Hyperlogarithmic functional identities on del Pezzo surfaces

Luc PIRIO

(Paris-Saclay - UVSQ & CNRS)

March 6th, 2023

Plan

I Intodaction

Polylogarithms

Functional identities

Theorem : $\mathbf{HLog}^w = 0$ for $w = 1, \ldots, 6$

II Proof

Del Pezzo surfaces

Hyperlogarithms

III Comparing $\mathbf{HLog}^2 = \boldsymbol{\mathcal{A}b}$ and $\mathbf{HLog}^3$

Theorem : $\mathbf{HLog}^3$ is cluster

The logarithm

- $\mathbf{Li}_1(z) = -\mathbf{Log}(1-z)$ $\qquad (z \in \mathbb{C})$

- **Integral formulas :**

$$\mathbf{Log}(z) = \int^z \frac{du}{u-0}$$

$$\mathbf{Li}_1(z) = -\int^z \frac{du}{u-1}$$

- **Series expansion :** $\mathbf{Li}_1(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$

- **Monodromy :** $\mathcal{M}_0\,(\,\mathbf{Log}\,) = \mathbf{Log} + 2i\pi$

- **Functional identity :** $\mathbf{Log}(\,x\,) - \mathbf{Log}(\,y\,) - \mathbf{Log}\left(\frac{x}{y}\right) = 0$

indoles logarithmorum hac aequatione fundamentali continetur [Pfaff 1788]

[The nature of logarithms is contained in this basic equation]

The dilogarithm $\mathbf{Li}_2$

- $\mathbf{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ $\qquad\qquad \left(|z| < 1 \right)$

- **Integral formulas :** $\mathbf{Li}_2(z) = \mathbf{L}_{01}(z) = -\int^z \log(1-u)\, \frac{du}{u-0}$

$$\mathbf{L}_{10}(z) = \int^z \log(u-0)\, \frac{du}{1-u}$$

- **Monodromy :** $\mathcal{M}_1\,(\mathbf{Li}_2) = \mathbf{Li}_2 - 2i\pi\, \mathbf{Log}$

- **Abel's functional identity** $(\mathcal{A}b)$ $\qquad \left(0 < x < y < 1 \right)$

$$\mathbf{Li}_2(x) - \mathbf{Li}_2(y) - \mathbf{Li}_2\left(\frac{x}{y}\right) - \mathbf{Li}_2\left(\frac{1-y}{1-x}\right) + \mathbf{Li}_2\left(\frac{x(1-y)}{y(1-x)}\right) =$$

$$\mathbf{Log}(y)\,\mathbf{Log}\left(\frac{1-y}{1-x}\right) - \frac{\pi^2}{6}$$

The dilogarithm $\mathbf{Li}_2$

- $\mathbf{Li}_2(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ $\qquad\qquad \Big(\ |z| < 1\ \Big)$

- **Integral formulas :** $\qquad \mathbf{Li}_2(z) \;=\; \mathbf{L}_{01}(z) \;=\; -\int^{z} \log(1-u)\,\frac{du}{u-0}$

$$\mathbf{L}_{10}(z) \;=\; \int^{z} \log(u-0)\,\frac{du}{1-u}$$

- **Monodromy :** $\qquad \mathcal{M}_1\,(\,\mathbf{Li}_2\,) \;=\; \mathbf{Li}_2 - 2i\pi\,\mathbf{Log}$

- **Abel's functional identity** $(\boldsymbol{\mathcal{A}b})$ $\qquad \Big(\ 0 < x < y < 1\ \Big)$

$$\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\Big(\frac{x}{y}\Big) - \mathbf{R}\Big(\frac{1-y}{1-x}\Big) + \mathbf{R}\Big(\frac{x(1-y)}{y(1-x)}\Big) = 0$$

$$\mathbf{R}(x) = \frac{1}{2}\Big(\mathbf{L}_{01}(x) - \mathbf{L}_{10}(x)\Big) = \mathbf{Li}_2(x) + \frac{1}{2}\mathbf{Log}(x)\mathbf{Log}(1-x) - \frac{\pi^2}{6}$$

The n-th polylogarithm $\mathbf{Li}_n$

- $\mathbf{Li}_n(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad\qquad\qquad \big(\; |z| < 1 \;\big)$

- **Integral formulas :** $\qquad \mathbf{Li}_n(z) \;=\; \int^z \mathbf{Li}_{n-1}(u) \frac{du}{u}$

$$\mathbf{Li}_n{}'(z) \;=\; \mathbf{Li}_{n-1}(z)/z$$

- **Monodromy :** $\qquad \mathcal{M}_1\,(\,\mathbf{Li}_n\,) \;=\; \mathbf{Li}_n - 2i\pi\,\frac{(\mathbf{Log})^{n-1}}{(n-1)!}$

- **Functional identities in one variable :**

$$\mathbf{Li}_n\Big(z^r\Big) \;=\; r^{n-1} \sum_{\omega^r=1} \mathbf{Li}_n(\omega\, z) \qquad\qquad \Big(\; |z| < 1 \;\Big)$$

$$\mathbf{Li}_n(z) + (-1)^n \mathbf{Li}_n\Big(z^{-1}\Big) = -\frac{(2i\pi)^n}{n!}\,\mathbf{B}_n\Big(\frac{\mathrm{Log}\, z}{2i\pi}\Big) \quad \Big(\; z \in \mathbb{C} \setminus [0, +\infty[\;\Big)$$

The n-th polylogarithm Li_n

- $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad\qquad \Big(\, |z| < 1 \, \Big)$

- **Integral formulas :**

$$\mathrm{Li}_n(z) = \int^z \mathrm{Li}_{n-1}(u) \frac{du}{u}$$

$$\mathrm{Li}_n{}'(z) = \mathrm{Li}_{n-1}(z)/z$$

- **Monodromy :**

$$\mathcal{M}_1 \left(\mathrm{Li}_n \right) = \mathrm{Li}_n - 2i\pi \frac{(\mathbf{Log})^{n-1}}{(n-1)!}$$

- **Functional identities in several variables ($\exists$?) :**

$$\sum_{i \in I} c_i \, \mathrm{Li}_n \big(U_i \big) = \mathbf{Elem}_{<n}$$

$\Big(\, I$ finite, $c_i \in \mathbb{Z}$, $U_i \in \mathbb{Q}(x_1, \ldots, x_N) \, \Big)$

The n-th polylogarithm $\mathbf{Li}_n$

- $\mathbf{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad \Big(\, |z| < 1 \, \Big)$

- **Integral formulas :**
$$\mathbf{Li}_n(z) = \int^z \mathbf{Li}_{n-1}(u) \frac{du}{u}$$
$$\mathbf{Li}_n{}'(z) = \mathbf{Li}_{n-1}(z)/z$$

- **Monodromy :**
$$\mathcal{M}_1 \left(\mathbf{Li}_n \right) = \mathbf{Li}_n - 2i\pi \frac{(\mathbf{Log})^{n-1}}{(n-1)!}$$

- **Functional identities in several variables ($\exists$?) :**

$$\sum_{i \in I} c_i \, \mathbf{Li}_n(U_i) = \mathbf{Elem}_{<n} \qquad \Longleftrightarrow \qquad \sum_{i \in I} c_i \, \boldsymbol{\mathcal{L}}_n(U_i) = \mathbf{0}$$

$\Big(\, I$ finite, $c_i \in \mathbb{Z}$, $U_i \in \mathbb{Q}(x_1, \ldots, x_N) \, \Big)$

Example : $\mathbf{L}\mathrm{i}_3$

- $\mathbf{L}\mathrm{i}_3(z) \;=\; \sum_{k=1}^{\infty} z^k/k^3 \;=\; \int^z \mathbf{L}\mathrm{i}_2(u)\frac{du}{u}$

- **Spence-Kummer identity $\mathcal{SK}$ (1809-1840) :**

$$2\,\mathbf{L}\mathrm{i}_3(x) + 2\,\mathbf{L}\mathrm{i}_3(y) - \mathbf{L}\mathrm{i}_3\left(\frac{x}{y}\right) + 2\,\mathbf{L}\mathrm{i}_3\left(\frac{1-x}{1-y}\right) + 2\,\mathbf{L}\mathrm{i}_3\left(\frac{x(1-y)}{y(1-x)}\right) - \mathbf{L}\mathrm{i}_3\left(xy\right)$$
$$+\, 2\,\mathbf{L}\mathrm{i}_3\left(-\frac{x(1-y)}{(1-x)}\right) + 2\,\mathbf{L}\mathrm{i}_3\left(-\frac{(1-y)}{y(1-x)}\right) - \mathbf{L}\mathrm{i}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right)$$
$$= 2\,\mathbf{L}\mathrm{i}_3(1) - \mathrm{Log}\,(y)^2\mathrm{Log}\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3}\mathrm{Log}\,(y) + \frac{1}{3}\mathrm{Log}\,(y)^3$$

Example : $\mathbf{Li}_3$

- $\mathbf{Li}_3(z) \;=\; \sum_{k=1}^{\infty} z^k/k^3 \;=\; \int^z \mathbf{Li}_2(u)\frac{du}{u}$

- **Spence-Kummer identity $\mathcal{SK}$ (1809-1840) :**

$$
\begin{aligned}
&2\,\mathbf{Li}_3(x) + 2\,\mathbf{Li}_3(y) - \mathbf{Li}_3\Big(\frac{x}{y}\Big) + 2\,\mathbf{Li}_3\Big(\frac{1-x}{1-y}\Big) + 2\,\mathbf{Li}_3\Big(\frac{x(1-y)}{y(1-x)}\Big) - \mathbf{Li}_3\big(xy\big)\\
&\quad + 2\,\mathbf{Li}_3\Big(-\frac{x(1-y)}{(1-x)}\Big) + 2\,\mathbf{Li}_3\Big(-\frac{(1-y)}{y(1-x)}\Big) - \mathbf{Li}_3\Big(\frac{x(1-y)^2}{y(1-x)^2}\Big)\\
&\quad = 2\,\mathbf{Li}_3(1) - \mathrm{Log}\,(y)^2\mathrm{Log}\Big(\frac{1-y}{1-x}\Big) + \frac{\pi^2}{3}\mathrm{Log}\,(y) + \frac{1}{3}\mathrm{Log}\,(y)^3
\end{aligned}
$$

$$
\begin{aligned}
&2\,\mathcal{L}_3(x) + 2\,\mathcal{L}_3(y) - \mathcal{L}_3\Big(\frac{x}{y}\Big) + 2\,\mathcal{L}_3\Big(\frac{1-x}{1-y}\Big) + 2\,\mathcal{L}_3\Big(\frac{x(1-y)}{y(1-x)}\Big) - \mathcal{L}_3\big(xy\big)\\
&\quad + 2\,\mathcal{L}_3\Big(-\frac{x(1-y)}{(1-x)}\Big) + 2\,\mathcal{L}_3\Big(-\frac{(1-y)}{y(1-x)}\Big) - \mathcal{L}_3\Big(\frac{x(1-y)^2}{y(1-x)^2}\Big) = \mathbf{0}
\end{aligned}
$$

$$
\mathcal{L}_3(z) = \mathbf{Li}_3(z) - \mathbf{Li}_2(z)\,\mathrm{Log}|z| + \tfrac{1}{3}\,\mathbf{Li}_1(z)\big(\mathrm{Log}|z|\big)^2
$$

Example : Li_4

- $\mathbf{Li}_4(x) = \sum_{k=1}^{\infty} x^k/k^4 \qquad \mathcal{L}_4(x) = \mathbf{Li}_4(x) + \mathbf{Elem}_{<4}(x)$

- **Kummer's identity $\mathcal{K}(4)$ (1840) :**

$$
\begin{aligned}
&\mathcal{L}_4\Big(-\frac{x^2y\eta}{\zeta}\Big) + \mathcal{L}_4\Big(-\frac{y^2x\zeta}{\eta}\Big) + \mathcal{L}_4\Big(\frac{x^2y}{\eta^2\zeta}\Big) + \mathcal{L}_4\Big(\frac{y^2x}{\zeta^2\eta}\Big)\\
&-6\,\mathcal{L}_4\big(xy\big) - 6\,\mathcal{L}_4\Big(\frac{xy}{\eta\zeta}\Big) - 6\,\mathcal{L}_4\Big(-\frac{xy}{\eta}\Big) - 6\,\mathcal{L}_4\Big(-\frac{xy}{\zeta}\Big)\\
&-3\,\mathcal{L}_4\big(x\eta\big) - 3\,\mathcal{L}_4\big(y\zeta\big) - 3\,\mathcal{L}_4\Big(\frac{x}{\eta}\Big) - 3\,\mathcal{L}_4\Big(\frac{y}{\zeta}\Big)\\
&-3\,\mathcal{L}_4\Big(-\frac{x\eta}{\zeta}\Big) - 3\,\mathcal{L}_4\Big(-\frac{y\zeta}{\eta}\Big) - 3\,\mathcal{L}_4\Big(-\frac{x}{\eta\zeta}\Big) - 3\,\mathcal{L}_4\Big(-\frac{y}{\eta\zeta}\Big)\\
&+6\,\mathcal{L}_4(x) + 6\,\mathcal{L}_4(y) + 6\,\mathcal{L}_4\Big(-\frac{x}{\zeta}\Big) + 6\,\mathcal{L}_4\Big(-\frac{y}{\eta}\Big) = \mathbf{0}
\end{aligned}
$$

$$(\ \zeta = 1 - x,\ \eta = 1 - y\ )$$

- **Abel 1881 (Spence 1809, Hill 1829, Rogers 1907)**

$$\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0 \qquad (\mathcal{A}b)$$

- **Spence-Kummer :** $\sum_{i=1}^{9} c_i \, \mathcal{L}_3\Big(U_i(x,y)\Big) = 0 \qquad (\mathcal{SK})$

- **Kummer 1840 :** $\sum_i c_i \, \mathcal{L}_n\Big(U_i(x,y)\Big) = 0 \qquad (\ n \leq 5\ ) \qquad (\mathcal{K}_n)$

- ...

- **Goncharov 1995 :** $\sum_{i=1}^{22} c_i \, \mathcal{L}_3\Big(U_i(a,b,c)\Big) = 0 \qquad (\mathcal{G}on)$

- **Gangl 2003 :** $\sum_i c_i \, \mathcal{L}_n\Big(U_i(x,y)\Big) = 0 \qquad (\ n = 6,7\ ) \qquad (\mathcal{G}an_n)$

- **Charlton, Gangl, Radchenko, Rudenko, Goncharov-Rudenko, ...**

- **Functional identities (FI) of polylogarithms Li_n :**
  - Hyperbolic geometry
  - Web geometry ($n \leq 3$)
  - K-theory of number fields (*'Zagier's conjecture'*) ($n \leq 4$)
  - Theory of periods (MZVs)
  - High energy particule physics ('*Scattering amplitudes*')
  - Mathematical physics ('*Y-systems*') ($n = 2$)
  - Cluster algebras ($n \leq 4$)
  - Mirror symmetry ('*Scattering diagrams*') ($n = 2$)

- **Functional identities (FI) of polylogarithms Li_n :**
  - Hyperbolic geometry
  - Web geometry ($n \leq 3$)
  - K-theory of number fields ($n \leq 4$)
  - Theory of periods (MZVs)
  - High energy particule physics ('*Scattering amplitudes*')
  - Mathematical physics : Y-systems ($n = 2$)
  - Cluster algebras ($n \leq 4$)
  - Mirror symmetry ('*Scattering diagrams*') ($n = 2$)

- $\boldsymbol{Ab}(x,y) = \mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) \equiv 0$

> **Thm [de Jeu 20]** $\forall\ I$ finite, $c_i \in \mathbb{Q}$ and $U_i \in \mathbb{Q}[x_1, \ldots, x_m]$:
>
> $\sum_{i\in I} c_i\, \mathbf{R}(U_i) \equiv \mathbf{cst} \iff$ $\sum_{i\in I} c_i\, \mathbf{R}(U_i)$ is a LC of specializations of $\boldsymbol{Ab}(X_s, Y_s)$ with $X_s, Y_s \in \mathbb{Q}[x_1, \ldots, x_m]\ \forall\ s$

- $\left(\mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}(x/y) = 0 \right)$ is the FFI of the logarithm
- $\boldsymbol{\mathcal{A}b} \iff \left(\boldsymbol{Ab}(x,y) \equiv 0 \right)$ is the FFI of the dilog ✓
- $\boldsymbol{\mathcal{G}on}_{22} \iff \sum_{i=1}^{22} c_i \mathcal{L}_3(U_i) = 0$ is the FFI of the trilog **?**
- $\mathbf{Q}_4$ **[Goncharov-Rudenko]** is the FFI of the tetralog **?**

[Griffiths 2002] *The legacy of Abel in algebraic geometry*

We do not attempt to formulate this question precisely – intuitively, we are asking whether or not for each n there is an integer $d(n)$ such that there is a "new" $d(n)$-web of maximum rank one of whose abelian relations is a (the ?) functional equation with $d(n)$ terms for $\mathbf{L}\mathrm{i}_n$? Here, 'new" means the general extension of the phenomena above for the logarithm when $n = 1$, where $d(1) = 3$, for the [= 5-term identity] *when $n = 2$ and $d(2) = 5$,...*

[Goncharov-Rudenko 2018] *'Motivic correlator, cluster algebras ...'*

Conclusion. *If $n > 3$, the problem of writing explicitly functional equations for the classical n-logarithms might not be the "right" problem. It seems that when n is growing the functional equations become so complicated that one can not write them down on a piece of paper.*

- **Problems about functional identities of polylogarithms :**
  - Finding **FI**'s for $\mathcal{L}_n$ $\big($ *e.g.* $\exists\ n \geq 8$? $\big)$
  - Is there a sequence $(\boldsymbol{FI}_n)_{n\geq 1}$ of **FI**'s for the polylogarithms ?
  - Is there a fundamental **FI**'s for $\mathcal{L}_n$ for each $n \geq 1$?
  - Better understand the polylogarithmic **FI**'s

- In this talk, allowing to deal with **hyperlogarithms** :
  - We describe a series of hyperlogarithmic identities

$$\mathbf{HLog}^1 \iff \Big(\mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}(x/y) = 0 \Big)$$

$$\mathbf{HLog}^2 \iff \Big(\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\Big(\frac{x}{y}\Big) - \mathbf{R}\Big(\frac{1-y}{1-x}\Big) + \mathbf{R}\Big(\frac{x(1-y)}{y(1-x)}\Big) = 0 \Big)$$

$$\vdots$$

$$\mathbf{HLog}^6 \qquad \Big(\text{weight 6 hyperlogarithmic identity} \Big)$$

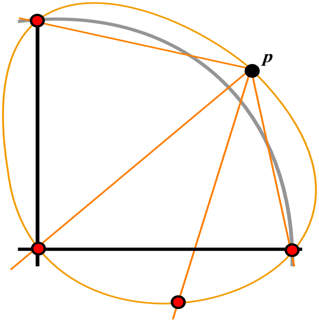
  - For $w = 1, \ldots, 6$, one has

$$\mathbf{HLog}^w \quad : \quad \sum_{i=1}^{\kappa} \boldsymbol{AH}_i^w(\phi_i) = 0$$

A geometric view on Abel's identity

- $(\mathcal{A}b)$ $\quad \mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$

$$\underset{U_1}{\underset{\|}{x}} \qquad \underset{U_2}{\underset{\|}{y}} \qquad \underset{U_3}{\underset{\|}{\tfrac{x}{y}}} \qquad \underset{U_4}{\underset{\|}{\tfrac{1-y}{1-x}}} \qquad \underset{U_5}{\underset{\|}{\tfrac{x(1-y)}{y(1-x)}}}$$



Base points of the U_i's :

- $p_1 = [1, 0, 0]$
- $p_2 = [0, 1, 0]$
- $p_3 = [0, 0, 1]$
- $p_4 = [1, 1, 1]$

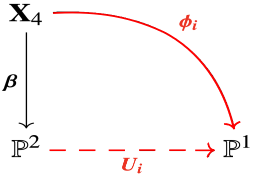
$\rightsquigarrow$ **Blow-up** $\quad \beta : X_4 = \mathbf{Bl}_{p_1,\ldots,p_4}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$

A geometric view on Abel's identity

- $(\mathcal{A}\mathbf{b})$ $\quad \mathsf{R}(x) - \mathsf{R}(y) - \mathsf{R}\left(\frac{x}{y}\right) - \mathsf{R}\left(\frac{1-y}{1-x}\right) + \mathsf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$

  where $U_1 = x$, $U_2 = y$, $U_3 = \frac{x}{y}$, $U_4 = \frac{1-y}{1-x}$, $U_5 = \frac{x(1-y)}{y(1-x)}$

- **Blow-up** $\quad \beta : X_4 = \mathsf{Bl}_{p_1,\ldots,p_4}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$



$\phi_1, \ldots, \phi_5 : X_4 \longrightarrow \mathbb{P}^1$
are the five conic fibrations on the del Pezzo surface X_4

- $(\mathcal{A}\mathbf{b}) \Longleftrightarrow \quad \exists\ (\epsilon_i)_{i=1}^{5} \in \{\pm 1\}^5$ unique up to sign $\quad$ st. $\quad \sum_{i=1}^{5} \epsilon_i\, \mathsf{R}(\phi_i) = 0$

Generalization to del Pezzo surfaces

- $p_1, \ldots, p_r \in \mathbb{P}^2$: points in general position, with $r \in \{3, \ldots, 8\}$
- **Blow-up** $\beta_r : X_r = \mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$ $\quad (\; X_r = \mathbf{dP}_{9-r} \;)$

Prop : 1. *There is a finite number κ of conic fibrations on X_r* $\quad \phi_1, \ldots, \phi_\kappa : X_r \longrightarrow \mathbb{P}^1$

2. *For each i, $\Sigma_i = \mathbf{Spectrum}(\phi_i) \subset \mathbb{P}^1$ has $r-1$ elements*

Def : *The complete antisymmetric weight $r-2$ hyperlogarithm* : $\quad AH^{r-2}_{\Sigma_i} : \widetilde{\mathbb{P}^1 \setminus \Sigma_i} \longrightarrow \mathbb{C}$

Thm [Castravet-P] $\exists \; (\epsilon_i)_{i=1}^{\kappa} \in \{ \pm 1 \}^{\kappa}$, $\pm$-unique, such that

$$(\mathbf{HLog}^{r-2}) \qquad \sum_{i=1}^{\kappa} \epsilon_i \, AH^{r-2}_{\Sigma_i}(\phi_i) = 0$$

$$(\mathbf{HLog}^{r-2}) \qquad \sum_{i=1}^{\kappa} \epsilon_i \, \boldsymbol{AH}^{r-2}_{\boldsymbol{\Sigma_i}}(\phi_i) = 0$$

- One identity $\mathbf{HLog}^{r-2}$ for each del Pezzo $\mathbf{dP}_d = \boldsymbol{X}_r \quad (d = 9 - r)$

[$d = 6$] $\mathbf{dP}_6$ is unique, $\boldsymbol{AH}^1_{\boldsymbol{\Sigma_i}} = \mathbf{Log}$ for any i

$$\mathbf{HLog}^1 = \Big(\mathbf{Log}(\, x\,) - \mathbf{Log}(\, y\,) - \mathbf{Log}(\, x/y\,) = \mathbf{0} \Big)$$

[$d = 5$] $\mathbf{dP}_5$ is unique : $\boldsymbol{AH}^2_{\boldsymbol{\Sigma_i}} = \frac{1}{2}\Big(\mathbf{L}_{01} - \mathbf{L}_{10}\Big) = \mathbf{R}$ for any i

$$\mathbf{HLog}^2 = \Big(\sum_{i=1}^{5} \epsilon_i \, \mathbf{R}(\, \phi_i\,) = \mathbf{0} \Big) \qquad (\mathcal{A}\mathbf{b})$$

$$(\mathbf{HLog}^{r-2}) \qquad \sum_{i=1}^{\kappa} \epsilon_i \, \boldsymbol{AH}_{\boldsymbol{\Sigma}_i}^{r-2}(\phi_i) = 0$$

$[\,d=4\,]$ $\mathbf{dP}_4$ ∞^2 moduli $\rightsquigarrow$ ∞^2 identities $\mathbf{HLog}^3$

$$\boldsymbol{AH}_1^3(\,x\,) + \boldsymbol{AH}_2^3\left(\frac{1}{y}\right) + \boldsymbol{AH}_3^3\left(\frac{y}{x}\right) + \cdots$$

$$\cdots + \boldsymbol{AH}_9^3\left(\frac{y(x-b)}{x(y-a)}\right) + \boldsymbol{AH}_{10}^3\left(\frac{a\,(b-x)}{by-ax}\right) = 0$$

$[\,d=3\,]$ $\mathbf{dP}_3$ = cubic surface in $\mathbb{P}^3$ $\rightsquigarrow$ ∞^4 identities $\mathbf{HLog}^4$

$$\sum_{i=1}^{27} \boldsymbol{AH}_i^4(\,\phi_i) = \mathbf{0}$$

Thm [Castravet-P. 2022]

$\exists\ (\epsilon_i)_{i=1}^{\kappa} \in \{\pm 1\}^{\kappa}$, unique op to sign, such that

$$(\mathbf{HLog}^{r-2}) \qquad \sum_{i=1}^{\kappa} \epsilon_i\, \boldsymbol{AH}^{r-2}_{\boldsymbol{\Sigma}_i}(\phi_i) = 0$$

$\longrightarrow$ Del Pezzo surfaces

$\longrightarrow$ Hyperlogarithms (Iterated integrals)

Del Pezzo surfaces I : properties

- $\mathbf{dP}_d \subset \mathbb{P}^d$ smooth surface, of degree d $\qquad (\, d = 9 - r \,)$
- $\mathbf{dP}_d = \mathbf{X}_r = \mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2)$ $\qquad \mathbf{Pic}(\mathbf{dP}_d) = \mathbb{Z}\,\boldsymbol{h} \oplus \big(\bigoplus_{i=1}^r \mathbb{Z}\,\boldsymbol{\ell}_i \big)$
- $-\boldsymbol{K}_{\mathbf{dP}_d} = 3\boldsymbol{h} - \sum_{i=1}^r \boldsymbol{\ell}_i$ ample $\rightsquigarrow$ $\varphi_{|-\boldsymbol{K}|} : \mathbf{dP}_d \hookrightarrow \mathbb{P}^d$ embedding
- $\mathbf{Pic}(\mathbf{dP}_d) \supset \boldsymbol{K}^\perp = \langle \, \rho_1, \ldots, \rho_r \, \rangle$ $\qquad \rho_i = \boldsymbol{\ell}_i - \boldsymbol{\ell}_{i+1} \quad i \leq r-1$

  $\rho_r = 3\boldsymbol{h} - \sum_{i=1}^3 \boldsymbol{\ell}_i$
- $-(\cdot,\cdot) + \{\rho_i\}_{i=1}^r$ $\rightsquigarrow$ Root system $E_r \subset R_r = \boldsymbol{K}^\perp \otimes \mathbb{R}$
- For any root ρ : $\qquad s_\rho : R_r \longrightarrow R_r$ $\qquad$ (reflection)

  $$d \longmapsto d + (d,\rho)\rho$$
- $\boldsymbol{W}_r = \boldsymbol{W}(E_r) = \langle \, s_{\rho_1}, \ldots, s_{\rho_r} \, \rangle \subset \mathrm{O}(R_r)$: Weyl group of type E_r

Del Pezzo surfaces I

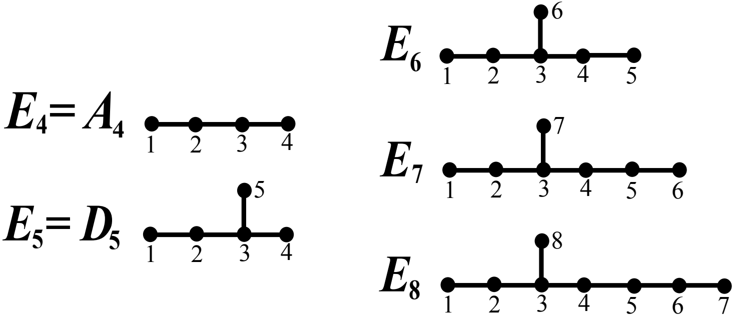


FIGURE – Dynkin diagram E_r $\big($ where k stands for ρ_k $\big)$

Lines and conics on $X_r = \mathbf{dP}_d$ $\quad \big(\, d = 9 - r \,\big)$

- Lines $\mathcal{L}_r = \Big\{\, \ell \in \mathbf{Pic}(X_r) \;\Big|\; (\ell, -K) = 1 \,,\; \ell^2 = -1 \,\Big\}$

  $\quad \Cup$

  $\delta \;\rightsquigarrow\;$ ' $\delta = [\,\delta\,]$ ' for $\mathbb{P}^1 \simeq \delta \subset \mathbf{dP}_d \quad \deg(\delta) = 1$

$$\mathcal{L}_r = W_r \cdot \ell_r \qquad\qquad \mathbf{W}'_{r-1} = \mathbf{Stab}(\ell_r) = \langle\, s_{\rho_i} \,|\, i \neq r-1 \,\rangle$$

$$= \mathbf{W}(E'_{r-1}) \quad (\text{ Weyl group })$$

- Clonics $\mathcal{K}_r = \Big\{\, \mathfrak{c} \in \mathbf{Pic}(X_r) \;\Big|\; (\mathfrak{c}, -K) = 2 \,,\; \mathfrak{c}^2 = 0 \,\Big\}$

  $\quad \Cup$

  $\mathfrak{c} \;\longleftrightarrow\;$ Fibration in conics $\phi_{\mathfrak{c}} : X_r \to \mathbb{P}^1$

$$\mathcal{K}_r = W_r \cdot (h - \ell_1) \qquad \mathbf{W}''_{r-1} = \mathbf{Stab}(h - \ell_1) = \langle\, s_{\rho_2}, \ldots, s_{\rho_r} \,\rangle$$

$$= W(E''_{r-1}) \quad (\text{ type } E''_{r-1} = D_{r-1} \text{ })$$

r	3	4	5	6	7	8
E_r	$A_2 \times A_1$	A_4	D_5	E_6	E_7	E_8
$W_r = W(E_r)$	$\mathfrak{S}_3 \times \mathfrak{S}_2$	$\mathfrak{S}_5$	$(\mathbf{Z}/2\mathbf{Z})^4 \ltimes \mathfrak{S}_5$	$W(E_6)$	$W(E_7)$	$W(E_8)$
$\omega_r = \|W_r\|$	12	5!	$2^4 \cdot 5!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
$l_r = \|\mathcal{L}_r\|$	6	10	16	27	56	240
$\kappa_r = \|\mathcal{K}_r\|$	3	5	10	27	126	2160

Exemple : the lines of dP_2 seen on the plane

- $\mathbf{dP}_2 = \boldsymbol{X}_7 = \mathbf{Bl}_{p_1,\ldots,p_7}(\mathbb{P}^2) \xrightarrow{\beta} \mathbb{P}^2$

- $\ell_i = \beta^{-1}(p_i) \subset \boldsymbol{X}_7 \quad \rightsquigarrow \quad \boldsymbol{\ell}_i \in \mathbf{Pic}(\boldsymbol{X}_7)$

  $\rightsquigarrow \quad \boldsymbol{\ell} = \sum_{i=1}^{7} \boldsymbol{\ell}_i$

Line	Class in $\mathrm{Pic}(X_7)$	Number of such lines	Model in $\mathbf{P}^2$
ℓ_i	ℓ_i	7	first infinitesimal neighbourhood $p_i^{(1)}$
ℓ_{ij}	$h - \ell_i - \ell_j$	21	line joining p_i to p_j
C_{ij}	$2h - \boldsymbol{\ell} + \ell_i + \ell_j$	21	conic through the p_k's, $k \notin \{i, j\}$
C_i^3	$3h - \boldsymbol{\ell} - \ell_i$	7	cubic through all the p_l's with a node at p_i

TABLE 2. Lines on dP_2 and the corresponding 'curves' in the projective plane

Exemple : clonics of dP_2

Conic class $\mathfrak{c}$	Number of such $\mathfrak{c}$	Linear system $\lvert\mathfrak{C}_{\mathfrak{c}}\rvert$	$\mathfrak{C}_{\mathfrak{c}}^{red}$
$h - \ell_i$	7	lines through p_i	$\ell_{ij} + \ell_j$
$2h - \sum_{i \in I} \ell_i$	35	conics through the p_i's, $i \in I$	$\ell_{i_1 i_2} + \ell_{i_3 i_4}$ $\ell_{i_3} + C_{i_1 i_2}$
$3h - \boldsymbol{\ell} + \ell_i - \ell_j$	42	cubics through the p_k's for $k \neq i$, with a node at p_j	$\ell_{jk} + C_{ik}$ $\ell_i + C_j^3$
$4h - \boldsymbol{\ell} - \sum_{j \in J} \ell_j$	35	quartics through the p_k's with a node at p_j for $j \in J$	$C_{k_1 k_2} + C_{k_3 k_4}$ $\ell_{j_1 j_2} + C_{j_3}^3$
$5h - 2\boldsymbol{\ell} + \ell_i$	7	quintics through the p_k's with a node at p_k except for $k = i$	$C_{ij} + C_j^3$

TABLE 3. Conic classes on dP_2 and their reducible fibers

Non-irreducible conics

- $\boldsymbol{L}_r = \cup_{\ell \in \mathcal{L}_r} \ell \subset \boldsymbol{X}_r \quad \rightsquigarrow \quad \boldsymbol{Y}_r = \boldsymbol{X}_r \setminus \boldsymbol{L}_r$

- $\mathcal{K}_r \ni \mathfrak{c} \quad \rightsquigarrow$ Conic fibration $\quad \phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1$

$$\Sigma_{\mathfrak{c}} = \mathbf{Spectrum}(\phi_{\mathfrak{c}}) = \left\{ \lambda \in \mathbb{P}^1 \;\middle|\; \phi_{\mathfrak{c}}^{-1}(\lambda) \text{ not irreducible } \right\}$$
$$= \left\{ \sigma_{\mathfrak{c},1} \, , \, \ldots \, , \, \sigma_{\mathfrak{c},r-1} \, , \, \sigma_{\mathfrak{c},r-1} = \infty \right\} \subset \mathbb{P}^1$$

- For $\quad \sigma_{\mathfrak{c},i} \in \Sigma_{\mathfrak{c}} : \;\; \phi_{\mathfrak{c}}^{-1}\big(\sigma_{\mathfrak{c},i}\big) = L'_{\mathfrak{c},i} + L''_{\mathfrak{c},i} \qquad \Big(\, L'_{\mathfrak{c},i}, L''_{\mathfrak{c},i} \in \mathcal{L}_r \, \Big)$

- $\mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\mathbb{P}^1 \, , \, \Omega^1_{\mathbb{P}^1}\big(\, \mathrm{Log}\, \Sigma_{\mathfrak{c}} \,\big)\Big) \;\; = \left\langle \, \frac{dz}{z-\sigma_{\mathfrak{c},i}} \, \right\rangle_{i=1}^{r-2}$

- $\mathbf{H}_{\mathfrak{c}} = \phi_{\mathfrak{c}}^*\Big(\mathcal{H}_{\mathfrak{c}} \Big) = \left\langle \, \frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}} - \sigma_{\mathfrak{c},i}} \, \right\rangle_{i=1}^{r-2} \subset \mathbf{H}^0\Big(\boldsymbol{X}_r \, , \, \Omega^1_{\boldsymbol{X}_r}(\mathrm{Log}\, \boldsymbol{L}_r) \Big) = \mathbf{H}_{\boldsymbol{X}_r}$

Del Pezzo's web $\mathcal{W}_{\mathbf{dP}_d}$

- $\mathcal{W}_{\mathbf{dP}_d} = \mathcal{W}\big(\phi_{\mathfrak{c}}\big)_{\mathfrak{c}\in\mathcal{K}_r}$: κ_r-web by conics on $\mathbf{dP}_d$

- **Quest°** : $\exists\ \Big(F_{\mathfrak{c}}(\phi_{\mathfrak{c}})\Big)_{\mathfrak{c}\in\mathcal{K}_r}$ such that $\sum_{\mathfrak{c}\in\mathcal{K}_r} F_{\mathfrak{c}}\big(\phi_{\mathfrak{c}}\big) = 0$
  with polylogarithmic $F_{\mathfrak{c}}$'s ?

Theorem : $\exists\ (\epsilon_{\mathfrak{c}})_{\mathfrak{c}\in\mathcal{K}_r} \in \{1, -1\}^{\mathcal{K}_r}$, $\pm$-unique such that

$$\big(\mathbf{HLog}^{r-2}\big) \qquad \sum_{\mathfrak{c}\in\mathcal{K}_r} \epsilon_{\mathfrak{c}}\, \boldsymbol{AH}_{\mathfrak{c}}^{r-2}\big(\phi_{\mathfrak{c}}\big) = 0$$

where $\forall\ \mathfrak{c}$: $\boldsymbol{AH}_{\mathfrak{c}}^{r-2}$ = complete antisymmetric hyperlogarithm of weight $r-2$ on $\mathbb{P}^1 \setminus \Sigma_{\mathfrak{c}}$.

III Iterated integrals

- **Poincaré (1884), Lappo-Danilevski (1928), Chen (1973)**

- $\boldsymbol{Y}$ complex manifold

- $\mathbf{H} = \Big\langle \omega_1 , \ldots , \omega_m \Big\rangle \subset \mathbf{H}^0\Big(\boldsymbol{Y}, \Omega^1_{\boldsymbol{Y}} \Big) \quad + \left[\begin{array}{r} d\omega_i = 0 \\ \omega_i \wedge \omega_j = 0 \end{array} \right]$

- **Ex :** $\phi : \boldsymbol{Y} \to C$ and $\omega_i \in \phi^*\Big(\mathbf{H}^0\big(C, \Omega^1_C \big)\Big) \quad i = 1, \ldots, m$

- Base point $y \in \boldsymbol{Y}$, path $\gamma^x : [0,1] \to \boldsymbol{Y}$ from y to x :
  - $\mathbb{I}_{\omega_i} : x \longmapsto \int_{\gamma^x} \omega_i \quad \rightsquigarrow \quad \mathbb{I}_{\omega_i} \in \mathcal{O}_y$
  - $\mathbb{I}_{\omega_j\omega_i} : x \longmapsto \int_{\gamma^x} \omega_j(u) \cdot \mathbb{I}_{\omega_i}(u) \quad \rightsquigarrow \quad \mathbb{I}_{\omega_j\omega_i} \in \mathcal{O}_y$
  - $\mathbb{I}_{\omega_k\omega_j\omega_i} : x \longmapsto \int_{\gamma^x} \omega_k(u) \cdot \mathbb{I}_{\omega_j\omega_i}(u) \quad \rightsquigarrow \quad \mathbb{I}_{\omega_k\omega_j\omega_i} \in \mathcal{O}_y$

III Iterated integrals : polylogarithms

$$\mathbf{II}^w \,:\, \mathbf{H}^{\otimes w} \longrightarrow \mathcal{O}_y$$

- $\underline{\omega} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_w} \longmapsto \mathbb{I}_{\underline{\omega}} : z \mapsto \int_{\gamma^z} \omega_{i_1}(u) \cdot \mathbb{I}_{\omega_{i_2}\cdots\omega_{i_w}}(u)$

- $\mathbf{II} : \Big(\bigoplus_{w\geq 0} \mathbf{H}^{\otimes w} \,,\, ш \Big) \longrightarrow \mathcal{O}_y$ monomorphism of $\mathbb{C}$-algebra

- $\forall\, \underline{\omega}$: $\mathbb{I}_{\underline{\omega}} \in \mathcal{O}_y \cap \widetilde{\mathcal{O}}(\boldsymbol{Y})$ unipot. monodromy $\longrightarrow$ Symbol $\boldsymbol{\mathcal{S}}(\mathbb{I}_{\underline{\omega}}) = \underline{\omega}$ ✓

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \{0, 1, \infty\}$

$$\mathbf{H} = \Big\langle \tfrac{dz}{z} \,,\, \tfrac{dz}{1-z} \Big\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 \,,\, \Omega^1_{\mathbb{P}^1}(\,\mathrm{Log}\,\Sigma\,) \Big)$$

$$\mathbf{Li}_n = \mathbf{II}^n\Big(\big(\tfrac{dz}{z}\big)^{\otimes(n-1)} \otimes \big(\tfrac{dz}{1-z}\big) \Big)$$ (**‘Polylogarithms’**)

III Iterated integrals : hyperlogarithms

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \left\{ \sigma_1 \, , \, \ldots \, , \, \sigma_{r-2} \, , \, \sigma_{r-1} = \infty \right\}$

$$\mathbf{H} = \left\langle \tfrac{dz}{z-\sigma_1}, \ldots, \tfrac{dz}{z-\sigma_{r-2}} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 \, , \, \Omega^1_{\mathbb{P}^1}\big(\mathrm{Log}\, \Sigma \big) \Big)$$

$$\mathbf{II}^n\Big(\big(\tfrac{dz}{z-\sigma_{i_1}}\big) \otimes \cdots \otimes \big(\tfrac{dz}{z-\sigma_{i_n}}\big) \Big)$$

'Hyperlogarithms'

- **Complete antisymmetric hyperlog of weight $r-2$ on $\mathbb{P}^1 \setminus \boldsymbol{\Sigma}$** :

$$\boldsymbol{AH}^{r-2}_{\Sigma} = \mathbf{II}^n\left(\mathbf{Asym}\left(\left(\frac{dz}{z-\sigma_1} \right) \otimes \cdots \otimes \left(\frac{dz}{z-\sigma_{r-2}} \right) \right) \right)$$

$$= \mathbf{II}^n\left(\tfrac{1}{(r-2)!} \textstyle\sum_{\nu \in \mathfrak{S}_{r-2}} (-1)^{\nu} \left(\tfrac{dz}{z-\sigma_{\nu(1)}} \right) \otimes \cdots \otimes \left(\tfrac{dz}{z-\sigma_{\nu(r-2)}} \right) \right)$$

- **Ex :** $\boldsymbol{AH}^2_{\{0,1,\infty\}} = \frac{1}{2} \mathbf{II}^2\left(\frac{dz}{z} \otimes \frac{dz}{(1-z)} - \frac{dz}{(1-z)} \otimes \frac{dz}{z} \right) = \mathbf{R}$

IV Identity HLog^{r-2} : proof(s)

$$\left(\mathbf{HLog}^{r-2}\right) \; : \quad \sum_{\mathfrak{c}\in\mathcal{K}_r} \epsilon_{\mathfrak{c}}\, \boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\varphi_{\mathfrak{c}}) = 0 \;\; \text{with} \quad \boldsymbol{AH}_{\mathfrak{c}}^{r-2} = \boldsymbol{AH}_{\Sigma_{\mathfrak{c}}}^{r-2}$$

- $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1 \supset \Sigma_{\mathfrak{c}} = \left\{ \sigma_{\mathfrak{c},i} \right\}_{i=1}^{r-1}$ $\qquad \mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\Omega^1_{\mathbb{P}^1}\big(\operatorname{Log} \Sigma_{\mathfrak{c}}\big)\Big)$
- $\mathbf{H}_{\mathfrak{c}} = \phi_{\mathfrak{c}}^*\big(\mathcal{H}_{\mathfrak{c}}\big) \subset \mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\operatorname{Log} \boldsymbol{L}_r\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r}$
- $\boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\phi_{\mathfrak{c}}) = \mathbf{II}\left(\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},1}}\right) \wedge \cdots \wedge \left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},r-2}}\right)\right) \in \mathbf{II}^{r-2}\left(\wedge^{r-2}\mathbf{H}_{\mathfrak{c}}\right)$

  $\downarrow \; \mathcal{S}$ (symbol)
- $\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = \left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},1}}\right) \wedge \cdots \wedge \left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},r-2}}\right) \;\; \in \wedge^{r-2}\mathbf{H}_{\mathfrak{c}} \subset \wedge^{r-2}\mathbf{H}_{X_r}$

$$\left(\mathbf{HLog}^{r-2}\right) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \text{in} \;\; \wedge^{r-2}\mathbf{H}_{X_r}$$

IV Identity HLog^{r-2} : proof(s)

$$(\mathbf{HLog}^{r-2}) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \text{in} \quad \wedge^{r-2}\mathbf{H}_{X_r}$$

- $\mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\,\boldsymbol{L}_r\,\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r} \xrightarrow{\;\oplus_\ell \mathbf{Res}_\ell\;} \mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ injective

  $\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} \in \wedge^{r-2}\mathbf{H}_{X_r} \hookrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r} \curvearrowleft \boldsymbol{W}(E_r)$

- $0 \to \mathbf{K}^{r-2} \longrightarrow \oplus_{\mathfrak{c}}\,(\mathbf{H}_{\mathfrak{c}})^{\wedge(r-2)} \longrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ SES $\mathbb{C}$-vect spaces

  $\wr|$

  $\mathrm{Ind}_{\boldsymbol{W}''_{r-1}}^{\boldsymbol{W}_r}(\mathbf{sign}''_{r-1}) \longrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ SES of $\boldsymbol{W}_r$-reps

- $\mathfrak{c}_0 \rightsquigarrow \boldsymbol{\Omega}_{\mathfrak{c}_0}^{r-2} \in \wedge^{r-2}\mathbf{H}_{\mathfrak{c}_0} \rightsquigarrow \boldsymbol{W}''_{r-1} = \mathbf{Stab}(\mathfrak{c}_0) \subset \boldsymbol{W}(E_r)$

  $\mathfrak{c}_0 = \big(\boldsymbol{h} - \boldsymbol{\ell}_1\big)$

IV Identity HLog^{r-2} : proof(s)

- $\mathfrak{c}_0 \quad \rightsquigarrow \quad \boldsymbol{\Omega}^{r-2}_{\mathfrak{c}_0} \in \wedge^{r-2}\mathbf{H}_{\mathfrak{c}_0} \quad \rightsquigarrow \quad \boldsymbol{W}''_{r-1} = \mathbf{Stab}(\mathfrak{c}_0) \subset \boldsymbol{W}(E_r)$
- $\boldsymbol{W}_r = \bigsqcup_{\mathfrak{c}} \gamma^{\mathfrak{c}}_{\mathfrak{c}_0} \cdot \boldsymbol{W}''_{r-1}$

$\rightsquigarrow$ **Def** :
$$\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = (-1)^{\gamma^{\mathfrak{c}}_{\mathfrak{c}_0}} \left(\gamma^{\mathfrak{c}}_{\mathfrak{c}_0} \bullet \boldsymbol{\Omega}^{r-2}_{\mathfrak{c}_0} \right) \in \wedge^{r-2}\mathbf{H}_{\mathfrak{c}}$$

- **Facts :**
  1. $\mathbb{C}\, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = \wedge^{r-2}\mathbf{H}_{\mathfrak{c}}$
  2. $\left(\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} \right)_{\mathfrak{c}\in\mathcal{K}} \in \oplus_{\mathfrak{c}\in\mathcal{K}} \left(\wedge^{r-2}\, \mathbf{H}_{\mathfrak{c}} \right)$ *is canonical*
  is $\boldsymbol{W}_r$*-stable*
  3. $\left\langle \left(\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} \right)_{\mathfrak{c}} \right\rangle \simeq \mathbf{sign}_r$ *as a* $\boldsymbol{W}_r$*-represent°*

- $$\mathbf{sign}_r \hookrightarrow \oplus_{\mathfrak{c}} (\mathbf{H}_{\mathfrak{c}})^{\wedge(r-2)} \longrightarrow \wedge^{r-2}\, \mathbf{C}^{\mathcal{L}_r} \qquad \text{in } \mathrm{Rep}(\boldsymbol{W}_r)$$
  $$\mathbf{1} \longmapsto \left(\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} \right)_{\mathfrak{c}} \longmapsto \textstyle\sum_{\mathfrak{c}} \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = \mathbf{hlog}^{r-2}$$

IV Identity HLog^{r-2} : proof

- $\mathbf{hlog}^{r-2} = \sum_{\mathfrak{c}} \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2}$

- One decomposes $\mathbf{hlog}^{r-2}$ in a natural basis of $\wedge^{r-2}\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$

- Typical element of the basis : $\underline{\boldsymbol{\ell}} = \ell_3 \wedge \cdots \wedge \ell_r$

- **Facts :** **1.** $\underline{\boldsymbol{\ell}}$ *appears in* $\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2}$ *only for* $\mathfrak{c} = \boldsymbol{h} - \ell_1$ *or* $\mathfrak{c} = \boldsymbol{h} - \ell_2$

  2. *Moreover, it appears with opposite sign*

- Using the $\boldsymbol{W}_r$-action $\Longrightarrow$ $\boxed{\mathbf{hlog}^{r-2} = \sum_{\mathfrak{c}} \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0}$ ■

IV Identity HLog^{r-2} : other proofs

$$\boxed{\mathbf{hlog}^{r-2} = \sum_{\mathfrak{c}} \mathbf{\Omega}_{\mathfrak{c}}^{r-2} = 0}$$

- $\mathbf{sign}_r \hookrightarrow \oplus_{\mathfrak{c}} \big(\mathbf{H}_{\mathfrak{c}}\big)^{\wedge(r-2)} \longrightarrow \wedge^{r-2}\, \mathbb{C}^{\mathcal{L}_r}$

$$\mathbf{1} \longmapsto \quad \big(\mathbf{\Omega}_{\mathfrak{c}}^{r-2}\big)_{\mathfrak{c}} \quad \longmapsto \quad \sum_{\mathfrak{c}} \mathbf{\Omega}_{\mathfrak{c}}^{r-2}$$

IV Identity HLog^{r-2} : other proofs

$$\boxed{\mathbf{hlog}^{r-2} = \sum_{\mathfrak{c}} \mathbf{\Omega}_{\mathfrak{c}}^{r-2} = 0}$$

- $\mathbf{sign}_r \hookrightarrow \oplus_{\mathfrak{c}} (\mathbf{H}_{\mathfrak{c}})^{\wedge(r-2)} \longrightarrow \wedge^{r-2}\,\mathbb{C}^{\mathcal{L}_r}$ $\qquad \left\langle \mathbf{sign}_r\,,\, \wedge^{r-2}\,\mathbb{C}^{\mathcal{L}_r} \right\rangle = 0$

  $\mathbf{1} \longmapsto \left(\mathbf{\Omega}_{\mathfrak{c}}^{r-2}\right)_{\mathfrak{c}} \longmapsto \sum_{\mathfrak{c}} \mathbf{\Omega}_{\mathfrak{c}}^{r-2}$ $\qquad$ (GAP3)

- Explicit description of lines $\longrightarrow$ $\mathbb{Z}^{\mathcal{L}_r} \simeq \mathbb{Z}^{l_r}$

  $+\ \mathbb{Z}$-linear algebra $\longrightarrow$ $\sum \mathbf{\Omega}_{\mathfrak{c}}^{r-2} = 0$ $\quad$ (Maple)

- Analytic constructive proof (?)

Comparing HLog^2 and HLog^3

- $\boxed{\mathbf{HLog}^2}$ $\quad \mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$

- $\boxed{\mathbf{HLog}^3}$ $\quad \sum_{i=1}^{10} \epsilon_i \, \boldsymbol{AH}^3_{\boldsymbol{\Sigma_i}}(\boldsymbol{\phi_i}) = 0$ with for $\Sigma = \{ b_1, \ldots, b_4 \}$

$$\boldsymbol{AH}^3_\Sigma(x) = \frac{1}{3} \sum_{k=1}^{3} (-1)^{k-1} \mathbf{Log}\left(1 - \frac{x}{b_k}\right) \mathbf{R}_{\Sigma \setminus \{b_k\}}(x)$$

$$\boldsymbol{AH}^3_{\mathbf{1}}(\boldsymbol{x}) + \boldsymbol{AH}^3_{\mathbf{2}}\left(\frac{\mathbf{1}}{\boldsymbol{y}}\right) + \boldsymbol{AH}^3_{\mathbf{3}}\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right) + \boldsymbol{AH}^3_{\mathbf{4}}\left(\frac{\boldsymbol{x-y}}{\boldsymbol{x-1}}\right) + \boldsymbol{AH}^3_{\mathbf{5}}\left(\frac{\boldsymbol{b(a-x)}}{\boldsymbol{ay-bx}}\right)$$

$$+\boldsymbol{AH}^3_{\mathbf{6}}\left(\frac{\boldsymbol{P}}{\boldsymbol{(x-1)(y-b)}}\right) + \boldsymbol{AH}^3_{\mathbf{7}}\left(\frac{\boldsymbol{(x-y)(y-b)}}{\boldsymbol{y\,P}}\right) + \boldsymbol{AH}^3_{\mathbf{8}}\left(\frac{\boldsymbol{x\,P}}{\boldsymbol{(x-y)(x-a)}}\right)$$

$$+\boldsymbol{AH}^3_{\mathbf{9}}\left(\frac{\boldsymbol{y(x-b)}}{\boldsymbol{x(y-a)}}\right) + \boldsymbol{AH}^3_{\mathbf{10}}\left(\frac{\boldsymbol{a\,(b-x)}}{\boldsymbol{by-ax}}\right) = \mathbf{0}$$

Webs $\mathcal{W}_{\mathbf{dP}_5}$ and $\mathcal{W}_{\mathbf{dP}_4}$

- $(\mathcal{A}\mathbf{b}) \quad \mathbf{R}(\phi_1) - \mathbf{R}(\phi_2) - \mathbf{R}\Big(\phi_3\Big) - \mathbf{R}\Big(\phi_4\Big) + \mathbf{R}\Big(\phi_5\Big) = 0$

- For each i : $\mathcal{F}_{\phi_i}$ = foliation by level subsets $\{\phi_i = \lambda\}$, $\lambda \in \mathbb{P}^1$

- **Web :** $\mathcal{W}_{\mathbf{dP}_5} = \Big(\mathcal{F}_{\phi_1}, \ldots, \mathcal{F}_{\phi_5}\Big)$: 5-tuple of foliations

- $\mathcal{W}_{\mathbf{dP}_5}$ = geometric object $\rightsquigarrow$ $(\mathcal{A}\mathbf{b})$

- $\mathcal{W}_{\mathbf{dP}_4} = \Big(\mathcal{F}_{\phi_k}\Big)_{\substack{\phi_k : \mathbf{dP_4} \to \mathbb{P}^1 \\ \text{conic fibrations}}} \rightsquigarrow \big(\mathbf{HLog}^3\big)$

Comparing the webs $\mathcal{W}_{\mathbf{dP}_5}$ and $\mathcal{W}_{\mathbf{dP}_4}$

- Both $\boldsymbol{\mathcal{W}}_{\mathbf{dP}_5}$ and $\boldsymbol{\mathcal{W}}_{\mathbf{dP}_4}$ satisfy similar remarkable properties :
  - non-linearizable webs
  - maximal rank (with all their ARs poly-/hyperlogarithmic)
  - characterized by the matroid of their hexagonal subwebs
  - can be constructed geometrically à la [Gelfand -MacPherson]
  - are cluster webs

- $\boldsymbol{\mathcal{W}}_{\mathbf{dP}_5}$ *is equivalent to the* $\boldsymbol{\mathcal{X}}$*-cluster web of type* A_2
- $\boldsymbol{\mathcal{W}}_{\mathbf{dP}_4}$ *is equivalent to a* $\boldsymbol{\mathcal{X}}$*-cluster web of type* D_4

Cluster algebras (mutations)

- **Seed :** $\boldsymbol{S} = (\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{B} \,)$ with $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{x} = (x_1, \ldots, x_n)$

  $\boldsymbol{B} = (\, b_{ij} \,) \in \mathrm{M}_n(\mathbb{Z})$ antisymm$^{\text{able}}$

- **Mutation :** $\boldsymbol{\mu_k} : (\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{B} \,) \longmapsto (\, \boldsymbol{a}', \boldsymbol{x}', \boldsymbol{B}' \,)$

  $\mathcal{A}$–mutation

$$a'_j = \begin{cases} a_k^{-1} \Big[\prod_{b_{k\ell}>0} a_\ell^{b_{k\ell}} + \prod_{b_{kl}<0} a_l^{-b_{kl}} \Big] & j = k \\ a_j & j \neq k \end{cases}$$

  $\mathcal{X}$ – mutation

$$x'_j = \begin{cases} x_j^{-1} & j = k \\ x_j \Big(1 + x_k^{\mathsf{s}(-b_{kj})} \Big)^{-b_{kj}} & j \neq k \end{cases}$$

Cluster algebra : type A_2

- **Matrix** $B = B_Q \quad \longleftrightarrow \quad$ **Quiver** $Q = Q_B$

$$B_{A_2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \longleftrightarrow \quad Q_{A_2} \; : \; \mathbf{1} \longrightarrow \mathbf{2}$$

$$\left(\left(\frac{1+x_2}{x_1}, \frac{1}{x_2}\right), \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \xleftarrow{\mu_2} \left(\left(\frac{1}{x_1}, x_2\right), \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \xrightarrow{\mu_1} \left(\left(x_1, \frac{x_2}{1+x_1}\right), \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$$

$$\downarrow \mu_1 \qquad\qquad\qquad\qquad \downarrow \mu_2$$

$$\left(\left(\frac{x_1}{1+x_2}, \frac{1+x_1+x_2}{x_1 x_2}\right), \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \qquad \left(\left(\frac{x_1 x_2}{1+x_1+x_2}, \frac{1+x_1}{x_2}\right), \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$$

- $\mathcal{X}$**-cluster variables** : $\quad x_1 \, , \; x_2 \, , \; \frac{1+x_2}{x_1} \, , \; \frac{1+x_1}{x_2} \, , \; \frac{1+x_1+x_2}{x_1 x_2}$

- For $\quad \mathbf{R}^c(x) = -\mathbf{Li}_2(-x) - \frac{1}{2}\,\mathbf{Log}(x)\,\mathbf{Log}(1+x)$:

$$\mathbf{R}^c(x_1) + \mathbf{R}^c(x_2) + \mathbf{R}^c\left(\frac{1+x_2}{x_1}\right) + \mathbf{R}^c\left(\frac{1+x_1}{x_2}\right) + \mathbf{R}^c\left(\frac{1+x_1+x_2}{x_1 x_2}\right) = 0 \quad (\mathcal{A}\mathbf{b})$$

Cluster periods and cluster dilogarithmic identities

- Notion of **period** of a cluster algebra $\mathcal{A}$ $\longleftrightarrow$ loop in the exchange graph $\Gamma_{\mathcal{A}}$

- $$\begin{array}{cccccccccccc} \boldsymbol{\mathcal{S}}_0 & \xrightarrow{\mu_{i_1}} & \boldsymbol{\mathcal{S}}_1 & \xrightarrow{\mu_{i_2}} & \boldsymbol{\mathcal{S}}_2 & \xrightarrow{\mu_{i_3}} & \cdots & \longrightarrow & \boldsymbol{\mathcal{S}}_{k-1} & \xrightarrow{\mu_{i_k}} & \boldsymbol{\mathcal{S}}_k \simeq \boldsymbol{\mathcal{S}}_0 & \\ & & \Cup & & \Cup & & & & \Cup & & \Cup & \\ & & \boldsymbol{x_{i_1}} & & \boldsymbol{x_{i_2}} & & \cdots & & \boldsymbol{x_{i_{k-1}}} & & \boldsymbol{x_{i_k}} & \longleftarrow \mathcal{X}\text{-variables} \end{array}$$

- $\boldsymbol{D}_0 = \mathrm{Diag}(d_1, \ldots, d_n) \in M_n(\mathbb{Z})$: right skew symmetrizer of $\boldsymbol{B}_0$

Thm [Nakanishi] For some $N \in \mathbb{N}_{>0}$, one has

$$\sum_{s=1}^{k} d_{i_s}\, \mathbf{R}^c\Big(\, \boldsymbol{x_{i_s}} \,\Big) \equiv N\pi^2/6$$

Cluster Varieties

- For any seed $\big(\,\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{B}\,\big) = \boldsymbol{\mathcal{S}} \overset{\text{mut}^o}{\sim} \boldsymbol{\mathcal{S}}_0 = \big(\,\boldsymbol{a}_0, \boldsymbol{x}_0, \boldsymbol{B}_0\big)$ initial seed :

$$\mathrm{Spec}\big[\boldsymbol{a}^{\pm 1}\big] = \boldsymbol{\mathcal{A}}\mathbb{T}_{\mathcal{S}} \xrightarrow{\ \boldsymbol{p}\ } \boldsymbol{\mathcal{X}}\mathbb{T}_{\mathcal{S}} = \mathrm{Spec}\big[\boldsymbol{x}^{\pm 1}\big] \simeq \left(\mathbb{C}^*\right)^{\boldsymbol{n}}$$

$$(a_i)_{i=1}^{n} \longrightarrow \Big(\prod_{j=1}^{n} a_j^{b_{ij}}\Big)_{i=1}^{n}$$

- **Cluster varieties** : **[GSV], [FG]**

$$_{\boldsymbol{\mathcal{A}}\text{-mut}^o}\Big\backslash\Big(\bigcup_{\mathcal{S}\sim\mathcal{S}_0} \boldsymbol{\mathcal{A}}\mathbb{T}_{\mathcal{S}}\Big) = \boldsymbol{\mathcal{A}} \xrightarrow{\ \boldsymbol{p}\ } \boldsymbol{\mathcal{X}} = \Big(\bigcup_{\mathcal{S}\sim\mathcal{S}_0} \boldsymbol{\mathcal{X}}\mathbb{T}_{\mathcal{S}}\Big)\Big/_{\boldsymbol{\mathcal{X}}\text{-mut}^o}$$

- **Secondary cluster variety** : $\boldsymbol{\mathcal{U}} = \mathbf{Im}(p) \subset \boldsymbol{\mathcal{X}}$

- <u>**Ex**</u> : $\boldsymbol{\mathcal{X}}_{A_n} \simeq \boldsymbol{\mathcal{M}}_{0,n+3} \sqcup \boldsymbol{\mathcal{B}}$ with $\boldsymbol{\mathcal{B}} \subsetneq \partial\boldsymbol{\mathcal{M}}_{0,n+3}$ **[FG]**

  $\boldsymbol{\mathcal{X}}_{A_{2n}} = \boldsymbol{\mathcal{U}}_{A_{2n}}$ $\qquad \boldsymbol{\mathcal{X}}_{A_{2n+1}} \supsetneq \boldsymbol{\mathcal{U}}_{A_{2n+1}}$ (codim 1)

Cluster functional identities

Def : A **Cluster functional identity** is an identity

$$\sum_{\sigma\in\Sigma} F_\sigma\big(\, x_\sigma \,\big) \equiv \mathrm{cst}$$

where – the F_σ's are some analytic functions

– $\big\{\, x_\sigma \,\big|\, \sigma \in \Sigma \,\big\}$ is a set of ($\boldsymbol{\mathcal{X}}$-)cluster variables

- **Cluster ensemble** :

$$\begin{array}{ccccc} \boldsymbol{T}_1 \circlearrowright \boldsymbol{\mathcal{A}} & \xrightarrow{\quad \boldsymbol{p} \quad} & \boldsymbol{\mathcal{X}} & \xrightarrow{\quad \boldsymbol{\lambda} \quad} & \boldsymbol{T}_2 \\ & & \cup & & \Cup \\ & & \boldsymbol{\mathcal{U}} & \longrightarrow & \mathbf{1} \end{array}$$

Def° : **secondary cluster variables** = restrictions of the $\boldsymbol{\mathcal{X}}$-cluster variables on $\boldsymbol{\mathcal{U}} = \lambda^{-1}(\mathbf{1})$

- **Ex** : $\overline{\boldsymbol{\mathcal{M}}}_{0,6} \supset \boldsymbol{\mathcal{X}}_{A_3} \supsetneq \boldsymbol{\mathcal{U}}_3 \quad \Big(\, \overline{\boldsymbol{\mathcal{U}}}_3 = \text{Keel-Vermeire divisor} \,\Big)$

- **Ex** : $\overline{\mathcal{M}}_{0,6} \supset \mathcal{X}_{A_3} \supsetneq \mathcal{U}_3 \quad \left(\overline{\mathcal{U}}_3 = \text{Keel-Vermeire divisor} \right)$

- Secondary cluster variables in type A_3 :

$$x_1 \;,\; x_2 \;,\; \frac{1+x_2}{x_1} \;,\; \frac{1+x_1}{x_2} \;,\; \frac{(1+x_1)^2}{x_2} \;,\; \frac{1+x_1+x_2}{x_1x_2} \;,$$
$$\frac{(1+x_1)^2+x_2}{x_1x_2} \;,\; \frac{1+x_1+x_2}{x_1(1+x_1)} \;,\; \frac{(1+x_1+x_2)^2}{x_1^2x_2}$$

- Cluster functional identity of weight 3 :

$$2\,\mathbf{L}_3(x_1) + 2\,\mathbf{L}_3\left(\frac{1+x_1}{x_2}\right) - \mathbf{L}_3(x_2) + 2\,\mathbf{L}_3\left(\frac{1+x_2}{x_1}\right) + 2\,\mathbf{L}_3\left(\frac{1+x_1+x_2}{x_1x_2}\right) - \mathbf{L}_3\left(\frac{(1+x_1)^2}{x_2}\right)$$
$$+\, 2\,\mathbf{L}_3\left(\frac{(1+x_1)^2+x_2}{x_1x_2}\right) + 2\,\mathbf{L}_3\left(\frac{1+x_1+x_2}{x_1(1+x_1)}\right) - \mathbf{L}_3\left(\frac{(1+x_1+x_2)^2}{x_1^2x_2}\right) = 0$$

$$\wr|$$

$$2\,\mathcal{L}_3(x) + 2\,\mathcal{L}_3(y) - \mathcal{L}_3\left(\frac{x}{y}\right) + 2\,\mathcal{L}_3\left(\frac{1-x}{1-y}\right) + 2\,\mathcal{L}_3\left(\frac{x(1-y)}{y(1-x)}\right) - \mathcal{L}_3(xy)$$
$$(\mathcal{SK}) \qquad + 2\,\mathcal{L}_3\left(-\frac{x(1-y)}{(1-x)}\right) + 2\,\mathcal{L}_3\left(-\frac{(1-y)}{y(1-x)}\right) - \mathcal{L}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right) = 0$$

$\longrightarrow$ Spence-Kummer identity $(\mathcal{SK})$ is cluster

$\mathbf{HLog}^3$ is cluster

- **Cluster ensemble of type D_4** :

$$\mathcal{A}_{D_4} \xrightarrow{\ p\ } \mathcal{X}_{D_4} \xrightarrow{\ \lambda\ } T_2 \simeq (\mathbb{C}^*)^2$$

$$\cup \qquad \Downarrow$$

$$\mathcal{U}_{\alpha,\beta} \to (\alpha, \beta)$$

- Restrictions on $\mathcal{U}_{\alpha,\beta}$ of the 52 $\mathcal{X}$-cluster variables of type D_4 $\rightsquigarrow$ 32 cluster variables $x^{\sigma}_{\alpha,\beta} : \mathcal{U}_{\alpha,\beta} \dashrightarrow \mathbb{P}^1$

Thm : $\exists$ *10 cluster variables* $x^{1}_{\alpha,\beta}, \ldots, x^{10}_{\alpha,\beta} : \mathcal{U}_{\alpha,\beta} \dashrightarrow \mathbb{P}^1$ *st* *up to an explicit birational isomorphism* $\mathbf{dP}_{a,b} \simeq \mathcal{U}_{\alpha,\beta}$:

$$\mathbf{HLog}^3(a,b) \iff \sum_{s=1}^{10} AH^3_s\left(x^{s}_{\alpha,\beta}\right) = 0$$

What next ?

- **Applications** – $\mathbf{HLog}^1 = \Big(\mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}\big(\tfrac{x}{y}\big) = 0\Big)$ ✓
  - $\mathbf{HLog}^2 = \Big(\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\Big(\frac{x}{y}\Big) - \mathbf{R}\Big(\frac{1-y}{1-x}\Big) + \mathbf{R}\Big(\frac{x(1-y)}{y(1-x)}\Big) = 0 \Big)$ ✓
  - $\mathbf{HLog}^3 = \Big(\sum_{i=1}^{10} \boldsymbol{AH}_i^3(U_i(x,y)) = 0 \Big)$?

- **Construction** of $\mathbf{HLog}^3$ à la Gelfand-MacPherson ?

- **Interpretation** of $\mathbf{HLog}^3$ in terms of the SC of $\mathbf{dP}_4$?

- **Versions** Unival. $\mathbf{HLog}^3_{\text{univ}}$? Quantum $\mathbf{HLog}^3_q$? Motivic $\mathbf{HLog}^3_{\text{mot}}$?

- **Singular del Pezzo's**

- **Blow-ups** $\mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2)$ with $r \geq 9$: $\sum_{\mathfrak{c}} \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0$?