Hyperlogarithmic functional identities on del Pezzo surfaces

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Plan

I Intoduction

Polylogarithms

Functional identities

Theorem : $\mathbf{HLog}^w = 0$ for $w = 1, \ldots, 6$

II Proof

Del Pezzo surfaces

Hyperlogarithms

III Comparing $\mathbf{HLog}^2 = \boldsymbol{\mathcal{A}b}$ and $\mathbf{HLog}^3$

Theorem : $\mathbf{HLog}^3$ is cluster

The logarithm

- $\mathbf{L}\mathrm{i}_1(z) = -\mathbf{Log}(1 - z) \qquad (z \in \mathbb{C})$

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indoles logarithmorum hac aequatione fundamentali continetur [Pfaff 1788]

[The nature of logarithms is contained in this basic equation]

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- **Abel's functional identity** $(\boldsymbol{\mathcal{A}b})$ $\qquad \Big(0 < x < y < 1 \Big)$

$$\mathbf{Li}_2(x) - \mathbf{Li}_2(y) - \mathbf{Li}_2\Big(\frac{x}{y}\Big) - \mathbf{Li}_2\Big(\frac{1-y}{1-x}\Big) + \mathbf{Li}_2\Big(\frac{x(1-y)}{y(1-x)}\Big) =$$

$$\mathbf{Log}(y)\,\mathbf{Log}\Big(\frac{1-y}{1-x}\Big) - \frac{\pi^2}{6}$$

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$$\mathbf{R}(x) = \frac{1}{2}\Big(\mathbf{L}_{01}(x) - \mathbf{L}_{10}(x)\Big) = \mathbf{Li}_2(x) + \frac{1}{2}\mathbf{Log}(x)\mathbf{Log}(1-x) - \frac{\pi^2}{6}$$

The n-th polylogarithm Li_n

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$$\mathbf{Li}_n(z) \;=\; \int^z \mathbf{Li}_{n-1}(u)\frac{du}{u}$$

$$\mathbf{Li}_n{}'(z) \;=\; \mathbf{Li}_{n-1}(z)/z$$

- **Monodromy :** $\qquad \mathcal{M}_1\left(\mathbf{Li}_n\right) \;=\; \mathbf{Li}_n - 2i\pi\,\frac{(\mathbf{Log})^{n-1}}{(n-1)!}$

- **Functional identities in one variable :**

$$\mathbf{Li}_n\Big(z^r\Big) \;=\; r^{n-1}\sum_{\omega^r=1}\mathbf{Li}_n(\omega\, z) \qquad\qquad \Big(\ |z|< 1\ \Big)$$

$$\mathbf{Li}_n(z) + (-1)^n\mathbf{Li}_n\Big(z^{-1}\Big) \;=\; -\frac{(2i\pi)^n}{n!}\,\mathbf{B}_n\bigg(\frac{\mathrm{Log}\, z}{2i\pi}\bigg) \quad \Big(\ z\in\mathbb{C}\setminus[0,+\infty[\ \Big)$$

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$$\Big(\ I \ \text{ finite}\,,\ c_i \in \mathbb{Z}\,,\ U_i \in \mathbb{Q}(x_1, \ldots, x_N)\ \Big)$$

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$\Big(\, I$ finite, $c_i \in \mathbb{Z}$, $U_i \in \mathbb{Q}(x_1, \ldots, x_N) \, \Big)$

Example : $\mathbf{Li}_3$

- $\mathbf{Li}_3(z) \;=\; \sum_{k=1}^{\infty} z^k/k^3 \;=\; \int^z \mathbf{Li}_2(u)\frac{du}{u}$

- **Spence-Kummer identity $\mathcal{SK}$ (1809-1840) :**

$$2\,\mathbf{Li}_3(x) + 2\,\mathbf{Li}_3(y) - \mathbf{Li}_3\Big(\frac{x}{y}\Big) + 2\,\mathbf{Li}_3\Big(\frac{1-x}{1-y}\Big) + 2\,\mathbf{Li}_3\Big(\frac{x(1-y)}{y(1-x)}\Big) - \mathbf{Li}_3\big(xy\big)$$

$$+\,2\,\mathbf{Li}_3\Big(-\frac{x(1-y)}{(1-x)}\Big) + 2\,\mathbf{Li}_3\Big(-\frac{(1-y)}{y(1-x)}\Big) - \mathbf{Li}_3\Big(\frac{x(1-y)^2}{y(1-x)^2}\Big)$$

$$= 2\,\mathbf{Li}_3(1) - \mathrm{Log}\,(y)^2\mathrm{Log}\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3}\mathrm{Log}\,(y) + \frac{1}{3}\mathrm{Log}\,(y)^3$$

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$$+ 2\,\mathbf{Li}_3\Big(-\frac{x(1-y)}{(1-x)}\Big) + 2\,\mathbf{Li}_3\Big(-\frac{(1-y)}{y(1-x)}\Big) - \mathbf{Li}_3\Big(\frac{x(1-y)^2}{y(1-x)^2}\Big)$$
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$$\mathcal{L}_3(z) = \mathbf{Li}_3(z) - \mathbf{Li}_2(z)\,\mathrm{Log}|z| + \tfrac{1}{3}\,\mathbf{Li}_1(z)\big(\mathrm{Log}|z|\big)^2$$

Example : Li_4

- $\mathbf{Li}_4(x) \; = \; \sum_{k=1}^{\infty} x^k/k^4 \qquad \boldsymbol{\mathcal{L}}_4(x) \; = \; \mathbf{Li}_4(x) + \mathbf{Elem}_{<4}(x)$
- **Kummer's identity $\mathcal{K}(4)$ (1840) :**

$$
\begin{aligned}
&\boldsymbol{\mathcal{L}}_4\Big(-\frac{x^2y\eta}{\zeta}\Big) + \boldsymbol{\mathcal{L}}_4\Big(-\frac{y^2x\zeta}{\eta}\Big) + \boldsymbol{\mathcal{L}}_4\Big(\frac{x^2y}{\eta^2\zeta}\Big) + \boldsymbol{\mathcal{L}}_4\Big(\frac{y^2x}{\zeta^2\eta}\Big)\\
&-6\,\boldsymbol{\mathcal{L}}_4\Big(xy\Big) - 6\,\boldsymbol{\mathcal{L}}_4\Big(\frac{xy}{\eta\zeta}\Big) - 6\,\boldsymbol{\mathcal{L}}_4\Big(-\frac{xy}{\eta}\Big) - 6\,\boldsymbol{\mathcal{L}}_4\Big(-\frac{xy}{\zeta}\Big)\\
&-3\,\boldsymbol{\mathcal{L}}_4\Big(x\eta\Big) - 3\,\boldsymbol{\mathcal{L}}_4\Big(y\zeta\Big) - 3\,\boldsymbol{\mathcal{L}}_4\Big(\frac{x}{\eta}\Big) - 3\,\boldsymbol{\mathcal{L}}_4\Big(\frac{y}{\zeta}\Big)\\
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&+6\,\boldsymbol{\mathcal{L}}_4(x) + 6\,\boldsymbol{\mathcal{L}}_4(y) + 6\,\boldsymbol{\mathcal{L}}_4\Big(-\frac{x}{\zeta}\Big) + 6\,\boldsymbol{\mathcal{L}}_4\Big(-\frac{y}{\eta}\Big) = \mathbf{0}
\end{aligned}
$$

$$(\ \zeta = 1 - x,\ \eta = 1 - y\ )$$

- **Abel 1881 (Spence 1809, Hill 1829, Rogers 1907)**

$$\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0 \qquad (\boldsymbol{\mathcal{A}b})$$

- **Spence-Kummer :** $\sum_{i=1}^{9} c_i\, \mathcal{L}_3\Big(U_i(x,y)\Big) = 0$ $\quad(\boldsymbol{\mathcal{SK}})$

- **Kummer 1840 :** $\sum_i c_i\, \mathcal{L}_n\Big(U_i(x,y)\Big) = 0 \quad (\ n \leq 5\ )$ $\quad(\boldsymbol{\mathcal{K}_n})$

- ...

- **Goncharov 1995 :** $\sum_{i=1}^{22} c_i\, \mathcal{L}_3\Big(U_i(a,b,c)\Big) = 0$ $\quad(\boldsymbol{\mathcal{G}on})$

- **Gangl 2003 :** $\sum_i c_i\, \mathcal{L}_n\Big(U_i(x,y)\Big) = 0 \quad (\ n = 6,7\ )$ $\quad(\boldsymbol{\mathcal{G}an_n})$

- **Charlton, Gangl, Radchenko, Rudenko, Goncharov-Rudenko, ...**

- **Functional identities (FI) of polylogarithms $\mathbf{Li}_n$:**
  - Hyperbolic geometry
  - Web geometry ($n \leq 3$)
  - K-theory of number fields (*'Zagier's conjecture'*) ($n \leq 4$)
  - Theory of periods (MZVs)
  - High energy particule physics ('*Scattering amplitudes*')
  - Mathematical physics ('*Y-systems*') ($n = 2$)
  - Cluster algebras ($n \leq 4$)
  - Mirror symmetry ('*Scattering diagrams*') ($n = 2$)

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Thm [de Jeu 20] $\forall\ I$ finite, $c_i \in \mathbb{Q}$ and $U_i \in \mathbb{Q}[x_1, \ldots, x_m]$:

$\sum_{i \in I} c_i\, \mathbf{R}(U_i) \equiv \mathbf{cst}$ $\iff$ $\sum_{i \in I} c_i\, \mathbf{R}(U_i)$ is a LC of specializations of $\boldsymbol{Ab}(X_s, Y_s)$ with $X_s, Y_s \in \mathbb{Q}[x_1, \ldots, x_m]\ \forall\ s$

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- $\boldsymbol{\mathcal{G}on}_{22} \Longleftrightarrow \sum_{i=1}^{22} c_i \mathcal{L}_3(U_i) = 0$ is the FFI of the trilog **?**

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- $\big(\, \mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}(x/y) = 0 \big)$ is the FFI of the logarithm
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- $\boldsymbol{\mathcal{G}on}_{22}$ $\iff$ $\sum_{i=1}^{22} c_i \mathcal{L}_3(U_i) = 0$ is the FFI of the trilog **?**
- $\mathbf{Q}_4$ **[Goncharov-Rudenko]** is the FFI of the tetralog **?**

[Griffiths 2002] *The legacy of Abel in algebraic geometry*

We do not attempt to formulate this question precisely – intuitively, we are asking whether or not for each n there is an integer $d(n)$ such that there is a "new" $d(n)$-web of maximum rank one of whose abelian relations is a (the ?) functional equation with $d(n)$ terms for $\mathbf{L}\mathrm{i}_n$ *? Here, 'new" means the general extension of the phenomena above for the logarithm when $n = 1$, where $d(1) = 3$, for the* [= 5-term identity] *when $n = 2$ and $d(2) = 5$,...*

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[Goncharov-Rudenko 2018] *'Motivic correlator, cluster algebras ...'*

> **Conclusion.** *If $n > 3$, the problem of writing explicitly functional equations for the classical n-logarithms might not be the "right" problem. It seems that when n is growing the functional equations become so complicated that one can not write them down on a piece of paper.*

- **Problems about functional identities of polylogarithms :**
  - Finding **FI**'s for $\mathcal{L}_n$ $\big($ *e.g.* $\exists\ n \geq 8$? $\big)$
  - Is there a sequence $(\boldsymbol{FI_n})_{n\geq 1}$ of **FI**'s for the polylogarithms ?
  - Is there a fundamental **FI**'s for $\mathcal{L}_n$ for each $n \geq 1$?
  - Better understand the polylogarithmic **FI**'s

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  - We describe a series of hyperlogarithmic identities

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$$\mathbf{HLog}^2 \Longleftrightarrow \Big(\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\Big(\frac{x}{y}\Big) - \mathbf{R}\Big(\frac{1-y}{1-x}\Big) + \mathbf{R}\Big(\frac{x(1-y)}{y(1-x)}\Big) = 0 \Big)$$

$$\vdots$$

$$\mathbf{HLog}^6 \qquad \Big(\text{weight 6 hyperlogarithmic identity} \Big)$$

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$$\mathbf{HLog}^6 \qquad \Big(\text{weight 6 hyperlogarithmic identity} \Big)$$

  - For $w = 1, \ldots, 6$, one has

$$\mathbf{HLog}^w \quad : \quad \sum_{i=1}^{\kappa} \boldsymbol{AH}_i^w(\phi_i) = 0$$

A geometric view on Abel's identity

- $(\mathcal{A}\mathbf{b})$ $\quad \mathsf{R}(x) - \mathsf{R}(y) - \mathsf{R}\left(\frac{x}{y}\right) - \mathsf{R}\left(\frac{1-y}{1-x}\right) + \mathsf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$

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A geometric view on Abel's identity

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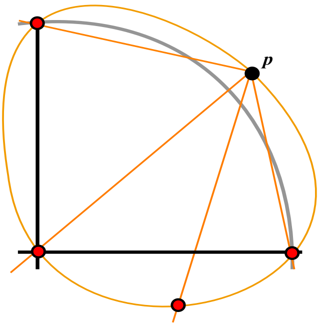

Base points of the U_i's :

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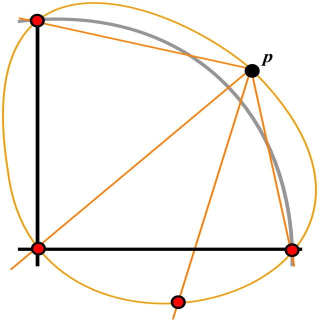

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\mathbf{X}_4 \arrow[d, "\beta"'] \arrow[rr, bend left, "\phi_i"] & & \\
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Generalization to del Pezzo surfaces

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Thm [Castravet-P] $\exists \ (\epsilon_i)_{i=1}^{\kappa} \in \{\pm 1\}^{\kappa}$, $\pm$-unique, such that

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$$\mathbf{HLog}^2 = \Big(\; \sum_{\boldsymbol{i=1}}^{\mathbf{5}} \epsilon_{\boldsymbol{i}} \, \mathbf{R}(\,\phi_{\boldsymbol{i}}\,) \boldsymbol{=} \mathbf{0} \; \Big) \qquad (\mathcal{A}\mathbf{b})$$

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$[\, d = 4 \,]$ $\mathbf{dP}_4$ ∞^2 moduli $\rightsquigarrow$ ∞^2 identities $\mathbf{HLog}^3$

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[$d = 3$] $\mathbf{dP}_3$ = cubic surface in $\mathbb{P}^3$ $\rightsquigarrow$ ∞^4 identities $\mathbf{HLog}^4$

$$\sum_{i=1}^{27} \boldsymbol{AH}^4_i(\phi_i) = \mathbf{0}$$

Thm [Castravet-P. 2022]

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$\longrightarrow$ Del Pezzo surfaces

$\longrightarrow$ Hyperlogarithms (Iterated integrals)

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- $\boldsymbol{W}_r = \boldsymbol{W}(E_r) = \langle\, s_{\rho_1}, \ldots, s_{\rho_r} \,\rangle \subset \mathrm{O}(R_r)$: Weyl group of type E_r

Del Pezzo surfaces I

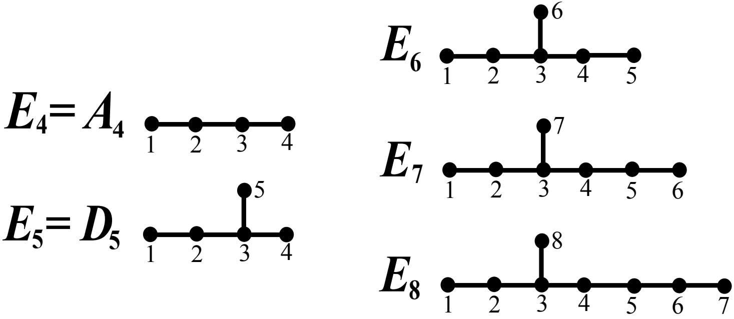

FIGURE – Dynkin diagram E_r $\left(k \text{ stands for } \rho_k \text{ for any } k = 1, \ldots, r\right)$

Lines and conics on $X_r = \mathrm{dP}_d$ $\quad \left(d = 9 - r \right)$

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- <u>Clonics</u> $\mathcal{K}_r = \Big\{ \, \mathfrak{c} \in \mathbf{Pic}(X_r) \;\Big|\; (\mathfrak{c}, -K) = 2 \, , \; \mathfrak{c}^2 = 0 \, \Big\}$

$$\begin{array}{l} \mathcal{K}_r \\ \Cup \\ \mathfrak{c} \quad \longleftrightarrow \quad \text{Fibration in conics} \;\; \phi_{\mathfrak{c}} : X_r \to \mathbb{P}^1 \end{array}$$

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$$\mathcal{L}_r = W_r \cdot \ell_r \qquad\qquad \mathbf{W}'_{r-1} = \mathbf{Stab}(\ell_r) = \langle\, s_{\rho_i} \,|\, i \neq r-1 \,\rangle$$

$$= \mathbf{W}(E'_{r-1}) \quad (\text{ Weyl group })$$

- Clonics $\mathcal{K}_r = \Big\{\, \mathfrak{c} \in \mathbf{Pic}(X_r) \ \Big|\ (\mathfrak{c}, -K) = 2\,,\ \mathfrak{c}^2 = 0 \,\Big\}$

$$\begin{array}{l} \mathcal{K}_r \\ \ \Cup \\ \ \mathfrak{c} \quad \longleftrightarrow \quad \text{Fibration in conics} \ \ \phi_{\mathfrak{c}} : X_r \to \mathbb{P}^1 \end{array}$$

$$\mathcal{K}_r = W_r \cdot (h - \ell_1)$$

Lines and conics on $X_r = \mathbf{dP}_d$ $\quad \big(\, d = 9 - r \,\big)$

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$$= W(E''_{r-1}) \quad (\text{ type } E''_{r-1} = D_{r-1} \;)$$

r	3	4	5	6	7	8
E_r	$A_2 \times A_1$	A_4	D_5	E_6	E_7	E_8
$W_r = W(E_r)$	$\mathfrak{S}_3 \times \mathfrak{S}_2$	$\mathfrak{S}_5$	$(\mathbf{Z}/2\mathbf{Z})^4 \ltimes \mathfrak{S}_5$	$W(E_6)$	$W(E_7)$	$W(E_8)$
$\omega_r = \lvert W_r \rvert$	12	5!	$2^4 \cdot 5!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
$l_r = \lvert \mathcal{L}_r \rvert$	6	10	16	27	56	240
$\kappa_r = \lvert \mathcal{K}_r \rvert$	3	5	10	27	126	2160

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Line	Class in $\mathrm{Pic}(X_7)$	Number of such lines	Model in $\mathbf{P}^2$
ℓ_i	ℓ_i	7	first infinitesimal neighbourhood $p_i^{(1)}$
ℓ_{ij}	$h - \ell_i - \ell_j$	21	line joining p_i to p_j
C_{ij}	$2h - \boldsymbol{\ell} + \ell_i + \ell_j$	21	conic through the p_k's, $k \notin \{i, j\}$
C_i^3	$3h - \boldsymbol{\ell} - \ell_i$	7	cubic through all the p_l's with a node at p_i

TABLE 2. Lines on dP_2 and the corresponding 'curves' in the projective plane

Exemple : clonics of dP_2

Conic class $\mathfrak{c}$	Number of such $\mathfrak{c}$	Linear system $\lvert\mathfrak{C}_\mathfrak{c}\rvert$	$\mathfrak{C}_\mathfrak{c}^{red}$
$h - \ell_i$	7	lines through p_i	$\ell_{ij} + \ell_j$
$2h - \sum_{i \in I} \ell_i$	35	conics through the p_i's, $i \in I$	$\ell_{i_1 i_2} + \ell_{i_3 i_4}$ $\ell_{i_3} + C_{i_1 i_2}$
$3h - \boldsymbol{\ell} + \ell_i - \ell_j$	42	cubics through the p_k's for $k \neq i$, with a node at p_j	$\ell_{jk} + C_{ik}$ $\ell_i + C_j^3$
$4h - \boldsymbol{\ell} - \sum_{j \in J} \ell_j$	35	quartics through the p_k's with a node at p_j for $j \in J$	$C_{k_1 k_2} + C_{k_3 k_4}$ $\ell_{j_1 j_2} + C_{j_3}^3$
$5h - 2\boldsymbol{\ell} + \ell_i$	7	quintics through the p_k's with a node at p_k except for $k = i$	$C_{ij} + C_j^3$

TABLE 3. Conic classes on dP_2 and their reducible fibers

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- $\boldsymbol{L_r} = \cup_{\ell \in \mathcal{L}_r} \ell \subset \boldsymbol{X}_r \quad \rightsquigarrow \quad \boldsymbol{Y_r} = \boldsymbol{X}_r \setminus \boldsymbol{L}_r$

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$$\Sigma_{\mathfrak{c}} = \textbf{Spectrum}(\,\phi_{\mathfrak{c}}) = \left\{ \lambda \in \mathbb{P}^1 \;\middle|\; \phi_{\mathfrak{c}}^{-1}(\lambda) \text{ not irreducible } \right\}$$
$$= \left\{ \sigma_{\mathfrak{c},1} \,,\; \dots \,,\; \sigma_{\mathfrak{c},r-1} \,,\; \sigma_{\mathfrak{c},r-1} = \infty \right\} \subset \mathbb{P}^1$$

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- For $\sigma_{\mathfrak{c},i} \in \Sigma_{\mathfrak{c}}$: $\phi_{\mathfrak{c}}^{-1}\big(\sigma_{\mathfrak{c},i}\big) = L'_{\mathfrak{c},i} + L''_{\mathfrak{c},i} \qquad \Big(L'_{\mathfrak{c},i}, L''_{\mathfrak{c},i} \in \boldsymbol{\mathcal{L}}_r \Big)$

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- $\mathbf{H}_{\mathfrak{c}} = \phi_{\mathfrak{c}}^*\Big(\mathcal{H}_{\mathfrak{c}}\Big) = \left\langle \dfrac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}} - \sigma_{\mathfrak{c},i}} \right\rangle_{i=1}^{r-2} \subset \mathbf{H}^0\Big(\boldsymbol{X}_r \,,\, \Omega^1_{\boldsymbol{X}_r}(\operatorname{Log} \boldsymbol{L}_r)\Big) = \mathbf{H}_{\boldsymbol{X}_r}$

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- **Quest$^{\mathbf{o}}$** : $\exists\ \Big(F_{\mathfrak{c}}(\phi_{\mathfrak{c}})\Big)_{\mathfrak{c}\in\mathcal{K}_r}$ such that $\sum_{\mathfrak{c}\in\mathcal{K}_r} F_{\mathfrak{c}}(\phi_{\mathfrak{c}}) = 0$
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Theorem : $\exists\ (\epsilon_{\mathfrak{c}})_{\mathfrak{c}\in\mathcal{K}_r} \in \{1, -1\}^{\mathcal{K}_r}$, $\pm$-unique such that

$$(\mathbf{HLog}^{r-2}) \qquad \sum_{\mathfrak{c}\in\mathcal{K}_r} \epsilon_{\mathfrak{c}}\, \boldsymbol{AH}_{\mathfrak{c}}^{r-2}\big(\phi_{\mathfrak{c}}\big) = 0$$

where $\forall\ \mathfrak{c}$: $\boldsymbol{AH}_{\mathfrak{c}}^{r-2}$ = complete antisymmetric hyperlogarithm of weight $r-2$ on $\mathbb{P}^1 \setminus \Sigma_{\mathfrak{c}}$.

III Iterated integrals

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- $\forall\, \underline{\omega} \,:$ $\mathbb{I}_{\underline{\omega}} \in \mathcal{O}_y \cap \widetilde{\mathcal{O}}(\boldsymbol{Y})$ unipot. monodromy $\longrightarrow$ Symbol $\boldsymbol{\mathcal{S}}(\mathbb{I}_{\underline{\omega}}) = \underline{\omega}$ ✓

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \{ 0, 1, \infty \}$

$$\mathbf{H} = \Big\langle \tfrac{dz}{z} \,,\, \tfrac{dz}{1-z} \Big\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 \,,\, \Omega^1_{\mathbb{P}^1}(\,\mathrm{Log}\,\Sigma\,) \Big)$$

$$\mathbf{Li}_n = \mathbf{II}^n\Big(\big(\tfrac{dz}{z}\big)^{\otimes(n-1)} \otimes \big(\tfrac{dz}{1-z}\big) \Big) \qquad \text{(‘Polylogarithms’)}$$

III Iterated integrals : hyperlogarithms

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- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \left\{ \sigma_1 \,, \,\ldots\,, \,\sigma_{r-2} \,, \,\sigma_{r-1} = \infty \right\}$

$$\mathbf{H} = \left\langle \tfrac{dz}{z-\sigma_1}, \ldots, \tfrac{dz}{z-\sigma_{r-2}} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 \,, \, \Omega^1_{\mathbb{P}^1}\big(\operatorname{Log} \Sigma \,\big) \Big)$$

III Iterated integrals : hyperlogarithms

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \left\{ \sigma_1 , \ldots , \sigma_{r-2} , \sigma_{r-1} = \infty \right\}$

$$\mathbf{H} = \left\langle \frac{dz}{z-\sigma_1}, \ldots, \frac{dz}{z-\sigma_{r-2}} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 , \Omega^1_{\mathbb{P}^1}(\operatorname{Log} \Sigma) \Big)$$

$$\mathbf{II}^n\Big(\big(\tfrac{dz}{z-\sigma_{i_1}}\big) \otimes \cdots \otimes \big(\tfrac{dz}{z-\sigma_{i_n}}\big) \Big)$$

'Hyperlogarithms'

III Iterated integrals : hyperlogarithms

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \left\{ \sigma_1 \, , \, \ldots \, , \, \sigma_{r-2} \, , \, \sigma_{r-1} = \infty \right\}$

$$\mathbf{H} = \left\langle \tfrac{dz}{z-\sigma_1}, \ldots, \tfrac{dz}{z-\sigma_{r-2}} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 \, , \, \Omega^1_{\mathbb{P}^1}(\, \mathrm{Log}\, \Sigma \,) \Big)$$

$$\mathbf{II}^n\Big(\big(\tfrac{dz}{z-\sigma_{i_1}}\big) \otimes \cdots \otimes \big(\tfrac{dz}{z-\sigma_{i_n}}\big) \Big)$$

'Hyperlogarithms'

- **Complete antisymmetric hyperlog of weight $r-2$ on $\mathbb{P}^1 \setminus \boldsymbol{\Sigma}$** :

$$\boldsymbol{AH}_{\Sigma}^{r-2} = \mathbf{II}^n\left(\mathbf{Asym}\left(\left(\frac{dz}{z-\sigma_1} \right) \otimes \cdots \otimes \left(\frac{dz}{z-\sigma_{r-2}} \right) \right) \right)$$

III Iterated integrals : hyperlogarithms

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \left\{ \sigma_1 , \ldots , \sigma_{r-2} , \sigma_{r-1} = \infty \right\}$

$$\mathbf{H} = \left\langle \frac{dz}{z-\sigma_1}, \ldots, \frac{dz}{z-\sigma_{r-2}} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 , \Omega^1_{\mathbb{P}^1}(\operatorname{Log}\Sigma) \Big)$$

$$\mathbf{II}^n\Big(\big(\tfrac{dz}{z-\sigma_{i_1}}\big) \otimes \cdots \otimes \big(\tfrac{dz}{z-\sigma_{i_n}}\big) \Big)$$ **'Hyperlogarithms'**

- **Complete antisymmetric hyperlog of weight $r-2$ on $\mathbb{P}^1 \setminus \boldsymbol{\Sigma}$** :

$$\boldsymbol{AH}^{r-2}_{\Sigma} = \mathbf{II}^n\left(\mathbf{Asym}\left(\left(\frac{dz}{z-\sigma_1}\right) \otimes \cdots \otimes \left(\frac{dz}{z-\sigma_{r-2}}\right)\right) \right)$$

$$= \mathbf{II}^n\left(\frac{1}{(r-2)!} \sum_{\nu\in\mathfrak{S}_{r-2}} (-1)^{\nu} \left(\frac{dz}{z-\sigma_{\nu(1)}}\right) \otimes \cdots \otimes \left(\frac{dz}{z-\sigma_{\nu(r-2)}}\right) \right)$$

III Iterated integrals : hyperlogarithms

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \left\{ \sigma_1 \,,\, \ldots \,,\, \sigma_{r-2} \,,\, \sigma_{r-1} = \infty \right\}$

$$\mathbf{H} = \left\langle \tfrac{dz}{z-\sigma_1}, \ldots, \tfrac{dz}{z-\sigma_{r-2}} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 \,,\, \Omega^1_{\mathbb{P}^1}(\,\mathrm{Log}\,\Sigma\,)\Big)$$

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- **Complete antisymmetric hyperlog of weight $r-2$ on $\mathbb{P}^1 \setminus \boldsymbol{\Sigma}$:**

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$$= \mathbf{II}^n\left(\tfrac{1}{(r-2)!} \textstyle\sum_{\nu \in \mathfrak{S}_{r-2}} (-1)^{\nu} \left(\tfrac{dz}{z-\sigma_{\nu(1)}}\right) \otimes \cdots \otimes \left(\tfrac{dz}{z-\sigma_{\nu(r-2)}}\right) \right)$$

- **Ex :** $\boldsymbol{AH}^2_{\{0,1,\infty\}} = \frac{1}{2}\,\mathbf{II}^2\Big(\frac{dz}{z} \otimes \frac{dz}{(1-z)} - \frac{dz}{(1-z)} \otimes \frac{dz}{z} \Big) = \mathbf{R}$

IV Identity $\mathbf{HLog}^{r-2}$: proof(s)

$$\boxed{\big(\mathbf{HLog}^{r-2}\big)\ :\quad \sum_{\mathfrak{c}\in\mathcal{K}_r}\epsilon_{\mathfrak{c}}\,\boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\varphi_{\mathfrak{c}})=0\ \text{ with }\quad \boldsymbol{AH}_{\mathfrak{c}}^{r-2}=\boldsymbol{AH}_{\Sigma_{\mathfrak{c}}}^{r-2}}$$

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- $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1$

IV Identity HLog^{r-2} : proof(s)

$$\boxed{\left(\mathbf{HLog}^{r-2}\right)\ :\quad \sum_{\mathfrak{c}\in\mathcal{K}_r}\epsilon_{\mathfrak{c}}\,\boldsymbol{AH}^{r-2}_{\mathfrak{c}}\big(\varphi_{\mathfrak{c}}\big)=0\ \text{ with }\quad \boldsymbol{AH}^{r-2}_{\mathfrak{c}}=\boldsymbol{AH}^{r-2}_{\Sigma_{\mathfrak{c}}}}$$

- $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1 \supset \Sigma_{\mathfrak{c}} = \left\{\,\sigma_{\mathfrak{c},i}\,\right\}_{i=1}^{r-1}$ $\qquad\qquad \mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\Omega^1_{\mathbb{P}^1}\big(\,\mathrm{Log}\,\Sigma_{\mathfrak{c}}\,\big)\Big)$

IV Identity $\mathbf{HLog}^{r-2}$: proof(s)

$$\big(\mathbf{HLog}^{r-2}\big) \ : \quad \sum_{\mathfrak{c}\in\mathcal{K}_r} \epsilon_{\mathfrak{c}}\, \boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\varphi_{\mathfrak{c}}) = 0 \ \text{ with } \quad \boldsymbol{AH}_{\mathfrak{c}}^{r-2} = \boldsymbol{AH}_{\Sigma_{\mathfrak{c}}}^{r-2}$$

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- $\mathbf{H}_{\mathfrak{c}} = \phi_{\mathfrak{c}}^*\big(\,\mathcal{H}_{\mathfrak{c}}\,\big) \subset \mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\,\boldsymbol{L}_r\,\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r}$

IV Identity HLog^{r-2} : proof(s)

$$\boxed{\left(\mathbf{HLog}^{r-2}\right)\ :\quad \sum_{\mathfrak{c}\in\mathcal{K}_r}\epsilon_{\mathfrak{c}}\,\boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\varphi_{\mathfrak{c}})=0\ \ \text{with}\quad \boldsymbol{AH}_{\mathfrak{c}}^{r-2}=\boldsymbol{AH}_{\Sigma_{\mathfrak{c}}}^{r-2}}$$

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- $\mathbf{H}_{\mathfrak{c}} = \phi_{\mathfrak{c}}^*\big(\,\mathcal{H}_{\mathfrak{c}}\,\big) \subset \mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\,\boldsymbol{L}_r\,\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r}$

- $\boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\phi_{\mathfrak{c}}) = \mathbf{II}\left(\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},1}}\right)\wedge\cdots\wedge\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},r-2}}\right)\right)$

IV Identity HLog^{r-2} : proof(s)

$$\boxed{\left(\mathbf{HLog}^{r-2}\right) \; : \quad \sum_{\mathfrak{c}\in\mathcal{K}_r} \epsilon_{\mathfrak{c}}\, \boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\varphi_{\mathfrak{c}}) = 0 \;\text{ with }\; \boldsymbol{AH}_{\mathfrak{c}}^{r-2} = \boldsymbol{AH}_{\Sigma_{\mathfrak{c}}}^{r-2}}$$

- $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1 \supset \Sigma_{\mathfrak{c}} = \left\{ \sigma_{\mathfrak{c},i} \right\}_{i=1}^{r-1} \qquad \mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\Omega^1_{\mathbb{P}^1}\big(\operatorname{Log}\Sigma_{\mathfrak{c}}\big)\Big)$

- $\mathbf{H}_{\mathfrak{c}} = \phi_{\mathfrak{c}}^*\big(\mathcal{H}_{\mathfrak{c}}\big) \subset \mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\operatorname{Log}\boldsymbol{L}_r\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r}$

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IV Identity $\mathbf{HLog}^{r-2}$: proof(s)

$$\boxed{\left(\mathbf{HLog}^{r-2}\right)\ :\quad \sum_{\mathfrak{c}\in\mathcal{K}_r}\epsilon_{\mathfrak{c}}\,\boldsymbol{AH}^{r-2}_{\mathfrak{c}}(\varphi_{\mathfrak{c}})=0\ \text{ with }\quad \boldsymbol{AH}^{r-2}_{\mathfrak{c}}=\boldsymbol{AH}^{r-2}_{\Sigma_{\mathfrak{c}}}}$$

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- $\mathbf{H}_{\mathfrak{c}}=\phi^*_{\mathfrak{c}}\big(\,\mathcal{H}_{\mathfrak{c}}\,\big)\subset\mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\,\boldsymbol{L}_r\,\big)\Big)=\mathbf{H}_{\boldsymbol{X}_r}$

- $\boldsymbol{AH}^{r-2}_{\mathfrak{c}}(\phi_{\mathfrak{c}})=\mathbf{II}\left(\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},1}}\right)\wedge\cdots\wedge\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},r-2}}\right)\right)\in\mathbf{II}^{r-2}\Big(\wedge^{r-2}\,\mathbf{H}_{\mathfrak{c}}\Big)$

  $\Big\downarrow\quad \boldsymbol{\mathcal{S}}$ (symbol)

- $\boldsymbol{\Omega}^{r-2}_{\mathfrak{c}}=\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},1}}\right)\wedge\cdots\wedge\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},r-2}}\right)$

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$$\boxed{\left(\mathbf{HLog}^{r-2}\right)\ :\quad \sum_{\mathfrak{c}\in\mathcal{K}_r}\epsilon_{\mathfrak{c}}\,\boldsymbol{AH}^{r-2}_{\mathfrak{c}}(\varphi_{\mathfrak{c}})=0\ \ \text{with}\quad \boldsymbol{AH}^{r-2}_{\mathfrak{c}}=\boldsymbol{AH}^{r-2}_{\Sigma_{\mathfrak{c}}}}$$

- $\phi_{\mathfrak{c}}:\boldsymbol{X}_r\to\mathbb{P}^1\supset\Sigma_{\mathfrak{c}}=\{\,\sigma_{\mathfrak{c},i}\,\}_{i=1}^{r-1}$ $\qquad$ $\mathcal{H}_{\mathfrak{c}}=\mathbf{H}^0\Big(\Omega^1_{\mathbb{P}^1}\big(\,\mathrm{Log}\,\Sigma_{\mathfrak{c}}\,\big)\Big)$

- $\mathbf{H}_{\mathfrak{c}}=\phi_{\mathfrak{c}}^*\big(\,\mathcal{H}_{\mathfrak{c}}\,\big)\subset\mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\,\boldsymbol{L}_r\,\big)\Big)=\mathbf{H}_{\boldsymbol{X}_r}$

- $\boldsymbol{AH}^{r-2}_{\mathfrak{c}}(\phi_{\mathfrak{c}})=\mathbf{II}\left(\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},1}}\right)\wedge\cdots\wedge\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},r-2}}\right)\right)\in\mathbf{II}^{r-2}\Big(\wedge^{r-2}\,\mathbf{H}_{\mathfrak{c}}\Big)$

$$\Big\downarrow\ \ \boldsymbol{\mathcal{S}}\ \ \text{(symbol)}$$

- $\boldsymbol{\Omega}^{r-2}_{\mathfrak{c}}=\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},1}}\right)\wedge\cdots\wedge\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},r-2}}\right)\ \in\ \wedge^{r-2}\mathbf{H}_{\mathfrak{c}}$

IV Identity $\mathbf{HLog}^{r-2}$: proof(s)

$$\boxed{\left(\mathbf{HLog}^{r-2}\right) \ : \quad \sum_{\mathfrak{c}\in\mathcal{K}_r} \epsilon_{\mathfrak{c}}\, \boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\varphi_{\mathfrak{c}}) = 0 \ \text{ with } \ \boldsymbol{AH}_{\mathfrak{c}}^{r-2} = \boldsymbol{AH}_{\Sigma_{\mathfrak{c}}}^{r-2}}$$

- $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1 \supset \Sigma_{\mathfrak{c}} = \left\{ \sigma_{\mathfrak{c},i} \right\}_{i=1}^{r-1} \qquad \mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\Omega^1_{\mathbb{P}^1}\big(\operatorname{Log}\Sigma_{\mathfrak{c}}\big)\Big)$

- $\mathbf{H}_{\mathfrak{c}} = \phi_{\mathfrak{c}}^*\big(\mathcal{H}_{\mathfrak{c}}\big) \subset \mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\operatorname{Log}\boldsymbol{L}_r\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r}$

- $\boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\phi_{\mathfrak{c}}) = \mathbf{II}\left(\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},1}}\right)\wedge\cdots\wedge\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},r-2}}\right)\right) \in \mathbf{II}^{r-2}\Big(\wedge^{r-2}\,\mathbf{H}_{\mathfrak{c}}\Big)$

  $\Big\downarrow\ \boldsymbol{\mathcal{S}}$ (symbol)

- $\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = \left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},1}}\right)\wedge\cdots\wedge\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},r-2}}\right) \ \in \wedge^{r-2}\mathbf{H}_{\mathfrak{c}} \subset \wedge^{r-2}\mathbf{H}_{X_r}$

IV Identity $\mathbf{HLog}^{r-2}$: proof(s)

$$\left(\mathbf{HLog}^{r-2}\right) \; : \quad \sum_{\mathfrak{c}\in\mathcal{K}_r} \epsilon_{\mathfrak{c}}\, \boldsymbol{AH}^{r-2}_{\mathfrak{c}}(\varphi_{\mathfrak{c}}) = 0 \quad \text{with} \quad \boldsymbol{AH}^{r-2}_{\mathfrak{c}} = \boldsymbol{AH}^{r-2}_{\Sigma_{\mathfrak{c}}}$$

- $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1 \supset \Sigma_{\mathfrak{c}} = \left\{ \sigma_{\mathfrak{c},i} \right\}_{i=1}^{r-1} \qquad \mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\Omega^1_{\mathbb{P}^1}\big(\operatorname{Log}\Sigma_{\mathfrak{c}}\big)\Big)$
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- $\boldsymbol{AH}^{r-2}_{\mathfrak{c}}(\phi_{\mathfrak{c}}) = \mathbf{II}\left(\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},1}}\right) \wedge \cdots \wedge \left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},r-2}}\right)\right) \in \mathbf{II}^{r-2}\Big(\wedge^{r-2}\mathbf{H}_{\mathfrak{c}}\Big)$

$\Big\downarrow\; \mathcal{S}$ (symbol)

- $\boldsymbol{\Omega}^{r-2}_{\mathfrak{c}} = \left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},1}}\right) \wedge \cdots \wedge \left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c},r-2}}\right) \in \wedge^{r-2}\mathbf{H}_{\mathfrak{c}} \subset \wedge^{r-2}\mathbf{H}_{X_r}$

$$\left(\mathbf{HLog}^{r-2}\right) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}^{r-2}_{\mathfrak{c}} = 0 \quad \text{in} \quad \wedge^{r-2}\mathbf{H}_{X_r}$$

IV Identity $\mathbf{HLog}^{r-2}$: proof(s)

$$\boxed{\big(\mathbf{HLog}^{r-2}\big) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \text{in} \quad \wedge^{r-2}\mathbf{H}_{X_r}}$$

IV Identity $\mathbf{HLog}^{r-2}$: proof(s)

$$\boxed{\left(\mathbf{HLog}^{r-2}\right) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}} \, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \text{in} \quad \wedge^{r-2} \mathbf{H}_{X_r}}$$

- $\mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\operatorname{Log} \boldsymbol{L}_r\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r} \xrightarrow{\oplus_\ell \mathbf{Res}_\ell} \mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$

IV Identity $\mathbf{HLog}^{r-2}$: proof(s)

$$\boxed{\left(\mathbf{HLog}^{r-2}\right) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \text{in} \quad \wedge^{r-2}\mathbf{H}_{X_r}}$$

- $\mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\,\boldsymbol{L}_r\,\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r} \xrightarrow{\ \oplus_\ell \mathbf{Res}_\ell\ } \mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ injective

IV Identity $\mathbf{HLog}^{r-2}$: proof(s)

$$\boxed{\left(\mathbf{HLog}^{r-2}\right) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \text{in} \quad \wedge^{r-2}\mathbf{H}_{X_r}}$$

- $\mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\operatorname{Log} \boldsymbol{L}_r\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r} \xrightarrow{\oplus_\ell \mathbf{Res}_\ell} \mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ injective

$$\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} \in \wedge^{r-2}\mathbf{H}_{X_r} \hookrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$$

IV Identity $\mathbf{HLog}^{r-2}$: proof(s)

$$\boxed{\left(\mathbf{HLog}^{r-2}\right) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \text{in} \quad \wedge^{r-2}\mathbf{H}_{X_r}}$$

- $\mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\operatorname{Log} \boldsymbol{L}_r\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r} \xrightarrow{\oplus_\ell \mathbf{Res}_\ell} \mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ injective

$$\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} \in \wedge^{r-2}\mathbf{H}_{X_r} \hookrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r} \curvearrowleft \boldsymbol{W}(E_r)$$

IV Identity HLog^{r-2} : proof(s)

$$\boxed{\big(\mathbf{HLog}^{r-2}\big) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \text{in} \quad \wedge^{r-2}\mathbf{H}_{X_r}}$$

- $\mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\operatorname{Log} \boldsymbol{L}_r\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r} \xrightarrow{\oplus_\ell \mathbf{Res}_\ell} \mathbb{C}^{\mathcal{L}_r}$ injective

$$\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} \in \wedge^{r-2}\mathbf{H}_{X_r} \hookrightarrow \wedge^{r-2}\,\mathbb{C}^{\mathcal{L}_r} \curvearrowleft \boldsymbol{W}(E_r)$$

- $0 \to \mathbf{K}^{r-2} \longrightarrow \oplus_{\mathfrak{c}} (\mathbf{H}_{\mathfrak{c}})^{\wedge(r-2)} \longrightarrow \wedge^{r-2}\,\mathbb{C}^{\mathcal{L}_r}$ SES $\mathbb{C}$-vect spaces

IV Identity HLog^{r-2} : proof(s)

$$\boxed{\left(\mathbf{HLog}^{r-2}\right) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \text{in} \quad \wedge^{r-2}\mathbf{H}_{X_r}}$$

- $\mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\,\boldsymbol{L}_r\,\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r} \xrightarrow{\ \oplus_\ell \mathbf{Res}_\ell\ } \mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ injective

$$\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} \in \wedge^{r-2}\mathbf{H}_{X_r} \hookrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r} \curvearrowleft \boldsymbol{W}(E_r)$$

- $0 \to \mathbf{K}^{r-2} \longrightarrow \oplus_{\mathfrak{c}}\big(\mathbf{H}_{\mathfrak{c}}\big)^{\wedge(r-2)} \longrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ SES $\mathbb{C}$-vect spaces

$$\wr|$$

$\mathrm{Ind}_{\boldsymbol{W}''_{r-1}}^{\boldsymbol{W}_r}\big(\mathbf{sign}''_{r-1}\big) \longrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ SES of $\boldsymbol{W}_r$-reps

IV Identity HLog^{r-2} : proof(s)

$$\boxed{(\mathbf{HLog}^{r-2}) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \text{in} \quad \wedge^{r-2}\mathbf{H}_{X_r}}$$

- $\mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\operatorname{Log} \boldsymbol{L}_r\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r} \xrightarrow{\oplus_\ell \mathbf{Res}_\ell} \mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ injective

$$\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} \in \wedge^{r-2}\mathbf{H}_{X_r} \hookrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r} \curvearrowleft \boldsymbol{W}(E_r)$$

- $0 \to \mathbf{K}^{r-2} \longrightarrow \oplus_{\mathfrak{c}}(\mathbf{H}_{\mathfrak{c}})^{\wedge(r-2)} \longrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ SES $\mathbb{C}$-vect spaces

$$\wr|$$

$\mathrm{Ind}_{\boldsymbol{W}''_{r-1}}^{\boldsymbol{W}_r}(\mathbf{sign}''_{r-1}) \longrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ SES of $\boldsymbol{W}_r$-reps

- $\mathfrak{c}_0 \rightsquigarrow \boldsymbol{\Omega}_{\mathfrak{c}_0}^{r-2} \in \wedge^{r-2}\mathbf{H}_{\mathfrak{c}_0} \rightsquigarrow \boldsymbol{W}''_{r-1} = \mathbf{Stab}(\mathfrak{c}_0) \subset \boldsymbol{W}(E_r)$

IV Identity $\mathbf{HLog}^{r-2}$: proof(s)

$$\boxed{\big(\mathbf{HLog}^{r-2}\big) \quad \Longleftrightarrow \quad \textstyle\sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \text{in} \quad \wedge^{r-2}\mathbf{H}_{X_r}}$$

- $\mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\,\boldsymbol{L}_r\,\big)\Big) = \mathbf{H}_{\boldsymbol{X}_r} \xrightarrow{\oplus_\ell \mathbf{Res}_\ell} \mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ injective

  $$\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} \in \wedge^{r-2}\mathbf{H}_{X_r} \hookrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r} \curvearrowleft \boldsymbol{W}(E_r)$$

- $0 \to \mathbf{K}^{r-2} \longrightarrow \oplus_{\mathfrak{c}}\big(\mathbf{H}_{\mathfrak{c}}\big)^{\wedge(r-2)} \longrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ SES $\mathbb{C}$-vect spaces

  $$\wr|$$

  $\mathrm{Ind}_{\boldsymbol{W}''_{r-1}}^{\boldsymbol{W}_r}\big(\mathbf{sign}''_{r-1}\big) \longrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ SES of $\boldsymbol{W}_r$-reps

- $\mathfrak{c}_0 \quad \rightsquigarrow \quad \boldsymbol{\Omega}_{\mathfrak{c}_0}^{r-2} \in \wedge^{r-2}\mathbf{H}_{\mathfrak{c}_0} \quad \rightsquigarrow \quad \boldsymbol{W}''_{r-1} = \mathbf{Stab}(\mathfrak{c}_0) \subset \boldsymbol{W}(E_r)$

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IV Identity $\mathbf{HLog}^{r-2}$: proof(s)

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- Analytic constructive proof (?)

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$$\boldsymbol{AH}^3_1(\boldsymbol{x}) + \boldsymbol{AH}^3_2\left(\frac{\boldsymbol{1}}{\boldsymbol{y}}\right) + \boldsymbol{AH}^3_3\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right) + \boldsymbol{AH}^3_4\left(\frac{\boldsymbol{x-y}}{\boldsymbol{x-1}}\right) + \boldsymbol{AH}^3_5\left(\frac{\boldsymbol{b(a-x)}}{\boldsymbol{ay-bx}}\right)$$

$$+\boldsymbol{AH}^3_6\left(\frac{\boldsymbol{P}}{\boldsymbol{(x-1)(y-b)}}\right) + \boldsymbol{AH}^3_7\left(\frac{\boldsymbol{(x-y)(y-b)}}{\boldsymbol{y\,P}}\right) + \boldsymbol{AH}^3_8\left(\frac{\boldsymbol{x\,P}}{\boldsymbol{(x-y)(x-a)}}\right)$$

$$+\boldsymbol{AH}^3_9\left(\frac{\boldsymbol{y(x-b)}}{\boldsymbol{x(y-a)}}\right) + \boldsymbol{AH}^3_{10}\left(\frac{\boldsymbol{a\,(b-x)}}{\boldsymbol{by-ax}}\right) = \boldsymbol{0}$$

Webs $\mathcal{W}_{\mathsf{dP}_5}$ and $\mathcal{W}_{\mathsf{dP}_4}$

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Comparing the webs $\mathcal{W}_{\mathsf{dP}_5}$ and $\mathcal{W}_{\mathsf{dP}_4}$

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- Both $\mathcal{W}_{\mathbf{dP}_5}$ and $\mathcal{W}_{\mathbf{dP}_4}$ satisfy similar remarkable properties :
  - non-linearizable webs
  - maximal rank (with all their ARs poly-/hyperlogarithmic)
  - characterized by the matroid of their hexagonal subwebs
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- $\boldsymbol{\mathcal{W}}_{\mathbf{dP}_5}$ *is equivalent to the* $\boldsymbol{\mathcal{X}}$*-cluster web of type* A_2
- $\boldsymbol{\mathcal{W}}_{\mathbf{dP}_4}$ *is equivalent to a* $\boldsymbol{\mathcal{X}}$*-cluster web of type* D_4

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$\mathcal{A}$–mutation

$$a_j' = \begin{cases} a_k^{-1}\Big[\prod_{b_{k\ell}>0} a_\ell^{b_{k\ell}} + \prod_{b_{kl}<0} a_l^{-b_{kl}}\Big] & j = k \\ a_j & j \neq k \end{cases}$$

$\mathcal{X}$ – mutation

$$x_j' = \begin{cases} x_j^{-1} & j = k \\ x_j \left(1 + x_k^{\mathsf{s}(-b_{kj})}\right)^{-b_{kj}} & j \neq k \end{cases}$$

Cluster algebra : type A_2

- **Matrix** $B = B_Q$ $\longleftrightarrow$ **Quiver** $Q = Q_B$

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$$\begin{array}{ccccc}
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- $$\textbf{Matrix } B = B_Q \quad \longleftrightarrow \quad \textbf{Quiver } Q = Q_B$$

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- For $\quad \mathbf{R}^c(x) = -\mathbf{Li}_2(-x) - \frac{1}{2}\,\mathbf{Log}(x)\,\mathbf{Log}(1+x)$:

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Cluster periods and cluster dilogarithmic identities

- Notion of **period** of a cluster algebra $\mathcal{A}$ $\longleftrightarrow$ loop in the exchange graph $\Gamma_{\mathcal{A}}$

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Thm [Nakanishi] For some $N \in \mathbb{N}_{>0}$, one has

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where – the F_σ's are some analytic functions

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$\longrightarrow$ Spence-Kummer identity $(\mathcal{SK})$ is cluster

HLog^3 is cluster

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- **Cluster ensemble of type D_4** :

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- **Blow-ups** $\mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2)$ with $r \geq 9$: $\sum_{\mathfrak{c}} \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0$?