Généralisations de l'équation du logarithme $\mathrm{Log}(x) - \mathrm{Log}(y) - \mathrm{Log}\left(x/y\right) = 0$ via les surfaces de del Pezzo

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Le logarithme

- $\mathrm{Li}_1(z) = -\mathbf{Log}(1-z)$ $\qquad (z \in \mathbb{C})$

- **Formule intégrale :** $\mathbf{Log}(z) \; = \; \int^z \frac{du}{u-0}$

  $\mathrm{Li}_1(z) \; = \; -\int^z \frac{du}{u-1}$

- **Développement en série :** $\mathrm{Li}_1(z) \; = \; \sum_{k=1}^{\infty} \frac{z^k}{k}$

- **Monodromie :** $\mathcal{M}_0\,(\,\mathbf{Log}\,) \; = \; \mathbf{Log} + 2i\pi$

- **Indentité fonctionnelle :** $\mathbf{Log}(\,x\,) - \mathbf{Log}(\,y\,) - \mathbf{Log}\left(\,\frac{x}{y}\,\right) = 0$

Indoles logarithmorum hac aequatione fundamentali continetur **[Pfaff 1788]**

[La nature des logarithmes est contenue dans cette éq° fondamentale]

Le dilogarithme Li_2

- $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ $\quad\quad \big(\, |z| < 1 \, \big)$

- **Formule Intégrale :** $\quad \mathbf{L}\mathrm{i}_2(z) = \mathbf{L}_{01}(z) = -\int^z \log(1-u)\,\frac{du}{u-0}$

$$\mathbf{L}_{10}(z) = \int^z \log(u-0)\,\frac{du}{1-u}$$

- **Monodromie :** $\quad \mathcal{M}_1\,(\mathbf{L}\mathrm{i}_2) = \mathbf{L}\mathrm{i}_2 - 2i\pi\,\mathbf{Log}$

- **Identité fonctionnelle d'Abel** $(\boldsymbol{\mathcal{A}b})$ $\quad\quad \big(\; 0 < x < y < 1 \; \big)$

$$\mathbf{L}\mathrm{i}_2(x) - \mathbf{L}\mathrm{i}_2(y) - \mathbf{L}\mathrm{i}_2\Big(\frac{x}{y}\Big) - \mathbf{L}\mathrm{i}_2\Big(\frac{1-y}{1-x}\Big) + \mathbf{L}\mathrm{i}_2\left(\frac{x(1-y)}{y(1-x)}\right) =$$

$$\mathbf{Log}(y)\,\mathbf{Log}\left(\frac{1-y}{1-x}\right) - \frac{\pi^2}{6}$$

Le dilogarithme Li_2

- $\mathbf{Li}_2(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ $\qquad\qquad \big(\; |z| < 1 \;\big)$

- **Formule Intégrale :** $\quad \mathbf{Li}_2(z) \;=\; \mathbf{L}_{01}(z) \;=\; -\int^z \log(1-u)\,\frac{du}{u-0}$

$$\mathbf{L}_{10}(z) \;=\; \int^z \log(u-0)\,\frac{du}{1-u}$$

- **Monodromie :** $\quad \mathcal{M}_1\left(\mathbf{Li}_2\right) \;=\; \mathbf{Li}_2 - 2i\pi\,\mathbf{Log}$

- **Identité fonctionnelle d'Abel** $(\boldsymbol{\mathcal{A}b})$ $\qquad \Big(\; 0 < x < y < 1 \;\Big)$

$$\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\Big(\frac{x}{y}\Big) - \mathbf{R}\Big(\frac{1-y}{1-x}\Big) + \mathbf{R}\Big(\frac{x(1-y)}{y(1-x)}\Big) = 0$$

$$\mathbf{R}(x) = \frac{1}{2}\Big(\mathbf{L}_{01}(x) - \mathbf{L}_{10}(x)\Big) = \mathbf{Li}_2(x) + \frac{1}{2}\mathbf{Log}(x)\mathbf{Log}(1-x) - \frac{\pi^2}{6}$$

Le n-ième polylogarithme $\mathbf{Li}_n$

- $\mathbf{Li}_n(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad\qquad \big(\ |z| < 1\ \big)$

- **Formule intégrale :**
$$\mathbf{Li}_n(z) \;=\; \int^{z} \mathbf{Li}_{n-1}(u) \frac{du}{u}$$
$$\mathbf{Li}_n{}'(z) \;=\; \mathbf{Li}_{n-1}(z)/z$$

- **Monodromie :** $\mathcal{M}_1\,(\,\mathbf{Li}_n\,) \;=\; \mathbf{Li}_n - 2i\pi\,\frac{(\mathbf{Log})^{n-1}}{(n-1)!}$

- **Identités fonctionnelles en une variable :**

$$\mathbf{Li}_n\Big(z^r\Big) \;=\; r^{n-1} \sum_{\omega^r=1} \mathbf{Li}_n(\omega\, z) \qquad \Big(\ |z| < 1\ \Big)$$

$$\mathbf{Li}_n(z) + (-1)^n \mathbf{Li}_n\Big(z^{-1}\Big) \;=\; -\frac{(2i\pi)^n}{n!}\,\mathbf{B}_n\Big(\frac{\mathrm{Log}\, z}{2i\pi}\Big) \quad \Big(\ z \in \mathbb{C} \setminus [0, +\infty[\ \Big)$$

Le n-ième polylogarithme $\mathbf{Li}_n$

- $\mathbf{Li}_n(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad\qquad \big(\; |z| < 1 \;\big)$

- **Formule intégrale :** $\qquad \mathbf{Li}_n(z) \;=\; \int^z \mathbf{Li}_{n-1}(u) \frac{du}{u}$

$$\mathbf{Li}_n{}'(z) \;=\; \mathbf{Li}_{n-1}(z)/z$$

- **Monodromie :** $\qquad \mathcal{M}_1\,(\,\mathbf{Li}_n\,) \;=\; \mathbf{Li}_n - 2i\pi\,\frac{(\mathbf{Log})^{n-1}}{(n-1)!}$

- **Identités fonctionnelles en plusieurs variables ($\exists$?) :**

$$\sum_{i \in I} c_i\, \mathbf{Li}_n\big(U_i\big) = \mathbf{Elem}_{<n}$$

$$\Big(\; I \;\text{ fini}\,,\; c_i \in \mathbb{Z}\,,\; U_i \in \mathbb{Q}(x_1, \ldots, x_N) \;\Big)$$

Le n-ième polylogarithme $\mathbf{Li}_n$

- $\mathbf{Li}_n(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad\qquad \big(\; |z| < 1 \;\big)$

- **Formule intégrale :** $$\mathbf{Li}_n(z) \;=\; \int^z \mathbf{Li}_{n-1}(u)\frac{du}{u}$$ $$\mathbf{Li}_n{}'(z) \;=\; \mathbf{Li}_{n-1}(z)/z$$

- **Monodromie :** $$\mathcal{M}_1\left(\mathbf{Li}_n\right) \;=\; \mathbf{Li}_n - 2i\pi\,\frac{(\mathbf{Log})^{n-1}}{(n-1)!}$$

- **Identités fonctionnelles en plusieurs variables ($\exists\,?$) :**

$$\sum_{i\in I} c_i\, \mathbf{Li}_n(U_i) = \mathbf{Elem}_{<n} \qquad \Longleftrightarrow \qquad \sum_{i\in I} c_i\, \mathcal{L}_n(U_i) = \mathbf{0}$$

$\Big(\; I \text{ fini}\,,\; c_i \in \mathbb{Z}\,,\; U_i \in \mathbb{Q}(x_1,\ldots,x_N) \;\Big)$

Exemple : $\mathbf{L}\mathrm{i}_3$

- $\mathbf{L}\mathrm{i}_3(z) \;=\; \sum_{k=1}^{\infty} z^k/k^3 \;=\; \int^z \mathbf{L}\mathrm{i}_2(u)\frac{du}{u}$

- **Identité de Spence-Kummer $\mathcal{SK}$ (1809-1840) :**

$$2\,\mathbf{L}\mathrm{i}_3(x) + 2\,\mathbf{L}\mathrm{i}_3(y) - \mathbf{L}\mathrm{i}_3\Big(\frac{x}{y}\Big) + 2\,\mathbf{L}\mathrm{i}_3\Big(\frac{1-x}{1-y}\Big) + 2\,\mathbf{L}\mathrm{i}_3\Big(\frac{x(1-y)}{y(1-x)}\Big) - \mathbf{L}\mathrm{i}_3\big(xy\big)$$
$$+\,2\,\mathbf{L}\mathrm{i}_3\Big(-\frac{x(1-y)}{(1-x)}\Big) + 2\,\mathbf{L}\mathrm{i}_3\Big(-\frac{(1-y)}{y(1-x)}\Big) - \mathbf{L}\mathrm{i}_3\Big(\frac{x(1-y)^2}{y(1-x)^2}\Big)$$
$$= 2\,\mathbf{L}\mathrm{i}_3(1) - \mathrm{Log}\,(y)^2\mathrm{Log}\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3}\mathrm{Log}\,(y) + \frac{1}{3}\mathrm{Log}\,(y)^3$$

Example : $\mathbf{Li}_3$

- $\mathbf{Li}_3(z) \;=\; \sum_{k=1}^{\infty} z^k/k^3 \;=\; \int^z \mathbf{Li}_2(u)\frac{du}{u}$

- **Identité de Spence-Kummer $\mathcal{SK}$ (1809-1840) :**

$$\begin{aligned}
2\,\mathbf{Li}_3(x) &+ 2\,\mathbf{Li}_3(y) - \mathbf{Li}_3\Big(\frac{x}{y}\Big) + 2\,\mathbf{Li}_3\Big(\frac{1-x}{1-y}\Big) + 2\,\mathbf{Li}_3\Big(\frac{x(1-y)}{y(1-x)}\Big) - \mathbf{Li}_3\big(xy\big)\\
&+ 2\,\mathbf{Li}_3\Big(-\frac{x(1-y)}{(1-x)}\Big) + 2\,\mathbf{Li}_3\Big(-\frac{(1-y)}{y(1-x)}\Big) - \mathbf{Li}_3\Big(\frac{x(1-y)^2}{y(1-x)^2}\Big)\\
&= 2\,\mathbf{Li}_3(1) - \mathrm{Log}\,(y)^2\mathrm{Log}\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3}\mathrm{Log}\,(y) + \frac{1}{3}\mathrm{Log}\,(y)^3
\end{aligned}$$

$$\begin{aligned}
2\,\mathcal{L}_3(x) &+ 2\,\mathcal{L}_3(y) - \mathcal{L}_3\Big(\frac{x}{y}\Big) + 2\,\mathcal{L}_3\Big(\frac{1-x}{1-y}\Big) + 2\,\mathcal{L}_3\Big(\frac{x(1-y)}{y(1-x)}\Big) - \mathcal{L}_3\big(xy\big)\\
&+ 2\,\mathcal{L}_3\Big(-\frac{x(1-y)}{(1-x)}\Big) + 2\,\mathcal{L}_3\Big(-\frac{(1-y)}{y(1-x)}\Big) - \mathcal{L}_3\Big(\frac{x(1-y)^2}{y(1-x)^2}\Big) = \mathbf{0}
\end{aligned}$$

$$\mathcal{L}_3(z) = \mathbf{Li}_3(z) - \mathbf{Li}_2(z)\,\mathrm{Log}|z| + \tfrac{1}{3}\,\mathbf{Li}_1(z)\big(\mathrm{Log}|z|\big)^2$$

Exemple : Li_4

- $\mathbf{Li}_4(x) \; = \; \sum_{k=1}^{\infty} x^k/k^4 \qquad \boldsymbol{\mathcal{L}}_4(x) \; = \; \mathbf{Li}_4(x) + \mathbf{Elem}_{<4}(x)$
- **Identité fonctionnelle de Kummer $\mathcal{K}(4)$ (1840) :**

$$
\begin{aligned}
&\boldsymbol{\mathcal{L}}_4\Big(-\frac{x^2y\eta}{\zeta}\Big) + \boldsymbol{\mathcal{L}}_4\Big(-\frac{y^2x\zeta}{\eta}\Big) + \boldsymbol{\mathcal{L}}_4\Big(\frac{x^2y}{\eta^2\zeta}\Big) + \boldsymbol{\mathcal{L}}_4\Big(\frac{y^2x}{\zeta^2\eta}\Big)\\
&-6\,\boldsymbol{\mathcal{L}}_4\Big(xy\Big) - 6\,\boldsymbol{\mathcal{L}}_4\Big(\frac{xy}{\eta\zeta}\Big) - 6\,\boldsymbol{\mathcal{L}}_4\Big(-\frac{xy}{\eta}\Big) - 6\,\boldsymbol{\mathcal{L}}_4\Big(-\frac{xy}{\zeta}\Big)\\
&-3\,\boldsymbol{\mathcal{L}}_4\Big(x\eta\Big) - 3\,\boldsymbol{\mathcal{L}}_4\Big(y\zeta\Big) - 3\,\boldsymbol{\mathcal{L}}_4\Big(\frac{x}{\eta}\Big) - 3\,\boldsymbol{\mathcal{L}}_4\Big(\frac{y}{\zeta}\Big)\\
&-3\,\boldsymbol{\mathcal{L}}_4\Big(-\frac{x\eta}{\zeta}\Big) - 3\,\boldsymbol{\mathcal{L}}_4\Big(-\frac{y\zeta}{\eta}\Big) - 3\,\boldsymbol{\mathcal{L}}_4\Big(-\frac{x}{\eta\zeta}\Big) - 3\,\boldsymbol{\mathcal{L}}_4\Big(-\frac{y}{\eta\zeta}\Big)\\
&+6\,\boldsymbol{\mathcal{L}}_4(x) + 6\,\boldsymbol{\mathcal{L}}_4(y) + 6\,\boldsymbol{\mathcal{L}}_4\Big(-\frac{x}{\zeta}\Big) + 6\,\boldsymbol{\mathcal{L}}_4\Big(-\frac{y}{\eta}\Big) = \mathbf{0}
\end{aligned}
$$

$$(\; \zeta = 1 - x, \; \eta = 1 - y \;)$$

- **Abel 1881 (Spence 1809, Hill 1829, Rogers 1907)**

$$\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0 \qquad (\boldsymbol{\mathcal{A}b})$$

- **Spence-Kummer :** $\sum_{i=1}^{9} c_i \,\mathcal{L}_3\Big(U_i(x,y)\Big) = 0$ $\quad(\boldsymbol{\mathcal{SK}})$

- **Kummer 1840 :** $\sum_i c_i \,\mathcal{L}_n\Big(U_i(x,y)\Big) = 0 \quad (\ n \leq 5\ ) \quad (\boldsymbol{\mathcal{K}_n})$

- ...

- **Goncharov 1995 :** $\sum_{i=1}^{22} c_i \,\mathcal{L}_3\Big(U_i(a,b,c)\Big) = 0$ $\quad(\boldsymbol{\mathcal{G}on})$

- **Gangl 2003 :** $\sum_i c_i \,\mathcal{L}_n\Big(U_i(x,y)\Big) = 0 \quad (\ n = 6,7\ ) \quad (\boldsymbol{\mathcal{G}an_n})$

- **Charlton, Gangl, Radchenko, Rudenko, Goncharov-Rudenko, ...**

- **Identités fonctionnelles (IF) des polylogarithmes Li_n :**

  - Géométrie hyperbolique
  - Géométrie des tissus $(\, n \leq 3 \,)$
  - K-théorie des corps de nombres $(\, n \leq 4 \,)$
  - Théorie des périodes (MZVs)
  - Physique des particules ('*Scattering amplitudes*')
  - Physique mathématique ('*Y-systèmes*') $(\, n = 2 \,)$
  - Algèbres amassées $(\, n \leq 4 \,)$
  - Symétrie miroir ('*Scattering diagrams*') $(\, n = 2 \,)$

- **Problèmes :** – trouver des **IF** pour $\mathcal{L}_n$ $\big(\textit{e.g.} \ \exists \ n \geq 8 \ ? \ \big)$

  – mieux comprendre les **FIs** polylogarithmiques

- **Identités fonctionnelles (IF) des polylogarithmes $\mathbf{Li}_n$:**
  - Géométrie hyperbolique
  - Géométrie des tissus ($n \leq 3$)
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- **Problèmes :** – trouver des **IF** pour $\mathcal{L}_n$ (*e.g.* $\exists\ n \geq 8$?)

  – mieux comprendre les **FIs** polylogarithmiques

K-théorie et identités polylogarithmiques

- $\boldsymbol{F}$ = corps de nombres $\rightsquigarrow K_n(\boldsymbol{F}) = \begin{cases} n \text{ pair} & \checkmark \quad \text{(Borel)} \\ n \text{ impair} & ? \end{cases}$

- **[Zagier]** $K_3(\boldsymbol{F}) \leftrightsquigarrow B_2(\boldsymbol{F})$ avec

$$B_2(\boldsymbol{F}) = \frac{\mathbb{Z}\Big[\boldsymbol{F}\backslash\{0,1\}\Big]}{\Big\langle [x]-[y]-\Big[\frac{x}{y}\Big]-\Big[\frac{1-y}{1-x}\Big]+\Big[\frac{x(1-y)}{y(1-x)}\Big], \; x,y\notin\{0,1\}\Big\rangle}$$

$$\left(\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0 \right)$$

- **Régulateurs (de Borel)** $\mathcal{R}_2^B = \boldsymbol{\mathcal{L}}_2 : K_3(\boldsymbol{F}) \longrightarrow \mathbb{R}$

  $\mathcal{R}_n^B = \boldsymbol{\mathcal{L}}_n : K_{2n-1}(\boldsymbol{F}) \longrightarrow \mathbb{R} \; (\, n \geq 2 \,)$

- Comprendre les IF de $\boldsymbol{\mathcal{L}}_n$ $\rightarrow$
  - Desc° de $K_{2n-1}(\boldsymbol{F})$ par générateurs et relat°
  - Applications à la "Conjecture de Zagier"

'Scattering amplitudes' **et identités fonctionnelles**

- *'Scattering amplitudes'* $\boldsymbol{I} = \int_\Delta \boldsymbol{\Psi}$ (important en HEPP)

$$\boldsymbol{I} = \boldsymbol{I}' + \mathcal{R}$$

$$\boldsymbol{I} = \boldsymbol{I}' + \sum_{i\in I} \mathbf{F}_i(\, x_i\,)$$

- **[dDDS]** *'The 2-loop hexagon Wilson loop in* $\mathcal{N} = 4$ SYM*'* (2010)

$\mathcal{R}^{(2)}_{6,WL} =$ 'remainder' : formule de 17 pages !

[GSVV] $\mathcal{R}^{(2)}_{6,WL} = \sum_{i=1}^{3}\left(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathbf{L}\mathrm{i}_4(v_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathbf{L}\mathrm{i}_2(v_i)\right)^2 + \cdots$

- Importance de simplifier $\sum_{i\in I} \mathbf{F}_i(\, x_i\,)$ pour $\mathbf{F}_i =$ polylogarithmes
  $\mathbf{F}_i =$ hyperlogarithmes
  $\mathbf{F}_i =$ polylogs elliptiques

- Justifie l'étude des identités fonctionnelles $\sum_{j\in J} \mathbf{F}_j(\, x_j\,) = \mathrm{cst}$

- $\boldsymbol{Ab}(x, y) = \mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) \equiv 0$

Thm [de Jeu 20] $\forall$ I fini, $c_i \in \mathbb{Q}$ et $U_i \in \mathbb{Q}[x_1, \ldots, x_m]$:

$\sum_{i \in I} c_i \, \mathbf{R}(U_i) \equiv \mathbf{cst}$ $\iff$ $\sum_{i \in I} c_i \, \mathbf{R}(U_i)$ est une CL de spécialisations de $\boldsymbol{Ab}(X_s, Y_s)$ avec $X_s, Y_s \in \mathbb{Q}[x_1, \ldots, x_m]$ $\forall$ s

- $\big(\mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}(x/y) = 0\big)$ est l'IFF du log ✓
- $\boldsymbol{\mathcal{A}b}$ $\iff$ $\big(\boldsymbol{Ab}(x, y) \equiv 0\big)$ est l'IFF du dilog ✓
- $\boldsymbol{\mathcal{G}on}_{22}$ $\iff$ $\sum_{i=1}^{22} c_i \mathcal{L}_3(U_i) = 0$ est l'IFF du trilog **?**
- $\mathbf{Q}_4$ **[Goncharov-Rudenko]** est l'IFF du tetralog **?**

[Griffiths 2002] *The legacy of Abel in algebraic geometry*

We do not attempt to formulate this question precisely – intuitively, we are asking ***whether or not for each n there is an integer $d(n)$ such that there is a "new" $d(n)$-web of maximum rank one of whose abelian relations is a (the ?) functional equation with $d(n)$ terms for $\mathbf{Li}_n$?*** *Here, 'new" means the general extension of the phenomena above for the logarithm when $n = 1$, where $d(1) = 3$, for the 5-term identity when $n = 2$ and $d(2) = 5, \ldots$*

[Goncharov-Rudenko 2018] *'Motivic correlator, cluster algebras ...'*

Conclusion. *If* ***$n > 3$, the problem of writing explicitly functional equations for $\mathbf{Li}_n$ might not be the "right" problem****. It seems that when n is growing the functional equations become so complicated that one can not write them down on a piece of paper.*

- **Principaux problèmes sur les IF des polylogarithmes :**

  – Trouver des **IF** pour $\mathcal{L}_n$ $\left(\textit{e.g.} \ \exists\ n \geq 8\ ? \ \right)$

  – Existe-t-il une suite $(\boldsymbol{IF}_n)_{n\geq 1}$ de **IF** pour les polylogarithmes ?

  – Existe-t-il une **IF** fondamentale pour $\mathcal{L}_n$ pour chaque $n \geq 1$?

  – Mieux comprendre les **IF** polylogarithmiques

- Dans cet exposé, en considérant des **hyperlogarithmes** :
  - On décrit une série d'identités hyperlogarithmiques

$$\mathbf{HLog}^1 \iff \Big(\ \mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}(x/y) = 0 \ \Big)$$

$$\mathbf{HLog}^2 \iff \left(\ \mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\Big(\frac{x}{y}\Big) - \mathbf{R}\Big(\frac{1-y}{1-x}\Big) + \mathbf{R}\Big(\frac{x(1-y)}{y(1-x)}\Big) = 0 \ \right)$$

$$\vdots$$

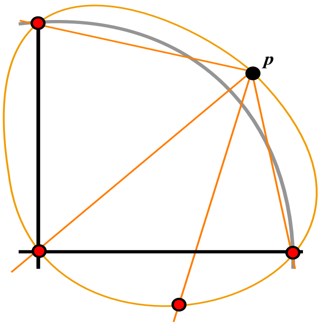
$$\mathbf{HLog}^6 \qquad \Big(\text{ IF hyperlogarithmique de poids 6 } \Big)$$

  - Pour $w = 1, \ldots, 6$, on a

$$\mathbf{HLog}^w \quad : \quad \sum_{i=1}^{\kappa} \boldsymbol{AH}_i^w(\phi_i) = 0$$

Une vue géométrique de l'identité d'Abel

- $(\mathcal{A}\mathbf{b})$ $\quad \underset{\displaystyle U_1}{\underset{\shortparallel}{\mathbf{R}(x)}} - \underset{\displaystyle U_2}{\underset{\shortparallel}{\mathbf{R}(y)}} - \underset{\displaystyle U_3}{\underset{\shortparallel}{\mathbf{R}\left(\frac{x}{y}\right)}} - \underset{\displaystyle U_4}{\underset{\shortparallel}{\mathbf{R}\left(\frac{1-y}{1-x}\right)}} + \underset{\displaystyle U_5}{\underset{\shortparallel}{\mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right)}} = \mathbf{0}$



Points base des U_i's :

- $p_1 = [\,1\,,\,0\,,\,0\,]$
- $p_2 = [\,0\,,\,1\,,\,0\,]$
- $p_3 = [\,0\,,\,0\,,\,1\,]$
- $p_4 = [\,1\,,\,1\,,\,1\,]$

$\rightsquigarrow$ **Éclatement** $\quad \beta\,:\,X_4 = \mathbf{Bl}_{p_1,\ldots,p_4}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$

Une vue géométrique de l'identité d'Abel

- $(\mathcal{A}\mathrm{b})$ $$\mathrm{R}\big(\underset{\substack{\| \\ U_1}}{x}\big) - \mathrm{R}\big(\underset{\substack{\| \\ U_2}}{y}\big) - \mathrm{R}\Big(\underset{\substack{\| \\ U_3}}{\tfrac{x}{y}}\Big) - \mathrm{R}\Big(\underset{\substack{\| \\ U_4}}{\tfrac{1-y}{1-x}}\Big) + \mathrm{R}\Big(\underset{\substack{\| \\ U_5}}{\tfrac{x(1-y)}{y(1-x)}}\Big) = 0$$

- **Éclatement** $\beta : X_4 = \mathrm{Bl}_{p_1,\ldots,p_4}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$

$$\begin{tikzcd} \mathbf{X}_4 \arrow[d, "\beta"'] \arrow[dr, bend left, "\phi_i"] & \\ \mathbb{P}^2 \arrow[r, dashed, "U_i"'] & \mathbb{P}^1 \end{tikzcd}$$

$\phi_1, \ldots, \phi_5 : X_4 \longrightarrow \mathbb{P}^1$ **sont les les cinq fibrations en coniques sur la surface de del Pezzo X_4**

- $(\mathcal{A}\mathrm{b}) \Longleftrightarrow$ $\exists\, (\epsilon_i)_{i=1}^{5} \in \{\pm 1\}^5$ ($\exists\,!$ au signe près) tq. $\sum_{i=1}^{5} \epsilon_i\, \mathrm{R}(\phi_i) = 0$

Généralisation aux surfaces de del Pezzo

- $p_1, \ldots, p_r \in \mathbb{P}^2$: points en position générale $\big(r \in \{3, \ldots, 8\} \big)$
- **Blow-up** $\beta_r : X_r = \mathrm{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$ $\big(X_r = \mathrm{dP}_{9-r} \big)$

Prop : 1. *Il y a un nombre fini κ_r de fibrations en coniques* $\phi_1, \ldots, \phi_\kappa : X_r \longrightarrow \mathbb{P}^1$

2. *Pour tout i :* $\Sigma_i = \mathbf{Spectre}(\phi_i) \subset \mathbb{P}^1$ *a $r-1$ éléments*

Def° : *L'* **hyperlogarithme complet antisymétrique de poids $r-2$** : $AH^{r-2}_{\Sigma_i} : \widetilde{\mathbb{P}^1 \setminus \Sigma_i} \longrightarrow \mathbb{C}$

Thm [Castravet-P] $\exists\ (\epsilon_i)_{i=1}^{\kappa} \in \{\pm 1\}^{\kappa}$, $\pm$-unique, tel que

$$\big(\mathbf{HLog}^{r-2}\big) \qquad \sum_{i=1}^{\kappa} \epsilon_i\, AH^{r-2}_{\Sigma_i}(\phi_i) = 0$$

$$\boxed{\left(\mathbf{HLog}^{r-2}\right) \qquad \sum_{i=1}^{\kappa} \epsilon_i \, \boldsymbol{AH}^{r-2}_{\boldsymbol{\Sigma_i}}(\phi_{\boldsymbol{i}}) = 0}$$

- Une identité $\mathbf{HLog}^{r-2}$ pour chaque del Pezzo $\mathbf{dP}_d = \boldsymbol{X}_r$ $(d = 9 - r)$

[$d = 6$] $\mathbf{dP}_6$ est unique, $\boldsymbol{AH}^1_{\boldsymbol{\Sigma_i}} = \mathbf{Log}$ pour tout i

$$\mathbf{HLog}^1 = \Big(\mathbf{Log}(\,x\,) - \mathbf{Log}(\,y\,) - \mathbf{Log}(\,x/y\,) = \mathbf{0} \Big)$$

[$d = 5$] $\mathbf{dP}_5$ est unique : $\boldsymbol{AH}^2_{\boldsymbol{\Sigma_i}} = \frac{1}{2}\Big(\mathbf{L}_{01} - \mathbf{L}_{10}\Big) = \mathbf{R}$ pour tout i

$$\mathbf{HLog}^2 = \Big(\sum_{i=1}^{5} \epsilon_i \, \mathbf{R}(\,\phi_i\,) = \mathbf{0} \Big) \qquad (\mathcal{A}\mathbf{b})$$

$$\boxed{\left(\mathbf{HLog}^{r-2}\right) \qquad \sum_{i=1}^{\kappa} \epsilon_i \, AH^{r-2}_{\Sigma_i}(\phi_i) = 0}$$

[$d = 4$] $\mathbf{dP}_4$ ∞^2 moduli $\rightsquigarrow$ ∞^2 identités $\mathbf{HLog}^3$

$$AH^3_1(\,x\,) + AH^3_2\left(\frac{1}{y}\right) + AH^3_3\left(\frac{y}{x}\right) + \cdots$$

$$\cdots + AH^3_9\left(\frac{y(x-b)}{x(y-a)}\right) + AH^3_{10}\left(\frac{a\,(b-x)}{by-ax}\right) = 0$$

[$d = 3$] $\mathbf{dP}_3$ = surface cubique dans $\mathbb{P}^3$ $\rightsquigarrow$ ∞^4 identités $\mathbf{HLog}^4$

$$\sum_{i=1}^{27} AH^4_i(\,\phi_i) = 0$$

Thm [Castravet-P. 2022]

$\exists\ (\epsilon_i)_{i=1}^{\kappa} \in \{\pm 1\}^{\kappa}$ unique au signe près tel que

$$\left(\mathbf{HLog}^{r-2}\right) \qquad\qquad \sum_{i=1}^{\kappa} \epsilon_i\, \boldsymbol{AH}_{\boldsymbol{\Sigma_i}}^{r-2}(\boldsymbol{\phi_i}) = 0$$

$\longrightarrow$ Surfaces de Del Pezzo

$\longrightarrow$ Hyperlogarithmes (aka "Intégrales itérées sur $\mathbb{P}^1$")

Surfaces de Del Pezzo I : proprietés

- $\mathbf{dP}_d \subset \mathbb{P}^d$ surface lisse, de degré d $\qquad (\, d = 9 - r \,)$
- $\mathbf{dP}_d = \mathbf{X}_r = \mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2)$ $\qquad \mathbf{Pic}(\mathbf{dP}_d) = \mathbb{Z}\,\boldsymbol{h} \oplus \big(\bigoplus_{i=1}^r \mathbb{Z}\,\boldsymbol{\ell}_i \big)$
- $-\boldsymbol{K}_{\mathbf{dP}_d} = 3\boldsymbol{h} - \sum_{i=1}^r \boldsymbol{\ell}_i$ ample $\rightsquigarrow$ $\varphi_{|-\boldsymbol{K}|} : \mathbf{dP}_d \hookrightarrow \mathbb{P}^d$ plongement
- $\mathbf{Pic}(\mathbf{dP}_d) \supset \boldsymbol{K}^\perp = \langle \, \rho_1, \ldots, \rho_r \, \rangle$ $\qquad \rho_i = \boldsymbol{\ell}_i - \boldsymbol{\ell}_{i+1} \quad i \leq r - 1$

  $$\rho_r = 3\boldsymbol{h} - \textstyle\sum_{i=1}^3 \boldsymbol{\ell}_i$$
- $-(\cdot,\cdot) + \{\rho_i\}_{i=1}^r$ $\rightsquigarrow$ Système de racines $E_r \subset R_r = \boldsymbol{K}^\perp \otimes \mathbb{R}$
- Pour toute racine ρ : $\qquad$ (reflexion orthog.)

  $$\begin{aligned} s_\rho : R_r &\longrightarrow R_r \\ d &\longmapsto d + (d,\rho)\rho \end{aligned}$$
- $\boldsymbol{W}_r = \boldsymbol{W}(E_r) = \langle \, s_{\rho_1}, \ldots, s_{\rho_r} \, \rangle \subset \mathrm{O}(R_r)$: gpe de Weyl de type E_r

Surfaces de Del Pezzo I

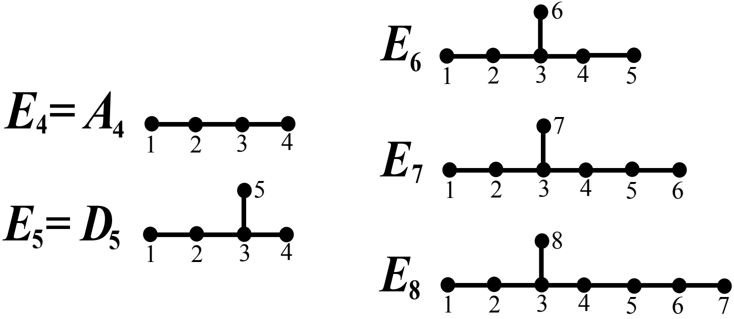


FIGURE – Dynkin diagram E_r $\big(k$ stands for ρ_k for any $k = 1, \ldots, r\big)$

Droites et coniques sur $X_r = \mathbf{dP}_d$ $\quad \big(\, d = 9 - r \, \big)$

- Droites $\mathcal{L}_r = \Big\{ \, \boldsymbol{\ell} \in \mathbf{Pic}(\boldsymbol{X}_r) \; \Big| \; (\, \boldsymbol{\ell} \, , -\boldsymbol{K} \,) = 1 \, , \; \boldsymbol{\ell}^2 = -1 \, \Big\}$

$$\begin{array}{l} \mathcal{L}_r \\ \Cup \\ \boldsymbol{\delta} \quad \rightsquigarrow \quad |\, \boldsymbol{\delta} \,| = \{\, \delta \,\} \quad \text{avec} \quad \mathbb{P}^1 \simeq \delta \subset \mathbf{dP}_d \qquad \deg(\delta) = 1 \end{array}$$

$$\mathcal{L}_r = \boldsymbol{W}_r \cdot \ell_r$$

- Cloniques $\mathcal{K}_r = \Big\{ \, \mathfrak{c} \in \mathbf{Pic}(\boldsymbol{X}_r) \; \Big| \; (\, \mathfrak{c} \, , -\boldsymbol{K} \,) = 2 \, , \; \mathfrak{c}^2 = 0 \, \Big\}$

$$\begin{array}{l} \mathcal{K}_r \\ \Cup \\ \mathfrak{c} \quad \longleftrightarrow \quad \text{Fibration en coniques} \;\; \phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1 \end{array}$$

$$\mathcal{K}_r = \boldsymbol{W}_r \cdot (\boldsymbol{h} - \ell_1)$$

r	3	4	5	6	7	8
E_r	$A_2 \times A_1$	A_4	D_5	E_6	E_7	E_8
$W_r = W(E_r)$	$\mathfrak{S}_3 \times \mathfrak{S}_2$	$\mathfrak{S}_5$	$(\mathbf{Z}/2\mathbf{Z})^4 \ltimes \mathfrak{S}_5$	$W(E_6)$	$W(E_7)$	$W(E_8)$
$\omega_r = \lvert W_r \rvert$	12	$5!$	$2^4 \cdot 5!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
$l_r = \lvert \mathcal{L}_r \rvert$	6	10	16	27	56	240
$\kappa_r = \lvert \mathcal{K}_r \rvert$	3	5	10	27	126	2160

Exemple : les droites de dP_2 vues sur le plan

- $\mathbf{dP}_2 = \boldsymbol{X}_7 = \mathbf{Bl}_{p_1,\dots,p_7}(\mathbb{P}^2) \xrightarrow{\beta} \mathbb{P}^2$

- $\ell_i = \beta^{-1}(p_i) \subset \boldsymbol{X}_7 \quad \rightsquigarrow \quad \boldsymbol{\ell}_i \in \mathbf{Pic}(\boldsymbol{X}_7)$

  $\rightsquigarrow \quad \boldsymbol{\ell} = \sum_{i=1}^{7} \boldsymbol{\ell}_i$

Line	Class in $\mathrm{Pic}(X_7)$	Number of such lines	Model in $\mathbf{P}^2$
ℓ_i	ℓ_i	7	first infinitesimal neighbourhood $p_i^{(1)}$
ℓ_{ij}	$h - \ell_i - \ell_j$	21	line joining p_i to p_j
C_{ij}	$2h - \boldsymbol{\ell} + \ell_i + \ell_j$	21	conic through the p_k's, $k \notin \{i, j\}$
C_i^3	$3h - \boldsymbol{\ell} - \ell_i$	7	cubic through all the p_l's with a node at p_i

Table 2. Lines on dP_2 and the corresponding 'curves' in the projective plane

Exemple : cloniques de dP_2

Conic class $\mathfrak{c}$	Number of such $\mathfrak{c}$	Linear system $\lvert\mathfrak{C}_\mathfrak{c}\rvert$	$\mathfrak{C}_\mathfrak{c}^{\mathrm{red}}$
$h - \ell_i$	7	lines through p_i	$\ell_{ij} + \ell_j$
$2h - \sum_{i \in I} \ell_i$	35	conics through the p_i's, $i \in I$	$\ell_{i_1 i_2} + \ell_{i_3 i_4}$ $\ell_{i_3} + C_{i_1 i_2}$
$3h - \boldsymbol{\ell} + \ell_i - \ell_j$	42	cubics through the p_k's for $k \neq i$, with a node at p_j	$\ell_{jk} + C_{ik}$ $\ell_i + C_j^3$
$4h - \boldsymbol{\ell} - \sum_{j \in J} \ell_j$	35	quartics through the p_k's with a node at p_j for $j \in J$	$C_{k_1 k_2} + C_{k_3 k_4}$ $\ell_{j_1 j_2} + C_{j_3}^3$
$5h - 2\boldsymbol{\ell} + \ell_i$	7	quintics through the p_k's with a node at p_k except for $k = i$	$C_{ij} + C_j^3$

TABLE 3. Conic classes on dP_2 and their reducible fibers

Coniques non irréductibles sur $X_r = \mathrm{dP}_d$

- $\boldsymbol{L}_r = \cup_{\ell \in \boldsymbol{\mathcal{L}}_r} \ell \subset \boldsymbol{X}_r \quad \rightsquigarrow \quad \boldsymbol{U}_r = \boldsymbol{X}_r \setminus \boldsymbol{L}_r$

- $\boldsymbol{\mathcal{K}}_r \ni \mathfrak{c} \quad \rightsquigarrow$ Fibration en coniques $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1$

$$\begin{aligned}\Sigma_{\mathfrak{c}} = \textbf{Spectre}(\phi_{\mathfrak{c}}) &= \left\{ \sigma \in \mathbb{P}^1 \;\middle|\; \phi_{\mathfrak{c}}^{-1}(\sigma) \text{ pas irreductible } \right\} \\ &= \left\{ \sigma_{\mathfrak{c}}^1 , \ldots , \sigma_{\mathfrak{c}}^{r-2} , \sigma_{\mathfrak{c}}^{r-1} = \infty \right\} \subset \mathbb{P}^1\end{aligned}$$

- Pour $\sigma_{\mathfrak{c}}^i \in \Sigma_{\mathfrak{c}} \; : \; \phi_{\mathfrak{c}}^{-1}(\sigma_{\mathfrak{c}}^i) = L_{\mathfrak{c}}^i + \widetilde{L}_{\mathfrak{c}}^i \qquad \left(L_{\mathfrak{c}}^i , \widetilde{L}_{\mathfrak{c}}^i \in \boldsymbol{\mathcal{L}}_r \right)$

- $\mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\mathbb{P}^1 , \Omega^1_{\mathbb{P}^1}(\mathrm{Log}\,\Sigma_{\mathfrak{c}})\Big) = \left\langle \dfrac{dz}{z-\sigma_{\mathfrak{c}}^i} \right\rangle_{i=1}^{r-2} \simeq \mathbb{C}^{r-2}$

$\wr\|$

- $\mathbf{H}_{\mathfrak{c}} = \phi_{\mathfrak{c}}^*\Big(\mathcal{H}_{\mathfrak{c}}\Big) = \left\langle \dfrac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c}}^i} \right\rangle_{i=1}^{r-2} \subset \mathbf{H}^0\Big(\boldsymbol{X}_r , \Omega^1_{\boldsymbol{X}_r}(\mathrm{Log}\,\boldsymbol{L}_r)\Big) = \mathbf{H}_{\boldsymbol{X}_r}$

Tissu de del Pezzo $\mathcal{W}_{\mathbf{dP}_d}$

- $\mathcal{W}_{\mathbf{dP}_d} = \mathcal{W}\big(\phi_{\mathfrak{c}}\big)_{\mathfrak{c}\in\mathcal{K}_r}$: κ_r-tissu en coniques sur $\mathbf{dP}_d$

- **Quest$^{\mathbf{o}}$** : $\exists \Big(F_{\mathfrak{c}}(\phi_{\mathfrak{c}})\Big)_{\mathfrak{c}\in\mathcal{K}_r}$ tel que $\sum_{\mathfrak{c}\in\mathcal{K}_r} F_{\mathfrak{c}}(\phi_{\mathfrak{c}}) = 0$

  avec les $F_{\mathfrak{c}}$ polylogarithmiques ?

Théorème : $\exists\ (\epsilon_{\mathfrak{c}})_{\mathfrak{c}\in\mathcal{K}_r} \in \{1\,,\,-1\}^{\mathcal{K}_r}$ $\pm$-unique tel que

$$\Big(\mathbf{HLog}^{r-2}\Big) \qquad \sum_{\mathfrak{c}\in\mathcal{K}_r} \epsilon_{\mathfrak{c}}\, AH_{\mathfrak{c}}^{r-2}\big(\phi_{\mathfrak{c}}\big) = 0$$

où $\forall\ \mathfrak{c}$: $AH_{\mathfrak{c}}^{r-2}$ = hyperlogarithme antisymétrique complet de poids $r-2$ sur $\mathbb{P}^1 \setminus \Sigma_{\mathfrak{c}}$

Intégrales itérées

- **Poincaré (1884), Lappo-Danilevski (1928), Chen (1973)**

- $\boldsymbol{Y}$ variété complexe

- $\mathbf{H} = \Big\langle \omega_1 \, , \, \ldots \, , \, \omega_m \Big\rangle \subset \mathbf{H}^0\Big(\boldsymbol{Y}, \Omega^1_{\boldsymbol{Y}} \Big) \; + \left[\begin{array}{r} d\omega_i = 0 \\ \omega_i \wedge \omega_j = 0 \end{array} \right]$

- **Ex :** $\phi : \boldsymbol{Y} \to \boldsymbol{C}$ and $\omega_i \in \phi^*\Big(\mathbf{H}^0\big(\boldsymbol{C}, \Omega^1_{\boldsymbol{C}} \big)\Big) \quad i = 1, \ldots, m$

- Point base $y \in \boldsymbol{Y}$, chemin $\gamma^x : [0,1] \to \boldsymbol{Y}$ de y à x :
  - $\mathbb{I}_{\omega_i} \; : \; x \longmapsto \int_{\gamma^x} \omega_i \quad \rightsquigarrow \quad \mathbb{I}_{\omega_i} \in \mathcal{O}_y$
  - $\mathbb{I}_{\omega_j\omega_i} \; : \; x \longmapsto \int_{\gamma^x} \omega_j(u) \cdot \mathbb{I}_{\omega_i}(u) \quad \rightsquigarrow \quad \mathbb{I}_{\omega_j\omega_i} \in \mathcal{O}_y$
  - $\mathbb{I}_{\omega_k\omega_j\omega_i} \; : \; x \longmapsto \int_{\gamma^x} \omega_k(u) \cdot \mathbb{I}_{\omega_j\omega_i}(u) \quad \rightsquigarrow \quad \mathbb{I}_{\omega_k\omega_j\omega_i} \in \mathcal{O}_y$

Intégrales itérées (polylogarithmes)

$$\mathbf{II}^w \,:\, \mathbf{H}^{\otimes w} \longrightarrow \mathcal{O}_y$$

- $\underline{\omega} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_w} \longmapsto \mathbb{I}_{\underline{\omega}} : z \mapsto \int_{\gamma^z} \omega_{i_1}(u) \cdot \mathbb{I}_{\omega_{i_2} \cdots \omega_{i_w}}(u)$

- $\mathbf{II} : \left(\bigoplus_{w \geq 0} \mathbf{H}^{\otimes w} \,,\, ш \right) \longrightarrow \mathcal{O}_y$ morphisme injectif de $\mathbb{C}$-algèbres

- $\forall\, \underline{\omega}$: $\mathbb{I}_{\underline{\omega}} \in \mathcal{O}_y \cap \widetilde{\mathcal{O}}(\boldsymbol{Y})$ monod. unipotente $\longrightarrow$ Symbole $\boldsymbol{\mathcal{S}}(\mathbb{I}_{\underline{\omega}}) = \underline{\omega}$ ✓

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \Sigma$ avec $\Sigma = \{0, 1, \infty\}$

$$\mathbf{H} = \left\langle \frac{dz}{z} \,,\, \frac{dz}{1-z} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 \,,\, \Omega^1_{\mathbb{P}^1}(\operatorname{Log}\Sigma) \Big)$$

$$\mathbf{Li}_n = \mathbf{II}^n\Big(\left(\tfrac{dz}{z}\right)^{\otimes(n-1)} \otimes \left(\tfrac{dz}{1-z}\right) \Big)$$ (**‘Polylogarithmes’**)

Intégrales itérées (hyperlogarithmes)

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \Sigma$ avec $\Sigma = \left\{ \sigma^1 , \ldots , \sigma^{r-2} , \sigma^{r-1} = \infty \right\}$

$$\mathbf{H} = \left\langle \tfrac{dz}{z-\sigma^1}, \ldots, \tfrac{dz}{z-\sigma^{r-2}} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 , \Omega^1_{\mathbb{P}^1}(\operatorname{Log}\Sigma) \Big)$$

$$\mathbf{II}^n\Big(\big(\tfrac{dz}{z-\sigma^{i_1}}\big) \otimes \cdots \otimes \big(\tfrac{dz}{z-\sigma^{i_n}}\big) \Big) \qquad \text{“Hyperlogarithme”}$$

- **Hyperlog antisymétrique complet de poids $r-2$ sur $\mathbb{P}^1 \setminus \boldsymbol{\Sigma}$:**

$$\boldsymbol{AH}_{\Sigma}^{r-2} = \mathbf{II}^n\left(\mathbf{Asym}\left(\left(\frac{dz}{z-\sigma^1}\right) \otimes \cdots \otimes \left(\frac{dz}{z-\sigma^{r-2}}\right)\right) \right)$$

$$= \mathbf{II}^n\left(\tfrac{1}{(r-2)!} \textstyle\sum_{\nu \in \mathfrak{S}_{r-2}} (-1)^\nu \left(\tfrac{dz}{z-\sigma^{\nu(1)}}\right) \otimes \cdots \otimes \left(\tfrac{dz}{z-\sigma^{\nu(r-2)}}\right) \right)$$

- **Ex :** $\boldsymbol{AH}^2_{\{0,1,\infty\}} = \tfrac{1}{2}\, \mathbf{II}^2\Big(\tfrac{dz}{z} \otimes \tfrac{dz}{(1-z)} - \tfrac{dz}{(1-z)} \otimes \tfrac{dz}{z} \Big) = \mathbf{R}$

III Identité $\mathbf{HLog}^{r-2}$: preuve(s)

$$\boxed{\left(\mathbf{HLog}^{r-2}\right) \; : \quad \sum_{\mathfrak{c}\in\mathcal{K}} \epsilon_{\mathfrak{c}}\, \boldsymbol{AH}^{r-2}_{\mathfrak{c}}(\phi_{\mathfrak{c}}) = 0 \;\; \text{avec} \;\; \boldsymbol{AH}^{r-2}_{\mathfrak{c}} = \boldsymbol{AH}^{r-2}_{\Sigma_{\mathfrak{c}}}}$$

- $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1 \supset \Sigma_{\mathfrak{c}} = \{\, \sigma^i_{\mathfrak{c}} \,\}_{i=1}^{r-1}$ $\qquad \mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\Omega^1_{\mathbb{P}^1}\big(\,\mathrm{Log}\,\Sigma_{\mathfrak{c}}\,\big)\Big)$

- $\phi^*_{\mathfrak{c}}\big(\,\mathcal{H}_{\mathfrak{c}}\,\big) = \mathbf{H}_{\mathfrak{c}} \subset \mathbf{H}_{\boldsymbol{X}_r} = \mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\,\boldsymbol{L}_r\,\big)\Big)$

- $\boldsymbol{AH}^{r-2}_{\mathfrak{c}}(\phi_{\mathfrak{c}}) = \mathbf{II}\left(\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma^1_{\mathfrak{c}}}\right)\wedge\cdots\wedge\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma^{r-2}_{\mathfrak{c}}}\right)\right) \in \mathbf{II}^{r-2}\Big(\wedge^{r-2}\,\mathbf{H}_{\mathfrak{c}}\Big)$

$$\Big\downarrow\; \boldsymbol{\mathcal{S}} \;\; \text{(symbole)}$$

- $\boldsymbol{\Omega}^{r-2}_{\mathfrak{c}} = \left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma^1_{\mathfrak{c}}}\right)\wedge\cdots\wedge\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma^{r-2}_{\mathfrak{c}}}\right) \;\in\; \wedge^{r-2}\mathbf{H}_{\mathfrak{c}} \subset \wedge^{r-2}\mathbf{H}_{X_r}$

$$\boxed{\left(\mathbf{HLog}^{r-2}\right) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}^{r-2}_{\mathfrak{c}} = 0 \quad \text{in} \;\; \wedge^{r-2}\mathbf{H}_{X_r}}$$

$$\boxed{\textbf{Preuves de :}\quad \mathbf{hlog}^{r-2} = \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \mathbf{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \textbf{dans } \wedge^{r-2}\mathbf{H}_{X_r}}$$

- $\mathbf{H}_{\boldsymbol{X}_r} = \mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\, \boldsymbol{L}_r\,\big)\Big) \xrightarrow{\ \oplus_\ell \mathbf{Res}_\ell\ } \mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ injective

$$\mathbf{\Omega}_{\mathfrak{c}}^{r-2} \in \wedge^{r-2}\mathbf{H}_{X_r} \hookrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r} \curvearrowleft \boldsymbol{W}(E_r)$$

[**P1**] On décompose $\mathbf{hlog}^{r-2}$ dans une base naturelle de $\wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$

[**P2**] $\mathbf{sign}_r \hookrightarrow \oplus_{\mathfrak{c}}(\mathbf{H}_{\mathfrak{c}})^{\wedge(r-2)} \longrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r} \quad \Big\langle \mathbf{sign}_r\,,\, \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r} \Big\rangle = 0$

$$\mathbf{1} \longmapsto \big(\mathbf{\Omega}_{\mathfrak{c}}^{r-2}\big)_{\mathfrak{c}} \longmapsto \sum_{\mathfrak{c}} \mathbf{\Omega}_{\mathfrak{c}}^{r-2}$$

(GAP3)

[**P3**] Descrip° explicite des droites $\longrightarrow$ $\mathbb{Z}^{\boldsymbol{\mathcal{L}}_r} \simeq \mathbb{Z}^{|\boldsymbol{\mathcal{L}}_r|}$

$+$ algèbre linéaire sur $\mathbb{Z}$ $\longrightarrow$ $\sum \epsilon_{\mathfrak{c}} \mathbf{\Omega}_{\mathfrak{c}}^{r-2} = 0$ (Maple)

[**P4**?] Preuve analytique inductive

IV Comparaison entre HLog^2 et HLog^3

- $\boxed{\mathbf{HLog}^2}$ $\quad \mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$

- $\boxed{\mathbf{HLog}^3}$ $\quad \sum_{i=1}^{10} \epsilon_i\, \boldsymbol{AH}^3_{\boldsymbol{\Sigma_i}}(\boldsymbol{\phi_i}) = 0$ avec pour $\Sigma = \{ b_1, \ldots, b_4 \}$

$$\boldsymbol{AH}^3_{\Sigma}(x) = \frac{1}{3} \sum_{k=1}^{3} (-1)^{k-1} \mathbf{Log}\left(1 - \frac{x}{b_k}\right) \mathbf{R}_{\Sigma \setminus \{b_k\}}(x)$$

$$\boldsymbol{AH}^3_{\mathbf{1}}(\boldsymbol{x}) + \boldsymbol{AH}^3_{\mathbf{2}}\left(\frac{\mathbf{1}}{\boldsymbol{y}}\right) + \boldsymbol{AH}^3_{\mathbf{3}}\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right) + \boldsymbol{AH}^3_{\mathbf{4}}\left(\frac{\boldsymbol{x-y}}{\boldsymbol{x-1}}\right) + \boldsymbol{AH}^3_{\mathbf{5}}\left(\frac{\boldsymbol{b(a-x)}}{\boldsymbol{ay-bx}}\right)$$

$$+\boldsymbol{AH}^3_{\mathbf{6}}\left(\frac{\boldsymbol{P(x,y)}}{\boldsymbol{(x-1)(y-b)}}\right) + \boldsymbol{AH}^3_{\mathbf{7}}\left(\frac{\boldsymbol{(x-y)(y-b)}}{\boldsymbol{y\,P(x,y)}}\right) + \boldsymbol{AH}^3_{\mathbf{8}}\left(\frac{\boldsymbol{x\,P(x,y)}}{\boldsymbol{(x-y)(x-a)}}\right)$$

$$+\boldsymbol{AH}^3_{\mathbf{9}}\left(\frac{\boldsymbol{y(x-b)}}{\boldsymbol{x(y-a)}}\right) + \boldsymbol{AH}^3_{\mathbf{10}}\left(\frac{\boldsymbol{a\,(b-x)}}{\boldsymbol{by-ax}}\right) = \mathbf{0}$$

Tissus de del Pezzo $\mathcal{W}_{\mathbf{dP}_5}$ et $\mathcal{W}_{\mathbf{dP}_4}$

- $(\mathcal{A}\mathbf{b})$ $\quad \mathbf{R}(\phi_1) - \mathbf{R}(\phi_2) - \mathbf{R}\Big(\phi_3\Big) - \mathbf{R}\Big(\phi_4\Big) + \mathbf{R}\Big(\phi_5\Big) = 0$

- Pour chaque i : $\mathcal{F}_{\phi_i}$ = feuilletage par les $\{\phi_i = \lambda\}$, $\lambda \in \mathbb{P}^1$

- **Tissu :** $\mathcal{W}_{\mathbf{dP}_5} = \Big(\mathcal{F}_{\phi_1}, \ldots, \mathcal{F}_{\phi_5}\Big)$: 5-uplet de feuilletages

- $\mathcal{W}_{\mathbf{dP}_5}$ = objet géometrique $\quad \rightsquigarrow \quad \big(\mathbf{HLog}^2\big) = (\mathcal{A}\mathbf{b})$

- $\mathcal{W}_{\mathbf{dP}_4} = \Big(\mathcal{F}_{\phi_k}\Big)_{\substack{\phi_k\,:\,\mathbf{dP_4} \to \mathbb{P}^1 \\ \text{fib}^\circ \text{ en coniques}}} \quad \rightsquigarrow \quad \big(\mathbf{HLog}^3\big)$

Comparaison des tissus $\mathcal{W}_{\mathbf{dP}_5}$ et $\mathcal{W}_{\mathbf{dP}_4}$

$\mathcal{W}_{\mathbf{dP}_5}$ et $\mathcal{W}_{\mathbf{dP}_4}$ satisfont des propriétés remarquables similaires :

- *formés par les pinceaux de coniques sur une del Pezzo*
- *non-linéarisables*
- *de rang maximal, toutes leurs RA hyperlogarithmiques*
- *on a* $$\mathbf{RA}(\mathcal{W}_{\mathbf{dP}_5}) = \mathbf{LogRA}^1(\mathcal{W}_{\mathbf{dP}_5}) \oplus \langle\, \mathbf{HLog}^2 \,\rangle$$ $$\mathbf{RA}(\mathcal{W}_{\mathbf{dP}_4}) = \mathbf{LogRA}^1(\mathcal{W}_{\mathbf{dP}_4}) \oplus \mathbf{HLogRA}^2(\mathcal{W}_{\mathbf{dP}_4}) \oplus \langle\, \mathbf{HLog}^3 \,\rangle$$ *(décompositions en $\boldsymbol{W}_r$-representations irréductibles)*
- *caracterisés par les matroïde de leurs sous-3-tissus hexagonaux*
- *sont "canoniquement algébrisables" (!)*
- *sont des "tissus modulaires"*
- *sont des "tissus cluster" ("amassés")*
- *s'obtiennent géometriquement à la* **[Gelfand-MacPherson]**

Comparaison des tissus $\mathcal{W}_{\mathbf{dP}_5}$ et $\mathcal{W}_{\mathbf{dP}_4}$

$\mathcal{W}_{\mathbf{dP}_5}$ et $\mathcal{W}_{\mathbf{dP}_4}$ satisfont des propriétés remarquables similaires :

- *formés par les pinceaux de coniques sur une del Pezzo*
- *non-linéarisables*
- *de rang maximal, toutes leurs RA hyperlogarithmiques*
- *on a*
$$\mathbf{RA}(\mathcal{W}_{\mathbf{dP}_5}) = \mathbf{LogRA}^1(\mathcal{W}_{\mathbf{dP}_5}) \oplus \langle\, \mathbf{HLog}^2 \,\rangle$$
$$\mathbf{RA}(\mathcal{W}_{\mathbf{dP}_4}) = \mathbf{LogRA}^1(\mathcal{W}_{\mathbf{dP}_4}) \oplus \mathbf{HLogRA}^2(\mathcal{W}_{\mathbf{dP}_4}) \oplus \langle\, \mathbf{HLog}^3 \,\rangle$$
(décompositions en $\boldsymbol{W}_r$-representations irréductibles)
- *caracterisés par les matroïde de leurs sous-3-tissus hexagonaux*
- *sont "canoniquement algébrisables" (!)*
- *sont des "tissus modulaires"*
- *sont des "tissus cluster" ("amassés")*
- *s'obtiennent géometriquement à la* **[Gelfand-MacPherson]**

Tissus de Gelfand-MacPherson

- $\boldsymbol{G} \supset \boldsymbol{P} \supset \boldsymbol{H}$: $\boldsymbol{G}$ = groupe de Lie simple, type Dynkin D, rang r
  $\boldsymbol{H} \simeq (\mathbb{C}^*)^r$ = sous-tore de Cartan
  $\boldsymbol{P}$ = sous-groupe parabolique standard (maximal)

- $\boldsymbol{X} = \boldsymbol{G}/\boldsymbol{P}$: variété algébrique projective $\boldsymbol{G}$-homogène

- $\boldsymbol{X} \subset \mathbb{P}(\boldsymbol{V}_\rho)$: $\boldsymbol{V}_\rho$ = rep° de $\boldsymbol{G}$ $\big(\rho : \boldsymbol{G} \hookrightarrow \mathbf{GL}(V_\rho) \big)$
  $\boldsymbol{X} = \boldsymbol{G} \cdot v_\rho$ avec v_ρ de plus haut poids $\omega = \omega_P$
  $\boldsymbol{P} = \mathrm{Stab}_{\boldsymbol{G}}(v_\rho)$: stabilisateur de v_ρ dans $\boldsymbol{G}$

- $\mathfrak{W}_\rho = \{\text{poids de } \rho\} \subset \mathfrak{h}^*_{\mathbb{R}} \simeq \mathbb{R}^r$ $\Big(\mathfrak{h} = \mathrm{Lie}(\boldsymbol{H}) \Big)$

  – $\boldsymbol{X} \ni x = \Big[\sum_{w \in \mathfrak{W}} p^w(x) v_w \Big] \in \mathbb{P}(\boldsymbol{V}_\rho)$ **$(p^w(x))_w$ "coordonnées de Plücker généralisées"**

- **"Application moment"** : $\mu = \mu_{D,\rho} : \boldsymbol{X} \longrightarrow \mathfrak{h}^*_{\mathbb{R}}$, $x \longmapsto \dfrac{\sum_w |p^w(x)|^2 w}{\sum_w |p^w(x)|^2}$

Tissus de Gelfand-MacPherson

- **Application moment :** $\mu : \boldsymbol{X} \longrightarrow \mathfrak{h}_{\mathbb{R}}^{*}$, $x \longmapsto \dfrac{\sum_w |p^w(x)|^2 w}{\sum_w |p^w(x)|^2}$

- **Polytope moment :** $\mu(\boldsymbol{X}) = \boldsymbol{\Delta} = \boldsymbol{\Delta}_{\boldsymbol{D},\boldsymbol{\rho}} = \mathrm{Conv}(\mathfrak{W}_\rho) \subset \mathfrak{h}_{\mathbb{R}}^{*}$

- On note $H_+ = \boldsymbol{H}(\mathbb{R}_{>0}) \simeq (\mathbb{R}_{>0})^r$

Thm [Atiyah, Guillemin-Sternberg, Gelfand-Serganova]

1. Pour tout $x \in \boldsymbol{X}$:
   - $\boldsymbol{\Delta}_{\boldsymbol{x}} = \mu(\overline{\boldsymbol{H}\cdot x})$ est un $\mathfrak{W}_\rho$-sous-polytope de $\boldsymbol{\Delta}$
   - μ induit un isom C^ω de variétés à coins $\overline{H_+ \cdot x} \simeq \boldsymbol{\Delta}_{\boldsymbol{x}}$
2. Pour x générique, *i.e.* $x \in \boldsymbol{X}^\circ = \mu^{-1}(\mathring{\boldsymbol{\Delta}})$, on a $\boldsymbol{\Delta}_{\boldsymbol{x}} = \boldsymbol{\Delta}_{D,\rho}$

Tissus de Gelfand-MacPherson

- F face de Δ : $\boldsymbol{X}_F = \mu^{-1}(F) \subset \boldsymbol{X}$

Prop : **1.** $\boldsymbol{X}_F = \boldsymbol{G}_F / \boldsymbol{P}_F$ avec $(\boldsymbol{G}_F, \boldsymbol{P}_F)$ de type (D_F, ω_F)
et $F \simeq \boldsymbol{\Delta}_{D_F, \omega_F}$

2. On a $\boldsymbol{V}_\rho = \boldsymbol{V}_F \oplus \boldsymbol{V}^F$ comme $\boldsymbol{G}_F$-rep° $\rightarrow$ Project° linéaire $\Pi_F : \boldsymbol{V}_\rho \to \boldsymbol{V}_F$

$$\boldsymbol{X}_F = \boldsymbol{X} \cap \mathbb{P}(\boldsymbol{V}_F) = \Pi_F(\boldsymbol{X}) \subset \mathbb{P}(\boldsymbol{V}_F)$$

3. $\Pi_F(\boldsymbol{X}^\circ) = \boldsymbol{X}_F^\circ$ et $\Pi_F : \boldsymbol{X}^\circ \longrightarrow \boldsymbol{X}_F^\circ$ est une fibration localem$^{\text{t}}$ triviale en espaces projectifs à poids

4. $\boldsymbol{H}$-torseur $\nu_{\boldsymbol{H}} : \boldsymbol{X}^\circ \longrightarrow \mathcal{Y}^\circ = \boldsymbol{X}^\circ / \boldsymbol{H}$ $\left(\mathcal{Y}^\circ \subset \mathcal{Y} = \boldsymbol{X}^{ss} /\!/ \boldsymbol{H} \right)$

5. $\Pi_F : \boldsymbol{X}^\circ \longrightarrow \boldsymbol{X}_F^\circ$ est $(\boldsymbol{H}, \boldsymbol{H_F})$-équivariante $\left(\boldsymbol{H} \twoheadrightarrow \boldsymbol{H_F} \right)$

Tissus de Gelfand-MacPherson

5. $\mathbb{P}(\boldsymbol{V}_\rho) \supset \boldsymbol{X}^\circ \xrightarrow{\Pi_F} \boldsymbol{X}_F^\circ \subset \mathbb{P}(\boldsymbol{V}_F)$ est $(\boldsymbol{H}, \boldsymbol{H_F})$-équivariante

6. $\exists\ \pi_F : \boldsymbol{\mathcal{Y}}^\circ \to \boldsymbol{\mathcal{Y}}_F^\circ = \boldsymbol{X}_F^\circ / \boldsymbol{H}_F$ tel que le diagramme

$$\begin{tikzcd} \boldsymbol{X}^\circ \arrow[r, "\Pi_F"] \arrow[d, "\nu_{\boldsymbol{H}}"'] & \boldsymbol{X}_F^\circ \arrow[d, "\nu_{\boldsymbol{H}_F}"] \\ \boldsymbol{\mathcal{Y}}^\circ \arrow[r, "\pi_F"'] & \boldsymbol{\mathcal{Y}}_F^\circ \end{tikzcd}$$

soit commutatif

• **Def$^\circ$: Tissus de Gelfand-MacPherson**

$$\mathcal{W}_{\boldsymbol{X}}^{GM} = \mathcal{W}\Big(\Pi_F \ \Big| \ F \text{ face de codim 1 de } \boldsymbol{\Delta} \Big) \leftarrow \boldsymbol{H}\text{-équivariant}$$

$$\mathcal{W}_{\boldsymbol{\mathcal{Y}}}^{GM} = \mathcal{W}\Big(\pi_F \ \Big| \ F \text{ face de codim 1 de } \boldsymbol{\Delta} \Big) = \Big(\mathcal{W}_{\boldsymbol{X}}^{GM}\Big)/\boldsymbol{H}$$

Tissus de Gelfand-MacPherson : $G_k(\mathbb{C}^N)$

- $\boldsymbol{G}_k(\mathbb{C}^N) \subset \mathbb{P}(\wedge^k \mathbb{C}^N)$ $\qquad\qquad \Big(\boldsymbol{G} = \mathbf{SL}(\mathbb{C}^N),\ D = A_{N-1}, \text{ etc } \Big)$

- $\mu : \boldsymbol{G}_k(\mathbb{C}^N) \longrightarrow \boldsymbol{\Delta}_k^N = \left\{ (t_i)_{i=1}^N \,\middle|\, \begin{array}{l} 0 \leq t_i \leq 1 \\ \sum_{i=1}^N t_i = k \end{array} \right\}$ **hypersimplexe**

- Faces de $\boldsymbol{\Delta}_k^N$ de codim 1 $= \left\{ \begin{array}{ll} \boldsymbol{\Delta}_k^N \cap \{t_i = 0\} = \boldsymbol{\Delta}_k^{N-1} & \leftsquigarrow \quad \boldsymbol{G_k}(\mathbb{C}^{\boldsymbol{N-1}}) \\ \boldsymbol{\Delta}_k^N \cap \{t_i = 1\} \simeq \boldsymbol{\Delta}_{k-1}^{N-1} & \leftsquigarrow \quad \boldsymbol{G_{k-1}}(\mathbb{C}^{\boldsymbol{N-1}}) \end{array} \right.$

- Pour chaque $i = 1, \ldots, N$, il y a deux "applications faces" :

$$\boldsymbol{G}_{\boldsymbol{k-1}}^\circ\Big(\mathbb{C}^{\boldsymbol{N-1}}_{\{\boldsymbol{x_i=0}\}}\Big) \xleftarrow{\Pi_{\{t_i=1\}}} \boldsymbol{G}_{\boldsymbol{k}}^\circ(\mathbb{C}^{\boldsymbol{N}}) \xrightarrow{\Pi_{\{t_i=0\}}} \boldsymbol{G}_{\boldsymbol{k}}^\circ\Big(\mathbb{C}^{\boldsymbol{N}}/_{\langle \boldsymbol{e_i} \rangle}\Big)$$

Tissus de Gelfand-MacPherson : $G_k(\mathbb{C}^N)$

- Pour chaque $i = 1, \ldots, N$, il y a deux "applications faces" :

$$
\begin{tikzcd}
\boldsymbol{G}^{\circ}_{\boldsymbol{k-1}}\Big(\mathbb{C}^{\boldsymbol{N-1}}_{\{\boldsymbol{x_i=0}\}}\Big) \arrow[d, "\nu_{i,0}"] & \boldsymbol{G}^{\circ}_{\boldsymbol{k}}(\mathbb{C}^{\boldsymbol{N}}) \arrow[l, "\Pi_{\{t_i=0\}}"'] \arrow[r, "\Pi_{\{t_i=1\}}"] \arrow[d, "\nu"] & \boldsymbol{G}^{\circ}_{\boldsymbol{k}}\Big(\mathbb{C}^{\boldsymbol{N}}/_{\langle \boldsymbol{e_i}\rangle}\Big) \arrow[d, "\nu_{i,1}"] \\
\mathbf{Conf}^{\circ}_{N-1}(\mathbb{P}^{k-2}) & \mathbf{Conf}^{\circ}_{N}(\mathbb{P}^{k-1}) \arrow[l, "\pi_{i,0}"'] \arrow[r, "\pi_{i,1}"] & \mathbf{Conf}^{\circ}_{N-1}(\mathbb{P}^{k-1}) \\
\Big[\mathbf{Proj}_{p_i}(p_k)\Big]_{k\neq i} & {[p_1, \ldots, p_N]} \arrow[l, mapsto] \arrow[r, mapsto] & {[p_1, \ldots, \widehat{p}_i, \ldots, p_N]}
\end{tikzcd}
$$

- **Tissu GM :** $\mathcal{W}^{GM}_{\mathbf{Conf}_N(\mathbb{P}^k)} = \mathcal{W}\left(\begin{array}{l} N \text{ applications d'oubli d'un point } + \\ N \text{ applications de proj}^{\circ} \text{ d'un point} \end{array}\right)$

- $\boldsymbol{k = 2}$ **:** $\mathbf{Conf}^{\circ}_N(\mathbb{P}^1) = \boldsymbol{\mathcal{M}}_{0,N}$ $\qquad \mathcal{W}^{GM}_{\mathcal{M}_{0,N}} = \mathcal{W}\Big(\boldsymbol{\mathcal{M}}_{0,N} \xrightarrow{f_i} \boldsymbol{\mathcal{M}}_{0,N-1}\Big)$

$\boldsymbol{+ N = 5}$ **:** $\overline{\boldsymbol{\mathcal{M}}}_{0,5} \simeq \mathbf{dP}_5 = \boldsymbol{X}_4$ $\qquad \mathcal{W}^{GM}_{\mathcal{M}_{0,5}} = \mathcal{W}_{\mathbf{dP}_5} \simeq \mathcal{B} \longleftarrow (\mathcal{A}\mathbf{b})$

Construction de ($\mathcal{A}b$) par Gelfand et MacPherson

$$\begin{tikzcd} \boldsymbol{G}_2^o(\mathbb{R}^5) \arrow[r, "F_i"] \arrow[d, "\nu=\nu_4"'] & \boldsymbol{G}_2^o\left(\mathbb{R}^5/\langle e_i\rangle\right) \arrow[l, bend right, "a_i"'] \arrow[d, "\nu_i=\nu_3"] \\ \boldsymbol{\mathcal{M}}_{0,5}(\mathbb{R}) \arrow[r, "f_i"] & \boldsymbol{\mathcal{M}}_{0,4}(\mathbb{R}) \end{tikzcd} \simeq \mathbb{R}\setminus\{0,1\}$$

$\mathbf{P}_1$: 1ère classe Pontryagin

$\mathbf{H}^4\Big(\boldsymbol{G}_2(\mathbb{R}^5)\Big) \ni \mathbf{P}_1 = \left[\,\boldsymbol{\Omega}\,\right]$

avec $\boldsymbol{\Omega} \in \Omega^4\Big(\boldsymbol{G}_2(\mathbb{R}^5)\Big)^{\mathbf{SO}_5(\mathbb{R})}$

- $\int$ sur les fibres de la 4-forme $\boldsymbol{\Omega}$ $\longrightarrow$
  - $\omega_{0,5} = \nu_*(\boldsymbol{\Omega}) \in \Omega^0(\boldsymbol{\mathcal{M}}_{0,5}(\mathbb{R}))$
  - $\omega_{0,4,i} = (\nu_i)_*(\boldsymbol{\Omega}) \in \Omega^1(\boldsymbol{\mathcal{M}}_{0,4}(\mathbb{R}))$

- En $\nu(\xi)$: $\omega_{0,5} = \int_{\overline{\boldsymbol{H}\cdot\xi}} \boldsymbol{\Omega} \quad +$

  $\overline{\boldsymbol{H}\cdot\xi} \simeq \boldsymbol{\Delta}_2^5$ via $\boldsymbol{G}_2(\mathbb{R}^5) \xrightarrow{\mu} \boldsymbol{\Delta}_2^5$

  $\partial\left[\,\boldsymbol{\Delta}_2^5\,\right] = \sum_{i=1}^5 (-1)^i a_{i*}\Big(\left[\,\boldsymbol{\Delta}_{2,i}^4\,\right]\Big)$

- Stokes pour $\int$ sur les fibres :

$$d\underbrace{\omega_{0,5}}_{=0} = \sum_{i=1}^5 (-1)^i \boldsymbol{f_i}^*\big(\underbrace{\omega_{0,4}}_{=\mathrm{d}\mathbf{R}}\big) \Longrightarrow \quad 0 = \sum_{i=1}^5 (-1)^i\, \boldsymbol{f_i}^*(\mathrm{d}\mathbf{R}) \quad (\boldsymbol{\mathcal{A}}\mathbf{b})$$

Variétés de Cox

- Comment obtenir $G_2(\mathbb{C}^5)$ à partir de $X_4 = \mathbf{dP}_5 = \overline{\mathcal{M}}_{0,5}$?

C'est sa variété de Cox !

- S = variété projective (lisse) telle que $\mathbf{Pic}_{\mathbb{Z}}(S) = \oplus_{i=0}^{r} \mathbb{Z}\,\ell_i$

$\Big(S = \mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2) \quad \ell_0 = h = [\,H\,]$ et $\ell_i = [\,E_i\,] \quad i = 1,\ldots,r \Big)$

- **Def°** : **Anneau de Cox**

$$\mathbf{Cox}(S) = \bigoplus_{n_0,\ldots,n_r \in \mathbb{Z}} \mathbf{H}^0\Big(S, \mathcal{O}_S(n_0H + n_1E_1 + \cdots + n_rE_r)\Big)$$

- **Faits** : – $\mathbf{Cox}(\mathbb{P}^n) = \mathbb{C}[x_0,\ldots,x_n]$ (polynômes homogènes)
  - $\mathbf{Cox}(S) = \mathbb{C}[y_1,\ldots,y_m] \iff S$ est torique
  - $\mathbf{Cox}(S)$ de type fini $=$ S "Mori Dream Space" (MMP ✓)

$$\mathbf{Cox}(S) = \frac{\mathbb{C}[\Gamma_1,\ldots,\Gamma_m]}{\mathcal{J}_S} \longrightarrow \mathbf{A}(S) = \mathrm{Spec}\Big(\mathbf{Cox}(S)\Big) \subset \mathbb{A}^m_\Gamma$$

Variétés de Cox des surfaces de del Pezzo

- Surface $\boldsymbol{S} \supset \ell$ avec $\ell \simeq \mathbb{P}^1$ et $\ell^2 = -1$ $\Longrightarrow$ $\sigma_\ell \in \mathbf{H}^0(\mathcal{O}_{\boldsymbol{X}_r}(\ell)) \setminus \{0\}$ générateur de $\mathbf{Cox}(\boldsymbol{S})$

- Dans $\mathbf{Bl}_{p_1,\dots,p_9}(\mathbb{P}^2)$: $\exists\ \infty$ de (-1)-droites $\Longrightarrow$ $\mathbf{Cox}\Big(\mathbf{Bl}_{p_1,\dots,p_9}(\mathbb{P}^2)\Big)$ pas de type fini

Thm [Batyrev, Popov] Pour $r = 3, \dots, 8$, on a

$$\mathbf{Cox}\Big(\mathbf{dP}_d = \boldsymbol{X_r} = \mathbf{Bl}_{p_1,\dots,p_r}(\mathbb{P}^2)\Big) = \mathbb{C}\Big[\,\sigma_\ell \,\Big|\, \ell \in \boldsymbol{\mathcal{L}}_r\,\Big] / \boldsymbol{\mathcal{J}}_{\mathbf{dP_d}}$$

- $\mathbf{T}_{\mathrm{NS}} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{Pic}_{\mathbb{Z}}(\boldsymbol{X}_r), \mathbb{C}^*) \circlearrowleft \mathbf{A}(\boldsymbol{X}_r) \rightsquigarrow \quad \boldsymbol{X}_r = \mathbf{A}(\boldsymbol{X}_r) /\!/ \,\mathbf{T}_{\mathrm{NS}}$

- $\mathbf{A}(\boldsymbol{X}_r) \hookrightarrow \mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ + $\mathbb{Z}$-graduation sur $\mathbf{Cox}(\boldsymbol{X_r})$ induite par $(-K, \cdot)$

$$\longrightarrow \mathbf{P}(\boldsymbol{X}_r) = \underbrace{\mathrm{Proj}\Big(\mathbf{Cox}(\boldsymbol{X_r})\Big)}_{=\left(\mathbf{A}(\boldsymbol{X}_r) - \{0\}\right)/_{\mathbb{C}^*}} \subset \mathbb{P}\Big(\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}\Big) \circlearrowleft \boldsymbol{\mathcal{T}}_{\mathrm{NS}} = \mathbf{T}_{\mathrm{NS}}/_{\mathbb{C}^*}$$

Variétés de Cox des surfaces de del Pezzo

- $\mathbf{P}(\boldsymbol{X}_r) = \mathrm{Proj}\Big(\mathbf{Cox}(\boldsymbol{X}_r)\Big) \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r}) \circlearrowright \left.\begin{array}{l} \mathcal{T}_{\mathrm{NS}} = \mathbf{T}_{\mathrm{NS}}/\mathbb{C}^* \\ \boldsymbol{W}(E_r) \end{array}\right\} \subset \boldsymbol{G}(E_r)$

Thm [Batyrev, Popov, Derenthal, Serganova-Skorobogatov]

1. Il y a un plongement $(\mathcal{T}_{\mathrm{NS}}, \boldsymbol{H}_r)$-équivariant :

$$\mathbf{P}(\boldsymbol{X}_r) \hookrightarrow \underbrace{\mathcal{X}_r = \boldsymbol{G}_r/\boldsymbol{P}_r \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r})}_{\text{espace et rep}^\circ\text{ minuscules}}$$

2. Il y a un plongement $f_{SS} : \boldsymbol{X}_r \hookrightarrow \mathcal{Y}_r = \mathcal{X}_r /\!/ \boldsymbol{H}_r$ tel que le diagramme suivant soit commutatif :

$$\begin{tikzcd} \mathbf{P}(\boldsymbol{X}_r) \arrow[r, hook] \arrow[d] & \mathcal{X}_r \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r}) \arrow[d] \\ \boldsymbol{X}_r \arrow[r, hook, "f_{SS}"] & \mathcal{Y}_r \end{tikzcd}$$

- $r=4,\ E_r=A_4$: $\quad \mathbf{P}(\boldsymbol{X}_4)=\boldsymbol{G}_2(\mathbb{C}^5)\hookrightarrow \mathbb{P}(\mathbb{C}^{\mathcal{L}_4})\simeq\mathbb{P}^9 \qquad$ (Plücker)

$$
\begin{tikzcd}
& \mathbf{P}(\boldsymbol{X}_4) \arrow[r, equal] \arrow[d] & \boldsymbol{G}_2(\mathbb{C}^5) \arrow[r, dashed, "\Pi_i"] \arrow[d] & \boldsymbol{G}_2(\mathbb{C}^4_i) \arrow[d] \\
\mathbf{dP}_5 = & \boldsymbol{X}_4 \arrow[r, hook, "f_{SS}", "\sim"'] & \boldsymbol{\mathcal{Y}}_4=\overline{\boldsymbol{\mathcal{M}}}_{0,5} \arrow[r, dashed, "\pi_i", "f_i"'] & \overline{\boldsymbol{\mathcal{M}}}_{0,4}\simeq\mathbb{P}^1
\end{tikzcd}
$$

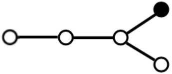
$$\longrightarrow\ \boldsymbol{\mathcal{W}}_{\mathbf{dP}_5}=f_{SS}^*\Big(\boldsymbol{\mathcal{W}}^{GM}_{\boldsymbol{\mathcal{Y}}_4}=\boldsymbol{\mathcal{W}}_{\boldsymbol{\mathcal{M}}_{0,5}}\Big)$$

- Pour $r\in\{4,\ldots,7\}$ ($r=8\,?$), on a :

$$
\begin{tikzcd}
& \mathbf{P}(\boldsymbol{X}_r) \arrow[r, hook] \arrow[d] & \boldsymbol{G}_r/\boldsymbol{P}_r \arrow[r, dashed, "\Pi_F"] \arrow[d] & \boldsymbol{\mathcal{X}}_F \arrow[d] \\
\mathbf{dP}_{9-r} = & \boldsymbol{X}_r \arrow[r, hook, "f_{SS}"] \arrow[ur, hook, "F_{SS}"] & \boldsymbol{\mathcal{Y}}_r \arrow[r, dashed, "\pi_F"] & \boldsymbol{\mathcal{Y}}_F
\end{tikzcd}
$$

Thm : $\quad$ On a $\quad \boldsymbol{\mathcal{W}}_{\mathbf{dP}_{9-r}}=f_{SS}^*\Big(\boldsymbol{\mathcal{W}}^{GM}_{\boldsymbol{\mathcal{Y}}_r}\Big)=F_{SS}^*\Big(\boldsymbol{\mathcal{W}}^{GM}_{\boldsymbol{G}_r/\boldsymbol{P}_r}\Big)$

Exemple : $\mathcal{W}_{\mathbf{dP}_4} = \boldsymbol{X}_5$

- Pour $r = 5$: type D_5  $\boldsymbol{G}_5 = \mathrm{Spin}_{10}(\mathbb{C})$

  $\boldsymbol{G}_5/\boldsymbol{P}_5 = \mathbb{S}_{\mathbf{5}} \subset \mathbb{P}(\boldsymbol{S}_{\mathbf{5}}^+)$

  – $\mathbb{S}_{\mathbf{5}} \simeq \boldsymbol{OG}_5^+(\mathbb{C}^{10}) =$ "*Spinor 10-fold*"
  – $\boldsymbol{S}_{\mathbf{5}}^+ \simeq \mathbb{C}^{16} =$ "*half-spin representation*"

- $\boldsymbol{\mu} : \mathbb{S}_{\mathbf{5}} \to \boldsymbol{\Delta}_{D_5} =$ **demihypercube** : enveloppe convexe des $\frac{1}{2}(\epsilon_1, \ldots, \epsilon_5) \in \frac{1}{2}\{\pm 1\}^5$ avec $\epsilon_1 \cdots \epsilon_5 = 1$

- Facettes : $\boldsymbol{\Delta}_{D_5,i}^{\varepsilon} = \boldsymbol{\Delta}_{D_5} \cap \{\, t_i = \frac{\varepsilon}{2} \,\} \simeq \boldsymbol{\Delta}_{D_4,\mathbb{Q}^6}$ $\quad \left(\begin{array}{l} i = 1, \ldots, 5 \\ \varepsilon \in \{\pm 1\} \end{array} \right)$

- Diagramme :

$$\begin{tikzcd}
 & \mathbf{P}(\boldsymbol{X}_5) \arrow[r, hook] \arrow[d] & \mathbb{S}_5 \arrow[r, dashed, "\Pi_F"] \arrow[d] & \mathbb{Q}^6_F \arrow[d] & \\
\mathbf{dP}_4 = & \boldsymbol{X}_5 \arrow[r, hook, "f_{SS}"] & \boldsymbol{\mathcal{Y}}_5 \arrow[r, dashed, "\pi_F"] & \boldsymbol{\mathcal{Y}}_F & \simeq \mathbb{P}^2
\end{tikzcd}$$

Exemple : $\mathcal{W}_{\mathbf{dP}_4} = \boldsymbol{X}_5$

- Diagramme :

$$\begin{tikzcd} & \mathbf{P}(\boldsymbol{X}_5) \arrow[r, hook] \arrow[d] & \mathbb{S}_5 \arrow[r, dashed, "\Pi_F"] \arrow[d] & \mathbb{Q}^6_F \arrow[d] & \\ \mathbf{dP}_4 = & \boldsymbol{X}_5 \arrow[r, hook, "f_{SS}"] & \boldsymbol{\mathcal{Y}}_5 \arrow[r, dashed, "\pi_F"] & \boldsymbol{\mathcal{Y}}_F & \simeq \mathbb{P}^2 \end{tikzcd}$$

- Tissu de Gelfand-MacPherson de $\mathbb{S}_5$:

$$\mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_5} = \mathcal{W}\Big(\pi_i^{\varepsilon} : \boldsymbol{\mathcal{Y}}_5 \dashrightarrow \mathbb{P}^2\Big) = \begin{array}{l}\text{10-tissu de codim 2} \\ \text{sur } \boldsymbol{\mathcal{Y}}_5 \simeq_{birat} \mathbb{C}^5\end{array}$$

Thm : Les tissus $\mathcal{W}_{\mathbf{dP}_5} = f^*_{SS}\Big(\mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_4}\Big)$ et $\mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_5}$ sont *encore plus similaires* que ne le sont $\mathcal{W}_{\mathbf{dP}_5}$ et $\mathcal{W}_{\mathbf{dP}_4}$!

- $\Big[\mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_4} \to \mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_5} \to \mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_6} \to \cdots\Big]$ "mieux que" $\Big[\mathcal{W}_{\mathbf{dP}_5} \to \mathcal{W}_{\mathbf{dP}_4} \to \mathcal{W}_{\mathbf{dP}_3} \to \cdots\Big]$

Questions

- $\mathbf{HLog}^{r-2} \in \boldsymbol{AR}(\mathcal{W}_{\mathbf{dP}_{9-r}})$ *peut-elle être obtenue à partir d'une RA* $\boldsymbol{\mathcal{A}}_r$ *de* $\mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_r}$? **Oui !**

- **Question :** *Peut-on construire géométriquement à la Gelfand et MacPherson la relation abelienne* $\boldsymbol{\mathcal{A}}_r$ *de* $\mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_r}$ *à partir d'une classe caracté- ristique sur (une forme réelle de)* $\boldsymbol{G}_r/\boldsymbol{P}_r$ *?*

- **Géométrie (différentielle) projective des surfaces :**
  Surface $\boldsymbol{S} \subset \mathbb{P}^d$ "assez générique" ($d = 3, 4, 5$)

$\mathbf{d} = \mathbf{3}$: [Moutard, Darboux - 1880] 27-tissu sur $\boldsymbol{S} \subset \mathbb{P}^3$

$\mathbf{d} = \mathbf{4}$: [Darboux ?] 10-tissu sur $\boldsymbol{S} \subset \mathbb{P}^4$

$\mathbf{d} = \mathbf{5}$: [C. Segre -1921] 5-tissu sur $\boldsymbol{S} \subset \mathbb{P}^5$

$\longrightarrow$ Plein de questions !

Beaucoup d'autres questions...

- **Applications** – $\mathbf{HLog}^1 = \Big(\mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}\big(\tfrac{x}{y}\big) = 0\Big)$ ✓
  - $\mathbf{HLog}^2 = \left(\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0 \right)$ ✓
  - $\mathbf{HLog}^3 = \left(\sum_{i=1}^{10} \boldsymbol{AH}_i^3(U_i(x,y)) = 0 \right)$?
- **Construction** de $\mathbf{HLog}^3$ à la Gelfand-MacPherson ?
- **Interprétation** de $\mathbf{HLog}^3$ en termes du SC de $\mathbf{dP}_4$?
- **Versions** Unival. $\mathbf{HLog}^3_{\mathrm{univ}}$? Quantique $\mathbf{HLog}^3_q$? Motivique $\mathbf{HLog}^3_{\mathrm{mot}}$?
- **Surfaces de del Pezzo singulières/réelles** ?
- **Blow-ups** $\mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2)$ avec $r \geq 9$: $\sum_{\mathfrak{c}\in\mathcal{K}} \boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\varphi_{\mathfrak{c}}) = 0$?