CUSPIDAL ℓ-MODULAR REPRESENTATIONS OF $\mathrm{GL}_n(F)$ DISTINGUISHED BY A GALOIS INVOLUTION

by

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Abstract. — Let F/F_0 be a quadratic extension of non-Archimedean locally compact fields of residual characteristic $p \neq 2$ with Galois automorphism σ, and let R be an algebraically closed field of characteristic $\ell \notin \{0,p\}$. We reduce the classification of $\mathrm{GL}_n(F_0)$-distinguished cuspidal R-representations of $\mathrm{GL}_n(F)$ to the level 0 setting. Moreover, under a parity condition, we give necessary conditions for a σ-selfdual cuspidal R-representation to be distinguished. Finally, we classify the distinguished cuspidal $\overline{\mathbb{F}}_\ell$-representations of $\mathrm{GL}_n(F)$ having a distinguished cuspidal lift to $\overline{\mathbb{Q}}_\ell$.

Contents

1. Introduction 1
2. Notation 5
3. Basic results 6
4. Reduction to level zero 10
5. The odd case 27
6. Distinguished lift theorems 33
References 40

1. Introduction

1.1. Let F/F_0 be a quadratic extension of non-Archimedean locally compact fields whose residual characteristic is a prime number p different from 2. Let σ be its non-trivial automorphism, and G be the general linear group $\mathrm{GL}_n(F)$ for some positive integer n. It is a totally disconnected, locally compact group, on which the involution σ acts componentwise, and the group G^σ of its σ-fixed points is equal to $\mathrm{GL}_n(F_0)$.

Now fix an algebraically closed field R of characteristic different from p. A (smooth) representation π of G on an R-vector space V is said to be *distinguished* (by G^σ) if V carries a non-zero G^σ-invariant linear form; more generally, if χ is a smooth character of G^σ with values in $R^\times$, the representation π is said to be χ-*distinguished* if V carries a non-zero linear form Λ such that

$$\Lambda(\pi(h)v) = \chi(h)\Lambda(v), \quad h \in G^\sigma, \quad v \in V.$$

1.2. In the case where R is the field of complex numbers, distinguished irreducible representations of G have been extensively studied:

(1) they are σ-selfdual, that is, the contragredient $\pi^\vee$ of a distinguished irreducible representation π of G is isomorphic to its σ-conjugate π^σ ([**14, 29, 30**]) and their central character is trivial on $F_0^\times$,

(2) a σ-selfdual discrete series representation of G is either distinguished, or $\varkappa$-distinguished ($\varkappa$ denotes the character of $F_0^\times$ whose kernel is the subgroup of F/F_0-norms), but not both: this is the Dichotomy and Disjunction Theorem ([**21, 2, 3**]),

(3) distinguished generic irreducible representations of G are classified in terms of their cuspidal support ([**4, 23, 24**]),

(4) distinguished cuspidal representations of G are characterized in terms of their Galois parameter ([**15**]) and in terms of type theory (see [**32**] and below).

1.3. Distinguished irreducible representations of G with coefficients in a field R of positive characteristic have been less well studied (see [**3, 32, 22, 11**]). As in the complex case, they are σ-selfdual, and their central character is trivial on $F_0^\times$. For σ-selfdual *supercuspidal* representations, that is, irreducible representations which do not occur as subquotients of parabolically induced representations from a proper Levi subgroup, one has a Dichotomy and Disjunction Theorem (see §3.2). One also has a characterization of distinction in terms of Galois parameters ([**11**] Proposition 3.15) and in terms of types ([**32**] Theorem 10.9). But there are explicit examples of σ-selfdual non-supercuspidal *cuspidal* representations that are neither distinguished nor $\varkappa$-distinguished (as in [**32**] Remark 2.18) and of Steinberg representations that are both distinguished and $\varkappa$-distinguished ([**11**] Remark 1.9). Also, there is no known classification of distinguished cuspidal representations of $\mathrm{GL}_n(F)$ for an arbitrary $n \geqslant 3$ (see [**11**] for $n = 2$).

In this paper, which can be considered as a sequel to [**32**], where all distinguished supercuspidal R-representations of G have been classified, we investigate the classification of distinguished cuspidal R-representations of G in terms of their supercuspidal support. We:

– reduce this classification to that of distinguished cuspidal representations of level 0, and from there to finite group theory (see Section 4),

– give a necessary condition of distinction for σ-selfdual cuspidal representations of G that satisfy a certain parity condition (see Section 5),

– classify the (distinguished, cuspidal) $\overline{\mathbb{F}}_\ell$-representations of G having a distinguished cuspidal lift to $\overline{\mathbb{Q}}_\ell$, where $\overline{\mathbb{Q}}_\ell$ is an algebraic closure of the field of ℓ-adic numbers with residue field $\overline{\mathbb{F}}_\ell$.

Let us explain these results in more detail.

1.4. Bushnell and Kutzko [**8**], in work extended to the modular setting by Vignéras [**37**], have given an explicit construction of a collection of pairs $(\mathbf{J}, \boldsymbol{\lambda})$ called *extended maximal simple types* (which we will abbreviate to *types* here), consisting of a compact-mod-centre open subgroup $\mathbf{J}$ of G and an irreducible R-representation $\boldsymbol{\lambda}$ of $\mathbf{J}$, such that the representations $\mathrm{ind}_{\mathbf{J}}^G(\boldsymbol{\lambda})$ are (irreducible and) cuspidal, and such that every cuspidal R-representation of G appears in the collection of $\mathrm{ind}_{\mathbf{J}}^G(\boldsymbol{\lambda})$.

We need the following invariants associated to a cuspidal R-representation of G following this explicit construction by compact induction (see §4.2 and §4.7):

(1) the *endo-class* $\boldsymbol{\Theta}$: a fine refinement of the level introduced by Bushnell-Henniart in [**5**] and which applies equally well to the modular setting,

(2) the *tame parameter field* T: a tamely ramified extension of F of degree dividing n, uniquely determined up to F-isomorphism by $\mathbf{\Theta}$,

(3) the *relative degree* m: a positive integer such that $m[T:F]$ divides n, uniquely determined by $\mathbf{\Theta}$ and n.

Suppose further that $\mathbf{\Theta}$ is σ-selfdual (which follows if for example the cuspidal representation itself is σ-selfdual), then there is a uniquely determined tamely ramified extension T_0 of F_0 contained in T such that T is isomorphic to $T_0 \otimes_{F_0} F$. The Galois group of T/T_0 canonically identifies with that of F/F_0, and the unique non-trivial automorphism of T/T_0 extending σ will be denoted by σ (see §4.3). Our main theorem on reduction to the level 0 setting is then (see Theorem 4.42):

Theorem 1.1. — (1) *There is a natural bijection:*

$$\pi \mapsto \pi_{\mathrm{t}} \tag{1.1}$$

from the set of isomorphism classes of cuspidal representations of G with endo-class $\mathbf{\Theta}$ to the set of isomorphism classes of cuspidal representations of level 0 *of* $\mathrm{GL}_m(T)$.

(2) *The representation π is σ-selfdual if and only if π_{t} is σ-selfdual.*

(3) *The representation π is $\mathrm{GL}_n(F_0)$-distinguished if and only if π_{t} is $\mathrm{GL}_m(T_0)$-distinguished.*

The map (1.1) is also compatible with supercuspidal support, see Proposition 4.44 for a precise statement.

1.5. Let us briefly explain how the map (1.1) above is defined. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a type inducing a cuspidal representation π of G with σ-selfdual endo-class $\mathbf{\Theta}$, tame parameter field T and relative degree m. Then:

(1) The group $\mathbf{J}$ has a unique maximal compact subgroup $\mathbf{J}^0$, and a unique maximal normal pro-p subgroup $\mathbf{J}^1$.

(2) There is a group isomorphism $\mathbf{J}^0/\mathbf{J}^1 \simeq \mathrm{GL}_m(\boldsymbol{l})$, where $\boldsymbol{l}$ is the residue field of T.

(3) The restriction of $\boldsymbol{\lambda}$ to $\mathbf{J}^1$ is isotypic for an irreducible representation η of $\mathbf{J}^1$, and this representation η extends (non-canonically) to $\mathbf{J}$.

(4) The choice of a representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η determines a decomposition $\boldsymbol{\lambda} \simeq \boldsymbol{\kappa} \otimes \boldsymbol{\tau}$, where $\boldsymbol{\tau}$ is a representation of $\mathbf{J}$ trivial on $\mathbf{J}^1$, uniquely determined up to isomorphism.

The fact that $\mathbf{\Theta}$ is σ-selfdual implies that there is a preferred choice for $(\mathbf{J}, \boldsymbol{\lambda})$: the group $\mathbf{J}$ is fixed by σ, the representation η is σ-selfdual and there exists a natural isomorphism between the space of G^σ-invariant linear forms on π and that of $\mathbf{J} \cap G^\sigma$-invariant linear forms on $\boldsymbol{\lambda}$. Such a type is called *generic* (see Definition 4.32). We prove (see Proposition 4.17):

Proposition 1.2. — *The representation η has a unique extension $\boldsymbol{\kappa}$ to $\mathbf{J}$ which is both σ-selfdual and $\mathbf{J} \cap G^\sigma$-distinguished, and whose determinant has order a power of p.*

The choice of the representation $\boldsymbol{\kappa}$ given by Proposition 1.2 thus uniquely determines a representation $\boldsymbol{\tau}$ of $\mathbf{J}$ trivial on $\mathbf{J}^1$.

Now there is a natural choice, as explained in §4.10, of a σ-fixed maximal compact subgroup $\mathbf{J}_{\mathrm{t}}^0$ of $\mathrm{GL}_m(T)$, with normalizer $\mathbf{J}_{\mathrm{t}}$ and pro-p-radical $\mathbf{J}_{\mathrm{t}}^1$, such that there is a σ-equivariant group isomorphism:

$$\mathbf{J}/\mathbf{J}^1 \simeq \mathbf{J}_{\mathrm{t}}/\mathbf{J}_{\mathrm{t}}^1.$$

The representation $\boldsymbol{\tau}$ then defines a representation of $\mathbf{J}_{\mathrm{t}}$ trivial on $\mathbf{J}_{\mathrm{t}}^1$, denoted $\boldsymbol{\tau}_{\mathrm{t}}$. The cuspidal representation π_{t} associated with π by (1.1) is then the compact induction of $\boldsymbol{\tau}_{\mathrm{t}}$ to $\mathrm{GL}_m(T)$.

1.6. Having reduced the classification of distinguished cuspidal R-representations to level 0, we further reduce this classification to the finite group setting. Let π be a σ-selfdual cuspidal R-representation of G of level 0 with central character c_π and generic type $(\mathbf{J}, \boldsymbol{\lambda})$. Restricting $\boldsymbol{\lambda}$ to $\mathbf{J}^0$ defines a cuspidal R-representation V of $\mathrm{GL}_n(\boldsymbol{k})$, where $\boldsymbol{k}$ is the residue field of F. We prove (see Theorem 4.46):

Theorem 1.3. — *Suppose $n \neq 1$. The representation π is $\mathrm{GL}_n(F_0)$-distinguished if and only if its central character c_π is trivial on $F_0^\times$ and*

(1) *if F/F_0 is unramified, then V is $\mathrm{GL}_n(\boldsymbol{k}_0)$-distinguished ($\boldsymbol{k}_0$ is the residue field of F_0);*

(2) *if F/F_0 is ramified, then n is even, V is $\mathrm{GL}_{n/2}(\boldsymbol{k}) \times \mathrm{GL}_{n/2}(\boldsymbol{k})$-distinguished, the vector space of $\mathrm{GL}_{n/2}(\boldsymbol{k}) \times \mathrm{GL}_{n/2}(\boldsymbol{k})$-invariant linear forms on V has dimension 1, and*

$$s = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \in \mathrm{GL}_n(\boldsymbol{k})$$

acts on this space by the sign $c_\pi(\varpi)$, where ϖ is any uniformizer of F.

1.7. Let π be a cuspidal non-supercuspidal R-representation of G. Following [**26**], we recall in §3.4 that there are a uniquely determined integer $r = r(\pi) \geqslant 2$ and a supercuspidal R-representation ρ of $\mathrm{GL}_{n/r}(F)$ such that π is isomorphic to $\mathrm{St}_r(\rho)$, where $\mathrm{St}_r(\rho)$ denotes the unique generic subquotient of the parabolically induced representation

$$\rho\nu^{-(r-1)/2} \times \cdots \times \rho\nu^{(r-1)/2}$$

(where ν denote the unramified character which is the absolute value of F composed with the determinant). The representation ρ is not unique in general, but, if π is σ-selfdual and r is odd, and if one further demands that ρ be σ-selfdual, then ρ is uniquely determined up to isomorphism (see Proposition 3.8). In this case, we obtain further necessary conditions for distinction (see Theorem 5.1):

Theorem 1.4. — *Let π be a σ-selfdual cuspidal non-supercuspidal R-representation of $\mathrm{GL}_n(F)$. Assume that the integer $r = r(\pi)$ is odd, thus π is isomorphic to $\mathrm{St}_r(\rho)$ for a uniquely determined σ-selfdual supercuspidal representation ρ of $\mathrm{GL}_{n/r}(F)$. If π is $\mathrm{GL}_n(F_0)$-distinguished, then*

(1) *the relative degree $m = m(\pi)$ and the ramification index of T/T_0 have the same parity,*

(2) *the representation ρ is $\mathrm{GL}_{n/r}(F_0)$-distinguished.*

As a corollary, we extend the Disjunction Theorem from the supercuspidal setting (that is, the statement that, if $\ell \neq 2$, a supercuspidal R-representation is not both distinguished and $\varkappa$-distinguished) to include cuspidal R-representations π with $r(\pi)$ odd.

1.8. Say that an irreducible $\overline{\mathbb{F}}_\ell$-representation π of G *lifts to* $\overline{\mathbb{Q}}_\ell$ if there exists a free $\overline{\mathbb{Z}}_\ell$-lattice L equipped with a linear action of G such that the $\overline{\mathbb{F}}_\ell$-representation of G on $L \otimes \overline{\mathbb{F}}_\ell$ is isomorphic to π. When this is the case, say that the smooth $\overline{\mathbb{Q}}_\ell$-representation of G on $L \otimes \overline{\mathbb{Q}}_\ell$ is a *lift* of π to $\overline{\mathbb{Q}}_\ell$. Following [**37**], any cuspidal $\overline{\mathbb{F}}_\ell$-representation of G lifts to $\overline{\mathbb{Q}}_\ell$ and any of its lifts is cuspidal.

According to [**22**] (see Theorem 3.3), any cuspidal $\overline{\mathbb{F}}_\ell$-representation of G having a G^σ-distinguished lift to $\overline{\mathbb{Q}}_\ell$ is G^σ-distinguished. The converse holds for supercuspidal representations (see [**32**] and [**11**]), but not for cuspidal representations in general. In the final section, we classify the G^σ-distinguished cuspidal $\overline{\mathbb{F}}_\ell$-representations of G having a G^σ-distinguished cuspidal lift to $\overline{\mathbb{Q}}_\ell$ (see Propositions 6.1 and 6.2 for a precise statement).

Structure of the paper

After setting some notation in Section 2, in Section 3 we collect together necessary background from the literature and prove some basic results on σ-selfdual cuspidal R-representations.

Section 4 constitutes the technical heart of the paper. It reduces the problem of classifying distinguished cuspidal R-representations to level 0.

In Section 5, under a parity condition, we provide necessary conditions for distinction, allowing us to deduce the Disjunction Theorem and a lifting theorem.

Finally, in Section 6, we classify those cuspidal $\overline{\mathbb{F}}_\ell$-representations having a distinguished cuspidal lift.

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2. Notation

2.1. Given any non-archimedean locally compact field F, we write $\mathcal{O}_F$ for its ring of integers, $\mathfrak{p}_F$ for the maximal ideal of $\mathcal{O}_F$, $\boldsymbol{k}_F$ for its residue field and q_F for the cardinality of $\boldsymbol{k}_F$.

We also write val_F for the valuation of F taking any uniformizer to 1, and $|\cdot|_F$ for the absolute value of F taking any uniformizer to the inverse of q_F.

Given any finite extension L of K, we write $\mathrm{N}_{L/K}$ and $\mathrm{tr}_{L/K}$ for the norm and trace maps.

2.2. Given a locally compact, totally disconnected topological group G and an algebraically closed field R of characteristic different from p, we consider smooth representations of G on R-vector spaces. We will abbreviate *smooth R-representation* to *R-representation*, or even *representation* if the coefficient field R is clear from the context.

An *R-character* (or *character*) of G is a group homomorphism from G to $R^\times$ with open kernel.

Let π be a representation of G. We write $\pi^\vee$ for its contragredient. Given a character χ of G, we write $\pi\chi$ for the representation $g \mapsto \chi(g)\pi(g)$ of G.

Let π be a representation of a closed subgroup H of G. Given any element $g \in G$, we write π^g for the representation $x \mapsto \pi(gxg^{-1})$ of $H^g = g^{-1}Hg$. Given any continuous involution σ of G, we write π^σ for the representation $\pi \circ \sigma$ of $\sigma(H)$. Given any character μ of $H \cap G^\sigma$, we say that π is *μ-distinguished* if the space $\mathrm{Hom}_{H\cap G^\sigma}(\pi, \chi)$ is non-zero. If μ is the trivial character, we will abbreviate *μ-distinguished* to *$H \cap G^\sigma$-distinguished*, or just *distinguished*.

2.3. Let us fix a separable quadratic extension F/F_0 of non-archimedean locally compact fields of residual characteristic p, and let σ denote its non-trivial automorphism. Let

$$\varkappa = \varkappa_{F/F_0} : F_0^\times \to \{-1, 1\} = \mathbb{Z}^\times \tag{2.1}$$

denote the $\mathbb{Z}$-valued character of $F_0^\times$ with kernel $\mathrm{N}_{F/F_0}(F^\times)$. When needed, we will consider $\varkappa$ as a character with values in any algebraically closed field R. We abbreviate $q = q_F$ and $q_0 = q_{F_0}$.

We fix a square root

$$q_0^{1/2} \in R \tag{2.2}$$

of q_0 in R an define

$$q^{1/2} = \begin{cases} q_0^{1/2} & \text{if } F/F_0 \text{ is ramified,} \\ q_0 & \text{if } F/F_0 \text{ is unramified,} \end{cases} \tag{2.3}$$

which we will use to normalize parabolic induction and restriction functors (see below).

2.4. Given a positive integer $n \geqslant 1$, the automorphism σ acts on the group $\mathrm{GL}_n(F)$ componentwise, thus defines a continuous involution of $\mathrm{GL}_n(F)$, still denoted σ. Its fixed points form the subgroup $\mathrm{GL}_n(F_0)$.

We denote by ν the unramified character "absolute value of the determinant" of $\mathrm{GL}_n(F)$ and by $\nu^{1/2}$ the unramified character taking any element whose determinant has valuation 1 to $q^{-1/2}$. We thus have $(\nu^{1/2})^2 = \nu$. Similarly, we define the characters ν_0 and $\nu_0^{1/2}$ of $\mathrm{GL}_n(F_0)$.

Given positive integers $n_1, \dots, n_r$ such that $n_1 + \cdots + n_r = n$ and, for each $i = 1, \dots, r$, given an R-representation π_i of $\mathrm{GL}_{n_i}(F)$, we write

$$\pi_1 \times \cdots \times \pi_r \tag{2.4}$$

for the representation of $\mathrm{GL}_n(F)$ obtained by normalized parabolic induction from $\pi_1 \otimes \cdots \otimes \pi_r$ along the parabolic subgroup generated by upper triangular matrices and the standard Levi subgroup $\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$.

An irreducible R-representation of $\mathrm{GL}_n(F)$ is said to be *cuspidal* (respectively, *supercuspidal*) if it does not occur as a subrepresentation (respectively, a subquotient) of any representation of the form (2.4) with $r \geqslant 2$. Any supercuspidal representation of $\mathrm{GL}_n(F)$ is cuspidal. When R has characteristic 0, any cuspidal representation of $\mathrm{GL}_n(F)$ is supercuspidal. When R has characteristic $\ell > 0$, the group $\mathrm{GL}_n(F)$ may have cuspidal non-supercuspidal representations (see §3.4).

Given a representation π of $\mathrm{GL}_n(F)$ and a character χ of $F^\times$, we will write $\pi\chi$ for $\pi(\chi \circ \det)$.

2.5. Let us fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field of ℓ-adic numbers. Let $\overline{\mathbb{Z}}_\ell$ denote its ring of integers, and $\overline{\mathbb{F}}_\ell$ denote the residue field of $\overline{\mathbb{Z}}_\ell$.

We call an irreducible representation π of a locally compact, totally disconnected group G on a $\overline{\mathbb{Q}}_\ell$-vector space V *integral* if it stabilizes a $\overline{\mathbb{Z}}_\ell$-lattice L in V. In this case, we obtain a smooth $\overline{\mathbb{F}}_\ell$-representation $L \otimes \overline{\mathbb{F}}_\ell$ of G whose isomorphism class may depend on the choice of L.

If G is either the group of rational points of a connected reductive linear algebraic F-group or a finite group (see [**38**, Theorem 1] and the Brauer–Nesbitt principle), the smooth $\overline{\mathbb{F}}_\ell$-representation $L \otimes \overline{\mathbb{F}}_\ell$ has finite length, and its semisimplification is independent of the choice of L. This semisimplification is called the *reduction modulo* ℓ of π, and is denoted by $\mathbf{r}_\ell(\pi)$.

Given an irreducible $\overline{\mathbb{F}}_\ell$-representation ρ of G, we call an irreducible integral $\overline{\mathbb{Q}}_\ell$-representation with reduction modulo ℓ equal to ρ a $\overline{\mathbb{Q}}_\ell$-*lift* of ρ.

3. Basic results

In this section, p is an arbitrary prime number, F/F_0 is a separable quadratic extension and R has characteristic $\ell \neq p$. We fix a positive integer $n \geqslant 1$.

3.1. Fundamental results of Flicker and Prasad [**14, 29, 30**] on irreducible complex representations of $\mathrm{GL}_n(F)$ distinguished by $\mathrm{GL}_n(F_0)$ have been extended to irreducible R-representations in [**32**] Theorem 4.1.

Theorem 3.1. — *Let π be an irreducible representation of $\mathrm{GL}_n(F)$ distinguished by $\mathrm{GL}_n(F_0)$.*

(1) *The central character c_π of π is trivial on $F_0^\times$.*
(2) *The R-vector space $\mathrm{Hom}_{\mathrm{GL}_n(F_0)}(\pi, R)$ has dimension 1.*
(3) *The contragredient $\pi^\vee$ of π is isomorphic to π^σ.*

We will say that a representation π of $\mathrm{GL}_n(F)$ is *σ-selfdual* if $\pi^\vee$ is isomorphic to π^σ.

3.2. For supercuspidal representations, we have the following Dichotomy and Disjunction Theorem ([**21**] Theorem 4, [**2**] Corollary 1.6 if $\ell = 0$, [**32**] Theorem 10.8 if $p \neq 2$ and [**11**] Theorem 3.14 if $\ell \neq 0, 2$).

Theorem 3.2. — *Let ρ be a σ-selfdual supercuspidal R-representation of $\mathrm{GL}_n(F)$.*

(1) *If $\ell = 2$, then ρ is distinguished.*
(2) *If $\ell \neq 2$, then ρ is either distinguished or $\varkappa$-distinguished, but not both.*

3.3. In this paragraph, ℓ is a prime number different from p and we will consider representations with coefficients in $\overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{F}}_\ell$. The following theorem is [**22**] Theorem 3.4.

Theorem 3.3. — *Let π be an integral σ-selfdual cuspidal $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_n(F)$. If π is distinguished by $\mathrm{GL}_n(F_0)$, then its reduction mod ℓ is (irreducible, cuspidal and) distinguished.*

It follows that any σ-selfdual cuspidal $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_n(F)$ having a distinguished lift to $\overline{\mathbb{Q}}_\ell$ is distinguished. For supercuspidal representations, one has the following converse (see [**32**] Theorem 10.11 if $p \neq 2$, and [**11**] Theorem 3.4):

Theorem 3.4. — *Any $\mathrm{GL}_n(F_0)$-distinguished supercuspidal $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_n(F)$ has a $\mathrm{GL}_n(F_0)$-distinguished lift to $\overline{\mathbb{Q}}_\ell$.*

We also have the following Distinguished Lift Theorem, making Theorem 3.4 more precise.

Theorem 3.5. — *Let ρ be a σ-selfdual supercuspidal $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_n(F)$.*

(1) *The representation ρ has a σ-selfdual lift to $\overline{\mathbb{Q}}_\ell$.*
(2) *Let μ be a σ-selfdual lift of ρ to $\overline{\mathbb{Q}}_\ell$ and suppose that $\ell \neq 2$. Then μ is distinguished if and only ρ is distinguished.*

Proof. — If $p \neq 2$, this is [**32**] Theorem 10.11. Assume now that $p = 2$, thus $\ell \neq 2$.

By Theorem 3.2, the representation ρ is either distinguished or $\varkappa$-distinguished. If it is distinguished, it has a σ-selfdual lift thanks to Theorem 3.4 and Theorem 3.1(3). If it is $\varkappa$-distinguished, fix a $\overline{\mathbb{Q}}_\ell$-character ξ of $F^\times$ extending the canonical $\overline{\mathbb{Q}}_\ell$-lift of $\varkappa$. The reduction mod ℓ of ξ is an $\overline{\mathbb{F}}_\ell$-character of $F^\times$ extending $\varkappa$, denoted χ. The representation $\rho\chi$ is distinguished and supercuspidal. It thus has a σ-selfdual lift π. Then $\pi\xi^{-1}$ is a distinguished lift of ρ. This proves (1).

Let μ be a σ-selfdual lift of ρ, and assume that ρ is distinguished. If μ is not distinguished, it must then be $\varkappa$-distinguished. By Theorem 3.3, this implies that ρ is $\varkappa$-distinguished, which contradicts the Dichotomy and Disjunction Theorem. Conversely, if μ is distinguished, then ρ is distinguished thanks to Theorem 3.3. □

3.4. From now on, we consider the case of cuspidal non-supercuspidal R-representations, thus ℓ is a prime number different from p. Let us recall how they are classified in terms of their supercuspidal support.

Recall that a representation π of $\mathrm{GL}_n(F)$ on an R-vector space V is *generic* if V carries a non-zero R-linear form Λ such that $\Lambda(\pi(u)v) = \theta(u)v$ for all $v \in V$ and all unipotent upper triangular matrices u, where $\theta(u) = \psi(u_{1,2} + \cdots + u_{n-1,n})$ and ψ is a non-trivial R-character of F.

Let $k \geqslant 1$ be a positive integer, and ρ be a supercuspidal R-representation of $\mathrm{GL}_k(F)$. According to **[26]** 8.1, for any $r \geqslant 1$, the induced representation

$$\rho\nu^{-(r-1)/2} \times \cdots \times \rho\nu^{(r-1)/2} \tag{3.1}$$

contains a unique generic irreducible subquotient, denoted $\mathrm{St}_r(\rho)$.

Let $e(\rho)$ be the smallest integer $i \geqslant 1$ such that $\rho\nu^i$ is isomorphic to ρ and $t(\rho)$ be the torsion number of ρ, that is, the number of unramified characters χ of $F^\times$ such that $\rho\chi$ is isomorphic to ρ. By **[28]** Lemme 3.6, these integers are related by the identity

$$e(\rho) = \text{order of } q^{t(\rho)} \bmod \ell. \tag{3.2}$$

By **[26]** Théorème 6.14, one has the following classification.

Proposition 3.6. — *Let π be a cuspidal non-supercuspidal R-representation of $\mathrm{GL}_n(F)$.*

(1) *There are a unique positive integer $r = r(\pi) \geqslant 2$ dividing n and a supercuspidal representation ρ of $\mathrm{GL}_{n/r}(F)$ such that π is isomorphic to $\mathrm{St}_r(\rho)$.*

(2) *There is a unique integer $v \geqslant 0$ such that $r = e(\rho)\ell^v$.*

(3) *Let ρ' be a supercuspidal representation of $\mathrm{GL}_{n/r}(F)$. The representation π is isomorphic to $\mathrm{St}_r(\rho')$ if and only if ρ' is isomorphic to $\rho\nu^i$ for some $i \in \mathbb{Z}$.*

Note that, conversely, by the same references, if ρ is a supercuspidal representation of $\mathrm{GL}_k(F)$ and $r = e(\rho)\ell^v$ for some $v \geqslant 0$, the representation $\mathrm{St}_r(\rho)$ is cuspidal.

3.5. We now classify σ-selfdual cuspidal representations.

Lemma 3.7. — *Let ρ be a supercuspidal R-representation of $\mathrm{GL}_k(F)$ for some $k \geqslant 1$. Let $r \geqslant 2$ be such that $\mathrm{St}_r(\rho)$ is cuspidal, and suppose that $\mathrm{St}_r(\rho)$ is σ-selfdual. Then there is an $i \in \mathbb{Z}$, uniquely determined mod $e(\rho)$, such that $\rho^{\vee\sigma}$ is isomorphic to $\rho\nu^i$.*

Proof. — The representation $\mathrm{St}_r(\rho)$ is σ-selfdual if and only if the representation

$$\mathrm{St}_r(\rho)^{\vee\sigma} \simeq \mathrm{St}_r(\rho^{\vee\sigma})$$

is isomorphic to $\mathrm{St}_r(\rho)$. The result then follows from Proposition 3.6. $\square$

Proposition 3.8. — *Let π be a cuspidal σ-selfdual representation of $\mathrm{GL}_n(F)$. Set $r = r(\pi)$ and write $k = n/r$.*

(1) *If r is odd or $\ell = 2$, there is a unique σ-selfdual supercuspidal representation ρ of $\mathrm{GL}_k(F)$ such that π is isomorphic to $\mathrm{St}_r(\rho)$.*

(2) *Suppose that r is even and $\ell \neq 2$.*

(a) *There are a supercuspidal representation ρ of $\mathrm{GL}_k(F)$ and an $i \in \{0,1\}$ such that π is isomorphic to $\mathrm{St}_r(\rho)$ and $\rho^{\vee\sigma} \simeq \rho\nu^i$.*

(b) *Let ρ' be a supercuspidal representation of $\mathrm{GL}_k(F)$ and $j \in \{0,1\}$ such that π is isomorphic to $\mathrm{St}_r(\rho')$ and $\rho'^{\vee\sigma} \simeq \rho'\nu^j$. Then $j = i$, and either $\rho' \simeq \rho$ or $\rho' \simeq \rho\nu^{r/2}$.*

Proof. — If $r = 1$, the result is trivial. Let us assume that $r \geqslant 2$. Fix a supercuspidal irreducible representation ρ of $\mathrm{GL}_k(F)$ such that π is isomorphic to $\mathrm{St}_r(\rho)$. By Lemma 3.7, there is an $i \in \mathbb{Z}$ such that $\rho^{\vee\sigma} \simeq \rho\nu^i$. Changing ρ to $\rho' = \rho\nu^s$ for some $s \in \mathbb{Z}$ does not change $\mathrm{St}_r(\rho)$, but changes i to $i - 2s$. If r is odd or $\ell = 2$, then $e(\rho)$ is odd, thus $2\mathbb{Z} + e(\rho)\mathbb{Z} = \mathbb{Z}$. This proves (1). Similarly, if r is even and $\ell \neq 2$, then $e(\rho)$ is even: we thus may assume that $i \in \{0, 1\}$, proving (2.a). Moreover, if ρ' and j are as in (2.b), then $j - i$ is even, thus $j = i$. Moreover, ρ' is isomorphic to $\rho\nu^s$ for some $0 \leqslant s < e(\rho)$ such that $\nu^{2st(\rho)} = 1$, thus $e(\rho)$ divides $2s$. □

3.6. We will need the finite field analogue of 3.4 (see **[37]** III.2.5 or **[9]** Theorem 19.3).

Proposition 3.9. — *Let $\boldsymbol{k}$ be a finite field of characteristic p.*

(1) *Let $f \geqslant 1$ be a positive integer and ϱ be a supercuspidal representation of* $\mathrm{GL}_f(\boldsymbol{k})$.

(a) *For all $u \geqslant 1$, the induced representation*

$$\varrho \times \cdots \times \varrho \quad (u\ \textit{times})$$

has a unique generic irreducible subquotient, denoted $\mathrm{st}_u(\varrho)$.

(b) *Let $e(\varrho)$ be the order of q^f mod ℓ. The representation $\mathrm{st}_u(\varrho)$ is cuspidal if and only if $u = 1$ or $u = e(\varrho)\ell^v$ for some $v \geqslant 0$.*

(2) *Let W be a cuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$. There exist a unique integer $u = r(\mathrm{W}) \geqslant 1$ dividing n and a unique supercuspidal representation ϱ of $\mathrm{GL}_{n/u}(\boldsymbol{k})$ such that $\mathrm{W} \simeq \mathrm{st}_u(\varrho)$.*

3.7. As in the previous paragraph, $\boldsymbol{k}$ is a finite field of characteristic p. Let us recall how to parametrize cuspidal representations of $\mathrm{GL}_n(\boldsymbol{k})$ by regular characters (**[17]**, **[12]** Theorem 3.5 and **[13, 19]**).

Let $\overline{\boldsymbol{k}}$ be an algebraic closure of $\boldsymbol{k}$. For any integer $s \geqslant 1$, let $\boldsymbol{k}_s$ be the extension of $\boldsymbol{k}$ of degree s contained in $\overline{\boldsymbol{k}}$. Let Δ denote the group $\mathrm{Gal}(\boldsymbol{k}_n/\boldsymbol{k})$. A character of $\boldsymbol{k}_n^\times$ is Δ-regular if it is fixed by no non-trivial element of Δ.

Proposition 3.10. — (1) *Associated with any Δ-regular $\overline{\mathbb{Q}}_\ell$-character ξ of $\boldsymbol{k}_n^\times$, there is a cuspidal $\overline{\mathbb{Q}}_\ell$-representation W_ξ of $\mathrm{GL}_n(\boldsymbol{k})$, unique up to isomorphism, such that*

$$\mathrm{tr}\,\mathrm{W}_\xi(x) = (-1)^{n-1} \cdot \sum_{\delta \in \Delta} \xi(x^\delta)$$

for all $x \in \boldsymbol{k}_n^\times$ of degree n over $\boldsymbol{k}$, where $\boldsymbol{k}_n^\times$ is considered as a maximal torus in $\mathrm{GL}_n(\boldsymbol{k})$.

(2) *The correspondence*

$$\xi \mapsto \mathrm{W}_\xi$$

induces a bijection from the set of Δ-conjugacy classes of Δ-regular $\overline{\mathbb{Q}}_\ell$-characters of $\boldsymbol{k}_n^\times$ to that of isomorphism classes of cuspidal $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{GL}_n(\boldsymbol{k})$.

By reduction mod ℓ, we get the following classification.

Proposition 3.11. — (1) *Given any Δ-regular $\overline{\mathbb{Q}}_\ell$-character ξ of $\boldsymbol{k}_n^\times$, the reduction mod ℓ of W_ξ, denoted $\overline{\mathrm{W}}_\xi$, is irreducible and cuspidal, and it only depends on the reduction mod ℓ of ξ.*

(2) *Reduction mod ℓ induces a bijection from the set of Δ-conjugacy classes of $\overline{\mathbb{F}}_\ell$-characters of $\boldsymbol{k}_n^\times$ having a Δ-regular lift to $\overline{\mathbb{Q}}_\ell$ to that of isomorphism classes of cuspidal $\overline{\mathbb{F}}_\ell$-representations of the group $\mathrm{GL}_n(\boldsymbol{k})$.*

(3) *The integer* $r(\overline{\mathrm{W}}_\xi)$ *is the greatest divisor* r *of* n *such that the reduction of* ξ *mod* ℓ *factorizes through a character of* $\boldsymbol{k}_{n/r}^\times$.

Definition 3.12. — A *parameter* of a cuspidal representation ρ of $\mathrm{GL}_n(\boldsymbol{k})$ is a character of $\boldsymbol{k}_n^\times$ whose Δ-conjugacy class corresponds to ρ by the bijection of either Proposition 3.10 or 3.11.

3.8. Finally, we will need the following distinction criterion for cuspidal $\overline{\mathbb{Q}}_\ell$-representations (see [**18**] Proposition 6.1 and [**10**] Lemme 3.4.10) of $\mathrm{GL}_n(\boldsymbol{k})$ when p is odd.

Proposition 3.13. — *Assume that* q *is odd,* n *is even and write* $n = 2u$. *We consider the group* $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$ *as a Levi subgroup of* $\mathrm{GL}_n(\boldsymbol{k})$. *Let* ξ *be a* Δ*-regular* $\overline{\mathbb{Q}}_\ell$*-character of* $\boldsymbol{k}_n^\times$.

(1) *The following assertions are equivalent.*

  (a) *The cuspidal* $\overline{\mathbb{Q}}_\ell$*-representation* W_ξ *is* $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$*-distinguished.*

  (b) *The space of* $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$*-invariant linear forms on* W_ξ *has* $\overline{\mathbb{Q}}_\ell$*-dimension* 1.

  (c) *The cuspidal* $\overline{\mathbb{Q}}_\ell$*-representation* W_ξ *is selfdual.*

  (d) *The character* ξ *is trivial on* $\boldsymbol{k}_u^\times$.

(2) *Assume that the conditions of* (1) *are satisfied, and fix an element* $\alpha \in \boldsymbol{k}_n^\times$ *such that* $\alpha \notin \boldsymbol{k}_u^\times$ *and* $\alpha^2 \in \boldsymbol{k}_u^\times$. *The element*

$$s = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \in \mathrm{GL}_n(\boldsymbol{k}), \tag{3.3}$$

where id *is the identity in* $\mathrm{GL}_u(\boldsymbol{k})$*, normalizes the group* $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$ *and acts on the* $\overline{\mathbb{Q}}_\ell$*-vector space of* $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$*-invariant linear forms on* W_ξ *by the sign* $-\xi(\alpha)$.

Remark 3.14. — Suppose that W_ξ is $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$-distinguished. By [**32**] Lemma 2.6, the cuspidal $\overline{\mathbb{F}}_\ell$-representation $\overline{\mathrm{W}}_\xi$ is $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$-distinguished as well. More precisely, if we fix a non-zero $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$-invariant $\overline{\mathbb{Q}}_\ell$-linear form Λ on W_ξ together with a $\mathrm{GL}_n(\boldsymbol{k})$-stable $\overline{\mathbb{Z}}_\ell$-lattice $L \subseteq \mathrm{W}_\xi$, then the associated $\overline{\mathbb{F}}_\ell$-linear form

$$\overline{\Lambda} : L \otimes \overline{\mathbb{F}}_\ell \to \Lambda(L) \otimes \overline{\mathbb{F}}_\ell$$

is non-zero and $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$-invariant. Moreover, if s acts on Λ by a sign $c \in \{-1, 1\} \subseteq \overline{\mathbb{Q}}_\ell^\times$, then s acts on $\overline{\Lambda}$ by the image of c in $\overline{\mathbb{F}}_\ell^\times$.

4. Reduction to level zero

In this section, p is odd, ℓ is any prime number different from p and R has characteristic 0 or ℓ. Let us fix a positive integer $n \geqslant 1$, and set $G = \mathrm{GL}_n(F)$. We fix a character

$$\psi : F \to R^\times \tag{4.1}$$

which is trivial on $\mathfrak{p}_F$ but not on $\mathcal{O}_F$.

4.1. Let us recall the definitions and main results of [**8, 7, 27, 3**] which we will need.

Let $[\mathfrak{a},\beta]$ be a simple stratum in the algebra $\mathbf{M}_n(F)$ of $n\times n$ matrices with entries in F. Recall that $\mathfrak{a}$ is a hereditary $\mathcal{O}_F$-order of $\mathbf{M}_n(F)$ and β is an element of $\mathbf{M}_n(F)$ such that

– the F-algebra $E = F[\beta]$ is a field, and
– the multiplicative group $E^\times$ normalizes $\mathfrak{a}$

(plus an extra technical condition on β which is not necessary to recall here: see [**8**] 1.5.5).

Let $\mathcal{K}_\mathfrak{a}$ be the normalizer of $\mathfrak{a}$ in G and $\mathfrak{p}_\mathfrak{a}$ be its Jacobson radical, and set $\mathrm{U}^1_\mathfrak{a} = 1+\mathfrak{p}_\mathfrak{a}$. Let B be the centralizer of E in $\mathbf{M}_n(F)$. The intersection $\mathfrak{b} = \mathfrak{a}\cap B$ is a hereditary order in B.

Associated with $[\mathfrak{a},\beta]$ in [**8**] Chapter 3, there are compact mod centre open subgroups

$$H^1(\mathfrak{a},\beta)\subseteq \mathbf{J}^1(\mathfrak{a},\beta)\subseteq \mathbf{J}^0(\mathfrak{a},\beta)\subseteq \mathbf{J}(\mathfrak{a},\beta)\subseteq \mathcal{K}_\mathfrak{a}$$

and a non-empty finite set $\mathcal{C}(\mathfrak{a},\beta)$ of characters of $H^1(\mathfrak{a},\beta)$ called *simple characters*, depending on the choice of (4.1). We write $\mathbf{J} = \mathbf{J}(\mathfrak{a},\beta)$, $\mathbf{J}^0 = \mathbf{J}^0(\mathfrak{a},\beta)$, $\mathbf{J}^1 = \mathbf{J}^1(\mathfrak{a},\beta)$ and $H^1 = H^1(\mathfrak{a},\beta)$ for simplicity.

We will only be interested in the case where $\mathfrak{b}$ is a maximal order in B, in which case the simple stratum $[\mathfrak{a},\beta]$ and the simple characters in $\mathcal{C}(\mathfrak{a},\beta)$ are said to be *maximal*. For the following result, see [**7**] 2.1, 3.2 and [**8**] 5.1.1.

Proposition 4.1. — *Let $[\mathfrak{a},\beta]$ be a maximal simple stratum.*

(1) *The group $\mathbf{J}^0$ is the unique maximal compact subgroup of $\mathbf{J}$, and $\mathbf{J}^1$ is its unique maximal normal pro-p-subgroup.*

(2) *One has $\mathbf{J} = E^\times\mathbf{J}^0 = (\mathbf{J}\cap B^\times)\mathbf{J}^1$ and*

$$\mathbf{J}\cap B^\times = \mathcal{K}_\mathfrak{b},\quad \mathbf{J}^0\cap B^\times = \mathfrak{b}^\times,\quad \mathbf{J}^1\cap B^\times = \mathrm{U}^1_\mathfrak{b}. \tag{4.2}$$

(3) *There is an isomorphism of E-algebras*

$$B\simeq \mathbf{M}_m(E),\quad m = n/[E:F], \tag{4.3}$$

sending $\mathfrak{b}^\times$ to the maximal compact open subgroup $\mathrm{GL}_m(\mathcal{O}_E)$, which induces isomorphisms

$$\mathbf{J}^0/\mathbf{J}^1\simeq \mathfrak{b}^\times/\mathrm{U}^1_\mathfrak{b}\simeq \mathrm{GL}_m(\boldsymbol{l}) \tag{4.4}$$

where $\boldsymbol{l}$ is the residue field of E.

(4) *Given any simple character $\theta\in\mathcal{C}(\mathfrak{a},\beta)$, we have*

(a) *the normalizer of θ in G is equal to $\mathbf{J}$, and*

(b) *there is an irreducible representation η of $\mathbf{J}^1$, unique up to isomorphism, whose restriction to H^1 contains θ, and such a representation extends to $\mathbf{J}$.*

The representation η of (4.b) is called the *Heisenberg representation* associated with θ. If $\boldsymbol{\kappa}$ is a representation of $\mathbf{J}$ extending η, any other extension of η to $\mathbf{J}$ has the form $\boldsymbol{\kappa}\boldsymbol{\xi}$ for a unique character $\boldsymbol{\xi}$ of $\mathbf{J}$ trivial on $\mathbf{J}^1$.

Remark 4.2. — (1) An isomorphism as in Proposition 4.1(3) comes from the choice of an $\mathcal{O}_E$-basis of an $\mathcal{O}_E$-lattice $\mathcal{L}$ in F^n whose endomorphism algebra is $\mathfrak{b}$.

(2) Changing the isomorphism (4.3), that is, changing the basis of $\mathcal{L}$ above, has the effect of conjugating the identification (4.4) by an inner automorphism of $\mathrm{GL}_m(\boldsymbol{l})$.

A character θ of an open pro-p-subgroup H of G will be called a *maximal simple character* if there is a maximal simple stratum $[\mathfrak{a},\beta]$ in $\mathbf{M}_n(F)$ such that $H = H^1(\mathfrak{a},\beta)$ and $\theta\in\mathcal{C}(\mathfrak{a},\beta)$. Given a maximal simple character θ of G, we will write H^1_θ for the group on which θ is defined, $\mathbf{J}_\theta$

for its G-normalizer, $\mathbf{J}_\theta^0$ for its unique maximal compact subgroup, $\mathbf{J}_\theta^1$ for its unique maximal normal pro-p-subgroup and T for the maximal tamely ramified extension of F in $E = F[\beta]$. The following result shows how the latter depends on the choice of $[\mathfrak{a}, \beta]$ (see **[7]** 2.1, 2.5 and 2.6).

Proposition 4.3. — *Let $[\mathfrak{a}', \beta']$ be a simple stratum such that $\theta \in \mathcal{C}(\mathfrak{a}', \beta')$, and set $E' = F[\beta']$.*

(1) *The orders $\mathfrak{a}$, $\mathfrak{a}'$ are equal and E, E' have the same degree over F.*
(2) *The simple stratum $[\mathfrak{a}, \beta']$ is maximal.*
(3) *The maximal tamely ramified extension of F in E' is $\mathbf{J}_\theta^1$-conjugate to T.*

It follows that the G-conjugacy class of the simple character θ uniquely determines the integer m in (4.3), as well as the extension T up to G-conjugacy (or equivalently up to F-isomorphism). However, the fields E, E' need not be isomorphic (see **[7]** Example 2.1).

4.2. In this paragraph only, we let n vary among all positive integers, and consider the set

$$\mathcal{C}_{\max}(F) = \bigcup_{[\mathfrak{a},\beta]} \mathcal{C}(\mathfrak{a}, \beta)$$

where the union is taken over all maximal simple strata of $\mathbf{M}_n(F)$, for any $n \geqslant 1$. It is endowed with an equivalence relation called *endo-equivalence* (**[5, 6]**). An equivalence class for this equivalence relation is called an *endo-class* (see **[3]** 3.2).

Given a maximal simple character $\theta \in \mathcal{C}_{\max}(F)$ with endo-class $\boldsymbol{\Theta}$, the degree $[E : F]$ and the F-isomorphism class of its tame parameter field T only depend on $\boldsymbol{\Theta}$. They are called the *degree* and *tame parameter field* of $\boldsymbol{\Theta}$, respectively.

For a given n, any two maximal simple characters of $\mathrm{GL}_n(F)$ are endo-equivalent if and only if they are $\mathrm{GL}_n(F)$-conjugate.

Remark 4.4. — Note that endo-equivalence is defined in **[5, 6]** for arbitrary simple characters, not only for maximal ones, but we will not need this extra generality.

4.3. We go back to the situation of Paragraph 4.1, assuming further that the character ψ of (4.1) is σ-invariant, which is possible since $p \neq 2$. As in **[32]**, we will say that:

– a simple stratum $[\mathfrak{a}, \beta]$ in $\mathbf{M}_n(F)$ is *σ-selfdual* if $\mathfrak{a}$ is σ-stable and $\sigma(\beta) = -\beta$,
– a simple character θ is *σ-selfdual* if the group H_θ^1 is σ-stable and $\theta^{-1} \circ \sigma = \theta$,
– an endo-class $\boldsymbol{\Theta}$ of (maximal) simple characters is *σ-selfdual* if for some (or equivalently for any) $\theta \in \boldsymbol{\Theta}$, the character $\theta^{-1} \circ \sigma$ is in $\boldsymbol{\Theta}$.

Proposition 4.5. — *Let θ be a σ-selfdual maximal simple character.*

(1) *There is a σ-stable simple stratum $[\mathfrak{a}, \beta]$ such that $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$.*
(2) *Let E_0 be the σ-fixed points of E and $\boldsymbol{l}_0$ be its residue field. There exists an isomorphism (4.3) inducing an isomorphism (4.4) which transports the action of σ on $\mathbf{J}^0/\mathbf{J}^1$ to*
  (a) *the action of the non-trivial element of* $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$ *on* $\mathrm{GL}_m(\boldsymbol{l})$ *if E/E_0 is unramified,*
  (b) *the adjoint action of*

$$\begin{pmatrix} -\mathrm{id}_i & 0 \\ 0 & \mathrm{id}_{m-i} \end{pmatrix} \in \mathrm{GL}_m(\boldsymbol{l}) \tag{4.5}$$

on $\mathrm{GL}_m(\boldsymbol{l})$ *if E/E_0 is ramified, for a uniquely determined integer $i \in \{0, \dots, \lfloor m/2 \rfloor\}$.*

Remark 4.6. — Choose a basis of $\mathcal{L}$ as in Remark 4.2, with the additional condition that its vectors are σ-invariant. The isomorphism (4.3) associated with it then transports the action of σ on B to that of the generator of $\mathrm{Gal}(E/E_0)$ on $\mathbf{M}_m(E)$. When E/E_0 is ramified, this corresponds to the case where $i = 0$ in (4.5).

If θ is a σ-selfdual maximal simple character, we will write T_0 for the maximal tamely ramified extension of F_0 in E_0, that is, $T_0 = T \cap E_0$. By [**3**] Lemma 4.10, the canonical homomorphism

$$T_0 \otimes_{F_0} F \to T \tag{4.6}$$

is an isomorphism. Also, T/T_0 and E/E_0 have the same ramification index. By [**3**] Lemma 4.29, the F_0-isomorphism class of T_0 is uniquely determined by the endo-class $\boldsymbol{\Theta}$ of θ. And it follows from (4.6) that the F_0-isomorphism class of T_0 determines the F-isomorphism class of T.

The following result is given by [**32**] Proposition 6.12, Lemma 6.20 and [**33**] Lemme 3.28. (The latter reference in [**33**] is for representations with coefficients in $R = \mathbb{C}$, but its proof is still valid in the ℓ-modular case.)

Proposition 4.7. — *Let θ be a σ-selfdual maximal simple character.*

(1) *The Heisenberg representation η of θ is σ-selfdual and $\mathbf{J}^1 \cap G^\sigma$-distinguished, and the space $\mathrm{Hom}_{\mathbf{J}^1 \cap G^\sigma}(\eta, R)$ has dimension* 1.

(2) *For any representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η, there are*

(a) *a unique character $\boldsymbol{\xi}$ of $\mathbf{J}$ trivial on $\mathbf{J}^1$ such that $\boldsymbol{\kappa}^{\vee\sigma}$ is isomorphic to $\boldsymbol{\kappa\xi}$,*

(b) *a unique character χ of $\mathbf{J} \cap G^\sigma$ trivial on $\mathbf{J}^1 \cap G^\sigma$ such that*

$$\mathrm{Hom}_{\mathbf{J}^1 \cap G^\sigma}(\eta, R) = \mathrm{Hom}_{\mathbf{J} \cap G^\sigma}(\boldsymbol{\kappa}, \chi^{-1}), \tag{4.7}$$

and the restriction of $\boldsymbol{\xi}$ to $\mathbf{J} \cap G^\sigma$ is equal to χ^2.

(3) *Given a representation $\boldsymbol{\kappa}$ as in* (2) *and an irreducible representation $\boldsymbol{\tau}$ of $\mathbf{J}$ trivial on $\mathbf{J}^1$, the canonical linear map*

$$\mathrm{Hom}_{\mathbf{J}^1 \cap G^\sigma}(\eta, R) \otimes \mathrm{Hom}_{\mathbf{J} \cap G^\sigma}(\boldsymbol{\tau}, \chi) \to \mathrm{Hom}_{\mathbf{J} \cap G^\sigma}(\boldsymbol{\kappa} \otimes \boldsymbol{\tau}, R) \tag{4.8}$$

is an isomorphism of R-vector spaces.

(4) *There exists a σ-selfdual representation of $\mathbf{J}$ extending η.*

Conversely, let $\boldsymbol{\Theta}$ be a σ-selfdual endo-class of degree dividing n. By [**3**] Section 4, it contains a σ-selfdual maximal simple character θ in G, and we have the following classification.

Proposition 4.8. — *Let T/T_0 be the quadratic extension associated with $\boldsymbol{\Theta}$ as above, and let us write $m = n/\deg(\boldsymbol{\Theta})$.*

(1) *If T/T_0 is unramified, the G-conjugacy class of θ contains a unique G^σ-conjugacy class of σ-selfdual simple characters.*

(2) *If T/T_0 is ramified, the number of G^σ-conjugacy classes of σ-selfdual simple characters in the G-conjugacy class of θ is equal to $\lfloor m/2 \rfloor + 1$, each class corresponding bijectively to an integer $i \in \{0, \dots, \lfloor m/2 \rfloor\}$ characterized by Proposition 4.5(2.b).*

Remark 4.9. — When T/T_0 is ramified, we define (as in [**3, 32**]) the *index* of a σ-selfdual maximal simple character θ to be the integer $i \in \{0, \dots, \lfloor m/2 \rfloor\}$ associated with its G^σ-conjugacy class. By [**3**] Remark 4.28 or [**32**] 5.D, if θ has index 0 and if

$$y_i = \mathrm{diag}(\varpi, \dots, \varpi, 1, \dots, 1) \in \mathrm{GL}_m(E) \simeq B^\times, \quad \varpi \text{ a uniformizer of } E \text{ occurring } i \text{ times},$$

for some $i \in \{0, \dots, \lfloor m/2 \rfloor\}$, then θ^{y_i} is a σ-selfdual maximal simple character of index i.

4.4. Let θ be a maximal simple character, and $[\mathfrak{a},\beta]$ be a simple stratum such that $\theta\in\mathcal{C}(\mathfrak{a},\beta)$. As in 4.1, write $\mathbf{J}=\mathbf{J}_\theta$, $\mathbf{J}^0=\mathbf{J}^0_\theta$, $\mathbf{J}^1=\mathbf{J}^1_\theta$ and $H^1=H^1_\theta$. Let η be the Heisenberg representation of $\mathbf{J}^1$ associated with θ. In this paragraph, we give a slightly generalized version of [**7**] 3.3.

Fix an $\mathcal{O}_E$-lattice $\mathcal{L}$ in $V=F^n$ whose endomorphism algebra is $\mathfrak{b}$. (It is uniquely determined up to homothety as $\mathfrak{b}$ is maximal.) Fix a divisor $u\geqslant 1$ of m and decompositions

$$\mathcal{L}=\mathcal{L}_*\oplus\cdots\oplus\mathcal{L}_*,\quad V=V_*\oplus\cdots\oplus V_*, \tag{4.9}$$

in which V_* is an E-vector space of dimension m/u and $\mathcal{L}_*$ is an $\mathcal{O}_E$-lattice of rank m/u in V_*. It defines a Levi subgroup M of G. Fix a pair (Q_-,Q_+) of M-opposite parabolic subgroups of G with Levi component M, and write N_-, N_+ for their unipotent radicals. Define $\mathfrak{a}_*=\mathrm{End}_{\mathcal{O}_F}(\mathcal{L}_*)$. It is a hereditary order, and $[\mathfrak{a}_*,\beta]$ is a maximal simple stratum in $\mathrm{End}_F(V_*)$. Write $\mathbf{J}_*$, $\mathbf{J}^0_*$, $\mathbf{J}^1_*$ and H^1_* for the subgroups associated with it. Compare with [**7**] 3.3, which corresponds to the particular case where $u=m$.

The next result follows from [**5**] Example 10.9 (compare with Lemma 1 in [**7**] 3.3).

Lemma 4.10. — (1) *There are Iwahori decompositions*

$$\begin{aligned} H^1 &= (H^1\cap N_-)\cdot(H^1\cap M)\cdot(H^1\cap N_+),\\ H^1\cap M &= H^1_*\times\cdots\times H^1_* \end{aligned}$$

and

$$\begin{aligned} \mathbf{J}^1 &= (\mathbf{J}^1\cap N_-)\cdot(\mathbf{J}^1\cap M)\cdot(\mathbf{J}^1\cap N_+),\\ \mathbf{J}^1\cap M &= \mathbf{J}^1_*\times\cdots\times \mathbf{J}^1_*. \end{aligned}$$

(2) *The character θ is trivial on $H^1\cap N_-$, $H^1\cap N_+$ and there exists a unique simple character $\theta_*\in\mathcal{C}(\mathfrak{a}_*,\beta)$ such that θ agrees with $\theta_*\otimes\cdots\otimes\theta_*$ on $H^1\cap M$.*

Moreover, the map $\theta\mapsto\theta_*$ defined by Lemma 4.10(2) is a bijection from $\mathcal{C}(\mathfrak{a},\beta)$ to $\mathcal{C}(\mathfrak{a}_*,\beta)$: this is an instance of the transfer of [**8**] 3.6.

Let η_* denote the Heisenberg representation of $\mathbf{J}^1_*$ associated with θ_*. Compare the next result with Lemma 2 in [**7**] 3.3 and the discussion after it.

Lemma 4.11. — *Let $\boldsymbol{\kappa}_*$ be a representation of $\mathbf{J}_*$ extending η_*.*

(1) *The set $\mathbf{J}_+=(H^1\cap N_-)\cdot(\mathbf{J}\cap Q_+)$ is a group, and there is a unique representation $\boldsymbol{\kappa}_+$ of $\mathbf{J}_+$ which is trivial on $H^1\cap N_-$, $\mathbf{J}^0\cap N_+$ and agrees with $\boldsymbol{\kappa}_*\otimes\cdots\otimes\boldsymbol{\kappa}_*$ on $\mathbf{J}\cap M$.*

(2) *The representation $\widetilde{\boldsymbol{\kappa}}_+$ of $(\mathrm{J}^1\cap N_-)\cdot(\mathbf{J}\cap Q_+)=\mathbf{J}^1\mathbf{J}_+$ induced by $\boldsymbol{\kappa}_+$ extends η.*

(3) *There is a unique irreducible representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending $\widetilde{\boldsymbol{\kappa}}_+$.*

Proof. — That $\mathbf{J}_+$ is a group follows from the fact that H^1 is normalized by $\mathbf{J}$, thus by $\mathbf{J}\cap Q_+$. The existence of $\boldsymbol{\kappa}_+$ follows from the containment

$$(\mathbf{J}^0\cap N_+)\cdot(H^1\cap N_-)\subseteq(H^1\cap N_-)\cdot(\mathbf{J}^1\cap M)\cdot(\mathbf{J}^0\cap N_+)$$

(see the argument of [**31**] 2.3). Mackey's formula implies that the restriction of $\widetilde{\boldsymbol{\kappa}}_+$ to $\mathbf{J}^1$ is

$$\mathrm{Ind}_{\mathbf{J}^1\cap\mathbf{J}_+}^{\mathbf{J}^1}(\boldsymbol{\kappa}_+).$$

The restriction of $\boldsymbol{\kappa}_+$ to $\mathbf{J}^1\cap\mathbf{J}_+=(H^1\cap N_-)\cdot(\mathbf{J}^1\cap Q_+)$ is the unique representation η_+ which is trivial on $H^1\cap N_-$, $\mathbf{J}^1\cap N_+$ and agrees with $\eta_*\otimes\cdots\otimes\eta_*$ on $\mathbf{J}^1\cap M$. The representation it induces to $\mathbf{J}^1$ is isomorphic to η: indeed, this representation contains θ by Lemma 4.10(2), and it has dimension

$$\begin{aligned}\dim(\eta_*\otimes\cdots\otimes\eta_*)\cdot(\mathbf{J}^1\cap N_-:H^1\cap N_-) &= (\mathbf{J}^1\cap M:H^1\cap M)^{1/2}\cdot(\mathbf{J}^1\cap N_-:H^1\cap N_-)\\ &= (\mathbf{J}^1:H^1)^{1/2}\end{aligned}$$

which is the dimension of η (see [**27**] 2.3).

It remains to prove (3). First, uniqueness follows from the fact that any two such extensions differ from a character of $\mathbf{J}$ trivial on $\mathbf{J}^1\mathbf{J}_+$, and such a character is trivial since $p\neq 2$. Existence follows from [**8**] 5.2.4 (see [**27**] 2.4 in the modular case). □

The reader will pay attention to the fact that $\mathbf{J}\cap M$ is not equal to $\mathbf{J}_*\times\cdots\times\mathbf{J}_*$ in general (unless $u=1$), but is generated by $\mathbf{J}^0_*\times\cdots\times\mathbf{J}^0_*$ and $E^\times$ (the latter being diagonal in M).

Lemma 4.12. — (1) *The map*

$$\boldsymbol{\kappa}_*\mapsto\boldsymbol{\kappa} \tag{4.10}$$

from isomorphism classes of representations of $\mathbf{J}_*$ extending η_* to isomorphism classes of representation of $\mathbf{J}$ extending η is surjective.

(2) *Any two representations of $\mathbf{J}_*$ extending η_* have the same image if and only if they are twists of each other by a character of $\mathbf{J}_*$ trivial on $\mathbf{J}^0_*$ and of order dividing u.*

Proof. — The case where $u=m$ is given by Corollary 1 in [**7**] 3.3, and the general case follows by transitivity. □

Remark 4.13. — Suppose that u is equal to m. Let $y\in M\cap B^\times$ and write $\theta'=\theta^y\in\mathcal{C}(\mathfrak{a}^y,\beta)$. The groups associated with θ' are $\mathbf{J}'=\mathbf{J}^y$, etc. The group isomorphism $B^\times\simeq\mathrm{GL}_m(E)$ identifies $M\cap B^\times$ with the diagonal torus $E^\times\times\cdots\times E^\times$, and $E^\times$ normalizes θ_*. The character θ' is thus trivial on $H^{1y}\cap N_-$, $H^{1y}\cap N_+$ and agrees with $\theta_*\otimes\cdots\otimes\theta_*$ on $H^{1y}\cap M=H^1\cap M$. If $\boldsymbol{\kappa}_*$ is a representation of $\mathbf{J}_*$ extending η_*, the representation of $\mathbf{J}'$ corresponding to it by (4.10) is $\boldsymbol{\kappa}^y$.

4.5. Suppose now that the simple character θ and the simple stratum $[\mathfrak{a},\beta]$ of 4.4 are σ-selfdual. The groups $\mathbf{J}$, $\mathbf{J}^0$, $\mathbf{J}^1$ and H^1 are thus σ-stable. Suppose also that the decompositions (4.9) are σ-stable. The Levi subgroup M is thus σ-stable, and we may assume that Q_-, Q_+ are σ-stable as well. Also, the simple stratum $[\mathfrak{a}_*,\beta]$ and the simple character θ_* given by Lemma 4.10 are σ-selfdual. Let G_* denote the group $\mathrm{Aut}_F(V_*)$. It is isomorphic to $\mathrm{GL}_{n/u}(F)$.

We may also assume that our choice of basis induces an isomorphism of groups (4.4) between $\mathbf{J}^0/\mathbf{J}^1$ and $\mathrm{GL}_m(\boldsymbol{l})$ as in Proposition 4.5(4), transporting the action of σ on $\mathbf{J}^0/\mathbf{J}^1$ to

– the action of the non-trivial element of $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$ on $\mathrm{GL}_m(\boldsymbol{l})$ if T/T_0 is unramified,
– the adjoint action of (4.5) on $\mathrm{GL}_m(\boldsymbol{l})$ for some $i\in\{0,\dots,\lfloor m/2\rfloor\}$, if T/T_0 is ramified.

Let $\boldsymbol{\kappa}_*$ be a representation of $\mathbf{J}_*$ extending η_*, and let $\boldsymbol{\kappa}$ correspond to it by (4.10).

Lemma 4.14. — *If $\boldsymbol{\kappa}_*$ is σ-selfdual, then $\boldsymbol{\kappa}$ is σ-selfdual.*

Proof. — First, $\boldsymbol{\kappa}_+^{\vee\sigma}$ is trivial on both $H^1\cap N_-$, $\mathbf{J}^1\cap N_+$ and agrees with $\boldsymbol{\kappa}_*^{\vee\sigma}\otimes\cdots\otimes\boldsymbol{\kappa}_*^{\vee\sigma}$ on $\mathbf{J}\cap M$. If $\boldsymbol{\kappa}_*$ is σ-selfdual, it follows by uniqueness that $\boldsymbol{\kappa}_+^{\vee\sigma}$ is σ-selfdual, thus $\widetilde{\boldsymbol{\kappa}}_+^{\vee\sigma}$ is σ-selfdual as well. The unique representation of $\mathbf{J}$ extending $\widetilde{\boldsymbol{\kappa}}_+^{\vee\sigma}$ is $\boldsymbol{\kappa}^{\vee\sigma}$, hence $\boldsymbol{\kappa}$ is σ-selfdual. □

We will need the following lemma. Let ϖ be a uniformizer of E such that

$$\sigma(\varpi) = \begin{cases} \varpi & \text{if } T/T_0 \text{ is unramified,} \\ -\varpi & \text{if } T/T_0 \text{ is ramified.} \end{cases} \tag{4.11}$$

Note that the group $\mathbf{J}$ is generated by $\mathbf{J}^0$ and ϖ.

Lemma 4.15. — *The group $\mathbf{J} \cap G^\sigma$ is generated by $\mathbf{J}^0 \cap G^\sigma$ and an element ϖ' such that*

(1) *$\varpi' = \varpi$ if T/T_0 is unramified,*
(2) *$\varpi' = \varpi^2$ if T/T_0 is ramified and $m \neq 2i$,*
(3) *$\varpi' \in \varpi\mathbf{J}^0$ if T/T_0 is ramified and $m = 2i$.*

Proof. — If T/T_0 is unramified, see **[32]** Lemma 9.1. Suppose that T/T_0 is ramified, and assume that there is an $x \in \mathbf{J} \cap G^\sigma$, $x \notin \langle\varpi^2, \mathbf{J}^0 \cap G^\sigma\rangle$. We have $x = \varpi^v y$ where $v \in \mathbb{Z}$ is odd and $y \in \mathbf{J}^0$ satisfies $\sigma(y) = -y$. Reducing mod $\mathbf{J}^1$, we get an $\alpha \in \mathrm{GL}_m(\boldsymbol{l})$ such that $\sigma(\alpha) = -\alpha$. Since the involution σ acts on $\mathrm{GL}_m(\boldsymbol{l})$ by conjugacy by

$$\delta = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1)$$

(where -1 occurs i times), this implies that δ and $-\delta$ are $\mathrm{GL}_m(\boldsymbol{l})$-conjugate, thus $m = 2i$. Conversely, if $m = 2i$, then

$$w = \begin{pmatrix} 0 & \mathrm{id}_i \\ \mathrm{id}_i & 0 \end{pmatrix} \in \mathbf{J}^0 \cap B^\times = \mathrm{GL}_{2i}(\mathcal{O}_E) \tag{4.12}$$

is σ-anti-invariant, and $\varpi' = \varpi w$ has the required property. □

We now investigate the behavior of the map (4.10) with respect to distinction. The case where $u = m$ will be sufficient for our purpose (see Paragraph 4.6).

Lemma 4.16. — *Suppose that $u = m$ and $\boldsymbol{\kappa}_*$ is $\mathbf{J}_* \cap G^\sigma_*$-distinguished.*

(1) *If T/T_0 is unramified, or if T/T_0 is ramified and $m \neq 2i$, the representation $\boldsymbol{\kappa}$ is $\mathbf{J} \cap G^\sigma$-distinguished.*

(2) *If T/T_0 is ramified and $m = 2i$, there exists a quadratic character $\boldsymbol{\xi}$ of $\mathbf{J}$ trivial on $\mathbf{J}^0$ such that $\boldsymbol{\kappa\xi}$ is $\mathbf{J} \cap G^\sigma$-distinguished.*

Proof. — The representation $\boldsymbol{\kappa}_+$ is $\mathbf{J}_+ \cap G^\sigma$-distinguished, thus $\widetilde{\boldsymbol{\kappa}}_+$ is $\mathbf{J}^1\mathbf{J}_+ \cap G^\sigma$-distinguished. It follows that $\boldsymbol{\kappa}$ is $\mathbf{J}^1\mathbf{J}_+ \cap G^\sigma$-distinguished. Let χ be the character of $\mathbf{J} \cap G^\sigma$ associated with $\boldsymbol{\kappa}$ by Proposition 4.7. It is trivial on $\mathbf{J}^1\mathbf{J}_+ \cap G^\sigma$. Restricting to $\mathbf{J}^0 \cap G^\sigma$, it is a character of

$$(\mathbf{J}^0 \cap G^\sigma)/(\mathbf{J}^1 \cap G^\sigma) \simeq \mathrm{GL}_m(\boldsymbol{l})^\sigma.$$

Since $\mathbf{J} \cap M \subseteq \mathbf{J}_+$, it is trivial on the image of $(\mathbf{J} \cap M) \cap (\mathbf{J}^0 \cap G^\sigma)$ in $\mathrm{GL}_m(\boldsymbol{l})^\sigma$, which is made of the σ-fixed points of the diagonal torus $\mathsf{M} = \boldsymbol{l}^\times \times \dots \times \boldsymbol{l}^\times$.

If T/T_0 is unramified, we have $\mathrm{GL}_m(\boldsymbol{l})^\sigma = \mathrm{GL}_m(\boldsymbol{l}_0)$ and $\mathsf{M}^\sigma = \boldsymbol{l}_0^\times \times \dots \times \boldsymbol{l}_0^\times$, thus χ is trivial on $\mathbf{J}^0 \cap G^\sigma$. If T/T_0 is ramified, we have $\mathrm{GL}_m(\boldsymbol{l})^\sigma = \mathrm{GL}_i(\boldsymbol{l}) \times \mathrm{GL}_{m-i}(\boldsymbol{l})$ and $\mathsf{M}^\sigma = \mathsf{M}$. Again, χ is trivial on $\mathbf{J}^0 \cap G^\sigma$.

By Lemma 4.15, it remains to consider the value of χ at ϖ'. If T/T_0 is unramified, or if T/T_0 is ramified and $m \neq 2i$, we have $\varpi' \in \mathbf{J}^1\mathbf{J}_+ \cap G^\sigma$, thus χ is trivial.

Now assume that T/T_0 is ramified and $m = 2i$. Let $\boldsymbol{\xi}$ be the quadratic character of $\mathbf{J}$ trivial on $\mathbf{J}^0$ defined by $\xi(\varpi') = \chi(\varpi')$. Then $\boldsymbol{\kappa\xi}$ is $\mathbf{J} \cap G^\sigma$-distinguished. □

We will prove in Paragraph 4.6 that the quadratic character $\boldsymbol{\xi}$ of Lemma 4.16(2) is always trivial: see Corollary 4.20.

4.6. As in Paragraph 4.5, the simple character θ and the simple stratum $[\mathfrak{a}, \beta]$ are both maximal σ-selfdual, and η is the Heisenberg representation of $\mathbf{J}^1$ associated with θ. The next proposition, which says that η has a canonical extension to $\mathbf{J}$, is the core of our proof of Theorem 4.42.

Proposition 4.17. — *There is, up to isomorphism, a unique representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η satisfying the following conditions:*

(1) *it is both σ-selfdual and $\mathbf{J} \cap G^\sigma$-distinguished,*
(2) *its determinant has order a power of p.*

This unique representation will be denoted $\boldsymbol{\kappa}_\theta$.

Remark 4.18. — This extends (and makes more precise) the results of **[32]** (see *ibid.*, Propositions 7.9, 9.4) where θ is assumed to be generic and either T/T_0 is unramified and m is odd, or T/T_0 is ramified and $m \in \{1, 2i\}$. See also **[32]** Remarks 9.5 and 9.9.

Proof. — Suppose first that there exists a representation satisfying (1). As in the proof of **[33]** Corollary 6.12, one then easily proves the existence of a representation $\boldsymbol{\kappa}$ satisfying (1) and (2). Let us prove that such a representation is unique. Any other representation of $\mathbf{J}$ satisfying the conditions of the proposition is of the form $\boldsymbol{\kappa\phi}$ for some character $\boldsymbol{\phi}$ of $\mathbf{J}$ which is σ-selfdual and trivial on $(\mathbf{J} \cap G^\sigma)\mathbf{J}^1$, and whose order is a power of p. The restriction of $\boldsymbol{\phi}$ to $\mathbf{J}^0$ can be considered as a character of $\mathrm{GL}_m(\boldsymbol{l})$. Since the latter group is not isomorphic to $\mathrm{GL}_2(\mathbb{F}_2)$ (for $p \neq 2$), this character factors through the determinant. Its order is thus prime to p, which implies that $\boldsymbol{\phi}$ is trivial on $\mathbf{J}^0$. It is thus a character of $\mathbf{J}/(\mathbf{J} \cap G^\sigma)\mathbf{J}^0$ which, by Lemma 4.15, has order at most 2. Uniqueness follows from the fact that $p \neq 2$.

We are now reduced to proving the existence of a representation $\boldsymbol{\kappa}$ satisfying (1). If $m = 1$, this follows from **[32]** Propositions 7.9, 9.4. (See also Remark 4.18.)

Now consider the constructions of 4.4 and 4.5 with $u = m$. Thanks to the case where m is equal to 1, there is a representation $\boldsymbol{\kappa}_*$ of $\mathbf{J}_*$ extending η_* which is both σ-selfdual and $\mathbf{J}_* \cap G_*^\sigma$-distinguished. Let $\boldsymbol{\kappa}$ be the representation of $\mathbf{J}$ extending η associated with it by (4.10). Lemma 4.14 implies that it is σ-selfdual, and Lemma 4.16 implies that there is a quadratic character $\boldsymbol{\xi}$ of $\mathbf{J}$ trivial on $\mathbf{J}^0$ such that $\boldsymbol{\kappa\xi}$ is $\mathbf{J} \cap G^\sigma$-distinguished. Since $\boldsymbol{\xi}$ is unramified and quadratic, $\boldsymbol{\kappa\xi}$ is also σ-selfdual and extends η. □

Remark 4.19. — Notice that this gives another proof of **[32]** Propositions 7.9, 9.4, based on the case $m = 1$ only.

Now we can improve Lemma 4.16. Suppose we are in the situation of Paragraphs 4.4 and 4.5, with $u = m$.

Corollary 4.20. — *Suppose that $u = m$. Let $\boldsymbol{\kappa}_*$ be a representation of $\mathbf{J}_*$ extending η_* and $\boldsymbol{\kappa}$ correspond to it by the map (4.10). If $\boldsymbol{\kappa}_*$ is $\mathbf{J}_* \cap G_*^\sigma$-distinguished, then $\boldsymbol{\kappa}$ is $\mathbf{J} \cap G^\sigma$-distinguished.*

Proof. — The result is given by Lemma 4.16, except when T/T_0 is ramified and $m = 2i$, which we assume now. Suppose that $\boldsymbol{\kappa}_*$ is $\mathbf{J}_* \cap G_*^\sigma$-distinguished. By Lemma 4.16, there is a quadratic character $\boldsymbol{\xi}$ of $\mathbf{J}$ trivial on $\mathbf{J}^0$ such that $\boldsymbol{\kappa\xi}$ is $\mathbf{J} \cap G^\sigma$-distinguished. Let $\boldsymbol{\kappa}_\theta$ be the representation given by Proposition 4.17 and write $\boldsymbol{\kappa\xi} = \boldsymbol{\kappa}_\theta\boldsymbol{\phi}$ for some character $\boldsymbol{\phi}$ of $\mathbf{J}$ trivial on $(\mathbf{J} \cap G^\sigma)\mathbf{J}^1$. Restricting to $\mathbf{J}^0$, the character $\boldsymbol{\phi}$ can be seen as a character of $\mathrm{GL}_m(\boldsymbol{l})$ of the form $\alpha \circ \det$, for some character α of $\boldsymbol{l}^\times$, which is trivial on $\mathrm{GL}_m(\boldsymbol{l})^\sigma = \mathrm{GL}_i(\boldsymbol{l}) \times \mathrm{GL}_i(\boldsymbol{l})$. This implies that α is trivial, thus $\boldsymbol{\phi}$ is trivial on $\mathbf{J}^0$. Also, $\boldsymbol{\phi}$ is trivial at $\varpi' \in \varpi\mathbf{J}^0$ by Lemma 4.15. It is thus trivial. In conclusion, we have $\boldsymbol{\kappa} = \boldsymbol{\kappa}_\theta\boldsymbol{\xi}$. Taking determinants, we get

$$\det \boldsymbol{\kappa} = \boldsymbol{\xi} \cdot \det \boldsymbol{\kappa}_\theta. \tag{4.13}$$

Now there is a $y \in M \cap B^\times$ such that $\theta' = \theta^y \in \mathcal{C}(\mathfrak{a}^y, \beta)$ is a σ-selfdual maximal simple character of index 0 (in the sense of Remark 4.9). By Remark 4.13, the simple character of $\mathcal{C}(\mathfrak{a}_*, \beta)$ associated with θ' by Lemma 4.10 is still θ_*, and the representation of $\mathbf{J}' = \mathbf{J}^y$ corresponding to $\boldsymbol{\kappa}_*$ by (4.10) is $\boldsymbol{\kappa}^y$. Let $\boldsymbol{\kappa}_{\theta'}$ be the representation associated with θ' by Proposition 4.17. By Lemma 4.16, $\boldsymbol{\kappa}'$ is distinguished. By the discussion above, it follows that

$$\det \boldsymbol{\kappa}^y = \det \boldsymbol{\kappa}_{\theta'}. \tag{4.14}$$

But the characters $\det \boldsymbol{\kappa}$, $\det \boldsymbol{\kappa}^y$ have the same order (since they are conjugate to each others), and the latter one has order a power of p thanks to (4.14). Now (4.13) implies that $\boldsymbol{\xi}$ has order a power of p. Since $\boldsymbol{\xi}$ is quadratic and $p \neq 2$, this character is trivial. □

We extract from the proof of Corollary 4.20 the following valuable corollary.

Corollary 4.21. — *Suppose that $u = m$. Let $\boldsymbol{\kappa}_{\theta_*}$ and $\boldsymbol{\kappa}_\theta$ be the representations associated with θ_* and θ by Proposition 4.17, respectively. Then the map (4.10) takes $\boldsymbol{\kappa}_{\theta_*}$ to $\boldsymbol{\kappa}_\theta$.*

We also have the following corollary, which extends **[32]** Lemma 7.10(3), Corollary 9.6(1).

Corollary 4.22. — *Any $\mathbf{J} \cap G^\sigma$-distinguished representation of $\mathbf{J}$ extending η is σ-selfdual.*

Proof. — Let $\boldsymbol{\kappa}$ be a $\mathbf{J} \cap G^\sigma$-distinguished representation of $\mathbf{J}$ extending η, and $\boldsymbol{\xi}$ be the unique character of $\mathbf{J}$ trivial on $\mathbf{J}^1$ such that $\boldsymbol{\kappa} = \boldsymbol{\kappa}_\theta\boldsymbol{\xi}$. We have to prove that $\boldsymbol{\xi}^{-1} \circ \sigma = \boldsymbol{\xi}$. The fact that $\boldsymbol{\kappa}$ is distinguished implies that $\boldsymbol{\xi}$ is trivial on $(\mathbf{J} \cap G^\sigma)\mathbf{J}^1$. Restricting to $\mathbf{J}^0$, the character $\boldsymbol{\xi}$ can be seen as a character of $\mathrm{GL}_m(\boldsymbol{l})$ of the form $\alpha \circ \det$, for some character α of $\boldsymbol{l}^\times$.

If T/T_0 is unramified, α is trivial on $\boldsymbol{l}_0^\times$, thus $\boldsymbol{\xi}^{-1} \circ \sigma$ and $\boldsymbol{\xi}$ coincide on $\mathbf{J}^0$. They also coincide on $\varpi \in \mathbf{J} \cap G^\sigma$, thus they are equal.

If T/T_0 is ramified, α is trivial, thus $\boldsymbol{\xi}^{-1} \circ \sigma$ and $\boldsymbol{\xi}$ are both trivial on $\mathbf{J}^0$. Since $\boldsymbol{\xi}$ is trivial on $\varpi^2 \in \mathbf{J} \cap G^\sigma$, we get $\boldsymbol{\xi}^{-1} \circ \sigma(\varpi) = \boldsymbol{\xi}^{-1}(-\varpi) = \boldsymbol{\xi}^{-1}(\varpi) = \boldsymbol{\xi}(\varpi)$, which finishes the proof. □

4.7. We now come to the type theoretic description of cuspidal representations of G. The following proposition follows from **[8]** Theorem 8.4.1, Corollary 6.2.3, Theorem 5.7.1 (see **[27]** Théorèmes 3.4, 3.7 and **[34]** Theorem 7.2 in the modular case).

Proposition 4.23. — *Let π be a cuspidal representation of G. There is, up to G-conjugacy, a unique simple character θ such that the restriction of π to H^1_θ contains θ, and it is maximal.*

Let π be a cuspidal representation of G, and let θ be a simple character occurring in π. Associated with it, there are:

– the positive integer $m(\pi) = m \geqslant 1$ defined by (4.3), called the *relative degree* of π,

– the G-conjugacy class (or equivalently the F-isomorphism class) of the tamely ramified extension T of F associated with θ, called the *tame parameter field* of π,

– the endo-class $\boldsymbol{\Theta}$ of θ, called the *endo-class* of π.

(Note that, when π has level 0, one has $m = n$ and $T = F$, and $\boldsymbol{\Theta}$ is the null endo-class.)

Write $\mathbf{J} = \mathbf{J}_\theta$, $\mathbf{J}^0 = \mathbf{J}^0_\theta$, $\mathbf{J}^1 = \mathbf{J}^1_\theta$ and let η be the Heisenberg representation of θ. The next proposition follows from **[27]** Lemme 5.3, Theorem 3.11.

Proposition 4.24. — *Let $\boldsymbol{\kappa}$ be a representation of $\mathbf{J}$ extending η, and define a representation of $\mathbf{J}$ on the space $\mathrm{Hom}_{\mathbf{J}^1}(\boldsymbol{\kappa},\pi)$ by making $x\in\mathbf{J}$ act on $f\in\mathrm{Hom}_{\mathbf{J}^1}(\boldsymbol{\kappa},\pi)$ by*

$$x\cdot f=\pi(x)\circ f\circ\boldsymbol{\kappa}(x)^{-1}.$$

This representation, denoted $\boldsymbol{\tau}$, has the following properties:

(1) *It is irreducible, and trivial on $\mathbf{J}^1$.*

(2) *If one identifies $\mathbf{J}^0/\mathbf{J}^1$ with a finite general linear group as in* (4.4)*, its restriction to $\mathbf{J}^0$ is the inflation of a cuspidal representation.*

(3) *The compact induction of $\boldsymbol{\kappa}\otimes\boldsymbol{\tau}$ from $\mathbf{J}$ to G is isomorphic to π.*

Any two representations of $\mathbf{J}$ extending η differ from a character of $\mathbf{J}$ trivial on $\mathbf{J}^1$. The pair

$$(\mathbf{J},\boldsymbol{\kappa}\otimes\boldsymbol{\tau}) \tag{4.15}$$

thus only depends on π and the choice of θ, and not on the choice of $\boldsymbol{\kappa}$.

When π varies among all cuspidal representations of G and θ varies among all maximal simple characters in π, the pairs (4.15) are called *extended maximal simple types* in **[8, 27]**, which we will abbreviate to *types* here. A given cuspidal representation of G thus contains, up to G-conjugacy, a unique type $(\mathbf{J},\boldsymbol{\lambda})$: there is a unique maximal simple character θ such that $\mathbf{J}_\theta=\mathbf{J}$ and θ occurs in the restriction of $\boldsymbol{\lambda}$ to H^1_θ, a representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ which restricts irreducibly to $\mathbf{J}^1$ and a representation $\boldsymbol{\tau}$ of $\mathbf{J}$ trivial on $\mathbf{J}^1$ such that $\boldsymbol{\lambda}$ is isomorphic to $\boldsymbol{\kappa}\otimes\boldsymbol{\tau}$.

Remark 4.25. — If $\mathfrak{a}$ is a maximal order in $\mathbf{M}_n(F)$, the trivial character of $\mathrm{U}^1_{\mathfrak{a}}$ is a maximal simple character, with $E=T=F$ and $m=n$. The cuspidal representations of G that contain such a simple character are precisely the cuspidal representations of level 0.

Fix a representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η and define $\boldsymbol{\tau}$ as in Proposition 4.24, and fix a simple stratum $[\mathfrak{a},\beta]$ such that $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ and an isomorphism (4.3). This gives a field E and an isomorphism $\mathbf{J}^0/\mathbf{J}^1\simeq\mathrm{GL}_m(\boldsymbol{l})$, where $\boldsymbol{l}$ is the residue field of T.

By Proposition 4.24(2), the restriction of $\boldsymbol{\tau}$ to $\mathbf{J}^0$ is the inflation of a cuspidal irreducible representation, denoted V.

On the other hand, the representation $\boldsymbol{\tau}$ has a central character : it is a character of the centre $E^\times\mathbf{J}^1/\mathbf{J}^1$ of $\mathbf{J}/\mathbf{J}^1$, or equivalently a tamely ramified character of $E^\times$. Since E is totally wildly ramified over its maximal tamely ramified subextension T, any tamely ramified character of $E^\times$ factors through the norm $\mathrm{N}_{E/T}$. The restriction of $\boldsymbol{\tau}$ to $E^\times$ is thus a multiple of $\omega\circ\mathrm{N}_{E/T}$ for a uniquely determined tamely ramified character ω of $T^\times$.

The data V and ω are subject to the compatibility condition that the restriction of V to $\boldsymbol{l}^\times$ is a multiple of the character of $\boldsymbol{l}^\times$ whose inflation to $\mathcal{O}_T^\times$ is the restriction of ω^{p^e}, with $p^e=[E:T]$. Associated with V by Proposition 3.9, there are a unique integer $u\geqslant 1$ dividing m and a unique supercuspidal representation ϱ of $\mathrm{GL}_{m/u}(\boldsymbol{l})$ such that V is isomorphic to $\mathrm{st}_u(\varrho)$. The next important result is **[28]** Lemma 3.2. The integer $r(\pi)$ has been defined in Paragraph 3.4.

Lemma 4.26. — *The integer u is equal to $r(\pi)$.*

It follows that $r(\pi)$ divides m, and that π is supercuspidal if and only if V is supercuspidal.

4.8. Write $r = r(\pi)$ and $k = n/r$, and let ρ be a supercuspidal representation of $\mathrm{GL}_k(F)$ such that π is isomorphic to $\mathrm{St}_r(\rho)$ given by Proposition 3.6. In this paragraph, we will compare the type theoretic description of π with that of ρ. As in 4.7, we fix a representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η. It defines an irreducible representation $\boldsymbol{\tau}$ of $\mathbf{J}$ trivial on $\mathbf{J}^1$, then a cuspidal representation V of $\mathrm{GL}_m(\boldsymbol{l})$ and a tamely ramified character ω of $T^\times$. There is also a (unique) supercuspidal representation ϱ of $\mathrm{GL}_{m/r}(\boldsymbol{l})$ such that V is isomorphic to $\mathrm{st}_r(\varrho)$.

Since r divides m, we may apply the results of 4.4 to the case where $u = r$, which we assume now. Let θ_* be the simple character associated with θ by Lemma 4.10.

Lemma 4.27. — *The representation ρ contains θ_*.*

Proof. — This follows from the description[(1)] of $\mathrm{St}_r(\rho)$ in **[26]** Section 6. □

Consequently, π and ρ have the same endo-class. We have the following immediate corollary.

Corollary 4.28. — *We have $m(\pi) = m(\rho)r$ and the representations π, ρ have the same tame parameter field.*

Let η_* be the Heisenberg representation associated with θ_*, and let $\boldsymbol{\kappa}_*$ be a representation of $\mathbf{J}_*$ extending η_* such that the representation of $\mathbf{J}$ associated with it by (4.10) is $\boldsymbol{\kappa}$. It defines an irreducible representation $\boldsymbol{\tau}_*$ of $\mathbf{J}_*$ trivial on $\mathbf{J}_*^1$, such that the pair $(\mathbf{J}_*, \boldsymbol{\kappa}_* \otimes \boldsymbol{\tau}_*)$ is a type in ρ. Associated with this, there are a cuspidal representation ϱ_* of $\mathrm{GL}_{m/r}(\boldsymbol{l})$ (which is supercuspidal thanks to the comment after Lemma 4.26) and a tamely ramified character ω_* of $T^\times$. The following proposition compares the pairs (ϱ, ω) and (ϱ_*, ω_*) associated with $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_*$.

Proposition 4.29. — *We have $\varrho \simeq \varrho_*$ and $\omega = \omega_*^r$.*

Proof. — Again, the fact that ϱ is isomorphic to ϱ_* follows from the description of $\mathrm{St}_r(\rho)$ in **[26]** Section 6. It thus remains to prove the second equality. For this, consider the action of $\mathbf{J}$ on

$$\mathrm{Hom}_{\mathbf{J}^1}(\boldsymbol{\kappa}, \mathscr{I}(\rho, r)) \tag{4.16}$$

where $\mathscr{I}(\rho, r)$ is the parabolically induced representation (3.1). By **[35]** Proposition 5.6, its restriction to $\mathbf{J}^0$ is the inflation of the induced representation $\varrho_* \times \cdots \times \varrho_*$ of $\mathrm{GL}_m(\boldsymbol{l})$. By tracking the action of $E^\times$ in the arguments of **[35]** Section 5, we see that it acts on the space (4.16) by the character

$$\omega_*^r \circ \mathrm{N}_{E/T}.$$

In particular, $E^\times$ acts through this character on the subquotient $\mathrm{Hom}_{\mathbf{J}^1}(\boldsymbol{\kappa}, \pi)$, which is $\boldsymbol{\tau}$. □

4.9. Suppose that the cuspidal representation π is σ-selfdual. We say a type $(\mathbf{J}, \boldsymbol{\lambda})$ is *σ-selfdual* if $\mathbf{J}$ is σ-stable and $\boldsymbol{\lambda}^{\vee\sigma}$ is isomorphic to $\boldsymbol{\lambda}$. The next result is **[3]** Theorem 4.1.

Proposition 4.30. — *The representation π contains a σ-selfdual type.*

A type $(\mathbf{J}, \boldsymbol{\lambda})$ contains a unique simple character θ such that $\mathbf{J}_\theta = \mathbf{J}$: it follows that, if $(\mathbf{J}, \boldsymbol{\lambda})$ is σ-selfdual, θ is σ-selfdual as well. In particular, π contains a σ-selfdual simple character.

Let θ be a σ-selfdual simple character occurring in π, and $[\mathfrak{a}, \beta]$ be a σ-selfdual simple stratum such that $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ (which exists by Proposition 4.5). The G^σ-conjugacy class (or equivalently

(1) Warning: the representation denoted $\mathrm{St}(\rho, r)$ in **[26]** corresponds to $\mathrm{St}_r(\rho\nu^{(r-1)/2})$, and the one denoted $\mathrm{St}_r(\rho)$ in **[26]** corresponds to $\mathrm{St}_v(\rho\nu^{(v-1)/2})$ with $v = e(\rho)\ell^r$.

the F_0-isomorphism class) of the tamely ramified extension T_0 of E_0 associated with θ only depends on π. Associated with π, there is thus a quadratic extension T/T_0.

Remark 4.31. — When π has level 0, one has $T_0 = F_0$.

If follows from Proposition 4.8 that π contains

– only one G^σ-conjugacy class of σ-selfdual types if T/T_0 is unramified,
– $\lfloor m/2 \rfloor + 1$ different G^σ-conjugacy classes of σ-selfdual types if T/T_0 is ramified.

Among these G^σ-conjugacy classes of σ-selfdual types, one is of particular importance.

Definition 4.32. — A σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$ is said to be *generic* if either T/T_0 is unramified, or T/T_0 is ramified and the integer i of Proposition 4.5(2.b) is equal to $\lfloor m/2 \rfloor$.

A σ-selfdual cuspidal representation of G thus contains, up to G^σ-conjugacy, a unique generic σ-selfdual type. The next result is **[32]** Theorem 10.3 (see also **[3]** Section 6).

Proposition 4.33. — *Let π be a σ-selfdual cuspidal representation of G and $(\mathbf{J}, \boldsymbol{\lambda})$ be its generic σ-selfdual type. Then π is distinguished if and only if $\boldsymbol{\lambda}$ is $\mathbf{J} \cap G^\sigma$-distinguished.*

If $(\mathbf{J}, \boldsymbol{\lambda})$ is a σ-selfdual type, and if θ is the unique simple character contained in $\boldsymbol{\lambda}$ such that $\mathbf{J}_\theta = \mathbf{J}$, we will write $\boldsymbol{\lambda}_{\mathrm{w}}$ for the unique representation $\boldsymbol{\kappa}_\theta$ of $\mathbf{J}$ extending the Heisenberg representation of θ given by Proposition 4.17. The next result extends **[32]** Propositions 7.9, 9.8 to the case of arbitrary σ-selfdual cuspidal representations.

Proposition 4.34. — *Let π be a σ-selfdual cuspidal representation of G. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a generic σ-selfdual type in π and $\boldsymbol{\tau}$ be the representation of $\mathbf{J}$ trivial on $\mathbf{J}^1$ such that $\boldsymbol{\lambda}$ is isomorphic to $\boldsymbol{\lambda}_{\mathrm{w}} \otimes \boldsymbol{\tau}$. Then π is distinguished if and only if $\boldsymbol{\tau}$ is $\mathbf{J} \cap G^\sigma$-distinguished.*

Proof. — This follows from Proposition 4.33 together with the fact that

$$\mathrm{Hom}_{\mathbf{J}\cap G^\sigma}(\boldsymbol{\lambda}, R) \simeq \mathrm{Hom}_{\mathbf{J}\cap G^\sigma}(\boldsymbol{\lambda}_{\mathrm{w}}, R) \otimes \mathrm{Hom}_{\mathbf{J}\cap G^\sigma}(\boldsymbol{\tau}, R)$$

and $\mathrm{Hom}_{\mathbf{J}\cap G^\sigma}(\boldsymbol{\lambda}_{\mathrm{w}}, R)$ has dimension 1 (see Proposition 4.7(4)). □

Fix isomorphisms

$$B \simeq \mathsf{M}_m(E), \quad \mathbf{J}^0/\mathbf{J}^1 \simeq \mathrm{GL}_m(\boldsymbol{l}), \tag{4.17}$$

as in Proposition 4.5(4). The restriction of $\boldsymbol{\tau}$ to $\mathbf{J} \cap B^\times$ is a generic σ-selfdual type of level 0 in $B^\times \simeq \mathrm{GL}_m(E)$ and $\mathbf{J}/\mathbf{J}^1$ is naturally isomorphic to $(\mathbf{J} \cap B^\times)/(\mathbf{J}^1 \cap B^\times)$. The representation $\boldsymbol{\tau}$ is thus distinguished by $\mathbf{J} \cap G^\sigma$ if and only if its restriction to $\mathbf{J} \cap \mathrm{GL}_m(E)$ is distinguished by $\mathbf{J} \cap \mathrm{GL}_m(E_0)$. Proposition 4.34 used twice thus implies that π is distinguished by G^σ if and only if the cuspidal representation of level 0 of $\mathrm{GL}_m(E)$ compactly induced from the restriction of $\boldsymbol{\tau}$ to $\mathbf{J} \cap \mathrm{GL}_m(E)$ is distinguished by $\mathrm{GL}_m(E_0)$.

However, the field extension E is not canonical. In 4.10, we will canonically associate with π a σ-selfdual cuspidal representation π_{t} of level 0 of $\mathrm{GL}_m(T)$, which is $\mathrm{GL}_m(T_0)$-distinguished if and only if π is G^σ-distinguished, where T/T_0 is the quadratic extension associated with π. Our strategy is inspired from **[7]** Section 3.

The following proposition relates the parity of m/r to the ramification of T/T_0.

Proposition 4.35. — *Let π be a σ-selfdual cuspidal representation of $\mathrm{GL}_n(F)$ with quadratic extension T/T_0, and write $m = m(\pi)$, $r = r(\pi)$. Then*

$$m/r \text{ is } \begin{cases} \text{odd if } T/T_0 \text{ is unramified,} \\ \text{either even or equal to } 1 \text{ if } T/T_0 \text{ is ramified.} \end{cases}$$

Proof. — Write π as $\mathrm{St}_r(\rho)$ as in Proposition 3.8 with $\rho^{\sigma\vee} \simeq \rho\nu^i$ for some $i \in \{0,1\}$. Then $\rho\nu^{i/2}$ is a σ-selfdual supercuspidal representation of $\mathrm{GL}_{n/r}(F)$, and the quadratic extension associated with it is T/T_0. Applying **[32]** Propositions 8.1, 9.8, we get the expected result. □

4.10. In order to prove Theorem 4.42, it will be useful to consider the slightly more general situation where π is a cuspidal representation of G with σ-selfdual endo-class $\boldsymbol{\Theta}$. Thus π itself needs not be σ-selfdual. However, it has a relative degree m and, since $\boldsymbol{\Theta}$ is σ-selfdual, there is a quadratic extension T/T_0 associated with it. Moreover, by Proposition 4.8, it contains, up to G^σ-conjugacy, a unique generic σ-selfdual maximal simple character θ. Let $\mathbf{J}$ be its normalizer in G and $\boldsymbol{\kappa}_\theta$ be the representation of $\mathbf{J}$ given by Proposition 4.17. Then π contains a unique type of the form

$$(\mathbf{J}, \boldsymbol{\kappa}_\theta \otimes \boldsymbol{\tau}) \tag{4.18}$$

for a uniquely determined irreducible representation $\boldsymbol{\tau}$ of $\mathbf{J}$ trivial on $\mathbf{J}^1$. Fix a σ-selfdual simple stratum $[\mathfrak{a}, \beta]$ and isomorphisms (4.17) as in Proposition 4.5.

First, we define an open and compact mod centre subgroup $\mathbf{J}_{\mathrm{t}}$ of $\mathrm{GL}_m(T)$ as follows:

– if T/T_0 is unramified, $\mathbf{J}_{\mathrm{t}}$ is the normalizer of $\mathrm{GL}_m(\mathcal{O}_T)$ in $\mathrm{GL}_m(T)$,

– if T/T_0 is ramified, and if t is a uniformizer of T such that $\sigma(t) = -t$, then $\mathbf{J}_{\mathrm{t}}$ is the normalizer in $\mathrm{GL}_m(T)$ of the conjugate of $\mathrm{GL}_m(\mathcal{O}_T)$ by the diagonal element

$$\mathrm{diag}(t, \dots, t, 1, \dots, 1) \in \mathrm{GL}_m(T)$$

where t occurs $\lfloor m/2 \rfloor$ times.

The group $\mathbf{J}_{\mathrm{t}}$ (which does not depend on the choice of t in the ramified case) has a unique maximal compact subgroup $\mathbf{J}_{\mathrm{t}}^0$ and a unique normal maximal pro-p-subgroup $\mathbf{J}_{\mathrm{t}}^1$. The natural group isomorphism

$$\mathbf{J}_{\mathrm{t}}^0/\mathbf{J}_{\mathrm{t}}^1 \simeq \mathrm{GL}_m(\boldsymbol{l}) \tag{4.19}$$

transports the action of $\sigma \in \mathrm{Gal}(T/T_0)$ on $\mathbf{J}_{\mathrm{t}}^0/\mathbf{J}_{\mathrm{t}}^1$ to

– the action of the non-trivial element of $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$ on $\mathrm{GL}_m(\boldsymbol{l})$ if T/T_0 is unramified,

– the adjoint action of

$$\begin{pmatrix} -\mathrm{id}_{\lfloor m/2 \rfloor} & 0 \\ 0 & \mathrm{id}_{m-\lfloor m/2 \rfloor} \end{pmatrix} \in \mathrm{GL}_m(\boldsymbol{l}), \tag{4.20}$$

on $\mathrm{GL}_m(\boldsymbol{l})$ if T/T_0 is ramified.

Remark 4.36. — When T/T_0 is ramified, the isomorphism (4.19) depends on the choice of t : changing t to another uniformizer t' conjugates the isomorphism by the σ-invariant element

$$\mathrm{diag}(\alpha, \dots, \alpha, 1, \dots, 1) \in \mathrm{GL}_m(\boldsymbol{l})$$

where α (which occurs i times) is the image of $t't^{-1}$ in $\boldsymbol{l}^\times$. This element is central in $\mathrm{GL}_m(\boldsymbol{l})^\sigma$.

We now associate to $\boldsymbol{\tau}$ an irreducible representation $\boldsymbol{\tau}_{\mathrm{t}}$ of $\mathbf{J}_{\mathrm{t}}$ trivial on $\mathbf{J}^1_{\mathrm{t}}$. On the one hand, the restriction of $\boldsymbol{\tau}$ to $\mathbf{J}^0$ is the inflation of an irreducible cuspidal representation V of $\mathrm{GL}_m(\boldsymbol{l})$. On the other hand, the restriction of $\boldsymbol{\tau}$ to $E^\times$ is a multiple of $\omega \circ \mathrm{N}_{E/T}$ for a uniquely determined tamely ramified character ω of $T^\times$: see 4.7. Note that $[E:T] = p^e$ for some $e \geqslant 1$.

Lemma 4.37. — *Let* V *and* ω *be as above.*

(1) *There is a unique representation* $\boldsymbol{\tau}_{\mathrm{t}}$ *of* $\mathbf{J}_{\mathrm{t}}$ *trivial on* $\mathbf{J}^1_{\mathrm{t}}$ *such that*

(a) *the restriction of* $\boldsymbol{\tau}_{\mathrm{t}}$ *to* $T^\times$ *is a multiple of the character* ω,

(b) *the restriction of* $\boldsymbol{\tau}_{\mathrm{t}}$ *to* $\mathbf{J}^0_{\mathrm{t}}$ *is the inflation of* $\mathrm{V}^{(p^{-e})}$, *where* $\mathrm{V}^{(p^{-e})}$ *is the representation of* $\mathrm{GL}_m(\boldsymbol{l})$ *obtained from* V *by applying the automorphism* $x \mapsto x^{p^{-e}}$.

(2) *The pair* $(\mathbf{J}_{\mathrm{t}}, \boldsymbol{\tau}_{\mathrm{t}})$ *is a level* 0 *type in* $\mathrm{GL}_m(T)$.

(3) *Up to isomorphism, the representation* $\boldsymbol{\tau}_{\mathrm{t}}$ *only depends on* $\boldsymbol{\tau}$, *and not on the choice of the* σ*-selfdual simple stratum* $[\mathfrak{a}, \beta]$, *the uniformizer* t *and the identification* $\mathbf{J}^0/\mathbf{J}^1 \simeq \mathrm{GL}_m(\boldsymbol{l})$.

Proof. — Uniqueness follows from the fact that $\mathbf{J}_{\mathrm{t}}$ is generated by $\mathbf{J}^0_{\mathrm{t}}$ and $T^\times$, and the existence of $\boldsymbol{\tau}_{\mathrm{t}}$ follows from the fact that the restriction of $\mathrm{V}^{(p^{-e})}$ to $\boldsymbol{l}^\times$ is a multiple of the character of $\boldsymbol{l}^\times$ defined by the restriction of ω to the units of $T^\times$. Since $\mathrm{V}^{(p^{-e})}$ is cuspidal, the pair $(\mathbf{J}_{\mathrm{t}}, \boldsymbol{\tau}_{\mathrm{t}})$ is a level 0 type by construction. It remains to prove (3). Since it will require techniques which are not used anywhere else in the paper, we will prove it apart, in Paragraph 4.13. ☐

It will be convenient to give another description of the representation $\boldsymbol{\tau}_{\mathrm{t}}$.

Lemma 4.38. — (1) *There is a unique group isomorphism* $\boldsymbol{\pi} : \mathbf{J}/\mathbf{J}^1 \to \mathbf{J}_{\mathrm{t}}/\mathbf{J}^1_{\mathrm{t}}$ *such that*

(a) *its restriction to* $\mathrm{GL}_m(\boldsymbol{l})$ *is the automorphism acting entrywise by* $\phi : x \mapsto x^{p^e}$,

(b) *for all* $x \in E^\times$, *the image of* $x\mathbf{J}^1$ *is* $\mathrm{N}_{E/T}(x)\mathbf{J}^1_{\mathrm{t}}$.

(2) *The isomorphism* $\boldsymbol{\pi}$ *is* σ*-equivariant.*

(3) *The representation* $\boldsymbol{\tau}_{\mathrm{t}}$ *is isomorphic to* $\boldsymbol{\tau} \circ \boldsymbol{\pi}^{-1}$.

Proof. — Again, uniqueness follows from the the fact that $\mathbf{J}$ is generated by $\mathbf{J}^0$ and $E^\times$. Existence follows from the fact that $\mathrm{N}_{E/T}(x) = x^{p^e}$ for all $x \in \mathcal{O}_E^\times$ of order prime to p, and that $\mathrm{N}_{E/T}$ induces a group isomorphism from $E^\times/(1+\mathfrak{p}_E)$ to $T^\times/(1+\mathfrak{p}_T)$.

Define $\boldsymbol{\pi}_1$ to be $\sigma \circ \boldsymbol{\pi} \circ \sigma^{-1}$. The restriction of $\boldsymbol{\pi}_1$ to $E^\times$ corresponds to $\sigma \circ \mathrm{N}_{E/T} \circ \sigma^{-1}$, which is equal to $\mathrm{N}_{E/T}$ since E and T are stable by σ. The restriction of $\boldsymbol{\pi}_1$ to $\mathrm{GL}_m(\boldsymbol{l})$ is

– the automorphism defined by making $\sigma \circ \phi \circ \sigma^{-1} = \phi \in \mathrm{Gal}(\boldsymbol{l}/\mathbb{F}_p)$ act entrywise if T/T_0 is unramified,

– the automorphism $\mathrm{Ad}(\delta^{-1}\phi(\delta)) \circ \phi = \phi$ if T/T_0 is ramified, where δ is the ϕ-invariant matrix defined by (4.5).

The fact that $\boldsymbol{\pi}$ is σ-equivariant now follows from its uniqueness, and (3) is immediate. ☐

Now let us describe the behavior of $\boldsymbol{\tau} \mapsto \boldsymbol{\tau}_{\mathrm{t}}$ with respect to duality and distinction.

Lemma 4.39. — (1) *The representation* $\boldsymbol{\tau}_{\mathrm{t}}$ *is* σ*-selfdual if and only if* $\boldsymbol{\tau}$ *is* σ*-selfdual.*

(2) *The representation* $\boldsymbol{\tau}_{\mathrm{t}}$ *is* $\mathbf{J}_{\mathrm{t}} \cap \mathrm{GL}_m(T_0)$*-distinguished if and only if* $\boldsymbol{\tau}$ *is* $\mathbf{J} \cap G^\sigma$*-distinguished.*

Proof. — Saying that $\boldsymbol{\tau}$ is σ-selfdual is equivalent to saying that the representation V and the character $\omega \circ \mathrm{N}_{E/T}$ are σ-selfdual. Assertion (1) follows from the fact that $(\mathrm{V}^{(p^{-e})})^{\vee\sigma}$ is isomorphic to $(\mathrm{V}^{\vee\sigma})^{(p^{-e})}$ and $(\omega \circ \mathrm{N}_{E/T})^{-1} \circ \sigma$ is equal to $(\omega^{-1} \circ \sigma) \circ \mathrm{N}_{E/T}$.

Assertion (2) follows from the fact that $\boldsymbol{\tau}_{\mathrm{t}} \circ \boldsymbol{\pi} = \boldsymbol{\tau}$ and $\boldsymbol{\pi}$ maps $(\mathbf{J}/\mathbf{J}^1)^\sigma$ to $(\mathbf{J}_{\mathrm{t}}/\mathbf{J}_{\mathrm{t}}^1)^\sigma$. □

Corollary 4.40. — *The pair $(\mathbf{J}_{\mathrm{t}}, \boldsymbol{\tau}_{\mathrm{t}})$ is a generic σ-selfdual type if and only if $(\mathbf{J}, \boldsymbol{\kappa}_\theta \otimes \boldsymbol{\tau})$ is a generic σ-selfdual type.*

Proof. — This follows from Lemma 4.39(1), thanks to our choice of $\mathbf{J}_{\mathrm{t}}$ (see (4.20)). □

4.11. We still are in the situation of Paragraph 4.10. Consider the compactly induced representation

$$\pi_{\mathrm{t}} = \mathrm{ind}_{\mathbf{J}_{\mathrm{t}}}^{\mathrm{GL}_m(T)}(\boldsymbol{\tau}_{\mathrm{t}}). \tag{4.21}$$

It satisfies the following properties.

Proposition 4.41. — (1) *The representation π_{t} is cuspidal, irreducible and has level* 0.

(2) *One has $m(\pi_{\mathrm{t}}) = m$ and $r(\pi_{\mathrm{t}}) = r$.*

(3) *The representation π_{t} is σ-selfdual if and only if π is σ-selfdual.*

(4) *The representation π_{t} is $\mathrm{GL}_m(T_0)$-distinguished if and only π is $\mathrm{GL}_n(F_0)$-distinguished.*

Proof. — Assertion (1) follows from the fact that π_{t} is compactly induced from a level 0 type in $\mathrm{GL}_m(T)$ (see Lemma 4.37 and Remark 4.25). The first equality of Assertion (2) follows from Remark 4.25, and the second one from Lemma 4.26.

Suppose that π is σ-selfdual. Then $\boldsymbol{\tau}$ is σ-selfdual (see 4.9). By Lemma 4.39, the representation $\boldsymbol{\tau}_{\mathrm{t}}$ is σ-selfdual as well. By compact induction, it follows that π_{t} is σ-selfdual. The argument also works the other way round, proving (3). Assertion (4) follows from Proposition 4.34 together with Lemma 4.39(2). □

Theorem 4.42. — (1) *The process*

$$\pi \mapsto \pi_{\mathrm{t}} \tag{4.22}$$

induces a bijection from the set of isomorphism classes of cuspidal representations of G with endo-class $\boldsymbol{\Theta}$ to that of cuspidal representations of level 0 of $\mathrm{GL}_m(T)$.

(2) *The bijection (4.22) maps σ-selfdual representations onto σ-selfdual representations and G^σ-distinguished representations onto $\mathrm{GL}_m(T_0)$-distinguished representations.*

(3) *For any cuspidal representation π with endo-class $\boldsymbol{\Theta}$ and any tamely ramified character χ of $F^\times$, the representation $(\pi\chi)_{\mathrm{t}}$ is isomorphic to $\pi_{\mathrm{t}}(\chi \circ \mathrm{N}_{T/F})$.*

Proof. — For (1), let π_0 be a cuspidal representation of level 0 of $\mathrm{GL}_m(T)$. It contains a level 0 type $(\mathbf{J}_{\mathrm{t}}, \boldsymbol{\tau}_0)$ for a uniquely determined representation $\boldsymbol{\tau}_0$ of $\mathbf{J}_{\mathrm{t}}$ trivial on $\mathbf{J}_{\mathrm{t}}^1$. It then suffices to check that the process

$$\pi_0 \mapsto \mathrm{ind}_{\mathbf{J}}^{G}(\boldsymbol{\kappa}_\theta \otimes (\boldsymbol{\tau}_0 \circ \boldsymbol{\pi}))$$

gives the inverse bijection. For (3), notice that if π contains the type $(\mathbf{J}, \boldsymbol{\kappa}_\theta \otimes \boldsymbol{\tau})$, then $\pi\chi$ contains the type $(\mathbf{J}, \boldsymbol{\kappa}_\theta \otimes \boldsymbol{\tau}\chi^{\mathbf{J}})$, where $\chi^{\mathbf{J}}$ is the unique character of $\mathbf{J}$ trivial on $\mathbf{J}^1$ whose restriction to $\mathbf{J} \cap B^\times \simeq \mathrm{GL}_m(E)$ is equal to $(\chi \circ \mathrm{N}_{E/F}) \circ \det_E$ where $\det_E$ is the determinant on $B \simeq \mathbf{M}_m(E)$. Then $(\boldsymbol{\tau}\chi^{\mathbf{J}})_{\mathrm{t}}$ is isomorphic to the representation $\boldsymbol{\tau}_{\mathrm{t}}$ twisted by the character of $\mathbf{J}_{\mathrm{t}}$ trivial on $\mathbf{J}_{\mathrm{t}}^1$ given by $(\chi \circ \mathrm{N}_{T/F}) \circ \det_T$, where $\det_T$ is the determinant on $\mathbf{M}_m(T)$. Assertion (2) is given by Proposition 4.41. □

Corollary 4.43. — *Let μ be a tamely ramified character of $F_0^\times$. A cuspidal representation π of $\mathrm{GL}_n(F)$ with endo-class $\boldsymbol{\Theta}$ is distinguished by μ if and only if π_{t} is distinguished by $\mu \circ \mathrm{N}_{T_0/F_0}$.*

Proof. — Fix a tamely ramified character ξ of $F^\times$ extending μ. Then π is μ-distinguished if and only if $\pi\xi^{-1}$ is distinguished, and $(\pi\xi^{-1})_{\mathrm{t}}$ is isomorphic to $\pi_{\mathrm{t}}(\xi^{-1}\circ\mathrm{N}_{T/F})$. Thus π is μ-distinguished if and only if π_{t} is distinguished by the character $\xi\circ\mathrm{N}_{T/F}|_{T_0^\times} = \mu\circ\mathrm{N}_{T_0/F_0}$. $\square$

Finally, let us describe the compatibility of the process (4.22) with the description of cuspidal representations in terms of supercuspidal ones of 4.8.

Proposition 4.44. — *Let π be a cuspidal representation of G with endo-class $\boldsymbol{\Theta}$ and $r = r(\pi)$. Let ρ be a supercuspidal representation of $\mathrm{GL}_{n/r}(F)$ such that π is isomorphic to $\mathrm{St}_r(\rho)$. Then π_{t} is isomorphic to $\mathrm{St}_r(\rho_{\mathrm{t}})$.*

Remark 4.45. — Note that this makes sense since, by Corollary 4.28, the representations π, ρ have the same endo-class $\boldsymbol{\Theta}$, thus the same quadratic extension T/T_0.

Proof. — The representation π contains a type of the form $(\mathbf{J}, \boldsymbol{\kappa}_\theta\otimes\boldsymbol{\tau})$ for a unique representation $\boldsymbol{\tau}$ of $\mathbf{J}$ trivial on $\mathbf{J}^1$. Fix a σ-selfdual simple stratum $[\mathfrak{a},\beta]$ and isomorphisms (4.17) as in Proposition 4.5. Associated with $\boldsymbol{\tau}$, there are a tamely ramified character ω of T, and a cuspidal representation $\mathrm{V} = \mathrm{st}_r(\varrho)$ of $\mathrm{GL}_m(\boldsymbol{l})$, for some supercuspidal representation ϱ of $\mathrm{GL}_{m/r}(\boldsymbol{l})$. The representation $\boldsymbol{\tau}$ is entirely determined by the fact that

– its restriction to $\mathbf{J}^0$ is the inflation of V,
– its restriction to $E^\times$ is a multiple of the character $\omega\circ\mathrm{N}_{E/T}$.

We now use the results of 4.4 and 4.5 for $u = r$. Let θ_* be the simple character associated with θ by Lemma 4.10. Thanks to Corollary 4.21, the representation $\boldsymbol{\kappa}_{\theta_*}$ corresponds to $\boldsymbol{\kappa}_\theta$ via the map (4.10). Paragraph 4.8 says that ρ contains the type $(\mathbf{J}_*, \boldsymbol{\kappa}_{\theta_*}\otimes\boldsymbol{\tau}_*)$, where $\boldsymbol{\tau}_*$ is the representation of $\mathbf{J}_*$ trivial on $\mathbf{J}_*^1$ determined by

– its restriction to $\mathbf{J}_*^0$ is the inflation of ϱ,
– its restriction to $E^\times$ is a multiple of $\omega_*\circ\mathrm{N}_{E/T}$, where ω_* is a tamely ramified character of $T^\times$ such that $\omega_*^r = \omega$.

Thus ρ_{t} is compactly induced from the level 0 type $(\mathbf{J}_{*,\mathrm{t}}, \boldsymbol{\tau}_{*,\mathrm{t}})$ where $\boldsymbol{\tau}_{*,\mathrm{t}}$ is determined by

– its restriction to $\mathbf{J}_{*,\mathrm{t}}^0$ is the inflation of $\varrho^{(p^{-e})}$,
– its restriction to $T^\times$ is a multiple of ω_*.

Thus $\mathrm{St}_r(\rho_{\mathrm{t}})$ is compactly induced from the level 0 type $(\mathbf{J}_{\mathrm{t}}, \boldsymbol{\delta})$ where $\boldsymbol{\delta}$ is determined by

– its restriction to $\mathbf{J}_{\mathrm{t}}^0$ is the inflation of $\mathrm{st}_r(\varrho^{(p^{-e})}) \simeq \mathrm{V}^{(p^{-e})}$,
– its restriction to $T^\times$ is a multiple of $\omega_*^r = \omega$.

It follows that $\boldsymbol{\delta}$ is isomorphic to $\boldsymbol{\tau}_{\mathrm{t}}$, whence $\mathrm{St}_r(\rho_{\mathrm{t}})$ is isomorphic to π_{t}. $\square$

4.12. Finally, let π be a σ-selfdual cuspidal representation of G, of level 0. It has a central character c_π and its generic type $(\mathbf{J}, \boldsymbol{\lambda})$ defines a cuspidal representation V of $\mathrm{GL}_n(\boldsymbol{k})$. Assume that $n \neq 1$. In the spirit of Proposition 4.34, we give a necessary and sufficient condition for π to be distinguished by $\mathrm{GL}_n(F_0)$ in terms of c_π and V.

Theorem 4.46. — *The representation π is $\mathrm{GL}_n(F_0)$-distinguished if and only if its central character c_π is trivial on $F_0^\times$ and*

(1) *if F/F_0 is unramified, then V is $\mathrm{GL}_n(\boldsymbol{k}_0)$-distinguished,*

(2) *if F/F_0 is ramified, then n is even,* V *is* $\mathrm{GL}_{n/2}(\boldsymbol{k})\times\mathrm{GL}_{n/2}(\boldsymbol{k})$*-distinguished, the vector space of* $\mathrm{GL}_{n/2}(\boldsymbol{k})\times\mathrm{GL}_{n/2}(\boldsymbol{k})$*-invariant linear forms on* V *has dimension* 1 *and*

$$s = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \in \mathrm{GL}_n(\boldsymbol{k}) \tag{4.23}$$

acts on this space by the sign $c_\pi(\varpi)$, where ϖ is any uniformizer of F.

Proof. — By Proposition 4.34, the representation π is $\mathrm{GL}_n(F_0)$-distinguished if and only if $\boldsymbol{\lambda}$ is $\mathbf{J}\cap G^\sigma$-distinguished. In the unramified case, the result follows from the fact that $\mathbf{J}\cap G^\sigma$ is generated by $F_0^\times$ and $\mathbf{J}^0\cap G^\sigma$ (see Lemma 4.15) and $(\mathbf{J}^0\cap G^\sigma)/(\mathbf{J}^1\cap G^\sigma)$ identifies with $\mathrm{GL}_n(\boldsymbol{k}_0)$.

Assume now that we are in the ramified case. Since c_π is trivial on $F_0^\times$ and $1+\mathfrak{p}_F$, its value at ϖ does not depend on the choice of a uniformizer ϖ of F. We thus may and will assume that $\sigma(\varpi)=-\varpi$, thus $c_\pi(\varpi)\in\{-1,1\}$. By Lemma 4.15 again, $\mathbf{J}\cap G^\sigma$ is generated by $F_0^\times$, $\mathbf{J}^0\cap G^\sigma$ and ϖw, where the element $w\in\mathbf{J}^0$ is defined by (4.12). The quotient $(\mathbf{J}^0\cap G^\sigma)/(\mathbf{J}^1\cap G^\sigma)$ identifies with $\mathrm{GL}_{n/2}(\boldsymbol{k})\times\mathrm{GL}_{n/2}(\boldsymbol{k})$, the image of w in $\mathbf{J}^0/\mathbf{J}^1\simeq\mathrm{GL}_n(\boldsymbol{k})$ is the element s, the space of $\mathrm{GL}_{n/2}(\boldsymbol{k})\times\mathrm{GL}_{n/2}(\boldsymbol{k})$-invariant linear forms on V has dimension 1 by **[32]** Corollary 2.16 and ϖs acts on this space as $c_\pi(\varpi)s$. □

Putting Theorems 4.42 and 4.46 together, we have thus reduced the problem of characterizing distinguished cuspidal representations of $\mathrm{GL}_n(F)$ to a problem about distinction of cuspidal representations of finite general linear groups.

4.13. In this paragraph, we prove Lemma 4.37(3). First, by Remarks 4.2, 4.36, changing (4.17) and t does not affect the isomorphism class of $\boldsymbol{\tau}_{\mathrm{t}}$. Let $[\mathfrak{a},\beta']$ be another σ-selfdual maximal simple stratum such that $\theta\in\mathcal{C}(\mathfrak{a},\beta')$. Conjugating by $\mathbf{J}^1$, we may and will assume that the maximal tamely ramified extension of F in $E'=F[\beta']$ is T. This gives us another isomorphism $\boldsymbol{\pi}'$ from $\mathbf{J}/\mathbf{J}^1$ to $\mathbf{J}_{\mathrm{t}}/\mathbf{J}^1_{\mathrm{t}}$. By construction, it coincides with $\boldsymbol{\pi}$ on $\mathbf{J}^0/\mathbf{J}^1$ and the image of $x\mathbf{J}^1$ by $\boldsymbol{\pi}'$ is equal to $\mathrm{N}_{E'/T}(x)\mathbf{J}^1$ for all $x\in E'^\times$. We are going to prove that $\boldsymbol{\pi}'$ is equal to $\boldsymbol{\pi}$. The result will then follow from the fact that $\boldsymbol{\tau}_{\mathrm{t}}$ is equal to $\boldsymbol{\tau}\circ\boldsymbol{\pi}$. For this, it suffices to prove that $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ take the same value at some given uniformizer of E'. Let ϖ, ϖ' be uniformizers of E, E' respectively.

The centre of $\mathbf{J}/\mathbf{J}^1$ is $E^\times\mathbf{J}^1/\mathbf{J}^1=E'^\times\mathbf{J}^1$, thus $E'^\times\subseteq E^\times\mathbf{J}^1$. We thus may write $\varpi'\in\varpi\zeta\mathbf{J}^1$ for some root of unity ζ of $T^\times$ of order prime to p. Changing ϖ' to $\varpi'\zeta^{-1}$, we may and will assume that $\varpi'\in\varpi\mathbf{J}^1$. It suffices to prove the following claim.

Claim 4.47. — *We have* $\mathrm{N}_{E'/T}(\varpi')\equiv\mathrm{N}_{E/T}(\varpi)\mod 1+\mathfrak{p}_T$.

First, this is true when $m=1$. Indeed, writing G_T for the centralizer of T in G and $\det_T$ for the determinant on G_T, we have $\det_T(x)=\mathrm{N}_{E/T}(x)$ for all $x\in E^\times$, thus

$$\mathrm{N}_{E'/T}(\varpi')=\det{}_T(\varpi')\in\det{}_T(\varpi)\cdot\det{}_T(\mathbf{J}^1\cap G_T)=\mathrm{N}_{E/T}(\varpi)\cdot(1+\mathfrak{p}_T).$$

Now assume that $m>1$. We use the results of 4.4 for $u=m$. Let $\theta_*\in\mathcal{C}(\mathfrak{a}_*,\beta)$ denote the transfer of θ as in Lemma 4.10. Fix a T-embedding

$$\iota: E'\to\mathrm{End}_T(V_*)\subseteq\mathrm{End}_F(V_*)$$

such that $\mathfrak{a}_*$ is normalized by $\iota E'^\times$, and transfer θ to $\theta_\bullet\in\mathcal{C}(\mathfrak{a}_*,\iota\beta')$ in the sense of **[8]** 3.6. The simple character θ is in $\mathcal{C}(\mathfrak{a},\beta)\cap\mathcal{C}(\mathfrak{a},\beta')$. It follows from **[5]** Theorem 8.7 that θ_*, $\theta_\bullet$ intertwine in G_*, and from [8] Theorem 3.5.11 that $\theta_\bullet = \theta_*^x$ for some $x \in \mathcal{K}_{\mathfrak{a}_*}$. Changing ι to $\mathrm{Ad}(x) \circ \iota$, we thus may assume that $\theta_\bullet = \theta_* \in \mathcal{C}(\mathfrak{a}_*, \beta) \cap \mathcal{C}(\mathfrak{a}_*, \iota\beta')$. By using ι, we get a diagonal embedding

$$E' \to \mathrm{End}_T(V_*) \times \cdots \times \mathrm{End}_T(V_*) \subseteq \mathrm{End}_T(V)$$

denoted ϕ, which is the identity on $T^\times$. The Skolem-Noether theorem implies that $\phi = \mathrm{Ad}(g)$ for some $g \in G_T$. Conjugating by g, we thus may assume that $E^\times$ and $E'^\times$ are both diagonal in M. The identity $\varpi' \in \varpi\mathbf{J}^1$ thus implies $\varpi' \in \varpi\mathbf{J}^1_*$. We are thus reduced to the case where $m = 1$.

Remark 4.48. — The fact that $\boldsymbol{\tau}_\mathrm{t}$ does not depend on the choice of β is claimed in [7] Lemma 3.6. However, Property (b) of this lemma does not hold: using the notation of *ibid.*, the restriction of $\lambda_\xi^{\mathbf{J}}$ to $P^\times$ is a multiple of the character $\xi \circ \mathrm{N}_{P/T}$, whereas the restriction of $(\xi|_{T^\times})^{\mathbf{J}}$ to $P^\times$ is $(\xi \circ \mathrm{N}_{P/T})^s$. (Note that P corresponds to our E, and s corresponds to our m.)

5. The odd case

In this section, p is odd, ℓ is any prime number different from p and the field R has characteristic ℓ. This section is devoted to the proof of the following theorem.

Theorem 5.1. — *Let π be a σ-selfdual cuspidal non-supercuspidal R-representation of $\mathrm{GL}_n(F)$. Assume that the integer $r = r(\pi)$ is odd, thus π is isomorphic to $\mathrm{St}_r(\rho)$ for a uniquely determined σ-selfdual supercuspidal representation ρ of $\mathrm{GL}_k(F)$, with $k = n/r$. If π is $\mathrm{GL}_n(F_0)$-distinguished, then*

(1) *the relative degree $m = m(\pi)$ and the ramification index of T/T_0 have the same parity,*
(2) *the representation ρ is $\mathrm{GL}_k(F_0)$-distinguished.*

Note that the fact that r is odd and $r \neq 1$ implies that $\ell \neq 2$.

5.1. Before we start the proof of Theorem 5.1, let us prove the following Disjunction Theorem.

Corollary 5.2. — *Let π be a σ-selfdual cuspidal R-representation of $\mathrm{GL}_n(F)$. Assume that $r(\pi)$ is odd. Then π cannot be both distinguished and $\varkappa$-distinguished.*

Proof. — Assume that π is both distinguished and $\varkappa$-distinguished, and let χ be a tamely ramified character of $F^\times$ extending $\varkappa$. Then $\pi\chi$ is distinguished, it is isomorphic to $\mathrm{St}_r(\rho\chi)$ and $\rho\chi$ is supercuspidal and σ-selfdual. Theorem 5.1 applied to both π and $\pi\chi$ implies that ρ is both distinguished and $\varkappa$-distinguished. This contradicts Theorem 3.2. □

We also have the following Distinguished Lift Theorem.

Corollary 5.3. — *Let π be a $\mathrm{GL}_n(F_0)$-distinguished cuspidal $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_n(F)$ with $r(\pi)$ odd. There is a $\mathrm{GL}_n(F_0)$-distinguished integral generic $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_n(F)$ whose reduction mod ℓ contains π.*

Proof. — Write π as $\mathrm{St}_r(\rho)$ with ρ distinguished. Let μ be a distinguished integral cuspidal lift of ρ, which exists by Theorem 3.5. Then the generic representation $\mathrm{St}_r(\mu)$ satisfies the required property (see [1] Theorem 1.3 or [23] Corollary 4.2 when F has characteristic zero, and observe that the argument in [23] holds verbatim in positive characteristic thanks to [20, Theorem 4.7]). □

Remark 5.4. — A $\mathrm{GL}_n(F_0)$-distinguished integral generic $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_n(F)$ as in Corollary 5.3 may not be cuspidal. See Section 6 for the classification of all distinguished cuspidal $\overline{\mathbb{F}}_\ell$-representations of $\mathrm{GL}_n(F)$ having a cuspidal distinguished lift to $\overline{\mathbb{Q}}_\ell$.

Finally, compare Theorem 5.1 with the following finite field analogue.

Proposition 5.5. — *Let $\boldsymbol{k}/\boldsymbol{k}_0$ be a quadratic extension of finite fields of characteristic p. Let ϱ be a supercuspidal R-representation of $\mathrm{GL}_f(\boldsymbol{k})$ for some $f \geqslant 1$, and r be an odd integer such that $\mathrm{st}_r(\varrho)$ is cuspidal. If $\mathrm{st}_r(\varrho)$ is distinguished by $\mathrm{GL}_{fr}(\boldsymbol{k}_0)$, then ϱ is distinguished by $\mathrm{GL}_f(\boldsymbol{k}_0)$.*

Proof. — First, **[32]** Remark 4.3 tells us that $\mathrm{st}_r(\varrho)$ is σ-selfdual (where σ is here the nontrivial automorphism of $\boldsymbol{k}/\boldsymbol{k}_0$). Proposition 3.9 implies that ϱ is σ-selfdual. By **[32]** Lemma 2.5, it is distinguished by $\mathrm{GL}_f(\boldsymbol{k}_0)$. □

5.2. Let us prove Theorem 5.1(1). Since r is odd, m has the same parity as m/r, and, since π is non-supercuspidal, we have $r > 1$, thus $m > 1$. It follows from **[32]** Proposition 7.1 that, if m is odd, T/T_0 is unramified, and from Proposition 4.35 that, if m is even, T/T_0 is ramified.

5.3. We now start the proof of Theorem 5.1(2). We thus have a distinguished cuspidal representation π of $\mathrm{GL}_n(F)$, which we write $\mathrm{St}_r(\rho)$ with ρ supercuspidal and σ-selfdual.

Associated with π, there are a positive divisor m of n, a quadratic extension T/T_0 and a cuspidal representation π_{t} of $\mathrm{GL}_m(T)$. By Proposition 4.41, the representation π_{t} has level 0, it is distinguished by $\mathrm{GL}_m(T_0)$ and it satisfies $r(\pi_{\mathrm{t}}) = r$.

Similarly, associated with ρ, there is a supercuspidal σ-selfdual representation ρ_{t} of $\mathrm{GL}_{m/r}(T)$, which has level 0, and is distinguished by $\mathrm{GL}_{m/r}(T_0)$ if and only if ρ is distinguished by $\mathrm{GL}_k(F_0)$. By Proposition 4.44, the representation π_{t} is isomorphic to $\mathrm{St}_r(\rho_{\mathrm{t}})$.

It follows that, in order to prove Theorem 5.1(2), we may assume that π has level 0.

5.4. Let π be a distinguished cuspidal representation of level 0 of $\mathrm{GL}_n(F)$. Associated with it, there are its central character c_π and a cuspidal representation V of $\mathrm{GL}_n(\boldsymbol{k})$ (see §4.7).

The representation π is isomorphic to $\mathrm{St}_r(\rho)$ for a unique σ-selfdual supercuspidal representation ρ, and ρ has level 0. Associated with ρ, there are its central character c_ρ and a supercuspidal representation ϱ of $\mathrm{GL}_k(\boldsymbol{k})$. We have the relation

$$c_\pi = (c_\rho)^r \tag{5.1}$$

and, by Proposition 4.29, the representation V is isomorphic to $\mathrm{st}_r(\varrho)$.

Since π is distinguished, its central character is trivial on $F_0^\times$. Since ρ is σ-selfdual, the restriction of c_ρ to $F_0^\times$ has order at most 2. Restricting the relation (5.1) to $F_0^\times$, and since r is odd, we deduce that c_ρ is trivial on $F_0^\times$.

5.5. In this paragraph, we will assume that F/F_0 is unramified. By Theorem 4.46, the representation V is distinguished by $\mathrm{GL}_n(\boldsymbol{k}_0)$. By **[32]** Remark 4.3, it is thus σ-selfdual, that is

$$\mathrm{st}_r(\varrho) \simeq \mathrm{V} \simeq \mathrm{V}^{\sigma\vee} \simeq \mathrm{st}_r(\varrho^{\sigma\vee}).$$

It follows from Proposition 3.9 that ϱ is σ-selfdual. By **[32]** Lemma 2.5, it is thus distinguished by $\mathrm{GL}_k(\boldsymbol{k}_0)$. Applying Theorem 4.46 again, we deduce that ρ is distinguished by $\mathrm{GL}_k(F_0)$. This proves Theorem 5.1 in the unramified case.

5.6. From now on, and until the end of this section, we assume that F/F_0 is ramified. By Theorem 4.46, we may write $n = 2u$ for some integer $u \geqslant 1$. We write $G = G_n = \mathrm{GL}_n(\boldsymbol{k})$, $H = H_n = \mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$ and $K = K_n$ for the normalizer of H in G, which is generated by H and

$$s = s_n = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \in G$$

where id is the identity in $\mathrm{GL}_u(\boldsymbol{k})$. It will be convenient to introduce the following definition.

Definition 5.6. — Let $c \in \{-1, 1\} \subseteq R^\times$. An irreducible R-representation V of G is said to be c-distinguished by H if V is H-distinguished and s acts on the space of H-invariant linear forms on V by multiplication by c.

By Theorem 4.46, the representation V is H-distinguished and s acts on the 1-dimensional vector space $\mathrm{Hom}_H(\mathrm{V}, R)$ by the sign $c = c_\pi(\varpi)$. In other words, V is c-distinguished by H. We are now reduced to proving the following result. (Note that k is even since n is even and r is odd.)

Proposition 5.7. — *The supercuspidal representation ϱ is c-distinguished by H_k.*

Indeed, since r is odd, the identity (5.1) together with Proposition 5.7 will give us $c = c_\rho(\varpi)$. It will then follow from Theorem 4.46 that ρ is $\mathrm{GL}_k(F_0)$-distinguished.

5.7. Let π be an irreducible $\overline{\mathbb{F}}_\ell$-representation of G. The natural map

$$\mathrm{Hom}_{\overline{\mathbb{F}}_\ell H}(\pi, \overline{\mathbb{F}}_\ell) \otimes R \to \mathrm{Hom}_{RH}(\pi \otimes R, \overline{\mathbb{F}}_\ell \otimes R)$$

defined by $f \otimes r \mapsto r(f \otimes \mathrm{id})$ is an isomorphism of R-vector spaces. Moreover, these spaces have dimension at most 1, and it follows from this isomorphism that π is c-distinguished by H if and only if $\pi \otimes R$ is c-distinguished by H.

Since G is finite, any irreducible R-representation of G is defined over $\overline{\mathbb{F}}_\ell$, that is, isomorphic to $\pi_0 \otimes R$ for some irreducible $\overline{\mathbb{F}}_\ell$-representation π_0 of G. In order to prove Proposition 5.7, we thus may assume that R is equal to $\overline{\mathbb{F}}_\ell$.

5.8. From now on, we assume that R is equal to $\overline{\mathbb{F}}_\ell$. The remaining part of the section will be devoted to the proof of Proposition 5.7.

Lemma 5.8. — *There exists a c-distinguished irreducible $\overline{\mathbb{Q}}_\ell$-representation of G whose reduction mod ℓ contains* V.

Proof. — Let χ denote the unique $\overline{\mathbb{F}}_\ell$-character of K trivial on H such that $\chi(s) = c$. Since V is c-distinguished, it embeds in $\mathrm{Ind}_K^G(\chi)$. Equivalently, the representation $\mathrm{Ind}_K^G(\chi)$, which is selfdual (as χ is equal to χ^{-1}), surjects onto the contragredient W of V. Let Π be a projective indecomposable $\overline{\mathbb{F}}_\ell$-representation of G whose unique irreducible quotient is isomorphic to W. Let $\widetilde{\Pi}$ be the unique projective $\overline{\mathbb{Z}}_\ell$-representation of G such that $\widetilde{\Pi} \otimes \overline{\mathbb{F}}_\ell$ is isomorphic to Π. Let Λ be a surjective homomorphism from $\mathrm{Ind}_K^G(\chi)$ to W. By projectivity, it defines a non-zero homomorphism Λ' from Π to $\mathrm{Ind}_K^G(\chi)$, then a non-zero homomorphism Λ'' from $\widetilde{\Pi}$ to $\mathrm{Ind}_K^G(\widetilde{\chi})$, where $\widetilde{\chi}$ is the canonical $\overline{\mathbb{Z}}_\ell$-lift of χ.

By inverting ℓ, we deduce that there is an irreducible $\overline{\mathbb{Q}}_\ell$-representation X of G occurring in each of the semi-simple representations $J = \mathrm{Ind}_K^G(\widetilde{\chi}) \otimes \overline{\mathbb{Q}}_\ell$ and $\widetilde{\Pi} \otimes \overline{\mathbb{Q}}_\ell$. It is thus c-distinguished and, by **[36]** 15.4, its reduction mod ℓ contains W.

Now observe that, since $\widetilde{\chi}$ is quadratic, J is selfdual. The contragredient of X is thus c-distinguished and its reduction mod ℓ contains V. $\square$

5.9. Let τ be a c-distinguished irreducible $\overline{\mathbb{Q}}_\ell$-representation as in Lemma 5.8. Consider its cuspidal support: there are positive integers $n_1, \dots, n_t$ such that $n_1 + \dots + n_t = n$ and, for each i in $\{1, \dots, t\}$, a cuspidal irreducible $\overline{\mathbb{Q}}_\ell$-representation ρ_i of $\mathrm{GL}_{n_i}(\boldsymbol{k})$, such that τ occurs as a component of the parabolically induced representation $\rho_1 \times \dots \times \rho_t$, denoted W. The representation W is thus c-distinguished. We claim the following.

Claim 5.9. — *There is an $i \in \{1, \dots, t\}$ such that n_i is even and ρ_i is c-distinguished by H_{n_i}.*

Before proving this claim in the next paragraph, let us explain how it implies Proposition 5.7.

Propositions 3.9 and 3.11 imply that, for each $i \in \{1, \dots, t\}$, the reduction mod ℓ of ρ_i is irreducible and cuspidal, of the form $\mathrm{st}_{r_i}(\varrho_i)$ for a unique positive integer r_i and a unique supercuspidal representation ϱ_i. Since the reduction mod ℓ of τ contains V, the representation V occurs as an irreducible component of the parabolically induced representation $\mathbf{r}_\ell(\rho_1) \times \dots \times \mathbf{r}_\ell(\rho_t)$. Uniqueness of the supercuspidal support implies that $\varrho_i \simeq \varrho$ for all i. It follows that either $r_i = 1$ or $r_i = e(\varrho)\ell^{v_i}$ for some $v_i \geqslant 0$. Observe that, as $r = e(\varrho)\ell^v$ for some $v \geqslant 0$ and r is odd, the integer $e(\varrho)$ is odd, thus r_i is odd in any case, for all i.

Fix an integer i as in Claim 5.9, and let ξ_i be a parameter for ρ_i in the sense of Definition 3.12. It is a $\mathrm{Gal}(\boldsymbol{k}_{n_i}/\boldsymbol{k})$-regular $\overline{\mathbb{Z}}_\ell$-character of $\boldsymbol{k}_{n_i}^\times$. By Proposition 3.13, it is trivial on $\boldsymbol{k}_{u_i}^\times$, where u_i is defined by $n_i = 2u_i$, and it takes the unique element of $\boldsymbol{k}_{n_i}^\times/\boldsymbol{k}_{u_i}^\times$ of order 2 to $-c$.

Since the reduction mod ℓ of ρ_i is $\mathrm{st}_{r_i}(\varrho)$, the reduction mod ℓ of ξ_i takes the form $\vartheta \circ \mathrm{N}_{\boldsymbol{k}_{n_i}/\boldsymbol{k}_k}$ where ϑ is a parameter for ϱ.

Since $n_i = r_i k$ and r_i is odd, ϑ is trivial on $\boldsymbol{k}_l^\times$ (where $k = 2l$) and takes the element of $\boldsymbol{k}_k^\times/\boldsymbol{k}_l^\times$ of order 2 to $-c$.

By Proposition 3.13, the canonical $\overline{\mathbb{Z}}_\ell$-lift of ϑ is the parameter of a c-distinguished $\overline{\mathbb{Q}}_\ell$-lift of ϱ, which implies that ϱ is c-distinguished (see Remark 3.14). This proves Proposition 5.7.

5.10. The remaining part of this section will be devoted to the proof of Claim 5.9. We follow the argument of **[25]** Section 3, which simplifies in our situation since we deal with finite groups. Let $\mathbf{A}$ denote the set of t-uples $\alpha = (\alpha_1, \dots, \alpha_t)$ where

(1) for each i, the element α_i is a family of $t+1$ non-negative integers of the form

$$\alpha_i = (n_{i,1}, \dots, n_{i,i-1}, n_{i,i}^+, n_{i,i}^-, n_{i,i+1}, \dots, n_{i,t})$$

of sum n_i,

(2) one has $n_{1,1}^+ + \dots + n_{t,t}^+ = n_{1,1}^- \dots + n_{t,t}^-$ and $n_{i,j} = n_{j,i}$ for all $i \neq j$.

For an $\alpha \in \mathbf{A}$, it will be convenient to set $n_{i,i} = n_{i,i}^+ + n_{i,i}^-$ for each integer $i \in \{1, \dots, t\}$.

As in **[25]** 3.1, the set $\mathbf{A}$ parametrizes the set of (P, H)-double cosets in G, where P in the parabolic subgroup of G generated by upper triangular matrices and the standard Levi subgroup M isomorphic to $G_{n_1} \times \dots \times G_{n_t}$. Let us explain how this parametrization works. Associated with any $\alpha \in \mathbf{A}$, there are

– a standard Levi subgroup

$$M_\alpha = (G_{n_{1,1}} \times G_{n_{1,2}} \times \dots \times G_{n_{1,t}}) \times \dots \times (G_{n_{t,1}} \times G_{n_{t,2}} \times \dots \times G_{n_{t,t}}) \subseteq M,$$

– a diagonal element

$$d_\alpha = \mathrm{diag}\left(\begin{pmatrix}\mathrm{id}_{n^+_{1,1}} & \\ & -\mathrm{id}_{n^-_{1,1}}\end{pmatrix}, \mathrm{id}_{n_{1,2}}, \dots, \mathrm{id}_{n_{1,t}}, \dots, \mathrm{id}_{n_{t,1}}, \mathrm{id}_{n_{t,2}}, \dots, \begin{pmatrix}\mathrm{id}_{n^+_{t,t}} & \\ & -\mathrm{id}_{n^-_{t,t}}\end{pmatrix}\right) \in M_\alpha,$$

– a permutation matrix $w_\alpha \in G$ defined as follows: decompose $\{1, \dots, n\}$ as the disjoint union of intervals $J_{i,j} = \{a_{i,j}, a_{i,j}+1, \dots, b_{i,j}\}$ of length $n_{i,j}$, for each $i, j \in \{1, \dots, t\}$, where $a_{1,1} = 1$, $a_{i,j+1} = b_{i,j} + 1$ if $j \neq t$ and $a_{i+1,1} = b_{i,t} + 1$ if $i \neq t$; then w_α is the involution which

- restricts to the identity on $J_{i,i}$ for each i,
- exchanges the intervals $J_{i,j}$ and $J_{j,i}$ if $i \neq j$, and sends the kth element of any of these intervals to the kth element of the other one, for all $k \in \{1, \dots, n_{i,j}\}$.

A system of representatives $(x_\alpha)_{\alpha \in \mathbf{A}}$ of (P, H)-double cosets in G is then obtained by any choice of $x_\alpha \in G$ such that

$$x_\alpha \begin{pmatrix}\mathrm{id}_u & \\ & -\mathrm{id}_u\end{pmatrix} x_\alpha^{-1} = e_\alpha, \tag{5.2}$$

where $e_\alpha = d_\alpha w_\alpha$.

Definition 5.10. — An $\alpha \in \mathbf{A}$ is called *admissible* if, for any i, there exists a unique j such that $n_{i,j} \neq 0$. This defines an involution $\sigma_\alpha : i \mapsto j$ on $\{1, \dots, t\}$.

When this is the case, let us write H_α for the subgroup of M made of the $\mathrm{diag}(g_1, \dots, g_t) \in M$ such that

$$\begin{array}{rcll} g_{\sigma_\alpha(i)} & = & g_i & \text{for all } i \in \{1, \dots, t\}, \\ g_i & \in & \mathrm{GL}_{n^+_{i,i}}(\boldsymbol{k}) \times \mathrm{GL}_{n^-_{i,i}}(\boldsymbol{k}) & \text{for all } i \text{ fixed by } \sigma_\alpha. \end{array}$$

Moreover, if $n^+_{i,i} = n^-_{i,i}$ for all $i \in \{1, \dots, t\}$, we define a matrix $k_\alpha = \mathrm{diag}(k_1, \dots, k_t) \in M$ by

$$k_i = \mathrm{id}_{n_i} = -k_{\sigma_\alpha(i)} \text{ if } i < \sigma_\alpha(i), \quad k_i = s_{n_i} \text{ if } i = \sigma_\alpha(i).$$

This matrix normalizes H_α, and we write K_α for the group generated by H_α and k_α.

We denote by θ_α the inner automorphism of the group $\mathrm{PGL}_n(\boldsymbol{k})$ induced by conjugacy by e_α (which normalizes M_α). It is not hard to check that:

Lemma 5.11. — *Let Z denote the centre of G.*

(1) *An $\alpha \in \mathbf{A}$ is admissible if and only if M/Z is θ_α-stable in $G/Z = \mathrm{PGL}_n(\boldsymbol{k})$.*

(2) *Suppose that $\alpha \in \mathbf{A}$ is admissible. The preimage of $(M/Z)^{\theta_\alpha}$ in G, denoted by L_α, is*

$$\begin{cases} K_\alpha & \text{if } n^+_{ii} = n^-_{ii} \text{ for all } i, \\ H_\alpha & \text{otherwise.} \end{cases}$$

When $L_\alpha = K_\alpha$, we denote by χ_α the character of K_α trivial on H_α and sending k_α to c. Otherwise, we set χ_α to be the trivial character of $L_\alpha = H_\alpha$. We have the following lemma.

Lemma 5.12. — *Suppose that α is admissible and $L_\alpha = K_\alpha$. Then there is a system of representatives $(x_\alpha)_{\alpha \in \mathbf{A}}$ of (P, H)-double cosets of G satisfying both* (5.2) *and $x_\alpha s_n x_\alpha^{-1} = k_\alpha$.*

Proof. — Let us set $m_i = n^+_{i,i} = n^-_{i,i} = n_{i,i}/2$ for any integer $i \in \{1, \dots, t\}$ such that $\sigma_\alpha(i) = i$. For each $\alpha \in \mathbf{A}$, we look for a matrix $x_\alpha \in G$ such that

$$x_\alpha \begin{pmatrix}\mathrm{id}_u & \\ & -\mathrm{id}_u\end{pmatrix} x_\alpha^{-1} = e_\alpha \quad \text{and} \quad x_\alpha \begin{pmatrix} & \mathrm{id}_u \\ \mathrm{id}_u & \end{pmatrix} x_\alpha^{-1} = k_\alpha.$$

To make an explicit choice of $x_\alpha \in G$, it will be convenient to introduce the matrix $v_\alpha \in G$ defined as follows: for all integers $i, j \in \{1, \dots, t\}$, the (i,j)-block of v_α in $\mathbf{M}_{n_i,n_j}(\boldsymbol{k})$ is

– the identity matrix id_{n_i} if $j = i$ or $j = \sigma_\alpha(i) < i$,
– its opposite $-\mathrm{id}_{n_i}$ if $j = \sigma_\alpha(i) > i$,
– and 0 otherwise.

Then we choose y_α the permutation matrix corresponding to the permutation of minimal length (with the usual generators of the symmetric group) satisfying

$$y_\alpha \begin{pmatrix} \mathrm{id}_u & \\ & -\mathrm{id}_u \end{pmatrix} y_\alpha^{-1} = l_\alpha$$

where $l_\alpha = \mathrm{diag}(l_1, \dots, l_t) \in M$ is defined by

$$l_i = \mathrm{id}_{n_i} = -l_{\sigma_\alpha(i)} \text{ if } i < \sigma_\alpha(i), \quad l_i = \begin{pmatrix} \mathrm{id}_{m_i} & \\ & -\mathrm{id}_{m_i} \end{pmatrix} \text{ if } i = \sigma_\alpha(i).$$

Finally we put $x_\alpha = v_\alpha y_\alpha$, which has the desired property thanks to the equality

$$\begin{pmatrix} \mathrm{id}_k & -\mathrm{id}_k \\ \mathrm{id}_k & \mathrm{id}_k \end{pmatrix} \begin{pmatrix} \mathrm{id}_k & \\ & -\mathrm{id}_k \end{pmatrix} \begin{pmatrix} \mathrm{id}_k & -\mathrm{id}_k \\ \mathrm{id}_k & \mathrm{id}_k \end{pmatrix}^{-1} = s_{2k} \tag{5.3}$$

valid for any $k \geqslant 1$. With this choice, the careful reader checks by a computation relying again on Equality (5.3), that $y_\alpha s_n y_\alpha^{-1} = v_\alpha^{-1} k_\alpha v_\alpha$, which is the desired equality. □

Now we have the following lemma.

Lemma 5.13. — *There is an admissible $\alpha \in \mathbf{A}$ such that $\mathrm{Hom}_{L_\alpha}(\rho_1 \otimes \cdots \otimes \rho_t, \chi_\alpha)$ is non-zero.*

Proof. — Given any subgroup X of G, we will write $\overline{X}$ for its image in $G/Z = \mathrm{PGL}_n(\boldsymbol{k})$. In particular, we have $\overline{G} = G/Z$. Note that $\overline{K} = K/Z$ is the subgroup of $\overline{G}$ made of all elements fixed by conjugacy by

$$\begin{pmatrix} \mathrm{id}_u & 0 \\ 0 & -\mathrm{id}_u \end{pmatrix} \bmod Z.$$

Let χ be the unique character of K trivial on H such that $\chi(s) = c$. The character that it induces on $\overline{K}$ will still be denoted by χ. Since W is c-distinguished, Mackey's formula implies that there is an $x \in G$ such that ρ, the representation of P inflated from $\rho_1 \otimes \cdots \otimes \rho_t$, is distinguished by the character $\chi^x|_{P \cap K^x}$. We derive from ρ a representation $\overline{\rho}$ of $\overline{P}$ distinguished by $\chi^x|_{\overline{P} \cap \overline{K}^x}$.

In fact, because H is a subgroup of K, we can chose x to be some x_α for $\alpha \in \mathbf{A}$. Now we claim that for all non admissible $\alpha \in \mathbf{A}$, the space

$$\mathrm{Hom}_{\overline{P} \cap \overline{K}^{x_\alpha}}(\overline{\rho}, \chi^{x_\alpha}) \tag{5.4}$$

is zero, so in particular x can only be of the form x_α for admissible α. Indeed, it follows from **[25]** Proposition 3.5 that, for a non admissible α, the group $P \cap H^{x_\alpha}$ contains a non trivial unipotent radical U_α of some parabolic subgroup of M, but the character χ_α is trivial on U_α, so if the space (5.4) were not reduced to zero, we would deduce that $\mathrm{Hom}_{U_\alpha}(\rho, R)$ is non-zero, contradicting the cuspidality of ρ. Hence we deduce $x = x_\alpha$ for an α which is admissible. In this case, $\overline{M} \cap \overline{K}^{x_\alpha}$ is equal to $\overline{M}^{\theta_\alpha}$, so that the space

$$\mathrm{Hom}_{\overline{M}^{\theta_\alpha}}(\overline{\rho}, \chi^{x_\alpha}) = \mathrm{Hom}_{L_\alpha}(\rho, \chi^{x_\alpha})$$

is non-zero. If $L_\alpha = K_\alpha$, then χ^{x_α} is equal to χ_α thanks to Lemma 5.12. Otherwise, χ^{x_α} and χ_α are trivial, thus equal. The statement now follows. □

Recall that, for any $i \in \{1, \dots, t\}$, either $r_i = 1$ or $r_i = e(\varrho)\ell^{v_i}$ for some $v_i \geqslant 0$.

Lemma 5.14. — *Let $\alpha \in \mathbf{A}$ be as in Lemma 5.13. Then the involution σ_α has a fixed point.*

Proof. — Let I_1 be the set of $i \in \{1, \dots, t\}$ such that $r_i > 1$, let t_1 be the cardinality of this set, and define $t_0 = t - t_1$. The identity $r = r_1 + \cdots + r_t$ implies

$$r = t_0 + e(\varrho) \cdot \sum_{i \in I_1} \ell^{v_i}.$$

Since r, $e(\varrho)$ and ℓ are odd, it follows that $t_0 + t_1 = t$ is odd. Thus σ_α has a fixed point. □

Claim 5.9 now follows from Lemmas 5.13, 5.14. Indeed, by Lemma 5.13, there is an admissible $\alpha \in \mathbf{A}$ such that $\mathrm{Hom}_{L_\alpha}(\rho_1 \otimes \cdots \otimes \rho_t, \chi_\alpha)$ is non-zero. Since L_α contains H_α, the representation ρ_i is, for all i fixed by σ_α, distinguished by the Levi subgroup

$$\mathrm{GL}_{n_{i,i}^+}(\boldsymbol{k}) \times \mathrm{GL}_{n_{i,i}^-}(\boldsymbol{k}).$$

By **[32]** Proposition 2.14, this implies that $n_{i,i}^+ = n_{i,i}^-$ for all i fixed by σ_α, thus $L_\alpha = K_\alpha$. By Lemma 5.14, there in an integer $i \in \{1, \dots, t\}$ fixed by σ_α. The ith block of $L_\alpha = K_\alpha$ is K_{n_i} and $\chi_\alpha(k_i) = c$. Thus ρ_i is c-distinguished.

6. Distinguished lift theorems

In this section, p is odd and ℓ is a prime number different from p. We look for a necessary and sufficient condition for an $\overline{\mathbb{F}}_\ell$-cuspidal representation of $\mathrm{GL}_n(F)$ to have a $\mathrm{GL}_n(F_0)$-distinguished lift to $\overline{\mathbb{Q}}_\ell$. Since the case of supercuspidal representations is treated by Theorems 3.3 and 3.4, we will concentrate on non-supercuspidal cuspidal representations.

6.1. We will prove the following two propositions.

Proposition 6.1. — *Let π be a σ-selfdual cuspidal $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_n(F)$ with quadratic extension T/T_0 and $m = m(\pi)$. Assume that $r = r(\pi) > 1$ is odd. Then π has a distinguished lift to $\overline{\mathbb{Q}}_\ell$ if and only if*

(1) *the representation π is isomorphic to $\mathrm{St}_r(\rho)$ for some $\mathrm{GL}_{n/r}(F_0)$-distinguished supercuspidal representation ρ of $\mathrm{GL}_{n/r}(F)$,*

(2) *if e, e_0 are the orders of the cardinalities of the residue fields $\boldsymbol{l}$, $\boldsymbol{l}_0$ of T, T_0 mod ℓ, then*

  (a) *either T/T_0 is unramified and e_0 is even,*

  (b) *or T/T_0 is ramified and m/e is odd.*

Note that the assumption "$r > 1$ is odd" in Proposition 6.1 implies that $\ell \neq 2$.

Proposition 6.2. — *Let π be a σ-selfdual cuspidal $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_n(F)$ with quadratic extension T/T_0 and $m = m(\pi)$. Assume that $r = r(\pi)$ is even. Then π has a distinguished lift to $\overline{\mathbb{Q}}_\ell$ if and only if*

(1) *the extension T/T_0 is ramified,*

(2) *one has $m = r$,*

(3) *if we denote by ν_0 the normalized absolute value of F_0, then*

  (a) *either $\ell \neq 2$ and π is isomorphic to $\mathrm{St}_m(\rho)$ for some supercuspidal representation ρ of $\mathrm{GL}_{n/m}(F)$ which is either $\varkappa$-distinguished or ν_0^{-1}-distinguished,*

(b) *or* $\ell = m = r = 2$*, the cardinality of the residue field of* T_0 *is congruent to* -1 *mod* 4 *and* π *is isomorphic to* $\mathrm{St}_2(\rho)$ *where* ρ *is a* $\mathrm{GL}_{n/2}(F_0)$*-distinguished supercuspidal representation of* $\mathrm{GL}_{n/2}(F)$*.*

We also formulate the following conjecture making Proposition 6.2 more precise.

Conjecture 6.3. — *If* π *is a* σ*-selfdual cuspidal* $\overline{\mathbb{F}}_\ell$*-representation of* $\mathrm{GL}_n(F)$ *such that the integer* $r(\pi)$ *is even, the following assertions are equivalent:*

(1) *the representation* π *is distinguished,*
(2) *the representation* π *has a distinguished lift to* $\overline{\mathbb{Q}}_\ell$*,*
(3) *the three conditions of Proposition 6.2 hold.*

By Proposition 6.2, Theorem 3.3, we know that (2) implies (1) and is equivalent to (3). We thus conjecturate that (1) implies (3). See **[11]** Theorem 4.6 for the case $n = r = 2$.

6.2. Let π be a σ-selfdual cuspidal $\overline{\mathbb{F}}_\ell$-representation of $G = \mathrm{GL}_n(F)$. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a generic σ-selfdual type in π, let $\boldsymbol{\lambda}_{\mathrm{w}}$ be the representation of $\mathbf{J}$ given by Proposition 4.17 (see Paragraph 4.9) and $\boldsymbol{\tau}$ be the representation of $\mathbf{J}$ trivial on $\mathbf{J}^1$ such that $\boldsymbol{\lambda}$ is isomorphic to $\boldsymbol{\lambda}_{\mathrm{w}} \otimes \boldsymbol{\tau}$. Associated with π by (4.21), there is also a σ-selfdual cuspidal $\overline{\mathbb{F}}_\ell$-representation π_{t} of $\mathrm{GL}_m(T)$.

Lemma 6.4. — *The following assertions are equivalent.*

(1) *The representation* π *has a* $\mathrm{GL}_n(F_0)$*-distinguished lift to* $\overline{\mathbb{Q}}_\ell$*.*
(2) *The representation* $\boldsymbol{\lambda}$ *has a* $\mathbf{J} \cap \mathrm{GL}_n(F_0)$*-distinguished lift to* $\overline{\mathbb{Q}}_\ell$*.*
(3) *The representation* $\boldsymbol{\tau}$ *has a* $\mathbf{J} \cap \mathrm{GL}_n(F_0)$*-distinguished lift to* $\overline{\mathbb{Q}}_\ell$*.*
(4) *The representation* π_{t} *has a* $\mathrm{GL}_m(T_0)$*-distinguished lift to* $\overline{\mathbb{Q}}_\ell$*.*

Proof. — Fix a σ-selfdual simple stratum $[\mathfrak{a}, \beta]$ as well as isomorphisms (4.17) as in Proposition 4.5. Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be the σ-selfdual maximal simple character associated with $\boldsymbol{\lambda}$, and $\widetilde{\theta}$ be its unique $\overline{\mathbb{Q}}_\ell$-lift: this is a σ-selfdual maximal simple character (with respect to the unique $\overline{\mathbb{Q}}_\ell$-lift $\widetilde{\psi}$ of the character ψ given by (4.1)) having the same G-normalizer $\mathbf{J}$ as θ.

Let $\widetilde{\boldsymbol{\lambda}}_{\mathrm{w}}$ be the $\overline{\mathbb{Q}}_\ell$-representation of $\mathbf{J}$ associated with $\widetilde{\theta}$ by Proposition 4.17. It is $\mathbf{J} \cap G^\sigma$-distinguished and σ-selfdual, and its determinant has order a power of p. It is thus integral. Let us consider its reduction mod ℓ. On the one hand, it is $\mathbf{J} \cap G^\sigma$-distinguished, σ-selfdual, and its determinant has order a power of p. On the other hand, **[27]** Proposition 2.37 implies that it is an irreducible representation extending the Heisenberg representation associated with θ. By uniqueness, we deduce that $\widetilde{\boldsymbol{\lambda}}_{\mathrm{w}}$ is a $\overline{\mathbb{Q}}_\ell$-lift of $\boldsymbol{\lambda}_{\mathrm{w}}$.

Suppose that π has a G^σ-distinguished $\overline{\mathbb{Q}}_\ell$-lift $\widetilde{\pi}$. Thus $\widetilde{\pi}$ is a σ-selfdual and cuspidal representation of G containing the maximal simple character $\widetilde{\theta}$. By Proposition 4.33, this representation $\widetilde{\pi}$ contains a distinguished generic σ-selfdual type, which we may assume to be of the form $(\mathbf{J}, \widetilde{\boldsymbol{\lambda}})$ with $\widetilde{\boldsymbol{\lambda}} = \widetilde{\boldsymbol{\lambda}}_{\mathrm{w}} \otimes \widetilde{\boldsymbol{\tau}}$ and the representation $\widetilde{\boldsymbol{\tau}}$ is $\mathbf{J} \cap G^\sigma$-distinguished. Reducing mod ℓ, we deduce that π contains the type $\boldsymbol{\lambda}_{\mathrm{w}} \otimes \boldsymbol{\delta}$ where $\boldsymbol{\delta}$ is the reduction mod ℓ of $\widetilde{\boldsymbol{\tau}}$. But π also contains the type $\boldsymbol{\lambda}_{\mathrm{w}} \otimes \boldsymbol{\tau}$, thus $\boldsymbol{\delta}$ is isomorphic to $\boldsymbol{\tau}$, and the reduction mod ℓ of $\widetilde{\boldsymbol{\lambda}}$ is isomorphic to $\boldsymbol{\lambda}$. Thus (1) implies both (2) and (3).

Conversely, suppose that $\boldsymbol{\tau}$ has a distinguished $\overline{\mathbb{Q}}_\ell$-lift $\widetilde{\boldsymbol{\tau}}$. Then the pair $(\mathbf{J}, \widetilde{\boldsymbol{\lambda}}_{\mathrm{w}} \otimes \widetilde{\boldsymbol{\tau}})$ is a distinguished type whose compact induction to G is a G^σ-distinguished $\overline{\mathbb{Q}}_\ell$-lift of π, and whose reduction mod ℓ is isomorphic to $\boldsymbol{\lambda}_{\mathrm{w}} \otimes \boldsymbol{\tau} \simeq \boldsymbol{\lambda}$. Thus (3) implies both (1) and (2).

Applying these results to the representation π_{t}, we get that π_{t} has a distinguished lift to $\overline{\mathbb{Q}}_\ell$ if and only if $\boldsymbol{\tau}_{\mathrm{t}}$ has a distinguished lift to $\overline{\mathbb{Q}}_\ell$. The fact that $\boldsymbol{\tau}$ is isomorphic to $\boldsymbol{\tau}_{\mathrm{t}} \circ \boldsymbol{\pi}$ (by Lemma 4.38) thus implies that (4) is equivalent to (3). $\square$

It follows from Lemma 6.4, together with Corollary 4.43 and Proposition 4.44, that, in order to prove Propositions 6.1 and 6.2, it suffices to prove them for σ-selfdual cuspidal $\overline{\mathbb{F}}_\ell$-representations of level 0. (For Proposition 6.2(3.a), it also follows from the fact that $\varkappa_{F/F_0} \circ \mathrm{N}_{T_0/F_0} = \varkappa_{T/T_0}$ and $\nu_{F_0} \circ \mathrm{N}_{T_0/F_0} = \nu_{T_0}$.)

6.3. We continue with the situation of Paragraph 6.2, assuming further that π has level 0. Thus π is a σ-selfdual cuspidal $\overline{\mathbb{F}}_\ell$-representation of G of level 0. We will also assume that π is non-supercuspidal, that is, $r = r(\pi) > 1$. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a generic σ-selfdual type in π. Associated with it in Paragraph 4.7, there are

– a σ-selfdual tamely ramified character ω of $F^\times$, which is the central character c_π of π,

– and a σ-selfdual cuspidal representation V of $\mathrm{GL}_n(\boldsymbol{k})$ of the form $\mathrm{st}_r(\varrho)$ for some supercuspidal representation ϱ of $\mathrm{GL}_{n/r}(\boldsymbol{k})$, uniquely determined up to isomorphism (thus V is non-supercuspidal).

Recall that the restriction of $\boldsymbol{\lambda}$ to $\mathbf{J}^0$ is the inflation of V, and that its restriction to $F^\times$ is a multiple of ω. Since V is σ-selfdual, Proposition 3.9 implies that ϱ is σ-selfdual.

The action of σ on $\mathrm{GL}_n(\boldsymbol{k})$ is described in Proposition 4.5: this is the action of the non-trivial automorphism of $\boldsymbol{k}/\boldsymbol{k}_0$ if F/F_0 is unramified, and the adjoint action of (4.5) with $i = \lfloor m/2 \rfloor$ otherwise.

Let us fix a uniformizer ϖ of F such that $\varpi \in F_0$ if F/F_0 is unramified, and $\varpi^2 \in F_0$ if F/F_0 is ramified. (One thus has $\sigma(\varpi) = -\varpi$ in the ramified case.)

Lemma 6.5. — *The representation π has a $\mathrm{GL}_n(F_0)$-distinguished lift to $\overline{\mathbb{Q}}_\ell$ if and only if V has a $\mathrm{GL}_n(\boldsymbol{k})^\sigma$-distinguished lift $\widetilde{\mathrm{V}}$ to $\overline{\mathbb{Q}}_\ell$ such that*

(1) *if F/F_0 is unramified, then $\omega(\varpi) = 1$,*

(2) *if F/F_0 is ramified, then n is even and* (4.23) *acts on the space of $\mathrm{GL}_{n/2}(\boldsymbol{k}) \times \mathrm{GL}_{n/2}(\boldsymbol{k})$-invariant linear forms on $\widetilde{\mathrm{V}}$ by a sign whose reduction mod ℓ is equal to $\omega(\varpi)$.*

Proof. — By Lemma 6.4, the representation π has a $\mathrm{GL}_n(F_0)$-distinguished lift if and only if the type $\boldsymbol{\lambda}$ has a $\mathbf{J} \cap \mathrm{GL}_n(F_0)$-distinguished lift to $\overline{\mathbb{Q}}_\ell$. Suppose $\boldsymbol{\lambda}$ has a distinguished lift $\widetilde{\boldsymbol{\lambda}}$. Then the pair $(\mathbf{J}, \widetilde{\boldsymbol{\lambda}})$ is the generic type of a distinguished cuspidal $\overline{\mathbb{Q}}_\ell$-representation $\widetilde{\pi}$, compactly induced from $\widetilde{\boldsymbol{\lambda}}$. Associated with it, there are

– a cuspidal $\overline{\mathbb{Q}}_\ell$-representation $\widetilde{\mathrm{V}}$ of $\mathrm{GL}_n(\boldsymbol{k})$ lifting V,

– a tamely ramified $\overline{\mathbb{Q}}_\ell$-character $\widetilde{\omega}$ of $F^\times$ lifting ω.

By Theorem 4.46, the character $\widetilde{\omega}$ is trivial on $F_0^\times$ and $\widetilde{\mathrm{V}}$ is distinguished by $\mathrm{GL}_n(\boldsymbol{k})^\sigma$. If F/F_0 is unramified, then $\widetilde{\omega}(\varpi) = 1$, thus $\omega(\varpi) = 1$. If F/F_0 is ramified, then $n = 2u$ for some $u \geqslant 1$ and $\widetilde{\mathrm{V}}$ is $\widetilde{\omega}(\varpi)$-distinguished (in the sense of Definition 5.6), and the reduction mod ℓ of $\widetilde{\omega}(\varpi)$ is equal to $\omega(\varpi)$.

Conversely, suppose that V has a $\mathrm{GL}_n(\boldsymbol{k})^\sigma$-distinguished lift $\widetilde{\mathrm{V}}$ satisfying the conditions of the lemma. Let $\widetilde{\omega}$ be a $\overline{\mathbb{Q}}_\ell$-lift of ω coinciding on the units of F with the inflation of the central character of $\widetilde{\mathrm{V}}$, and

(1) if F/F_0 is unramified, then $\widetilde{\omega}(\varpi) = 1$,

(2) if F/F_0 is ramified, then $\widetilde{\omega}(\varpi) \in \{-1, 1\}$ and the representation $\widetilde{\mathrm{V}}$ is $\widetilde{\omega}(\varpi)$-distinguished.

Inflate $\widetilde{\mathrm{V}}$ to $\mathbf{J}^0$, and extend it to a representation $\widetilde{\boldsymbol{\lambda}}$ of $\mathbf{J}$ by demanding that the restriction of $\widetilde{\boldsymbol{\lambda}}$ to $F^\times$ is a multiple of $\widetilde{\omega}$. The representation $\widetilde{\boldsymbol{\lambda}}$ is then a $\mathbf{J} \cap \mathrm{GL}_n(F_0)$-distinguished lift of $\boldsymbol{\lambda}$. □

6.4. In this paragraph, we assume that F/F_0 is unramified.

Lemma 6.6. — *Let* W *be a σ-selfdual cuspidal $\overline{\mathbb{F}}_\ell$-representation of* $\mathrm{GL}_n(\boldsymbol{k})$. *It has a* $\mathrm{GL}_n(\boldsymbol{k}_0)$-*distinguished lift to $\overline{\mathbb{Q}}_\ell$ if and only if n is odd and*

(1) *either* W *is supercuspidal,*

(2) *or* W *is non-supercuspidal and the order of the cardinality of $\boldsymbol{k}_0$ mod ℓ is even (thus $\ell \neq 2$).*

Proof. — By **[16]** Theorem 3.6, an irreducible $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_n(\boldsymbol{k})$ is $\mathrm{GL}_n(\boldsymbol{k}_0)$-distinguished if and only if it is σ-selfdual.

First, the condition on the parity of n is necessary: see **[32]** Lemma 2.3 for instance. Now assume that n is odd. If $\ell \neq 2$, the result is given by **[22]** Proposition 4.6. If $\ell = 2$, then W has the form $\mathrm{st}_r(\varrho)$, where ϱ is a supercuspidal representation of $\mathrm{GL}_{n/r}(\boldsymbol{k})$ and $r = 2^v$ for some $v \geqslant 0$. Since n is odd, W must be supercuspidal, and the result is given by **[32]** Remark 2.7. □

Remark 6.7. — Let q be the cardinality of $\boldsymbol{k}$ and q_0 be that of $\boldsymbol{k}_0$. Let e and e_0 be the orders of q and q_0 mod ℓ, respectively. Note that $r = e(\varrho)\ell^v$ for some $v \geqslant 0$, where $e(\varrho)$ is the order of q^f mod ℓ with $f = n/r$. If n is odd, then f and r are odd, thus $e(\varrho)$ is odd. But $e(\varrho) = e/(e, f)$. It follows that $e = e_0/(e_0, 2)$ is odd. Thus e_0 is not divisible by 4.

Example 6.8. — Let W be the σ-selfdual cuspidal $\overline{\mathbb{F}}_\ell$-representation $\mathrm{st}_e(1)$ of $\mathrm{GL}_e(\boldsymbol{k})$. We have $e = e_0/(e_0, 2)$, which is odd if and only if e_0 is not divisible by 4. Thus W has a $\mathrm{GL}_e(\boldsymbol{k}_0)$-distinguished lift to $\overline{\mathbb{Q}}_\ell$ if and only if e_0 is divisible by 2 but not by 4.

Suppose first that π has a distinguished lift to $\overline{\mathbb{Q}}_\ell$. On the one hand, the generic type of such a lift defines a σ-selfdual cuspidal $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_n(\boldsymbol{k})$, and **[32]** Lemma 2.3 implies that n is odd, thus r is odd. On the other hand, Theorem 3.3 implies that π is distinguished. It thus follows from Theorem 5.1 that π is isomorphic to $\mathrm{St}_r(\rho)$ for some distinguished supercuspidal representation ρ of $\mathrm{GL}_{n/r}(F)$. Finally, Lemma 6.5 says that V has a distinguished lift. It follows from Lemma 6.6 that the order e_0 of the cardinality of $\boldsymbol{k}_0$ mod ℓ is even.

We thus proved that, when F/F_0 is unramified, if π has a distinguished lift, then r is odd and Conditions (1), (2.a) of Proposition 6.1 are satisfied.

Conversely, suppose that the conditions (1), (2.a) of Proposition 6.1 are satisfied. Then V has a distinguished lift $\widetilde{\mathrm{V}}$. By Lemma 6.5, the representation π has a $\mathrm{GL}_n(F_0)$-distinguished lift to $\overline{\mathbb{Q}}_\ell$ if and only if $\omega(\varpi) = 1$. By Paragraph 4.8, the central character ω_* of ρ satisfies $\omega_*^r = \omega$. Since ρ is distinguished, we have $\omega_*(\varpi) = 1$, thus $\omega(\varpi) = \omega_*(\varpi)^r = 1$.

We proved Proposition 6.1 in the case when F/F_0 is unramified.

6.5. In this paragraph, we assume that F/F_0 is ramified. Let q denote the cardinality of $\boldsymbol{k}$, and let e denote the order of q mod ℓ.

Lemma 6.9. — *Let* W *be a selfdual cuspidal $\overline{\mathbb{F}}_\ell$-representation of* $\mathrm{GL}_n(\boldsymbol{k})$, *isomorphic to* $\mathrm{st}_r(\varrho)$ *for some selfdual supercuspidal representation of* $\mathrm{GL}_{n/r}(\boldsymbol{k})$. *Write $u = \lfloor n/2 \rfloor$. Then* W *has a lift to $\overline{\mathbb{Q}}_\ell$ which is distinguished by* $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_{n-u}(\boldsymbol{k})$ *if and only if*

(1) *either* W *is supercuspidal,*

(2) *or* W *is non-supercuspidal, n is even and*
   (a) *either $\ell \neq 2$ and r, n/e are odd,*
   (b) *or $\ell \neq 2$ and $r = n$,*
   (c) *or $\ell = n = r = 2$ and q is congruent to -1 mod 4, and ϱ is trivial.*

Proof. — First note that n must be either even or equal to 1: see **[32]** Lemma 2.17 for instance. Also, the supercuspidal case is given by **[32]** Remark 2.21. Let us assume that W is non-supercuspidal (thus n is even, and we will write $n = 2u$). We use the notation of Paragraph 3.7.

Set $f = n/r$ and let α be a $\mathrm{Gal}(\boldsymbol{k}_f/\boldsymbol{k})$-regular $\overline{\mathbb{F}}_\ell$-character of $\boldsymbol{k}_f^\times$ of order A which is a parameter of ϱ in the sense of Definition 3.12. Let $\widetilde{\upsilon}$ be the canonical $\overline{\mathbb{Q}}_\ell$-lift of

$$\upsilon = \alpha \circ \mathrm{N}_{\boldsymbol{k}_n/\boldsymbol{k}_f},$$

that is, its unique lift of order A. Let $\widetilde{\mathrm{W}}$ be a cuspidal lift of W. It is parametrized by a $\mathrm{Gal}(\boldsymbol{k}_n/\boldsymbol{k})$-regular character of $\boldsymbol{k}_n^\times$ lifting υ, that is, of the form $\widetilde{\upsilon}\phi$, where ϕ is a $\overline{\mathbb{Q}}_\ell$-character of $\boldsymbol{k}_n^\times$ of order ℓ^s for some $s \geqslant 0$. Since W is not supercuspidal, one has $s \geqslant 1$. The character $\widetilde{\upsilon}\phi$ has order $\mathrm{A}\ell^s$.

By Proposition 3.13, the representation $\widetilde{\mathrm{W}}$ is distinguished by $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$ if and only if it is selfdual, which is also equivalent (see for instance **[32]** (2.7)) to $\mathrm{A}\ell^s$ dividing q^u+1. Similarly, the fact that ϱ is selfdual is equivalent to

– either $f = 1$ and ϱ is a quadratic character (thus A is equal to 1 or 2),
– or f is even and A divides $q^{f/2}+1$ (thus $\mathrm{A} > 2$ since q has order $f \geqslant 2$ mod A).

Suppose that $\ell \neq 2$ and f is even. If $\widetilde{\mathrm{W}}$ is distinguished, then A divides $q^{f/2}+1$ and q^u+1. Since $u = rf/2$, we have

$$q^u + 1 = 1 + (-1)^r + \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i} (q^{f/2}+1)^i$$

thus A divides $1+(-1)^r$. Since $\mathrm{A} > 2$, it follows that r is odd. Also, ℓ divides q^u+1, that is, the order of q^u mod ℓ is $e/(e,u) = 2$, which implies that n/e is odd. Conversely, suppose that r and n/e are odd. The fact that A divides $q^{f/2}+1$ and r is odd implies that A divides q^u+1. Now ℓ^s divides $q^n - 1 = (q^u+1)(q^u-1)$. If ℓ divides q^u-1, then e divides $u = n/2$, thus n/e is even: contradiction. Thus ℓ^s divides q^u+1, thus $\widetilde{\mathrm{W}}$ is distinguished.

Suppose that $\ell \neq 2$ and $f = 1$. Then ϱ is a character of $\boldsymbol{k}^\times$, thus $r = e\ell^v$ for some $v \geqslant 0$. This gives $n/e = \ell^v$, which is odd. The same argument as above implies that q^u+1 is a multiple of ℓ^s. It is also a multiple of $\mathrm{A} \in \{1,2\}$ since it is even. Thus $\widetilde{\mathrm{W}}$ is distinguished.

Now suppose that $\ell = 2$. If $\widetilde{\mathrm{W}}$ is distinguished and f is even, then, as in the case where $\ell \neq 2$, the integer $\mathrm{A} > 2$ divides $q^{f/2}+1$ and $q^{rf/2}+1$, thus r is odd. But the fact that W is cuspidal implies that r is a power of 2. It follows that $r = 1$: contradiction. Thus $f = 1$, that is W is the representation $\mathrm{st}_n(1)$ with $n = 2^t$ for some $t \geqslant 1$. Moreover, q has order m mod 2^s, that is, 2^s divides $q^n - 1$ but not $q^u - 1$. Set

$$a = \upsilon_2(q^u+1), \quad b = \upsilon_2(q^u-1).$$

We have $b < s \leqslant a+b$ and $\min(a,b) = 1$. The fact that $\widetilde{\mathrm{W}}$ is distinguished implies $s \leqslant a$, which gives $b = 1 < a$, that is 4 divides q^u+1. Since u is a power of 2, we deduce that 4 divides $q+1$ and $u = 1$.

Conversely, suppose that $\ell = n = r = 2$ and 4 divides $q+1$ (hence $b = 1 < a$). Then any $\overline{\mathbb{Q}}_2$-character of $\boldsymbol{k}_2^\times$ of order 2^a parametrizes a distinguished cuspidal $\overline{\mathbb{Q}}_2$-representation of $\mathrm{GL}_2(\boldsymbol{k})$ lifting $\mathrm{W} = \mathrm{st}_2(1)$. □

Example 6.10. — The fact that $\mathrm{GL}_f(\boldsymbol{k})$ has a selfdual supercuspidal $\overline{\mathbb{F}}_\ell$-representation is equivalent to the fact that there is an $\boldsymbol{k}$-regular $\overline{\mathbb{F}}_\ell$-character of $\boldsymbol{k}_f^\times$ which is trivial on $\boldsymbol{k}_{f/2}^\times$, that is, there exists an integer A with the following properties:

(1) A is prime to ℓ and the order of q mod A is equal to f,

(2) A divides $q^{f/2}+1$.

Now suppose that $\ell > 2$ and $f = 2$. Thus $\mathrm{GL}_2(\boldsymbol{k})$ has a selfdual supercuspidal $\overline{\mathbb{F}}_\ell$-representation if and only if there exists an integer A prime to ℓ dividing $q+1$ but not $q-1$, that is, if and only if $q+1$ has a prime divisor different from 2 and ℓ. Assume this is the case, and let ϱ be a selfdual supercuspidal $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_2(\boldsymbol{k})$. Let W be the selfdual cuspidal $\overline{\mathbb{F}}_\ell$-representation $\mathrm{st}_r(\varrho)$ of $\mathrm{GL}_n(\boldsymbol{k})$ with $r = e/(e,2)$ and $m = 2r$. Then r is odd if and only if e is not divisible by 4, and $n/e = 2/(e,2)$ is odd if and only if e is even. If we take $q = 9$ and $\ell = 7$, we get $r = 3$ and $n/e = 2$. If we take $q = 5$, we get $q-1 = 4$ and $q^6 - 1 = 1953 \times 8$. Thus if ℓ is a prime divisor of 1953, we get $r = 3$ and $n/e = 6$.

In addition, we have the following result. We assume that $n = 2u$ for some $u \geqslant 1$.

Lemma 6.11. — *Let* W *be a selfdual cuspidal* $\overline{\mathbb{F}}_\ell$*-representation of* $\mathrm{GL}_n(\boldsymbol{k})$ *of the form* $\mathrm{st}_n(\varrho)$ *for some quadratic character* ϱ *of* $\boldsymbol{k}^\times$. *Assume* W *is distinguished by* $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$. *Then* (4.23) *acts on the space of* $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$*-invariant linear forms on* W *by*

$$\begin{cases} -1 & \text{if } \varrho \text{ is trivial,} \\ (-1)^{u(q-1)/2} & \text{if } \varrho \text{ is non-trivial.} \end{cases}$$

Proof. — Let c be the sign such that W is c-distinguished. If $\ell = 2$, the result is immediate since the only sign is 1. Assume that $\ell \neq 2$. By Lemma 6.9, the representation W has a distinguished cuspidal $\overline{\mathbb{Q}}_\ell$-lift. Let $\widetilde{\mathrm{W}}$ be such a $\overline{\mathbb{Q}}_\ell$-lift and ξ be a parameter for $\widetilde{\mathrm{W}}$. Let α be an element of $\boldsymbol{k}_n$ such that $\alpha \notin \boldsymbol{k}_u$ and $\alpha^2 \in \boldsymbol{k}_u$. By Proposition 3.13, the representation $\widetilde{\mathrm{W}}$ is $-\xi(\alpha)$-distinguished by $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$. Since $\widetilde{\mathrm{W}}$ lifts W, we have

– the reduction mod ℓ of the parameter ξ is equal to $(\varrho \circ \mathrm{N}_{\boldsymbol{k}_n/\boldsymbol{k}})\phi$ where ϕ is a character whose order is a power of ℓ (see Proposition 3.11),

– the reduction mod ℓ of $-\xi(\alpha)$ is equal to c (see Remark 3.14).

On the one hand, the character ξ is trivial on $\boldsymbol{k}_u^\times$ since $\widetilde{\mathrm{W}}$ is selfdual (see Proposition 3.13). On the other hand, $\varrho \circ \mathrm{N}_{\boldsymbol{k}_n/\boldsymbol{k}}$ is trivial on $\boldsymbol{k}_u^\times$ since ϱ is quadratic and the index of $\boldsymbol{k}_u^\times$ in $\boldsymbol{k}_n^\times$ is even. We deduce that ϕ is trivial on $\boldsymbol{k}_u^\times$, thus $\phi(\alpha)$ is a sign. Since it has order a power of $\ell \neq 2$, it is trivial. It follows that

$$c = -\varrho(\mathrm{N}_{\boldsymbol{k}_n/\boldsymbol{k}}(\alpha)).$$

If ϱ is trivial, this gives $c = -1$, as expected. Assume now that ϱ is non-trivial. It thus coincides with $\varkappa$ on $\boldsymbol{k}^\times$. Since α^2 is not a square in $\boldsymbol{k}_u^\times$, its $\boldsymbol{k}_u/\boldsymbol{k}$-norm is not a square in $\boldsymbol{k}^\times$. Thus

$$c = -\varkappa(\mathrm{N}_{\boldsymbol{k}_u/\boldsymbol{k}}(\alpha^{q^u+1})) = -(-1)^{(q^u+1)/2}$$

and one verifies that this is equal to $\varkappa(-1)^u = (-1)^{u(q-1)/2}$ as expected. □

6.6. Let us prove Proposition 6.1 when F/F_0 is ramified. Assume that r is odd, and suppose that π has a distinguished lift to $\overline{\mathbb{Q}}_\ell$. By Theorem 3.3, it is distinguished. Thus Theorem 5.1 implies that n/r is even and π is isomorphic to $\mathrm{St}_r(\rho)$ for some distinguished supercuspidal representation ρ of $\mathrm{GL}_{n/r}(F)$. Lemma 6.5 says that V has a distinguished lift. It follows from Lemma 6.9 that n/e is odd.

Conversely, assume that r is odd, π is isomorphic to $\mathrm{St}_r(\rho)$ for some distinguished supercuspidal representation ρ of $\mathrm{GL}_{n/r}(F)$ of level 0, n is even and n/e is odd. It follows from Lemma 6.9 that V has a distinguished lift $\widetilde{\mathrm{V}}$. Let $\varepsilon \in \{-1,1\}$ be the unique sign such that $\widetilde{\mathrm{V}}$ is ε-distinguished by $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$ in the sense of Definition 5.6, with $n = 2u$. By Lemma 6.5, the representation π has a distinguished lift if and only if $\omega(\varpi)$ is equal to the image of ε in $\overline{\mathbb{F}}_\ell^\times$, denoted c. We are going to prove that this is the case. Let ω_* be the central character of ρ. By Theorem 4.46, we have

– the representation ϱ is $\omega_*(\varpi)$-distinguished by $\mathrm{GL}_{k/2}(\boldsymbol{k}) \times \mathrm{GL}_{k/2}(\boldsymbol{k})$.

(Note k is even since n is even and r is odd.) By Proposition 4.29, we have

– the sign $\omega(\varpi)$ is equal to $\omega_*(\varpi)^r = \omega_*(\varpi)$.

Let α be the unique sign such that V is α-distinguished by $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$. By Remark 3.14, we have $\alpha = c$. On the other hand, we have $\alpha = \omega_*(\varpi)$ by Proposition 5.7. Putting these facts together, we get $\omega(\varpi) = \omega_*(\varpi) = \alpha = c$ as expected. This proves Proposition 6.1 if F/F_0 is ramified. Together with Paragraph 6.4, this finishes the proof of Proposition 6.1.

6.7. In this paragraph and the next one, we prove Proposition 6.2. Assume that r is even, and let q be the cardinality of $\boldsymbol{k}$. Since r divides n, we have $n = 2u$ for some $u \geqslant 1$.

Suppose that π has a distinguished lift. By Paragraph 6.4, this implies that F/F_0 is ramified. By Lemma 6.5, the representation V has a distinguished $\overline{\mathbb{Q}}_\ell$-lift. By Lemma 6.9, one has $r = n$, thus V is isomorphic to $\mathrm{st}_n(\varrho)$ for a character ϱ of $\boldsymbol{k}^\times$ of order at most 2. Besides, if $\ell = 2$, then $n = 2$ and q is congruent to -1 mod 4. Since $r = n$, the representation π is isomorphic to $\mathrm{St}_n(\rho)$ for a tamely ramified character ρ of $F^\times$ whose restriction to the units of F is the inflation of ϱ.

Suppose first that $\ell = 2$. Since π is distinguished (by Theorem 3.3), it is σ-selfdual (by Theorem 3.1). It follows from Proposition 3.8 that the representation ρ is σ-selfdual and from Theorem 3.2 that it is $F_0^\times$-distinguished, as expected.

Suppose now that $\ell \neq 2$. By Proposition 3.8, we may choose ρ so that $\rho^{-1} \circ \sigma = \rho\nu^i$ for some $i \in \{0,1\}$, that is, $\rho \circ \mathrm{N}_{F/F_0} = \nu^{-i}$. It remains to prove that the restriction of ρ to $F_0^\times$ is either $\varkappa$ or ν_0^{-1}.

Let c be the sign by which the element (4.23) acts on the space of $\mathrm{GL}_u(\boldsymbol{k}) \times \mathrm{GL}_u(\boldsymbol{k})$-invariant linear forms on V, which is given by Lemma 6.11. Remind that we have fixed a uniformizer ϖ of F such that $\sigma(\varpi) = -\varpi$, thus $\varpi_0 = \varpi^2$ is a uniformizer of F_0. The representation π is distinguished (by Theorem 3.3) and it follows from Theorem 4.46 that $c = c_\pi(\varpi)$. We have

$$c_\pi(\varpi) = \rho(\varpi)^n = \rho(\varpi_0)^u. \tag{6.1}$$

On the other hand, the identity $\rho \circ \mathrm{N}_{F/F_0} = \nu^{-i}$ implies that $\rho(-\varpi_0) = q^i$.

Lemma 6.12. — *We have $q^u \equiv -1$ mod ℓ.*

Proof. — Since $r \geqslant 2$ and π is cuspidal, r has the form $e(\rho)\ell^v$ for some $v \geqslant 0$, where $e(\rho)$ is the order of q^k mod ℓ by (3.2). In particular, $(q^k)^r = q^n$ is congruent to 1 mod ℓ. Moreover, since ℓ is odd, one has $q^u \equiv -1 \neq 1$ mod ℓ. □

It follows from Lemma 6.12 and (6.1) that

$$c = \begin{cases} (-1)^i & \text{if } \varrho \text{ is trivial,} \\ (-1)^i \cdot \varkappa(-1)^u & \text{otherwise (that is, if } \varrho = \varkappa). \end{cases}$$

Comparing with Lemma 6.11, we get the following corollary.

Corollary 6.13. — *We have $i = 1$ if ϱ is trivial, and $i = 0$ if ϱ is non-trivial.*

If $i = 0$, then ρ is selfdual. By Theorem 3.2, it is either distinguished or $\varkappa$-distinguished. Since its restriction to the units of F is the inflation of $\varrho = \varkappa$, we deduce that ρ is $\varkappa$-distinguished.

If $i = 1$, then $\rho\nu^{1/2}$ is unramified and selfdual. By Theorem 3.2, it is distinguished. Thus the restriction of ρ to $F_0^\times$ is equal to $\nu^{-1/2}|_{F_0^\times} = \nu_0^{-1}$.

6.8. Let us finish the proof of Proposition 6.2. Assume that $n = r = 2u$ for some $u \geqslant 1$, the extension F/F_0 is ramified and π is isomorphic to $\mathrm{St}_n(\rho)$ for some tamely ramified character ρ of $F^\times$. We also assume that

– either $\ell \neq 2$ and the restriction of ρ to $F_0^\times$ is either $\varkappa$ or ν_0^{-1},

– or $\ell = n = r = 2$, q is congruent to -1 mod 4 and ρ is trivial on $F_0^\times$.

It follows from Lemma 6.9 that V has a distinguished $\overline{\mathbb{Q}}_\ell$-lift $\widetilde{\mathrm{V}}$, which is ε-distinguished for some sign $\varepsilon \in \{-1, 1\}$. By Lemma 6.5, the representation π has a distinguished lift to $\overline{\mathbb{Q}}_\ell$ if and only if the reduction of ε mod ℓ, denoted c, is equal to $\omega(\varpi)$. Let us prove that this is the case. On the one hand, we have $\omega(\varpi) = \rho(\varpi)^n = \rho(\varpi_0)^u$. If $\ell = 2$, we have $\omega(\varpi) = 1$. Otherwise, we have

$$\omega(\varpi) = \begin{cases} q^u & \text{if the restriction of } \rho \text{ to } F_0^\times \text{ is } \nu_0^{-1}, \\ \varkappa(-1)^u & \text{if the restriction of } \rho \text{ to } F_0^\times \text{ is } \varkappa. \end{cases}$$

On the other hand, V is distinguished, and it is isomorphic to $\mathrm{st}_n(\varrho)$ where ϱ is the character of $\boldsymbol{k}^\times$ defined by the restriction of ρ to the units of F. One thus may apply Lemma 6.11, which says that V is α-distinguished, with

$$\alpha = \begin{cases} -1 & \text{if } \varrho \text{ is trivial,} \\ (-1)^{u(q-1)/2} & \text{if } \varrho \text{ is non-trivial.} \end{cases}$$

Note that $\alpha = c$ by Remark 3.14, and that ϱ is trivial if and only if the restriction of ρ to $F_0^\times$ is equal to ν_0^{-1}. Together with Lemma 6.12, this gives $\omega(\varpi) = c$ as expected.

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