Equivariant cobordism of horospherical varieties

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Notations

Let k be a field of characteristic zero.

1. We will consider the category of smooth quasi-projective k-schemes and denote it by $\mathbf{Sm}_k$.
2. G will denote a linear algebraic connected reductive group of dimension g over k.
3. Furthermore, all representations of G will be finite-dimensional.
4. Let $B \subseteq G$ be a Borel subgroup, $T \subseteq B$ a maximal torus, W the Weyl group of (G, T) and U the unipotent radical of B.

Cobordism cycles

Definition

Let $X \in \mathbf{Sm}_k$. A **cobordism cycle** over X is a family $(f : Y \to X, L_1, ..., L_r)$ consisting of

1. a projective morphism $f : Y \to X$ where $Y \in \mathbf{Sm}_k$ is integral and
2. a finite sequence $(L_1, ..., L_r)$ of r line bundles over Y (this might be empty).

The dimension of this cobordism cycle is $\dim_k(Y) - r \in \mathbb{Z}$.

Cobordism cycles

1. Let $\mathcal{Z}_*(X)$ be the free graded abelian group generated by the isomorphism classes of cobordism cycles over X.
2. Smooth pullbacks and projective push-forwards exist and have the same properties as in CH_*.
3. $\Omega_*(X)$ is given by imposing three axioms on $\mathcal{Z}_*(X)$.

Main differences between Ω_* and CH_*

Classes

1. In CH_* we allow any closed subscheme to define a class.
2. In Ω_* we allow only **smooth** schemes to define a class.

Formal group laws for Chern classes

1. **Additive formal group law:** $F_{\mathrm{CH}}(u, v) = u + v$
2. **Multiplicative formal group law:** $F_K(u, v) = u + v - buv$
3. **Universal formal group law:** $F_{\mathbb{L}} = \sum_{i,j} a_{i,j} u^i v^j \in \mathbb{L}[[u, v]]$

Properties of algebraic cobordism

Proposition (Levine-Morel)

The functor $X \mapsto \Omega^*(X)$ is the universal oriented cohomology theory on $\mathbf{Sm}_k$. Thus, given an oriented cohomology theory A^* on $\mathbf{Sm}_k$, there is a unique morphism $\theta : \Omega^* \to A^*$ of oriented cohomlogy theories.

Proposition (Levine-Morel)

The canonical homomorphism $\Psi : \mathbb{L}^* \to \Omega^*(k)$ is an isomorphism.

Proposition (Levine-Morel)

The canonical morphism $\Omega^* \to \mathrm{CH}^*$ induces an isomorphism $\Omega^* \otimes_{\mathbb{L}^*} \mathbb{Z} \xrightarrow{\cong} \mathrm{CH}^*$.

Equivariant cobordism

Proposition (Krishna)

Let $\{(V_j, U_j)\}_{j\geq 0}$ be a "nice enough" sequence of l_j-dimensional good pairs. Then for any scheme $X \in G - \mathbf{Sch}_k$ of dimension d and any $i \in \mathbb{Z}$, one has

$$\Omega_i^G(X) \xrightarrow{\cong} \varprojlim_j \Omega_{i+l_j-g}\left(X \times^G U_j\right).$$

Moreover, such a sequence of good pairs always exists.

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Horospherical varieties

Definition

Let X be a normal G-variety.

1. We call a closed subgroup $H \subseteq G$ containing U **horospherical**. In this case, the homogeneous space G/H is said to be **horospherical**.
2. We call X **horospherical** if it contains an open orbit isomorphic to a horospherical homogeneous space.

Remark

G/H horospherical can be described as torus bundle over flag variety G/P with fiber P/H for $P = N_G(H)$. (Pasquier)

Examples

1. Normal toric varieties are horospherical.
2. Flag varieties are horospherical.

Example $\mathrm{IG}(2,5)$

Let ω be an antisymmetric form of maximal rank on a complex vector space V of dimension 5. We denote by $\mathrm{IG}(2,5)$ the Grassmannian of vector subspaces of V which are isotropic for ω, i.e.

$$\mathrm{IG}(2,5) := \{\Sigma \in V \mid \dim \Sigma = 2,\ \omega|_{\Sigma} = 0\}.$$

In fact, $\mathrm{IG}(2,5)$ is Sp_4-horospherical.

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3 important known results

Proposition 1 (Krishna)

Let T be of rank n. Then there is a graded $\mathbb{L}$-algebra isomorphism

$$\mathbb{L}[[t_1, ..., t_n]]_{\mathrm{gr}} \xrightarrow{\cong} \Omega^*_T(k).$$

3 important known results

Proposition 2 (Krishna)

Let $X \in T - \mathbf{Sm}_k$ be projective. Further, let X^T consist of finitely many fixed points $x_1, ..., x_s$ and let $i : X^T \hookrightarrow X$ denote the inclusion. Then $i^* : \Omega^*_T(X)_{\mathbb{Q}} \to \Omega^*_T(X^T)_{\mathbb{Q}}$ is injective and its image is the intersection of the images of

$$i^*_{T'} : \Omega^*_T(X^{T'})_{\mathbb{Q}} \to \Omega^*_T(X^T)_{\mathbb{Q}}$$

where T' runs over all subtori of codimension one in T.

3 important known results

Proposition 3 (Krishna)

Let $X \in T - \mathbf{Sm}_k$ be projective with finitely many fixed points $x_1, ..., x_s$ and finitely many invariant curves. Then the image of

$$i^* : \Omega_T^*(X)_{\mathbb{Q}} \to \Omega_T^*(X^T)_{\mathbb{Q}}$$

is the set of $(f_1, ..., f_s) \in \Omega_T^*(k)_{\mathbb{Q}}^s$ s.t. $f_i \equiv f_j \mod c_1^T(L_\chi)$ when x_i and x_j are connected by an invariant irreducible curve where T acts through the weight χ.

Equivariant cobordism of spherical varieties

Goal

We want to describe to image of the pullback map

$$i^* : \Omega^*_T(X)_{\mathbb{Q}} \to \Omega^*_T(X^T)_{\mathbb{Q}}$$

for any projective, smooth and (horo-)spherical G-variety X using the previous localisation theorems also for **infinitely many** T-invariant curves.

Fact

Any (horo-)spherical G-variety has finitely many T-fixed points.

Main Theorem

For any projective, smooth and (horo-)spherical G-variety X, the pullback map $i^* : \Omega^*_T(X)_{\mathbb{Q}} \to \Omega^*_T(X^T)_{\mathbb{Q}}$ is injective and its image consists of all families $(f_x)_{x \in X^T}$ s.t.

1. $f_x \equiv f_y \mod c_1^T(L_\chi)$ whenever x and y are connected by a T-invariant curve with weight χ.
2. $(f_x - f_y) + \rho_{1/2} c_1^T(L_\alpha)(f_z - f_x) \equiv 0 \mod c_1^T(L_\alpha)^2$ whenever α is a positive root and x, y and z lie in a connected component of $X^{\mathrm{Ker}(\alpha)^0}$ isomorphic to a projective plane $\mathbb{P}^2$.
3. $f_x - f_y + f_z - f_w \equiv 0 \mod c_1^T(L_\alpha)^2$ whenever α is a positive root and x, y, z and w lie in a connected component of $X^{\mathrm{Ker}(\alpha)^0}$ isomorphic to $\mathbb{F}_0$.
4. $(f_y - f_x)\rho_{n/2} c_1^T(L_\alpha) + \rho_{-n/2} c_1^T(L_\alpha)(f_z - f_w) \equiv 0 \mod c_1^T(L_\alpha)^2$ whenever α is a positive root and x, y, z and w lie in a connected component of $X^{\mathrm{Ker}(\alpha)^0}$ isomorphic to $\mathbb{F}_n$ for $n \geq 1$.

Questions?

Thank you for your attention! Questions?