(Webs of maximal rank)
n-covered varieties
and Jordan algebras

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Luc PIRIO
CNRS & Université de Versailles (프랑스)

Plan of the talk

0. Motivation (webs of maximal rank)

1. Varieties n-covered by curves

2. Case $n = 3$ (Jordan algebra)

3. The XJC-correspondence

4. Questions

Motivation

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Theorem : [Pirio-Trépreau 2013]

$\mathcal{W}_d$ = germ of d-web of codimension $r > 1$ on $(\mathbb{C}^{nr}, 0)$:

$$\mathcal{W}_d \text{ has max. rank} \quad \Longrightarrow \quad \begin{array}{c} \mathcal{W}_d \text{ is ‘algebraizable’} \\ (\text{ generalized sense if } d = d_{n,r} \ ) \end{array}$$

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$\bullet\!\!\rightsquigarrow$ **Geometric problem :** determination of the $X \subset \mathbb{P}^N$'s :

– n-covered by **RNC**s

– extremal

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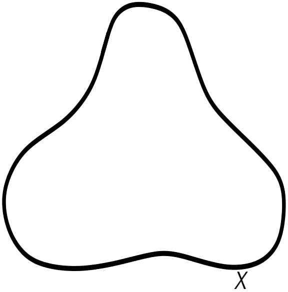
- X irreducible projective variety (over $\mathbb{C}$)



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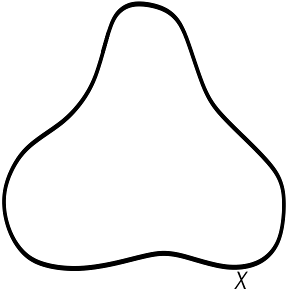

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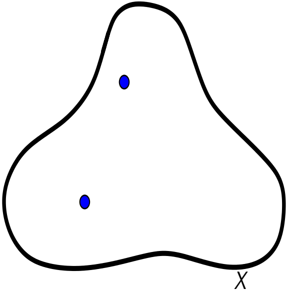

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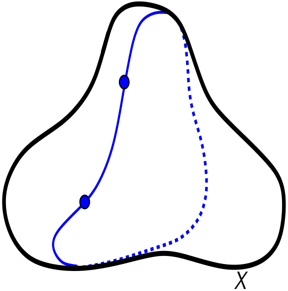

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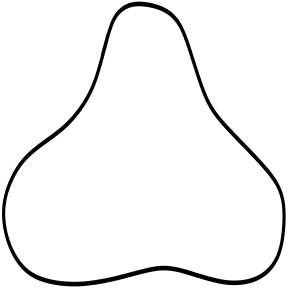

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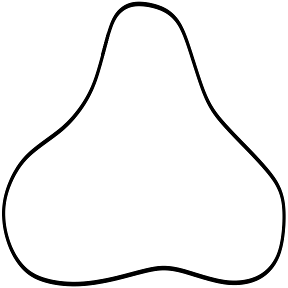

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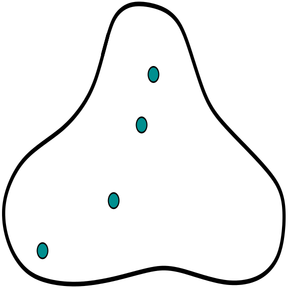

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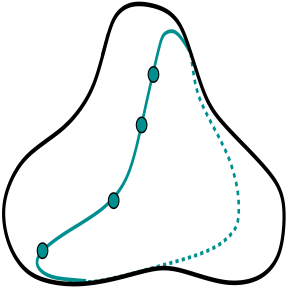

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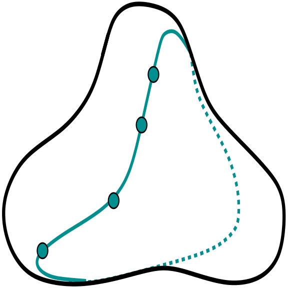

Proposition : X RC $\iff$ X n-RC for all $n \geq 2$

Notations

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- over $\mathbb{C}$
- X irreducible projective variety
- $X \subset \mathbb{P}^N$ fixed embedding
- $r = \dim X \geq 1$
- $n =$ number of points ≥ 2
- $\delta =$ degree with respect to $\mathcal{O}_X(1) \quad \geq n - 1$

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Remarks :

- 1. is sharp
- $\pi(r,n,\delta)$ is '*Castelnuovo-Harris bound*' on the genus
- 1. implies 2.
- 2. is due to Fano when $n=2$ and $\Sigma \subset \mathrm{RatCurves}(X)$

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- $\dim\langle X\rangle + 1 \leq \sum_{i=1}^{m} \binom{r+\rho}{r} + \sum_{j=1}^{m'} \binom{r+\rho-1}{r} =: \pi(r,n,\delta)$ $\square$

Proposition : Let $X = \mathbf{X}^r(n, \delta)$

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Definition : an *‘osculating projection’* is a linear projection

$$\Pi_S : X \subset \mathbb{P}^{\pi-1} \dashrightarrow T^{(\rho)}_{X,x_1} \simeq \mathbb{P}^{\binom{r+\rho}{r}-1}$$

from $$S = \Big(\oplus_{i=2}^{m} T^{(\rho)}_{X,x_i}\Big) \oplus \Big(\oplus_{j=1}^{m'} T^{(\rho-1)}_{X,x_{m+j}}\Big)$$

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- **Def :** V is a *'Castelnuovo variety'* if $\mathbf{p}_g(V) = \pi(r, n, \delta) > 0$

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  [Castelnuovo-Harris] : $\mathbf{p}_g(V) \leq \pi(r,n,\delta)$

- **Def :** V is a *'Castelnuovo variety'* if $\mathbf{p}_g(V) = \pi(r,n,\delta) > 0$

- **[Harris] :** $\big|I_V(2)\big|$ cuts out $Y_V \subset \mathbb{P}^{n+r-2}$: $\begin{cases} \dim Y_V = r \\ \deg Y_V = n-1 \end{cases}$

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- For some $L_V \in \mathbf{Pic}(Y_V)$, the following diagram commutes

$$\begin{tikzcd}
V \arrow[r, dashed, "\phi_{|K_V|}"] \arrow[d, hook] & \mathbb{P}^{\pi-1} \arrow[d, equal] \\
Y_V \arrow[r, dashed, "\psi_{|L_V|}"'] & \mathbb{P}^{\pi-1}
\end{tikzcd}$$

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Facts :

- for all r, n, δ : $\exists\ X = \mathbf{X}(n,\delta)$ of C-type
- $X = \mathbf{X}^r(2,\delta)$ C-type $\Longrightarrow$ $X = v_\delta(\mathbb{P}^r)$
- $X = \mathbf{X}^r\big(n, \rho(n-1)\big)$ C-type $\Longrightarrow$ $X = v_\rho(Y)$, deg $Y = n-1$

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Problem : *to classify the* $X = \mathbf{X}(n, \delta)$ *: is* X *of C-type ?*

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$n \geq 2$: **Pirio-Trépreau** (2013), **Pirio-Russo** (2013 - 2016)

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$$\text{Let } X = \mathbf{X}^r(2,\delta) : \begin{cases} \bullet\ X \subset \mathbb{P}^{\binom{r+\delta}{r}-1} \text{ non degenerate} \\ \bullet\ \text{2-covered by a family } \Sigma_X \text{ of} \\ \quad \text{irreducible curves of degree } \delta \end{cases}$$

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Proof : by induction on $r = \dim X$ using osculating projections

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- $\left. \begin{array}{l} \deg C' = \rho \quad \forall C' \in \Sigma' \\ + \ \Sigma' \text{ is 2-covering} \end{array} \right\} \quad \Longrightarrow \quad X' = \mathbf{X}^r(2,\rho) = v_\rho(\mathbb{P}^r)$

Varieties $X(n, \delta)$: general case $n \geq 3$

Theorem : [Pirio-Trépreau]

For $X = \mathbf{X}^r(n, \delta)$ and $C \in \Sigma$ general

1. C is a RNC of degree δ
2. X smooth along C & $N_{C/X} = \mathcal{O}_C(n-1)^{\oplus(r-1)}$
3. $\exists\,\mathbf{!}$ RNC of degree δ through $x_1, \ldots, x_n \in X$ general
4. a general $\Pi_S : X \dashrightarrow \Pi_S(X) \subset T^{(\rho)}_{X,x}$ is birational
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Rks :

- 2. $\Longrightarrow$ intersection along $C \in \Sigma$ general is well-defined
- 3. $\Longrightarrow$ n-covering family Σ is unique

For $C \in \Sigma_X$ general $\left(C = \text{RNC of degree } \delta \simeq \mathbb{P}^1 \right)$:

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For $C \in \Sigma_X$ general $\Big(\ C = \text{RNC of degree } \delta \simeq \mathbb{P}^1 \ \Big)$:

$$\boxed{\text{2. } X \text{ smooth along } C \ \& \ N_{C/X} = \mathcal{O}_C(n-1)^{\oplus(r-1)}}$$

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- $AG_{G_n(\mathbb{C}^{n+r-1})}$ flat-model AG_{Σ_X} on Σ_X is half-flat

Varieties $X = \mathbf{X}(n, \delta)$: general case $n \geq 3$

$$\boxed{\text{4. a general } \Pi_S : X \dashrightarrow \Pi_S(X) = v_\rho(\mathbb{P}^r) \subset T^{(\rho)}_{X,x} \text{ is birational}}$$

Varieties $X = \mathbf{X}(n, \delta)$: general case $n \geq 3$

$$\boxed{\text{4. a general } \Pi_S : X \dashrightarrow \Pi_S(X) = v_\rho(\mathbb{P}^r) \subset T^{(\rho)}_{X,x} \text{ is birational}}$$

For a general $(n-2)$-tuple $\mathbf{x} \in X^{n-2}$:

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For a general $(n-2)$-tuple $\mathbf{x} \in X^{n-2}$:

- $$\begin{tikzcd} X \arrow[r, dashed, two heads, "\Pi_{\mathbf{x}}=\Pi_S"] & X'_{\mathbf{x}} = v_\rho\left(\mathbb{P}^r\right) \arrow[r, "v_\rho^{-1}"] & \mathbb{P}^r \end{tikzcd}$$

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- $\mathcal{H}_X = \cup_{\mathbf{x} \in X^{n-2}} \mathcal{H}_{\mathbf{x}}$: algebraic system of divisors on X

Theorem : [Pirio-Trépreau] For $X = \mathbf{X}^r(n, \delta)$:

I. The following assertions are equivalent

1. X is of C-type
2. $\mathcal{H}_X$ is a linear system
3. AG_{Σ_X} is flat

II. This is the case if

- $n = 2$ [Bompiani]
- $r = 2$ [(Enriques-)Bol-Bompiani]
- $n \geq 3$ and $\delta \neq 2n - 3$

III. $\exists\ X = \mathbf{X}(n, 2n-3)$'s not of C-type for $n = 3, 4, 5, 6$

Varieties $\mathbf{X}(n, \delta)$ with $n \geq 3$ and $\delta = 2n - 3$

Problem : **classify the $X = \mathbf{X}(n, 2n-3)$'s not of C-type**

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First case : $n = 3$

classify the $X = \mathbf{X}^r(3,3)$'s *i.e.* the varieties such that

- $X^r \subset \mathbb{P}^{2r+1}$ non-degenerate
- 3-covered by twisted cubic curves

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First case : $n = 3$

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Question : are there other $X = \mathbf{X}(3,3)$'s ?

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Examples :

- $\mathbf{A}$ associative non-commutative (e.g. $\mathbf{A} = M_n(\mathbb{C})$)
  $$x * y = \frac{xy+yx}{2} \quad \Longrightarrow \quad \mathbf{A}^+ = (\mathbf{A}, *) \text{ Jordan algebra}$$

- $\forall\ q \in \mathbf{Sym}^2\big(W^*\big)$, $\mathbf{J}_q = \mathbb{C} \oplus W$ is Jordan with the product
  $$(\lambda, w) \bullet (\lambda', w') = \Big(\lambda\lambda' - q(w, w'), \lambda\, w' + \lambda'\, w\Big)$$

- $B = (\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}) \otimes \mathbb{C} \Longrightarrow \mathbf{Herm}_3(B)$ with $M * N = \frac{(MN+NM)}{2}$

Jordan algebra of rank 3

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Jordan algebra of rank 3

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Remark : $\mathbb{P}\mathbf{J} \circlearrowleft : [x] \longmapsto [x^{-1}] = \big[x^{\#}\big]$ belongs to $\mathbf{Cr}_{2,2}\big(\mathbb{P}\mathbf{J}\big)$

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– $X_{\mathbf{J}}$ not a scroll $\Longrightarrow$ not of C-type

Semi-simple Jordan algebras

- There is a notion of (semi-)simplicity for Jordan algebras
- Simple Jordan algebras are classified [Albert 1947]

Jordan algebra J	**Cubic curve** $X_{\mathbf{J}}$
$\mathbb{C} \times \mathbf{J}'$ with $\mathbf{J}'$ ssimple $\mathbf{rk}(\mathbf{J}') = 2$ dim $\mathbf{J}' = r - 1$	$\mathbf{Seg}\left(\mathbb{P}^1 \times Q^{r-1}\right) \subset \mathbb{P}^{2r+1}$
$\mathbf{Herm}_3\left(\mathbb{R}_{\mathbb{C}}\right)$	$LG_3\left(\mathbb{C}^6\right) \quad \subset \mathbb{P}^{13}$
$\mathbf{Herm}_3\left(\mathbb{C}_{\mathbb{C}}\right)$	$G_3\left(\mathbb{C}^6\right) \quad \subset \mathbb{P}^{19}$
$\mathbf{Herm}_3\left(\mathbb{H}_{\mathbb{C}}\right)$	$OG_6\left(\mathbb{C}^{12}\right) \subset \mathbb{P}^{31}$
$\mathbf{Herm}_3\left(\mathbb{O}_{\mathbb{C}}\right)$	$E_7/P_7 \quad \subset \mathbb{P}^{55}$

TABLE : s-simple rank 3 Jordan algebras and their associated cubic curves

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6. In fact $\varphi_x \in \mathbf{Cr}_{22}\big(E, E'\big)$ $\rightsquigarrow$ $\varphi_x \in \mathbf{Cr}_{22}\big(\mathbb{P}^{r-1}\big)$

$\mathbf{X}(3,3)$: towards a classification

$$\varphi_x \in \mathbf{Cr}_{22}(\mathbb{P}^{r-1}) \quad \rightsquigarrow \begin{cases} \mathcal{B}_x = \mathbf{Baseloc}(\varphi_x) \quad \subset E = \mathbb{P}\big(T_x X\big) \\ \mathcal{B}'_x = \mathbf{Baseloc}(\varphi_x^{-1}) \subset E' = \mathbb{P}^{r-1} \end{cases}$$

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Theorem : [Pirio-Russo]

A. X of C-type $\quad \Longleftrightarrow \quad \varphi_x \in \mathbf{Lin} \subset \mathbf{Cr}_{22}(\mathbb{P}^{r-1})$

B. If X not of C-type then

1. $\Pi_x^{-1} : \mathbb{P}^r \overset{Cr.}{\dashrightarrow} X$ induced by $\left|3H' - 2\mathcal{B}'_x\right|$
2. $\mathcal{B}_x = \mathbf{Hilb}^{t+1}(X,x) \sim_{proj} \mathcal{B}'_x$
3. X smooth $\quad \Longleftrightarrow \quad \mathcal{B}_x$ and $\mathcal{B}'_x$ smooth

$\mathbf{X}(3,3)$: classification

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Theorem : [Pirio-Russo]

If $X = \mathbf{X}(3,3)$ is smooth then

- either X is of Castelnuovo type

$$X = S_{1\dots13} \qquad \text{or} \qquad X = S_{1\dots122}$$

- either $X = X_{\mathbf{J}}$ for a semi-simple Jordan algebra $\mathbf{J}$

$$X = \mathbf{Seg}\big(\mathbb{P}^1 \times Q^{r-1}\big),\ LG_3\big(\mathbb{C}^6\big),\ G_3\big(\mathbb{C}^6\big),\ OG_6\big(\mathbb{C}^{12}\big),\ E_7/P_7$$

Three worlds

$\mathbf{X}^r(3,3)$

$\mathbf{Jordan}_3^r$

$\mathbf{Cr}_{22}\left(\mathbb{P}^{r-1}\right)$

Three worlds

$$\mathbf{X}^r(3,3)\Big/_{\textit{projective equivalence}}$$

$$\mathbf{Jordan}_3^r\Big/_{\textit{isotopy}}$$

$$\mathbf{Cr}_{22}(\mathbb{P}^{r-1})\Big/_{\textit{linear equivalence}}$$

Three worlds

$$\left[\mathbf{X}^r(3,3)\right]$$

$$\left[\mathbf{Jordan}^r_3\right]$$

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Three worlds

$$\begin{tikzcd}
\left[\mathbf{X}^r(3,3)\right] & & \left[\mathbf{Jordan}_3^r\right] \arrow[ll, "X_J \longleftarrow J"'] \\
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\end{tikzcd}$$

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Three worlds

$$\begin{tikzcd}
{\left[\mathbf{X}^r(3,3)\right]} \arrow[dr, equal] \arrow[ddrr, bend right] & & & {\left[\mathbf{Jordan}^r_3\right]} \arrow[lll] \arrow[ddl, bend left] \\
& {\left[C\text{-}\mathbf{type}\right]} & & \\
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\left[\mathbf{X}^{r}(3,3)\right] \arrow[dr, equal] \arrow[ddr, bend right] & & \left[\mathbf{Jordan}^{r}_{cub}\right] \arrow[ll] \arrow[ddl, bend left] \\
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& \left[\overline{\mathbf{Cr}_{22}}\left(\mathbb{P}^{r-1}\right)\right] &
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Three worlds

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\left[\mathbf{X}^r(3,3)\right] \arrow[dr, dash, double] \arrow[dddrr, bend right] & & & & \left[\mathbf{Jordan}^r_{cub}\right] \arrow[llll] \arrow[dl, dash, double] \arrow[dddll, bend left] \\
& \left[C\text{-}\mathbf{type}\right] & & \left[\mathbf{rank} < \mathbf{3}\right] & \\
& & \left[\mathbf{Lin}\right] & & \\
& & \left[\overline{\mathbf{Cr}_{22}}\left(\mathbb{P}^{r-1}\right)\right] \arrow[u, dash, double] & &
\end{tikzcd}$$

Three worlds

$$\begin{tikzcd}
\left[\mathbf{X}^r(3,3)\right] \arrow[dddrr, bend right] & & & & \left[\mathbf{Jordan}^r_{cub}\right] \arrow[llll] \arrow[dddll, bend left] \\
& \left[C\text{-}\mathbf{type}\right] \arrow[ul, phantom, sloped, "\subset"] \arrow[rr, phantom, "\simeq"] \arrow[dr, phantom, sloped, "\simeq"] & & \left[\mathbf{rank} < \mathbf{3}\right] \arrow[ur, phantom, sloped, "\supset"] \arrow[dl, phantom, sloped, "\simeq"] & \\
& & \left[\mathbf{Lin}\right] \arrow[d, phantom, sloped, "\subset"] & & \\
& & \left[\overline{\mathbf{Cr}}_{22}\left(\mathbb{P}^{r-1}\right)\right] & &
\end{tikzcd}$$

The XJC-correspondence [Pirio-Russo]

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Theorem :

- the *'XJC-diagram'* below is commutative
- all maps are bijections
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The XJC-principle : any notion / construction / result concerning the **X**, **J** or **C**-world admits counterparts in the two other worlds

The *XJC*-principle I

$$\text{Let} \left\{ \begin{array}{l} X \in \mathbf{X}(3,3) \\ J \text{ cubic Jordan algebra} \\ \varphi \text{ (2,2) Cremona map} \end{array} \right\} \text{be corresponding objects}$$

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Theorem : the following assertions are equivalent :

- X is smooth
- J is semi-simple
- φ is semi-special

The *XJC*-principle II

The XJC-principle II

Let $\mathbf{J}$ be a Jordan algebra

- Radical $\mathbf{R} = \mathbf{Rad}(\mathbf{J}) < \mathbf{J}$
- Exact sequence : $(\mathscr{R}) \quad 0 \to \mathbf{R} \to \mathbf{J} \to \mathbf{J}/\mathbf{R} \to 0$

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Wedderburn's type theorem : [Albert-Penico $\sim$ 1950]

1. $\mathbf{J}_{ss} = \mathbf{J}/\mathbf{R}$ semi-simple
2. $(\mathscr{R})$ splits : $\mathbf{J} \simeq \mathbf{R} \ltimes \mathbf{J}_{ss}$
3. $\mathbf{R}$ solvable : $\quad 0 = \mathbf{R}^{(s)} \subsetneq \mathbf{R}^{(s-1)} \subsetneq \cdots \subsetneq \mathbf{R}^{(2)} \subsetneq \mathbf{R}^{(1)} = \mathbf{R}$

$$\text{with } \left(\frac{\mathbf{R}^{(i-1)}}{\mathbf{R}^{(i)}}\right)^2 = 0 \text{ for } i = 1, \ldots, s$$

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Theorem :

1. $X_{ss} = \Pi_{R_X}(X) = \mathbf{X}^{r_{ss}}(3,3)$ is semi-simple
2. $\Pi_{R_X} : X \dashrightarrow X_{ss}$ splits :

$$\exists\, \sigma : \mathbb{P}^{2r_{ss}+1} \overset{linear}{\hookrightarrow} \mathbb{P}^{2r+1} \quad \textit{such that} \quad \big(\Pi_{R_X} \circ \sigma\big)\big|_{X_{ss}} = \mathbf{Id}_{X_{ss}}$$

Theorem :

1. $X_{ss} = \Pi_{R_X}(X) \in \left\{ \textbf{Seg}(\mathbb{P}^1 \times Q),\ LG_3(\mathbb{C}^6),\ G_3(\mathbb{C}^6),\ OG_6(\mathbb{C}^{12}),\ E_7/P_7 \right\}$

2. Π_{R_X} splits :

$$\begin{tikzcd} X \arrow[rr, dashed, bend left, "\Pi_{R_X}"] & & X_{ss} \arrow[ll, hook', "\textit{linear embedding}" description] \end{tikzcd}$$

3. Π_{R_X} is *'solvable'* :

$$\begin{tikzcd} X = X^{(s)} \arrow[r, dashed, two heads] \arrow[rrrr, dashed, bend left, "\Pi_{R_X}"] & X^{(s-1)} \arrow[r, dashed, two heads] & \cdots \arrow[r, dashed, two heads] & X^{(2)} \arrow[r, dashed, two heads] & X^{(1)} = X_{ss} \end{tikzcd}$$

with for $i = 1, \ldots, s$:

- $X^{(i)} = \mathbf{X}^{r_i}(3,3)$
- $X^{(i)} \dashrightarrow\!\!\!\rightarrow X^{(i-1)}$ lin. projection, linear fibers, *admissible*

Some questions about $X = \mathbf{X}(n, 2n-3)$'s

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  - construction/classification of $X = \mathbf{X}(n, 2n-3)$'s
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  - relation between $\mathbf{Rad}(\mathbf{J})$ and $\mathbf{sing}(X_{\mathbf{J}})$?
  - desingularization of $X_{\mathbf{J}}$?
  - description of $\mathbf{Pic}(X_{\mathbf{J}})$?

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  - first case : $X_\mathbf{J} = \mathbf{Seg}\big(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\big) \subset \mathbb{P}^7$
- Same questions for $n \geq 3$, *e.g.* $n = 6$: $v_3(\mathbb{P}^3) = \mathbf{X}(6,9)$

Extending the XJC correspondence ?

Extending the *XJC* correspondence ?

- $\mathbf{A} = \mathbb{A} \otimes_{\mathbb{R}} \mathbb{C} \qquad \rightsquigarrow \qquad \mathbf{H}_3(\mathbf{A}) = \mathbf{Herm}_3(\mathbf{A})$

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- **Tits-Freudenthal magic square :**

B A	**R**	**C**	**H**	**O**
R	$\mathfrak{so}_3$	$\mathfrak{sl}_3$	$\mathfrak{sp}_6$	$\mathfrak{f}_4$
C	$\mathfrak{sl}_3$	$\mathfrak{sl}_3 \times \mathfrak{sl}_3$	$\mathfrak{sl}_6$	$\mathfrak{e}_6$
H	$\mathfrak{sp}_6$	$\mathfrak{sl}_6$	$\mathfrak{so}_{12}$	$\mathfrak{e}_7$
O	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

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$\mathfrak{sl}_3$	$\mathfrak{sl}_3 \times \mathfrak{sl}_3$	$\mathfrak{sl}_6$	$\mathfrak{e}_6$
$\mathfrak{sp}_6$	$\mathfrak{sl}_6$	$\mathfrak{so}_{12}$	$\mathfrak{e}_7$
$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

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- **Projective magic square :**

$v_2(Q^1)$	$\mathbb{P}(T_{\mathbb{P}^2})$	$G_2(\mathbb{C}^6)_0$	$\mathbf{O}\mathbb{P}^2_0$
$v_2(\mathbb{P}^2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$G_2(\mathbb{C}^6)$	$\mathbf{O}\mathbb{P}^2$
$LG_3(\mathbb{C}^6)$	$G_3(\mathbb{C}^6)$	$OG_6(\mathbb{C}^{12})$	E_7/P_7
F_4^{ad}	E_6^{ad}	E_7^{ad}	E_8^{ad}

Extending the *XJC* correspondence ?

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$\mathbf{X}(3,3)$'s

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$v_2(\mathbb{P}^2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$G_2(\mathbb{C}^6)$	$\mathbb{OP}^2$
$LG_3(\mathbb{C}^6)$	$G_3(\mathbb{C}^6)$	$OG_6(\mathbb{C}^{12})$	E_7/P_7
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$\mathbf{X}(3,3)$'s $\quad$ $\mathbf{Jordan}_3$ $\quad$ $\mathbf{Cr}_{22}$

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Geom	**Alg**	**Crem**
$\mathbf{X}(3,3)$'s	**Jordan**$_3$	**Cr**$_{22}$

Extending the XJC correspondence ?

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$v_2(\mathbb{P}^2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$G_2(\mathbb{C}^6)$	$\mathbf{O}\mathbb{P}^2$	Severi varieties		
$LG_3(\mathbb{C}^6)$	$G_3(\mathbb{C}^6)$	$OG_6(\mathbb{C}^{12})$	E_7/P_7	$\mathbf{X}(3,3)$'s	**Jordan**$_3$	**Cr**$_{22}$
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Geom	**Alg**	**Crem**
Severi varieties	Compos° algebras	
$\mathbf{X}(3,3)$'s	**Jordan**$_3$	**Cr**$_{22}$

Extending the XJC correspondence ?

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$LG_3(\mathbb{C}^6)$	$G_3(\mathbb{C}^6)$	$OG_6(\mathbb{C}^{12})$	E_7/P_7	$\mathbf{X}(3,3)$'s	**Jordan**$_3$	**Cr**$_{22}$
F_4^{ad}	E_6^{ad}	E_7^{ad}	E_8^{ad}		Structurable algebras rk 4	

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$LG_3(\mathbb{C}^6)$	$G_3(\mathbb{C}^6)$	$OG_6(\mathbb{C}^{12})$	E_7/P_7	**X**$(3,3)$'s	**Jordan**$_3$	**Cr**$_{22}$
F_4^{ad}	E_6^{ad}	E_7^{ad}	E_8^{ad}		Structurable algebras rk 4	**Cr**$^*_{33}$

Extending the *XJC* correspondence ?

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F_4^{ad}	E_6^{ad}	E_7^{ad}	E_8^{ad}	$?_{\mathbf{G}}$	Structurable algebras rk 4	$\mathbf{Cr}_{33}^*$

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- $?_{\mathbf{G}}$ = geometric characterization of adjoint varieties (??)

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$LG_3(\mathbb{C}^6)$	$G_3(\mathbb{C}^6)$	$OG_6(\mathbb{C}^{12})$	E_7/P_7	$\mathbf{X}(3,3)$'s	**Jordan$_3$**	$\mathbf{Cr}_{22}$
F_4^{ad}	E_6^{ad}	E_7^{ad}	E_8^{ad}	$?_{\mathbf{G}}$	Structurable algebras rk 4	$\mathbf{Cr}^*_{33}$

- $?_{\mathbf{G}}$ = geometric characterization of adjoint varieties (??)
- $\mathbf{Cr}^*_{33}$ = $(3,3)$ Cremona maps with F-locus = $\big(\text{quartic hypersurface}\big)^2$

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F_4^{ad}	E_6^{ad}	E_7^{ad}	E_8^{ad}	$?_{\mathbf{G}}$	Structurable algebras rk 4	$\mathbf{Cr}_{33}^*$

- $?_{\mathbf{G}}$ = geometric characterization of adjoint varieties (? ?)
- $\mathbf{Cr}_{33}^*$ = $(3,3)$ Cremona maps with F-locus = $\big(\text{quartic hypersurface}\big)^2$
- $\exists$? $\big($**Geom** – **Alg** – **Crem**$\big)$ correspondance for the 4th line ?

Extending the XJC correspondence.... further ?

				Geom	**Alg**	**Crem**
$v_2(Q^1)$	$\mathbb{P}(T_{\mathbb{P}^2})$	$G_2(\mathbb{C}^6)_0$	$\mathbf{O}\mathbb{P}^2_0$			
$v_2(\mathbb{P}^2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$G_2(\mathbb{C}^6)$	$\mathbf{O}\mathbb{P}^2$	Severi varieties	Compos° algebras	$?_{\mathbf{C}}$
$LG_3(\mathbb{C}^6)$	$G_3(\mathbb{C}^6)$	$OG_6(\mathbb{C}^{12})$	E_7/P_7	$\mathbf{X}(3,3)$'s	**Jordan**$_3$	**Cr**$_{22}$
F_4^{ad}	E_6^{ad}	E_7^{ad}	E_8^{ad}	$?_{\mathbf{G}}$	Structurable algebras rk 4	**Cr**$^*_{33}$

Extending the *XJC* correspondence.... further ?

			Geom	**Alg**	**Crem**
…	$G_2(\mathbb{C}^6)_0$	$\mathbf{O}\mathbb{P}^2_0$			
…	$G_2(\mathbb{C}^6)$	$\mathbf{O}\mathbb{P}^2$	Severi varieties	Compos° algebras	$?_{\mathbf{C}}$
…	$OG_6(\mathbb{C}^{12})$	E_7/P_7	$\mathbf{X}(3,3)$'s	$\mathbf{Jordan}_3$	$\mathbf{Cr}_{22}$
…	E_7^{ad}	E_8^{ad}	$?_{\mathbf{G}}$	Structurable algebras rk 4	$\mathbf{Cr}^*_{33}$

Extending the XJC correspondence.... further ?

			Geom	**Alg**	**Crem**
…	$G_2(\mathbb{C}^6)_0$	$\mathbf{O}\mathbb{P}^2_0$			
…	$G_2(\mathbb{C}^6)$	$\mathbf{O}\mathbb{P}^2$	Severi varieties	Compos° algebras	$?_{\mathbf{C}}$
…	$OG_6(\mathbb{C}^{12})$	E_7/P_7	$\mathbf{X}(3,3)$'s	**Jordan**$_3$	$\mathbf{Cr}_{22}$
…	E_7^{ad}	E_8^{ad}	$?_{\mathbf{G}}$	Structurable algebras rk 4	$\mathbf{Cr}^*_{33}$

- **[Freudenthal]** : magic square $\longleftrightarrow$ synthetic *“Freudental geometries”*

Extending the *XJC* correspondence.... further ?

			Geom	**Alg**	**Crem**	**Incid**
…	$G_2(\mathbb{C}^6)_0$	$\mathbf{O}\mathbb{P}^2_0$				
…	$G_2(\mathbb{C}^6)$	$\mathbf{O}\mathbb{P}^2$	Severi varieties	Compos° algebras	$?_{\mathbf{C}}$	
…	$OG_6(\mathbb{C}^{12})$	E_7/P_7	$\mathbf{X}(3,3)$'s	**Jordan**$_3$	$\mathbf{Cr}_{22}$	
…	E_7^{ad}	E_8^{ad}	$?_{\mathbf{G}}$	Structurable algebras rk 4	$\mathbf{Cr}^*_{33}$	

- **[Freudenthal]** : magic square ⟷ synthetic *"Freudental geometries"*

Extending the *XJC* correspondence.... further ?

			Geom	**Alg**	**Crem**	**Incid**
…	$G_2(\mathbb{C}^6)_0$	$\mathbf{O}\mathbb{P}^2_0$				
…	$G_2(\mathbb{C}^6)$	$\mathbf{O}\mathbb{P}^2$	Severi varieties	Compos° algebras	$?_{\mathbf{C}}$	
…	$OG_6(\mathbb{C}^{12})$	E_7/P_7	$\mathbf{X}(3,3)$'s	$\mathbf{Jordan}_3$	$\mathbf{Cr}_{22}$	
…	E_7^{ad}	E_8^{ad}	$?_{\mathbf{G}}$	Structurable algebras rk 4	$\mathbf{Cr}^*_{33}$	

- **[Freudenthal]** : magic square ⟷ synthetic *“Freudental geometries”*

Extending the XJC correspondence.... further ?

			Geom	**Alg**	**Crem**	**Incid**
…	$G_2(\mathbb{C}^6)_0$	$\mathbf{O}\mathbb{P}^2_0$				
…	$G_2(\mathbb{C}^6)$	$\mathbf{O}\mathbb{P}^2$	Severi varieties	Compos° algebras	$?_{\mathbf{C}}$	**[S-VanM]**
…	$OG_6(\mathbb{C}^{12})$	E_7/P_7	$\mathbf{X}(3,3)$'s	$\mathbf{Jordan}_3$	$\mathbf{Cr}_{22}$	
…	E_7^{ad}	E_8^{ad}	$?_{\mathbf{G}}$	Structurable algebras rk 4	$\mathbf{Cr}^*_{33}$	

- **[Freudenthal]** : magic square ⟷ synthetic *"Freudental geometries"*

Extending the XJC correspondence.... further ?

			Geom	**Alg**	**Crem**	**Incid**
…	$G_2(\mathbb{C}^6)_0$	$\mathbf{O}\mathbb{P}^2_0$				
…	$G_2(\mathbb{C}^6)$	$\mathbf{O}\mathbb{P}^2$	Severi varieties	Compos° algebras	$?_{\mathbf{C}}$	**[S-VanM]**
…	$OG_6(\mathbb{C}^{12})$	E_7/P_7	$\mathbf{X}(3,3)$'s	**Jordan**$_3$	**Cr**$_{22}$	
…	E_7^{ad}	E_8^{ad}	$?_{\mathbf{G}}$	Structurable algebras rk 4	**Cr**$^*_{33}$	

- **[Freudenthal]** : magic square ⟷ synthetic *"Freudental geometries"*

- [**Schillewaert Van Maldeghem**] *"On the varieties of the second row of the split Freudenthal-Tits Magic Square"* (2016)

Extending the XJC correspondence.... further ?

			Geom	Alg	Crem	Incid
…	$G_2(\mathbb{C}^6)_0$	$\mathbf{O}\mathbb{P}^2_0$				
…	$G_2(\mathbb{C}^6)$	$\mathbf{O}\mathbb{P}^2$	Severi varieties	Compos$^\circ$ algebras	$?_{\mathbf{C}}$	[S-VanM]
…	$OG_6(\mathbb{C}^{12})$	E_7/P_7	$\mathbf{X}(3,3)$'s	**Jordan**$_3$	$\mathbf{Cr}_{22}$	$?_{\mathrm{I}3}$
…	E_7^{ad}	E_8^{ad}	$?_{\mathbf{G}}$	Structurable algebras rk 4	$\mathbf{Cr}^*_{33}$	$?_{\mathrm{I}4}$

- **[Freudenthal]** : magic square ⟷ synthetic *“Freudental geometries”*
- [**Schillewaert Van Maldeghem**] *“On the varieties of the second row of the split Freudenthal-Tits Magic Square”* (2016)