Endo-parameters of p-adic classical groups

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Classical groups

We are given:

- F non-Arch. local field of odd residual characteristic, e.g. $\mathbb{Q}_p = \{\sum_{i=m}^{\infty} a_i p^i \mid m \in \mathbb{Z},\ a_i \in \{0, \ldots, p-1\}\}$ (locally compact, connected components are points)
- $\mathrm{F}/\mathrm{F}_\mathrm{o}$ a field extension, $[\mathrm{F} : \mathrm{F}_\mathrm{o}] \leqslant 2$ and $\langle(\bar{\ })\rangle = \mathrm{Gal}(\mathrm{F}/\mathrm{F}_\mathrm{o})$.
- $\dim_{\mathrm{F}} \mathrm{V} < \infty$, $h : \mathrm{V} \times \mathrm{V} \to (\mathrm{F}, \bar{\ })$ an $\epsilon \in \{\pm 1\}$-hermitian form.

We consider the groups:

$$\tilde{\mathrm{G}} = \mathrm{Aut}_{\mathrm{F}}(\mathrm{V})$$

$$\mathrm{G} = \mathrm{U}(h) \subseteq \tilde{\mathrm{G}}$$

the group of isometries of h.

Complex smooth representations

Let H be a locally compact group.

$\mathbb{C}$-representation: action of H on a $\mathbb{C}$-vector space W

$$\mathrm{H} \times \mathrm{W} \to \mathrm{W}, \ (x, w) \mapsto \rho(x)w$$

with a group homomorphism

$$\rho : \mathrm{H} \to \mathrm{Aut}_{\mathbb{C}}(\mathrm{W}).$$

smooth: Every w is fixed by some open subgroup (depending on w)

Example: Character: $\chi : \mathrm{H} \to \mathbb{C}^\times$, $((x, z) \mapsto \chi(x)z, \ z \in \mathbb{C})$

$$\chi \text{ is smooth} \Leftrightarrow \ker(\chi) \text{ is open.}$$

From now on: We only consider smooth representations

Semisimple characters for $\tilde{\mathrm{G}} = \mathrm{Aut}_{\mathrm{F}}(\mathrm{V})$ (Bushnell–Kutzko, Stevens)

Certain characters on pro-p subgroups of $\tilde{\mathrm{G}}$

$$\theta : \mathrm{K} \to \mathbb{C}^\times .$$

One needs arithmetic data ("semisimple stratum"):

- $\beta \in \mathrm{End}_{\mathrm{F}}(\mathrm{V}) = \mathrm{Lie}(\tilde{\mathrm{G}})$ such that

$$\mathrm{F}[\beta] = \text{product of fields} = \prod_{i \in \mathrm{I}} \mathrm{F}[\beta_i]$$

- Λ $\mathfrak{o}_{\mathrm{F}}$-lattice sequence

$$\Lambda : \mathbb{Z} \to \{\mathfrak{o}_{\mathrm{F}} - \text{lattices of } \mathrm{V}\}$$

  ("$\subseteq$"-decreasing, $\varpi_{\mathrm{F}}\Lambda(z) = \Lambda(z+e)$ for all $z \in \mathbb{Z}$ for some $e \in \mathbb{N}$ "ϖ_{F} translates Λ")
- condition 1: $\Lambda = \oplus_{i \in \mathrm{I}} \Lambda^i$, $x \in \mathrm{F}[\beta_i]^\times$ translates Λ^i (β_i in negative direction if $\beta_i \neq 0$.)
- condition 2: condition on "critical exponent"

Example

Take

$$\mathrm{V} = v_1\mathrm{F} \oplus v_2\mathrm{F} \oplus v_3\mathrm{F},\ \beta = \begin{pmatrix} & \varpi^{-1} & \\ & & \varpi^{-1} \\ 1 & & \end{pmatrix}$$

and Λ :

$$\ldots \supseteq v_1\mathfrak{p}^{-1} \oplus v_2\mathfrak{o} \oplus v_3\mathfrak{o} \supseteq v_1\mathfrak{o} \oplus v_2\mathfrak{o} \oplus v_3\mathfrak{o}$$

$$\supseteq v_1\mathfrak{o} \oplus v_2\mathfrak{o} \oplus v_3\mathfrak{p} \supseteq v_1\mathfrak{o} \oplus v_2\mathfrak{p} \oplus v_3\mathfrak{p} \supseteq \ldots$$

($\mathfrak{p} = \mathfrak{p}_\mathrm{F} = \{x \in \mathrm{F} |\ |x| < 1\}$, the valuation ideal of F)

$\mathrm{F}[\beta]$ is a field, $[\mathrm{F}[\beta] : \mathrm{F}] = 3$.

Set of semisimple characters:

- $(\beta, \Lambda) \rightsquigarrow \mathrm{K}(\beta,\Lambda) \leqslant \tilde{\mathrm{G}} = \mathrm{Aut}_{\mathrm{F}}(\mathrm{V})$ copen (compact open), pro-p
- $\mathscr{C}(\Lambda, \beta)$ set of certain characters on $\mathrm{K}(\beta,\Lambda)$.

Example: β, Λ

$$\mathrm{K}(\beta,\Lambda) = (1+\mathfrak{p}_{\mathrm{F}[\beta]})^{\times} \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$$

$$\mathscr{C}(\Lambda, \beta) = \left\{ (\theta, \mathrm{K}(\beta,\Lambda)) \,\middle|\, \theta|_{\left(\ddots\right)} \text{ is given from } \beta \text{ by Pontryagin duality} \right\}$$

Facts

Fact 1 ($\tilde{\mathrm{G}}$): (Bushnell–Kutzko 93, Dat 09, Stevens 05)

Every representation of $\tilde{\mathrm{G}}$ contains a semisimple character.

Fact 2 ($\tilde{\mathrm{G}}$): (Mackey theory)

If (θ_1, K_1), (θ_2, K_2) are contained in some irreducible representation of $\tilde{\mathrm{G}}$, then θ_1 and θ_2 intertwine:

$$\exists_{g \in \tilde{\mathrm{G}}} : \ {}^{g}\theta_1 = \theta_2 \text{ on } g\mathrm{K}_1 g^{-1} \cap \mathrm{K}_2.$$

We write $\theta_1 \overset{\tilde{\mathrm{G}}}{\sim} \theta_2$.

Fact 3 ($\tilde{\mathrm{G}}$): (Kurinczuk-S-Stevens)

Let $\theta_1, \theta_2, \theta_3$ be semisimple characters. $\theta_1 \overset{\tilde{\mathrm{G}}}{\sim} \theta_2 \overset{\tilde{\mathrm{G}}}{\sim} \theta_3 \implies \theta_1 \overset{\tilde{\mathrm{G}}}{\sim} \theta_3$.

(important for classification of irr. representations)

Simple restrictions

Simple restrictions: $\mathrm{F}[\beta] = \prod_{i\in \mathrm{I}} \mathrm{F}[\beta_i]$; $\mathrm{V} = \oplus_{i\in\mathrm{I}} \mathrm{V}^i$

$$\mathrm{M} = \{g \in \tilde{\mathrm{G}} |\ g\mathrm{V}^i \subseteq \mathrm{V}^i,\ i \in \mathrm{I}\}$$

Then for $\theta \in \mathscr{C}(\Lambda, \beta)$

$$\theta|_{\mathrm{M}\cap\mathrm{K}(\beta,\Lambda)} = \otimes_{i\in\mathrm{I}}\theta_i$$

with

$$\theta_i \in \mathscr{C}(\Lambda^i, \beta_i),$$

a simple character. ($\mathrm{F}[\beta_i]$ is a field)

For $\mathrm{G} = \mathrm{U}(h)$: Self-dual semisimple char.

h defines a duality: Take L, an $\mathfrak{o} = \mathfrak{o}_{\mathrm{F}}$-lattice in V.

$$\mathrm{L}^{\#} = \{v \in \mathrm{V} |\ h(\mathrm{L}, v) \subseteq \mathfrak{p}\}$$

$\rightsquigarrow \Lambda \mapsto \Lambda^{\#}$

$$\ldots \supseteq (\Lambda(1))^{\#} \supseteq (\Lambda(0))^{\#} \supseteq (\Lambda(-1))^{\#} \supseteq (\Lambda(-2))^{\#} \supseteq \ldots$$

(β, Λ) is called self-dual if

- $\Lambda, \Lambda^{\#}$ differ by a translation
- $-\beta = \sigma_h(\beta)$, i.e. $\beta \in \mathrm{Lie}(\mathrm{G})$

σ_h is the adjoint-anti-involution of h

Self-dual semisimple characters:

$$\mathscr{C}_-(\Lambda, \beta) = \{\theta|_{\mathrm{K}(\beta,\Lambda)\cap\mathrm{G}} |\ \theta \in \mathscr{C}(\Lambda, \beta)\}$$

Fact 1(G), Fact 2(G), Fact 3(G) hold.

Transfer between semisimple characters

- Fix β. Take Λ, Λ'. (giving semis. strata)
  1. $\exists!\ \tau_{\Lambda',\Lambda} : \mathscr{C}(\Lambda,\beta) \xrightarrow{\sim} \mathscr{C}(\Lambda',\beta)$, s.t. θ and $\tau_{\Lambda',\Lambda}(\theta)$ coincide on $\mathrm{K}\cap\mathrm{K}'$.
  2. $\mathscr{C}(\Lambda\oplus\Lambda',\beta\oplus\beta) \xrightarrow{\mathscr{C}} (\Lambda,\beta)$ via $\tilde{\theta}\mapsto\tilde{\theta}|_{\mathrm{K}}$, denoted as $\tau_{\Lambda,\Lambda\oplus\Lambda'}$.
- more general: Embeddings $\varphi : \mathrm{F}[\beta] \hookrightarrow \mathrm{End}_{\mathrm{F}}(\mathrm{V})$, s.t. $(\Lambda,\varphi(\beta))$ is a stratum. We have maps:

$$\tau_{\Lambda',\varphi',\Lambda,\varphi} : \mathscr{C}(\Lambda,\varphi(\beta)) \xrightarrow{\sim} \mathscr{C}(\Lambda',\varphi'(\beta))$$

transitivity of transfer

$\tau_{\Lambda_3,\Lambda_2}\circ\tau_{\Lambda_2,\Lambda_1} = \tau_{\Lambda_3,\Lambda_1}$.

Pss-character (Bushnell–Henniart 96, K-S-S)

Fix β.

Domain: $\mathscr{Q}(\beta) = \{(\mathrm{V},\varphi,\Lambda) |\ (\varphi(\beta),\Lambda) \text{ semisimple stratum}\}$

Range: $\mathfrak{C}(\beta) = \bigcup_{(\mathrm{V},\varphi,\Lambda)\in\mathscr{Q}(\beta)} \mathscr{C}(\Lambda,\varphi(\beta))$.

Pss-character (potentially semisimple character):

$$\Theta:\ \mathscr{Q}(\beta) \to \mathfrak{C}(\beta),$$

s.t. values are related by transfer:

$$\Theta(\mathrm{V}',\varphi',\Lambda') = \tau_{\Lambda',\Lambda}(\Theta(\mathrm{V},\varphi,\Lambda)) \in \mathscr{C}(\Lambda',\varphi'(\beta)).$$

Endo-equivalence: Given Θ supported on β and Θ' supported on β'.

$$\Theta \approx \Theta' \Leftrightarrow_{\text{Def.}} \mathrm{im}(\Theta) \cap \mathrm{im}(\Theta') \neq \emptyset.$$

GL-endo-parameter

Notation: $[\Theta]$ endo-class of Θ.

$$[\Theta] \widehat{=} \{[\Theta_i] |\ i \in \mathrm{I}\} \text{ (simple restrictions), "ps-characters".}$$

We collect simple endo-classes: $\mathscr{E} = \{[\Theta] |\ \Theta \text{ ps-char.}\}$.
We need the degree: $\deg([\Theta]) = [\mathrm{F}[\beta] : \mathrm{F}]$.
GL-**endo-parameter:**

$$\mathfrak{f} : \mathscr{E} \to \mathbb{N}_0$$

of finite support.

(K-S-S)

There is a canonical bijection from the set of **intertwining classes of semisimple characters** of $\tilde{\mathrm{G}}$ to the set of **those** GL-**endo-parameters** $\mathfrak{f}$ which satisfy

$$\dim_{\mathrm{F}} \mathrm{V} = \sum_{[\Theta] \in \mathscr{E}} \deg([\Theta]) \mathfrak{f}([\Theta])$$

We want to parametrize G-intertwining classes of self-dual semisimple characters. Fix β self-dual and $(\varepsilon, \mathrm{F}/\mathrm{F}_o)$.

Domain and range:

- $\mathscr{Q}_-(\beta) = \{((\mathrm{V}, h), \varphi, \Lambda) |\ (\mathrm{V}, \varphi, \Lambda) \in \mathscr{Q}(\beta),\ \varphi, \Lambda \text{ self-dual}\}$
- $\mathfrak{C}_-(\beta) = \bigcup_{((\mathrm{V},h),\varphi,\Lambda)} \mathscr{C}_-(\Lambda, \varphi(\beta))$

Self-dual pss-char. supported on β:

$$\Theta_- :\ \mathscr{Q}_-(\beta) \to \mathfrak{C}_-(\beta)$$

s.t.

$$\Theta_-((\mathrm{V}, h), \varphi, \Lambda) \in \mathscr{C}_-(\Lambda, \varphi(\beta))$$

and the values are related by transfer.

Self-dual endo-equivalence

Endo-equivalence: Given Θ_- supp. on β and Θ'_- supp. on β':

$$\Theta_- \approx \Theta'_- \Leftrightarrow_{\mathrm{Def}} \mathrm{im}(\Theta_-) \cap \mathrm{im}(\Theta'_-) \neq \emptyset$$

Θ_- has a lift Θ via restriction (Glauberman)

$$\mathscr{C}(\Lambda, \varphi(\beta)) \to \mathscr{C}_-(\Lambda, \varphi(\beta)).$$

K-S-S

$\Theta_- \approx \Theta'_- \Leftrightarrow \Theta \approx \Theta'$

Action on $\mathscr{E}$: $\Sigma = \{1, \sigma\}$ an abstract group. There is a map of order 2:

$$\mathscr{E} \to \mathscr{E},\ [\Theta] \mapsto \sigma([\Theta])$$

Suppose Θ_- is supp. on β with lift Θ and simple restrictions Θ_i. Then $\{[\Theta_i] |\ i \in \mathrm{I}\}$ is Σ-stable.

$$[\Theta_-] \text{ is simple (i.e. } \mathrm{F}[\beta] \text{ is a field)} \implies \sigma([\Theta]) = [\Theta].$$

Concordance: Concordance is an equivalence relation on pairs (φ, h), $\varphi : \mathrm{F}[\beta] \hookrightarrow \mathrm{End}_{\mathrm{F}}(\mathrm{V})$ simple self-dual. (for different β) Write: $(\varphi, h) \overset{\text{conc}}{\sim} (\varphi', h')$

Concordance

Typical application for concordance: Suppose θ_- and θ'_- are self-dual simple and $h = h'$ (possibly $\beta \neq \beta'$). Then

$$\theta_- \overset{\mathrm{G}}{\sim} \theta'_- \Leftrightarrow \theta \overset{\tilde{\mathrm{G}}}{\sim} \theta' \text{ and } (\varphi, h) \overset{\text{conc}}{\sim} (\varphi', h).$$

Special case $\beta = \beta'$:
$(\varphi, h) \overset{\text{conc}}{\sim} (\varphi', h) \Leftrightarrow \exists_{g \in \mathrm{G}}\ g\varphi(\beta)g^{-1} = \varphi'(\beta)$

Witt types: Take $\mathcal{O} \in \mathscr{E}/\Sigma$. $\mathrm{WT}(\mathcal{O})$ is the set of concordance data for $\mathcal{O}$. (elements are called Witt types). For $|\mathcal{O}| = 2$ it has no relevant data.

Self-dual endo-parameter

Self-dual endo-parameter: A section $\mathfrak{f}_-$ of

$$\bigsqcup_{\mathcal{O}\in\mathscr{E}/\Sigma} (\mathrm{WT}(\mathcal{O}) \times \mathbb{N}_0) \to \mathscr{E}/\Sigma$$

of finite support.

(Kurinczuk-S-Stevens)

There is a canonical bijection from the **set of G-intertwining classes of self-dual semisimple characters** for G to the set of **self-dual endo-parameters adapted to** h.