}$, where p_i, q_j are paths in Q such that p_i, q_j are non-trivial for $1 \leq i < r$ and $1 < j \leq r$ respectively. In case q_1 is finite and does not start in a peak, define x_1 to be the maximal path such that $x_1 q_1 p_1^{-1}$ is a string. In case p_r is finite and does not start in a peak, define y_r to be the maximal path such that $q_r p_r^{-1} y_r^{-1}$ is a string. For each $1 < i \leq r$,* *let x_i and y_{i-1} be the maximal paths such that $y_{i-1}p_{i-1}$ and x_iq_i are strings. Then*

$$\Omega^-(M(w)) = K_1 \oplus K_2 \oplus \cdots \oplus K_r \oplus K_{r+1},$$

where

1. *$K_1 = 0$ if x_1 is not defined; and otherwise, $K_1 = M(x)$ with x the path such that $x_1 = x\alpha$ for an arrow α;*

2. *$K_i = M(x_iy_{i-1}^{-1})$ for $2 \leq i \leq r$;*

3. *$K_{r+1} = 0$ if y_r is not defined; and otherwise, $K_{r+1} = M(y)$ with y the path such that $y_r = y\beta$ for an arrow β.*

Proof. We denote by l_i and d_i the length of p_i and q_i respectively, for $i = 1, \ldots, r$. Write $w = \prod_{i\in S} c_i$, where S is an interval of $\mathbb{Z}$ and $c_i : a_i \to a_{i+1}$ are edges in Q. Let $m_1, \ldots, m_r$ be the deeps for w such that $a_{m_i} = e(p_i)$. Since $e(q_{i-1}) = e(p_{i-1})$ and $s(q_i) = s(p_{i-1})$, we see that $m_{i-1} + l_{i-1} = m_i - d_i$, which is a peak for w, for $2 \leq i \leq r$. Let $\{v_i \mid i \in \bar{S}\}$ be a w-string basis for $M(w)$. By Lemma 6.2.1, $M(w)$ admits an injective envelope

$$g_w : M(w) \to \oplus_{i\in\nabla} I_{a_i}.$$

We will construct a surjective homomorphism $\pi : \oplus_{i\in\nabla} I_{a_i} \to \oplus_{i=1}^{r+1} K_i$ and show that the kernel of this map is $\mathrm{Im}(g_w)$. For $1 \leq i \leq r$, let s_i, t_i be the longest paths ending at a_{m_i} such that $s_i t_i^{-1}$ is a string. Since s_i and t_i have an endpoint, they are either trivial or we can write them as $s_i = \cdots\alpha_{2,i}\alpha_{1,i}$ and $t_i = \cdots\beta_{2,i}\beta_{1,i}$, where $\alpha_{j,i}, \beta_{j',i}$ are arrows. We note that if s_i and t_i are both trivial, then w would have to be a trivial string. Now set $v_{0,i} := e^*_{a_{m_i}}$, $v_{-j,i} := (\alpha_{j,i}\cdots\alpha_{2,i}\alpha_{1,i})^*$ for $j \leq l(s_i)$, and $v_{j,i} := (\beta_{j,i}\cdots\beta_{2,i}\beta_{1,i})^*$ for $j \leq l(t_i)$. Using Lemma 4.4.2, we have that the set $\{v_{-j,i} \mid j \leq l(s_i)\} \cup \{v_{0,i}\} \cup \{v_{j,i} \mid j \leq l(t_i)\}$ is a k-basis of $I_{a_{m_i}}$. Since q_i, p_i are paths ending at a_i, we can assume, without loss of generality, that they are terminal subpaths of s_i and t_i respectively.

Now if q_1 is infinite or starts in a peak, $q_1 = s_1$ and we define $\pi(v_{-j,1}) := 0$ for all $1 \le j \le l(s_i)$. Otherwise, x_1 is a non-trivial path such that $x_1 q_1 = s_1$. So we get that $x = \cdots \alpha_{d_1+2,1}$. We define $\pi(v_{-j,1}) := 0$ for $1 \le j \le d_1$, $\pi(v_{-d_1-1,1}) := e^*_{e(x)}$, and $\pi(v_{-j,1}) := (\alpha_{j,1} \cdots \alpha_{d_1+2,1})^*$ for $d_1 + 2 \le j \le l(s_1)$. Since the set $\{(\alpha_{j,1} \cdots \alpha_{d_1+2,1})^* \mid d_1 + 2 \le j \le l(s_1)\} \cup \{e^*_{e(x)}\}$ is an x-string basis of $M(x)$, we get that $K_1 \subset \mathrm{Im}(\pi)$.

Similarly if p_r is infinite or starts in a peak, $q_r = t_r$ and we define $\pi(v_{j,r}) := 0$ for all $0 \le j \le l(t_i)$. Otherwise, y_r is a non-trivial path such that $y_r p_r = t_r$. So we get that $y = \cdots \beta_{l_r+2,r}$. We define $\pi(v_{j,r}) := 0$ for $0 \le j \le l_r$, $\pi(v_{l_r+1,r}) := e^*_{e(y)}$ and $\pi(v_{j,r}) := (\beta_{j,r} \cdots \beta_{l_r+2,r})^*$ for $l_r + 2 \le j \le l(t_r)$. Since the set $\{(\beta_{j,r} \cdots \alpha_{l_r+2,r})^* \mid l_r + 2 \le j \le l(t_r)\} \cup \{e^*_{e(y)}\}$ is a y-string basis of $M(y)$, we get that $K_{r+1} \subset \mathrm{Im}(\pi)$.

Now suppose $2 \le i \le r$. Then either x_i is trivial or $x_i = \cdots \alpha_{d_i+2,i} \alpha_{d_i+1,i}$. Similarly either y_{i-1} is trivial or $y_{i-1} = \cdots \beta_{l_{i-1}+1,i-1}$. Now we define $\pi(v_{-j,i}) := 0$ for all $1 \le j \le d_i - 1$, $\pi(v_{t,i-1}) := 0$ for all $0 \le t \le l_{i-1} - 1$, $\pi(v_{-d_i,i}) = \pi(v_{l_{i-1},i-1}) := e^*_{s(q_i)}$, $\pi(v_{-j,i}) := (\alpha_{j,i} \cdots \alpha_{d_i+1,i})^*$ for all $d_i + 1 \le j \le l(s_i)$, and $\pi(v_{j,i-1}) := (\beta_{j,i-1} \cdots \beta_{l_{i-1}+1,i-1})^*$ for all $l_{i-1} + 1 \le j \le l(t_{i-1})$. Note that this defines π on all of $\oplus_{i \in \nabla} I_{a_i}$. Since the module $M(x_i y_{i-1}^{-1})$ has as a k-basis the set $\{\ldots, \alpha^*_{d_i+1,i}, e^*_{s(q_i)}, \beta^*_{l_{i-1}+1,i-1}, \ldots\}$, $K_i \subset \mathrm{Im}(\pi)$ for all $2 \le i \le r$. Therefore, we get that π is a surjective homomorphism.

We now show that $\mathrm{Im}(g_w) \subset \mathrm{Ker}(\pi)$. Let $i \in \bar{S}$. Then by Lemma 6.1.1, i is a predecessor of some deep m_{z_i} for $1 \le z_i \le r$. If $i = m_{z_i}$, then $\pi(g_w(v_i)) = \pi(e^*_{a_{m_{z_i}}}) = \pi(v_{0,z_i}) = 0$. Suppose $i < m_{z_i}$. Then $c_i \cdots c_{m_{z_i}-1}$ is a terminal subpath of q_{z_i}, and hence a terminal subpath of s_{z_i}. Now if $i - 1 \notin S$, then c_i is the starting point of the string and $z_i = 1$. By the definition of g_w, we get that $\pi(g_w(v_i)) = \pi((\alpha_{d_1,1} \cdots \alpha_{1,1})^*) = 0$. If $i - 1 \in S$ and c_{i-1} is an arrow, then i is a predecessor of a unique deep. Therefore, by the definition of g_w, we get that $\pi(g_w(v_i)) = \pi((\alpha_{m_{z_i}-i,z_i} \cdots \alpha_{1,z_i})^*) = 0$, since $0 < m_{z_i} - i < d_{z_i}$. On the other hand, if $i - 1 \in S$ and c_{i-1} is the inverse of an arrow, then i is a predecessor of m_{z_i-1} as well. This gives that $c_{m_{z_i-1}} \cdots c_{i-1}$ is the inverse of p_{z_i-1} and $\alpha_{m_{z_i}-i,z_i} \cdots \alpha_{1,z_i}$ is the path q_{z_i}.

Therefore, by the definition of g_w, we get that

$$\begin{aligned}\pi(g_w(v_i)) &= \pi((\alpha_{d_{z_i},z_i}\cdots\alpha_{1,z_i})^* - (\beta_{l_{z_i-1},z_i-1}\cdots\beta_{1,z_i-1})^*)\\ &= \pi(v_{-d_{z_i},z_i} - v_{l_{z_i-1},z_i-1}) = e*_{s(q_{z_i})} - e*_{s(q_{z_i})} = 0.\end{aligned}$$

Now we suppose that $i > m_{z_i}$. Then $c_{m_{z_i}}\cdots c_{i-1}$ is the inverse of a terminal subpath of p_{z_i}, and hence of a terminal subpath of t_{z_i}. Now if $i \notin S$, then c_{i-1} is the endpoint of the string and $z_i = r$. By the definition of g_w, we get that $\pi(g_w(v_i)) = \pi((\beta_{l_r,r}\cdots\beta_{1,r})^*) = 0$. If $i \in S$, and c_i is the inverse of an arrow, then i is a predecessor of a unique deep. Therefore, by the definition of g_w, we get that $\pi(g_w(v_i)) = \pi((\beta_{i-m_{z_i},z_i}\cdots\beta_{1,z_i})^*) = 0$, since $0 < i - m_{z_i} < l_{z_i}$. Finally, if c_i is an arrow, then i is a predecessor of m_{z_i+1} as well and we are in the same case as before. Therefore, $\mathrm{Im}(g_w) \subset \mathrm{Ker}(\pi)$. Now suppose $z \in \mathrm{Ker}(\pi)$ such that $z = \sum_{i=1}^{r}\sum_{j=-l(s_i)}^{l(t_i)} \lambda_{j,i}v_{j,i}$. Then $\pi(z) = 0$ implies that

$$\begin{gathered}\lambda_{-d_1-1,1}e^*_{e(x)} + \textstyle\sum_{b=2}^{l(s_1)-d_1} \lambda_{-d_1-b,1}(\alpha_{d_1+b,1}\cdots\alpha_{d_1+2,1})^* +\\ \textstyle\sum_{i=2}^{r}\sum_{b=1}^{l(t_{i-1})-l_{i-1}} \lambda_{l_{i-1}+b,i-1}(\beta_{l_{i-1}+b,i-1}\cdots\beta_{l_{i-1}+1,i-1})^* +\\ \textstyle\sum_{i=2}^{r}\sum_{b=1}^{l(s_i)-d_i} \lambda_{-d_i-b,i}(\alpha_{d_i+b,i}\cdots\alpha_{d_i+1,i})^* +\\ \textstyle\sum_{i=2}^{r}(\lambda_{-d_i,i} + \lambda_{l_{i-1},i-1})e^*_{s(q_i)} +\\ \textstyle\sum_{b=2}^{l(t_r)-l_r} \lambda_{l_r+b,r}(\beta_{l_r+b,r}\cdots\beta_{l_r+2,r})^* + \lambda_{l_r+1,r}e^*_{e(y)} = 0.\end{gathered}$$

This gives that

$$\begin{aligned} \lambda_{-d_i-j,i} &= 0 && 1 \le j \le l(s_i) - d_i,\quad 1 \le i \le r\\ \lambda_{l_i+j,i} &= 0 && 1 \le j \le l(t_i) - l_i,\quad 1 \le i \le r\\ \lambda_{-d_i,i} &= -\lambda_{l_{i-1},i-1} && 2 \le i \le r\end{aligned}$$

This gives that $z = \sum_{i=1}^{r}\sum_{-d_i}^{l_i} \lambda_{j,i}v_{j,i}$ such that $\lambda_{-d_i,i} = -\lambda_{l_{i-1},i-1}$ for $2 \le i \le r$. Therefore we get that $z \in \mathrm{Im}(g_w)$ and $\mathrm{Ker}(\pi) = \mathrm{Im}(g_w)$. Thus

$$\Omega^-(M(w)) \cong \oplus_{i\in\nabla}I_{a_i}/\mathrm{Im}(g_w) = \oplus_{i\in\nabla}I_{a_i}/\mathrm{Ker}(\pi) \cong \oplus_{i=1}^{r+1}K_i.$$

□

Example 6.2.1. *Let* Λ *be the string algebra from Example 4.4.2. Let* $w = a_1 b_1^{-1}$*. Then* w *has one deep,* 2*. Therefore, the injective envelope for* $M(w)$ *is given by* I_2*, where*

$$I_2 = M(\cdots a_2 a_1 b_1^{-1} b_2^{-1} \cdots).$$

By Theorem 6.2.1, $K_1 = M(\cdots a_4 a_3)$ *and* $K_2 = M(\cdots b_4 b_3)$*, and hence*

$$\Omega^{-}(M(w)) = M(\cdots a_4 a_3) \oplus M(\cdots b_4 b_3).$$

We immediately get the following theorem as a corollary of the previous theorem.

Theorem 6.2.2. *Let* Λ *be a locally bounded string algebra. Then a string module over* Λ *is finitely cogenerated if and only if it is finitely copresented.*

Proof. Clearly, if a module is finitely copresented, then it is finitely cogenerated. Conversely, suppose $M(w)$ is a finitely cogenerated module for a string w in Λ. Theorem 6.1.1 along with Lemma 6.1.2 implies that w is of the form $w = q_1 p_1^{-1} \cdots q_r p_r^{-1}$, where p_i, q_i are paths in Q such that p_i is non-trivial for $1 \leq i < r$ and q_i is non-trivial for $1 < i \leq r$. Using Theorem 6.2.1, we get that $\Omega^{-}(M(w))$ is a finite direct sum of string modules that are themselves finitely cogenerated. Hence $\Omega^{-}(M(w))$ is finitely cogenerated, which implies that $M(w)$ is finitely copresented. □

CONCLUSION

In this dissertation, we worked with locally semi-perfect algebras and proved the existence of almost split sequences in the category of locally finite-dimensional unital modules over certain locally semi-perfect algebras.

In the second half, we worked with locally finite-dimensional string algebras. We characterized the strings for which the associated string modules are finitely presented or finitely copresented and calculated their syzygies and cosyzygies respectively.

In order to give an explicit description of the almost split sequences, the next step would be to give a combinatorial description of the irreducible maps and almost split sequences for locally finite-dimensional string algebras. This would involve a generalization of the functorial factorization method used by Butler and Ringel in [11].

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