MSU Cluster algebras seminar 28/05/2020

Realizations of associahedra and minimal relations between g-vectors

(arXiv:1906.06861)

Report on a joint work with:

Arnau Padrol
Sorbonne Université

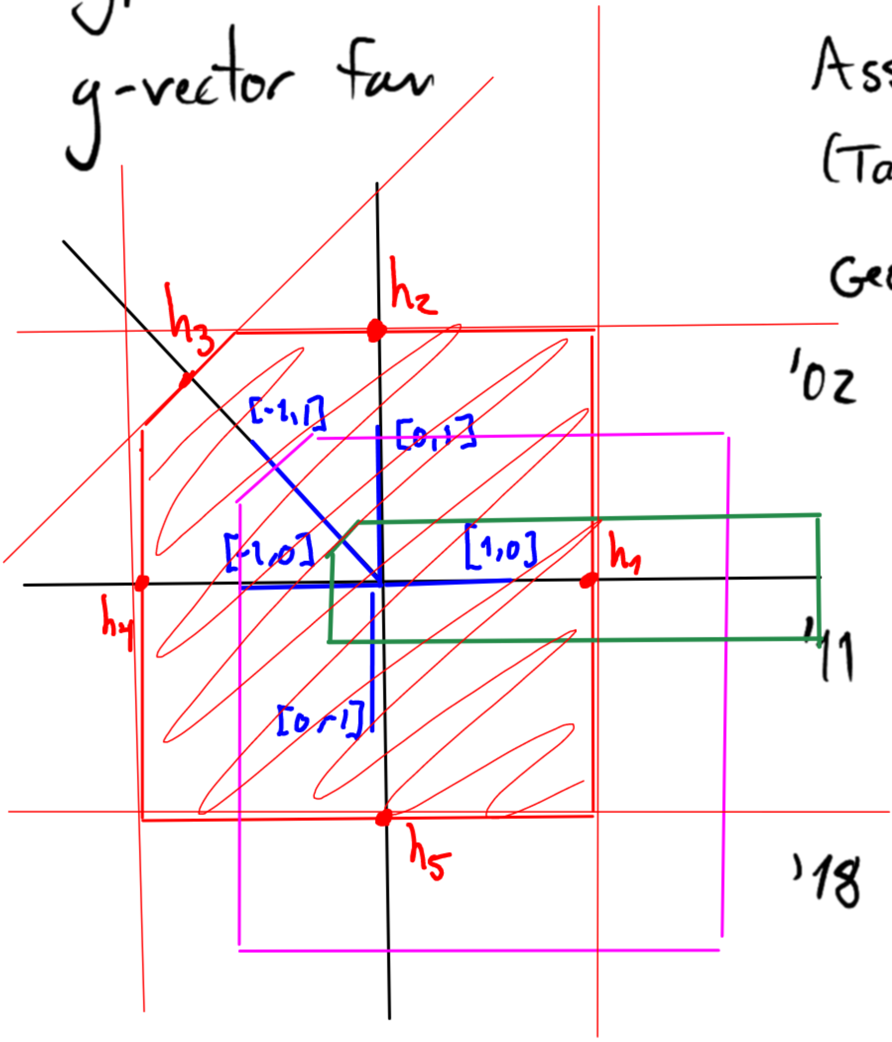
Yann Palu
Université UPJV Amiens

Vincent Pilaud
CNRS, École Polytechnique

1. Motivation: realizations of associahedra

Type A_2: $Q: 1 \to 2$

g-vector fan



Associahedron
(Tamari '51, Stasheff *1963)

Geometric realizations:

'02 Chapoton-Fomin-Zelevinsky (initial bipartite seeds).

'11 Hohlweg-Lange-Thomas (acyclic seeds)

'18 Hohlweg-Pilaud-Stella (any initial seed).

We are interested in the space of all realizations of the generalized associahedra.

'18 Arkani-Hamed, Bai, He & Yan described this space for $1 \to 2 \to \cdots \to n$

'18 Bazier-Matte, Douville, Mousavand, Thomas & Yildirim: generalized to any Dynkin type and any acyclic seed.

Our aim: Generalize to any Dynkin type and any initial seed, using a different approach.

Today

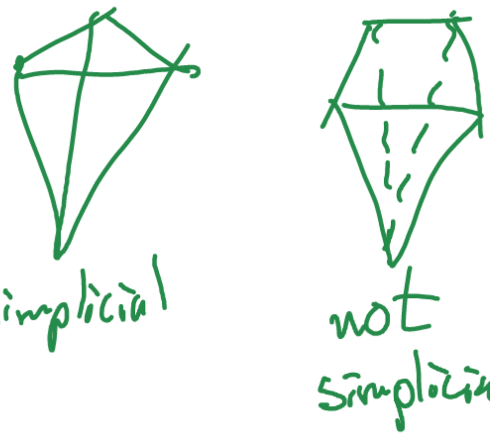
Two ingredients: ① Type cone of a fan.

② Categorification of g-vectors using extriangulated categories.

1. Type cones

Let $\mathcal{F}$ an $\overbrace{\text{essential}}^{\{0\}\text{ is a cone}}$, $\underbrace{\text{complete}}_{\text{covers } \mathbb{R}^n}$, $\overbrace{\text{simplicial}}^{\text{cones are simplicial}}$ fan in $\mathbb{R}^n$.

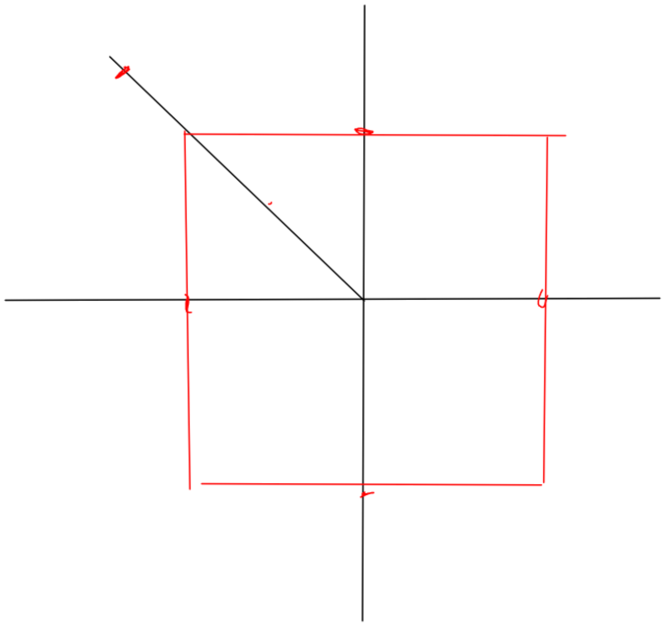
($\mathcal{F}$ has finitely many cones).



Let G be the matrix whose N rows are generators of the rays of $\mathcal{F}$.

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

A *polytopal realization* of $\mathcal{F}$ is a polytope whose normal fan is $\mathcal{F}$.



For any $h \in \mathbb{R}^N$, define

$$P_h = \{x \in \mathbb{R}^n \mid Gx \le h\}$$

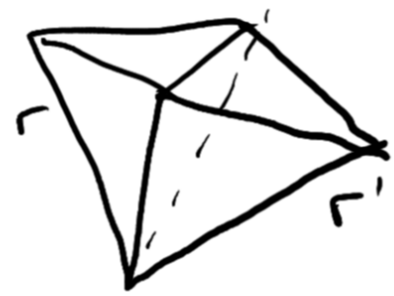
Def [McMullen, 1973] The *type cone* of $\mathcal{F}$ is

$$TC(\mathcal{F}) = \{h \in \mathbb{R}^N \mid P_h \text{ is a polytopal realization of } \mathcal{F}\}.$$

this is what we want to describe.

Facts:
- $TC(\mathcal{F})$ is an open cone in $\mathbb{R}^N$
- $TC(\mathcal{F}) + G\mathbb{R}^n = TC(\mathcal{F})$. It's often useful to project to a smaller space to get rid of $G\mathbb{R}^n$.

$\mathcal{F}$



• If R, R' are sets of rays of two adjacent cones in $\mathcal{F}$, and

$$R \setminus \{r\} = R' \setminus \{r'\}$$

and

exchange relation

$$\circledast \quad \alpha_{R,R'}(r)\cdot r + \alpha_{R,R'}(r')\cdot r' + \sum_{s \in R \setminus \{r\}} \alpha_{R,R'}(s)\cdot s = 0$$

with $\alpha_{R,R'}(r), \alpha_{R,R'}(r') > 0$,

then

$$TC(\mathcal{F}) = \left\{ h \in \mathbb{R}^N \;\middle|\; \sum_{s \in R \cup R'} \alpha_{R,R'}(s)\cdot h_s > 0 \quad \forall R, R' \text{ adjacent} \right\}.$$

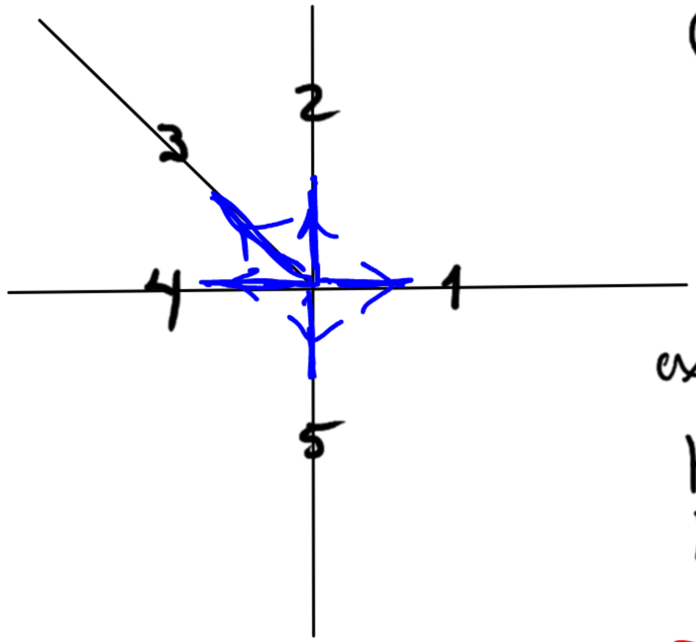
Thus the facets of $TC(\mathcal{F})$ are defined by exchange relations.

__Q.__ Which ones?

In other words: which exchange relations positively generate all the others?

3. An example

g-vector fan $\mathcal{F}$ in type A_2. $Q = 1 \longrightarrow 2$



$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$1 \cdot g_1 - 1 \cdot g_2 + 1 \cdot g_3 = 0$

exch. relations:

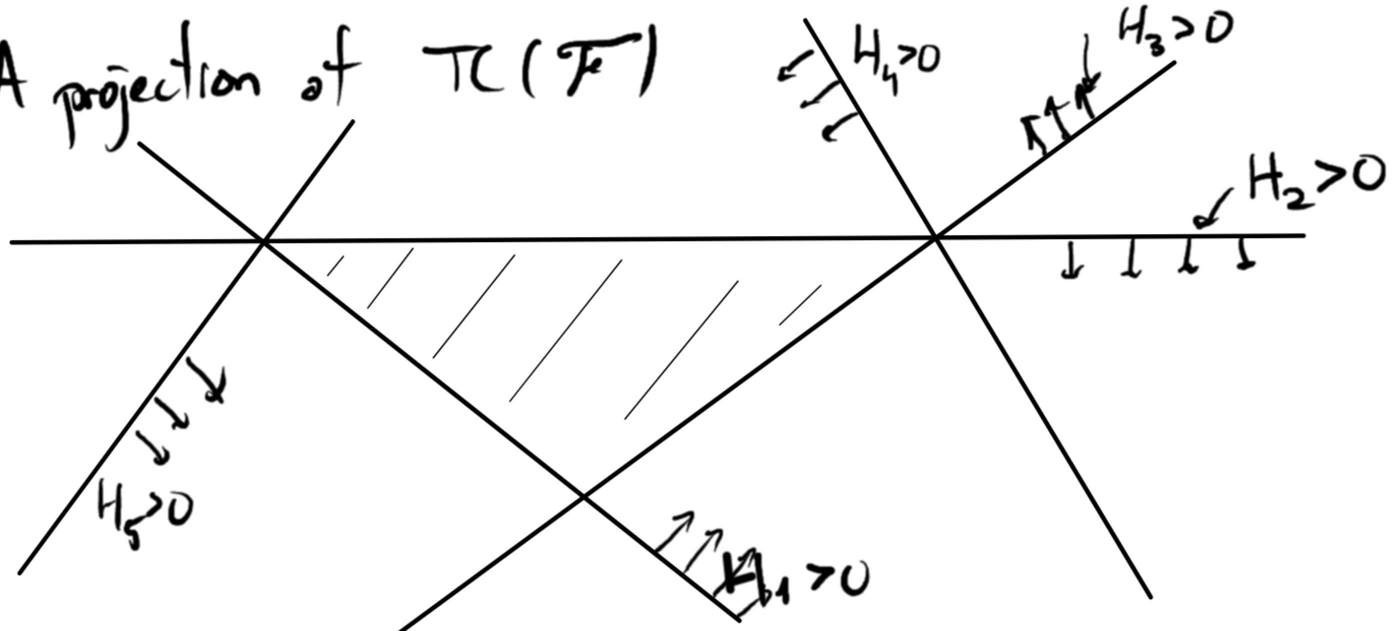
$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{matrix} & \left[\begin{matrix} \boxed{\begin{matrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{matrix}} \\ \begin{matrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{matrix} \end{matrix}\right] \end{array} = K$$

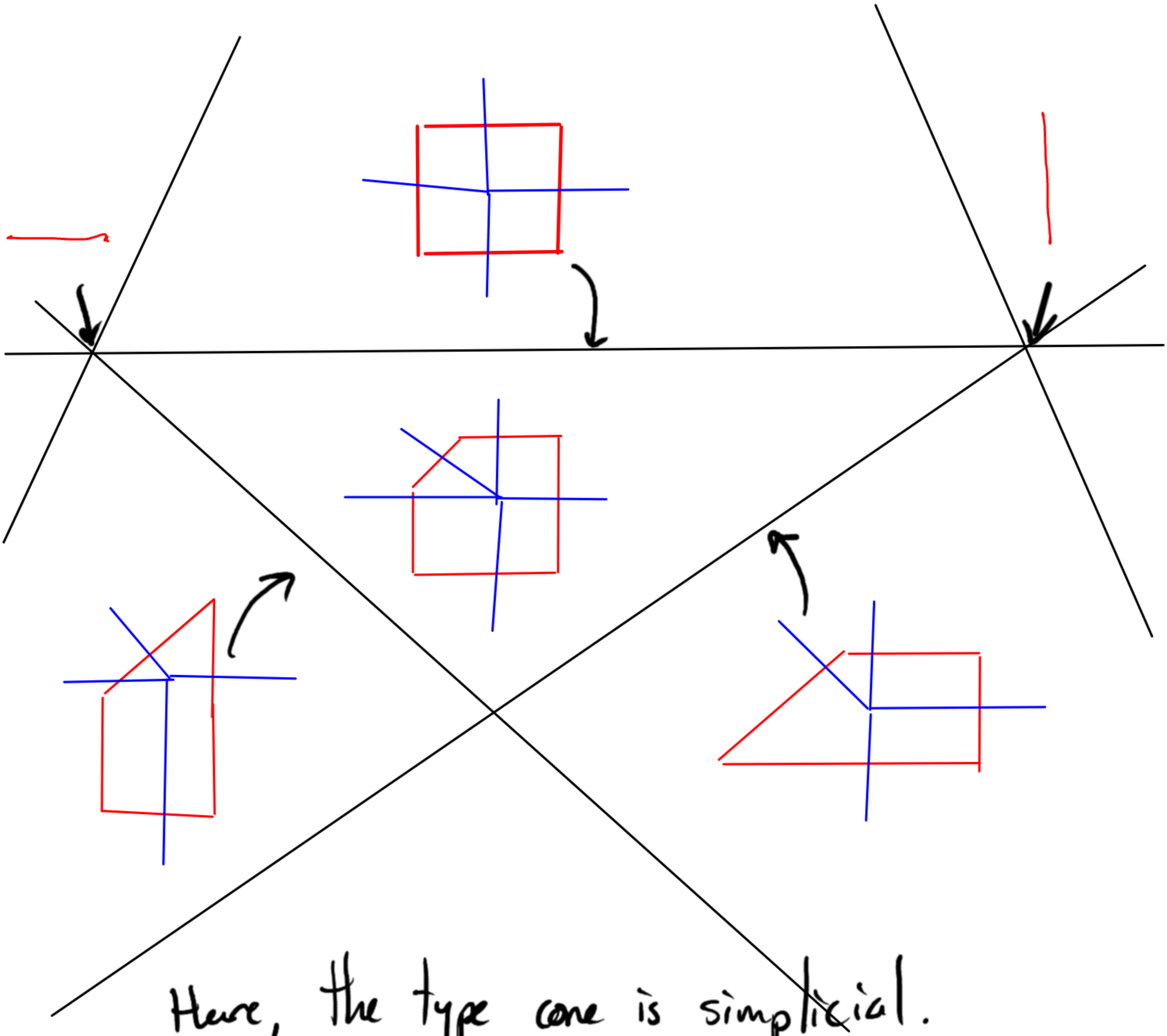
redundant (H_4, H_5)

$$TC(\mathcal{F}) = \left\{ h \in \mathbb{R}^5 \;\middle|\; \begin{matrix} h_1 - h_2 + h_3 > 0 \\ h_2 - h_3 + h_4 > 0 \\ h_3 - h_4 + h_5 > 0 \end{matrix} \right\}$$

$$\left.\begin{matrix} h_2 + h_5 > 0 \\ h_1 + h_4 > 0 \end{matrix}\right\} \text{redundant}$$

A projection of $TC(\mathcal{F})$



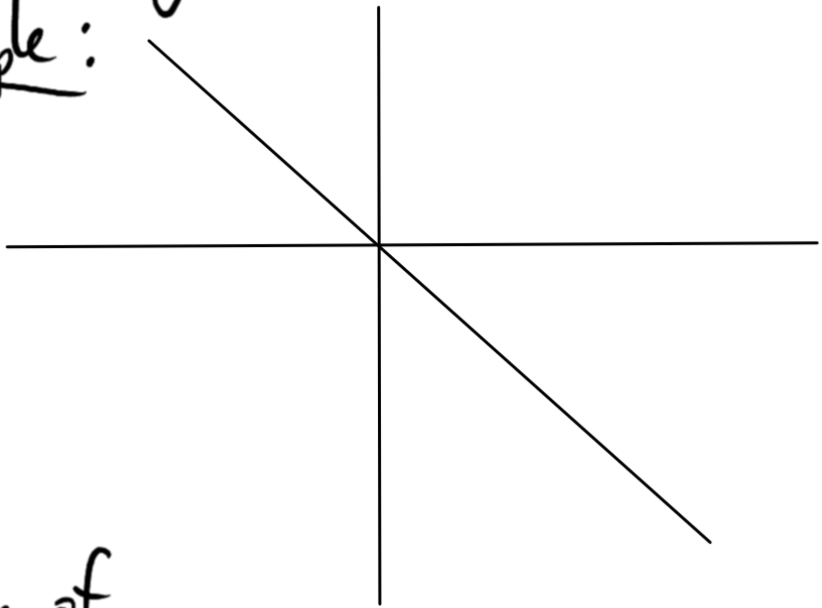


Here, the type cone is simplicial.

<u>Thm 1</u> [PPPP '19] The (relatively open) faces of $\overline{TC(\mathcal{F})}$ are the type cones of all the coarsenings of $\mathcal{F}$.

Remark: The type cone of a simplicial fan is not necessarily simplicial.

Counter-example:



(g-vector fan of

).

Thm 2 [PPPP '19] If $TC(\mathcal{F})$ is a simplicial cone, and if K is an $(N-n)\times N$-matrix whose rows are the exchange relations defining the facets of $TC(\mathcal{F})$, then the polytopal realizations of $\mathcal{F}$ are the

$$R_\ell = \{ z \in \mathbb{R}^N \mid Kz = \ell,\ z \geq 0 \},\quad \ell \in \mathbb{R}^{N-n}_{>0}.$$

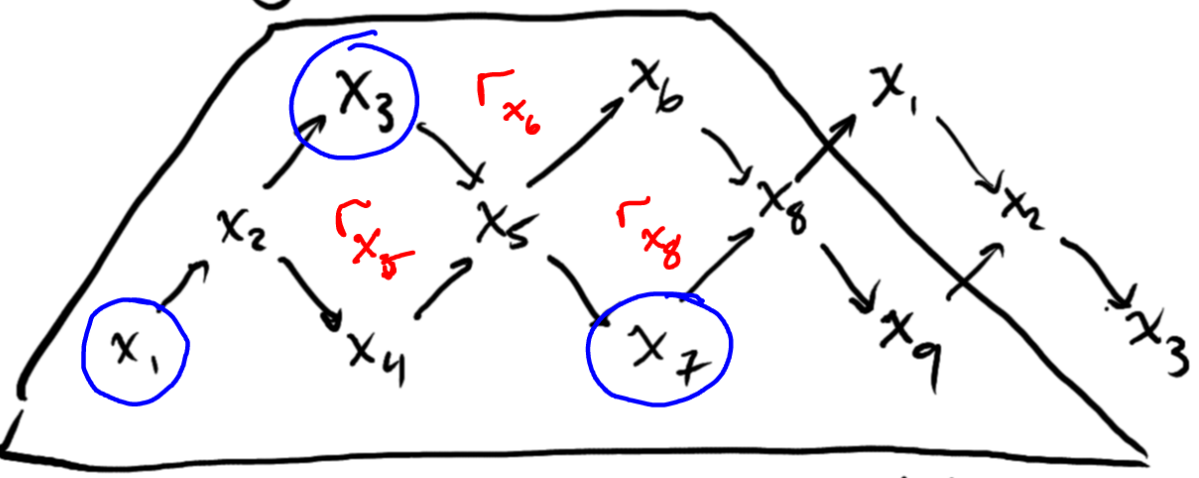
<u>Consequence</u>: If we can find $N-n$ exchange relations (= rows of K) that non-negatively generate all the others, then we can describe all polytopal realizations of $\mathcal{F}$.
(i.e. if $TC(\mathcal{F})$ is a simplicial cone).

4. Application to finite type cluster algebras

Let B be a skew-symmetrizable matrix of Dynkin type. Then one can organize all cluster variables on the bipartite belt.

<u>ex</u>: Type A_3

$r_{x_5}: g_2 + g_5 - g_3 - g_4 = 0$



$\rightsquigarrow$ get "mesh" exchange relations r_x $\forall x$.

Fix a cluster C.

$$\textbf{Thm 3}: [\overbrace{\text{ABHY}}^{1\to\cdots\to n}, \overbrace{\text{BDMTY}}^{\text{acyclic}}, \text{PPPP}]$$

The exchange relations

$$r_x, \quad x \notin C$$

non-negatively generate all other g-vector exchange relations. In particular, the type cone of the g-vector fan is simplicial.

Final remarks:

- Thm 2 + Thm 3 $\Rightarrow$ all realizations of generalized associahedra.
- Can apply this approach to other fans.
- Infinite types?