Erratum to "Springer fiber components in the two columns case for types A and D are normal"

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1 Short description of the error

In this short note we correct an error in our paper "Springer fiber components in the two columns case for types A and D are normal" [PS12]. In that paper we consider N an element in $\mathfrak{gl}_n$ or $\mathfrak{so}_{2n}$ such that $N^2 = 0$ and an irreducible component X of the Springer fiber of N. Recall that we worked over k an algebraicaly closed field. The following was the main result and stays unchanged.

Theorem 1.1. *The variety X is normal and admits a rational resolution.*

The main idea was to construct partial resolution $p_X : \widehat{X} \to X$ and prove that $\widehat{X}$ admit a Frobenius splitting. The results then follows from general arguments on Frobenius splittings.

It was recently pointed to us by Lucas Fresse and Simon Jacques that $\widehat{X}$ is not always irreducible. In this short erratum we explain that $\widehat{X}$ admits a unique irreducible component $\widehat{X}'$ such that the restriction $p_X : \widehat{X}' \to X$ of p_X to X is birational. The rest of the proof in [PS12] works with $\widehat{X}'$ in place of $\widehat{X}$.

Let us also emphasize that Simon Jacques has an explicit description of the irreducible component $\widehat{X}'$ in type A. This will appear in his forthcoming PhD Thesis.

Acknowledgements We are thankful to Lucas Fresse and Simon Jacques for pointing out this problem and for helpful discussions.

2 The variey $\widehat{X}'$ in type A

We start with case of nilpotent elements in $\mathfrak{gl}_n$ and will explain in the next section the changes needed for $N \in \mathfrak{so}_{2n}$. Let $N \in \mathfrak{gl}_n$ be a nilpotent element of order 2. The Springer fiber $\mathcal{F}_N$ is defined by $\mathcal{F}_N = \{(V_i)_{i\in[0,n]} \mid \dim V_i = i \text{ and } N(V_i) \subset V_i\}$, where $\mathcal{F}$ is the variety of complete flags in k^n. The variety $\mathcal{F}_N$ is reducible and if r is the rank of N, the irreducible components are indexed by the choice of sequences of integers $\tau = (0 = p_0 < p_1 < \cdots < p_r < p_{r+1})$ with $p_{r+1} = n$. The corresponding irreducible components $X = X_\tau \subset \mathcal{F}_N$ can be described as follows:

$$X = \overline{\{(V_i)_{i\in[0,n]} \in \mathcal{F}_N \mid \dim(\mathrm{Im} N \cap V_i) = k \text{ for all } k \in [0,r] \text{ and } i \in [p_k, p_{k+1})\}}.$$

To understand this variety, we constructed the variety $\widehat{X}$ as follows

$$\widehat{X} = \{((F_k)_{k\in[0,r]}, (V_i)_{i\in[0,n]}) \in \mathcal{F}(\mathrm{Im}N) \times \mathcal{F} \mid F_k \subset V_{p_k} \subset N^{-1}(F_{k-1}) \text{ for all } k \in [1,r]\},$$

where $\mathcal{F}(\mathrm{Im}N)$ is the variety of complete flags in $\mathrm{Im}N$. There are natural maps $q_X : \widehat{X} \to \mathcal{F}(\mathrm{Im}N)$ and $p_X : \widehat{X} \to \mathcal{F}$ induced by the first and the second projection. In [PS12, Proposition 2.6], we describe the structure of the maps q_X and p_X. As stated Proposition 2.6 in [PS12] is false. We give a corrected statement.

Proposition 2.1. *(ı) The map q_X is dominant and a locally trivial fibration over $\mathcal{F}(\mathrm{Im}N)$. Its fiber over $(F_k)_{k\in[0,r]}$ is isomorphic to the variety*

$$\mathcal{F}_w = \{(V_i)_{i\in[0,n]} \in \mathcal{F} \mid F_k \subset V_{p_k} \subset N^{-1}(F_{k-1}), \text{ for all } k \in [1,r]\}.$$

(ıı) The map p_X sends $\widehat{X}$ onto X and p_X maps a unique irreducible component $\widehat{X}'$ of $\widehat{X}$ birationally onto X.

Remark 2.2. Note that in (ı), the only difference with Proposition 2.6.(ı) in [PS12] is that we do not claim that $\mathcal{F}_w$ is a Schubert variety. The variety $\mathcal{F}_w$ is stable under the action of the stabiliser of $(F_k)_{k\in[0,r]}$ which is a Borel subgroup of $\mathrm{GL}(\mathrm{Im}N)$ but $\mathcal{F}_w$ is not irreducible in general as shows the following example (this example was pointed to us by Lucas Fresse and Simon Jacques).

Example 2.3. Let $n = 5$ and denote by $(e_i)_{i\in[1,5]}$ the canonical basis of k^5. Let N be given by

$$N = \begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix},$$

where I_2 is the identity matrix of size 2. Set $p_1 = 2$ and $p_2 = 3$. The variety X is given as follows: $X = \overline{\{(V_i)_{i\in[0,5]} \in \mathcal{F}_N \mid \dim(\mathrm{Im}N \cap V_2) = 1 \text{ and } \dim(\mathrm{Im}N \cap V_3) = 2\}}$. Note that $\mathrm{Im}N = \langle e_1, e_2\rangle \subset \mathrm{Ker}N = \langle e_1, e_2, e_3\rangle$. Fix $F_0 = 0 \subset F_1 = \langle e_1\rangle \subset F_2 = \langle e_1, e_2\rangle = \mathrm{Im}N$ a full flag in $\mathrm{Im}N$. The variety $\mathcal{F}_w$ of Proposition 2.1 is the following variety:

$$\mathcal{F}_w = \{(V_i)_{i\in[0,5]} \mid F_1 \subset V_2 \subset \mathrm{Ker}N \text{ and } F_2 \subset V_3 \subset N^{-1}(F_1) = \langle e_1, e_2, e_3, e_4\rangle\}.$$

We claim that this variety is not irreducible. Indeed, we have two possibilities for V_2: either $V_2 = \mathrm{Im}N = F_2$ or $\dim(V_2 \cap \mathrm{Im}N) = \dim(V_2 \cap F_2) = 1$. In the first case V_3 can be freely chosen such that $\mathrm{Im}N = \langle e_1, e_2\rangle \subset V_3 \subset \langle e_1, e_2, e_3, e_4\rangle$, this is isomorphic to $\mathbb{P}^1$ and V_1 and V_4 can also be freely chosen. In the second case, V_2 can be freely chosen such that $F_1 = \langle e_1\rangle \subset V_2 \subset \langle e_1, e_2, e_3\rangle = \mathrm{Ker}N$ with $V_2 \neq \langle e_1, e_2\rangle$ while $V_3 = V_2 + \mathrm{Im}N$. This is isomorphic to $\mathbb{A}^1$ and V_1 and V_4 can be freely chosen. This shows that $\mathcal{F}_w$ has two irreducible components, both Schubert varieties. In particular $\widehat{X}$ has two irreducible components $\widehat{X}'$ and $\widehat{X}''$ given by

$$\widehat{X}'' = \{((F_k)_{k\in[0,2]}, (V_i)_{i\in[0,5]}) \in \mathcal{F}(\mathrm{Im}N) \times \mathcal{F} \mid V_2 = \mathrm{Im}N \text{ and } \mathrm{Im}N \subset V_3 \subset N^{-1}(F_1)\} \text{ and}$$

$$\widehat{X}' = \overline{\left\{((F_k)_{k\in[0,2]}, (V_i)_{i\in[0,5]}) \in \mathcal{F}(\mathrm{Im}N) \times \mathcal{F} \;\middle|\; \begin{array}{l} F_1 \subset V_2 \subset \mathrm{Ker}N,\ V_2 \neq \mathrm{Im}N \\ \text{and } V_3 = \mathrm{Im}N + V_2 \end{array}\right\}}.$$

We have $p_X(\widehat{X}'') \subsetneq X$. Indeed, the elements of $p_X(\widehat{X}'')$ satisfy $V_2 = \mathrm{Im}N$ but the following flag: $V_1 = \langle e_1\rangle \subset V_2 = \langle e_1, e_3\rangle \subset V_3 = \langle e_1, e_3, e_2\rangle \subset V_4 = \langle e_1, e_3, e_2, e_4\rangle$ lies in $X \setminus p_X(\widehat{X}'')$. Proposition 2.1 implies that $p_X(\widehat{X}') = X$.

We now prove Proposition 2.1.

Proof. The proof is similar to the proof given in [PS12, Proposition 2.6] (where the wrong fact that $\mathcal{F}_w$ is a Schubert variety was not proved). In particular, we proved that p_X maps $\widehat{X}$ into X and that for an explicit open subset $X^0 \subset X$, the map p_X is an isomorphism. This in particular implies the statement that the map p_X sends $\widehat{X}$ onto X and p_X maps a unique irreducible component $\widehat{X}'$ of $\widehat{X}$ birationally onto X.

The only incorrect statement in the proof is the sentence *It is easy to check that for* $(V_i)_{i\in[0,n]}$ *general in the Schubert variety, we have* $\mathrm{Im}N \cap V_i = F_k$ *for* $i \in [p_k, p_{k+1})$. This is only true in the irreducible component $\widehat{X}'$. □

The rest of the paper works with $\widehat{X}'$ in place of $\widehat{X}$.

3 The variey $\widehat{X}'$ in type D

The problem is similar than in type A. Let N be a nilpotent element of order 2 and rank $2r$. The irreducible components of the Springer fiber are indexed by sequences of intergers $0 = p_1 < \cdots < p_r < p_{r+1} = 2r + 1$. We choose such a sequence and denote by X the corresponding irreducible component of the Springer fiber. In [PS12], we first remarked that the symmetric form ω on k^{2n} and the nilpotent element N induce a symplectic form α on $\mathrm{Im}N$ defined by $\alpha(x, y) = \omega(x, y')$ with $N(y') = y$. We then constructed the variety $\widehat{X}$ as follows

$$\widehat{X} = \{((F_k)_{k\in[0,r]}, (V_i)_{i\in[0,n]}) \in \mathrm{Sp}\mathcal{F}(\mathrm{Im}N)\times \mathrm{O}\mathcal{F} \mid F_k \subset V_{p_k} \subset N^{-1}(F_{k-1}) \text{ for all } k \in [1, r]\},$$

where $\mathrm{Sp}\mathcal{F}(\mathrm{Im}N)$ is the variety of flags $(F_k)_{k\in[0,r]}$ in $\mathrm{Im}N$ such that for all $k \in [0, r]$, $\dim F_k = k$ and the subspace F_k is isotropic for α in $\mathrm{Im}N$. There are maps $q_X : \widehat{X} \to \mathrm{Sp}\mathcal{F}(\mathrm{Im}N)$ and $p_X : \widehat{X} \to \mathrm{O}\mathcal{F}$ induced by the first and the second projection. In [PS12, Proposition 3.15], we describe the structure of the maps q_X and p_X. As stated Proposition 3.15 in [PS12] is false. We give a corrected statement.

Proposition 3.1. *(ı) The map q_X is dominant and a locally trivial fibration over $\mathcal{F}(\mathrm{Im}N)$. Its fiber over $(F_k)_{k\in[0,r]}$ is isomorphic to the variety*

$$\mathcal{F}_w = \{(V_i)_{i\in[0,n]} \in \mathrm{O}\mathcal{F} \mid F_k \subset V_{p_k} \subset N^{-1}(F_{k-1}), \text{ for all } k \in [1, r]\}.$$

(ıı) The map p_X sends $\widehat{X}$ onto X and p_X maps a unique irreducible component $\widehat{X}'$ of $\widehat{X}$ birationally onto X.

Remark 3.2. Note that in (ı), the only difference with Proposition 3.15.(ı) in [PS12] is that we do not claim that $\mathcal{F}_w$ is a Schubert variety. The variety $\mathcal{F}_w$ is stable under the action of the stabiliser of $(F_k)_{k\in[0,r]}$ which is a Borel subgroup of $\mathrm{Sp}(\mathrm{Im}N, \alpha)$ but $\mathcal{F}_w$ is not irreducible in general as shows the following example (this example was build-up from Example 2.3 pointed to us by Lucas Fresse and Simon Jacques).

Example 3.3. Let $n = 5$ and denote by $(e_i)_{i\in[1,10]}$ the canonical basis of k^{10}. Choose ω such that $\omega(e_i, e_j) = \delta_{i,11-j}$ for $i, j \in [1, 10]$. Let N be given by $\mathrm{Ker}N = \langle e_1, e_2, e_3, e_4, e_5, e_6\rangle$, $N(e_7) = e_1$, $N(e_8) = e_2$, $N(e_9) = -e_3$ and $N(e_{10}) = e_4$. We have $\mathrm{rk}(N) = 4 = 2r$ with $r = 2$.

Set $p_1 = 2$ and $p_2 = 3$. The variety X is given as follows:

$$X = \overline{\{(V_i)_{i\in[0,5]} \in \mathcal{F}_N \mid \dim(\mathrm{Im}N \cap V_2) = 1 \text{ and } \dim(\mathrm{Im}N \cap V_3) = 2\}}.$$

Note that $\mathrm{Im}N = \langle e_1, e_2, e_3, e_4\rangle \subset \mathrm{Ker}N = \langle e_1, e_2, e_3, e_4, e_5, e_6\rangle$ and $\alpha(e_i, e_j) = \delta_{i,5-j}$ for $1 \leq i < j \leq 4$. Fix $F_0 = 0 \subset F_1 = \langle e_1\rangle \subset F_2 = \langle e_1, e_2\rangle \subset \mathrm{Im}N$ an isotropic flag in $\mathrm{Im}N$. The variety $\mathcal{F}_w$ of Proposition 2.1 is the following variety:

$$\mathcal{F}_w = \{(V_i)_{i\in[0,5]} \in \mathrm{O}\mathcal{F} \mid F_1 \subset V_2 \subset \mathrm{Ker}N \text{ and } F_2 \subset V_3 \subset N^{-1}(F_1) = \langle e_i \mid i \in [1, 7]\rangle\}.$$

We claim that this variety is not irreducible. Indeed, we have two possibilities for V_2: either $V_2 = F_2 \subset \mathrm{Im}N$ or $\dim(V_2 \cap F_2) = 1$. In the first case V_3 can be freely chosen such that $F_2 = \langle e_1, e_2\rangle \subset V_3 \subset \langle e_i \mid i \in [1, 7]\rangle$ and V_3 is isotropic. This is isomorphic to $\mathbb{P}^4 = \mathbb{P}\langle e_3, e_4, e_5, e_6, e_7\rangle$ and V_1 and V_4 can also be freely chosen. In the second case, V_2 can be freely chosen such that $F_1 = \langle e_1\rangle \subset V_2 \subset \mathrm{Ker}N$ with $V_2 \neq \langle e_1, e_2\rangle$ while $V_3 = V_2 + \langle e_1, e_2\rangle$. This is an open subset of $\mathbb{P}^4 = \mathbb{P}\langle e_2, e_3, e_4, e_5, e_6\rangle$ and V_1 and V_4 can be freely chosen. This shows that $\mathcal{F}_w$ has two irreducible components, both Schubert varieties. In particular $\widehat{X}$ has two irreducible components $\widehat{X}'$ and $\widehat{X}''$ given by

$$\widehat{X}'' = \{((F_k)_{k\in[0,2]}, (V_i)_{i\in[0,5]}) \in \mathrm{Sp}\mathcal{F}(\mathrm{Im}N) \times \mathrm{O}\mathcal{F} \mid \ V_2 = F_2 \text{ and } F_2 \subset V_3 \subset N^{-1}(F_1)\}$$

$$\widehat{X}' = \overline{\left\{((F_k)_{k\in[0,2]}, (V_i)_{i\in[0,5]}) \in \mathrm{Sp}\mathcal{F}(\mathrm{Im}N) \times \mathrm{O}\mathcal{F} \mid \begin{array}{l} V_1 \subset V_2 \subset \mathrm{Ker}N, \ V_2 \neq F_2 \\ \text{and } V_3 = F_2 + V_2 \end{array}\right\}}.$$

We have $p_X(\widehat{X}'') \subsetneq X$. Indeed, the elements of $p_X(\widehat{X}'')$ satisfy $V_2 = F_2 \subset \mathrm{Im}N$ but the following flag: $V_1 = \langle e_1\rangle \subset V_2 = \langle e_1, e_6\rangle \subset V_3 = \langle e_1, e_6, e_2\rangle \subset V_4 = \langle e_1, e_6, e_2, e_{10}\rangle$ lies in $X \setminus p_X(\widehat{X}'')$. Proposition 3.1. impy that $p_X(\widehat{X}') = X$.

We now prove Proposition 3.1.

Proof. The proof is similar to the proof given in [PS12, Proposition 3.15] (where the wrong fact that $\mathcal{F}_w$ is a Schubert variety was not proved). In particular, we proved that p_X maps $\widehat{X}$ into X and that for an explicit open subset $X^0 \subset X$, the map p_X is an isomorphism. This in particular implies the statement that the map p_X sends $\widehat{X}$ onto X and p_X maps a unique irreducible component $\widehat{X}'$ of $\widehat{X}$ birationally onto X. $\square$

The rest of the paper works with $\widehat{X}'$ in place of $\widehat{X}$.

References

[PS12] N. Perrin and E. Smirnov *Springer fiber components in the two-columns case for types A and D are normal.* Bulletin de la SMF **140** (2012) no. 3, 309–333.