CNRS / Université de Cergy-Pontoise
Laboratoire Analyse, Géométrie, Modélisation

Volumes of moduli spaces of flat surfaces

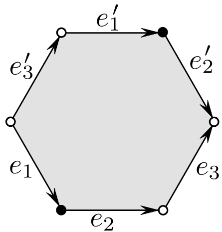
Adrien Sauvaget

*"I call our world Flatland, not because we call it so,
but to make its nature clearer to you, my happy readers,
who are privileged to live in Space."*

Edwin A. Abbott, *Flatland, a romance of many dimensions.*

Séminaire d'algèbre et géométrie du LMV, Versailles, le 21/04/2020

Flat surfaces

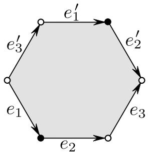


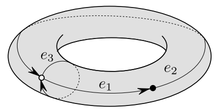
Definition (Flat surface with conical singularities)

Simply connected polygon in the euclidean plane with edges $(e_1, \ldots, e_k, e_1', \ldots, e_k')$ s.t.

$$\mathrm{length}(e_i) = \mathrm{length}(e_i').$$

Flat surfaces

<table>
<tr><td>

P, flat surface</td><td>

$C = P/\sim$, compact surface</td></tr>
<tr><td>• Vertices</td><td>markings: $x_1, \ldots, x_n$</td></tr>
<tr><td>• Edges</td><td>base of $H_1(C, \{x_1, \ldots, x_n\}, \mathbb{Z})$</td></tr>
<tr><td>• Euclidean metric</td><td>flat metric on $C \setminus \{x_1, \ldots, x_n\}$</td></tr>
<tr><td>• $2\pi\alpha_i = \sum_{v \mapsto x_i} \mathrm{angle}(v)$</td><td>conical singularities</td></tr>
<tr><td>• plane $\simeq \mathbb{C}$</td><td>Structure of Riemann surface</td></tr>
</table>

Gauss-Bonnet: $\sum_{i=1}^{n} \alpha_i = 2g(C) - 2 + n$

Flat surfaces with finite holonomy

Definition (translation surface)

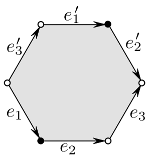
Flat surface s.t. transformations are *translation*

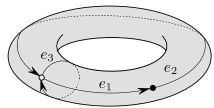
Flat surfaces with finite holonomy

Definition ($1/k$-translation surface)

Flat surface s.t. transformations are *translation* $+$ *rotation in* $\mathbb{U}_k$ *(kth root of unity)*

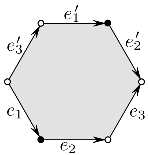
Flat surfaces with finite holonomy

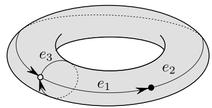




P, (1/k)-translation surface	$C = P/\sim$, compact surface
• plane $\simeq \mathbb{C}$	Structure of Riemann surface
• dz^k	$\eta \in H^0(C, \omega_{\log}^{\otimes k})$
• $2\pi\alpha_i = \sum_{v \mapsto x_i} \mathrm{angle}(v)$	$\mathrm{ord}_{x_i}(\eta) = k(\alpha_i - 1)$

Flat surfaces: an example



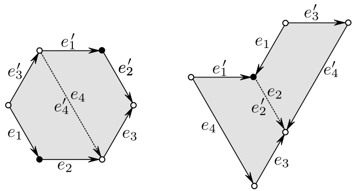


- $g = 1, n = 2 \rightsquigarrow 2g - 2 + n = 2$
- conical singularities $(8\pi/3, 4\pi/3)$
- 1/6-translation surfaces: $\omega^{\otimes 6} \simeq +4x_1 - 4x_2$.

Flat surfaces: equivalence

There are two types of transformation of flat surfaces:

1. Isometries of the plane: translation, rotation, scaling.



2. Cut-and-paste

$$P \sim P' \quad \Leftrightarrow \quad P \text{ and } P' \text{ related by a sequence of transformations}$$

$$\Leftrightarrow \quad \exists \varphi : C \overset{\sim}{\to} C', \mathrm{s.t.} \varphi(x_i) = x_i',$$

$$\text{and } \varphi \text{ is } \begin{cases} \bullet\text{isometry}/\mathbb{R}^* \\ \text{or} \\ \bullet\text{biholomorphism} + \text{same angles} \end{cases}$$

Flat surfaces: moduli spaces

Definition

Let $\alpha \in \mathbb{R}^n_{>0}$, s.t. $|\alpha| = 2g - 2 + n$. Then $\mathcal{M}_{g,n}(\alpha)$= *moduli space of flat surfaces of type* α

Moduli spaces of curves: quick tour

$2g - 2 + n > 0$.

Definition

- A *marked smooth curve* $/\mathbb{C}$: $(C, x_1, \ldots x_n)$, C smooth projective curve, $x_1, \ldots, x_n$ distinct points of C.
- $(C, x_1, \ldots, x_n) \sim (C', x_1, \ldots, x_n)$ iff $\exists \varphi : C \to C'$, s.t. $\varphi(x_i) = x_i'$.
- $\mathcal{M}_{g,n}$ = *moduli space of n-marked smooth curves of genus g*

Moduli space curves: quick tour

Properties of $\mathcal{M}_{g,n}$:

- Representable as a smooth DM-stack or orbifold;
- admits a smooth, irreducible compactification $\overline{\mathcal{M}}_{g,n}$;
- $\dim_{\mathbb{C}} = 3g - 3 + n$;
- $\exists$ universal curve: $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$.

WARNING! the DM stack structure reflects the deformation theory of smooth curves/Riemann surfaces !

Moduli space curves: several realizations

$\mathcal{M}_{g,n}(\mathbb{C}) =$

- $\{$smooth marked curves$\}/\sim$ or $\{$marked Riemann surfaces$\}/\sim$
- $\mathcal{M}^{\mathrm{hyp}}_{g,n}(L_1, \ldots, L_n)$, for $(L_1, \ldots, L_n) \in \mathbb{R}^n_{\geq 0})=$

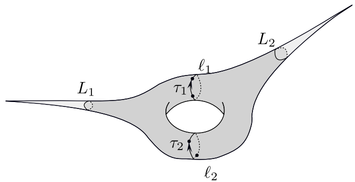
  $\{$ hyperbolic surfaces with geodesic boundaries of lengths $L_1, \ldots, L_n\}$

  $\rightsquigarrow$ structure of symplectic orbifold

- $\mathcal{M}_{g,n}(\alpha)$, for $\alpha = (\alpha_1, \ldots, \alpha_n)$ s.t. $|\alpha| = 2g - 2 + n$:

  $\rightsquigarrow$ what geometric structure?

Basics of $\mathcal{M}_{g,n}^{\mathrm{hyp}}(L_1, \ldots, L_n)$



- **Coordinates.** (Franchel-Nielsen) $(\ell_i, \tau_i)_{i \leq 3g-3+n}$
- **Symplectic form.** (Weil-Petersson) $\omega_{WP}(L_1, \ldots, L_n) = \sum_i \ell_i \wedge \tau_i$

$$\rightsquigarrow V_{g,n}^{WP}(L_1, \ldots, L_n) = \int_{\mathcal{M}_{g,n}(L_1,\ldots,L_n)} \omega_{WP}(L_1, \ldots, L_n)^{3g-3+n}$$

$$= \int_{\overline{\mathcal{M}}_{g,n}} \left(\frac{\kappa_1}{2\pi} + \frac{L_1^2 \psi_1}{2} + \ldots + \frac{L_n^2 \psi_n}{2} \right)^{3g-3+n}$$

Computable by induction (Topological Recursion) [Mirzakhani 04, 07]

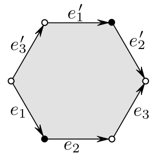
Moduli space of flat surfaces: $\mathcal{M}_{g,n}(\alpha)$

We want ν_α on $\mathcal{M}_{g,n}(\alpha) \Rightarrow$ definition for $V_{g,n}(\alpha)$

Warning. We will not define it as $\omega_\alpha^{\mathrm{top}}$ for some symplectic form!

$\rightsquigarrow$ Instead, we follow [Veech, *Flat surfaces* 93].

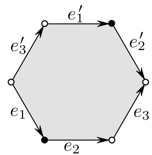
Moduli space of flat surfaces: $\mathcal{M}_{g,n}(\alpha)$



- **Coordinates.** $(e_i, \theta_i)_{i \leq 2g-1+n}$ (Example: $\theta_1 = -\theta_2 = \pi/6, \theta_3 = 0$).
- **Dimension count.** $\mathcal{M}_{g,n}(\mathbb{C}) \simeq \mathcal{M}_{g,n}(\alpha)$.

$$\begin{aligned}\dim_{\mathbb{R}}(\mathcal{M}_{g,n}) &= 2 \times (3g - 3 + n) \\ &= \underbrace{(2+1)}_{\text{vector+rotation}} \times \underbrace{(2g - 1 + n)}_{h^1(C,\{x_i\},\mathbb{Z})} \\ &\quad - \underbrace{(n-1)}_{\text{angle condition}} \underbrace{-2}_{\text{closure of polygonal}} \underbrace{-2}_{\text{modulo isometry}}\end{aligned}$$

Moduli space of flat surfaces: $\mathcal{M}_{g,n}(\alpha)$

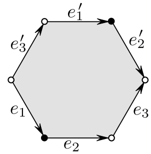


- **Coordinates.** $(e_i, \theta_i)_{i \leq 2g-1+n}$
- **Holonomy foliation.** $\exists\, \mathrm{hol} : \mathcal{T}_{g,n}(\alpha) \to (\mathbb{S}_1)^{2g}$ $(= \text{chose } \theta = (\theta_1, \ldots \theta_{2g-n+1}))$. Leave above θ locally modeled on complex projective space:

$$\left(\sum_{i=1}^{n} v_i (1 + e^{i\theta_i}) = 0 \right) / \mathbb{C}^*.$$

$$\Rightarrow \exists \mathcal{L}(\alpha) \to \mathcal{M}_{g,n}(\alpha) (\underset{\text{loc.}}{\simeq} \mathcal{O}(-1) \to \mathbb{P}^d)$$

Moduli space of flat surfaces: $\mathcal{M}_{g,n}(\alpha)$



- **Coordinates.** $(e_i, \theta_i)_{i \leq 2g-1+n}$
- **Holonomy foliation.** $\exists\, \mathrm{hol} : \mathcal{T}_{g,n}(\alpha) \to (\mathbb{S}_1)^{2g}$
- if $\alpha \in (\mathbb{R} \setminus \mathbb{Z})^n \Rightarrow U(p,q)$-**structure** on $\mathcal{L}(\alpha)|_{\mathrm{hol}^{-1}(\theta)}$:

$$h_{\alpha,\theta}(P,P) = \mathrm{area}(P),$$

with

$$\det(h_{\alpha,\theta}) = q(\alpha) = (-1)^{g+n-1} 2^{-2g} \cdot \prod_{i=1}^{n} \sin(\pi \alpha_i).$$

Interlude on $U(p,q)$-structures

Let h be a (p,q)-form on $\mathbb{C}^{p+q}$
$\mathcal{C}_h = \{x \in \mathbb{C}^{p+q}, \text{s.t. } h(x,x) > 0\}$. $\rightsquigarrow$ 2 volume forms on $\mathbb{P}\mathcal{C}_h$:

- $\nu_1' =$ Lebesgue volume form on $\mathbb{C}^{p+q}$
  $\mathcal{C}_{h,1}\{x \in \mathcal{C}(h), \text{s.t. } h(x,x) \leq 1\}$.

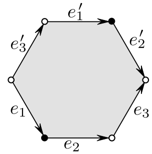
$$\nu_1 = \pi_* \nu_1' \text{ (where } \pi : \mathcal{C}_{h,1} \to \mathbb{P}\mathcal{C}_h)$$

- $h =$ hermitian metric on $\mathcal{O}(-1) \to \mathbb{P}\mathcal{C}_h$:

$$\nu_2 = (-\omega_h)^{p+q-1} (\omega_h = \frac{\partial\overline{\partial} h(x,x)}{2i\pi})$$

$$\Rightarrow \boxed{\nu_1 = \frac{\pi^{p+q}}{\det(h)(p+q)!}\nu_2} \text{ think to volume of a ball!}$$

Moduli space of curves: $\mathcal{M}_{g,n}(\alpha_1, \ldots, \alpha_n)$



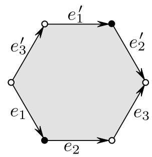
- **Coordinates.** $(e_i, \theta_i)_{i \leq 2g-1+n}$
- **Holonomy foliation.** $\exists\,\mathrm{hol} : \mathcal{T}_{g,n}(\alpha) \to (\mathbb{S}_1)^{2g}$
- if $(\alpha \in \mathbb{R} \setminus \mathbb{Z})^n \Rightarrow$ $U(p,q)$**-structure** on $\mathcal{L}(\alpha)|_{\mathrm{hol}^{-1}(\theta)}$:

$$h_{\alpha,\theta}(P,P) = \mathrm{area}(P),$$

with

$$\det(h_{\alpha,\theta}) = q(\alpha) = (-1)^{g+n-1} 2^{-2g} \cdot \prod_{i=1}^{n} \sin(\pi \alpha_i).$$

Moduli space of curves: $\mathcal{M}_{g,n}(\alpha_1, \ldots, \alpha_n)$



- **Coordinates.** $(e_i, \theta_i)_{i \leq 2g-1+n}$
- **Holonomy foliation.** $\exists\, \mathrm{hol} : \mathcal{T}_{g,n}(\alpha) \to (\mathbb{S}_1)^{2g}$
- if $(\alpha \in \mathbb{R} \setminus \mathbb{Z})^n \Rightarrow U(p,q)$**-structure** on $\mathcal{L}(\alpha)|_{\mathrm{hol}^{-1}(\theta)}$:

$$\nu_{\alpha,\theta} = \frac{\pi^{2g-2+n}}{q(\alpha)(2g-2+n)!}(-\omega_{h_{\alpha,\theta}})^{2g-3+n}$$

volume form in $\mathcal{L}(\alpha)|_{\mathrm{hol}^{-1}(\theta)}$

$$\boxed{\nu_\alpha = \nu_{\alpha,\theta} \wedge \mathrm{hol}^*(\text{haar volume form})}$$

Volume of $\mathcal{M}_{g,n}(\alpha_1, \ldots, \alpha_n)$?

Theorem (Veech, 93)

ν_α *is invariant under the action of the mapping class group .*

($\Rightarrow$ it defines a volume form on $\mathcal{M}_{g,n}(\alpha)$)

$$V_{g,n}(\alpha) = \nu_\alpha(\mathcal{M}_{g,n}) \in \mathbb{R}_{>0} \cup \infty$$

Computed in genus 0 with $0 < \alpha_i < 1$ by [McMullen, 06], [Koziarz, Nguyen, 16], and in genus 1, with $n = 2$ by [Gazhouani, Pirio, 18].

Volume of $\mathcal{M}_{g,n}(\alpha_1, \ldots, \alpha_n)$?

Sketch of computation of $V_{g,n}$. Let $\alpha = \mu/k \in (\mathbb{Q}^n \setminus \mathbb{Z}^n)$.

$$\begin{aligned}
\mathcal{M}_{g,n}(k,\mu) &= \{(C, x_1, \ldots, x_n), \text{s.t. } \omega_{\log}^{\otimes k} \simeq \sum m_i(x_i)\} \hookrightarrow \mathcal{M}_{g,n} \\
&= \{1/k\text{-translation surfaces with singularities } \alpha\ \} \hookrightarrow \mathcal{M}_{g,n}(\alpha) \\
&= \mathrm{hol}^{-1}(\mathbb{U}_k^{2g}) \\
&\Rightarrow \quad V_{g,n}(k,\mu) \underset{\mathrm{def}}{=} \mathrm{Vol}(\mathcal{M}_{g,n}(k,\mu))
\end{aligned}$$

1. $V_{g,n}(\alpha) = \lim_{\ell \to \infty} \frac{V_{g,n}(\ell k, \ell \mu)}{(k\ell)^{2g}}$
2. $V_{g,n}(k,\mu) \sim \int_{\mathcal{M}_{g,n}(k,\mu)} \omega_{h_\alpha}^{\mathrm{top}} = \int_{\overline{X}_{g,n}(k,\mu)} c_1(\mathcal{O}(1))^{\mathrm{top}}$ (Costantini, Moëller, Zachhuber, 2019).
3. Study the intersection theory of $\mathcal{M}_{g,n}(k,\mu)$ for fixed $k \Rightarrow$ procedure to compute $V_{g,n}(k,\mu)$ by induction on g and n.
4. Study the large k limit of these relations. **Here** we need new results: value of $[\overline{\mathcal{M}}_{g,n}(k,\mu)]$ in $H^*(\overline{\mathcal{M}}_{g,n})$.

Why do we need new results?

How to compute $[\overline{\mathcal{M}}_{g,n}(k,\mu)] \in A^g(\overline{\mathcal{M}}_{g,n},\mathbb{Q}), H^{2g}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$?

$$\begin{array}{c} \mathrm{Pic}^0_{g,n} \\ \sigma_0 \uparrow \downarrow \uparrow \sigma(k,\mu) \\ \mathcal{M}_{g,n}, \end{array} \qquad \sigma(k,\mu) = \omega^k_{\log}(\textstyle\sum_{i=1}^n m_i\sigma_i)$$

$$\mathcal{M}_{g,n}(k,\mu) = \mathcal{M}_{g,n} \times_{\mathrm{Pic}^0_{g,n}} \mathcal{M}_{g,n}$$

Why do we need new results?

How to compute $[\overline{\mathcal{M}}_{g,n}(k,\mu)] \in A^g(\overline{\mathcal{M}}_{g,n},\mathbb{Q}), H^{2g}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$?

$$\begin{array}{c} \overline{\mathrm{Pic}}^0_{g,n} \\ \sigma_0 \uparrow \downarrow \uparrow \sigma(k,\mu) \\ \overline{\mathcal{M}}_{g,n}, \end{array} \qquad \sigma(k,\mu) = \omega_{\log}^k(\textstyle\sum_{i=1}^n m_i\sigma_i)$$

Definition

$$\mathrm{DR}_{g,n}(k,\mu) = \sigma_0^*(\sigma(k,\mu)(\overline{\mathcal{M}}_{g,n}))$$

Why do we need new results?

Properties of $\mathrm{DR}_{g,n}(k,\mu)$:

- $\mathrm{DR}_{g,n}(k,\mu) = [\overline{\mathcal{M}}_{g,n}(k,\mu)] +$ *boundary* (geometrically described by [Schmitt, Holmes,19])
- Conjecture of [Schmitt, 17]: $\mathrm{DR}_{g,n}(k,\mu) = P_{g,n}(k,\mu)$ (Pixton's class)
- $P_{g,n}(k,\mu)$ is explicitly computable, and polynomial of degree $2g$ in k and m_i's

If this conjecture holds

$$\lim_{\ell\to\infty} \frac{1}{(k\ell)^{2g}} \mathrm{DR}_{g,n}(k\ell, \ell\mu) = \widetilde{P}_{g,n}(1, \mu/k)$$

(where $\widetilde{P}_{g,n}$ = degree $2g$ part of $P_{g,n}$).

Why do we need new results?

Properties of $\mathrm{DR}_{g,n}(k,\mu)$:

- $\mathrm{DR}_{g,n}(k,\mu) = [\overline{\mathcal{M}}_{g,n}(k,\mu)] +$ *boundary* (geometrically described by [Schmitt, Holmes,19])
- Conjecture of [Schmitt, 17]: $\mathrm{DR}_{g,n}(k,\mu) = P_{g,n}(k,\mu)$ (Pixton's class)
- $P_{g,n}(k,\mu)$ is explicitly computable, and polynomial of degree $2g$ in k and $m_i's$

If this conjecture holds, then

$$\lim_{\ell\to\infty} \frac{1}{(k\ell)^{2g}} \mathrm{DR}_{g,n}(k\ell,\ell\mu) = \widetilde{P}_{g,n}(1,\mu/k) = \lim_{\ell\to\infty} \frac{1}{(k\ell)^{2g}} [\overline{\mathcal{M}}_{g,n}(k\ell,\ell\mu)]$$

(where $\widetilde{P}_{g,n} =$ degree $2g$ part of $P_{g,n}$).

Outcome

Theorem ((S. 20))

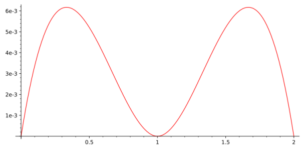
Assuming that Schmitt's conjecture is valid, then $\exists v_{g,n}$ *(computable) piece-wise polynomial s.t.*

$$V_{g,n}(\alpha) = \frac{v_{g,n}(\alpha)}{\sin(\pi\alpha_1)\dots\sin(\pi\alpha_n)}$$

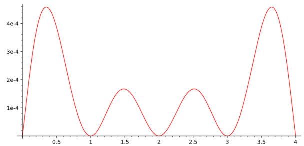
for almost all α.

Outcome

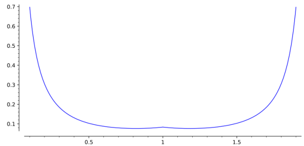
$$V_{g,n}(\alpha) = \frac{v_{g,n}(\alpha)}{\sin(\pi\alpha_1)\ldots\sin(\pi\alpha_n)}$$



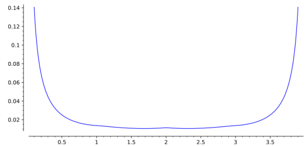
$v_{1,2}$



$v_{2,2}$



$V_{1,2}$



$V_{2,2}$

Thank you for your attention!