Exact structures, Hall algebras, and quantum groups

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Quivers and their representations

A **quiver** Q = oriented graph.

Q is **finite** if the set of its vertices and the set of its arrows are finite.

Fix a field k. A **representation** of Q is a diagram of vector spaces of the form given by Q.

Example

The quiver $\vec{A}_3 : 1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3$ is an orientation of the Dynkin diagram $A_3 : \bullet - \bullet - \bullet$

A representation of $\vec{A}_3$ is a diagram

$$V_1 \xleftarrow{f_\alpha} V_2 \xrightarrow{f_\beta} V_3,$$

where V_1, V_2, V_3 are $k-$vector spaces, f_α, f_β are linear maps.

Quiver representations and path algebras

Fin.-dim. representations of Q form an abelian category $\mathrm{rep}_k(Q)$.

The **path algebra** kQ is the k-algebra with basis given by the paths in Q, and with multiplication given by concatenation of paths.

Let Q be a acyclic quiver. We have an equivalence of categories

$$\mathrm{mod}\, kQ \xrightarrow{\sim} \mathrm{rep}_k(Q).$$

This category is **Krull-Schmidt**: every representation of Q can be written as a direct sum of **indecomposable** representations in a unique way (up to isomorphism and permutation of summands).

Gabriel's theorems

Theorem (Gabriel)

An (acyclic, connected) quiver has finitely many indecomposable representations if and only if it is an orientation of an ADE Dynkin diagram.
If Q is an orientation of the ADE diagram Δ, then there is a 1-to-1 correspondence between the isomorphism classes of non-trivial indecomposable objects in $\mathrm{rep}_k(Q) \xrightarrow{\sim} \mathrm{mod}\, kQ$ and the positive roots in the root system of Δ.

Theorem (Gabriel)

Let A be a finite-dimensional $k-$algebra. Then there exists a quiver Q and an ideal I in kQ such that $\mathrm{mod}\, A \xrightarrow{\sim} \mathrm{mod}(kQ/I)$.

Hall algebras

Fix $k = \mathbb{F}_q$. Let $\mathcal{C}$ be a small $k-$linear abelian category such that

$$|\operatorname{Hom}(A,B)| < \infty, \quad |\operatorname{Ext}^1(A,B)| < \infty, \qquad \forall A, B \in \mathcal{C}.$$

Definition-Theorem (Ringel)

The Hall algebra $\mathcal{H}(\mathcal{C})$ is the $\mathbb{Q}-$algebra with a basis $\{u_X \mid X \in \operatorname{Iso}(\mathcal{C})\}$ and multiplication

$$u_A * u_C = \sum_{B \in \operatorname{Iso}(\mathcal{C})} \frac{|\operatorname{Ext}^1(A,C)_B|}{|\operatorname{Hom}(A,C)|} u_B.$$

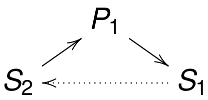
$\mathcal{H}(\mathcal{C})$ is associative and unital. It is usually not $q-$commutative.

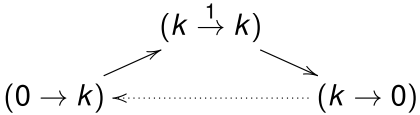
Here $\operatorname{Ext}^1(A,C)_B \subset \operatorname{Ext}^1(A,C)$ is given by short exact sequences

$$C \rightarrowtail B' \twoheadrightarrow A$$

with $B' \xrightarrow{\sim} B$.

Example: mod kA_2 = mod $k(1 \longrightarrow 2)$





$$u_{S_2} * u_{S_1} = u_{S_1 \oplus S_2};$$
$$u_{S_1} * u_{S_2} = u_{S_1 \oplus S_2} + (q-1)u_{P_1}.$$

$$u_{P_1} = \frac{1}{q-1}[u_{S_1}, u_{S_2}]. \qquad (1)$$

$$\mathfrak{g}(A_2) = \mathfrak{sl}_3; \qquad \mathfrak{n}^+(A_2) = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

Gabriel : $\alpha_1 \mapsto S_1,\ \alpha_2 \mapsto S_2,\ \alpha_1 + \alpha_2 \mapsto P_1.$

Ringel : $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right] \rightsquigarrow (1)$

Hall algebras and quantum groups

Theorem (Ringel-Green)

Let Q be a finite acyclic quiver. Then there is a Hopf algebra map

$$U_{\sqrt{q}}(\mathfrak{b}^-(Q)) \hookrightarrow \mathcal{H}_{tw}^{ex}(\operatorname{mod} kQ).$$

This is an isomorphism if and only if Q is of Dynkin type.

- $U_{\sqrt{q}}(\mathfrak{b}^-(Q))$ is the Borel part of the quantized Kac-Moody algebra associated to Q.
- $\mathcal{H}_{tw}^{ex}(\operatorname{mod} kQ)$ is $\mathcal{H}(\operatorname{mod} kQ)$ extended by $\mathbb{Q}K_0(\operatorname{mod} kQ)$, with the multiplication twisted by the square root of the Euler form (one should consider it over $\mathbb{Q}(\sqrt{q})$). It has a Hopf algebra structure.

Green and Xiao endowed the (twisted extended) Hall algebra of any **hereditary abelian** category with a Hopf algebra structure.

Exact structures

Quillen: *Exact categories*. Axiomatize extension-closed subcategories of abelian categories.

Examples

- The full subcategory of projective objects in an abelian category.
- Categories of vector bundles and of torsion sheaves on a scheme.
- Torsion and torsion free subcategories of abelian categories.

Theorem (Hubery)

Let $\mathcal{E}$ be a $\mathrm{Hom}-$ and Ext^1-finite, $k-$linear small exact category. The Hall algebra $\mathcal{H}(\mathcal{E})$ defined in the same way is associative and unital.

Exact structures II

Axiomatics suggests that an additive category may admit many different exact structures: one can choose different classes of *admissible short exact sequences* (= *conflations*).

Let $(\mathcal{A}, \mathcal{E})$ be an additive category endowed with an exact structure. Then $\mathrm{Ext}^1_{\mathcal{E}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \to \mathbf{Ab}$ is an additive bifunctor.

Upshot: $\mathcal{E}$ is uniquely determined by $\mathrm{Ext}^1_{\mathcal{E}}(-,-)$.

- Any *extension-closed* full subcategory of $(\mathcal{A}, \mathcal{E})$ has an induced exact structure (with the same $\mathrm{Ext}^1_{\mathcal{E}}(-,-)$).
- Any *closed* additive sub-bifunctor $\mathbb{F} \subset \mathrm{Ext}^1(-,-)$ defines a "smaller", or *relative*, exact structure on $\mathcal{A}$. This is equivalent to taking a sub-class of conflations (satisfying Quillen's axioms).

Hall algebras II

The Hall algebra of an exact category depends not only on the underlying additive category. It depends on the choice of exact structure!

Example

- Ringel-Green: $\mathcal{H}_{tw}(\mathrm{mod}\, kQ, \mathrm{ab}) \xleftarrow{\sim} U_{\sqrt{q}}(\mathfrak{n}^+)$.
- For any additive category $\mathcal{A}$, the Hall algebra $\mathcal{H}(\mathcal{A}, \mathrm{add})$ of the split exact structure is a polynomial algebra in $q-$commuting variables.
- $\mathcal{H}(\mathrm{mod}\, kA_2, \mathrm{add})$ is the polynomial algebra in u_{S_1}, u_{S_2}, and u_{P_1}, modulo relations:

$$u_{S_2} * u_{S_1} = u_{S_1 \oplus S_2} = u_{S_1} * u_{S_2};$$

$$u_{S_1} * u_{P_1} = u_{S_1 \oplus P_1} = \frac{1}{q} u_{P_1} * u_{S_1};$$

$$u_{S_2} * u_{P_1} = q u_{S_2 \oplus P_1} = q u_{P_1} * u_{S_2}.$$

Degree functions and filtrations

Definition

Consider a function $w : \mathrm{Iso}(\mathcal{A}) \to \mathbb{N}$. We say that w is

- *additive* if $w(M \oplus N) = w(M) + w(N)$ for all M and N;
- an $\mathcal{E}-$*quasi-valuation* if $w(X) \leq w(M \oplus N)$ whenever there exists a conflation $N \rightarrowtail X \twoheadrightarrow M$ in $\mathcal{E}$.
- an $\mathcal{E}-$*valuation* if it is an additive $\mathcal{E}-$quasi-valuation.

If $\mathcal{A}$ is Krull-Schmidt, an additive function is the same as a function on indecomposables: $\mathrm{Ind}(\mathcal{A}) \to \mathbb{N}$. Suppose $\mathcal{A}$ is $\mathrm{Hom} -$ finite.

Example

- $w_X := \dim \mathrm{Hom}(X, -)$ is a valuation for any exact structure on $\mathcal{A}$.. If X is $\mathcal{E}-$projective, it is additive on conflations in $\mathcal{E}$.
- $\dim \mathrm{End}(-)$ is a quasi-valuation for any exact structure on $\mathcal{A}$. But it is usually not additive.

Main Theorems

Let $\mathcal{A}$ be a Hom $-$finite $k-$linear idempotent complete additive category. Let $\mathcal{E}$ be an Ext^1 $-$finite exact structure on $\mathcal{A}$.

Theorem I (F.-G.)

Each $\mathcal{E}-$valuation $w : \mathrm{Iso}(\mathcal{A}) \to \mathbb{N}$ induces a filtration $\mathcal{F}_w$ on $\mathcal{H}(\mathcal{E})$. The associated graded is $\mathcal{H}(\mathcal{E}')$ for a smaller exact structure $\mathcal{E}' \leq \mathcal{E}$ on $\mathcal{A}$.

$\mathcal{A}$ is *locally finite* if $\forall X \in \mathcal{A}$, there exists only finitely many $Y, Z \in \mathrm{Ind}(\mathcal{A})$ s.t. $\quad \mathrm{Hom}(X, Y) \neq 0, \quad \mathrm{Hom}(Z, X) \neq 0.$

Theorem II (F.-G.)

Suppose $\mathcal{A}$ is locally finite. Then for each exact substructure $\mathcal{E}' < \mathcal{E}$, there exists an $\mathcal{E}-$valuation w such that

$$\mathbf{gr}_{\mathcal{F}_w}(\mathcal{H}(\mathcal{E})) = \mathcal{H}(\mathcal{E}').$$

As w, one can take a (formal) sum of $\dim(\mathrm{Hom}(X, -))$.

Lattice of exact structures I

Exact structures on an additive category form a poset.

Theorem (Brüstle-Hassoun-Langford-Roy)

This is a bounded complete lattice.

For any conflation $\delta : A \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} C$ in $\mathcal{E}$, one has an exact sequence of right $\mathcal{A}-$modules $\mathcal{A}^{op} \to \mathbf{Ab}$

$$0 \to \mathrm{Hom}(-, A) \overset{\mathrm{Hom}(-,f)}{\longrightarrow} \mathrm{Hom}(-, B) \overset{\mathrm{Hom}(-,g)}{\longrightarrow} \mathrm{Hom}(-, C).$$

The *contravariant defect* of δ is $\mathsf{Coker}(\mathrm{Hom}(-, g))$.
The category $\mathbf{def}\,\mathcal{E}$ of contravariant defects of conflations in $\mathcal{E}$ is an abelian category. Its simple objects are the defects of *Auslander-Reiten* (= *almost split*) conflations.
If $\mathcal{A}$ is Krull-Schmidt and locally finite, each object in $\mathbf{def}\,\mathcal{E}$ (for each $\mathcal{E}$!) has finite length.

Lattice of exact structures II

Theorem (..., Buan, Rump, Enomoto, F.-G.)

Each additive category $\mathcal{A}$ admits a unique maximal exact structure $(\mathcal{A}, \mathcal{E}^{\text{max}})$. There is a lattice isomorphism between

- *The lattice of exact structures on $\mathcal{A}$;*
- *The lattice of Serre subcategories of the category* $\mathbf{def}(\mathcal{A}, \mathcal{E}^{\text{max}})$.

If $\mathcal{A}$ is locally finite, these lattices are Boolean: they are isomorphic to the power set of $\text{AR}-$*conflations of* $\mathcal{E}^{\text{max}}$.

Sketches of the proofs

Proof of Theorem I

Each $\mathcal{E}-$valuation w induces a function $\widetilde{w} : \mathrm{Iso}(\mathbf{def}\,\mathcal{E}) \to \mathbb{N}$. This function is additive on short exact sequences.
Then $\mathrm{Ker}(\widetilde{w})$ is a Serre subcategory of $\mathbf{def}\,\mathcal{E}$. So it defines an exact substructure $\mathcal{E}' \leq \mathcal{E}$. Then $\mathbf{gr}_{\mathcal{F}_w}(\mathcal{H}(\mathcal{E})) = \mathcal{H}(\mathcal{E}')$.

Proof of Theorem II

Let $\mathrm{Ex}_+(\mathcal{E})$ be a sub-semigroup of $K_0^{\mathrm{add}}(\mathcal{A})$ generated by alternating sums $[X] - [Y] + [Z]$ for all conflations $X \rightarrowtail Y \twoheadrightarrow Z$.
Let $\mathrm{AR}_+(\mathcal{E})$ be its sub-semigroup generated by alternating sums for all AR $-$conflations.
If $\mathcal{A}$ is locally finite, then $\mathrm{Ex}_+(\mathcal{E}) = \mathrm{AR}_+(\mathcal{E})$ for each exact structure $\mathcal{E}$ on $\mathcal{A}$. Using this, we can prove that $\mathbf{gr}_{\mathcal{F}_w}(\mathcal{H}(\mathcal{E})) = \mathcal{H}(\mathcal{E}')$, for

$$w := \sum_{X \in \mathrm{Ind}(\mathrm{proj}(\mathcal{E}')) \setminus \mathrm{Ind}(\mathrm{proj}(\mathcal{E}))} \dim \mathrm{Hom}(X, -).$$

Cones

Assume that $\mathcal{A}$ has finitely many indecomposables. Consider $\Lambda^{\mathcal{E},\mathcal{E}'} := \mathrm{Ker}\,(K_0(\mathcal{E}') \twoheadrightarrow K_0(\mathcal{E}))$. Let

$$\mathcal{C}^{\mathcal{E},\mathcal{E}'} \subseteq \Lambda^{\mathcal{E},\mathcal{E}'} \otimes_{\mathbb{Z}} \mathbb{R}$$

be the polyhedral cone generated by $[X] - [Y] + [Z]$, for all conflations $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E} \setminus \mathcal{E}'$.

Proposition

$\mathcal{C}^{\mathcal{E},\mathcal{E}'}$ is simplicial. Its extremal rays are given by AR-conflations in $\mathcal{E} \setminus \mathcal{E}'$. Its face lattice is isomorphic to the interval $[\mathcal{E}', \mathcal{E}]$.

Cones II

For a pair of exact structures $\mathcal{E}' < \mathcal{E}$, we define the (Hall algebra) *degree cone*:

$$\mathcal{D}^{\mathcal{E},\mathcal{E}'} := \{\mathbf{d} \in \mathbb{R}^{\mathrm{Ind}(\mathcal{A})} \mid \mathbf{d} \text{ induces an algebra filtration}, \mathrm{gr}_{\mathbf{d}}(\mathcal{H}(\mathcal{E})) = \mathcal{H}(\mathcal{E}')\}.$$

From Theorems I and II, we have:

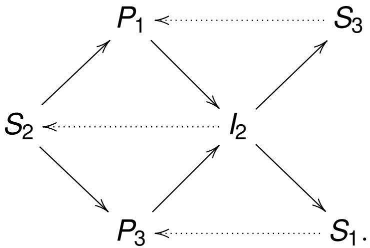
$$\mathcal{D}^{\mathcal{E},\mathcal{E}'} = \{\varphi \in (K_0^{\mathrm{add}}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R})^* \mid \text{for any } x \in \mathcal{C}^{\mathcal{E},\mathcal{E}'}, \varphi(x) > 0; \text{ for any } y \in \mathcal{C}^{\mathcal{E}'},\ \varphi(y) = 0\}.$$

Up to linearity subspace, the cones $\mathcal{C}^{\mathcal{E},\mathcal{E}'}$ and $\mathcal{D}^{\mathcal{E},\mathcal{E}'}$ are polar dual to each other.

Example: $A_3 \leftrightarrow \mathfrak{sl}_4$

$Q = 1 \longrightarrow 2 \longleftarrow 3.$

Auslander-Reiten quiver:



AR-conflations:

$$(1)\ S_2 \rightarrowtail P_1 \oplus P_3 \twoheadrightarrow I_2$$

$$(2)\ P_1 \rightarrowtail I_2 \twoheadrightarrow S_3$$

$$(3)\ P_3 \rightarrowtail I_2 \twoheadrightarrow S_1$$

We have $2^3 = 8$ different exact structures.

We identify $K_0^{\mathrm{add}}(\mathcal{A})$ with $\mathbb{Z}^6$. Let $x_1, \cdots, x_6$ be coordinates of $\mathbb{Z}^6$. Then

$$K_0(\mathcal{E}_1) = \mathbb{Z}^6/(x_1 - x_2 - x_3 + x_4)$$

and

$$K_0(\mathcal{E}_{12}) = \mathbb{Z}^6/(x_1 - x_2 - x_3 + x_4, x_2 - x_4 + x_5).$$

The kernel of $K_0(\mathcal{E}_1) \twoheadrightarrow K_0(\mathcal{E}_{12})$:

$$\mathbb{Z}(x_2 - x_4 + x_5).$$

$\mathcal{C}^{\mathcal{E}_{12},\mathcal{E}_1} = \mathbb{R}_{\geq 0}(x_2 - x_4 + x_5)$.

The dual cone $\mathcal{D}^{\mathcal{E}_{12},\mathcal{E}_1}$: $\mathbf{d} = (d_1, \cdots, d_6) \in \mathbb{R}^6$ s.t. $d_2 + d_5 > d_4$ and $d_1 + d_4 = d_2 + d_3$.

Its closure is a 5-dimensional cone in $\mathbb{R}^6$ having a 4-dimensional linearity space.

We denote $\diamond_{12}$ and $\diamond_1$ the multiplications in the Hall algebra $\mathcal{H}(Q, \mathcal{E}_{12})$ and $\mathcal{H}(Q, \mathcal{E}_1)$ respectively.
The only products of generators that are not given simply by the multiples of classes of their direct sums are the following:

$$[I_2] \diamond_{12} [S_2] = [I_2 \oplus S_2] + (q-1)[P_1 \oplus P_3],$$
$$[S_3] \diamond_{12} [P_1] = [S_3 \oplus P_1] + (q-1)[I_2].$$

$$[I_2] \diamond_1 [S_2] = [I_2 \oplus S_2] + (q-1)[P_1 \oplus P_3],$$
$$[S_3] \diamond_1 [P_1] = [S_3 \oplus P_1].$$

$$[I_2] \diamond_{12} [S_2] - q[S_2] \diamond_{12} [I_2] = (q-1)[P_1] \diamond_{12} [P_3],$$

$$[S_3] \diamond_{12} [P_1] - [P_1] \diamond_{12} [S_3] = (q-1)[I_2].$$

$$[I_2] \diamond_1 [S_2] - q[S_2] \diamond_1 [I_2] = (q-1)[P_1] \diamond_1 [P_3],$$

but the second relation transforms to

$$[S_3] \diamond_1 [P_1] - [P_1] \diamond_1 [S_3] = 0.$$

Each point $\mathbf{d} \in \mathcal{D}^{\mathcal{E}_{12},\mathcal{E}_1}$ gives rise to a filtration on $\mathcal{H}(Q,\mathcal{E}_{12})$ whose associated graded algebra is exactly $\mathcal{H}(Q,\mathcal{E}_1)$.

Comultiplication, quantum groups and Hall algebras

$\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ)$ is the category of 2-periodic complexes:

$$M^0 \underset{d^1}{\overset{d^0}{\rightleftarrows}} M^1 \,, \quad d^1 \circ d^0 = d^0 \circ d^1 = 0.$$

Theorem (Ringel-Green,...,Bridgeland, G., Lu-Peng,...)

Let Q be a finite acyclic quiver. Then

$$U_{\sqrt{q}}(\mathfrak{g}(Q)) \hookrightarrow \Big(\big(\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), ab)/\mathcal{I} \big) [S^{-1}] \Big)_{red} \,.$$

This is an isomorphism if and only if Q is of Dynkin type.

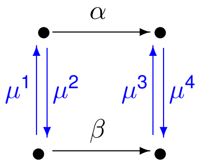
This is only an algebra map!

$$\text{gldim}(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), \text{ab}) = \infty.$$

So Green's comultiplication is not compatible with the multiplication.
Can we recover the comultiplication?

$\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kA_2)$

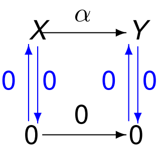
$\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ) \xrightarrow{\sim} \text{mod}\, k\widetilde{Q}/I$, for certain $\widetilde{Q}$, I.
$Q = A_2$. Then $\widetilde{Q}$:

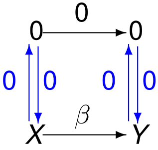


I is generated by:

$$\mu^1\mu^2, \quad \mu^2\mu^1, \quad \mu^3\mu^4, \quad \mu^4\mu^3,$$
$$\beta\mu^2 - \mu^4\alpha, \quad \alpha\mu^1 - \mu^3\beta.$$

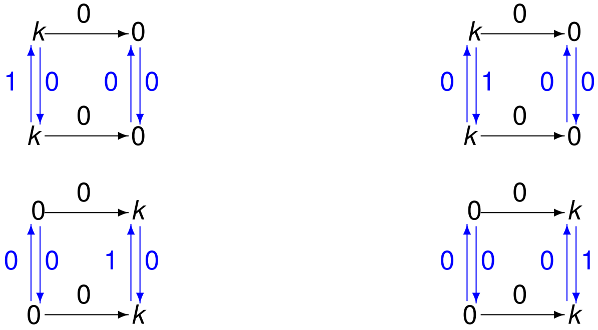
Nilpotent parts:





2 copies of $U_{\sqrt{q}}(\mathfrak{h})$: Group algebra of the Grothendieck group of acyclic complexes.

Generated by:



Define an exact structure $\mathcal{E}_{CE}$ on $\mathcal{C}_{\mathbb{Z}/2}(\operatorname{mod} kQ)$ as follows:

$$A^\bullet \to B^\bullet \to C^\bullet$$

is a conflation if

$$A^i \to B^i \to C^i \quad \text{and} \quad H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet)$$

are short exact for $i = 0, 1$.

$(\mathcal{C}_{\mathbb{Z}/2}(\operatorname{mod} kQ), \mathcal{E}_{CE})$ is **hereditary.** But Green's theorem used the abelian exact structure, so it doesn't apply. Still...

Theorem

$$U_{\sqrt{q}}(\mathfrak{g}(Q)) \hookrightarrow \left(\left(\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), \mathcal{E}_{CE})/\mathcal{I} \right) [S^{-1}] \right)_{red}$$

is a coalgebra homomorphism.

- The RHS is a twisted extended Hall algebra of $(\mathrm{gr}_{\mathbb{Z}/2}(\text{mod}\, kQ), \text{ab})$. This category is **hereditary and abelian**!
- This induces a comultiplication on the RHS compatible with the multiplication. It coincides with Green's comultiplication w.r.t. $\mathcal{E}$.
- The RHS is an **algebra degeneration** of $\left(\left(\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), \text{ab})/\mathcal{I} \right) [S^{-1}] \right)_{red}$.

The comultiplication above is compatible with the multiplication of $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), \text{ab})$.

Generalized quantum doubles

$U_{\sqrt{q}}(\mathfrak{g}(Q))$ is the (reduced) **Drinfeld double** of $U_{\sqrt{q}}(\mathfrak{g}(Q))$.

With A, B Hopf algebras and $\varphi : A \times B \to k$ a **Hopf pairing**, one can associate a Hopf algebra called the **generalized quantum double** $D_\varphi(A, B)$. As a coalgebra, it is just $A \otimes B$.

- The Drinfeld double: φ is non-degenerate.
- Tensor product: φ is as degenerate as possible.

Conjecture

All generalized quantum doubles of $U_{\sqrt{q}}(\mathfrak{b}^-(Q))$ are realized by Hall algebras of some of exact structures in $[\mathcal{E}_{CE}, ab]$.

Further directions

- Prove Theorem II in general case without using Auslander-Reiten theory. We have a conjectural approach, but it's too early to say anything.
- Degenerations of derived Hall algebras of triangulated categories (defined by Toën and Xiao-Xu). Involves **extriangulated structures** defined by Nakaoka-Palu.
- (w. X. Fang, Y. Palu, P.-G. Plamondon, M. Pressland) Use some relative structures on **cluster categories** to study degenerations of **quantum cluster algebras**.
- Cohomological HA, K-theoretic HA,... The PBW theorem is known for them, but it is proved differently.