Brauer algebras of complex reflection groups

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Schur-Weyl duality

Let V be a finite dimensional vector space, $\mathcal{S}_k$ the symmetric group on k elements and $GL(V)$ the group of invertible transformations of V. We have two actions on the tensor product $V^{\otimes k}$:

$$\mathcal{S}_k \curvearrowright V \otimes \cdots \otimes V \curvearrowleft GL(V)$$

$$\sigma \in \mathcal{S}_k, \sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

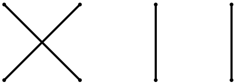
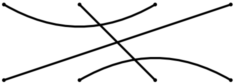
$$g \in GL(V), g(v_1 \otimes \cdots \otimes v_k) = g(v_1) \otimes \cdots \otimes g(v_k)$$

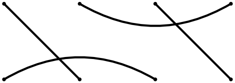
Theorem (Schur-Weyl duality)

The subalgebras of $End(V^{\otimes k})$ generated by the two above actions are mutual centralizers.

The classical case - definition and properties

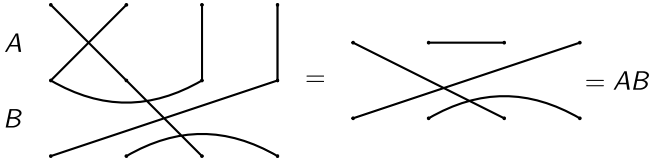
Let n be a positive integer and δ an indeterminate. The Brauer algebra $Br_n(\delta)$ (Brauer, 1937) is the $\mathbb{Z}[\delta]$-algebra with a basis consisting of the graphs of two (ordered) rows of n points and n edges between these points, no two of them meeting. Examples (for $n = 4$):

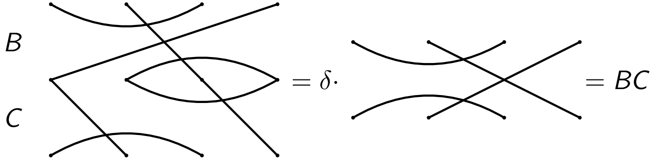
$A =$  , $B =$ 

$C =$ 

The classical case - definition and properties

Multiplication of two graphs is given by 'concatenation'. Any cycle that occurs gives a factor of δ:

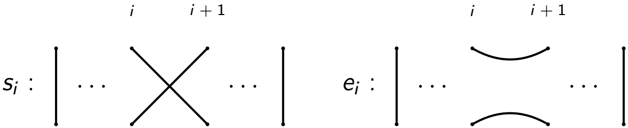




The classical case - definition and properties

It can also be defined as the $\mathbb{Z}[\delta]$-algebra with generators $s_i, e_i, i = 1, \ldots, n-1$ subject to the relations:

$s_i^2 = 1$	$e_i e_{i\pm1} e_i = e_i$
$s_i s_j = s_j s_i$, for $\lvert i-j\rvert > 1$	$s_i s_{i\pm1} e_i = e_{i\pm1} e_i$
$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$	$e_i s_{i\pm1} s_i = e_i e_{i\pm1}$
$e_i^2 = \delta e_i$	$s_i e_i = e_i s_i = e_i$
$e_i e_j = e_j e_i$, for $\lvert i-j\rvert > 1$	$e_i s_{i\pm1} e_i = e_i$
$s_i e_j = e_j s_i$, for $\lvert i-j\rvert > 1$	



The classical case - Schur-Weyl duality

The algebra $Br_n(\delta)$ is a natural extension of the group algebra of the symmetric group, which is generated by the elements s_i, or equivalently, the graphs with no horizontal lines. For integer d, $Br_n(d)$ acts on $V^{\otimes n}$ where V is a d-dimensional vector space, the action of an element e_i defined by:

$$e_i(v_1 \otimes \dots v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \dots v_n)$$

$$= v_1 \otimes \dots v_{i-1} \otimes (\sum_{k=1}^{d} \langle v_i, v_{i+1} \rangle u_k \otimes u_k) \otimes \cdots \otimes v_n,$$

where $u_1, \dots, u_d$ is an orthonormal basis for V.

Theorem (Schur-Weyl duality - Brauer, 1937)

The image of $Br_n(d)$ inside $End(V^{\otimes n})$ is the centralizer of the subalgebra generated by the action of $O(V)$, the group of orthogonal transformations of V.

The classical case - definition and properties

The representation theory of $Br_n(d)$ was clarified by Wenzl in 1988:

Theorem (Wenzl, 1988)

The algebra $Br_n(d)$ is semisimple if and only if d is not an integer or if $d > n$. As a consequence, for generic δ, $Br_n(\delta)$ is semisimple.

Remarks: In contrast to the following,

- this result describes the case for specific values of δ as well. He also, gave an inductive description of the representations of the algebra.
- it is in a large part based on the existence and nodegeneracy of certain traces (conditional expectations) on these algebras.

Complex reflection groups

Definition

Let V be a complex vector space. A finite subgroup $W \subset GL(V)$ is called a complex reflection group if it is generated by reflections, i.e. transformations that fix a 1-codimensional subspace (reflecting hyperplane).

- Symmetric group as all finite real reflection groups or, otherwise known, Coxeter groups are complex reflections groups. Coxeter groups can be classified by means of dynkin diagrams.
- Irreducible complex reflection groups were classified by Shephard and Todd in 1954. They can be distinguished into:
    1. the imprimitive groups which form an infinite series with three parameters $G(m, p, n)$,
    2. the 34 exceptional primitive complex reflection groups.

Generalizations to other reflection groups

There have been many definitions of Brauer-type algebras associated to real and complex reflection groups other that the symmetric group, such as:

- the cyclotomic Brauer algebra by Häring-Oldenburg (2001) associated to groups of type $G(m, 1, n)$ and,
- its generalization to type $G(m, p, n)$ by Bowman (2013).
- The Brauer algebra for diagrams of simply laced type by Cohen-Gijsbers-Wales (2005),
- the Brauer algebra defined by Chen (2011) associated to every complex reflection group.

The first two algebras are defined diagrammaticaly in a way similar to the original case. We will be concerned with the last two cases.

The Brauer-Chen algebra - definition

Let W be a complex reflection group with set of reflections R. The Brauer-Chen algebra over a field k is defined by generators $w \in W$ and e_H, indexed by the hyperplanes of the reflections of W subject to the relations:

- $e_H^2 = \delta e_H$
- $we_H = e_{wH} w$
- $we_H = e_H w = e_H$, if $H \subset ker(w-1)$
- $e_{H_1} e_{H_2} = e_{H_2} e_{H_1}$, if H_1, H_2 are transverse
- $e_{H_1} e_{H_2} = \sum_{s \in R, sH_2 = H_1} \mu_s s e_{H_2}$, if H_1, H_2 are not transverse,

where δ is an indeterminate and $\mu_s \in k, s \in R$ is a set of invariants. The hyperplanes of two reflections of W are called transverse if they are the only ones that contain their intersection.

The Brauer-Chen algebra - results

As Chen originally proved, among other properties the Brauer-Chen algebra:

- is finite dimensional,
- is isomorphic to the algebra of Cohen-Gijsbers-Wales for Coxeter groups of simply laced type,
- contains the cyclotomic Brauer algebra as a direct component, in the case of groups of type $G(m, 1, n)$,
- is semisimple in the case of dihedral groups, (a basis is provided as well).

The Brauer-Chen algebra - results

In 2007, Cohen-Frenk-Wales completely determined the representations of the Brauer algebra of simply laced type.

Theorem (Cohen-Frenk-Wales, 2007)

Let W be a coxeter group of simply laced type. Then:

- *The algebra $Br(W)$ is semisimple.*
- *There is a bijection*

$$Irr(Br(W)) \leftrightarrow (\mathcal{B}, \rho),$$

$\mathcal{B}$: *orbits of certain 'admissible' collections of orthogonal roots*
ρ: *irreducible representations of $W(C_{\mathcal{B}})$, a subgroup of W associated to $\mathcal{B}$.*

- *The dimension of $Br(W)$ is:*

$$\sum_{\mathcal{B}} |\mathcal{B}|^2 |W(C_{\mathcal{B}})|.$$

The Brauer-Chen algebra - results

In 2019, Marin defined a natural truncation of $Br(W)$ and determined the representations of the first quotient algebra $Br_1(W)$.

Theorem (Marin, 2019)

- *The algebra $Br_1(W)$ is semisimple.*
- *There is a bijection*

$$Irr(Br_1(W)) \leftrightarrow (\mathcal{H}, \theta),$$

$\mathcal{H}$: orbits of hyperplanes of W
θ: irreducible representations of $N_W(W_H)/W_H$ where W_H is the reflection subgroup associated to some $H \in \mathcal{H}$.

- *Restricted to W, the representation corresponding to the pair $(\mathcal{H}, \theta)$ is the $Ind_{N(W_H)}^{W}\theta$.*
- *The dimension of $Br_1(W)$ is:*

$$|W| + \sum_{\mathcal{H}} |\mathcal{H}| \cdot |W|/|W_H|.$$

The Brauer-Chen algebra - results

Two important aspects of this result:

- it covers the cases of groups with no pair of transverse hyperplanes,
- it generalizes the contruction of Cohen-Frenk-Wales for collections of roots of cardinality 1 to all complex reflection groups.

The Brauer-Chen algebra - results

A generalization of the two previous results is the following:

Theorem (A.)

- *The algebra $Br(W)$ is semisimple.*
- *There is a bijection*

$$Irr(Br(W)) \leftrightarrow (\mathcal{B}, \rho)$$

  $\mathcal{B}$: orbit of collections of transverse hyperplanes of W
  ρ: 'admissible' representations of $Stab(B)$ for some $B \in \mathcal{B}$

- *Restricted to W, the representation corresponding to $(\mathcal{B}, \rho)$ is the induced representation $Ind^{W}_{Stab(B)}\rho$.*

Remarks:

- Orthogonal roots $\rightarrow$ transverse hyperplanes.
- Admissible collections $\rightarrow$ admissible representations.

The Brauer-Chen algebra - results

Special cases

Once the previous result has been established, we can explicitly treat every case manually or computanionally. In particular:

- For complex reflection groups of the infinite series it is easily done manually.
- For the 34 exceptional reflection groups it can be done by computer. It comes down to solving some linear systems.

Possible future goals on the subject

To complete the picture one may want:

- to show that the necessary relations for a representation to be admissible can always be 'contained' in a subgroup;
- to identify this subgroup in general.

The form of the linear systems that occur in the admissibility of the representations satisfy certain properties, which seem to be very limiting. One way to answer these questions would be to give a general way of solving such systems.

Thank you!