Introduction to spherical varieties

N. Perrin

Contents

Part I

Some results on algebraic group actions

Chapter 1

Principal bundles

1.1 Reminders on linear algebraic groups

Let us recall few definitions. For more details we refer to our Wintersemester lecture or to one of the classical books on the subject [Bor91], [Hum75] and [Spr09].

Definition 1.1.1 *($\imath$) An algebraic group is a variety G which is a group such that the multiplication map $\mu : G \times G \to G$ and the inverse map $i : G \to G$ are morphisms.*

($\imath\imath$) An algebraic group is linear when it is an affine variety.

Let X be a variety with an action of G.

Definition 1.1.2 *The action of G on X is called rational or algebraic if the map $G \times X \to X$ induced by the action is a morphism.*

We will only deal with algebraic action therefore we shall only say action for an algebraic action.

Definition 1.1.3 *A subgroup H of G is called a closed subgroup if it is a closed subvariety of G.*

Theorem 1.1.4 *Let H be a closed subgroup of a linear algebraic group, then the quotient G/H has a unique structure of algebraic variety such that the quotient map $\pi : G \to G/H$ is a morphism. This map is flat and separable.*

In the next two sections we will prove that in several other situations we are able to prove the existence of quotients. This will be futher developped in chapter 3 and is one of the main topic of Geometric Invariant Theory (GIT), see [MuFoKi94] or [Dol94].

The very first situation we want to deal with is the case of finite groups.

1.2 Galois and unramified coverings

1.2.1 Existence of quotients by finite groups

Lemma 1.2.1 *Let A be a finitely generated algebra and G a finite group acting on A. Then A^G is finitely generated.*

Remark 1.2.2 We shall see several generalisations of this result in chapter 3.

Proof. The ring A is integrally closed over A^G. Indeed, for $a \in A$, we have the equation

$$\prod_{g \in G} (a - g \cdot a).$$

Let $a_1, \cdots a_n$ be generators of A as an algebra and let P_i be equations for the a_i over B. Let C be the subalgebra of B spanned by the coefficients of the P_i. The elements a_i are integral over C thus A is a finite module over C. But B is a sub-C-module of A and C is noetherian (because $C = k$[coeficients of the P_i]. Thus B is also finite over C and $B = C[b_1, \cdots, b_k]$, the result follows. □

We will prove the next proposition in a more general setting in chapter 3.

Proposition 1.2.3 *Let $X = \mathrm{Spec}(A)$ be affine and K be a finite group acting on X. Let $X/G = \mathrm{Spec}(A^G)$ and $\pi : X \to X/G$ be the induced morphism.*

(ı) The morphism π is constant on the orbits of K and any morphism $X \to Z$ constant on the K-orbits factors through X/G.

(ıı) The morphism π is finite.

(ııı) The variety X/G has the quotient topology.

(ıv) The fibers of π contain a unique closed orbit.

Remark 1.2.4 The above proposition will be true if we replace K by a reductive group G (see chapter 3). Note that as any orbit if finite and therefore closed, the last condition implies that X/G is the set of orbits justifying *a posteriori* the notation.

Corollary 1.2.5 *Assume that X is a variety such that any finite set is contained in an affine open subset (for example X is quasi-projective). Then there exists an algebraic variety X/G with a morphism $\pi : X \to X/G$ constant on the G orbits and such that for any morphism $\phi : X \to Z$ constant on the G-orbits, there exists a morphism $\psi : X/G \to Z$ with $\phi = \psi \circ \pi$.*

Proof. The condition tells us that X can be covered by affine subsets (U_i) stable under the action of X. The quotient exists on U_i and by the universal property is unique. Therefore on $U_i \cap U_j$ the two quotients coming from U_i and U_j are isomorphic. We can glue them to get the quotient of X. □

Remark 1.2.6 If we replace K by a reductive group G, then it is harder to find a mild condition to replace the condition that any finite subset of X is contained in an affine open subset which ensured the existence of an affine covering stable under the action. This is where the so called stability conditions are needed. We shall not enter this subject. For more details, see for example [MuFoKi94].

1.2.2 Unramified covers

Definition 1.2.7 *A morphism $f : X \to Y$ is called unramified if the condition $\Omega^1_{X/Y} = 0$ is satisfied. Unramified finite morphism are called an unramified cover.*

Remark 1.2.8 (ı) There are several equivalent conditions for a morphism to be unramified (see for example [Gro60, IV$_4$ Théorème 17.4.1]): $f : X \to Y$ is unramified if and only if one of the following equivalent conditions are satisfied

- the diagonal morphism $X \to X \times_Y X$ is open;
- for any point y in Y, the fiber of π over y is a disjoint union of reduced points (here we assume that k is algebraic closed).

(ıı) Note that such a finite cover is not necessarily étale. For this one need to add the assumption that the morphism is flat. Note also that if $f : X \to Y$ is a non ramified cover between varieties (reduced and irreducible schemes), then there is an open subset where f is flat and therefore étale.

(ııı) Note also that for $f : X \to Y$ a separable morphism between irreducible varieties (*i.e.* such that $k(X)/k(Y)$ is separable) then set of point in Y such that the morphism is separable is open and dense. We proved this statement for affine varieties last semester and one easily retricts to that case.

Let us state the following two facts that we shall use without proof.

Fact 1.2.9 *(ı) Unramified covers are stable under base change (see [Gro60,* IV$_4$ *Proposition 17.3.3]).*

(ıv) If $\pi : X \to X/K$ is the quotient of a variety X by a finite groups K, then π is an unramified cover if and only if K acts freely on X (see [Ser58, Section 1.4]). These unramified covers are called Galois covers.

Lemma 1.2.10 *Let $f : X \to Y$ be an unramified cover. There exists a Galois cover $\pi : Z \to Y$ such that X is a partial quotient of Z.*

Proof. Let n be the degree of f and consider X^n_Y the n-fold fibered product over X. Remove the (open because of the unramification and closed) subset of point fixed by at least one non trivial element in $\mathfrak{S}_n$ acting by permuting the points. Then the complementary Z maps to Y and this map $Z \to Y$ is an unramified covering. We have $X = Z/\mathfrak{S}_{n-1}$ and $Y = Z/\mathfrak{S}_n$ (the degree of the maps $Z/\mathfrak{S}_{n-1} \to X$ and $Z/\mathfrak{S}_n \to Y$ are both 1). □

Lemma 1.2.11 *Let $\pi : X \to X/K$ be a Galois cover of group K and let $f : Y \to X/K$ be a morphism, then the base change morphism $X \times_{X/K} Y \to Y$ is again a galois cover.*

Proof. Define the action on $X \times_{X/K} Y$ by $\sigma \cdot (x, y) = (\sigma(x), y)$ and let Z be the quotient. The map $X \times_{X/K} Y \to Y$ is constant on the K-orbits thus we have a morphism $Z \to Y$ and therefore a commutative diagram:

$$\begin{tikzcd} X \times_{X/K} Y \arrow[r] \arrow[d] & Y \\ Z. \arrow[ur] & \end{tikzcd}$$

Both maps starting from $X \times_{X/K} Y$ are of degre $|K|$ thus $Z \to Y$ is of degree 1 and therefore an isomorphism. □

1.3 Principal bundles

1.3.1 isotrivial bundles and special groups

Definition 1.3.1 *A principal bundle of group G over X is a morphism $f : P \to X$ with a faithful right action of G on P such that f is G-equivariant for the trivial action of G on X.*

Definition 1.3.2 *Let $f : P \to X$ be a G-principal bundle.*

(ı) The fibration is called trivial if there is an isomorphism $P \simeq X \times G$ such that f is the first projection.

(ıı) The fibration is called isotrivial if there exists an unramified cover $X' \to X$ such that the pull-back of the fibration to X' (obtained by base change) is trivial.

(iii) A fibration is called locally trivial if there exists a open covering $(U_i)_{i\in I}$ of X (for the Zariski topology) such that the restriction of the fibration to U_i is trivial for all $i \in I$.

(iv) A fibration is called locally isotrivial if there exists a open covering $(U_i)_{i\in I}$ (for the Zariski topology) and unramified maps $U'_i \to U_i$ such that pull-back to U'_i of the restriction of the fibration to U_i is trivial for all $i \in I$.

Remark 1.3.3 It can be proved, see [Gro95] that if G is a linear algebraic group, then any principal bundle is locally isotrivial.

Lemma 1.3.4 *Let $f : P \to X$ be a principal G-bundle and let $\pi : X' \to X$ be a Galois cover of group K.*

(i) Assume that the pull-back $X' \times_X P$ is trivial over X', then the action of K on $X' \times G$ is given by morphisms $f_\sigma : X' \to G$ for $\sigma \in K$ such that

$$\sigma \cdot (x, g) = (\sigma(x), f_\sigma(x)g).$$

(ii) Furthermore, the principal bundle $f : P \to X$ is trivial if (and only if) there is a morphism $a : X' \to G$ such that

$$f_\sigma(x) = a(\sigma(x))^{-1}a(x).$$

Remark 1.3.5 The classes of families $(f_\sigma)_{\sigma\in K}$ such that the above formula gives an action modulo the classes of such functions of the form $f_\sigma(x) = a(\sigma(x))^{-1}a(x)$ is a pointed set usually denoted by $H^1(K, \mathrm{Hom}(X', G))$.

Proof. (i) We know that the base change of the Galois cover is again a Galois cover thus we have an action of K on $X' \times G$. This action has to induce an equivariant map $X' \times G \to X'$ thus $\sigma(x, g) = (\sigma(x), a_\sigma(x, g))$. Furthermore, the action has to respect the G-action *i.e.*

$$\sigma(x, gh) = \sigma(x, g)h.$$

In particular $\sigma(x, g) = \sigma(x, e)g$ therefore $\sigma(x, g) = (\sigma(x), a_\sigma(x, e)g)$ proving (i) by setting $f_\sigma(x) = a_\sigma(x, e)$. Note that the associativity of the action gives the cocycle condition $f_{\sigma\tau} = f_\tau \circ \sigma \cdot f_\sigma$.

(ii) Consider the composition $X' \times G \to X' \times G \to (X \times G)/K$ whose first map is given by $(x, g) \mapsto (x, a(x)^{-1}g)$ and second map is given by the quotient of the action given by $f_\sigma(x) = a(\sigma(x))^{-1}a(x)$. This map is constant on the orbits of the action $\sigma \cdot (x, g) = (\sigma(x), g)$ and therefore factors through $X/K \times G$. The same argument gives the inverse map. □

Exercice 1.3.6 (i) Prove the converse statement of (ii) in the previous lemma.

(ii) Prove that is we define an action on $X' \times G$ as in the above lemma with the cocycle condition $f_{\sigma\tau} = f_\tau \circ \sigma \cdot f_\sigma$, then the quotient is a principal G-bundle over X.

Definition 1.3.7 *A group G is called special if any isotrivial principal bundle of group G is locally trivial.*

Remark 1.3.8 One can prove the following results on special groups, see [Ser58] and [Gro58].

(i) Any special group is connected and linear.

(ii) Connected solvable groups are special.

(iii) The groups GL, SL or Sp are special.

(iv) The groups PGL, SO or Spin are not special.

(v) A subgroup G of GL is special if and only if the fibration $\mathrm{GL} \to \mathrm{GL}/G$ is locally trivial.

(iii) There is a complete classification of special groups, see [Gro58].

Let us now prove that GL is special.

Theorem 1.3.9 *Any isotrivial* GL*-principal fibration is locally trivial.*

Proof. Let $P \to X$ be a locally isotrivial principal GL fibration. We thus have an unramified covering $\pi : X' \to X$ such that $X' \times_X P$ is trivial *i.e.* isomorphic to $X' \times \mathrm{GL}$. We want to prove that P is trivial. It is enough to prove this for $X' \to X$ a Galois covering in view of Lemma 1.2.10.

Let $X' \to X$ be a Galois covering trivialising P *i.e.* $X' \times_X P \simeq X' \times \mathrm{GL}$. We need to prove that locally the *cocycle* $(\varphi_\sigma)_{\sigma \in K} \in \mathrm{Hom}(X', \mathrm{GL})^K$ defining the action of K on $X' \times \mathrm{GL}$ comes from a boundary *i.e.* is of the form $\varphi_\sigma(x) = a(\sigma(x))^{-1} a(x)$ for $a \in \mathrm{Hom}(X', \mathrm{GL})$.

For this let $x \in X$, we will work locally around x. Consider the scheme $\pi^{-1}(x)$. This is a discrete disjoint union of 0-dimensional irreducible schemes. Let $A(x)$ be the (semi)local ring $\mathcal{O}_{X', \pi^{-1}(x)}$. Let x' be a point in $\pi^{-1}(x)$ and pick an element h in $\mathrm{GL}(A(x))$ with $h(x') = \mathrm{Id}$ and $h(y) = 0$ for $y \in \pi^{-1}(x)$ and $y \neq x'$. Define

$$a = \sum_{\sigma \in K} h \circ \sigma \cdot \varphi_\sigma \in \mathrm{GL}(A(x)).$$

We can now check the following equalities:

$$a \circ \sigma \cdot \varphi_\sigma = \sum_{\tau \in K} h \circ \tau \circ \sigma \cdot \varphi_\tau \circ \sigma \cdot \varphi_\sigma = \sum_{\tau \in K} h \circ \tau \circ \sigma \cdot \varphi_{\tau\sigma} = a$$

the second equality coming from the cocycle condition. $\square$

1.3.2 Existence of some quotients

Let G be an algebraic group and let H be a closed subgroup.

Proposition 1.3.10 *The quotient morphism $\pi : G \to G/H$ is a locally isotrivial H-principal bundle.*

In other words, there exists a covering of G/H by open subsets $(U_i)_{i \in I}$ and unramified coverings $\varphi_i : U_i' \to U_i$ such that the map $\pi : G \to G/H$ trivialises when pulled-back to U_i'.

Proof. Because the morphism is equivariant and G/H homogeneous, it is enough to check that there exists a non trivial open subset U of G/H with an unramified covering $\varphi : U' \to U$ such that the fibration π trivialises on U'.

Let G^0 be the connected component of G and let $H_0 = H \cap G^0$. The variety G^0/H_0 is irreducible and it is the connected component of G/H at the image of e the unit of G. Because π is separable, the extension $k(G^0) \to k(G^0/H_0)$ is separable. Consider the map on local rings $(\mathcal{O}_{G/H,\bar{e}}, \mathfrak{m}_{G/H,\bar{e}}) \to (\mathcal{O}_{G,e}, \mathfrak{m}_{G,e})$. Because the morphism is separable, the corresponding map $\mathfrak{m}_{G/H,\bar{e}}/\mathfrak{m}^2_{G/H,\bar{e}} \to \mathfrak{m}_{G,e}/\mathfrak{m}^2_{G,e}$ is injective. Pick a subspace $\mathfrak{n}$ of $\mathfrak{m}_{G,e}$ such that its image in the quotient is a supplementary of the image of this injection. Let I be the ideal in $\mathcal{O}_{G,e}$ spanned by $\mathfrak{n}$. Then the local ring $(\mathcal{O}_{G,e}/I, \mathfrak{m}_{G,e}/I)$ is the local ring of a subvariety X in G containing e whose tangent space is supplementary to that of H. The map $\pi : X \to G/H$ is therefore separable at e and $\dim X = \dim G/H$. Thus the map is quasi-finite and this implies that there exist an open dense subset U of G/H such that if we set $U' = X \cap \pi^{-1}(U)$, the morphism $\varphi = \pi|_{U'} : U' \to U$ is finite and thus an unramified covering (see last semester lecture Theorem 6.2.25).

We now only need to check that π trivialises when restricted to U'. We look at the pull-back diagram

$$\begin{tikzcd} U' \times_{G/H} G \arrow[r] \arrow[d] & G \arrow[d, "\pi"] \\ U' \arrow[r, "\varphi"] & U \subset G/H. \end{tikzcd}$$

We want to prove that $U' \times H$ is isomorphic to $U' \times_{G/H} G$. For this we check the universal property of the product. We have a natural map $\phi : U' \to G$ (the inclusion) such that $\varphi \circ \pi = \mathrm{Id}_U$. We may thus define maps $U' \times H \to G$ and $U' \times H \to U'$ by $(u, h) \mapsto \phi(u)h$ and $(u, h) \mapsto u$. This map obviously factors through the fibered product. If we have maps $a : Z \to G$ and $b : Z \to U'$ with $\pi \circ a = \varphi \circ b$ then we define $Z \to U' \times H$ by $z \mapsto (a(z), a(z)^{-1}\phi(b(z)))$. This concludes the proof. □

Corollary 1.3.11 *Let H be a closed subgroup of an algebraic group G and let X be a variety with a left action of H. Assume furthermore that any finite set of points in X is contained in an affine open subset (for example X quasi-projective). Let us define a right action of H on $G \times X$ by $h \cdot (g, x) = (gh, h^{-1}x)$.*

($\imath$) Then there exists an unique structure of algebraic variety on the set $G \times^H X$ of H-classes in $G \times X$. The morphism $G \times X \to G \times^H X$ is flat and separable.

($\imath\imath$) There is an action of G on $G \times^H X$.

($\imath\imath\imath$) There is a G-equivariant morphism $G \times^H X \to G/H$ which is isotrivial with fibers isomorphic to X.

Proof. The quotient being the solution of an universal problem. If it exists it is unique therefore we only need to construct it locally. By uniqueness the resulting quotients will glue together.

Since the map $\pi : G \to G/H$ it is locally isotrivial, we first consider an open subset U and an unramified covering $\varphi : U' \to U$ such that we have a trivialisation $\pi^{-1}(U') \simeq U' \times H$ and thus we get an isomorphism $\pi^{-1}(U') \times X = U' \times H \times X$. Furthermore, the action is given by $h \cdot (u, h', x) = (u, h'h, h^{-1}x)$. In particular on this open set, there is a quotient isomorphic to $U' \times X$. Indeed, we have a morphism $\phi : U' \times H \times X \to U' \times X$ defined by $(u, h, x) \mapsto (u, hx)$. This morphism is contant on the H-orbits. Furthermore, for any morphism $\psi : U' \times H \times X \to Z$ which is constant on the H-orbits, we may define $\bar\psi : U' \times X \to Z$ simply by composition with the map $U' \times X \to U' \times H \times X$ given by $(u, x) \mapsto (u, e, x)$. This map is a section of the quotient map ϕ thus $\bar\psi$ factorises ψ.

To prove the existence of the quotient on U, we only need to *descent* from U' to U. But the morphism $\varphi : U' \to U$ is an unramified cover. By taking another covering, we may assume that $U = U'/K$ with K a finite group (see Lemma 1.2.10). We may thus assume that $U' \to U$ is given as the quotient by a finite group K. Therefore the pull-back $U' \times H \to \pi^{-1}(U)$ is also given by a quotient of an action of K. Because of the compatibility with the first projection and the action of H, the action is given by $\sigma \cdot (u, h) = (\sigma(u), f_\sigma(u)h)$ with $f_\sigma : U' \to H$ a morphism. We may therefore define an action of K on $U' \times X$ by $\sigma \cdot (u, x) = (\sigma(u), f_\sigma(u) \cdot x)$. By our assumption on X there is a quotient of $U' \times X$ by K. For this quotient we have the diagram

$$\begin{tikzcd}
U' \times H \times X \arrow[r] \arrow[d] & \pi^{-1}(U) \times X \arrow[d, dashed] \\
U' \times X \arrow[r] \arrow[d] & (U' \times X)/K \arrow[d, dashed] \\
U' \arrow[r] & U
\end{tikzcd}$$

that we want to complete with the dashed arrows to get a commutative diagram. But the composition morphism $U' \times H \times X \to U' \times X \to (U' \times X)/K$ is constant on the K-orbits thus factors through $(U' \times H \times X)/K = \pi^{-1}(U) \times X$. This gives the first right vertical arrow. Now because the top square is commutative we get that the map $\pi^{-1}(U) \times X$ is constant on the H-orbits. We need to check that it satisfies the universal property of the quotient. If $\psi : \pi^{-1}(U) \times X \to Z$ is constant on the H-orbits, then the composition $U' \times H \times X \to \pi^{-1}(U) \times X \to Z$ is constant on the H-orbits and thus factors through $U' \times X$. Furthermore the above composition and therefore the induced map $U' \times X \to Z$ is constant on the K-orbits thus it factors through $(U' \times X)/K$. The existence of the last dashed arrow comes from the universal property of the quotient $U = U'/K$.

Again, because of the universal property of the quotient, the quotients on open subsets with trivialisation on an unramified covering will patch together to give a global quotient $G \times^H X$ which furthermore has a morphism to G/H (because it is the case locally) which is locally isotrivial.

Note that the morphism $G \times G \times X \to G \times X$ defined by left multiplication on G is equivariant under the H-action thus by the same construction for the H-action on $G \times G \times X$, this induces a morphism $G \times G \times^H X \to G \times^H X$ and it is easy to check that this morphism defines an action. $\square$

Remark 1.3.12 (i) If H is a parabolic subgroup, then the map $G \to G/H$ is locally trivial for the Zariski topology and the result is even easier.

(ii) This result is a special case of faithfully flat descent (see [Gro95]): indeed the map $G \to G/H$ is faithfully flat and there is a locally trivial fibration with fiber isomorphic to X over G: the trivial fibration $G \times X \to G$ therefore by faithfully flat descent, there exists a fibration $G \times^H X \to G/H$ with fibers isomorphic to X such that the following diagram is Cartesian:

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & G \arrow[d] \\ G \times^H X \arrow[r] & G/H. \end{tikzcd}$$

Corollary 1.3.13 *Let $X' \to X$ be a Galois covering of Galois group K and let $\rho : K \to \mathrm{GL}(V)$ be a representation of K. Consider the action of K on $X' \times V$ defined by $\sigma(x, v) = (\sigma(x), \rho(\sigma)(v))$.*

Then the quotient $X' \times^K V := (X' \times V)/K$ is a vector bundle over $X'/K = X$ i.e. *locally trivial.*

Proof. Consider the trivial principal $\mathrm{GL}(V)$ bundle $X' \times \mathrm{GL}(V)$ and the action of K on it induced by the representation ρ. The quotient $X' \times^K \mathrm{GL}(V)$ has a morphism to $X'/K = X$ and is an isotrivial principal $\mathrm{GL}(V)$ bundle. By the above result, we may assume that this principal bundle is trivial over X (by restriction to an open subset). The above fibration $X' \times^K V \to X'/K$ is obtained from $X' \times^K \mathrm{GL}(V) \to X'/K$ as follows:

$$X' \times^K V = (X' \times^K \mathrm{GL}(V)) \times^{\mathrm{GL(V)}} V \simeq (X'/K) \times (\mathrm{GL}(V) \times^{\mathrm{GL(V)}} V) \simeq X'/K \times V.$$

Proving the result. $\square$

Example 1.3.14 A very special case of the above construction is the following. Let V be a linear representation of H, then $G \times^H V \to G/H$ is a vector bundle over G/H with fibers isomorphic to V. This is the very first example of linearised vector bundle.

Note that if the action of H on V extends to an action of G, then the bundle is trivial. Indeed, we have the trivialisation morphisms given by $(\bar{g}, v) \mapsto \overline{(g, g^{-1} \cdot v)}$ and $\overline{(g, v)} \mapsto (\bar{g}, g \cdot v)$.

Note also that we only proved that the fibration $G \times^H V \to G/H$ is isotrivial. But as GL is special it is locally trivial and thus a vector bundle.

Chapter 2

Linearisation of line bundles

2.1 First definitions

Let G be a linear algebraic group and let X be a variety acted on by G.

Definition 2.1.1 *A G-linearisation of a vector (line) bundle $\pi : L \to X$ is a G-action on L given by $\Phi : G \times L \to L$ such that*

($\imath$) the morphism $\pi : M \to X$ is G-equivariant and

($\imath\imath$) the action of G on the fibers is linear i.e. *for all $x \in X$ and $g \in G$, the map $\phi_{g,x} : L_x \to L_{gx}$ is linear.*

We shall mainly consider G-linearised line bundles but in the following lemma we get G-linearised vector bundles as well.

Lemma 2.1.2 *($\imath$) Let V be a representation of H, then $G \times^H V \to G/H$ is a G-linearised vector bundle.*

($\imath\imath$) In particular for $\chi \in X^*(H)$ a character of H we get a linearised line bundle $L_\chi = G \times^H k$ with action $h \cdot (g, z) = (gh, \chi(h)^{-1}z)$. Any linearised line bundle is of that form. We thus have a group morphism

$$X^*(H) \to \mathrm{Pic}(G/H)$$

whose image is the subgroup $\mathrm{Pic}_G(G/H)$ of linearised line bundles.

Proof. ($\imath$) As observed in the construction of $G \times^H V$, this variety has a G-equivariant map to G/H whose fibers are isomorphic to V and furthermore the local trivialisation shows that the action is linear on the fibers.

($\imath\imath$) Assume conversely that $\pi : L \to G/H$ is a G-linearised line bundle. Then H acts linearly on the fiber L_e over the class of the identity element e. In particular, we get a character of H via the action map $\chi : H \to \mathrm{GL}(L_e)$ defined by $h \cdot l = \chi(h)l$ for $l \in L_e$. We may consider the morphism $G \times L_e \to L$ defined by $(g, l) \mapsto g \cdot l$ and the action of H on $G \times L_e$ defined by $h \cdot (g, l) \mapsto (gh, \chi^{-1}(h)l)$. The map is then constant on the H-orbits thus factors through $L_\chi \to L$. The morphism $G \times L_e \to L$ being surjective, so is $L_\chi \to L$ and therefore this is an isomorphism of line bundles. $\square$

We denote by p_X and p_G the projections from $G \times X$ to X and G respectively and by $\varphi : G \times X \to X$ and $\Phi : G \times L \to L$ and action of G on X and a linearisation of this action on a line bundle L.

Lemma 2.1.3 *($\imath$) For $\Phi : G\times L \to L$ a linearisation of a line bundle, there is a commutative diagram:*

$$\begin{tikzcd} G \times L \arrow[r, "\Phi"] \arrow[d, "\mathrm{Id}\times\pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\varphi"] & X \end{tikzcd}$$

which is furthermore cartesian. In otherwords, we have an isomorphism of line bundles

$$p_X^*(L) \simeq \varphi^*(L).$$

($\imath\imath$) The restriction of Φ to $\{e\} \times L$ is the identity.

Proof. The commutativity of the diagram is equivalent to the fact that $\pi : L \to X$ is equivariant. To prove that the diagram is cartesian, let us check the universal property of the product. Let $\alpha : Z \to L$ and $\beta : Z \to G \times X$ such that $\pi \circ \alpha = \varphi \circ \beta$. We define $\gamma : Z \to G \times L$ by $\gamma = (p_G \circ \beta, \Phi(i(p_G \circ \beta), \alpha))$. We need to check the equalities $(\mathrm{Id} \times \pi) \circ \gamma = \beta$ and $\Phi \circ \gamma = \alpha$. We compute $\pi(\Phi(i(p_G \circ \beta), \alpha)) = \varphi(\pi(\Phi(i(p_G \circ \beta), \alpha))) = \varphi(i(p_G \circ \beta), \varphi(p_G \circ \beta, p_X \circ \beta)) = p_X \circ \beta$ giving the first equality. The second equality is obvious.

The last assertion is obvious. □

Proposition 2.1.4 *Conversely, assume that L is a line bundle together with a morphism $\Phi : G\times L \to L$ satisfying the two conditions. (a) There is a commutative diagram:*

$$\begin{tikzcd} G \times L \arrow[r, "\Phi"] \arrow[d, "\mathrm{Id}\times\pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\varphi"] & X \end{tikzcd}$$

which is furthermore cartesian. In otherwords, we have an isomorphism of line bundles

$$p_X^*(L) \simeq \varphi^*(L).$$

(b) The restriction of Φ to $\{e\} \times L$ is the identity and $\Phi(g, \cdot) : L \to L$ maps the zero section to itself for all $g \in G$.

Then Φ is a linearisation of L.

Proof. We only need to check that this defines an action which is linear on the fibers. For $g \in G$, the morphism $\Phi(g, \cdot) : L_x \to L_{gx}$ is bijective and map 0 to 0. It is therefore a linear isomorphism. We thus has a function $f : G \times G \times L \to \mathbb{G}_m$ such that for all $g\, h \in G$ and $z \in L$ we have the equality

$$\Phi(gh, z) = f(g, h, z)\Phi(g, \Phi(h, z)).$$

But looking at trivialisations, we easily see that this function is regular.

Lemma 2.1.5 *Let X and Y be irreducible varieties, then the map $\mathcal{O}(X)^\times \times \mathcal{O}(Y)^\times \to \mathcal{O}(X \times Y)^\times$ is surjective.*

Proof. Let x_0 and y_0 be normal points on X and Y and let $f \in \mathcal{O}(X \times Y)^\times$. We may define the function $F : X \times Y \to \mathbb{G}_m$ by

$$f(x, y) = f(x_0, y_0)^{-1} f(x, y_0) f(x_0, y).$$

We only need to prove that $f = F$. For this it is sufficient to prove that these functions coincide in a neighbourhood $U \times V$ of (x_0, y_0). We may therefore assume that X and Y are affine and normal.

Let $\bar{X}$ and $\bar{Y}$ be normal projective compactifications of X and Y such that they are dense open subsets in these compactifications. We may consider f and F as rational functions on $\bar{X} \times \bar{Y}$ and the divisor $\div(\frac{f}{F})$ has a support contained in $((\bar{X} \setminus X) \times \bar{Y}) \cup (\bar{X} \times (\bar{Y} \setminus Y))$. It is therefore a sum of divisors of the form $D \times \bar{Y}$ and $\bar{X} \times D'$ with D and D' irreducible components of the boundary of X and Y. If $\frac{f}{F}$ has a zero on a divisor $D \times \bar{Y}$, then it is regular on an open set meeting $D \times \{y_0\}$. But $f(x, y_0) = F(x, y_0)$ for all $x \in X$ and therefore also on $\bar{X}$ leading to a contradiction. The same argument prove that $\frac{f}{F}$ has no pole and is therefore in $\mathcal{O}(\bar{X} \times \bar{Y})^\times$. It has to be a constant and the value a (x_0, y_0) proves tat this constant is 1. $\square$

Exercice 2.1.6 Prove the following consequence of this lemma: any invertible function $f \in \mathcal{O}(G)^\times$ over a group G with $f(e) = 1$ is a character.

This lemma implies that the function f above has the form $f(g, h, z) = r(g)r(h)t(z)$ for some functions $r \in \mathcal{O}(G)^\times$, $s \in \mathcal{O}(G)^\times$ and $t \in \mathcal{O}(L)^\times$. Now the equality $\Phi(e, z) = z$ gives the equalities

$$r(e)s(h)t(z) = 1 \text{ and } r(g)s(e)t(z) = 1.$$

for all $g, h \in G$ and $z \in L$. We then get the equalities

$$\begin{aligned} f(g, h, z) = \ & r(g)s(h)t(z) = (r(g)s(h)t(z))(r(e)s(e)t(z)) \\ & (r(g)s(e)t(z))(r(e)s(h)t(z)) = 1. \end{aligned}$$

The result follows. $\square$

Corollary 2.1.7 *A line bundle L over X with a G-action $\varphi : G \times X \to X$ is linearisable is and only if there exists an isomorphism $\varphi^*(L) \simeq p_X^*(L)$.*

Proof. The former Lemma implies that if L is linearisable, then such an isomorphism exists. Conversely such an isomorphism induces a pull-back diagram

$$\begin{tikzcd} G \times L \arrow[r, "\Phi"] \arrow[d, "\mathrm{Id}\times\pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\varphi"] & X \end{tikzcd}$$

such that for all $g \in G$ the map $\Phi(g, \cdot)$ sends the zero section to itself (because it is a pull-back diagram). Furthermore, the restriction of ϕ to $\{e\} \times L$ is an isomorphism of L. Therefore there is a regular function $\lambda : X \to \mathbb{G}_m$ defined by $\lambda(\pi(z)) \cdot z = \Phi(e, z)$. Replacing Φ by $\lambda^{-1}\Phi$ we obtain a morphism satisfying the conditions of the previous proposition and the result follows. $\square$

2.2 The Picard group of homogeneous spaces

Let us recall the following fact on reductive algebraic groups.

Fact 2.2.1 *A reductive algebraic group contains an open dense affine subset isomorphic to $\mathbb{G}_a^p \times \mathbb{G}_m^{2q}$.*

Proof. Use Bruhat decomposition to write the dense open cell as UTU^- where T is a maximal torus and U a maximal unipotent subgroup with U^- its opposite. Then we have seen that $T \simeq \mathbb{G}_m^q$ while $U \simeq \mathbb{G}_a^q \simeq U^-$. □

Remark 2.2.2 The fact that an open subset of G is isomorphic to a product of $\mathbb{G}_a$ and $\mathbb{G}_m$ is true for any connected algebraic group: by a result of Grothendieck [Gro58], if G is connected then as variety we have $G \simeq R(G) \times (G/R(G))$ and $G/R(G)$ is reductive. The group $R(G)$ is unipotent and one can prove that it is isomorphic to $\mathbb{G}_a^s$.

Let us prove the following result.

Lemma 2.2.3 *Let X be a normal variety with an action of G and let L be a line bundle on $G \times X$. Then we have an isomorphism*

$$L \simeq p_G^*(L|_{G\times\{x_0\}}) \otimes p_X^*(L|_{\{e\}\times X}),$$

for some $x_0 \in X$.

Proof. Let $M = L^{-1} \otimes p_G^*(L|_{G\times\{x_0\}}) \otimes p_X^*(L|_{\{e\}\times X})$.

Let us assume first that X is smooth. The Picard group of $G \times X$ is then isomorphic to the group of Weil divisors $\mathrm{Cl}(G \times X)$ (cf. [Har77, Chapter II, section 6]). By *loc. cit.* Proposition II.6.6, the pull-back gives identifications $\mathrm{Cl}(\mathbb{G}_a \times X) \simeq \mathrm{Cl}(X)$ and $\mathrm{Cl}(\mathbb{G}_m \times X) \simeq \mathrm{Cl}(X)$. Therefore on an affine open space U of G, we have $M|_U \simeq \mathcal{O}_U$.

Therefore the divisor class corresponding to M is represented by a divisor D supported in $(G \setminus U) \times X$. Therefore we have $D = p_G^{-1}(D')$ with D' a divisor in G. We thus have

$$M \simeq p_G^*(M_{G\times\{x_0\}})$$

but $M_{G\times\{x_0\}}$ is trivial therefore so is M.

If X is normal but not necessarily smooth, then X^{sm} the smooth locus of X has complementary in codimension 2 and every function defined on X^{sm} extends to a regular function on X. By the previous argument $M|_{X^{\mathrm{sm}}}$ is trivial therefore so is M. □

Proposition 2.2.4 *Let L be a line bundle on G and denote by $L^\times$ the complement of the zero section. Then $L^\times$ has a structure of a linear algebraic group such that the following two conditions hold.*

(ı) The projection $p : L \to G$ induces a group morphism $L^\times \to G$ with kernel central in $L^\times$ and isomorphic to $\mathbb{G}_m$.

(ıı) The line bundle L is $L^\times$-linearisable.

Proof. We need to define the multiplication map $\mu : L^\times \to L^\times$. Let us denote by $m : G \times G \to G$ the multiplication in G and by p_1 and p_2 the two projections on $G \times G$. We know by the previous lemma that there is an isomorphism $\psi : p_1^*(L) \otimes p_2^*(L) \to m^*(L)$. We construct via ψ a morphism $\mu : L \times L \to L$ as the composition:

$$\begin{tikzcd} L \times L \arrow[r] \arrow[d, "p\times p"'] & p_1^*(L) \otimes p_2^*(L) \arrow[r, "\psi"] \arrow[d] & m^*(L) \arrow[r] \arrow[d] & L \arrow[d, "p"] \\ G \times G \arrow[r, "\mathrm{Id}"] & G \times G \arrow[r, "\mathrm{Id}"] & G \times G \arrow[r, "m"] & G. \end{tikzcd}$$

If M is the locally free $k[G]$-module corresponding to L, then this map is given as follows

$$M \xrightarrow{m^\sharp} M \otimes^m_{k[G]} k[G \times G] \xrightarrow{\psi^\sharp} M \otimes^{p_1}_{k[G]} k[G \times G] \otimes_{k[G\times G]} M \otimes^{p_2}_{k[G]} k[G \times G] \longrightarrow M \otimes_k M$$

where $\otimes^f_{k[G]} k[G \times G]$ is the tensor product using the map $f : G \times G \to G$ and where $m^\sharp$ is given by the formula $m^\sharp((m \otimes (a \otimes b)) \otimes (m' \otimes (a' \otimes b'))) = aa'm \otimes bb'm'$.

We want to modify ψ so that μ will induce the desired multiplication map. Let us fix an identification $L_e \simeq \mathbb{G}_a$ and fix $1 \in L_e$ be the element corresponding to the unit in $\mathbb{G}_a$. The composition $L \to L \times \{1\} \xrightarrow{\mu} L$ is an isomorphism (check on the modules!) inducing the identity on G *i.e.* an isomorphism of vector bundles. Therefore there is an invertible function $r \in k[G]^\times$ with

$$\mu(l, 1) = r(p(l))l$$

for all $l \in L$. The same argument gives an invertible function $s \in k[G]^\times$ with

$$\mu(1, l') = s(p(l'))l'$$

for all $l' \in L$. Let us replace ψ by $\psi \circ (r^{-1} \otimes s^{-1})$ and denote by $\Delta : L \times L \to L$ the corresponding morphism. Then $1 \in L_e$ is a unit for this morphism:

$$\Delta(l, 1) = \mu(r^{-1}(p(l))l, 1) = l \text{ and } \Delta(1, l') = \mu(1, s^{-1}(p(l'))l') = l'.$$

Let us now prove that Δ is associative. Indeed, by linearity (use the same arguments as in Proposition 2.1.4) of the maps, there is an invertible function $t \in k[G \times G \times G]^\times$ with

$$\Delta(\mathrm{Id} \times \Delta)(l, l', l'') = t(p(l), p(l'), p(l''))\Delta(\Delta \times \mathrm{Id}(l, l', l'')).$$

As usual we can write $t(g, g', g'') = u(g)v(g')w(g'')$ with $u, v, w \in k[G]^\times$. We have (because 1 is a unit) the equality $t(e, e, e) = 1$ therefore we may assume $u(e) = v(e) = w(e)$ (replace u by $u(e)^{-1}u$ and do the same for v and w). Because 1 is a unit we have $t(g, e, e) = t(e, g', e) = t(e, e, g'') = 1$. We obtain the equalities $u(g) = v(g') = w(g'') = 1$ for all $g, g', g'' \in G$ therefore Δ is associative. Furthermore the morphism Δ being bilinear, the subset $L^\times$ is contained in the locus of invertible elements this proves the existence of the group structure on $L^\times$.

Furthermore by construction the map $p : L \to G$ induces a group morphism $L^\times \to G$. The kernel of this map is $L_e^\times \simeq \mathbb{G}_m$. This groups acts by scalar multiplication on the fibers and is therefore in the center of $L^\times$.

Finally, the restriction of Δ give a group action $L^\times \times L \to L$ which is a linearisation of the action of $L^\times$ on G ($L^\times$ acts on G via G and the map $L^\times \to G$), in other words the central kernel $L_e^\times \simeq \mathbb{G}_m$ acts trivially on G. □

Corollary 2.2.5 *Let G be a linear algebraic group and let $L \in \mathrm{Pic}(G)$. There exists a finite covering $\pi : G' \to G$ such that π^*L is trivial.*

Proof. We may assume G to be connected since all the connected components of G are isomorphic.

We consider $L^\times$ as a linear algebraic group and denote by $L_e^\times$ the kernel of the map $L^\times \to G$. Choose a representation V of $L^\times$ such that $L_e^\times$ does not act trivially (for example take a faithful representation, see [Spr09, Theorem 2.3.7]). Replacing V by a submodule W, we may assume that $L_e^\times$ acts by a nontrivial scalar on W (take an eigenspace V_χ of V with χ a non trivial character of $L_e^\times \simeq \mathbb{G}_m$, because $L_e^\times$ is central, this is again a sub-$L^\times$-module). We may furthermore assume that the representation V is also faithful on the Lie algebra level (*i.e.* $d_e\rho : \mathrm{Lie}(L^\times) \to \mathfrak{gl}(V)$ is injective, see [Spr09, Lemma 5.5.1]). The character χ corresponds to an integer n, the action is $t \cdot v = \chi(t)v = t^n v$. By the condition on the Lie algebra, the integer n has to be prime to $p = \mathrm{char}(k)$. Denote by $\rho : L^\times \to \mathrm{GL}(W)$ this new representation.

Let G' be the identity component of $\rho^{-1}(\mathrm{SL}(W))$. Then the restriction $\pi : G' \to G$ of $p : L^\times \to G$ is surjective and with finite fibers (the dimension of G' is $\dim L^\times - 1 = \dim G$ and G is connected). The map π is quasi-finite and affine thus finite. Note that the kernel of the map $\pi : G' \to G$ is isomorphic to the intersection $\mathbb{G}_m \cap \rho^{-1}(\mathrm{SL}(W))$ and therefore is isomorphic to the finite group of n-th root of the unit and therefore is a *reduced* finite group K and the map π is unramified.

Now the restriction of the action of $L^\times$ to G' induces a G'-linearisation of L. Therefore the line bundle π^*L is G'-linearised on G' and by Lemma 2.1.2 it is trivial. □

Corollary 2.2.6 *The Picard group* $\mathrm{Pic}(G)$ *is finite.*

Proof. Let $L \in \mathrm{Pic}(G)$ and let $\pi : G' \to G$ be a covering such that L is linearised and $\pi^*(L)$ is trivial. If K is the kernel of π, then K acts on L and if n is the order of K, the action of K on $L^{\otimes n}$ is trivial. Therefore G acts on $L^{\otimes n}$ and thus $L^{\otimes n}$ is G-linearisable. As above we get that $L^{\otimes n}$ is trivial. Therefore $\mathrm{Pic}(G)$ is a torsion group.

We are left to prove that $\mathrm{Pic}(G)$ is of finite type. But we have seen that there is an open subset U of G isomorphic to $\mathbb{G}_a^p \times \mathbb{G}_m^q$. We thus have $\mathrm{Pic}(U) = 0$ and the non trivial elements in $\mathrm{Pic}(G)$ are supported by divisors corresponding to irreducible components of $G \setminus U$. There are only finitely many of them concluding the proof (note that we use here the fact that G is smooth and thus that $\mathrm{Pic}(G)$ coincides with $\mathrm{Cl}(G)$ the group of Weil divisors. □

Corollary 2.2.7 *There is a finite covering* $\pi : G' \to G$ *such that* $\mathrm{Pic}(G') = 0$.

In particular if G *is simply connected, then* $\mathrm{Pic}(G) = 0$.

Proof. Again we may assume that G (and G') are connected.

By what we proved, it is enough to check that if $\pi : G' \to G$ is a finite covering, then the morphism $\pi^* : \mathrm{Pic}(G) \to \mathrm{Pic}(G')$ is surjective. Let $L' \in \mathrm{Pic}(G')$ and let $\phi : G'' \to G'$ be a finite covering such that L' is G''-linearisable and $\phi^*(L')$ is trivial. Let K be the kernel of ϕ. As L' is G''-linearisable, there exists a representation k_χ of K such that $L' \simeq G'' \times^K k_\chi$.

Let K' be the kernel of the composition $\pi \circ \phi : G'' \to G$. Then K is a subgroup of K'. But K' is finite and abelian, thus we may extend the representation of K in k_χ in a K'-representation k_η (we act by roots of the unit). We may set $L = G'' \times^{K'} k_\eta$. Then $L' = \pi^*L$ and the result follows.

If G is simply connected, then there are only trivial finite coverings. □

Remark 2.2.8 Note that we used here the fact that any non trivial covering of G comes from an abelian kernel or equivalentely that the fundamental group of G is abelian. Here is a proof for $\mathrm{char}(k) = 0$.

Let $f : [0,1] \to G$ and $g : [0,1] \to G$ be loops in G with $f(1) = g(1) = e$. Define the product $f \cdot g$ of loops by $(f \cdot g)(t) = f(t)g(t)$ and the concatenation of loops by:

$$(f \tilde{\cdot} g)(e^{2\pi i x}) := \begin{cases} f(e^{4\pi i x}) & , \quad 0 \leq x \leq \frac{1}{2} \\ g(e^{4\pi i x}) & , \quad \frac{1}{2} \leq x \leq 1 \ . \end{cases}$$

We construct a homotopy of loops $f \tilde{\cdot} g \approx f \cdot g \approx g \tilde{\cdot} f$. For each $-1 \leq \epsilon \leq 1$, let $p_\epsilon : [0,1] \to [0,1] \times [0,1]$ be a path in the unit square starting at $(0,0)$ and ending at $(1,1)$, such that p_{-1} goes along the left and top boundaries, p_0 goes along the diagonal, and p_1 goes along the bottom and right boundaries. Define $H : [0,1] \times [0,1] \to K$ by $H(x_1, x_2) := f(e^{2\pi i x_1})\, g(e^{2\pi i x_2})$. Then defining $h_\epsilon : S^1 \to K$ by $h_\epsilon(e^{2\pi i x}) := H(p_\epsilon(x))$ gives a continuous family of loops with $h_{-1} = f \tilde{\cdot} g$, $h_0 = f \cdot g$, and $h_1 = g \tilde{\cdot} f$. $\square$

Proposition 2.2.9 *Let G be a connected algebraic group and let H be a closed subgroup and denote by $\pi : G \to G/H$ the quotient map. Let us also denote by $\psi : X^*(H) \to \mathrm{Pic}(G/H)$ the group morphism defined in Lemma 2.1.2. Then we have an exact sequence*

$$X^*(G) \xrightarrow{\mathrm{res}} X^*(H) \xrightarrow{\psi} \mathrm{Pic}(G/H) \xrightarrow{\pi^*} \mathrm{Pic}(G).$$

Recall that the image of ψ is the subgroup of linearisable line bundles.

Proof. Let us start with the exactness at $X^*(H)$. If χ is a character of G, then we have already seen that the line bundle $L_\chi = G \times^H k_\chi$ is trivial (see Example 1.3.14). Conversely, if χ is a character of H such that $L_\chi = G \times^H k_\chi$ is trivial, then we have a trivialisation $\psi : G/H \times k \simeq L_\chi$ but G acts on $G \times^H k_\chi$ therefore it acts on k and this action extends the action of H.

Consider the exactness at $\mathrm{Pic}(G/H)$. Let $L \in \mathrm{Pic}(G/H)$ and denote by $\varphi : G \times G/H \to G/H$ the action of G. By Lemma 2.2.3 we have an isomorphism $\varphi^* L \simeq p_G^* M \otimes p_{G/H}^* N$ with $M = \varphi^* L|_{G \times \{\bar{e}\}} = \pi^* L$ and $N = \varphi^* L|_{\{e\} \times G/H} = L$. In other words, we have an isomorphism

$$\varphi^* L \simeq p_G^* \pi^*(L) \otimes p_{G/H}^* L.$$

The image of ψ is composed of the G-linearisable line bundles. We know that L is linearisable if and only if there is an isomorphism $\varphi^* L \simeq p_{G/H}^* L$ which in turn is equivalent to the fact that $\pi^*(L)$ is trivial. $\square$

Corollary 2.2.10 *Assume that G is semisimple and simply connected. Let P be a parabolic subgroup of G, then we have* $\mathrm{Pic}(G/P) \simeq X^*(P)$.

In particular $\mathrm{rank}(\mathrm{Pic}(G/B)) = \mathrm{rank}(G)$ *for B a Borel subgroup of G.*

Proof. If G is semisimple, then $X^*(G) = 0$ (we have $G = D(G)$ for example) and if it is semisimple then $\mathrm{Pic}(G) = 0$. The result follows from the above exact sequence. $\square$

2.3 Existence of linearisations and a result of Sumihiro

In this section we present a result of Sumuhiro [Sum74] and [Sum75].

Proposition 2.3.1 *Let L be a line bundle on a normal G-variety. There exists a positive integer n such that $L^{\otimes n}$ is G-linearisable.*

Proof. By Lemma 2.2.3 we have an isomorphism $\varphi^*L \simeq p_X^*M \otimes p_G^*N$ (this uses the normality assumption). Note that we have $M = \varphi^*L|_{\{e\}\times X} = L$.

But the Picard group of G is finite therefore there exists a positive integer n such that $N^{\otimes n}$ is trivial. We get an isomorphism $\varphi^*(L^{\otimes n}) \simeq p_X^*(M^{\otimes n}) \otimes p_G^*(M^{\otimes n}) \simeq p_G^*(L^{\otimes n})$. Therefore $L^{\otimes n}$ is linearisable. □

Lemma 2.3.2 *Let L be a G-linearisable line bundle on X. Then G acts on $H^0(X, L)$ via*

$$g \cdot \sigma(x) = g(\sigma(g^{-1} \cdot x))$$

for all $g \in G$, $\sigma \in H^0(X, L)$ and $x \in X$. Furthermore this representation is locally finite and rational.

Recall that a representation V of G is locally finite and rational if for all $v \in V$, there is a finite dimensional G-subspace W of V containing v such that the action is given by an algebraic group morphism $G \to \mathrm{GL}(W)$.

Proof. Note that there is an isomorphism $H^0(G \times X, p_X^*L) \simeq k[G] \otimes H^0(X, L)$ defined by $s \mapsto (f, \sigma)$ with $\sigma(x) = s(e, x)$ and $s(g, x) = f(g)s(e, x)$ (such an f exists because we take the pull-back of a line bundle on X). The inverse is defined by $f \otimes \sigma \mapsto [(g, x) \mapsto (g, f(g)\sigma(x))]$.

But because L is G-linearisable, the linearisation $\Phi : G \times L \to L$ proves that p_X^*L is also isomorphic to φ^*L. Pulling back sections, we get a morphism

$$\Phi^* : H^0(X, L) \to H^0(G \times X, \varphi^*L) \simeq H^0(G \times X, p_X^*(L)) \simeq k[G] \otimes H^0(X, L)$$

defined by $\sigma \mapsto s$ with $s(g, x) = g^{-1} \cdot \sigma(gx) = \Phi(g^{-1}, \sigma(gx))$. We may write $\Phi^*(\sigma) = \sum_i f_i \otimes \sigma_i$ with $f_i \in k[G]$ and $\sigma_i \in H^0(X, L)$, the sum being finite. We get $g \cdot \sigma = \sum_i f_i(g^{-1})\sigma_i$ and the result follows. □

Definition 2.3.3 *(ı) A G-variety is called linear if there exists a representation V of G and a G-equivariant isomorphism of X to a G-stable locally closed subvariety of $\mathbb{P}(V)$.*

(ıı) A G-variety is called locally linear if there exists a covering of X by linear G-stable open subsets.

The next result proves that normal G-varieties are locally linear.

Theorem 2.3.4 (Sumihiro's Theorem) *Let X be a normal variety with an action of an algebraic group G. Let Y be a G-orbit in X.*

There exists a finite dimensional representation V of G and a G-stable neigbourhood U of Y in X such that U is G-equivariantly isomorphic to a G-stable locally closed subvariety in $\mathbb{P}(V)$.

Proof. Let U_0 be an affine open subset in X meeting Y non trivially. Consider the divisor $D = X \setminus U_0$ and the invertible sheaf $\mathcal{O}_X(mD)$. Recall that $\mathcal{O}_X(mD)$ is the sheaf of rational functions with pole of order at most m at D.

Let $f_0 = 1, f_1, \cdots, f_n$ be generators of the algebra $k[U_0] \subset k(X)$ and let N be the linear span of these elements in $k(X)$. Then for some $m \geq 0$, we have the inclusion $N \subset H^0(X, \mathcal{O}_X(mD))$.

Now there exists an integer $n \geq m$ such that $\mathcal{O}_X(nD)$ is linearisable. Therefore we have a locally finite and rational action of G on $H^0(X, \mathcal{O}_X(nD))$. The space N is contained in $H^0(X, \mathcal{O}_X(nD))$ and we denote by W the (finite dimensional) subspace spaned by all the G-translates of N. We get a rational map

$$\psi : X \dashrightarrow \mathbb{P}(W^\vee)$$

defined by $x \mapsto [\ell_x]$ with $\ell_x(s) = s(x)$ and where $[\ell_x]$ is the class in the projective space of ℓ_x. This map is G-equivariant. Indeed, we have $(g \cdot \ell_x)(s) = \ell_x(g^{-1} \cdot s) = \ell_x(g^{-1}sg) = g^{-1}s(gx) = g^{-1}\ell_{gx}(s)$. But g acts by scalar multiplication thus there exists an invertible function (it is even a character) $f \in \mathcal{O}(G)^\times$ with $g \cdot \ell_x = f(g)\ell_{gx}(s)$ therefore $[g \cdot \ell_x] = [\ell_{gx}]$.

The map ψ induces an isomorphism on U_0 and by G-equivariance on $U = GU_0$ concluding the proof. □

Remark 2.3.5 The normality assumption is important as shows the following example. Consider X a plane nodal cubic. It has a $\mathbb{G}_m$ action with 2 orbits: the node and the complement of the node.

But the closure of any non trivial $\mathbb{G}_m$-orbit in a projective space is isomorphic to $\mathbb{P}^1$ with 3 orbits. Therefore X does not satisfy the conclusion of the previous proposition.

In these note we shall always assume that the varieties are locally linear (for example normal).

2.4 Quotient of projective varieties by reductive groups

We have seen in the previous chapter how to construct the quotient of an affine variety by a reductive group G. Let now X be a projective variety with a G-action. In this section we briefly explain how to construct quotients for this action and stable some of its properties.

First we choose an ample line bundle L and by previous results, we may assume that L is G-linearised and very ample (otherwise replace L by some tensor power $L^{\otimes n}$). We may therefore G-equivariantely embed X in some projective space $\mathbb{P}(V)$. We construct a quotient for $\mathbb{P}(V)$, the quotient for X is obtained by restriction.

Definition 2.4.1 *(ı) A point $[v] \in \mathbb{P}(V)$ is called semistable if there exists an invariant $f \in S^*(V^\vee)$ such that $f(v) \neq 0$. Denote by $\mathbb{P}(V)^{ss}$ the set of semistable points (resp. X^{ss} its intersection with X).*

(ıı) A point is unstable if it is not semistable. Denote by $\mathbb{P}(V)^u$ (resp. X^u) the set of unstable points.

Note that a point $[v]$ is unstable if the closure of $G \cdot v$ in V contains 0.

Now if $[v]$ is semistable and if f is invariant with $f(v) \neq 0$, then the open subset $D(f)$ is affine G-invariant and contains $[v]$. We may therefore construct the quotient $D(f)/G$ by taking the invariants. Because quotients are solution of an universal property, these constructions for any invariant f glue together to give a quotient

$$\pi : \mathbb{P}(V)^{ss} \to \mathbb{P}(V)^{ss}//G.$$

This construction satisfies the following properties:

Proposition 2.4.2 *The variety $\mathbb{P}(V)^{ss}//G$ is normal and projective. Furthermore for $U \subset \mathbb{P}(V)^{ss}//G$ an open affine subset, then $\pi^{-1}(U)$ is again an open affine subset and*

$$k[U] = k[\pi^{-1}(U)]^G.$$

Note that the quotient is the $\mathrm{Proj}(k[V]^G)$.

Definition 2.4.3 *A point is stable if it is semistable, if its orbit is closed and its stabiliser if finite. The set of stable points is denoted by $\mathbb{P}(V)^s$.*

Proposition 2.4.4 *The quotient restricts to a geometric quotient on $\mathbb{P}(V)^s$ (*i.e.* the quotient is an algebraic structure on the set of G-orbits on $\mathbb{P}(V)^s$).*

Chapter 3

Some results on invariants

I have not written yet the chapter on this classical topic.

Chapter 4

More quotients and U-invariants

In this chapter we assume that the base field k is of characteristic zero.

4.1 Isotypical decomposition

Definition 4.1.1 *Let G be an algebraic group.*

($\imath$) Let V and W be representations of G, we denote by $\mathrm{Hom}_G(V,W)$ the group of morphisms of G-modules from V to W.

($\imath\imath$) Let X and Y be G-varieties, we denote by $\mathrm{Mor}_G(X,Y)$ the set of equivariant morphisms from X to Y.

Fact 4.1.2 *Let X be a G-variety and V be a G-module. Then we have an identification*

$$\mathrm{Hom}_G(V,k[X]) \simeq (k[X]\otimes V^\vee)^G \simeq \mathrm{Mor}_G(X,V^\vee).$$

Proof. Let $(e_i)_{i\in[1,n]}$ be a basis of V and $(e_i^\vee)_{i\in[1,n]}$ be the dual basis. Define the map $f\mapsto\sum_i f(e_i)\otimes e_i^\vee$. One can easily check that this does not depend on the choice of the base. The action on the tensor product is given by the diagonal action thus $g\cdot\sum_i f(e_i)\otimes e_i^\vee=\sum_i gf(e_i)\otimes ge_i^\vee=\sum_i f(e_i)\otimes ge_i^\vee=\sum_i f(e_i)\otimes e_i^\vee$ thus we get an invariant. The converse map is $\sum_i a_i\otimes l_i\mapsto f$ with $f(x)=\sum_i a_il_i(x)$ (complete the l_i in a basis of $V^\vee$ and take the dual basis to get the expression as $\sum_i f(e_i)\otimes e_i^\vee$).

Define a map $\mathrm{Hom}_G(V,k[X])\to\mathrm{Mor}_G(X,V^\vee)$ by $f\mapsto\phi_f$ with $\phi_f(x)=(v\mapsto f(v)(x))$ and the converse map $\phi\mapsto f_\phi$ with $f_\phi(v)(x)=\phi(x)(v)$. One easily checks the compatibility of the actions. □

Fact 4.1.3 *The group $\mathrm{Hom}_G(V,k[X])$ is a $k[X]^G$-module.*

Proof. The action is simply given by $\phi\cdot f(v)=\phi f(v)$. □

Assume that G is reductive and let $\hat{G}$ be the set of irreducible representations of G.

Lemma 4.1.4 *Any G-module M (*i.e. *rational representation) admits a canonical decomposition*

$$\bigoplus_{V\in\hat{G}}\mathrm{Hom}_G(V,M)\otimes V\simeq M$$

the map being defined by $f\otimes v\mapsto f(v)$.

In particular for any G-variety X, we have a canonical decomposition

$$k[X]\simeq\bigoplus_{V\in\hat{G}}\mathrm{Mor}_G(X,V^\vee)\otimes V$$

and each of the $k[X]^G$-modules $\mathrm{Mor}_G(X,V^\vee)$ are finitely generated.

Proof. Because M is a rational representation, it has to be the direct sum of its irreducible finite dimensional sub-G-modules. We therefore only have to deal with M a simple module. The assertion is then easy to verify.

The module $\mathrm{Mor}_G(X, V^\vee)$ is isomorphic to $(k[X] \otimes V^\vee)^G$. But the algebra $k[X \times V]^G$ is isomorphic to

$$k[X \times V]^G \simeq \bigoplus_{n \geq 0} (k[X] \otimes S^n V^\vee)^G.$$

Therefore this algebra is finitely generated as a k-algebra and has a grading. Let $f_1, \cdots, f_n$ be generators of this algebra. The algebra $k[X]^G$ is generated by the elements f_i of degree 0 while the $k[X]^G$-module $(k[X] \otimes V^\vee)^G = (k[X \times V]^G)_1$ is generated by the elements of degree 1. □

Lemma 4.1.5 *There is a canonical decomposition as $G \times G$-module*

$$k[G] \simeq \bigoplus_{V \in \hat{G}} V \otimes V^\vee = \bigoplus_{V \in \hat{G}} \mathrm{End}_k(V).$$

Proof. The previous Lemma gives the $G \times G$-equivariant decomposition

$$k[G] = \bigoplus_{V \in \hat{G}} \mathrm{Mor}_G(G, V^\vee) \otimes V,$$

where the right (resp. left) factor of $G \times G$ acts on the right (resp. left by multiplication in G) factor.

Fact 4.1.6 *We have an isomorphism of G-modules:* $\mathrm{Mor}_G(G, V^\vee) \simeq V^\vee$.

Proof. Let us define the map from the left to the right by $\phi \mapsto \phi(e_G)$ and from the right to the left by $f \mapsto (g \mapsto g \cdot f)$. These are inverse maps and G-equivariant. □

This concludes the proof. □

Assume that G is connected and let B be a Borel subgroup, T a maximal torus in B and U be the unipotent part of B. We can write $B = TU$ and U being normal we have an exact sequence

$$1 \to U \to B \to T \to 1$$

giving on the level of characters the identification $X^*(B) \simeq X^*(T)$ since U has no nontrivial character (being unipotent). Let us denote by R the root system associated to T and by R^+ the set of positive roots associated to B. Recall the following result from the representation theory of G:

Theorem 4.1.7 *Let V be a simple representation of G, then V^U is a line where B acts by a character λ and V is uniquely determined by λ.*

Furthermore, the set $X^*(T)^+$ of all possible characters λ for a simple module V is the set of dominant characters i.e.

$$X^*(T)^+ = \{\lambda \in X^*(T) \ / \ \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+\}.$$

Corollary 4.1.8 *In particular $X^*(T)^+$ is a finitely generated monoid.*

Definition 4.1.9 *(i) We denote by $V(\lambda)$ the irreducible representation of highest weight λ .*

(ii) Let M be any G-module, we denote by $M_\lambda^{(B)}$ the subspace of semi-B-invariants where B acts by the character λ.

Corollary 4.1.10 *We have a G-equivariant isomorphism $M \simeq \bigoplus_{\lambda \in \hat{G}} M_\lambda^{(B)} \otimes V(\lambda)$ and isomorphisms*

$$\mathrm{Hom}_G(V(\lambda), M) \simeq M_\lambda^{(B)} \textit{ for all } \lambda \in X^*(G)^+.$$

Corollary 4.1.11 *The G-module M is uniquely determined by the T-module M^U.*

Theorem 4.1.12 *Let X be an affine G-variety, then $k[X]^U$ is finitely generated.*

Proof. We first reduce this problem to the case where $X = G$. Indeed, consider the principal U-bundle $\pi : G \to G/U$ and the action of G on $X \times G/U$ defined by $h \cdot (x, g) = (hx, hg)$.

Lemma 4.1.13 *There is an isomorphism $k[X]^U \simeq k[X \times G/U]^G$.*

Proof. Indeed, define the map $f \mapsto \varphi_f$ defined by $\varphi_f(x, \bar{g}) = f(g^{-1}x)$. This is well defined since f is U-invariant thus $(x, g) \mapsto f(g^{-1}x)$ is constant on the U-orbits. The converse map is defined by $\varphi \mapsto f_\varphi$ with $f_\varphi(x) = \varphi(x, \bar{e})$. □

By results of the previous chapter, we only need to check that $k[X \times G/U]$ is finitely generated *i.e.* that $k[G/U]$ is finitely generated or that $k[G]^U$ is finitely generated. For this recall the decomposition $k[G] = \oplus_{V \in \hat{G}} V \otimes V^\vee$ as $G \times G$-module thus

$$k[G]^U = \bigoplus_{V \in \hat{G}} V^\vee \simeq \bigoplus_{V \in \hat{G}} V.$$

But the monoid of dominant character is finitely generated, this concludes the proof because the span of $V(\lambda)V(\mu)$ is a G-module contained in $V(\lambda + \mu)$ thus equal to that module. □

Definition 4.1.14 *Let X be an affine G-variety, we can therefore define the* categorical *quotient $\pi : X \to X//U$ induced by the map $k[X]^U \to k[X]$.*

Remark 4.1.15 The above quotient may not be surjective. Indeed, let $X = G = \mathrm{SL}_2$. Check as an exercise that the quotient X/U is isomorphic to $\mathbb{A}^2 \setminus \{0\}$ while $X//U \simeq \mathbb{A}^2$.

Many of the properties of X can be detected on $X//U$. Here is an example.

Proposition 4.1.16 *Let G be an algebraic group, U a maximal unipotent subgroup and X an irreducible affine G-variety.*

*(i) The field $k(X)^U$ is the field of fractions of $k[X]^U$ and any B-eigenvector in $k(X)$ (*i.e. *element of $k(X)^{(B)}$ is the quotient of two eigenvectors in $k[X]$.*

(ii) The variety X is normal if and only $X//U$ is normal.

Proof. (i) The fraction field of $k[X]^U$ is contained in $k(X)^U$ and also the quotient of any two B-eigenvectors of $k[X]$ is a B-eigenvector of $k(X)$. Conversely, let $f \in k(X)^U$ (resp. in $k(X)^{(B)}$) and consider the vector space:

$$V_f = \{f' \in k[X] \ / \ f'f \in k[X]\}.$$

Since f is U-stable (resp. a B-eigenvector), then V_f is U-stable. But because U is unipotent, there is a U-invariant element f' in V_f (and even a B-eigenvector). This proves the result.

(ii) If X is normal then so is $X//U$ (this was proved in the previous chapter). Recall the proof: we know that $k[X]$ is integrally closed in $k(X)$ and we consider $k[X]^U$ in its field of fractions which is $k(X)^U$ by (1). If $f \in k(X)^U$ is such that $P(f) = 0$ with P a monic polynomial with coefficients in $k[X]^U$. Then f is in $k[X]$ and the result follows.

Conversely, suppose that $X//U$ is normal. Let $\pi : X' \to X$ be the normalisation of X. We define a G-action on X'. Indeed, the action morphism $G \times X \to X$ gives a morphism $G \times X' \to G \times X \to X$ and since $G \times X'$ is normal it factors through X' *i.e.* we have a commutative diagram:

$$\begin{tikzcd} G \times X' \arrow[r] \arrow[d] & X' \arrow[d] \\ G \times X \arrow[r] & X. \end{tikzcd}$$

Because this is an action on an open subset (where π is an isomorphism) and the varieties are normal, this is an action. Thus we also have a quotient $X'//U$ and a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ X'//U \arrow[r, "\bar{\pi}"] & X//U \end{tikzcd}$$

with $X'//U$ and $X//U$ normal varieties with $k(X')^U = k(X)^U$ *i.e.* $k[X']^U$ and $k[X]^U$ have the same field of fractions.

The algebra $k[X']$ is the integral closure of $k[X]$ in $k(X)$. Let us consider the ideal

$$I = \{f \in k[X] \ / \ fk[X'] \subset k[X]\}.$$

This ideal is stable under the action of G and therefore stable under U and thus contains an U-invariant element $f \in k[X]^U$. This implies the inclusion $fk[X']^U \subset k[X]^U$. The subspace $fk[X']^U$ is thus an ideal of $k[X]^U$ and thus a finite $k[X]^U$-module. Therefore $k[X']^U$ is also a finite $k[X]^U$-module but since $X//U$ is normal we get $k[X']^U = k[X]^U$. Finally since the U-invariants determine the module we get $k[X] = k[X']$. □

Corollary 4.1.17 *Let G be a connected reductive group, U a maximal unipotent subgroup and X an affine irreducible G-variety. Then for any $\lambda \in X^*(T)^+$, the space $k[X]^{(B)}_\lambda$ is a finitely generated $k[X]^G$-module and the set*

$$\{\lambda \in X^*(T) \ / \ k[X]^{(B)}_\lambda \neq 0\}$$

is a finitely generated monoid.

Proof. The first statement comes from the identifications

$$k[X]^{(B)}_\lambda \simeq \mathrm{Hom}_G(V(\lambda), k[X]) \simeq \mathrm{Mor}_G(X, V(\lambda)^\vee).$$

The second assertion from the decomposition

$$k[X]^U = \bigoplus_\lambda k[X]^{(B)}_\lambda$$

and the fact that this algebra is finitely generated. □

Definition 4.1.18 *Let X be an affine G variety and let B be a Borel subgroup of G.*
($\imath$) The monoid of X, denoted $C(X)$, is the finitely generated monoid $\{\lambda \in X^*(T)\ /\ k[X]^{(B)}_\lambda \neq 0\}$.
($\imath\imath$) The subgroup $\mathbb{X}^*(T)$ of $X^*(T)$ spanned by $C(X)$ is called the weight lattice of X.
($\imath\imath\imath$) The rank of X is the rank of $\mathbb{X}^*(X)$.

Remark 4.1.19 Note that the weight lattice of X is the set of weights of the field $k(X)$, in symbols $\mathbb{X}^*(X) = \{\lambda \in X^*(T)\ /\ k(X)^{(B)}_\lambda \neq 0\}$. Indeed, any weight of this field is the difference between two weights of $k[X]$ and conversely the difference between two weights of $k[X]$ is a weight of $k(X)$.

4.2 The cone of an affine G-variety

Let G be a reductive group. Recall the definition of a spherical variety.

Definition 4.2.1 *An irreducible normal G-variety X is called spherical if there exists a Borel subgroup of G with a dense orbit in X.*

Let us give simple simple representation theoretic characterisations of affine spherical varieties.

Lemma 4.2.2 *Let X be an affine irreducible G-variety. The following propositions are equivalent.*
($\imath$) The variety X contains a dense B-orbit.
($\imath\imath$) Any B-invariant rational function is constant.
($\imath\imath\imath$) The G-module is a direct sum of pairwise disctinct simple G-modules.

Proof. ($\imath$) $\Rightarrow$ ($\imath\imath$). If f is a B-invariant rational function, then it has to be constant on an open subset thus it is constant.

($\imath\imath$) $\Rightarrow$ ($\imath\imath$). By Rosenlicht's Theorem, there exists an open B-invariant subset U of X such that U has a geometric quotient U/B. In paricular $k(X)^B = k(U)^B$ is the field of fractions of $k[U]^B$. But $k(X)^B = k$ by definition thus $k[U]^B = k[U/B]$ is of dimension 0 thus U is a dense B-orbit.

($\imath\imath$) $\Rightarrow$ ($\imath\imath\imath$). Assume that the representation $V(\lambda)$ appears with multiplicity at least 2 in $k[X]$. Then there exists two B-eigenfunctions f and f' with eigenvalue λ. The quotient f/f' is a B-invariant non trivial rational function, a contradiction.

($\imath\imath\imath$) $\Rightarrow$ ($\imath\imath$). Let f be a B-invariant rational function. Then by Proposition 4.1.16 it is the quotient f_1/f_2 of two B-eigenfunctions. Their eigenvalue have to be the same and by assumption f_1 and f_2 must be colinear and f must be constant. □

We will see that these properties remain true in general. Recall the definition of a toric variety.

Definition 4.2.3 *An irreducible variety X is toric if there exists a torus T acting on X with a dense orbit isomorphic to T.*

Proposition 4.2.4 *Let X be an irreducible affine G-variety. The following conditions are equivalent.*
($\imath$) The variety X is spherical.
($\imath\imath$) The G-module $k[X]$ is multiplicity free and the weight monoid $C(X)$ is saturated i.e. *we have the equality $C(X) = \mathcal{C}(X) \cap \mathbb{X}(X)$ where $\mathcal{C}(X)$ is the cone spanned by $C(X)$ in $\mathbb{X}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.*
($\imath\imath\imath$) The affine T-variety $X//U$ is a toric normal variety.

Proof. ($\imath$)$\Rightarrow$($\imath\imath\imath$) We already know that if X is normal so is $X//U$. The T-module $k[X//U]$ is $k[X]^U$ and therefore multiplicity free as T-module. Therefore, there is a dense T-orbit in $X//U$. This orbit is isomorphic to $T/T' \simeq T''$ which is a torus thus $X//U$ is toric.

($\imath\imath\imath$)$\Rightarrow$($\imath\imath$) If $X//U$ is toric then $k[X]^U$ is multiplicity free as T-module thus $k[X]$ is multiplicity free as G-module. We are left to check that the normality corresponds to the saturation of the monoid.

Lemma 4.2.5 *Let Y be an affine irreducible variety with an action by a torus T such that $k[Y]$ is multiplicity free. Then the following are equivalent.*

(ı) The variety Y is normal.

(ıı) The monoid $C(Y)$ is saturated.

Proof. If the variety is normal, then $k[Y]$ is integrally closed. If $C(Y)$ was not saturated, there would be an element $\lambda \in \mathcal{C}(Y)\mathbb{X}(Y)$ and not in $C(Y)$. We would however have $n\lambda \in C(Y)$ for some n large enough. But then for $f \in k(Y)_\lambda$ we would have $f^n \in k[Y]$ thus f is integral on Y and $f \in k[Y]$. A contradiction.

Conversely, assume that $C(Y)$ is saturated and let $f \in k(Y)$ be integral on $k[Y]$. By decomposing f in sum of eigenvectors, we only need to check that if f is an eigenvector then it lies in $k[Y]$. We thus get that a multiple of λ, the weight of f, lies in $C(Y)$. But as $C(Y)$ is saturated, the weight λ already lies in $C(Y)$. Because $k[Y]$ is multipllicity free we get that $f \in k[Y]$. ☐

(ıı)⇒(ı) Because $k[X]$ is multiplicity free, the variety X contains a dense B-orbit. Furthermore, since $C(X)$ is saturated, the quotient $X//U$ is normal thus the variety X is normal. ☐

Remark 4.2.6 If one already knows the classification of toric varieties, one easily gets the following result.

If X satisfies one of properties of the previous proposition, then X has finitely many G-orbits, the closure of these orbits are again spherical varieties and they correspond to certain faces of the weight cone $C(X)$. Their weight group are direct factor of $\mathbb{X}(X)$.

Definition 4.2.7 *Let us denote by $\mathcal{C}(X)$ the union of half lines in $\mathbb{X}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ spanned by the elements in $C(X)$. This is the cone of X.*

Example 4.2.8 Let $X = G$, then we have the equalities $\mathbb{X}(G) = X^*(T)$ and $C(G) = X^*(T)^+ = \mathcal{C}(G)$. This last term is the cone of dominant characters. The rank $\mathrm{rk}(G)$ is the rank of G as reductive group.

Proposition 4.2.9 *For any $x \in X$, we have the inclusion $\mathcal{C}(\overline{Gx}) \subset \mathcal{C}(X)$ with equality for x in a non empty open subset of X.*

Proof. Let $x \in X$, because $\overline{Gx}$ is closed, any eigenvector of B in $k[\overline{Gx}]$ extends to $k[X]$ (the irreducible reprensations of $k[\overline{Gx}]$ appear in $k[X]$, in characteristic zero). We get the desired inclusion. In positive characteristic, modulo some power we may also lift an element in $k[\overline{Gx}]$ (see Lemma 4.3.2).

Furthermore $\mathcal{C}(X)$ is finitely generated (because $C(X)$ is finitely generated). Let (λ_i) be some generators and let (f_i) be some eigenfunctions in $k[X]$. Let $x \in X$ with $f_i(x) \neq 0$ for all i. Then f_i is a non trivial function on $\overline{Gx}$ thus the weight λ_i appears in $\mathcal{C}(\overline{Gx})$. As the weights (λ_i) generate $\mathcal{C}(X)$ we have the converse inclusion for this element x. ☐

Remark 4.2.10 More generally, if Y is a closed G-stable subvariety, we have the inclusion $\mathcal{C}(Y) \subset \mathcal{C}(X)$.

Definition 4.2.11 *(ı) We denote by* $\mathrm{Lin}(\mathcal{C}(X))$ *the linear part of the cone* i.e. *the maximal vector subspace contained in the cone.*

(ıı) For G a linear algebraic group and H a closed subgroup, we denote by $X^*(G)^H$ the kernel of the restriction map $X^*(G) \to X^*(H)$.

Proposition 4.2.12 *Let x such that Gx is a closed orbit and let G_x be the stabiliser of x. Then we have the inclusion $X^*(G)^{G_x} \subset \mathrm{Lin}(\mathcal{C}(X))$ with equality for some elements $x \in X$.*

Proof. By the above argument, $\mathcal{C}(X)$ contains $\mathcal{C}(Gx) = \mathcal{C}(G/G_x)$. This set contains $X^*(G)^{G_x}$ proving the inclusion.

For the second assertion, let us first remark that by the former proposition, we may assume that x has a dense orbit in X. Indeed, replace X by $\overline{Gx}$ with x such that $\mathcal{C}(X) = \mathcal{C}(\overline{Gx})$. We have the following lemma.

Lemma 4.2.13 *Let X be an affine G-variety with a dense orbit, then X has a unique closed orbit.*

Proof. Consider $\pi : X \to X//G$ the quotient and recall that π is sujective and that there is a unique closed orbit in each fiber of π. We are therefore left to prove that the quotient is reduced to one point. But π is constant on the G-orbits, therefore it is constant on a dense subset thus π is constant and the result follows. □

Let $x \in X$ such that Gx is the unique closed orbit and let $\lambda \in \mathrm{Lin}(\mathcal{C}(X))$. There exists an integer n such that $\pm n\lambda$ are weights of $k[X]$. Let f and f' be some eigenfunctions associated to these weigths. The function ff' is again an eigenfunction with weight 0. As the only representation with highest weight 0 is the trivial representation the product ff' is a invariant function for G. It is therefore constant on an open subset of X (the dense orbit) and thus on the all of X. Therefore f (and f') are non vanishing functions on $Gx = G/G_x$. They come from a non vanishing function $\bar{f}$ on G constant on G_x. But such a non vanishing function on G is a character (modulo a constant scalar: use Lemma 2.1.5). The result follows. □

Corollary 4.2.14 *Let X be an affine irreducible G variety.*

($\imath$) The cone $\mathcal{C}(X)$ contains no non trivial linear subspace if and only if we have the equality $G = D(G)G_x$ for all $x \in X$ such that Gx is closed.

($\imath\imath$) The cone $\mathcal{C}(X)$ is a vector space if and only if $D(G)$ acts trivially and for x in a non empty open subset of X, the orbit Gx is closed.

Proof. ($\imath$) The space $\mathrm{Lin}(\mathcal{C}(X))$ is trivial if and only if for any $x \in X$ such that Gx is closed the character group $X^*(G)^{G_x}$ is trivial. But recall that any character is trivial on $D(G)$ therefore $X^*(G) = X^*(G/D(G))$. Therefore the previous condition reads $\ker(X^*(G/D(G)) \to X^*(G_x/(D(G)\cap G_x))) = 0$. But $G/D(G)$ is a torus therefore $G_x/D(G) \cap G_x$ has to be equal to $G/D(G)$ *i.e.* $G = D(G)G_x$.

($\imath\imath$) The characters of $D(G)$ form a stricly convex cone. Therefore $D(G)$ has to act trivially on X. As $G/D(G)$ is a torus, we may assume that G is a torus. Let x be in the open subset such that $\mathcal{C}(\overline{Gx}) = \mathcal{C}(X)$. For λ a weight in this cone, then $-\lambda$ is in the cone therefore there exists an integer n such that $\pm n\lambda$ are weights of functions on X. Let f and f' in $k[X]$ be functions of weights $n\lambda$ and $-n\lambda$ respectively. Then ff' is weight 0 thus invariant for G and therefore constant on $\overline{Gx}$. This in particular implies that the ideal of $\overline{Gx} \setminus Gx$ is trivial: any B-eigenfunction f in this ideal is constant on $\overline{Gx}$ and therefore vanishes. By Lie-Kolchin this implies that the ideal is trivial. The orbit Gx is therefore closed.

Conversely, there exists $x \in X$ with Gx closed and $\mathcal{C}(X) = \mathcal{C}(Gx)$. But $k[Gx] = k[G/G_x]$ and since $D(G)$ acts trivally we have G/G_x is a torus quotient of $G/D(G)$. In particular the weights of $k[Gx]$ form a group proving the result. □

4.3 The weight lattice of a G-variety

We want to generalise the above definition of $\mathbb{X}^*(X)$ for any G-variety X. For this we need some affine subsets having nice properties with respect to the action.

We shall again prove our results in characteritic zero. However, we shall explain using the following lemma how to prove this result in positive characteristic also.

Let us first recall the definition of a geometrically reductive group.

Definition 4.3.1 *A group G is called geometrically reductive if for any finite dimensional representation V and any invariant vector $v \in V$, there exists an invariant polynomial $f \in S^n(V^\vee)^G$ with $f(v) \neq 0$.*

Lemma 4.3.2 *Let G be geometrically reductive and let A be an algebra acted on by G. Let I be a G-invariant ideal in A. Then for any element $\bar{a} \in (A/I)^{(B)}$, there exists an integer $n > 0$ and an element $a' \in A^{(B)}$ such that $\bar{a}' = \bar{a}^n$.*

In characteristic zero, we proved even more that this result, namely we proved the following lemma.

Lemma 4.3.3 *Let G be linearly reductive group and let A be an algebra acted on by G. Let I be a G-invariant ideal in A. Then for any element $\bar{a} \in (A/I)^{(B)}$, there exists an element $a' \in A^{(B)}$ such that $\bar{a}' = \bar{a}$.*

In other words, the map $A^{(B)} \to (A/I)^{(B)}$ is surjective.

As the example of projective homogeneous spaces (for example $\mathbb{P}^n$) shows, we may not hope for the existence of affine G-stable open subsets. The following proposition is a substitute for this.

Proposition 4.3.4 *Let X be a locally linear G-variety and Y a non empty closed G-stable subvariety. Then there exists an affine B-stable open subset X_0 of X meeting Y non trivially such that*

($\imath$) in characteristic zero, for any $f \in k[X_0 \cap Y]^{(B)}$, there exists $f' \in k[X_0]^{(B)}$ with $f = f'|_Y$.

($\imath\imath$) In positive characteristic, for any $f \in k[X_0 \cap Y]$, there exists $f' \in k[X_0]$ and $n \geq 0$ such that $f = f'|_Y{}^n$.

Proof. The proof in both cases is similar. Because X is locally linear, we may assume that X is contained in a projective space $\mathbb{P}(V)$ with V a representation of G. Let $\bar{X}$ and $\bar{Y}$ be the closures of X and Y in $\mathbb{P}(V)$ and let ∂X be the complement of X in its closure. In symbols $\partial X = \bar{X} \setminus X$. Let I and J be the homogeneous ideals of $k[V]$ corresponding to $\bar{Y}$ and ∂X in $\mathbb{P}(V)$ respectively.

Because Y is not contained in ∂X, the ideal J is not contained in I. All these ideals are G-invariants and we may consider the quotient $J/(I \cap J)$. There exists a B-eigenvector in this quotient *i.e.* an element $\phi \in (J/(J \cap I))^{(B)}$.

In characteristic zero, we can lift this element ϕ to an element $\phi' \in J^{(B)}$. In positive characteristic, we can lift a power ϕ^m of ϕ to $\phi' \in J^{(B)}$.

Define $X_0 = D(\phi') \cap \bar{X} = D(\phi') \cap X$, this is a B-invariant affine open subset of X which meets Y.

Let us prove the restriction results. Let $\tilde{X}$ and $\tilde{Y}$ be the affine cones in V over X and Y. For $f \in k[X_0 \cap Y]^{(B)}$, there exists $k \geq 0$ such that $f\phi'^k \in k[\tilde{Y}]^{(B)}$ and is homogeneous. By linear or geometric reductivity, we get the existence of $n \neq 1$ ($n = 1$ for linearly reductive groups) and $\varphi' \in k[\tilde{X}]^{(B)}$ such that $\varphi'|_{\tilde{Y}} = (f\phi'^k)^n$. Setting $f' = \varphi'\phi'^{-kn} \in k[X_0]^{(B)}$ gives the result. □

Remark 4.3.5 In the previous proposition, if G is a torus, then there is a covering of X by G-invariant affine open subsets.

Definition 4.3.6 *($\imath$) Let X be a locally linear B-variety, the weight lattice of X, denoted by $\mathbb{X}^*(X)$ is the set of non trivial eigenvalues of $k(X)$. In symbols*

$$\mathbb{X}^*(X) = \{\chi \in X^*(T) \ / \ k(X)^{(B)}_\chi \neq 0\}.$$

($\imath\imath$) As a subgroup of $X^*(T)$, the weight lattice of X is a free abelian group of finite rank. The rank of $\mathbb{X}^*(X)$ is called the rank of X and denoted by $\mathrm{rk}(X)$.

Remark 4.3.7 (ı) We have the exact sequence $1 \to k(X)^B \to k[X]^{(B)} \to \mathbb{X}(X) \to 0$ where the right map is given by sending a function to its weight.

(ıı) With X_0 as in the former proposition we have that $k(X)$ is the fraction field of $k[X_0]$ and any element of $k(X)^{(B)}$ is the quotient of two elements in $k[X_0]^{(B)}$ (for $\phi \in k(X)^{(B)}$ look at $\{f \in k[X_0] \ / \ f\phi \in k[X_0]\}$ which is a B-submodule and thus contains a B-eigenvector).

This implies the equality $\mathbb{X}^*(X) = \mathbb{X}^*(X_0)$.

Corollary 4.3.8 *Let X be a G-variety and Y be a closed subvariety stable under G. Then we have the inclusion $\mathbb{X}(Y) \subset \mathbb{X}(X)$. In particular* $\mathrm{rk}(Y) \leq \mathrm{rk}(X)$.

Proof. If X_0 is the affine open subset given by the previous proposition, then $k[Y \cap X_0]$ is a quotient of $k[X_0]$ and thus its weights are contained in those of $k[X_0]$. The result follows. □

Example 4.3.9 (ı) Let $X = G/P$ with P a parabolic subgroup. Then $\mathrm{rk}(X) = 0$. Indeed, because of the Bruhat decomposition, the variety has a dense U orbit. Therefore, any B-eigenfunction is constant on that orbit and therefore constant. Thus $\mathbb{X}(X) = \{0\}$.

(ıı) Let $X = T$ be a torus. Then $k[X] = k[X^*(T)]$ thus $\mathbb{X}(X) = X^*(T)$ and $\mathrm{rk}(X) = \mathrm{rk}(T) = \dim T$.

4.4 The cone of a projective G-variety

In this section we briefly explain, without proof, how to define an analog of the cone of X for a projective variety.

Definition 4.4.1 *Let X be a projective G-variety and let L be an ample G-linearised line bundle (recall that such a line bundle exists as soon as X is normal for example).*

The convex polytope of X and L, denoted by $P(X, L)$, is the subset of $X^*(T) \otimes \mathbb{Z}\mathbb{Q}$ defined by the equality

$$P(X, L) = \left\{ \frac{\lambda}{n} \in X^*(T) \otimes \mathbb{Z}\mathbb{Q} \ / \ H^0(X, L^{\otimes n})^{(B)}_{\lambda} \neq 0 \right\}.$$

Remark 4.4.2 The spectrum of the graded algebra $\oplus_n H^0(X, L^{\otimes n})$ is the affine cone $\hat{X}$ over X and the above set is a *projection* of its monoid $C(\hat{X})$.

Proposition 4.4.3 *(ı) The set $P(X, L)$ is a convex polytope.*

(ıı) The affine space generated by $P(X, L)$ has for vector space direction $\mathbb{X}(X) \otimes_Z \mathbb{Q}$ the weight vector space of X. In particular the dimension of $P(X, L)$ is the rank of X.

(ııı) For any $x \in X$, we have an inclusion $P(\overline{Gx}, L) \subset P(X, L)$ with equality for x in some open subset of X.

Proof. (ı) The line bundle L being ample, the algebra $R = \oplus_n H^0(X, L^{\otimes n})$ is finitely generated therefore the algebra R^U is also finitely generated. Choose $f_1, \cdots, f_k$ some generators of this algebra which are B-eigenvectors. Let (λ_i, n_i) be the weight and the degree of these elements. Let P be the convex hull of the (λ_i, n_i). We claim that $P = P(X, L)$.

Indeed, let $f \in H^0(X, L^{\otimes n})^{(B)}$ be an eigenvector for B. Let λ be its weight. Then f is a linear combination of monomials $f_1^{a_1} \cdots f_k^{a_k}$ with $\sum_i a_i n_i = n$ and $\sum_i a_i \lambda_i = \lambda$. We thus have

$$\frac{\lambda}{n} = \sum_i \frac{a_i n_i}{n} \frac{\lambda_i}{n_i}.$$

This proves the inclusion $P(X, L) \subset P$.

Conversely, let $\lambda = \sum_i c_i \lambda_i$ with $c_i \in \mathbb{Q}_{\geq 0}$ and $\sum_i c_i n_i = 1$. Let n be such that $nc_i \in \mathbb{Z}$ for all i, then we have, setting $a_i = nc_i$ the equality

$$\frac{\lambda}{n} = \sum_i \frac{a_i n_i}{n} \frac{\lambda_i}{n_i}.$$

The function $f_1^{a_1} \cdots f_k^{a_k} \in H^0(X, L^{\otimes n})$ has weight λ proving the converse inclusion.

(ii) The quotient of any two vectors of the same degree in $R^{(B)}$ obviously defines an element in $k(X)^{(B)}$. Conversely, for $f \in k(X)^{(B)}$, the space $\{f' \in R \ / \ ff' \in R\}$ is not empty and therefore contains a homogeneous B-eigenvector. Therefore $k(X)^{(B)} = \mathrm{Frac}_0(R^{(B)})$ with Frac_0 denoting the quotients of elements of the same degree. The result follows from this.

(iii) Let Y be a closed G-stable subvariety of X. We have the inclusion $P(Y, L) \subset P(X, L)$. Indded, this comes from the fact that $P(X, L)$ does only depend on $H^0(X, L^{\otimes n})$ for large n and the fact that the restriction map $H^0(X, L^{\otimes n}) \to H^0(Y, L^{\otimes n})$ is surjective for n large enough.

The equality occurs on the open subset of points x with $f_i(x) \neq 0$ where the f_i are as above such that their normalised weights span the polytope $P(X, L)$. $\square$

4.5 Complexity of a G-variety

Definition 4.5.1 *Let G be any linear algebraic group and let X be a G-variety. The complexity of X with respect to G, denoted by $c_G(X)$ is the minimal codimension of a G-orbit in X.*

Example 4.5.2 (i) If X is homogeneous under G, then $c_G(X) = 0$.

(ii) If X is a spherical variety for the reductive group G, then $c_G(X) = 0$ and even we have $c_B(X) = 0$. Actually, a G-variety X is spherical if and only if it is normal with $c_B(X) = 0$.

Lemma 4.5.3 *Let X be a G-variety, then $c_G(X) = \mathrm{Trans.deg}(k(X)^G)$.*

Proof. By Rosenlicht's Theorem, there exists an non empty open subset X_0 and a geometric quotient morphism $\pi : X_0 \to X_0/G$. We furthermore have $k(X_0/G) = k(X_0)^G = k(X)^G$. Therefore $c_G(X)$ is the dimension of the familly of general orbits in X and the dimension of the quotient X_0/G. $\square$

Example 4.5.4 Let $X = \mathfrak{gl}_n$ and $G = \mathrm{GL}_n$ acting on G by left multiplication. Let Y be the closed subset of X of matrices of rank at most one. We have $c_G(X) = 0$ since invertible matrices form a dense orbit while $c_G(Y) = n - 1$: after left multiplication a rank one matrice is determined by its kernel.

Theorem 4.5.5 *Let X be a G-variety and Y a B-stable closed subvariety. Then $c_B(Y) \leq c_B(X)$ and* $\mathrm{rk}(Y) \leq \mathrm{rk}(X)$.

Proof. Recall that the group G is generated by its minimal parabolic subgroups (a minimal parabolic subgroup is a subgroup generated by a Borel subgroup B and $U_{-\alpha}$ for α a simple root – the roots are simple with respect to B). Therefore, there exists a sequence of minimal parabolic subgroup $(P_i)_{i \in [1,r]}$ such that $P_1 \cdots P_r Y$ is stable under G. We are left to prove the following two assertions.

(i) If Y is a closed B-stable subvariety of X and P is a minimal parabolic subgroup of G containing B, then PY is a closed subvariety of X with $c_B(Y) \leq c_B(PY)$ and $\mathrm{rk}(Y) \leq \mathrm{rk}(PY)$.

(ii) If Y is a closed G-stable subvariety of X, then $c_B(Y) \leq c_B(X)$ and $\mathrm{rk}(Y) \leq \mathrm{rk}(X)$ (we already proved the statement on the rank).

Let us first prove (i). Consider the contracted product $P\times^B Y$. We have a morphism $P\times^B Y \to PY$ defined by $\overline{(p,y)} \mapsto py$. We also have the commutative diagram

$$\begin{tikzcd} P\times^B X \arrow[r] & X \\ P\times^B Y \arrow[u] \arrow[r] & PY \arrow[u] \end{tikzcd}$$

Note that because P acts on X, the contracted product $P\times^B X$ is isomorphic to $P/B\times X$ and the top horizontal arrow is then given by the second projection. Since P/B is proper, the closed subvariety $P\times^B Y$ is mapped to a closed subvariety PY in X.

Note that we may also assume that $Y \neq PY$, therefore $\dim(P\times^B Y) = \dim Y + 1 = \dim PY$. Thus the map $P\times^B Y \to PY$ is surjective with generically finite fibers. In particular we have $C_B(PY) = c_B(P\times^B Y)$. Let $s \in W_P$ be the non trivial element. Then $BsB/B \simeq B/B\cap B^s$ with $B^s = sBs^{-1}$ is open in P/B. It implies that $BsB\times^B Y \simeq B\times^{B^s} sY$ is open in $P\times^B Y$. We get the inequalities $c_B(PY) = c_B(P\times^B Y) \geq c_B(B\times^{B\cap B^s} sY) = c_{B\cap B^s}(sY) = c_{B\cap B^s}(Y) \geq c_B(Y)$. Similarly, we have $\mathbb{X}_B(Y) \subset \mathbb{X}_{B\cap B^s}(Y) = \mathbb{X}_{B\cap B^s}(sY) = \mathbb{X}_{B\cap B^s}(B\times^{B\cap B^s} Y) \subset \mathbb{X}_B(P\times^B Y)$. But as the map $P\times^B Y \to PY$ is generically finite, we get that $\mathbb{X}_B(PY)$ is a subgroup of finite index of $\mathbb{X}_B(P\times^B Y)$ proving the inequality on ranks as well.

Let us prove (ii). Let X_0 be an open B-stable affine subset meeting Y non trivially such that any element in $k[X_0\cap Y]^{(B)}$ can be lifted to an element in $k[X_0]^{(B)}$ (modulo taking some higher power in positive characteristic) as given in Proposition 4.3.4. Let $f \in k(Y)^B$ be a non trivial B-invariant rational function on Y. Write f as u/v with $u, v \in k[X_0\cap Y]^{(B)}$ with the same weight. Then there exists $n > 0$ and $u', v' \in k[X_0]^{(B)}$ such that $\bar{u}' = u^n$ and $\bar{v}' = v^n$. We thus get $\bar{u}'/\bar{v}' = f^n$ and the transcendence degree of $k(X)^B$ is bigger than the transcendence degree of $k(Y)^B$. ☐

Corollary 4.5.6 *Let X be a G-variety. If B has a open orbit in X, then B has only finitely many orbits in X.*

Proof. Let Y be a closed G-stable subvariety of X containing infinitely many B-orbits and of minimal dimension for this property. We have $c_B(X) = 0$ thus $c_B(Y) = 0$ and therefore Y has an open B-orbit. The complement Z of this orbit is closed G-stable and must have infinitely many B-orbits; A contradiction to minimality. ☐

Corollary 4.5.7 *Any spherical variety has finitely many B-orbits (and therefore finitely many G-orbits).*

Chapter 5

A characterisation of spherical varieties

5.1 Definition

Let G be a connected reductive group.

Definition 5.1.1 *A G-variety is called spherical if it is normal and has a dense orbit under some Borel subgroup of G.*

Example 5.1.2 (ı) The quotient G/P with P a parabolic subgroup of G is a spherical variety. Indeed, by Bruhat decomposition, even U a maximal unipotent subgroup has a dense orbit.

(ıı) More generally, by Bruhat decomposition again, any homogeneous space G/H with H a closed subgroup containing U a maximal unipotent subgroup is a spherical variety.

(ııı) Normal toric varieties are spherical.

(ıv) Symmetric varieties are spherical. A variety X is symmetric if $X = G/H$ with $G^\sigma \subset H \subset N_G(G^\sigma)$ where σ is an algebraic group involution of G and G^σ is the subgroup of fixed points. The proof of this fact is not completely obvious let us give two particular examples.

(v) The group G as $G \times G$ variety is symmetric and spherical. Indeed, the group G is isomorphic to the quotient $G \times G/G$ with G diagonally embedded in $G \times G$. A Borel subgroup of $G \times G$ is $B \times B$ and again Bruhat decomposition proves that G is spherical.

(vı) The quotient $\mathrm{GL}_n/\mathrm{O}_n$ is also a symmetric variey and thus spherical. Indeed, let GL_n act by conjuguation on the non degenerate quadratic form $q(x_1, \cdots, x_n) = \sum_i x_i^2$. The orbit is isomorphic to the quotient $\mathrm{GL}_n/\mathrm{O}_n$. The involution σ is defined by $\sigma(g) = {}^t g^{-1}$.

5.2 A characterisation

Definition 5.2.1 *Let X and Y be G-varieties, then X and Y are called G-birationnal if there exists non empty open subsets subsets stable under G which are G-isomorphic.*

Definition 5.2.2 *Let X be a spherical G-variety, then X' is an embedding of X if X' is also a spherical G-variety and there exists a G-equivariant morphism $X \to X'$ inducing an isomoprhism of X onto an open subset of X'.*

Theorem 5.2.3 *Let X be a normal quasi-projective G-variety. The following conditions are equivalent.*

(ı) The variety X is spherical.

(ıı) Any G-variety G-birational to X has finitely many G-orbits.

(ııı) For any G-linearised line bundle L on X, the G-module $H^0(X, L)$ is multiplicity free.

Proof. (i)⇒(ii). A spherical variety has B-complexity 0. As complexity is a birational invariant, the same is true for any G-birational variety. Furthermore, varieties with B-complexity 0 have finitely many B-orbits. In particular finitely many G-orbits.

(ii)⇒(iii). Let G/H be an open dense orbit of X. The G-module $H^0(X, L)$ is a submodule of $H^0(G/H, L)$ therefore we may assume that X is homogeneous of the form G/H. Then L is of the form $G \times^H k_\chi$ for some character χ of H. The group $H^0(G/H, L)$ is the group of sections of the map $p : G \times^H k_\chi \to G/H$ induced by the first projection on $G \times k_\chi$. In particular we get that $H^0(G/H, L) = k[G]^{(H)}_{-\chi}$. By the decomposition $k[G] = \oplus_{\lambda \in \hat{G}} V(\lambda)^\vee \otimes V(\lambda)$, we get that the multiplicity of $V(\lambda)^\vee$ in $H^0(G/H, L)$ is $\dim V(\lambda)^{(H)}_{-\chi}$.

Assume that this dimension is at least 2 *i.e.* $\dim V(\lambda)^{(H)}_{-\chi} \geq 2$. Let v and w be two linearly independent vectors of this eigenspace and consider $y = [v \oplus w] \in \mathbb{P}(V(\lambda) \oplus V(\lambda))$. Let Y be the closure of Gy in this projective space.

Lemma 5.2.4 *The variety Y has finitely many closed G-orbits.*

Proof. Let B be a Borel subgroup and η be a (unique up to scalar) eigenvector of B in $V(\lambda)^\vee$. This defines an hyperplane $H_\eta = \ker \eta$ in $V(\lambda)$. Since $V(\lambda)^\vee$ is simple, the G-orbit of η spans $V(\lambda)^\vee$ as a vector space therefore, there exists $g \in G$ sich that $(g \cdot \eta)(v) = 1$. Replacing B by a conjugate we may assume that $\eta(v) = 1$.

We may then define a rational function f on $\mathbb{P}(V(\lambda) \oplus V(\lambda))$ by

$$f(v_1 \oplus v_2) = \frac{\eta(v_2)}{\eta(v_1)}.$$

This function is defined on y and B-invariant. Let us check that it is not constant on Y. Otherwise we would have $z \in k$ with $\eta(gw) = z\eta(gv)$ for all $g \in G$. This in turn implies $(g^{-1} \cdot \eta)(w - zv) = 0$ for all $g \in G$ but since the orbit of η spans $V(\lambda)^\vee$ we get $\nu(w - zv) = 0$ for all $\nu \in V(\lambda)^\vee$ *i.e.* $w = zv$ a contradiction.

The image if f is locally closed in k thefore for any $z \in k$ except finitely many values, there exists $g_z \in G$ with $\eta(g_z w) = z\eta(g_z v)$. Excluding finitely many more values of z we may even assume that $\eta(g_z v) \neq 0$.

Let T be a maximal torus in B and B^- the opposite Borel subgroup *i.e.* the Borel subgroup of G such that $B \cap B^- = T$. We proved last semester that there is a unique B^--highest weight vector t_λ such that $\eta(t_\lambda) = 1$ (pick a basis (t_μ) of eigenvectors in $V(\lambda)$ and the dual basis $(t_\mu^\vee)$ in $V(\lambda)^\vee$, then $\eta = t_\lambda^\vee$). Because η is the dual basis element corresponding to t_λ, we may then write

$$g_z v = c_z t_\lambda + \sum_\mu v_\mu$$

with v_μ eigenvectors of eigenvalue μ and such that $\lambda - \mu$ is a non negative linear combination of simple roots. Furthermore we have $c \neq 0$ for z avoiding our finite set of values. We may also write

$$g_z w = d_z t_\lambda + \sum_\mu w_\mu$$

with w_μ eigenvectors of eigenvalue μ and such that $\lambda - \mu$ is a non negative linear combination of simple roots. We have $d_z = zc_z$.

Let $\theta \in X_*(T)$ be a cocharacter (or a one parameter subgroup) such that $\langle \theta, \alpha \rangle > 0$ for any simple root α. We have the equalities:

$$\begin{aligned} \theta(s) \cdot y &= \left[cs^{\langle \theta, \lambda \rangle}(t_\lambda \oplus zt_\lambda) + \textstyle\sum_\mu s^{\langle \theta, \mu \rangle}(v_\mu \oplus w_\mu) \right] \\ &= \left[c(t_\lambda \oplus zt_\lambda) + \textstyle\sum_\mu s^{\langle \theta, \mu - \lambda \rangle}(v_\mu \oplus w_\mu). \right] \end{aligned}$$

In particular we get the limit $\lim_{s\to\infty} \theta(s) \cdot y = [v_\lambda \oplus z v_\lambda]$.

The variety Y therefore contains the G-orbit of $[v_\lambda \oplus z v_\lambda]$ for all $z \in k$ except maybe for a finite number of values of z. Because $[v_\lambda \oplus z v_\lambda]$ is a highest weight vector for B, the stabiliser of this point contains B and the orbit is therefore projective thus compact and in particular closed. $\square$

Let us use Y to construct an embedding of G/H with infinitely many G-orbits. First note that since H acts on $v \oplus w$ via a character, it acts trivially on y therefore $G_y \supset H$.

Let X' be a compact embedding of $X = G/H$. Let us denote by x' the element of $G/H = X \subset X'$ corresponding to the identity element $e \in G$. Let us now consider X'' to be the nomalisation of the closure of $G \cdot (x', y)$ in $X' \times Y$. The G orbit $G \cdot (x', y)$ is isomorphic to G/H (since $G_y \subset H$) thus X'' is an embedding of X and the projection $X'' \to Y$ is proper since X' is compact. Therefore X'' maps surjectively on Y thus contains infinitely many orbits.

(ııı)⇒(ı). In view of Rosenlicht's Theorem, we only have to check that any B-invariant rational function on X is constant. Let $f \in k(X)^B$. Then there exists an integer $n > 0$ and elements $u, v \in H^0(X.L^{\otimes n})$ with $f = u/v$. Looking at the B-module $\{v \in H^0(X, L^{\otimes n}) \ / \ fv \in H^0(X, L^{\otimes n})$ we get by Lie-Kolchin the existence of V which is a B-eigenvector and in this case u is also a B-eigenvector for the same weight. By the multiplicity condition u and v have to be colinear proving the result. $\square$

Part II

Classification of embeddings of spherical varieties

4

Chapter 6

Valuations

6.1 Definitions

Definition 6.1.1 *A valuation of a normal variety X is a map $\nu : k(X) \to \mathbb{Q} \cup \{\infty\}$ satisfying the following four properties.*

(V1) We have the equality $\nu(0) = \infty$.

(V2) For $f_1, f_2 \in k(X)$, we have the inequality $\nu(f_1 + f_2) \geq \min(\nu(f_1), \nu(f_2))$.

(V3) For $f_1, f_2 \in k(X)$, we have the equality $\nu(f_1 f_2) = \nu(f_1) + \nu(f_2)$.

(V4) We have the equality $\nu(k^*) = 0$.

Example 6.1.2 (ı) The map ν defined by $\nu(k(X)^*) = 0$ and $\nu(0) = \infty$ is a valuation called the trivial valuation.

(ıı) Let D be a prime divisor $D \subset X$. Then D defines a valuation ν_D as follows. Let R be the local ring of the generic point of D. This is a discrete valuation ring (since X is normal it is smooth in codimension 1 thus R is smooth of codimension thus a discrete valuation ring). Its field of fraction is also $k(X)$ thus for $f \in k(X)$, write $f = u/v$ with $u, v \in R$ and $u = z^a u'$, $v = z^b v'$ with z an uniformising element of R not divising u' and v'. Set $\nu_D(f) = a - b$.

Definition 6.1.3 *Let ν be a valuation of $k(X)$. A center for ν is a subvariety Z of X such that $\mathcal{O}_{X,Z} \subset R_\nu = \{f \in k(X) \ / \ \nu(f) \geq 0\}$ and $\mathfrak{m}_{X,Z} \subset \mathfrak{m}_\nu = \{f \in k(X) \ / \ \nu(f) > 0\}$.*

Fact 6.1.4 *(ı) If X is affine, a valuation ν has a center if and only if ν is non negative on $k[X]$ and in that case its center is defined by the ideal $\mathfrak{m}_\nu \cap k[X]$.*

(ıı) A center if it exists is unique.

Proof. (ı) If Z is a center, then $k[X] \subset \mathcal{O}_{X,Z} \subset \mathcal{O}_\nu$ thus ν is non negative on $k[X]$. Conversely, if ν is non negative on $k[X]$, then we may define Z by its ideal $I(Z) = k[X] \cap \mathfrak{m}_\nu$. This is indeed a center. If Y is another center, then $I(Y) \subset k[X] \cap \mathfrak{m}_\nu = I(Z)$ thus $Z \subset Y$. Let $f \in I(Z) \setminus I(Y)$, then f has an inverse in $\mathcal{O}_{X,Y} \subset \mathcal{O}_\nu$ but $f \in \mathfrak{m}_\nu$ thus cannot be invertible in $\mathcal{O}_\nu$. A contradition thus $I(Z) = I(Y)$ and $Z = Y$.

(ıı) Assume that Y and Z are two centers for ν. Take an affine cover (U_i) of X. Then on each U_i, the center of ν is unique thus $Y \cap U_i = Z \cap U_i$ proving the uniqueness. $\square$

Definition 6.1.5 *Assume that X is a G-variety. A valuation ν is called invariant if $\nu(g \cdot f) = \nu(f)$ for all $g \in G$ and $f \in k(X)$. The set of invariant valuations of X is denoted by $\mathcal{V}(X)$.*

Fact 6.1.6 *(i) Let ν be a G-stable valuation and Z a center of ν, then Z is G-stable.*

(ii) Conversely, every G-stable subvariety Z is the center of a G-invariant valuation.

Proof. (i) Let Z be the center of ν. Then gZ is again a center of ν by invariance of ν thus $gZ = Z$ and Z is G-stable.

(ii) Let Z be a G-stable subvariety and let E be a component of the exceptional divisor of the normalisation of the blow-up Y of X along Z. Take the valuation v_E which is a valuation on $k(Y) = k(X)$. It is G-invariant with center Z. □

6.2 Existence of invariant valuations

Let G be a connected reductive group.

Lemma 6.2.1 *Let K be a field extension of k and L be an extension of K. Assume that $\nu : K \to \mathbb{Q}$ is a valuation, then there exists a valuation $\nu' : L \to \mathbb{Q}$ such that $\nu'|_K = \nu$.*

Proof. Let us prove this result in two steps. First, we prove that if L/K is a purely transcendental extension then the result holds. For this we only need to deal with the case $L = K(x)$ with x non algebraic over k. Any element in L is of the form $P(x)/Q(x)$ with $P, Q \in K[X]$. Let $a \in \mathbb{Q}_{\geq 0}$ be any non negative element and define $\nu'(P(x)) = a \deg P + \nu(\mathrm{dom}(P))$ where $\mathrm{dom}(P)$ is the leading term of P. We define more generally $\nu'(P(x)/Q(x)) = a(\deg P - \deg Q) + \nu(\mathrm{dom}(P)) - \nu(\mathrm{dom}(Q))$. This obviously satisfy the properties of a valuation and its restriction to K is ν.

If L/K is algebraic, we may assume that $L = K[X]/(R)$ with

$$R(X) = \sum_{i=1}^{n} a_i X^i$$

an irreducible polynomial. Any element of L is of the form $P(x)$ with $P \in K[X]$ of degree smaller than n. We define

$$\nu'(P(x)) = \deg P \cdot \frac{\nu(a_0)}{n} + \nu(\mathrm{dom}(P)).$$

This satisfies the multiplicative rule for the only relation in L given by $R(x) = 0$ and extends the valuation ν. □

Lemma 6.2.2 *Let ν be a valuation of $k(G)$, there exists a nuique invariant valuation $\bar{\nu}$ of $k(G)$ such that $\bar{\nu}(f) = \nu(g \cdot f)$ for any $f \in k(G)$ and all g in a non empty open subset U_f of G.*

Proof. We claim that the result will follow if we prove that for any $f \in k(G)$, the value of $\nu(g \cdot f)$ is constant on an open subset U_f of G.

Assume that this holds, then define $\bar{\nu}(f) = \nu(g \cdot f)$ for $g \in U_f$ and $\bar{\nu}(0) = \infty$. By definition $\bar{\nu}$ satisfies $(V1)$ and for $f, f' \in k(G)$, let $g \in U_f \cap U_{f'} \cap U_{f+f'}$ (this intersection is non empty since G is connected and the subsets U_f, $U_{f'}$ and $U_{f+f'}$ are open). We have

$$\begin{aligned}&\bar{\nu}(f + f') = \nu(g \cdot f + g \cdot f') \geq \min(\nu(g \cdot f), \nu(g \cdot f')) = \min(\bar{\nu}(f), \bar{\nu}(f')) \text{ and}\\ &\bar{\nu}(ff') = \nu((g \cdot f)(g \cdot f')) = \nu(g \cdot f)\nu(g \cdot f') = \bar{\nu}(f)\bar{\nu}(f').\end{aligned}$$

This proves $(V2)$ and $(V3)$ for $\bar{\nu}$. The condition $(V4)$ is obviously satisfied.

We are thus left to prove the first claim. We may assume that f is regular *i.e.* $f \in k[G]$. Let $V(n) = \{f' \in k[G] \ / \ \nu(f') \geq n\}$. This is a linear subspace of $k[G]$. Let V be a finite dimensional subrepresentation of $k[G]$ containing f. There exists an integer n such that $V \subset V(n)$. Let n_0 be the smallest such n. Define $U_f = \{g \in G \ / \ g \cdot f \notin V(n_0-1)\}$ and $Z_f = \{g \in G \ / \ g \cdot f \in V(n_0-1)\} = G \backslash U_f$. We have the following diagram:

$$\begin{tikzcd} Z_f \arrow[r] \arrow[d] & V(n_0-1) \arrow[d] \\ G \arrow[r] & V(n_0). \end{tikzcd}$$

Which is cartesian. Since $V(n_0-1)$ is closed in $V(n_0)$ so is Z_f in G therefore U_f is open and the result follows. $\square$

Corollary 6.2.3 *Let H be a closed subgroup of H.*

(ı) Any valuation $\nu \in \mathcal{V}(G)$ induces, by restriction, a valuation $\nu' = \mathrm{res}(\nu) \in \mathcal{V}(G/H)$. In other words, we have a restriction map $\mathrm{res} : \mathcal{V}(G) \to \mathcal{V}(G/H)$.

(ıı) Conversely, for any valuation $\nu' \in \mathcal{V}(G/H)$, there exists a valutation $\nu \in calV(G)$ such that $\mathrm{res}(\nu) = \nu'$. In other words, the map res is surjective.

Proof. (ı) The field of rational functions $k(G/H)$ is $k(G)^H$ and we define res as the restriction of the valuation.

(ıı) By Lemma 6.2.1 we can lift ν' to a valuation $\bar{\nu}$ of $k(G)$ and by Lemma 6.2.2 we can find an invariant valuation $\nu \in \mathcal{V}(G)$ with $\nu(f) = \bar{\nu}(g \cdot f)$ for g in some open subset U_f of G. Since ν is G-invariant, for $f \in k(G/H) = k(G)^H$, we have $\nu(f) = \nu(g \cdot f) = \bar{\nu}(g \cdot f)$ for all $g \in G$ and thus $\nu(f) = \nu'(f)$ by taking $g \in U_f$. $\square$

For $V \subset k[G]$ a vector subspace and n an integer, we define V^n to be the vector space spanned by all the products of n elements of V.

Proposition 6.2.4 *Let $\nu \in \mathcal{V}(G/H)$, let $f \in k(G/H)$ and let $h \in k(G)^{(B\times H)}$. Assume that fh is regular* i.e. *$fh \in k[G]$, and let V be the sub-G-module of $k[G]$ spanned by fh. Then the following conditions hold.*

(ı) We have the inclusion $V^n h^{-n} \subset k(G/H)$ for all $n \in \mathbb{N}$.

(ıı) We have the equality $\nu(f) = \min\{\frac{1}{n}\nu(f'/h^n) \ / \ n \in \mathbb{N}, \ f' \in (V^n)^{(B)}\}$.

Proof. (ı) Let λ be the H-character of h. The left G-action commutes with the right H-action therefore all the elements in V^n are H-eigenfunctions with eigenvalue $n\lambda$. Indeed, any such function is a linear combination of elements of the form $F = \prod_{i=1}^n (g_i \cdot fh)$ and for $u \in H$, we have $F \cdot u = \prod_{i=1}^n (g_i \cdot fh \cdot u) = \lambda(u)^n F$. We thus have $V^n h^{-n} \subset k(G)^H = k(G/H)$.

(ıı) Let $\nu' \in calV(G)$ be a lifting of ν and consider $V(q) = \{f' \in k[G] \ / \ \nu'(f') \geq q\}$. This is a vector subspace of $k[G]$ stable under G since ν' is G-invariant. It is therefore a G-submodule pf $k[G]$. Considering $V(n\nu'(fh))$ we have the inclusion $V^n \subset V(n\nu'(fh))$. Indeed an element in V^n is a linear combination of elements of the form $F = \prod_{i=1}^n (g_i \cdot fh)$ and we have $\nu'(F) = n\nu'(fh)$. Therefore for $f' \in V^n$ we have $\nu(f') \geq n\nu'(fh) = n(\nu(f) + \nu'(h))$ and $\nu(f) \leq \frac{1}{n}\nu'(f') - \nu'(h) = \frac{1}{n}\nu'(f'/h^n)$. Thus proves that the left hand side is smaller that the right hand side in (ıı).

Let us prove the equality. For this define $R = \oplus_{n\geq 0} V^v$. This is a graded integral k-algebra. For $r \in R$, we denote by r_n the n-th component of r in the grading. Let us define

$$\nu''(r) = \min\{\nu'(r_n) - n\nu'(fh) \ / \ n \in \mathbb{N}, \ r_n \neq 0\}$$

for all $r \in R \setminus \{0\}$ and $\nu''(0) = \infty$.

Lemma 6.2.5 *The map ν'' is a G-invariant valuation with $\nu''(r) \geq 0$ for all $r \in R$.*

Proof. The group G preserves the grading and ν' therefore ν'' is G-invariant. The conditions $(V1)$ and $(V4)$ are obviously satisfied. So is the condition $(V3)$ since the product preserves the grading. The condition $(V2)$ is also given by the fact that we took the direct sum of the V^n.

By what we proved before, we have $\nu'(r_n) \geq n\nu'(fh)$ proving the non negativity of ν'' on R. □

Let I be the ideal defined by ν'' *i.e.* $I = \{r \in R \ / \ \nu''(r) > 0\}$. This is an homogeneous ideal which is prime and G invariant. Note that $\nu''(fh) = 0$ thus the quotient R/I is non trivial. There exists therefore a B-eigenvector in $R/I^{(B)}$ which can be lifted to a B-eigenvector say f' in $R^{(B)}$. We may even choose f' to be homogneous that is to say in $(V^n)^{(B)}$. We get $\nu''(f') = 0$ or $\nu'(f') = n\nu'(fh)$ concluding the proof. □

Let $x_0 \in G/H$ and B a Borel subgroup of G such that Bx_0 is dense in G/H.

Definition 6.2.6 *Let us denote by $D(G/H)$ the set of B-stable prime divisors in G/H. This set is finite since G/H contains a dense orbit.*

Corollary 6.2.7 *Let $f \in k[Bx_0]$ and $\nu_0 \in \mathcal{V}(G/H)$. Then there exists $n \geq 0$ and $f' \in k(G/H)^{(B)}$ such that the following three conditions hold.*

($\imath$) $\nu_0(f') = \nu_0(f^n)$.

($\imath\imath$) $\nu(f') \geq \nu_0(f^n)$ for all $\nu \in cV(G/H)$.

($\imath$) $\nu_D(f') = \nu_D(f^n)$ for all $D \in D(G/H)$.

Proof. We may replace G by a finite cover G' of G. Therefore we may assume that $k[G]$ is factorial *i.e.* $\mathrm{Pic}(G) = 0$. Let $\pi : G \to G/H$ be the quotient map. Consider the divisor $1/f \circ \pi$ on G. Its B-stable part is again a divisor which is B-stable and is the divisor of a rational function h since $\mathrm{Pic}(G) = 0$. Since the divisor is $B \times H$-stable, this function is a $B \times H$-eigenfunction *i.e.* $h \in k(G)^{(B\times H)}$. Note that $h(f \circ \pi)$ is defined on BH and on all the B-stable divisors of G by definition of h. Since G is normal and $G \setminus BH$ contains only B-stable divisors, the function $h(f \circ \pi)$ is defined on a open subset whose complement has codimension at least 2. Since G is normal we have $h(f \circ \pi) \in k[G]$.

By the previous proposition, there exists an $n \geq 0$ and $\varphi \in k[G]^{(B)}$ with $\nu_0(\varphi/h^n) = \nu_0(f)$. Note also that for any $\nu \in \mathcal{V}(G/H)$ we also have $\nu(\varphi/h^n) \geq \nu(f^n)$. Proving the first two parts of the statement. Now for $D \in D(G/H)$ a B-stable divisor and D' any component of $\pi^{-1}(D)$, we have $\nu_D(\varphi/h^n) = \nu_{D'}(\varphi/h^n) \geq \nu_{D'}(1/h^n) = \nu_{D'}((f \circ \pi)^n) = \nu_D(f^n)$. This concludes the proof by setting $f' = \varphi/h^n$. □

The set $k(G/H)^{(B)}$ is a subgroup of the multiplicative group. The map $k(G/H)^{(B)} \to \mathbb{X}(G/H)$ sending f to its weight χ_f is a group morphism and since G/H has a desne B-orbit, any invariant function is constant therefore we have the exact sequence

$$1 \to k^* \to k(G/H)^{(B)} \to \mathbb{X}(G/H) \to 0.$$

Definition 6.2.8 *Denote by $\mathbb{X}^\vee(X)$ the dual lattice of $\mathbb{X}(X)$. In symbols $\mathbb{X}^\vee(X) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{X}(X), \mathbb{Z})$.*

Any valuation $\nu \in \mathcal{V}(G/H)$ induces a homomorphism $\rho_\nu : k(G/H)^{(B)} \to \mathbb{Q}$ defined by $\rho_\nu(f) = \nu(f)$. Therefore we have a map

$$\rho : \mathcal{V}(G/H) \to \mathbb{X}^\vee(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Lemma 6.2.9 *ICI AJOUTER UNE PREUVE DU FAIT QUE LA B-ORBITE EST AFFINE.*

Corollary 6.2.10 *The map ρ is injective.*

Proof. Let ν and ν' be two different elements of $\mathcal{V}(G/H)$. Since BH/H is affine, we have the equalities $k(G/H) = k(BH/H) = \mathrm{Frac}(k[BH/H])$ and since any valuation on $\mathrm{Frac}(R)$ is determined by its value on R, there exists $f \in k[BH/H]$ with $\nu(f) < \nu'(f)$. By the previous corollary, there exists $f' \in k(G/H)^{(B)}$ with $\nu(f') = \nu(f^n) < \nu'(f) \leq \nu'(f')$ proving the injectivity. $\square$

Chapter 7

Simple spherical embeddings

In this chapter, the variety X will be a spherical variety. We fix $x_0 \in X$ a base point such that $Gx_0 \simeq G/H$ is a dense G-orbit of X and we fix B a Borel subgroup of G such that Bx_0 is dense in X.

7.1 Simple spherical varieties

Let us start with the following lemma. Let X be a spherical variety and Y be a G-orbit in X. Define the subset $X_{Y,G}$ of X by

$$X_{Y,G} = \{x \in X \ / \ Y \subset \overline{Gx}\}.$$

Lemma 7.1.1 *The subset $X_{Y,G}$ is G-stable, open in X and Y is the unique closed G-orbit of $X_{Y,G}$. In particular X can be covered by G-stable open subsets with a unique closed orbit.*

Proof. The subset $X_{Y,G}$ is obviously G-stable. Let $x \in X \setminus X_{Y,G}$. Then $\overline{Gx} \not\supset Y$ thus for any $x' \in \overline{Gx}$ we have $\overline{Gx'} \not\supset Y$ *i.e.* $x' \notin X_{Y,G}$. Thus $\overline{Gx} \subset X \setminus X_{Y,G}$. Since there are only finitely many G-orbits and $X \setminus X_{Y,G}$ is the union of the closure of G-orbits of its point, this set is closed and $X_{Y,G}$ is open.

Let $x \in X_{Y,G}$ and assume that Gx is closed. Then $Gx = \overline{Gx} \supset Y$ thus $Gx = Y$. $\square$

Definition 7.1.2 *A spherical variety is called simple if it contains a unique closed orbit.*

The former lemma proves that spherical varieties are covered by simple spherical varieties.

7.2 Some canonical B-stable open affine subsets

Definition 7.2.1 *Let X be a spherical variety.*

(ı) We denote by $D(X)$ the finite set of B-stable prime divisors of X.

(ıı) For Y a G-orbit of X, we denote by $D_Y(X)$ the subset of $D(X)$ of divisors D with $Y \subset D$.

(ııı) We define the open subset $X_{Y,B}$ of X as follows:

$$X_{Y,B} = X \setminus \bigcup_{D \in D(X) \setminus D_Y(X)} D.$$

Let us start with the following facts.

Fact 7.2.2 *Let $D \in D(X)$ be an irreducible B-stable divisor of X. Then we have the alternative*

(ı) the divisor D has a dense open B-orbit contained in G/H the dense open G-orbit of X or

(ıı) the divisor D is G-stable.

Proof. Assume that D meets non trivially the orbit $Gx \simeq G/H$. Then $D \cap G/H$ is open in D and meets the dense B-orbit By of D. Therefore y is in the orbit of X and $By \subset Gx$. We are in the first case.

Conversely if D does not meet Gx, then D is contained in the complement $X \backslash Gx$. This complement is G-stable and of codimension at least 1. Therefore D has to be an irreducible component of this complement and because G is connected, D is mapped to itself by G thus D is G-stable. □

Fact 7.2.3 *Let X be a spherical variety and Y be a G-orbit. Let X_0 be a B-stable open affine subset of X which meets Y non trivially as constructed in Proposition 4.3.4. Then $X \setminus X_0$ is a union of divisors.*

Proof. The affine open subset X_0 was constructed from a projective compactification $\overline{X}$ of X by removing divisors. Therefore $\overline{X} \backslash X_0$ is a union of divisors. But since X is locally closed, the complement $\overline{X} \setminus X$ has to be closed. This implies that the subset $X \setminus X_0$ of X has irreducible components of codimension 1. □

Theorem 7.2.4 *Let X be a spherical variety and let Y be a G orbit of X.*

(i) The open set $X_{Y,B}$ is B-stable open and affine.

(ii) The variety Y is the unique closed orbit of $GX_{Y,B}$.

(iii) We have the equality

$$X_{Y,B} = \{x \in X \ / \ \overline{Bx} \supset Y\}.$$

In particular, the intersection $Y \cap X_{Y,B}$ is a B-orbit.

(iv) The G-orbit of $X_{Y,B}$ is the simple spherical variety $X_{Y,G}$.

(v) The complement $X_{Y,G} \setminus X_{Y,B}$ is the union of all B-stable irreducible divisors of $X_{Y,G}$ not containing Y. These divisors are not G-stable and are Cartier and globally generated in $X_{Y,G}$.

Proof. (i) Let X_0 be an open affine B-stable subset meeting Y non trivially and such that modulo taking powers, the elements of $k[X_0 \cap \overline{Y}]^{(B)}$ can be lifted to $k[X_0]^{(B)}$. By the previous fact, the open set X_0 is obtained from X by removing some B-stable divisors not containing Y.

Furthermore, there exists a function $f \in k[X_0]$ which does not vanish on Y but does vanish on any $D \in D(X) \setminus D_Y(X)$. Let ν_0 be a valuation with center Y, we have $\nu_0(f) = 0$ and $\nu_D(f) > 0$ for all $D \in D(X) \setminus D_Y(X)$. By Corollary 6.2.7, there exist $n \in \mathbb{N}$ and function $f' \in k(G/H)^{(B)}$ with $\nu_0(f') = \nu_0(f^n)$, $\nu(f') \geq \nu(f^n)$ for all $\nu \in \mathcal{V}(G/H)$ and $\nu_{D'}(f') \geq \nu_{D'}(f^n)$ for all $D' \in D(G/H)$. Note that for a divisor $D \in D(X)$, if D has a dense B orbit contained in the dense G-orbit of X, we have the equality $\nu_D = \nu_{D'}$ with $D' \in D(G/H)$ while if D is G-stable, we have $\nu_D \subset \mathcal{V}(G/H)$. Therefore for all $D \in D(X) \setminus D_Y(X)$, we have $\nu_D(f') > 0$. Note that f' is therefore defined where f is defined therefore $f' \in k[X_0]$. In particular, if we set $X_0' = X_0 \cap \{x \in X \ / \ f'(x) \neq 0\}$ we get an affine open subset meeting Y non trivially and contained in $X_{Y,B}$.

Let us prove that $X_0' = X_{Y,B}$. Indeed, the complement of X_0 in X is a union of B-stable divisors. These divisors do not contain Y since X_0 meets Y non trivially. The same is true for X_0'. This proves the converse inclusion.

(ii) Let Z be a G-orbit in $X_{Y,B}$ non containing Y in its closure. Then there exists $f \in [X_{Y,B}]$ such that f vanishes on Z and not on Y. We may again assume that f is a B eigenfunction (for example using Corollary 6.2.7 again). Then Z is contained in the B-stable divisor $D = \{x \in X_{Y,B} \ / \ f(x) = 0\}$ not containing Y. If thus divisor is non empty, then its closure in X is a B-stable divisor D' not containing Y *i.e.* $D' \in D(X) \setminus D_Y(X)$. A contradiction since D' does not meet $X_{Y,B}$.

(iii) Let $x \in X$ such that $\overline{Bx} \supset Y$. Then x is not contained in any $D \in D(X) \setminus D_Y(X)$ otherwise Bx would also be contained in D and therefore $Y \subset D$. A contradiction. Conversely, let $x \in X_{Y,B}$

and assume that Y is not contained in $\overline{Bx}$. Then there exists a B-eigenfunction $f \in k[X_{Y,B}]$ such that f vanishes on $\overline{Bx}$ but not on Y. The divisor D of the vanishing locus of f would be in $D(X) \setminus D_Y(X)$ and meet $X_{Y,B}$ in x a contradiction.

Let Y_B^0 be the dense B-orbit in Y. It is contained in $X_{Y,B}$. If there is another B-orbit in Y, then the elements of that orbit are not in $X_{Y,B}$ by what we just proved.

(ɪv) This is clear.

(v) The complement of $X_{Y,B}$ in X being a divisor and $X_{Y,B}$ being open in $X_{Y,G}$, the proof of Fact 7.2.3 applies thus the complement of $X_{Y,B}$ in $X_{Y,G}$ is a divisor which has to be B-stable. Let D be such a divisor, then $D = \overline{Bx}$ and $x \notin X_{Y,B}$ is equivalent to $Y \not\subset \overline{Bx} = D$ proving that these divisors are the divisors not containing Y. Furthermore, if D is such a divisor and is G-stable, then $D = \overline{Bx} = \overline{Gx}$ for some $x \in X_{Y,G}$ thus $Y \subset \overline{Gx} = \overline{Bx} = D$. A contradiction. Therefore these divisors are not G-stable.

Let $\mathcal{D}$ be the union of all the above divisors and let $X^{\rm sm} \to X$ be the inclusion of the smooth locus in X. Let $L^{\rm sm} = \mathcal{O}_{X^{\rm sm}}(D \cap X^{\rm sm})$. This is an invertible sheaf on X. Replacing G by a finite cover, we may assume that $L^{\rm sm}$ is G-linearised. The group G acts on $L = i_* L^{\rm sm}$. Since X is normal we get $L = \mathcal{O}_X(D)$. The locus in X where L is not locally free is a closed G-stable subset contained in D. It has to contain a closed G-orbit. Its intersection with $X_{Y,G}$ has to contain a closed G-orbit as well but the only closed G-orbit is Y which is not contained in D thus D is Cartier.

Let s be the canonical section of $L = \mathcal{O}_X(D)$. The group G acts on the sections of L thus gs is again a section and the locus where all these section vanish is a closed G-orbit therefore empty in $X_{Y,G}$. ☐

7.3 Classification of simple spherical embeddings

As already noticed. A B-stable divisor D in X is either G-stable or its intersection with the dense G-orbit is a non trivial B-stable divisor of G/H.

Definition 7.3.1 *For any G-orbit Y of a spherical variety X, we define the sets $\mathcal{F}_Y(X)$ and $G_Y(X)$ as follows:*

$$\mathcal{F}_Y(X) = \{D \cap G/H \in D(G/H) \ / D \in D_Y(X) \text{ is not } G\text{-stable}\} \text{ and}$$
$$G_Y(X) = \{\nu_D \in \mathcal{V}(G/H) \ / \ D \in D_Y(X) \text{ is } G\text{-stable}\}.$$

Theorem 7.3.2 *A simple spherical embedding X of G/H with closed orbit Y is completely determined by the pair $(\mathcal{F}_Y(X), G_Y(X))$.*

Proof. Let X' another simple embedding of G/H with closed orbit Y' and with the same data: $(\mathcal{F}_{Y'}(X'), G_{Y'}(X')) = (\mathcal{F}_Y(X), G_Y(X))$. Let $X_{Y,B}$ and $X_{Y',B}$ the corresponding B-stable affine subsets. Let us define the open set X_0 by

$$X_0 = G/H \qquad \bigcup_{D \in D(G/H) \setminus \mathcal{F}_Y(X)} .$$

This set is the same for X and X'. Now since X and X' are normal, so are open subsets in them. Furthermore, for a normal variety any function can be extended in codimension 2. We therefore get the equalities

$$k[X_{Y,B}] = \{f \in k[X_0] \ / \ \nu(f) \geq 0 \text{ for all } \nu \in G_Y(X)\} = k[X_{Y',B}].$$

Therefore, the G-birational isomorphism between X and X' induces an isomorphism $X_{Y,B} \simeq X_{Y',B}$ and therefore an isomorphism $X = X_{Y,G} \simeq X_{Y',G} = X'$. ☐

We will now replace the pair $(\mathcal{F}_Y(X), G_Y(X))$ by a colored cone. Let us recall some basic fact on convex geometry.

Definition 7.3.3 *(ı) Let V be a $\mathbb{Q}$-vector space, a cone C is a subset of V stable under addition and multiplication by elements in $\mathbb{Q}_{\geq 0}$.*

(ıı) The dual of a cone $C \subset V$ is the subset $C^\vee$ of $V^\vee$ defined by

$$C^\vee = \{f \in V^\vee \ / \ f(v) \geq 0 \text{ for all } v \in C\}.$$

(ııı) A cone is strictly convex if $C \cap -C = 0$ or equivalentely C contains no line.

(ıv) A cone is polyedral if C can be written $C = \mathbb{Q}_{\geq 0}v_1 + \cdots \mathbb{Q}_{\geq 0}v_n$.

(v) A face of the cone is a subset of the form $F_f = \{v \in C \ / \ f(v) = 0\}$ for some $f \in C^\vee$.

(vı) The dimension of a cone is the dimension of its linear span.

(vıı) An extremal ray is a face of dimension one.

(vııı) The relative interior C^0 of C is the complement of all proper faces of C.

In the sequel we shall consider the vector space $\mathbb{V} = \mathbb{X}^\vee(G/H)_\mathbb{Z} \otimes \mathbb{Q}$ and its dual $\mathbb{V}^\vee = \mathbb{X}(G/H)_\mathbb{Z} \otimes \mathbb{Q}$. Recall that we have an injective map $\rho : \mathcal{V}(G/H) \to \mathbb{V}$ defined by $\rho(\nu(f)) = \nu(f)$ and therefore an inclusion $G_Y(X) \subset \mathbb{V}$. We also have a map

$$D(G/H) \to \mathbb{V}$$

defined by $D \mapsto \rho(\nu_D)$. This map is *a priori* not injective and is indeed not injective in general.

Definition 7.3.4 *We define the cone $\mathcal{C}_Y^\vee(X)$ contained in $\mathbb{V}$ as the cone generated by $G_Y(X)$ and $\rho(\mathcal{F}_Y(X))$.*

Lemma 7.3.5 *The sets $\mathbb{Q}_{\geq 0}v$ with $v \in G_Y(X)$ are exactely the extremal rays of $\mathcal{C}_Y^\vee(X)$ which do not contain an element of $\rho(\mathcal{F}_Y(X))$.*

Proof. Let D be a G-stable divisor of X and let ν_D be the corresponding element in $G_Y(X)$. By Corollary 6.2.7 there exists a function $f \in k(G/H)^{(B)}$ such that f vanishes on all $D' \in D(X)$ except D. Therefore the face defined by f is an extremal ray containing D and not containing any element of $\rho(\mathcal{F}_Y(X))$. The converse is obvious by definition of the cone. □

Proposition 7.3.6 *Let X be a simple spherical embedding of G/H with closed orbit Y. Let $\nu \in \mathcal{V}(G/H)$.*

(ı) We have the equality $k[X_{Y,B}]^{(B)} = \{f \in k(G/H)^{(B)} \ / \ \chi_f \in \mathcal{C}_Y^\vee(X)^\vee\}$.

(ıı) The center of ν exists if and only if ν is in $\mathcal{C}_Y^\vee(X)$.

(ııı) The center of ν is Y if and only if ν is in $\mathcal{C}_Y^\vee(X)^0$.

Proof. (ı) A function f in $k[X_{Y,B}]^{(B)}$ is an element in $k(G/H)^{(B)}$ and is defined on the divisors $D \in D_Y(X)$ thus non negative on $\mathcal{C}_Y^\vee(X)$.

Conversely, a function $f \in k(G/H)^{(B)}$ non negative on $\mathcal{C}_Y^\vee(X)$ is defined on BH/B and on all $D \in D_Y(X)$. The union of those form an open subset of $X_{Y,B}$ of codimension bigger than 2. By normality it is defined on $X_{Y,B}$.

(ıı) Because $X_{Y,B}$ is affine, because ν is G-invariant and because $X = GX_{Y,B}$, the valuation ν has a center if and only if ν is non negative on $k[X_{Y,B}]$. By (ı) this is equivalent to $\nu \in \mathcal{C}_Y^\vee(X)$.

(ııı) If Y is the center of ν, then any $f \in k[X_{Y,B}]^{(B)}$ with $\nu(f) = 0$ will not vanish on Y and therefore will not vanish at all (since the zero locus of f is a B-stable divisor of $X_{Y,B}$ thus has to contain Y). Thus for any $\rho(\nu') \in \mathcal{C}_Y^\vee(X)$, we have $\nu'(f) = 0$ thus the weight of f is in the vertex of $\mathcal{C}_Y^\vee(X)^\vee$. This proves that ν is in the interior of $\mathcal{C}_Y^\vee(X)$.

Now let $\nu' \in \mathcal{C}_Y^\vee(X)$ with center Z strictly bigger that Y. Let ν_Y be the valuation associated to Y. There exists $f \in k[X_{Y,B}]$ with $\nu(f) = 0$ and $\nu_Y(f) > 0$. By Corollary 6.2.7 we may assume f to be a B-eigenfunction and we get that ν' is not in the interior of the cone. $\square$

Definition 7.3.7 *A colored cone for G/H, is a pair $(\mathcal{C}, \mathcal{F})$ with $\mathcal{C} \subset \mathbb{V}(G/H)$ and $\mathcal{F} \subset D(G/H)$ having the following properties.*

(CC1) The set $\mathcal{C}$ is a cone generated by $\rho(\mathcal{F})$ and finitely many elements of $\mathcal{V}(G/H)$ (seen as a subset of $\mathbb{V}(G/H)$).

(CC2) The intersection $\mathcal{C}^0 \cap \mathcal{V}(G/H)$ is non empty.

The colored cone is called strictly convex if the following condition holds.

(SCC) The cone $\mathcal{C}$ is strictly convex and $0 \notin \rho(\mathcal{F})$.

Theorem 7.3.8 *The map $X \mapsto (\mathcal{C}_Y^\vee(X), \mathcal{F}_Y(X))$ is a bijection between the isomorphism classes of simple spherical embeddings X of G/H with closed orbit Y and strictly convex colored cones.*

Proof. Let us first check that the map is well defined. Let X be such a simple embedding. Then we already know that (CC1) is satisfied and by the previous proposition, since Y induces an invariant valuation in the interior of the cone, we know that (CC2) is also satisfied. Finaly, to prove that the cone is strictly convex we need to prove that there is no linear subspace in it. Let us denote by $\mathcal{D}$ the union of B-stable divisors containing Y, these are exactly the B-stable divisors in $X_{Y,B}$. Then there exists $f \in k[X_{Y,B}]$ with $\mathcal{D} \subset \{x \in X_{Y,B} \ / \ f(x) = 0\}$. By Corollary 6.2.7, we may choose $f \in k[X_{Y,B}]^{(B)}$ and for any ν_D with $D \in D_Y(X)$ we have $\nu_D(f) > 0$. Therefore the generators (resp. the cone $\mathcal{C}_Y^\vee(X)$) are (is) in the half-space with positive (non negative value) on f proving (SCC).

The injectivity of the map follows from Theorem 7.3.2. Let us prove the surjectivity. Let $(\mathcal{C}, \mathcal{F})$ be a colored cone with the above three properties (CC1), (CC2) and (SCC). By (CC1), there exists a finite set of elements $g_1, \cdots, g_n \in k(G/H)^{(B)}$ such that the weights of the g_i span $\mathcal{C}^\vee \cap \mathbb{X}(G/H)$ as a monoid. Denote by $\pi : G \to G/H$ the quotient map and let $\mathcal{D}$ be the union of all divisors $D \in D(G/H) \setminus \mathcal{F}$. This is a B-stable divisor of G/H therefore there exists an element $f_0 \in k(G)^{(B \times H)}$ with $\mathcal{D} = \{x \in G/H \ / \ f_0(x) = 0\}$ and $f_i = f_0 g_i \in k[G]$ for all $i \in [1, n]$. Let W be the G-submodule of $k[G]$ spanned by the $(f_i)_{i \in [0,n]}$. Since the g_i are H-invariants while f_0 is a H-eigenfunction of weight $\chi \in X^*(H)$, we get that all the elements in W are H-eigenfunctions of weight χ. Therefore we have a G-equivariant morphism

$$\varphi : G/H \to \mathbb{P}(W^\vee)$$

defined by $x \mapsto [f_0(x) : \cdots : f_n(x)]$. Let $D(f_0)$ be the open subset of $\mathbb{P}(W^\vee)$ defined by the non vanishing of f_0. Let us define X_0 and X by

$$X_0 = \overline{\varphi(G/H)} \cap D(f_0) \text{ and } X = GX_0.$$

Lemma 7.3.9 *We have the equality $k[X_0]^{(B)} = \{f \in k(G/H)^{(B)} \ / \ \nu(f) \geq 0$ for all $\nu \in \mathcal{C}\}$.*

Proof. Let $\mathcal{M} = \{f \in k(G/H)^{(B)} \ / \ \nu(f) \geq 0$ for all $\nu \in \mathcal{C}\}$ and let $f \in \mathcal{M}$. Let us prove that f induces a function on X_0. Indeed, the weight of f is a linear combinaison with non negative integer coefficients $a_1, \cdots, a_n$ of the weights of $g_1, \cdots, g_n$. Consider $F = \prod_{i=1}^n g_i^{a_i}$. The functions f and F have the same weight thus f/F is B-invariant and thus constant on BH/H and thus on G/H. Thus f is a constant multiple of F which is defined on $D(f_0)$ and thus on X_0.

Now let $f \in k[X_0]^{(B)}$. To prove the converse inclusion we only have to consider $\rho(\nu) \in \mathcal{C}$ with $\nu \in \mathcal{V}(G/H)$ or $\nu = \nu_D$ with $D \in \mathcal{F}$. In the second case, since $\varphi(D)$ meets non trivially $D(f_0)$, then

f is defined on an open subset of D thus $\nu_D(f) \geq 0$. In the first case, we simply remark that $k[X_0]$ is a quotient of the symmetric algebra $S^*(W^\vee)$. But an element $f' \in W$ satisfies $\nu(f') \geq 0$ proving the result. $\square$

Note that this implies that the weights of elements in $k[X_0]^{(B)} = \mathcal{M}$ span $\mathbb{X}(G/H)$. Indeed, the orthogonal of the linear span of these weights has to be contained in $\mathcal{C}$. As $\mathcal{C}$ contains no linear subspace by (SCC), this orthogonal is trivial. But as X_0 is an affine open subset of X we get that $\mathbb{X}(X) = \mathbb{X}(G/H)$.

Lemma 7.3.10 *The fibers of φ are finite.*

Proof. Let us first check that these fibers are affine. Let $D \in \mathcal{F}$ be a B-stable divisor. Then $\nu_D(f) \geq 0$ for all $f \in k[X_0]^{(B)}$ and by (SCC) there exists f with $\nu_D(f) > 0$. Thus $D \subset \{x \in X \ / \ f(x) = 0\}$ and $\varphi|_D$ is not dominant on X. If $D \in D(G/H) \setminus \mathcal{F}$, then $\varphi(D) \subset \{x \in X \ / \ f_0(x) = 0\}$ and again $\varphi|_D$ is not dominant on X. Since φ is equivariant, this implies that the proper B-orbits of G/H are mapped to proper B-orbits in X thus $\varphi^{-1}(\varphi(BH/H)) = BH/H$. This B-orbit and its image are affine thus the restriction of φ is a morphism between affine varieties and the fibers are therefore affine.

Let us now prove that the fibers are proper. For this consider X' a proper embedding of G/H, let $x = \varphi(H/H)$ and $x' \in X$ the point of X' corresponding to H/H. Denote by $X'' = \overline{G(x, x')} \subset X \times X'$. Since $\mathrm{Stab}(x) \supset H$, we have that G/H is an open subset of X''. Furthermore, the projection $p : X'' \to X$ is proper since X' is proper. Assume that therefore exists Z a G-stable closed subset disjoint from G/H in X'' with $p|_Z$ dominant on X. Let $\nu_Z \in \mathcal{V}(G/H)$ the corresponding valuation. The map p induces an injection of fields $k(X) \to k(X'') = k(G/H)$ and let $f \in k(X)$ a non trivial element, the corresponding function induced on Z is also non trivial since Z is dominant. In particular $\nu_Z(f) = 0$. In particular, for any $f \in k[X_0]^{(B)}$, we have $\nu_Z(f) = 0$. And by our remark ν_Z vanishes on $\mathbb{X}(G/H)$ and also on $\mathbb{V}^\vee$. This implies since ρ is injective on $\mathcal{V}(G/H)$ that ν_Z is trivial. A contradiction. As above, since p is G-equivariant and X'' has finitely many G-orbits (there is a dense B-orbit) we get that $p^{-1}(p(G/H)) = G/H$ thus $\varphi|_{G/H} = p|_{G/H}$ is proper thus has proper fibers.

The fibers of φ are proper and affine thus finite. $\square$

We may therefore factor the map φ into the following composition

$$\begin{tikzcd} G/H \arrow[r] \arrow[dr, "\varphi"'] & X''' \arrow[d, "\psi"] \\ & X \end{tikzcd}$$

where the horizontal map is an open embedding while the map ψ is finite. Note that X''' has a structure of G-variety so that all the maps are G-equivariant. We claim that X''' is a simple embedding with colored cone $(\mathcal{C}, \mathcal{F})$. Let $X_0''' = \psi^{-1}(X_0)$ which is an affine B-stable open subset of X'''. The same proof as in Lemma 7.3.9 gives that $k[X_0''']^{(B)} = \mathcal{M} = k[X_0]^{(B)}$. Note also that $X''' = GX_0'''$.

Lemma 7.3.11 *The variety X''' has a unique closed G-orbit Y which is contained in all B-stable divisors of X_0'''.*

Proof. Let $\nu_Y \in \mathcal{C}^0 \cap \mathcal{V}(G/H)$. Such an element exists by (CC2). By definition ν_Y is non negative on $k[X_0''']^{(B)}$. By Corollary 6.2.7 it is non negative on $k[X_0''']$. Therefore it has a center Y_0 in X_0''' and by G-invariant it has a center Y in X'''.

Let Z be a G-stable closed subvariety of X''' and assume that $Y \not\subset Z$. The open subset X_0''' has to meet Z (since $X''' = GX_0'''$) thus ν_Z is non negative on $k[X_0''']$ and therefore $\nu_Z \in \mathcal{C}$. On the other hand, there exists a function $f \in k[X_0''']$ with $\nu_Z(f) > 0$ and $\nu_Y(f) = 0$. By Corollary 6.2.7, we may assume f to be a B-eigenfunction. But ν_Y is in $\mathcal{C}_Y^\vee(X)^0$ thus $\nu_y(f) = 0$ implies $\nu_Z(f) = 0$. A contradiction. Thus $Y \subset Z$ and Y is the only closed orbit of G in X'''.

The same argument proves that Y is contained in any B-stable divisor of X_0'''. □

Note that by construction X''' is normal and by the previous lemma we have $X_0''' = X_{Y,B}'''$. This implies that $\mathcal{C}_Y^\vee(X''') \cap \mathbb{X}(G/H)$ is the set of weights of $k[X_0'''] = \mathcal{M}$ therefore is equal to $\mathcal{C} \cap \mathbb{X}(G/H)$. In particular we get the equality $\mathcal{C}_Y^\vee(X''') = \mathcal{C}$. This also proves the inclusion $\mathcal{F} \subset \mathcal{F}_Y(X''')$. Conversely, if D is not in $\mathcal{F}$, then we have $\varphi(D) \subset \{x \in X \ / \ f_0(x) = 0\}$ which is not in X_0 therefore D is not in X_0''' which means that D is not in $\mathcal{F}_Y(X''')$. This concludes the proof. □

Definition 7.3.12 *Let $(\mathcal{C}, \mathcal{F})$ be a colored cone. A pair $(\mathcal{C}_0, \mathcal{F}_0)$ is called a colored face of $(\mathcal{C}, \mathcal{F})$ is the following conditions are satisfied.*
(a) The set $\mathcal{C}_0$ is a face of the cone $\mathcal{C}$.
(b) The intersection $\mathcal{C}_0 \cap \mathcal{V}(G/H)$ is non empty.
(c) We have the equality $\mathcal{F}_0 = \mathcal{F} \cap \mathcal{C}_0$.

Lemma 7.3.13 *Let X be a spherical embedding of G/H and let Y be an orbit. Then there is a bijection $Z \mapsto (\mathcal{C}_Z^\vee(X), \mathcal{F}_Z(X))$ between the set of G-orbits in X such that $\overline{Z} \supset Y$ and the set of faces of $(\mathcal{C}_Y^\vee(X), \mathcal{F}_Y(X))$.*

Proof. First note that if Z is a G-orbit with $\overline{Z} \supset Y$, then we have the inclusions $\mathcal{F}_Z(Y) \subset \mathcal{F}_Y(X)$ and $G_Z(X) \subset G_Y(X)$. Therefore also an inclusion $\mathcal{C}_Z^\vee(X) \subset \mathcal{C}_Y^\vee(X)$. Obviously this implies the inclusion $X_{Z,B} \subset X_{Y,B}$ corresponding to the localisation morphism $k[X_{Y,B}] \to k[X_{Z,B}]$. Let $D \in D_Y(X)$ and let f be a function (we can choose it to be a B-eigenfunction) such that f vanishes on the elements $D' \in D_Z(X)$ but not on D (*i.e.* $\nu_D(f) = 0$ but $\nu_{D'}(f) > 0$). Then $\mathcal{C}_Z^\vee(X)$ is in the face defined by f while D is not. As we can do this for any $D \in D_Y(X)$, we get that $\mathcal{C}_Z^\vee(X)$ is the intersection of all these faces. Furthermore ν_Z is in the interior of $\mathcal{C}_Z(X)$ thus $(\mathcal{C}_Z(X), \mathcal{F}_Z(X))$ is indeed a colored face of $(\mathcal{C}_Y^\vee(X), \mathcal{F}_Y(X))$.

Now if $(\mathcal{C}_0, \mathcal{F}_0)$ is a colored face of $(\mathcal{C}_Y^\vee(X), \mathcal{F}_Y(X))$, an element $\nu \in \mathcal{C}_0^0$ gives rise to a center which is closed and G-stable. This center has a unique dense G-orbit which we denote by Z. Note that $Y \subset \overline{Z}$. If D is an element in $D_Y(X)$, then Z is not contained in D if and only if there exists $f_D \in k[X_{Y,B}]$ such that $\nu(f_D) = 0$ while $\nu_D(f_D) > 0$. Therefore ν is in the faces F_D defined by all these functions f_D and since ν is in the interior of $\mathcal{C}_0$, this implies that $\mathcal{C}_0$ is contained in all the faces F_D. On the other hand, the element $\rho(\nu_D)$ is not in the face F_D therefore not in $\mathcal{C}_0$. This $\mathcal{C}_0$ does not contain any $\rho(\nu_D)$ for $D \in D_Y(X) \setminus D_Z(X)$ which in turn implies the inclusion $\mathcal{C}_0 \subset \mathcal{C}_Z^\vee(X)$. Conversely, if $D \in D_Z(X)$, then for $f \in k[X_{Y,B}]^{(B)}$ we have the implication $\nu(f) = 0 \Rightarrow \nu_D(f) = 0$. Therefore $\rho(\nu_D)$ is in any face containing ν. But ν being in the interior of $\mathcal{C}_0$ this implies the inclusion $\rho(\nu_D) \in \mathcal{C}_0$ and thus $\mathcal{C}_Z^\vee(X) \subset \mathcal{C}_0$. This proves the equality $\mathcal{C}_0 = \mathcal{C}_Z^\vee(X)$. The equality $\mathcal{F}_0 = \mathcal{F}_Z(X)$ follows by definition of colored faces. □

7.4 Some examples

Example 7.4.1 Consider the simplest simple embedding: $X = G/H$. Then the unique closed orbit Y is G/H itself. The open subset $X_{Y,B}$ is the dense B-orbit BH/H. In this case the cones $\mathcal{C}_Y^\vee(X)$ and $\mathcal{C}_Y^\vee(X)^\vee$ are trivial. The trivial valuation has the all variety *i.e.* Y as center. Note that $\mathcal{V}(G/H)$ is $\{0\}$ in this case.

Example 7.4.2 Let $G = \mathrm{SL}_2$ and $H = U$ where U is the maximal unipotent subgroup of the subgroup of upper-triangular matrices. Then it is easy to check that G/H is isomorphic to $\mathbb{A}^2 \setminus \{0\}$. Consider

$X = \mathbb{A}^2$. The SL_2-orbits are $\mathbb{A}^2 \setminus \{0\}$ and $Y = \{0\}$. The variety Y is the unique closed orbit. The B-orbits are $B(1,0) = \{(a,b) \in \mathbb{A}^2 \ / \ a \neq 0\}$, $B(0,1) = \{(a,b) \in \mathbb{A}^2 \ / \ a = 0 \text{ and } b \neq 0\}$ and $B(0,0) = \{0\}$. There is a unique B-stable divisor and its closure contains Y. We thus have $X_{Y,B} = X$.

We have $k(G/H)^{(B)} = k(\mathbb{A}^2)^{(B)} = \{\text{monomials in } a\}$ with a the function given by the first coordinate. The cone

$$\mathcal{C}_Y^\vee(X) = \mathbb{Z}_{\geq 0}\nu_a$$

is one dimensional generated by the valuation ν_a with respect to a. The image of the G-invariant valuation ν_Y is the valuation ν_a. We recover the fact that the valuation is in the interior if and only if its center is Y. We have the equality $\mathcal{V}(G/H) = \mathbb{V}(G/H)$. The valuations form a vector space of dimension 1.

Example 7.4.3 We can consider the same variety $X = \mathbb{A}^2$ as a spherical variety for $G = B = T = \mathbb{G}_m^2$. In that case the G-orbits are the same as the B-orbits and are described by $\{(a,b) \in \mathbb{A}^2 \ / \ a \neq 0,\ b \neq 0\}$, $\{(a,b) \in \mathbb{A}^2 \ / \ a \neq 0,\ b = 0\}$, $\{(a,b) \in \mathbb{A}^2 \ / \ a = 0,\ b \neq 0\}$ and $\{(a,b) \in \mathbb{A}^2 \ / \ a = b = 0\} = Y$. The variety Y is the only closed orbit. We have two B-stable divisors and again $X_{Y,B} = X$.

On the level of invariants we have $k(G/H)^{(B)} = k(\mathbb{A}^2)^{(T)} = \{\text{monomials in } a \text{ and } b\}$ with a, b the coordinate functions. The cone

$$\mathcal{C}_Y^\vee(X) = \mathbb{Z}_{\geq 0}\nu_a \oplus \mathbb{Z}_{\geq 0}\nu_b$$

is of dimension 2. In this case we again have the equality $\mathcal{V}(G/H) = \mathbb{V}(G/H)$.

Example 7.4.4 Consider the vector space V of quadratic forms on k^n and set $X = \mathbb{P}(V)$. The group $G = \mathrm{PGL}_n$ acts on X and the stabiliser of the element $q_n = \sum_{i \leq n} x_i^2$ is the group $H = PO_n$. The group G has a dense orbit $G/H = Gq$ in X.

Furthermore, it is easy to compute the G-orbits in X: there orbits are given by the rank. Denote by Y_k the G-orbit of quadratic forms of rank k, then $Y_n = G/H$ is dense while $Y_1 \simeq \mathbb{P}^{n-1}$ is the unique closed G-orbit in X. In particular X is a simple embedding.

Let us describe some B-orbits. Write an element in V as a matrix $A = (a_{i,j})$. Then the polynomial function Δ_k defined by the principal k-th minor of A is a B-eigenfunctions with weight $2\sum_{i \leq k} \varepsilon_i$. On ecan easily check that the B-orbit Bq_n is the locus defined by the non vanishing of these functions:

$$Bq_n = \bigcap_{i=1}^n D(\Delta_i).$$

This orbit is therefore dense in X. The complement of Bq_n is therefore the union of the irreducible divisor defined by $\Delta_k = 0$ for $k \in [1, n]$. Denote by D_k the divisor defined by Δ_k. The only G-stable divisor in D_n. Furthermore, since $q_1 = x_1^2$ is in Y and since we have $Y = \overline{Bq_1}$, we deduce that $Y \subset D_k$ for $k > 1$ but $Y \not\subset D_1$. We deduce $X_{Y,B} = D(\Delta_1)$.

Let us now compute the B-eigenfunctions of $X_{Y,B}$. Let f be such a function, then f is of the form

$$f = \mathrm{cst} \cdot \prod_{i=1}^n {\Delta_i}^{a_i}$$

with $a_i \in \mathbb{Z}_{\geq 0}$ for $i > 1$ and $a_1 \in \mathbb{Z}$. The generators of the weight monoid are therefore the weights of the functions Δ_k/Δ_1^k for $k > 1$. These weights are

$$2\sum_{i=2}^k (\varepsilon_i - \varepsilon_1) = -2\sum_{i=1}^k (k+1-i)\alpha_i.$$

The cone $\mathcal{C}_Y^\vee(X)$ is the dual of the cone spanned by these weights. It is a simplicial cone of dimension $n-1 = \dim V(G/H) = \mathrm{rk}(X)$. On this cone there are $n-1$ colors which span each of the extremal rays. If $(\varepsilon_i)_{i\in[1,n]}$ is an orthonormal basis, then the cone $\mathcal{C}_Y^\vee(X)$ is spanned by ε_n and $\varepsilon_i - \varepsilon_{i+1}$ for $i \in [2, n-1]$.

Note that in this case not all the faces are colored faces (ortherwise there would be 2^{n-1} irreducible closed G-stable subvarieties. This comes from the fact that the cone $\mathcal{V}(G/H)$ is not the all of $\mathbb{V}(G/H)$. It is strictly contained $\mathcal{C}_Y^\vee(X)$ and described as follows. The only G-orbits Y_k are given by the rank therefore the only irreducible closed G-stable subvarieties are their closure. But we have the equality

$$\overline{Y_k} = \bigcap_{i=k+1}^{n} D_i.$$

The cone of invariant valuations is therefore generated by the elements $\nu_k = \nu_{\overline{Y_k}}$. If $(\varepsilon_i)_{i\in[1,n]}$ is an orthonormal basis, then the cone $\mathcal{V}(G/H)$ is spanned by

$$\nu_k = \sum_{i=k+1}^{n} \varepsilon_i.$$

Chapter 8

Classification of spherical embeddings

8.1 Colored fans

Definition 8.1.1 *A colored fan $\mathbb{F}$ is a finite collection of colored cones $(\mathcal{C}, \mathcal{F})$ satisfying the following properties.*

(CF1) Every colored face of a colored cone $(\mathcal{C}, \mathcal{F})$ of $\mathbb{F}$ is in $\mathbb{F}$.

(CF2) For every $\nu \in \mathcal{V}(G/H)$ there is at most one colored cone $(\mathcal{C}, \mathcal{F})$ of $\mathbb{F}$ such that $\nu \in \mathcal{C}^0$.

A colored cone is called stricly convex if any cones of $\mathbb{F}$ are strictly convex. This is equivalent to the fact that $(0, \emptyset)$ is in $\mathbb{F}$.

Definition 8.1.2 *Let X be an embedding of G/H we define $\mathbb{F}(X)$ to be the set of all colored cones $(\mathcal{C}_Y^\vee(X), \mathcal{F}_Y(X))$ for Y a G-orbit in X.*

Theorem 8.1.3 *The map $X \mapsto \mathbb{F}(X)$ is a bijection between isomorphism classes of embedding and strictly colored fans.*

Proof. Assume first that X is a spherical embedding of G/H and let us prove that $\mathbb{F}(X)$ is a colored fan. By Lemma 7.3.13, we now that any face of a cone of $\mathbb{F}(X)$ is again a cone of $\mathbb{F}(X)$. If Y_1 and Y_2 are such that a valuation $\nu \in \mathcal{V}(G/H)$ lies in $\mathcal{C}_{Y_1}^\vee(X) \cap \mathcal{C}_{Y_2}^\vee(X)$, then their closure are the center of ν and must be equal. Thus $Y_1 = Y_2$. Finally, all the colored cones are strictly convex.

Conversely, let $\mathbb{F}$ be a colored fan. Then for any cone $(\mathcal{C}, \mathcal{F})$, there exists a simple spherical embedding $X(\mathcal{C}, \mathcal{F})$. These embedding are isomorphic on the smaller simple spherical embedding given by colored faces. We can therefore glue these embedding along there intersection to get X. This is a priori not a separated scheme. If we prove that it is separated, then it will be a spherical embedding with colored fan $\mathbb{F}$. So we are left to prove that X is indeed separated. By definition we need to prove that the diagonal embedding $X \to X \times X$ is closed. Let Y be an orbit of the closure, we want to prove that Y is contained in the diagonal. We may assume that Y is contained in a product $X_1 \times X_2$ with $X = X(\mathcal{C}, \mathcal{F})$ and $X_2 = X(\mathcal{C}', \mathcal{F}')$ with $(\mathcal{C}, \mathcal{F}) \neq (\mathcal{C}', \mathcal{F}')$ (otherwise Y will obviously be contained in the diagonal). Let X_3 be the normalisation of the closure of the diagonal embedding of G/H in $X_1 \times X_2$. The variety X_3 is a spherical embedding. Let Y_3 be an orbit in X_3 mapping onto Y. The valuation $\nu_{Y_3} \in \mathcal{V}(G/H)$ has $\overline{Y_3}$ for center. Let Y_1 and Y_2 be the orbits image of Y under the projections to X_1 and X_2. Then ν has Y_1 and Y_2 for center therefore $\nu \in \mathcal{C}_{Y_1}^\vee(X)^0 \cap \mathcal{C}_{Y_2}^\vee(X)^0 \cap \mathcal{V}(G/H)$. By (CF2) we get that the colored cones $\mathcal{C}_{Y_1}^\vee(X)^0$ and $\mathcal{C}_{Y_2}^\vee(X)^0$ are equal thus $(X_1)_{Y_1,G} = (X_2)_{Y_2,G}$ and Y is in the diagonal. □

8.2 Morphisms

Let $\varphi : G/H \to G/H'$ a dominant (surjective) G-equivariant morphism between homogeneous spherical varieties. This morphism induces a field extension $k(G/H') \to k(G/H)$ which in turn induces an injection $k(G/H')^{(B)} \to k(G/H)^{(H)}$. Taking weight leads to the injection

$$\varphi^* : \mathbb{X}(\mathbb{G}/H') \longrightarrow \mathbb{X}(G/H).$$

Taking duals induces a surjection

$$\varphi_* : \mathbb{X}^\vee(\mathbb{G}/H') \longrightarrow \mathbb{X}^\vee(G/H).$$

Fact 8.2.1 *We have the equality $\varphi_*(\mathcal{V}(G/H)) = \mathcal{V}(G/H')$.*

Proof. Proceed as in Corollary 6.2.3 using Lemma 6.2.1 and Lemma 6.2.2. □

Definition 8.2.2 *(ı) Denote by $\mathcal{F}_\varphi$ the set of $D \in D(G/H)$ such that φ maps D dominantly onto G/H'. Let $\mathcal{F}_\varphi^c$ be its complement in $D(G/H)$.*

(ıı) Denote again by $\varphi_* : \mathcal{F}_\varphi^c \to D(G/H')$ the map defined by $D \mapsto \varphi(D)$.

Definition 8.2.3 *(ı) Let $(\mathcal{C}, \mathcal{F})$ and $(\mathcal{C}', \mathcal{F}')$ be colored cones of G/H and G/H' respectively. We say that $(\mathcal{C}, \mathcal{F})$ mapso to $(\mathcal{C}', \mathcal{F}')$ is the following conditions holds.*

(CM1) We have the inclusion $\varphi_*(\mathcal{C}) \subset \mathcal{C}'$.

(CM2) We have the inclusion $\varphi_*(\mathcal{F} \setminus \mathcal{F}_\varphi) \subset \mathcal{F}'$.

(ıı) Let $\mathbb{F}$ and $\mathbb{F}'$ be colored fans of embeddings of G/H and G/H' respectively. The colored fan $\mathbb{F}$ maps to $\mathbb{F}'$ if every colored cone of $\mathbb{F}$ is mapped to a colored cone of $\mathbb{F}'$.

Theorem 8.2.4 *Let $\varphi : G/H \to G/H'$ be a surjective morphism between spherical homogeneous spaces. Let X and X' be embeddings of G/H and G/H' respectively.*

Then φ extends to a morphism $X \to X'$ if and only if $\mathbb{F}(X)$ maps to $\mathbb{F}(X')$.

Proof. We may and will assume that X and X' are simple embedding.

Assume that φ extends to such a morphism and let Y be the closed orbit in X. It is mapped to an orbit Y' in X'. This is the closed orbit: if Z is the closed orbit, then $Y \subset \varphi^{-1}(Z)$ thus $Y' \subset Z$ and both are orbit proving the equality. Furthermore, if $D \in D_Y(X)$, then if $\varphi(D)$ is not dense we have $\overline{\varphi(D)} \in D_{Y'}(X')$. This proves (CM2). Let us compare $X_{Y,B}$ and $X'_{Y',B}$. We have the equalities

$$X_{Y,B} = X \setminus \bigcup_{D \not\supset Y} D \text{ and } X'_{Y',B} = X' \setminus \bigcup_{D' \not\supset Y'} D'.$$

Let $x \in X_{Y,B}$, we claim that $\varphi(x) \in X'_{Y',B}$. If not, then there exists D' not containing Y' such that $\varphi(x) \in D'$. Let D be an irreducible component of $\varphi^{-1}(D')$ containing x. Then we have $Y \not\subset D$ (otherwise Y' would be contained in D') thus $\notin X_{Y,B}$ a contradiction. Therefore, if $f' \in k[X'_{Y',B}]$, then $\varphi^* f \in k[X_{Y,B}]$. This implies $\varphi^* \mathcal{C}_{Y'}(X')^\vee \subset \mathcal{C}_Y^\vee(X)^\vee$ and thus $\varphi_* \mathcal{C}_Y^\vee(X) \subset \mathcal{C}_{Y'}(X')$ which is (CM1).

Conversely, assume that (CM1) and (CM2) hold and consider the open subsets of the homogeneous spaces defined by

$$X_0 = G/H \setminus \bigcup_{D(G/H) \ni D \not\supset Y} D \text{ and } X'_0 = G/H' \setminus \bigcup_{D(G/H') \ni D' \not\supset Y'} D'.$$

As above, the map φ maps X_0 to X'_0 but we have seen the description of the B-stable affine open subsets and especially their algebras of functions:

$$k[X_{Y,B}] = \{f \in k[X_0] \ / \ \nu(f) \geq 0 \text{ for } \nu \in \mathcal{V}(G/H) \cap \mathcal{C}_Y^\vee(X)\} \text{ and}$$
$$k[X'_{Y',B}] = \{f' \in k[X'_0] \ / \ \nu'(f') \geq 0 \text{ for } \nu' \in \mathcal{V}(G/H') \cap \mathcal{C}_{Y'}^\vee(X')\}.$$

By (CM1), if $f' \in k[X'_0]$, then $\varphi^*(f') \in k[X_0]$ with $\nu(\varphi^*(f')) = \varphi_*(\nu)(f') \geq 0$ for $\nu \in \mathcal{V}(G/H) \cap \mathcal{C}_Y^\vee(X)$. Therefore the map φ extends on these affine subspaces and by G-invariance, it extends to X (since $X = GX_{Y,B}$). □

Definition 8.2.5 *Let $\mathbb{F}$ be a colored fan. The support of $\mathbb{F}$ denoted by* Supp$(\mathbb{F})$ *is the intersection of $\mathcal{V}(G/H)$ with the locus covered by the cones of $\mathbb{F}$. In symbols:*

$$\mathrm{Supp}(\mathbb{F}) = \mathcal{V}(G/H) \bigcap \left(\bigcup_{(\mathcal{C},\mathcal{F}) \in \mathbb{F}} \mathcal{C} \right).$$

Theorem 8.2.6 *Let $\varphi : X \to X'$ be a dominant morphism extending a surjective morphism $G/H \to G/H'$ between spherical embeddings.*

Then φ is proper if and only if $\mathrm{Supp}(\mathbb{F}(X)) = \varphi_*^{-1}(\mathrm{Supp}(\mathbb{F}(X')))$.

In particular X is proper if and only if $\mathrm{Supp}(\mathbb{F}(X)) = \mathcal{V}(G/H)$.

Note that the inclusion $\mathrm{Supp}(\mathbb{F}(X)) \subset \varphi_*^{-1}(\mathrm{Supp}(\mathbb{F}(X')))$ is always satisfied.

Proof. Let us recall the valuative criterion of properness (see for example [Har77, Theorem II.4.7]). A morphism $\varphi : X \to X'$ is proper if and only if for every valuation ring R with field of fractions K and for every commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "\varphi"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed, "\psi"] & X' \end{tikzcd}$$

we can complete the diagram with a unique morphism $\psi : \mathrm{Spec}R \to X$. Note that to get properness we may even only assume that the morphism $\mathrm{Spec}(K) \to X$ is dominant.

Let us first assume that φ is proper and let $\nu' \in \mathrm{Supp}(\mathbb{F}(X'))$. Let us first extend ν' to a valuation $\nu \in \mathcal{V}(G)$. Then $\nu : k(G) \to \mathbb{Q}$ is a valuation and we denote by R the associated ring defined by $R = \{f \in k(G) \ / \ \nu(f) \geq 0\}$. This valuation can be restricted to $k(G/H)$ (in other words ν' can be extended to $\nu \in \mathcal{V}(G/H)$. We are left to prove that $\nu \in \mathrm{Supp}(\mathbb{F}(X))$. Note that since ν' is in $\mathrm{Supp}(\mathbb{F}(X'))$, there exists an open affine subset $X'_{Y',B}$ such that ν' is non negative on $k[X'_{Y',B}]$. We thus have the inclusion $k[X'_{Y',B}] \subset R$ giving a morphism $\mathrm{Spec}(R) \to X'_{Y',B}$ which we can extend to $\mathrm{Spec}(R) \to X'$. By properness, we can lift this morphism to a morphism $\mathrm{Spec}(R) \to X$. The valuation ν above has therefore a center in X (take the image of the closed point of $\mathrm{Spec}(R)$). Therefore it has a center in any open subset $X_{Y,G}$ meeting this center non trivially. By Proposition 7.3.6 we have $\nu \in \mathcal{C}_Y^\vee(X)$ proving the equality $\mathrm{Supp}(\mathbb{F}(X)) = \varphi_*^{-1}(\mathrm{Supp}(\mathbb{F}(X')))$.

Conversely, assume that the equality $\mathrm{Supp}(\mathbb{F}(X)) = \varphi_*^{-1}(\mathrm{Supp}(\mathbb{F}(X')))$ holds and assume we have a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "\varphi"] \\ \mathrm{Spec}(R) \arrow[r] & X' \end{tikzcd}$$

as above. We want to lift the second horizontal map to X. By classical arguments (see [Gro60, Corollaire II.7.3.10(ii)]) we may assume that the top map meets the dense orbit.

Let us first produce an invariant valuation out of the map $\mathrm{Spec}(R) \to X'$. Consider the commutative diagram

$$\begin{tikzcd} G \times \mathrm{Spec}(K) \arrow[r] \arrow[d] & G \times X \arrow[r] \arrow[d] & X \arrow[d] \\ G \times \mathrm{Spec}(R) \arrow[r] & G \times X' \arrow[r] & X'. \end{tikzcd}$$

These maps are dominant inducing injections $k(G/H) \to k(G) \otimes_k K$ and $k(G/H') \to k[G] \otimes_k R$. Since R has a valuation say ν_0, this also defines valuations ν and ν' on $k(G/H)$ and $k(G/H')$. Explicitely this valuation is given by

$$\nu(f) = \min\{\nu_0(r \mapsto f(g\phi(r)) \ / \ g \in G\} \text{ and } \nu'(f') = \min\{\nu_0(r \mapsto f'(g\phi'(r)) \ / \ g \in G\}$$

where $\phi : \mathrm{Spec}(R) \to X$ and $\phi' : \mathrm{Spec}(R) \to X'$ are the given maps. These valuations are clearly invariant and ν lifts ν'. Furthermore, the valuation ν' is non negative on an open G-invariant subset *i.e.* on the union of some simple embeddings $X'_{Y',G}$ for some closed orbits Y'. In particular ν' is in $\mathrm{Supp}(\mathbb{F}(X'))$ and by assumption ν is in $\mathrm{Supp}(\mathbb{F}(X))$. There is therefore a center of ν in X thus an extension of $G \times \mathrm{Spec}(K) \to X$ to $G \times \mathrm{Spec}(R) \to X$ and taking the fiber at the identity of G we get the desired lifting. □

8.3 Integral submersions and colored fans

In this section we shall see than we can deal with (almost) all morphisms between sherical embeddings extending $\varphi : G/H \to G/H'$ for all H' at the same time.

Definition 8.3.1 *A colored subspace is a colored cone* $(\mathcal{C}, \mathcal{F})$ *such that* $\mathcal{C}$ *is a subspace.*

Remark 8.3.2 Note that the condition (CC2) is always satisfied since 0 is in the interior of the cone and is an invariant valuation.

Definition 8.3.3 *Let* $\varphi : G/H \to G/H'$ *be a surjective morphism between homogeneous spherical varieties. Define the set* $\mathcal{C}_\varphi^\vee$ *as follows:*

$$\mathcal{C}_\varphi^\vee = \{\nu \in \mathbb{X}^\vee(G/H) \ / \ \nu(f) = 0 \textit{ for all } f \in k(G/H')^{(B)}\}.$$

Definition 8.3.4 *An integral submersion of a homogeneous spherical variety* G/H *is a normal* G*-variety* G *together with a dominant* G*-morphism* $\varphi : G/H \to X$ *such that all fibers are reduced and irreducible.*

Lemma 8.3.5 *Let* $\varphi : G/H \to G/H'$ *be a surjective morphism between homogeneous spherical varieties and let*

$$X_0 = G/H \setminus \bigcup_{D \in D(G/H) \setminus \mathcal{F}_\varphi} D.$$

(i) We have the equality $k[BH'/H'] = \{f \in k[X_0] \ / \ \nu(f) \geq 0 \textit{ for all } \nu \in \mathcal{C}_\varphi \cap \mathcal{V}(G/H)\}$.

(ii) The pair $(\mathcal{C}_\varphi, \mathcal{F}_\varphi)$ *is a colored subspace.*

Proof. (ı) Choose an embedding $\psi : G/H \to X$ such that φ extends to a proper morphism $\bar{\varphi} : X \to G/H'$ (for this we only need to find a colored fan whose colored cones cover $\mathcal{C}_\varphi^\vee \cap \mathcal{V}(G/H)$, take for example $(\mathcal{C}_\varphi^\vee \cap \mathcal{V}(G/H), \emptyset)$). Write $X' = G/H' = Y'$, then BH'/H' is the B-stable affine open subset $X'_{Y,B}$. Let $\bar{X}_0 = \bar{\varphi}(X'_{Y',B})$. Consider the Stein factorisation $X \to \mathrm{Spec}(\bar{\varphi}_*\mathcal{O}_X) \to X'$ of $\bar{\varphi}$. Because $\bar{\varphi}$ has generically irreducible and generially reduced fibers the second morphism is finite of degree one. Since $X' = G/H'$ is normal this is an isomorphism thus $\bar{\varphi}_*\mathcal{O}_X = \mathcal{O}_{X'}$. Restricting to $X'_{Y',B}$ we get $\bar{\varphi}_*\mathcal{O}_{\bar{X}_0} = \mathcal{O}_{X'_{Y',B}}$. Taking global sections yields the equality $k[\bar{X}_0] = k[X'_{Y',B}]$.

On the other hand, we have the equality $\bar{X}_0 \cap G/H = X_0$ and the closure of $\bar{X}_0 \setminus X_0$ is G-stable. Therefore, if $f \in k[X_0]$ satisfies $\nu(f) \geq)$ for all $\nu \in \mathcal{V}(G/H)$ for all ν with a center in $\bar{X}_0$, then f extends to $\bar{X}_0$. These valuations are exactly those in $\mathcal{C}_\varphi^\vee \cap \mathcal{V}(G/H)$. In symbols

$$k[\bar{X}_0] = \{f \in k[X_0] \ / \ \nu(f) \geq 0 \text{ for all } \nu \in \mathcal{C}_\varphi^\vee \cap \mathcal{V}(G/H)\}.$$

This proves (ı).

(ıı) We know that (CC2) is always satisfied. To prove (CC1), we have to prove that $\mathcal{C}_\varphi^\vee \cap \mathcal{V}(G/H)$ and $\rho(\mathcal{F}_\varphi)$ span $\mathcal{C}_\varphi^\vee$ as a cone. Let $f \in k(G/H)^{(B)}$ such that $\nu(f) \geq 0$ for $\nu \in \mathcal{C}_\varphi^\vee \cap \mathcal{V}(G/H)$ or $\nu \in \rho(\mathcal{F}_\varphi)$. These rational functions f are exactly those with weight in the dual cone $\mathcal{C}^\vee$ of the cnoe $\mathcal{C}$ generated by $\mathcal{C}_\varphi^\vee \cap \mathcal{V}(G/H)$ and $\rho(\mathcal{F}_\varphi)$. Such a function f is defined on X_0 and by (ı) it is in $k[BH'/H']$. By definition of $\mathcal{C}_\varphi^\vee$ we have $\nu'(f) = 0$ for $\nu' \in \mathcal{C}_\varphi^\vee$. Therefore $\mathcal{C}_\varphi^\vee$ is in $(\mathcal{C}^\vee)^\vee = \mathcal{C}$ and $\mathcal{C} = \mathcal{C}_\varphi^\vee$ proving the result. □

Theorem 8.3.6 *(ı) The map $\varphi \mapsto (\mathcal{C}_\varphi^\vee, \mathcal{F}_\varphi)$ is a bijection between surjective integral submersions and colored subspaces.*

(ıı) For such a map φ we have an exact sequence $0 \to \mathcal{C}_\varphi^\vee \to \mathbb{X}^\vee(G/H) \xrightarrow{\varphi} \mathbb{X}^\vee(G/H') \to 0$ and we have the equalities $\mathcal{V}(G/H') = \varphi_*\mathcal{V}(G/H)$ and $D(G/H') = \mathcal{F}_\varphi^c$.

Proof. (ı) We have already seen that if φ is such an integral morphism, then $(\mathcal{C}_\varphi, \mathcal{F}_\varphi)$ is a colored subspace.

Let us prove the injectivity of the map. If $\varphi' : G/H \to G/H'$ and $\varphi'' : G/H \to G/H''$ are two integal morphisms with the same data: $(\mathcal{C}_{\varphi'}, \mathcal{F}_{\varphi'}) = (\mathcal{C}_{\varphi''}, \mathcal{F}_{\varphi''})$. Define the open set X_0 of G/H as above by

$$X_0 = G/H \bigcup_{D \in D(G/H) \setminus \mathcal{F}} D.$$

The above lemma proves that the dense B-orbits in X and X' are isomorphic thus the dense G-orbits are also isomorphic proving the injectivity of the map.

For the surjectivity, assume that $(\mathcal{C}, \mathcal{F})$ is a colored subspace, we will construct the corresponding morphism. Denote by $\mathbb{X}$ the kernel of the map $\mathbb{X}(G/H) \to \mathcal{C}^\vee$. Let $g_1, \cdots, g_n$ be rational functions on G/H such that their weights span $\mathbb{X}$. Note that these functions have at most poles on the B-stable divisors $D \in D(G/H)$ and by assumption their are defined on D with $\nu_D \in \mathcal{C}$ *i.e.* on $D \in \mathcal{F}$. Let $f_0 \in k[G]$ such that the vanishing locus of f_0 is $\pi^{-1}(\cup_{D \in D(G/H) \setminus \mathcal{F}} D)$ with $\pi : G \to G/H$ the quotient map. We may assume that f_0 is a $B \times H$-eigenfunction and that $f_i = f_0 g_i \circ \pi$ is defined on G *i.e.* $f_i \in k[G]$. Let W be the G-module spanned by the $f_0, \cdots, f_n$, since these functions are H-eigenfunctions with the same weight, we get a G-equivariant morphism

$$\varphi : G/H \to \mathbb{P}(W^\vee).$$

Consider the normalisation X of the closure $\overline{\varphi(G/H)}$ of the image. This is a spherical variety and the morphism φ factors through X. The image of G/H in X is an orbit $G \cdot x$. This orbit is not always isomorphic to G/H' for the closed (reduced) subgroup $H' = \mathrm{Stab}(x)$ since the orbit map $\phi : G \to G\cdot x$ may be non separable. However, the map ψ is constant on the H'-orbits therefore it factors through the quotient and we get the following commutative diagram

$$\begin{tikzcd} G/H \arrow[r, "\psi"] \arrow[dr] & G/H' \arrow[d] \\ & G\cdot x \subset X. \end{tikzcd}$$

Note that the fibers of ψ are isomorphic to H'/H which is integral if and only if it is connected. But by construction these fibers are connected.

Let $D(f_0)$ be the open subset of $\mathbb{P}(W^\vee)$ defined by the non vanishing of f_0. Let us define X_0 its inverse image in X and X'_0 its inverse image in G/H'.

Lemma 8.3.7 *We have the equality* $k[X'_0]^{(B)} = \{f \in k(G/H)^{(B)} \ / \ \nu(f) = 0$ *for all* $\nu \in \mathcal{C}\}$.

Proof. Let $\mathcal{M} = \{f \in k(G/H)^{(B)} \ / \ \nu(f) = 0$ for all $\nu \in \mathcal{C}\}$ and let $f \in \mathcal{M}$. Let us prove that f induces a function on X'_0. Indeed, the weight of f is a linear combinaison with non negative integer coefficients $a_1, \cdots, a_n$ of the weights of $g_1, \cdots, g_n$. Consider $F = \prod_{i=1}^n g_i^{a_i}$. The functions f and F have the same weight thus f/F is B-invariant and thus constant on BH'/H' and thus on G/H'. Thus f is a constant multiple of F which is defined on $D(f_0)$ and thus on X'_0.

Now let $f \in k[X'_0]^{(B)}$. Since $\mathcal{C}$ is a subspace, we only need to prove that $\nu(f) \geq 0$ for all $\nu \in \mathcal{C}$, taking the opposite will give the vanishing statement. To prove this we only have to consider $\rho(\nu) \in \mathcal{C}$ with $\nu \in \mathcal{V}(G/H)$ or $\nu = \nu_D$ with $D \in \mathcal{F}$. In the second case, since $\varphi(D)$ meets non trivially $D(f_0)$, then f is defined on an open subset of D thus $\nu_D(f) \geq 0$. In the first case, we simply remark that $k[X'_0]$ is a finite extension of a quotient of the symmetric algebra $S^*(W^\vee)$. But an element $f' \in W$ satisfies $\nu(f') \geq 0$ proving the result. $\square$

This lemma proves the equality $\mathcal{C}_\psi = \mathcal{C}$. We are left to prove the equality $\mathcal{F}_\psi = \mathcal{F}$. We have seen that for $D \in \mathcal{F}$, we have $\nu_D(f) = 0$ for all $f \in k[X'_0]^{(B)}$ and thus also for $f \in k[X'_0]$. This proves that D dominates G/H' *i.e.* $D \in \mathcal{F}_\psi$. Conversely, let $D \in \mathcal{F}_\psi$ *i.e.* $\psi(D)$ is dense in G/H', then $\nu_D(f) = 0$ for all $f \in k[X'_0]$. In particular $1/f_0$ is defined on an non empty open subset of D thus $D \in \mathcal{F}$ (since f_0 vanishes on all the divisors $D' \notin \mathcal{F}$). $\square$

Definition 8.3.8 *Let* $\varphi : G/H \to X$ *be a general (not necessarily surjective) integral submersion.*

(ı) If Y *is a* G*-orbit in* X *we may define* $(\mathcal{C}^\vee_{Y,\varphi}(X), \mathcal{F}_{Y,\varphi}(X)))$ *by the following equalities:*

$$\mathcal{C}^\vee_{Y,\varphi}(X) = \varphi_*^{-1}(\mathcal{C}_Y(X)) \text{ and } \varphi_*^{-1}(\mathcal{F}_{Y,\varphi}(X)) \cup \mathcal{F}_\varphi.$$

(ıı) We define $\mathbb{F}_\varphi(X)$ *as the collection of all* $(\mathcal{C}^\vee_{Y,\varphi}(X), \mathcal{F}_{Y,\varphi}(X)))$ *for* Y *a* G*-orbit in* X.

Combining the previous results we obtain.

Theorem 8.3.9 *The functor* $(\varphi : G/toX) \mapsto \mathbb{F}_\varphi(X)$ *is an equivalence of category between the category of integral submersions and the category of colored fans.*

Corollary 8.3.10 *Let* H *be a closed connected subgroup. There is a non decreasing bijection between the intermediate connected subgroups* $H \subset H' \subset G$ *and colored subspaces.*

Example 8.3.11 Consider the case $H = B$. Then we know that $\mathbb{X}(G/H) = \mathbb{X}^\vee(G/H) = \mathcal{V}(G/H) = 0$. Furthermore $D(G/H)$ is given by the divisorial Schubert varieties which are in bijection with simple roots. Therefore, the intermediate connected subgroups, *i.e.* the parabolic subgroups, are indexed by subsets of simple roots as is well known.

Part III

Geometry of spherical varieties

Chapter 9

Invariant valuations

9.1 The cone of valuations and toroidal embeddings

We have seen that the invariant valuation play an important role in the definition of colored cones and fans. In this section we may to prove that this set is a polyedral cone.

Let $g_1, \cdots, g_r \in k(G/H)^{(B)}$ and let $h \in k[G]^{(B\times H)}$ such that $f_i = g_i h \in k[G]$ for all i. Let W_i be the G-submodule of $k[G]$ generated by f_i. The same proof as in Proposition 6.2.4 shows that for any $f' \in (W_1 \cdots W_r)^{(B)}$ we have

$$f'/h^r \in k(G/H)^{(B)} \text{ and } \nu(f'/h^r) \geq \sum_i \nu(g_i) \text{ for all } \nu \in \mathcal{V}(G/H).$$

Definition 9.1.1 *For any tuple of elements $(g_1, \cdots, g_r, h, f')$ as above we define*

$$\delta = \delta(g_1, \cdots, g_r, h, f') = \sum_i w(g_i) - w(f'/h^r)$$

where $w(f)$ is the weight of a B-eigenfunction f. We define by $\Delta \subset \mathbb{X}(G/H)$ the set of all these weights.

Remark 9.1.2 The above inequality implies that for $\nu \in \mathcal{V}(G/H)$ we have $\nu(\delta) \leq 0$ for all $\delta \in \Delta$.

Proposition 9.1.3 *We have the equality $\mathcal{V}(G/H) = \{\nu \in \mathbb{X}^\vee(G/H) \ / \ \nu(\delta) \geq 0$ for all $\delta \in \Delta\}$.*

Proof. We have already seen one inclusion. Conversely, assume that $\nu \in \mathbb{X}^\vee(G/H)$ satisfies $\nu(\delta) \geq 0$ for all $\delta \in \Delta$. Define $(\mathcal{C}, \mathcal{F})$ by $\mathcal{C} = \mathbb{Q}_{\geq 0}\nu$ and $\mathcal{F} = \emptyset$. Let $g_1, \cdots, g_r$ be generators of the cone $\mathcal{C}^\vee$ and let $h = f_0 \in k[G]^{(B\times H)}$ such that f_0 vanishes on all divisors $D \in D(G/H)$ and $f_i = hg_i = f_0 g_i \circ \pi \in k[G]$. Let W_i for $i \in [0, r]$ be the G-module spanned by f_i in $k[G]$ and let W be the G-module spanned by the W_i. We may define $\varphi : G/H \to \mathbb{P}(W^\vee)$ as usual. Let X' be the closure of the image and let $X'_0 = X' \cap D(f_0)$ the locus where f_0 does not vanish.

Lemma 9.1.4 *We have the equality $k[X'_0]^{(B)} = \{f \in k(G/H)^{(B)} \ / \ \nu(f) \geq 0$ for all $\nu \in \mathcal{C}\}$.*

Proof. Let $\mathcal{M} = \{f \in k(G/H)^{(B)} \ / \ \nu(f) \geq 0$ for all $\nu \in \mathcal{C}\}$ and let $f \in \mathcal{M}$. Let us prove that f induces a function on X'_0. Indeed, the weight of f is a linear combinaison with non negative integer coefficients $a_1, \cdots, a_n$ of the weights of $g_1, \cdots, g_n$. Consider $F = \prod_{i=1}^n g_i^{a_i}$. The functions f and F have the same weight thus f/F is B-invariant and thus constant on BH/H and thus on G/H. Thus f is a constant multiple of F which is defined on $D(f_0)$ and thus on X'_0.

Now let $f \in k[X_0']^{(B)}$, we have to prove $\nu(f) \geq 0$. But $k[X_0']$ is a quotient of the symmetric algebra $S^*(W^\vee)$ thus any of its element is of the form f'/h^s with $f' \in (W_0^{n_0} \cdots W_r^{n_r})^{(B)}$ and $s = \sum_i n_i$. But the weight $\delta = \sum_i n_i w(g_i) - w(f'/h^s)$ is in Δ thus $\nu(\delta) \leq 0$ which implies $\nu(f'/h^s) \geq \sum_i n_i \nu(g_i) \geq 0$ proving the converse inclusion. □

Since the cone $\mathcal{C}$ is convex, we may prove as in Theorem 7.3.8 that the fibers are finite and therefore construct a commutative diagram

$$\begin{tikzcd} G/H \arrow[r, "i"] \arrow[dr, "\varphi"'] & X \arrow[d, "\psi"] \\ & X' \end{tikzcd}$$

such that i is an open embedding and ψ is a finite morphism. If X_0 is the inverse image of X_0' in X, then we have the equality $k[X_0]^{(B)} = k[X_0']^{(B)}$. We thus have $k[X_0]^{(B)} \neq k(G/H)^{(B)}$ and $D \cap X_0 = \emptyset$ for all $D \in D(G/H)$. Therefore X_0 is bigger than $G/H \setminus \cup_{D \in D(G/H)} D$ and must therefore meet a G-orbit different from G/H. Call Y such a G-orbit which is closed. We have $\nu_Y(f \geq 0$ for all $f \in k[X_0]^{(B)}$ *i.e.* $\nu_Y \in \mathcal{C} \cap \mathcal{V}(G/H)$ and ν_Y is non trivial thus it is in the interior and ν is a positive multiple of ν_Y proving the fact that ν is also a valuation. □

Corollary 9.1.5 *The set $\mathcal{V}(G/H)$ is a convex cone.*

Definition 9.1.6 *An embedding is called toroidal if no $D \in D(G/H)$ contains a G-orbit in its closure. In symbols $\mathcal{F}_Y(X) = \emptyset$ for all G-orbits Y in X.*

Remark 9.1.7 Note that if X is a toroidal embedding of G/H, then the fan $\mathbb{F}(X)$ is contained in $\mathcal{V}(G/H)$. In other words we have the equality $\mathbb{F}(X) = \mathrm{Supp}(\mathbb{F}(X))$.

Proposition 9.1.8 *There exists a complete toroidal embedding of G/H.*

Proof. Let us proceed as usual. Choose $f_0 \in k[G]^{(B\times H)}$ which vanishes on all the divisors $\pi^{-1}(D)$ for $D \in D(G/H)$ (and with $\pi : G \to G/H$ the quotient map) and let W be the G-submodule of $k[G]$ spanned by f_0. We have a morphism $\varphi : G/H \to \mathbb{P}(W^\vee)$ and let X' be the closure of its image. Let H be the hyperplane section defined by f_0. Since f_0 spans W as a G-module, there is no G-orbit contained in H (otherwise we would have for $[v]$ a point in that orbit the vanishings $(g \cdot f_0)(v) = 0$ for all $g \in G$ thus $v = 0$ a contradiction). Let X'' be a complete embedding of G/H and let X be the normalisation of the closure of G/H embedded diagonaly in $X' \times X''$. This is a spherical embedding of G/H and the morphism $X \to X'$ induced by projection is proper. Since X' is itself proper this is a proper embedding of G/H. On the other hand, a G-orbit Y in X is mapped to a G-orbit Y' in X' which is not contained in H thus Y is contained in no divisor $D \in D(G/H)$ proving that $\mathcal{F}_Y(X)$ is empty. □

Corollary 9.1.9 *The set of invariant valuations $\mathcal{V}(G/H)$ is a polyedral cone.*

Proof. We already know that it is a convex cone. Furthermore, for X a toroidal proper embedding of G/H we have that the union of the cones of $\mathbb{F}(X)$ is exactly $\mathcal{V}(G/H)$. As their finitely many such cones all of which are polyedral the result follows. □

Definition 9.1.10 *(i) Consider $X_*(T)$ the group of one parameter subgroups or cocharacters of the maximal torus T. This is the dual of $X^*(T) = X^*(B)$ the group of characters of T or of B. We define*

$$C = \{\nu \in X_*(T) \ / \ \nu(\alpha) \leq 0 \text{ for all positive roots } \alpha\}.$$

This is the co-dominant chamber which is spanned by the dominant coweights.

(ii) Define $C_{G/H}$ as the image of C under the surjective morphism $X_*(T) \to \mathbb{X}^\vee(G/H)$ coming from the inclusion $\mathbb{X}(G/H) \subset X^*(T)$.

Corollary 9.1.11 *The cone $C_{G/H}$ is contained in $\mathcal{V}(G/H)$.*

Proof. Let $g_1, \cdots, g_r$, h and f' be as in the definition of the set Δ. Then the element f_i is the highest weight vector of W_i the G-module spanned by $f_i = hg_i$ (because f_i is a B-eigenvector). Let Λ_i be the corresponding weight. But the element f' is in $(W_1 \cdots W_r)^{(B)}$ thus its weight μ is $\sum_i \lambda_i - A$ where A is a non negative linear combinaison of positive roots. Therefore $\delta = \mu - \sum_i \lambda_i$ is a non negative linear combinaison of positive roots. It is therefore in $C_{G/H}$. □

Example 9.1.12 For symmetric varieties *i.e.* spherical embeddings of a quotient G/H with $G^\sigma \subset H \subset N_G(G^\sigma)$ where σ is an involution of G, one can prove the equality

$$\mathcal{V}(G/H) = C_{G/H}.$$

See also example 7.4.4.

9.2 Horospherical varieties

Theorem 9.2.1 *The G-automorphism group $A = N_G(H)/H$ of G/H is an extension of a diagonalisable group by a finite p-group. In particular, the connected component of the identity A^0 is a central torus.*

Furthermore we have $\dim A = \dim \mathcal{V}(G/H) \cap (-\mathcal{V}(G/H))$.

Proof. Let $L \subset A$ be a connected subgroup and let H' be its preimage in $N_G(H)$. Let $\varphi : G/H \to G/H'$ be the quotient map. The group H' corresponds to the colored subspace $(\mathcal{C}_\varphi, \mathcal{F}_\varphi)$. Any G-equivariant automorphism has to map the dense B-orbit to itself. This implies $\varphi^{-1}(B\varphi(H'/H')) = BH/H$ therefore we have $\mathcal{F}_\varphi = \emptyset$. We get the inclusion $\mathcal{C}_\varphi \subset \mathcal{V}(G/H) \cap (-\mathcal{V}(G/H))$. Furthermore, if L is non trivial then $\mathcal{C}_\varphi \neq 0$.

Let L be one dimensional and let $f \in k(G/H)^{(B)}$such that its weight $w(f)$ does not vanish on $\mathcal{C}_\varphi$, which is possible since $\mathcal{C}_\varphi \neq 0$. This in particular implies that $f \notin k(G/H')^{(B)}$. This function is defined on the B-orbit BH/H and therefore on the fiber $\varphi^{-1}(\varphi(H/H))$ which is isomorphic to L. We thus have by resctriction a function on L which is non constant. Since $1/f$ satisfies the same condition, it is also a function on L and f is non vanishing. Therefore f is a multiple of a non trivial character. In particular L is isomorphic to $\mathbb{G}_m$ (since $\mathbb{G}_a$ has no non-trivial character). This implies that A^0 is a torus. Because A is contained in $\mathrm{Aut}_B(BH/H)$ we get that A is a subquotient of B proving the result on A.

Let L and H' as above. Note that since H'/H is a torus, there exists a sequence of connected subgroups $T_0 = H \subset T_1 \subset \cdots \subset T_n = H'$ with $\dim H_{i+1}/H_i = 1$ and $n = \dim H'/H$. There is therefore a sequence of colored subspaces with empty set of colors $\mathcal{C}_0 = 0 \subsetneq \mathcal{C}_1 \subsetneq \cdots \subsetneq \mathcal{C}_n = \mathcal{C}$. This implies that $\dim \mathcal{C} \geq n$ and proves the inequality $\dim A \leq \dim \mathcal{V}(G/H) \cap (-\mathcal{V}(G/H))$.

If $\nu \in \mathcal{V}(G/H) \cap (-\mathcal{V}(G/H))$ is non trivial, define the colored subspace $(\mathcal{C}, \mathcal{F}) = (\mathbb{Q}_{\geq 0}\nu, \emptyset)$ and let H' be the corresponding subgroup. Let $f \in k(G/H)^{(B)}$ non vanishing on ν, then $f \notin k(G/H')^{(B)}$. This defines a non constant non vanishing function on H'/H and therefore on H' via the composition $H' \to H'/H \xrightarrow{f} \mathbb{A}^1$. This function is a multiple of a character $\chi \in X^*(H')$ on H'. Its kernel $\ker \chi$ corresponds to a proper colored subspace of $(\mathcal{C}, \mathcal{F})$ thus $H = \ker \chi$. Therefore H' normalises H. Furthermore because the group of characters is discrete, we have that $N_G(H')^0$ fixes χ. Therefore $N_G(H')^0 \subset N_G(H)$. To prove the dimension equality, we proceed by induction. We therefore get the inequalities

$$\begin{aligned} \dim \mathcal{V}(G/H) \cap (-\mathcal{V}(G/H)) & \geq \dim N_G(H)/H \\ & \geq \dim N_G(H')/H' + 1 = \dim \mathcal{V}(G/H') \cap (-\mathcal{V}(G/H')) + 1. \end{aligned}$$

The equality $\dim \mathcal{V}(G/H') \cap (-\mathcal{V}(G/H')) + 1 = \dim \mathcal{V}(G/H) \cap (-\mathcal{V}(G/H))$ concludes the proof. □

Corollary 9.2.2 *The homogeneous spherical variety G/H has a simple completion if and only if its automorphism group $N_G(H)/H$ is finite.*

Proof. We have a simple completion if and only if $\mathcal{V}(G/H)$ is a striclty convex cone and in that case the completion is given by the colored cone $\mathcal{V}(G/H), \emptyset)$. We conclude by the previous result. □

Definition 9.2.3 *A spherical homogeneous variety G/H is called horospherical if H contains a maximal unipotent subgroup U.*

Corollary 9.2.4 *The variety G/H is horospherical if and only if $\mathcal{V}(G/H) = \mathbb{X}^\vee(G/H)$*

Proof. Let us first consider the case $H = U$. Note that $N_G(U) = B$ thus $N_G(U)/U = B/U = T$ and $\dim \mathcal{V}(G/U) \cap (-\mathcal{V}(G/U)) = \dim T \geq \dim \mathbb{X}^\vee(G/U)$ and we get the equality $\mathcal{V}(G/U) = \mathcal{V}(G/U) \cap (-\mathcal{V}(G/U)) = \mathbb{X}^\vee(G/U)$. Now if H is connected and contains U, let $\varphi : G/U \to G/H$ be the corresponding surjective integral morphism. If $\mathcal{C}_\varphi$ is the corresponding subspace we have $\mathbb{X}^\vee(G/H) = \mathbb{X}^\vee(G/U)/\mathcal{C}$ and $\mathcal{V}(G/H)$ is the image of $\mathcal{V}(G/U)$ under the projection. Thus again $\mathcal{V}(G/H) = \mathbb{X}^\vee(G/H)$. Finally, if H is not connected, let H^0 be its connected component, then $\mathbb{X}(G/H^0) \to \mathbb{X}(G/H)$ is an injection of lattices of the same rank with finite index thus the subspaces are the same and the result follows.

Conversely, if we have the equality $\mathcal{V}(G/H) = \mathbb{X}^\vee(G/H)$, then the subspace $(\mathbb{X}^\vee(G/H), \emptyset)$ is a colored subspace. Let P be the connected subgroup corresponding to this subspace. We have $\mathbb{X}^\vee(G/P) = 0$ thus by the properness criterion, the quotient G/P is proper and therefore projective (since G/P is quasi-projective). The subgroup P is then a parabolic subgroup. Furthermore, the proof of the previous Theorem implies that the inverse image H' in $N_G(H)$ of the connected component of $A = N_G(H)/H$ has $(\mathcal{V}(G/H), \emptyset)$ for cone and thus $H' = P$. Therefore H is normal in P and the quotient P/H is a torus. This implies that $U \subset H$. □

Chapter 10

Local structure results

In this chapter we assume that k is of characteristic 0, in symbols $\mathrm{char} k = 0$.

10.1 Local structure for G-varieties

Let G be reductive and let B be a Borel subgroup in G. In this section we prove a general result on the structure of G-varieties at the neighbourhood of a compact G-orbit. We start with the case of the projective space. Let V be a G-module and let Y be a closed orbit of $\mathbb{P}(V)$.

The stabiliser of any point in Y is a parabolic subgroup of G and there exists an element $y \in Y$ such that By is open and dense in Y. Let $v \in V$ such that $[v] = y$.

Fact 10.1.1 *There exists a B-eigenvector $\eta \in (V^\vee)^{(B)}$ such that $\langle \eta, v\rangle = 1$.*

Proof. If for any $\eta \in (V^\vee)^{(B)}$ we have $\langle \eta, v\rangle = 0$, then $\langle \eta, bv\rangle = 0$ for all $b \in B$ thus $\langle \eta, w\rangle = 0$ for all $w \in Y$ and because Y is a G-orbit we get $\langle g\eta, w\rangle = 0$ for all $g \in G$ and $w \in Y$ therefore Y is anihilated by $V^\vee$ (since $V^\vee$ is spanned by its highest weight vectors $(V^\vee)^{(B)}$). This implies $Y = \emptyset$ a contradiction.

Linear reductivity of G gives also this result. $\square$

Let η as in the previous fact and let P be its stabiliser.

Fact 10.1.2 *The stabiliser G_y of y and P are opposite parabolic subgroups and we have $By = Py$.*

Proof. The orbit Y is the quotient G/G_y and is projective. The subgroup G_y is therefore a parabolic subgroup and thus contains a Borel subgroup B' of G. Since any two Borel subgroups contain a maximal torus in their intersection, there exists T a maximal torus in $B \cap B'$. But By is open and dense in $Y = G/G_y$, thus BG_y is open and dense in G. This implies the decomposition $\mathfrak{g} = \mathfrak{b} + \mathfrak{g}_y$ on the level of Lie algebras and therefore $\mathfrak{g}_y$ contains $\mathfrak{b}^-$ the opposite Borel Lie algebra with respect to the torus T. Thus G_y contains B^- the Borel opposite to B and the vector v with $[v] = y$ is a T-fixed point (since $T \subset B' \subset G_y$). It is therefore a highest weight vector for B^-. Let λ_v be the T-weight of v and consider the decompositions

$$V = \bigoplus_{\lambda \in \hat{G}} V_\lambda^{m_\lambda} \text{ and } V^\vee = \bigoplus_{\lambda \in \hat{G}} (V_\lambda^\vee)^{m_\lambda},$$

where the weights $\lambda \in \hat{G}$ are considered as dominant weight for the Borel B (thus $-\lambda_v$ is dominant). The element η must lie in $(V_{-\lambda_v}^\vee)^{m_{-\lambda_v}}$. The stabilisers of y and $[\eta] \in \mathbb{P}(V^\vee)$ are therefore respectively spanned by T and the unipotent subgroups U_α associated to the roots α such that $\langle\alpha^\vee, \lambda_v\rangle \geq 0$, respectively $\langle\alpha^\vee, -\lambda_v\rangle \geq 0$ and are therefore opposite parabolic subgroups.

For the second statement, let us remark that P is spanned by B and the unipotent subgroups U_α with $\langle\alpha^\vee, -\lambda_v\rangle = 0$. But these subgroups U_α are also in G_y proving the result. □

Let us denote by L the intersection $P \cap G_y$. This is a Levi subgroup of both P and G_y. The group L is reductive and is a maximal reductive subgroup of both P and G_y. We denote by $R_u(P)$ the unipotent radical of P. Recall that we have the equality $P = LR_u(P)$.

Denote by $\mathbb{P}(V)_\eta = D(\eta)$ the open subset where η does not vanish. This subset is stable by P and contains Py.

Theorem 10.1.3 *There exists a closed subvariety S of $\mathbb{P}(V)_\eta$ stable under L and containing y such that the morphism*

$$R_u(P) \times S \to \mathbb{P}(V)_\eta$$

defined by $(p, x) \mapsto px$ is a P-equivariant isomorphism.

Proof. We first prove the following result.

Lemma 10.1.4 *If the statement is true for V simple, then it is true for any G-module V.*

Proof. Let us denote by $\langle Gv\rangle$ and $\langle G\eta\rangle$ the G-submodules of V and $V^\vee$ spanned by v and η. Note that $\langle Gv\rangle$ is isomorphic to $V_{-\lambda}$ while $\langle G\eta\rangle$ is isomorphic to $V_{-\lambda}^\vee$. The orthogonal $\langle G\eta\rangle^\perp$ is therefore of codimension $\dim V_{-\lambda}$ in V and in direct sum with $\langle Gv\rangle$. We thus get a decomposition

$$V = \langle Gv\rangle \oplus \langle G\eta\rangle^\perp.$$

We may define the projection p onto $\mathbb{P}\langle Gv\rangle$ from $\langle G\eta\rangle^\perp$. This is a rational morphism which is defined on $\mathbb{P}(V)_\eta$:

$$p : \mathbb{P}(V)_\eta \to \mathbb{P}\langle Gv\rangle.$$

This morphism is G equivariant and restrict to the identity on Y since Y is contained in $\mathbb{P}\langle Gv\rangle$. If the statement is true for $\langle Gv\rangle$ which is simple, then there exists S as above and we get the Cartesian diagram

$$\begin{tikzcd} R_u(P) \times p^{-1}(S) \arrow[r] \arrow[d, "p"'] & \mathbb{P}(V)_\eta \arrow[d] \\ R_u(P) \times S \arrow[r] & \mathbb{P}\langle Gv\rangle. \end{tikzcd}$$

Since the bottom horizontal arrow is an isomorphism, the same is true for the top horizontal arrow and the result follows. □

We are left to prove the result for V simple. Let $T_v = T_vGv$ be the tangent space of Gv at v. We consider T_v as a vector subspace of V. Since v is a G_y-eigenvector of non trivial weight (otherwise V would be trivial), the group G_y acts on T_v and contains the line kv. The space T_v is thus a sub-L-representation of V and since L is reductive there is a decomposition

$$V = T_v \oplus E$$

with E a representation of L. Define $S = \mathbb{P}(kv \oplus E)_\eta$. This is a closed subvariety of $\mathbb{P}(V)_\eta$ which is stable by L and contains y. Note that S is isomorphic to the affine space $y + E$ and that S meets Y tranversaly in y. Indeed, the tangent spaces of Y and S at y are T_v/kv and E which are suplementary in V/kv.

Lemma 10.1.5 *The variety $\mathbb{P}(V)_\eta$ has a unique closed orbit, the fixed point y.*

Proof. Now since V is simple, the weight λ_v is the smallest weight (for B) of V while $-\lambda_v$ is the highest weight of $V^\vee$. Any weight of V is therefore of the form $\lambda_v + \mu$ with μ a non negative linear combination of simple roots of B. This implies that the variety $\mathbb{P}(V)_\eta$ has a unique closed orbit, namely y: indeed any element $z = [w] \in \mathbb{P}(V)_\eta$ can be written $w = \sum_\mu w_{\lambda_v+\mu}$ with $w_{\lambda_v+\mu}$ of weight $\lambda_v + \mu$ and $t \in T$ acts via $t \cdot w = \sum_\mu (\lambda_v + \mu)(t) w_{\lambda_v+\mu}$. Let θ be a dominant cocharacter, then

$$\theta(s) \cdot w = \sum_\mu s^{\langle \theta, \lambda_v+\mu\rangle} w_{\lambda_v+\mu} = s^{\langle\theta,\lambda_v\rangle}\left(w_{\lambda_v} + \sum_{\mu\neq 0} s^{\langle\theta,\mu\rangle} w_{\lambda_v+\mu}\right)$$

and when s goes to 0 we get $[\theta(s) \cdot w] \to [w_{\lambda_v}] = y$. $\square$

Consider $R_u(P) \times S$ as a T-variety via the action $t \cdot (p, z) = tpt^{-1}, t \cdot z)$.

Lemma 10.1.6 *The variety $R_u(P) \times S$ has a unique closed T-orbit, the fixed point (e, y).*

Proof. The proof is similar as the proof of the previous lemma since we know all the weights of the action of T on $R_u(P)$ and on S. $\square$

Consider the multiplication morphism $m : R_u(P) \times S \to \mathbb{P}(V)_\eta$. We want to prove that this morphism is an isomorphism.

Lemma 10.1.7 *The differential $d_{(e,y)}m$ is injective.*

Proof. Let $\mathfrak{r}_u(P)$ be the Lie algebra of $R_u(P)$. We know that $Py = R_u(P)Ly = R_u(P)y$ is open in $Y = Gy$ therefore the morphism $R_u(P) \times kv \to Y$ is dominant and tangent map $\mathfrak{r}_u(P) \times kv \to T_yY$ is surjective. We get the equality $T_yY = \mathfrak{r}_u(P)v/kv$. The same argument gives $T_vGv = \mathfrak{r}_u(P)v + kv$.

Furthermore, since η is fixed by $R_u(P)$, we have $\langle \eta, pv\rangle = \langle p^{-1}\eta, v\rangle = \langle \eta, v\rangle = 1$ for all $p \in R_u(P)$ and therefore η is constant on $R_u(P)y$. This implies by derivation that η vanishes on $\mathfrak{r}_u(P)v$. In particular $\mathfrak{r}_u(P)v$ and kv are complement and $T_vGv = \mathfrak{r}_u(P)v \oplus kv$. This also implies the equality $T_vV = kv \oplus \mathfrak{r}_u(P)v \oplus E$.

These equalities lead to the identifications of S and $\mathbb{P}(V)_\eta$ with the affine spaces $v + E$ and $v + (\mathfrak{r}_U(P)v \oplus E)$. The morphism m is given by $m((p, (v + x)) = p \cdot (v + x)$. We may now compute the differential: $d_{(e,v)}m(\xi, x) = v + \xi \cdot v + x$ for $\xi \in \mathfrak{r}_u(P)$ and $x \in E$. Indeed, the first two terms come from the idfferentiation of the action of $R_u(P)$ on v while the second term comes from the differential of the action on E which is linear.

We are left to prove that the map $\mathfrak{r}_u(P) \to \mathfrak{r}_u(P)v$ given by the action on v is injective. This is trus since the intersection of $R_u(P)$ with the stabiliser G_y of y is trivial thus $R_u(P)$ acts freely on y and v thus by differentiation the same is true on the Lie algebra level. $\square$

Let Z be the locus in $R_u(P) \times S$ where the differential of m is not surjective. This is a closed subset of $R_u(P) \times S$. It it is non empty, then it contains a closed T-orbit which has to be (e, y), a contradiction. The morphism $m : R_u(P) \times S \to \mathbb{P}(V)_\eta$ is therefore open.

Let Z be the complement of the image, then Z is closed and T-stable. If it is non empty, then it contains a closed T-orbit which has to be y a contradiction. Thus m is surjective.

Thus m is a covering but since both varieties are affines spaces which are simply connected, the map m is an isomorphism. $\square$

Example 10.1.8 Let us consider V to be the vector space of quadratic forms on k^n *i.e.* $V = (S^2k^n)^\vee$. We have seen that there is a unique closed orbit of $G = \mathrm{GL}_n$ in $\mathbb{P}(V)$ given by the quadratic forms of rank one. Pick $y = x_1^2$. Then the stabiliser of y is the stabiliser of the hyperplane given by the last $n-1$ coordinate vectors. The linear form $\eta \in V^\vee$ can be chosen to be $\eta(q) = q(1, 0, \cdots, 0)$. We have for B given by the lower triangular matrices the fact that $\eta \in (V^\vee)^{(B)}$.

A quadratic form q satisfies $\eta(q) \neq 0$ if and only if it can be writen in the form

$$q(x_1, \cdots, x_n) = \lambda(x_1 + a_2x_2 + \cdots + a_nx_n)^2 + q'(x_2, \cdots, x_n)$$

where $\lambda \neq 0$ and q' is a quadratic form on the last $n-1$ variables.

Now $R_u(P)$ maps e_1 to $e_1 + a_2e_2 + \cdots + a_ne_n$ and is the identity on the other vectors, therefore the above Theorem boils down the the fact that there exists a unique $u \in R_u(P)$ such that

$$q = \lambda u(y + q'')$$

with $q'' = u^{-1}q'$ is a quadratic form in the last $n-1$ variables thus here S is the set of quadratic forms in these last $n-1$ variables.

Remark 10.1.9 The above result may seem not attractive but it enables to replace locally the study of quasi-projective G-varieties to quasi-affine G-varieties and of projective G-varieties to affine G-varieties. We will see an example of the use of this in the next result.

Corollary 10.1.10 *Let X be a G-variety, the following consitions are equivalent.*
(ı) We have the vanishing $\mathrm{rk}(X) = 0$.
(ıı) Any G-orbit in X is compact.

Proof. Assume that any G-orbit is compact and let $f \in k(X)^{(B)}$. Let Y be a G-orbit meeting the locus where f is defined. This orbit is compact it is of the form G/P for some parabolic subgroup P. Thus there is a dense U-orbit in Y. In particular we have $f|_Y \in k(G/P)^{(B)} = k(G/P)^B$ thus the weight of f is trivial.

Conversely, assume that $\mathrm{rk}(X) = 0$. Let Y be a G-orbit and let $\overline{Y}$ be its closure in X. Then we know that $\mathrm{rk}(\overline{Y}) \leq \mathrm{rk}(X)$ and since the rank is a birational invariant we get $\mathrm{rk}(Y) = 0$. We may therefore assume that X is homogeneous *i.e.* $X = Y$.

If $X = G/H$, we know that X is quasi-projective and that there exists V a G-module such that G/H is a locally closed subset of $\mathbb{P}(V)$. let $\overline{X}$ be the closure of X in $\mathbb{P}(V)$ and let $x \in \overline{X}$ be an element in a closed G-orbit Y. Then we get a parabolic subgroup P (opposite to $\mathrm{Stab}(x)$) with Levi factor L and a closed subvariety S of $\mathbb{P}(V)_\eta$ stable under L such that $R_u(P) \times S \to \mathbb{P}(V)_\eta$ is an isomorphism. Let $Z = \overline{X} \cap S$. We get an isomorphism $R_u(P) \times Z \to \overline{X}_\eta$ thus an open immersion

$$R_u(P) \times Z \to \overline{X}.$$

Furthermore Z contains x which is fixed by L.

Lemma 10.1.11 *We have* $k(X)^{(B)} = k(Z)^{(L\cap B)}$.

Proof. If $f \in k(Z)^{(L\cap B)}$ is of weight λ, then the composition $R_u(P) \times Z \xrightarrow{f} \mathbb{A}^1$ is a rational function $\bar{f}$ on X. Furthermore, for $b \in B$ we may write $b = uc$ with $u \in R_u(P)$ and $c \in L \cap B$ thus we get $(b \cdot \bar{f})(u', z) = \bar{f}(c^{-1}u'cu^{-1}, cz) = f(c^{-1}z) = \lambda(c)f(z) = \lambda(b)\bar{f}(u', z)$ proving that $\bar{f}$ is indeed in $k(X)^{(B)}$.

Conversely for $f \in k(X)^{(B)}$, then f defines a rational function on $R_u(P)\times Z$ such that if $b = uc$ is a decomposition with $u \in R_u(P)$ and $c \in B\cap L$ we have $b \cdot f(u', z) = f(c^{-1}u'cu^{-1}, c^{-1}z) = \lambda(b)f(u', z)$. In particular for $c = 1$ and $u = u'$ we have $f(u', z) = f(e, z)$. We may define $\tilde{f}(z) = f(e, z) = f(u', z)$ for all $u' \in R_u(P)$ which will therefore be a rational function on Z. We obviously have $\tilde{f} \in k(Z)^{(L\cap B)}$.

Furthermore one checks that $\overline{\tilde{f}} = f$ and $\tilde{\overline{f}} = f$ proving the result. □

As a consequence we get that $\mathrm{rk}(Z) = 0$ as a L-variety. We get that in $k[Z]$, any $B\cap L$-eigenfunction has a trivial weight. This implies that $k[Z]$ is a trivial L-module. But notice that Z is affine thus L acts trivially on Z. Therefore the maximal torus T of G, which is contained in L acts trivially on Z. But Y was closed thus compact and of the form G/P with P parabolic. Thus T only has finitely many fixed points in Y. Therefore Z must be finite and since $R_u(P) \times Z$ is open in $\overline{X}$ is it irreducible thus Z is one point. Then Y is the G-orbit of Z and has to be dense in $\overline{X}$ thus $Y = X$ which is compact. □

10.2 Local structure for spherical varieties

In the case of spherical varieties, we want to describe the local structure not only along proper orbits but along any G-orbit. Let X be a spherical variety and let Y be a G-orbit in X. Recall the definition of $X_{Y,B}$ which is

$$X_{Y,B} = X \setminus \bigcup_{D\in D(X)\setminus D_Y(X)} D = \{x \in X\ \overline{Bx} \supset Y\}.$$

Let us denote by P the stabiliser of $X_{Y,B}$ *i.e.* the set of elements $g \in G$ with $g \cdot X_{Y,B} = X_{Y,B}$. This is a parabolic subgroup containing B.

Theorem 10.2.1 *(ı) With the above notation, there exists L a Levi subgroup of P and S a closed subvariety of $X_{Y,B}$ such that*

(a) the variety S is stable under L and

(b) The map $R_u(P) \times S \to X_{Y,B}$ defined by $(p, x) \mapsto p \cdot x$ is a P-equivariant isomorphism.

(ıı) The variety S is an affine spherical variety for L and $S\cap Y$ is a L-orbit whose isotropy subgroup L_y for any point y in $S\cap Y$ contains $D(L)$ the derived subgroup of L. This subgroup L_y is independent of the chosen point y and we denote it by L_Y.

(ııı) There exists a closed L_Y-stable subvariety S_Y of S containing a fixed point for L_Y such that the morphism

$$L \times^{L_Y} S_Y \to S$$

defined by $\overline{(l, y)} \mapsto l\cdot y$ is a L-equivariant isomorphism. The variety S_Y is an affine spherical L_Y-variety whose rank is $\mathrm{rk}(X) - \mathrm{rk}(Y)$.

Proof. Since $X_{Y,B}$ is contained in $X_{Y,G}$ we may assume that X is simple with Y as closed orbit.

(ı) We want to apply Theorem 10.1.3. Let $\mathcal{D}$ be the complement of $X_{Y,B}$ in X. We know that $\mathcal{D}$ is Cartier and globally generated therefore there exists a canonical section η of the line bundle $\mathcal{O}_X(\mathcal{D})$ given by $\mathcal{O}_X \xrightarrow{\eta} \mathcal{O}_X(\mathcal{D})$. This section is an element of $H^0(X, \mathcal{O}_X(\mathcal{D}))$. We may assume, replacing G by a covering G' that $\mathcal{D}$ is G-linearised thus G acts on $H^0(X, \mathcal{O}_X(\mathcal{D}))$. Let $V^\vee$ be the G-submodule spanned by η. We have a G-equivariant morphism (this is indeed a morphism since $\mathcal{D}$ is globally generated):

$$\varphi : X \to \mathbb{P}(V)$$

defined by $x \mapsto [\sigma \mapsto \sigma(x)]$ for $\sigma \in V^\vee \subset H^0(X, \mathcal{O}_X(\mathcal{D}))$. By definition of η, we have $X_{Y,B} \to \mathbb{P}(V)_\eta$. Note also that since P stabilises $X_{X,B}$ it also stabilises $\mathcal{D}$ and thus η is a P-eigenfunction therefore P also stabilises $[\eta]$ or $\mathbb{P}(V)_\eta$. The map $X_{Y,B} \to \mathbb{P}(V)_\eta$ is therefore P-equivariant.

Since V is simple, the intersection of all translates gH_η of the vanising divisor H_η of η is empty. Therefore if Z is a closed orbit in $\mathbb{P}(V)$, it will not be contained in H_η (otherwise it would be contained in the intersection of the translates which is empty). Choose z in the dense P-orbit in Z and B a Borel subgroup such that Bz is dense in Z. We may apply Theorem 10.1.3 to get a closed L-stable subvariety S' of $\mathbb{P}(V)_\eta$ such that the morphism $R_u(P) \times S' \to \mathbb{P}(V)_\eta$ is a P-equivariant isomorphism. In particular S' meets the image of $X_{Y,B}$. Let $S = \varphi^{-1}(S')$. This is a closed L-stable subvariety of $X_{Y,B}$ and we have a Cartesian diagram

$$\begin{tikzcd} R_u(P) \times S \arrow[r] \arrow[d, "\mathrm{Id}\times\varphi"'] & X_{Y,B} \arrow[d, "\varphi"] \\ R_u(P) \times S' \arrow[r] & \mathbb{P}(V)_\eta. \end{tikzcd}$$

This proves that the top map is an isomorphism.

(ii) We have finitely many B-orbits in $X_{Y,B}$ since X is spherical thus B also has finitely many orbits in $R_u(P) \times S$. Recall that $P = R_u(P)L$ thus $B = R_u(P)(L \cap B)$ and that $L \cap B$ is a Borel subgroup of L. Recall also that the action of $P = R_u(P)L$ on $R_u(P) \times S$ is given by $ul \cdot (u', x) = (ulu'l^{-1}, l \cdot x)$ thus $B \cap L$ must have finitely many orbits in S. Since $X_{Y,B}$ is normal (recall that X is spherical thus normal) the variety S is also normal thus S is L-spherical and obviously affine. The inverse image of $Y \cap X_{Y,B}$, which is the dense B-orbit in Y, under the map $R_u(P) \times S \to X_{Y,B}$ is $R_u(P) \times (S \cap Y)$. Thus we have an isomorphism $R_u(P) \times (S \cap Y) \to Y \cap X_{Y,B}$ and since the right hand side is a P-orbit and a B-orbit, the intersection $S \cap Y$ has to be an L-orbit and a $(B \cap L)$-orbit as well.

Let $y \in Y \cap S$ and let L_y its stabiliser. As $S \cap Y$ is an L-orbit but also a $B \cap L$-orbit we have $S \cap Y = L/L_y = Ly = (L \cap B)y$ thus $L = (B \cap L)L_y$.

Lemma 10.2.2 *Let H be a closed subgroup of a connected reductive group G such that $G = BH$ for B a Borel subgroup of G, then H contains $D(G)$.*

Proof. We may assume that H is connected since $D(G)$ is. We claim that we may assume that G is semisimple. Indeed, if we know the result for G semisimple then consider the quotient $\pi : G \to G/R(G)$ where $R(G)$ is the radical of G. Then $\pi(D(G)) = D(G/R(G)) = G/R(G)$ since $G/R(G)$ is semisimple. Now $H/H \cap R(G)$ satisfies the hypothesis of the lemma in $G/R(G)$ thus $H/H \cap R(G)$ contains $D(G/R(G)) = G/R(G)$. We thus have a surjective map $H \to G/R(G)$ and thus a surjective map $D(H) \to D(G/R(G)) = G/R(G)$. But the map $D(G) \cap R(G)$ is finite thus $\dim D(G) = \dim D(G/R(G)) \leq \dim D(H) \leq \dim D(G)$. We have equality and $D(H) = D(G)$ thus $D(G) \subset H$.

If G is semisimplle, then consider the quotient $B\backslash G$ which is equal to $B \cap H\backslash H$ since $G = BH$. In particular G and the semisimple part H_s of H have the same rank (the rank of the Picard group of the previous homogeneous space). Furthermore, we have $\dim U_G = \dim B\backslash G = \dim B \cap H\backslash H = \dim U_H = \dim U_{H_s}$ thus $\dim H_s = \dim G$ therefore $H = G$ and the result follows. □

We deduce from this lemma that L_Y contains $D(L)$. We claim that this implies that L_y does not depend on y. Indeed, let $l \cdot y$ be a point in the orbit of y. Then the stabiliser of $l \cdot y$ is lL_yl^{-1} but for any $h \in L_y$ we have $lhl^{-1}h^{-1} \in D(L) \subset L_y$ thus $lhl^{-1} \in L_y$ and $L_{l \cdot y} \subset L_y$. The converse inclusion is proved the same way (since $D(L)$ is normal in L it is contained in $L_{l \cdot y}$). Denote by L_Y this stabilisor. We have $D(L) \subset L_Y$ thus L/L_Y is a quotient of $L/D(L)$ which is a torus (recall that the map $Z(L) \times D(L) \to L$ is surjective with finite kernel). The orbit $S \cap L = Ly$ is therefore isomorphic to the torus L/L_Y.

(iii) Let $(\chi_1, \cdots, \chi_n)$ a basis of the group of characters of L/L_Y. Since $S \cap Y$ is closed in S which is affine, we can extend these functions to functions $(f_1, \cdots, f_n)$ on S. We may furthermore assume that these functions are $L \cap B$ eigenfunctions of weights $(\chi_1, \cdots, \chi_n)$. These functions do not vanish on the closed orbit $S \cap Y$ in S thus they do not vanish at all on S. These functions therefore define a L-equivariant morphism $\varphi : S \to (\mathbb{G}_m)^n \simeq L/L_Y$. Let S_Y be the fiber over the identity element of this morphism. The natural map defined by the action: $L \times S_Y \to S$ factors through $L \times^{L_Y} S_Y \to S$. This map is bijective. Indeed, if $s \in S$, then there exists $l \in L$ such that $\bar{l} = \varphi(s)$ and if l' satisfies the same condition, then $l' = lh$ with $h \in L_Y$. Define $s \mapsto \overline{(l, l^{-1}s)} \in L \times^{L_Y} S_Y$. This is well defined and an inverse map. But since S is normal, this morphism must be an isomorphism.

Finally, we have $\mathrm{rk}(X) = \mathrm{rk}(S) = \dim(L/L_Y) + \mathrm{rk}(S_Y) = \mathrm{rk}(S \cap Y) + \mathrm{rk}(S_Y) = \mathrm{rk}(Y) + \mathrm{rk}(S_Y)$.
□

Example 10.2.3 Let us consider again the case of quadratic forms: V is the vector space of quadratic forms on k^n *i.e.* $V = (S^2k^n)^\vee$. We have seen that the G-orbits for $G = \mathrm{GL}_n$ in $\mathbb{P}(V)$ are given by the rank. Pick $y = x_1^2 + \cdot + x_i^2$ and let Y be its G-orbit. Then $X_{Y,B}$ is the set of quadratic forms such that the first i principal minors are non zero. Let P be the stabiliser of $\langle e_1\rangle, \cdots, \langle e_1, \cdots, e_i\rangle$, this is a parabolic subgroup and is the stabilor of $X_{Y,B}$.

Let S be the set of quadratic forms of the form $a_1x_1^2 + \cdots + a_ix_i^2 + q'(x_{i+1}, \cdots, x_n)$ with $a_k \in \mathbb{G}_m$. Then S is stable under $L = \mathbb{G}_m^i \times \mathrm{GL}_{n-i}$ which is the Levi subgroup of P. We see that the natural map

$$R_u(P) \times S \to X_{Y,B}$$

is an isomorphism.

The intersection $S \cap Y$ is the set of quadratic forms of the form $a_1x_1^2 + \cdots + a_ix_i^2$ with $a_k \in \mathbb{G}_m$. This is an L-orbit and the stabiliser of y is $L_Y = \{\pm 1\} \times \mathrm{GL}_{n-i}$ which contains $D(L) = \mathrm{SL}_{n-i}$.

If S_Y is the set of elemenets of the form $x_1^2 + \cdots + x_i^2 + q'(x_{i+1}, \cdots, x_n)$ we see that S_Y is stable under L_Y and meets Y in the unique point $y = x_1^2 + \cdots + x_i^2$. Furthermore, we have $L \times^{L_Y} S_Y \simeq S$.

Corollary 10.2.4 *Let X be a spherical G-variety, then any closed G-stable subvariety X' is again a spherical G-variety.*

Proof. We only have to prove that X' is normal. By the previous result, we may assume that X is affine. Then $X//U$ is a normal affine toric variety whose algebra $k[X]^U = k[X//U]$ is saturated *i.e.* the cone of weights of $k[X]^U$ is equal to the monoid of weights. Let X' be a closed subvariety, then $X'//U$ is a closed subvariety of $X//U$ and the algebra $k[X']^U$ is a quotient of the algebra $k[X]^U$. Let χ be such that $n\chi$ is the weight of an eigenfunction f' on X'. Then there exists an eigenfunction f on X such that $n\chi$ is the weight of that function. By normality there exists an eigenfunction F on X with weight χ. If F is in the kernel of the map $k[X] \to k[X']$, then so is F^n which has to be a multiple of f (by the multiplicity free property) thus $n\chi$ is not a weight of $k[X']$ a contradiction. Thus F is not in the kernel and its image is non trivial in $k[X']$ proving the saturation of $k[X']^U$. Thus $X'//U$ is normal and thus so is X'. □

Corollary 10.2.5 *Let L be an ample line bundle on X a spherical G-variety and let X' be a closed G-subvariety. Then the restriction*

$$H^0(X, L) \to H^0(X', L)$$

is surjective.

Proof. Replace G by a finite covering such that any line bundle is G-linearised. The algebra

$$\bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})$$

is of finite type (since L is ample, this is the algebra of the affine cone over X embedded by L), normal (since X is normal) and multiplicity free for the action of $G \times \mathbb{G}_m$ (since X is G-spherical). This is therefore the algebra of an affine spherical $G \times \mathbb{G}_m$ variety $\tilde{X}$ with a fixed point 0. The same argument gives us a $G \times \mathbb{G}_m$-spherical variety $\tilde{X}'$ whose algebra is

$$\bigoplus_{n=0}^{\infty} H^0(X', L^{\otimes n}).$$

There is a morphism $f : \tilde{X}' \to \tilde{X}$ defined by the restrictions $H^0(X, L^{\otimes n}) \to H^0(X', L^{\otimes n})$ and we have $f(0) = 0$. Since L is ample, the affine varieties $\tilde{X} \setminus \{0\}$ and $\tilde{X}' \setminus \{0\}$ are isomorphic to the varieties $L^\vee \setminus L_0^\vee$ and $L_{X'}^\vee \setminus {L_{X'}}_0^\vee$ where $L^\vee$ and its restriction ${L_{X'}}^\vee$ to X' are considered here as vector bundles over X and X' and where $L_0^\vee$ and ${L_{X'}}_0^\vee$ are the corresponding zero sections. In particular we get that f is an ismorphism of $\tilde{X}' \setminus \{0\}$ onto its image. Thus f induces a birational bijective map onto its image. But the image of f is G-stable and closed therefore normal and f is an isomorphism. Therefore $\tilde{X}'$ is a closed subvariety of $\tilde{X}$ proving the surjectivity. $\square$

Chapter 11

Line bundles on spherical varieties

11.1 Simple spherical varieties

11.1.1 Picard group

Let us start with the following general result.

Lemma 11.1.1 *Assume char$(k) = 0$ and let G be an affine normal G-variety containing a unique closed G-orbit Y. Then the restriction map* $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ *is injective.*

Proof. Replacing G by a finite cover we may assume that any line bundle L on X is G-linearised. If furthermore L has a trivial restriction to Y, then there is a nowhere vanishing element $s \in H^0(Y, L|_Y)$. Since $H^0(Y, L|_Y)$ is a G-representation, this element has to be a G-eigenfunction (the composition $G \to Y \to \mathbb{A}^1$ defined by $g \mapsto s(g \cdot y)$ with $y \in Y$ is nowhere vanishing thus has to be a multiple of a character). But we have a surjective map $H^0(X, L) \to H^0(Y, L)$ therefore we can lift s to a section $s' \in H^0(X, L)^{(G)}$. The locus where s' vanishes is then closed and G-stable therefore either empty or containing Y. The last case is impossible thus s' is nowhere vanishing and L is trivial $\square$

Remark 11.1.2 If $\mathrm{char}(k) = p > 0$, we cannot lift s to a section $s' \in H^0(X, L)$ but only a power (in fact a p^n-th power) of s *i.e.* there exists $s' \in H^0(X, L^{\otimes p^n})$ such that $s'|_Y = s^{p^n}$. We obtain that the kernel of the map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is p-divisible.

Theorem 11.1.3 *Let X be a simple spherical variety with closed orbit Y and colored cone $(\mathcal{C}_Y^\vee(X), \mathcal{F}_Y(X))$.*

($\imath$) Any Cartier divisor on X is linearly equivalent to a linear combination with integer coefficients

$$\sum_{D \in D(X) \setminus D_Y(X)} n_D D.$$

Furthermore, the map $\mathcal{C}_Y^\vee(X)^\perp \to \mathbb{Z}(D(X) \setminus D_Y(X))$ defined by $\chi \mapsto \sum_{D \in D(X) \setminus D_Y(X)} \langle \rho(\nu_D), \chi \rangle D$ induces an exact sequence:

$$\mathcal{C}_Y^\vee(X)^\perp \to \mathbb{Z}(D(X) \setminus D_Y(X)) \to \mathrm{Pic}(X) \to 0.$$

($\imath\imath$) A Cartier divisor is globally generated (resp. ample) if and only if it is a linear combination $\sum_{D \in D(X) \setminus D_Y(X)} n_D D$ with $n_D \geq 0$ (resp. $n_D > 0$) for all D.

Proof. (i) By the structure theorem for spherical varieties, there exists a parabolic subgroup P and a closed subvariety S of $X_{Y,B}$ such that $X_{Y,B}$ is isomorphic to $R_u(P) \times S$ where $R_u(P)$ is the unipotent radical of P while L is a Levi factor of P. The variety S is furthermore a L-affine with a unique closed orbit $Y \cap S$ which is isomorphic to L/L_Y such that $L_Y \supset D(L)$. In particular $S \cap Y$ is a torus and $\mathrm{Pic}(Y \cap S)$ is trivial. By the above Lemma we get that $\mathrm{Pic}(X_{Y,B})$ is trivial.

In particular, any divisor has support on the complement of $X_{Y,B}$ which is the union of the divisors $D \in D(X) \setminus D_Y(X)$. Since we know that the divisors are Cartier we get the first description of Cartier divisors.

Let $\delta = \sum_{D \in D(X) \setminus D_Y(X)} n_D D$ be a divisor linearly equivalent to zero. Then there exists $f \in k(X)$ such that $\mathrm{Div}(f) = \delta$. Since δ is a union of B-stable divisors, the function f is a B-eigenfunction. Furthermore, this function does not vanish on $X_{Y,B}$ thus it does not vanish uniformly on Y thus $\nu_D(f) = 0$ for divisor D containing Y. In particular the weight χ of f lies in $\mathcal{C}_Y^\vee(X)^\perp$ proving that $\mathrm{Div}(f)$ is in the image of the left map of the above exact sequence.

Conversely, let $f \in k(X)^{(B)}$ be such that the weight χ of f is in $\mathcal{C}_Y^\vee(X)^\perp$. Then the divisor of f has to be a linear combination of B-stable divisors but not vanishing on Y thus of the above form.

(ii) Let $\delta = \sum_{D \in D(X) \setminus D_Y(X)} n_D D$ be a Cartier divisor and let us assume (by taking a covering of G if necessary) that δ is G-linearised. If all n_D are non negative, then δ is effective and we can consider the canonical section η of this line bundle. We proved already that the G-module spanned by η generates δ (the locus where δ will not be globally generated is closed and G-stable thus contains Y but is contained in divisors not containing Y).

If all n_D are positive, we want to prove that δ is ample. It is enough to prove that there exists a collection (s_i) of section of δ which span δ and such that X_{s_i} is affine and any element f in $k[X_{s_i}]$ is of the form $f = f'/s_i^n$ for some some n and some $f' \in H^0(X, n\delta)$ (*i.e.* you have surjective morphism $(\oplus_n H^0(X, n\delta))_{(s_i)} \to k[X_{s_i}]$ giving an embedding on X_{s_i}). As above, let δ be the canonical section of δ. We have $X_\delta = X_{Y,B}$. Furthermore, since $X = X_{Y,G} = GX_{Y,B}$ is it enough to prove our statement for $X_\eta = X_{Y,B}$, the statement will then follows for anu $X_{g\eta}$ via action of $g \in G$. We know that X_η is affine, let $f \in k[X_\eta] = k[X_{Y,B}]$. Since f is defined on this open set, the divisor of poles of f is of the form

$$\mathrm{Div}_\infty(f) = \sum_{D \in D(X) \setminus D_Y(X),\ \nu_D(f) < 0} \nu_D(f) D$$

thus for n large enough $n\delta + \mathrm{Div}_\infty(f)$ is effective. Therefore $\eta^n f$ is a global section of $n\delta$ proving the result.

Conversely, assume first that δ is globally generated, then the exists $s \in H^0(X, \delta)$ with $s|_Y$ non trivial. We may choose s which is a B-eigenfunction. We then have $f \in k(X)^{(B)}$ such that the effective divisor defined by s is of the form $\mathrm{Div}(f) + \delta$ and does not contain Y. Therefore it is a linear combinaison of elements in $D(X) \setminus D_Y(X)$ with non negative coefficients.

If δ is ampmle, then the divisor

$$n\delta - \sum_{D \in D(X) \setminus D_Y(X)} D$$

is globally generated for n large enough and we may apply the previous result. □

Example 11.1.4 Let G/H be a homogeneous spherical variety. It is called sober if its cone of valuations $\mathcal{V}(G/H)$ is stricly convex. Recall that this is equivalent to the fact that the automorphism group $\mathrm{Aut}(G/H) = N_G(H)/H$ is finite.

There exists a unique simple toroidal proper compactification X of G/H: its colored cone is $(\mathcal{V}(G/H), \emptyset)$. For such a variety, the Picard group $\mathrm{Pic}(X)$ is freely generated by $D(X) \setminus D_Y(X)$ *i.e.*

$$\mathrm{Pic}(X) = \mathbb{Z}(D(X) \setminus D_Y(X)).$$

11.1.2 Weil divisors

In this subsection we describe the group $\mathrm{Cl}(X)$ of Weil divisors of any spherical variety.

Theorem 11.1.5 *Let X be a spherical variety. Any Weil divisor on X is linearly equivalent to a linear combination with integer coefficients*

$$\sum_{D\in D(X)} n_D D.$$

Furthermore, the map $\mathbb{X}(G/H)\to\mathbb{Z}(D(X))$ defined by $\chi\mapsto\sum_{D\in D(X)}\langle\rho(\nu_D),\chi\rangle D$ induces an exact sequence:

$$\mathbb{X}(G/H)\to\mathbb{Z}(D(X))\to\mathrm{Pic}(X)\to 0.$$

Proof. Let us first consider the dense B-orbit BH/H. We claim that we have the equality $\mathrm{Cl}(BH/H)=0$.

This is a direct consequence of the local structure Theorem. Indeed, let Y be the dense G-orbit. The open affine subspace $X_{Y,B}$ is $X\setminus\cup_{D\in D(X)}D$ since no divisor can contain Y. The intersection of Y with $X_{Y,B}$ is the dense B-orbit *i.e.* it is BH/H. Now there exists P a parabolic subgroup of G, L a Levi factor of P and S a closed subvariety such that $X_{Y,B}\simeq R_u(P)\times S$ and thus $BH/H=Y\cap X_{Y,B}\simeq(Y\cap S)\times R_u(P)$. Furthermore, we also proved that $Y\cap S$ is isomorphic to L/L_Y which is a torus. We thus get

$$BH/H\simeq R_u(P)\times(L/L_Y)$$

which is a product of $\mathbb{G}_a$ and $\mathbb{G}_m$ thus $\mathrm{Pic}(BH/H)$ is trivial. Since BH/H is an orbit thus smooth the same holds for $\mathrm{Cl}(BH/H)$.

In particular, any divisor has support on the complement of BH/H which is the union of the divisors $D\in D(X)$. We get the description of Weil divisors.

Let $\delta=\sum_{D\in D(X)}n_D D$ be a Weil divisor linearly equivalent to zero. Then there exists $f\in k(X)$ such that $\mathrm{Div}(f)=\delta$. Since δ is a union of B-stable divisors, the function f is a B-eigenfunction.

Conversely, let $f\in k(X)^{(B)}$ of weight $\chi\in\mathbb{X}(X)$. Then the divisor of f has to be a linear combination of B-stable divisors. □

Corollary 11.1.6 *Let X be a simple spherical variety, then X is locally factorial if and only if for any $D\in D_Y(X)$, there exists $\chi\in\mathbb{X}(G/H)$ such that*

$$\langle\rho(\nu_{D'}),\chi\rangle=\begin{cases}1 & \text{if } D'=D\\ 0 & \text{if } D'\neq D\end{cases}.$$

In other words X is locally factorial if and only if the elements $(\rho(\nu_D))_{D\in D_(X)}$ form a basis of $\mathbb{X}(G/H)^\vee$.

Remark 11.1.7 If X is toroidal, this is equivalent to the fact that $\mathcal{C}_Y^\vee(X)$ is spanned by a basis of $\mathbb{X}^\vee(G/H)$. It is actually equivalent to the smoothness of X in this case.

In general there is no nice (known) smoothness criterion.

Example 11.1.8 Let X be a smooth quadric of dimension at least 3 which is homogeneous under the group SO. Let $\hat{X}$ be the cone over X. This is a spherical variety under the group $\mathrm{SO}\times\mathbb{G}_m$. This variety is locally factorial but not smooth.

11.2 General case

Recall that any Weil divisor δ can be written in the form

$$\delta = \sum_{D \in D(X)} n_D D.$$

Lemma 11.2.1 *The divisor δ is Cartier if and only if for any orbit Y of X, there exists $\chi_{\delta,Y} \in \mathbb{X}(G/H)$ such that $\langle \rho(\nu_D), \chi_{\delta,Y} \rangle = n_D$ for all $D \in D_Y(X)$.*

Proof. This follows directly from the case of simple spherical varieties and the fact that the $X_{Y,G}$ form a covering of X when Y decribes all the G-orbits. □

Remark 11.2.2 Note that the elements $D \in D(X)$ are of two different nature: the G-stable divisors and the B-stable non G-stable divisors.

Definition 11.2.3 *(ı) Let us denote by $\mathcal{F}(X)$ the union of the set $\mathcal{F}_Y(X)$ for all G-orbit Y and by $\mathcal{F}^c(X)$ the set $D(G/H) \setminus \mathcal{F}(C)$. This is the set of B-stable divisors, which are not G-stable and do not contain any G-orbit. Note also that we have the equality*

$$\mathcal{F}^c(X) = D(X) \setminus \bigcup_{Y\ G\text{-orbits}} D_Y(X)$$

which is the set of the B-stable divisor which do not contain any G-orbit (a G-stable divisor always contains a G-orbit). We denote by $D_o(X)$ the set $D(X) \setminus \mathcal{F}^c(X)$, in others words we have the equality

$$D_o(X) = \bigcup_{Y\ G\text{-orbits}} D_Y(X).$$

This is the set of B-stable orbits which contain a G-orbit.

(ıı) We denote by $\mathcal{C}^\vee(X)$ the union of the cones $\mathcal{C}_Y^\vee(X)$ for Y a G-orbit.

Definition 11.2.4 *(ı) We define the set $PL(X)$ of piecewise linear functions as the subgroup of functions l on $\mathcal{C}^\vee(X)$ such that*

- *for any G-orbit Y, the restriction l_Y to $\mathcal{C}_Y^\vee(X)$ is the restriction of an element of $X(G/H)$;*
- *for any G-orbit Z with $Z \subset \overline{Y}$, we have $l_Z|_{\mathcal{C}_Y^\vee(X)} = l_Y$.*

(ıı) We denote by $L(X)$ the group of linear functions on $\mathbb{F}(X)$ i.e. *of functions l on $\mathbb{X}^\vee(G/H)$ such that the exists $\chi \in \mathbb{X}(G/H)$ such that for any G-orbit Y we have $l_Y = \chi|_Y$.*

Remark 11.2.5 Note that this only depends on the values of l on the maximal cones therefore on the cones of the closed orbits in X. Furthermore it only depends on the fan $\mathbb{F}(X)$ and even only on the union of the cones.

Definition 11.2.6 *(ı) Let $\mathrm{Car}^B(X)$ the group of B-stable Cartier divisors. By the previous lemma, we have a morphism*

$$\mathrm{Car}^B(X) \to PL(X)$$

defined by $\delta \mapsto l_\delta$ where $(l_\delta)_Y = \chi_{\delta,Y}$ with notation as in the previous lemma.

(ıı) Let $\mathrm{Div}^B(X)$ the subgroup of Cartier divisors $\mathrm{Div}(f)$ for $f \in k(X)^{(B)}$.

Lemma 11.2.7 *(ı) We have an exact sequence* $0 \to \mathbb{Z}(D(G/H) \setminus \mathcal{F}(X)) \to \mathrm{Car}^B(X) \to PL(X) \to 0$.
(ıı) This exact sequence restricts to $\mathcal{C}^\vee(X)^\perp \to \mathrm{Div}^B(X) \to L(X) \to 0$.

Proof. (ı) For the surjectivity, for $l \in PL(X)$, set $\delta = \sum_D n_D D$ with $n_D = \langle \rho(\nu_D), l\rangle$. By the very definition of $PL(X)$ and the previous lemma, this is a Cartier divisor. The kernel of the map is given by the Cartier divisors such that $n_D = 0$ for all $D \in D_Y(X)$ for some orbit Y. Since any G-stable divisor of X contains a G-orbit, we only have B-stable divisors of G/H which do not contain such a Y *i.e.* divisors $D \in D(G/H) \setminus \mathcal{F}(X)$. The inclusion is obvious.

(ıı) If the Cartier divisor is the divisor of a function its image is indeed in $L(X)$. If f is such that $\mathrm{Div}(f)$ is in the kernel, then for all Y and $D \in D_Y(X)$, we have $\langle \rho(\nu_D), f\rangle = 0$ thus $f \in \mathcal{C}^\vee(X)^\perp$. □

Theorem 11.2.8 *Let X be a spherical variety.*
(ı) There is an exact sequence

$$\mathcal{C}^\vee(X)^\perp \to \mathbb{Z}(D(G/H) \setminus \mathcal{F}(X)) \to \mathrm{Pic}(X) \to PL(X)/L(X) \to 0.$$

(ıı) The group $\mathrm{Pic}(X)$ *is free of finite rank.*
(ııı) For any Cartier B-stable divisor $\delta = \sum_{D \in D_o(X)} \langle \rho(\nu_D), l_\delta\rangle D + \sum_{D \in \mathcal{F}^c(X)} n_D D$, *the following conditions are equivalent.*

- *The divisor δ is globally generated.*
- *For any G-orbit Y, there exists an element $\chi_Y \in \mathbb{X}(G/H)$ such that* $\chi_Y|_{\mathcal{C}^\vee_Y(X)} = l_\delta|_{\mathcal{C}^\vee_Y(X)}$ *and* $\chi_Y|_{\mathcal{C}^\vee(X)\setminus\mathcal{C}^\vee_Y(X)} \leq l_\delta|_{\mathcal{C}^\vee(X)\setminus\mathcal{C}^\vee_Y(X)}$ *and such that* $\langle \rho(\nu_D), \chi_Y\rangle \leq n_D$ *for all* $D \in \mathcal{F}^c(X)$.

Proof. (ı) The exact sequence follows from the two previous exact sequences and snakes lemma.

(ıı) The abelian group $PL(X)$ is a subgroup of a product of free finitely generated abelian groups thus it is a free finitely generated abelian group. Since $L(X)$ is a subgroup of $PL(X)$ it is also finitely generated. We have to prove that the quotient is torsion free. If $l \in PL(X)$ is such that $nl \in L(X)$, then l itself is linear thus in $L(X)$. The quotient $PL(X)/L(X)$ is therefore torsion free. On the other hand, the groups $Z(D(G/H) \setminus \mathcal{F}(X))$ and $\mathcal{C}^\vee(X)^\perp$ are free and finitely generated. Furthermore, if $x \in Z(D(G/H) \setminus \mathcal{F}(X))$ is such that $nx \in \mathcal{C}^\vee(X)^\perp$, then obviously $x \in \mathcal{C}^\vee(X)^\perp$ and the quotient is also torsion free. Thus proves that $\mathrm{Pic}(X)$ is torsion free and finitely generated.

(ııı) Assume that δ is globally generated and let Y be a G-orbit. Since δ is globally generated, there exists $s \in H^0(X, \delta)$ such that $s|_Y$ does not constantly vanish. We may furthermore assume that s is a B-eigenvector. We then have $\delta = A + \mathrm{Div}(f)$ with A effectif and f a B-eigenfunction. Let χ_Y be the weight of f. Because A is effectif and does not contain Y in its support, we have $\langle \rho(\nu_D), l_\delta\rangle \geq \langle \rho(\nu_D), \chi_Y\rangle$ for all $D \in D_o(X)$ with equality for $D \supset Y$ and $\langle \rho(\nu_D), \chi_Y\rangle \leq n_D$ for all $D \in \mathcal{F}^c(X)$.

Following the above proof in the reverse order proves the converse. □

Remark 11.2.9 (ı) Note that the elements $l \in PL(X)$ do only depend on the linear functions l_Y such that $l|_{\mathcal{C}^\vee_Y(X)} = l_Y|_{\mathcal{C}^\vee_Y(X)}$ on a maximal cone $\mathcal{C}^\vee_Y(X)$. Therefore l is determined by its values on the closed orbits.

(ıı) Note also that if X is complete, then the maximal cones have maximal dimension therefore the weights χ_Y of the previous Theorem for Y closed are determined by the value of l_Y.

Definition 11.2.10 *A function $l \in PL(X)$ is called convex if* $l_Y \leq (l_Z)|_{\mathcal{C}^\vee_Y(X)}$ *for any orbit Y and any closed orbit Z. If the inequality is strict, then l is called striclty convex.*

Corollary 11.2.11 *Let X be a proper spherical variety and let δ be a B-stable Cartier divisor writen as*

$$\delta = \sum_{D\in D_o(X)} \langle\rho(\nu_D), l_\delta\rangle D + \sum_{D\in\mathcal{F}^c(X)} n_D D.$$

Then δ is globally generated (resp. ample) if and only if l_δ is convex (resp. strictly convex) and if for all Y closed orbit and for all $D \in \mathcal{F}^c(X)$ we have $\langle\rho(\nu_D), l_Y\rangle \leq n_D$ (resp. $\langle\rho(\nu_D), l_Y\rangle < n_D$).

Proof. The result for the globally generated line bundles follows from the previous Theorem. To prove the ampleness criterion, let us first make the following remark.

Lemma 11.2.12 *Let Y be a closed G-orbit and let f_Y be a B-eigenfunction with $w(f) = -l_Y$. Any B-stable divisor δ' which is linearly equivalent to δ and whose support does not contain Y is of the form*

$$\delta' = \delta + \mathrm{Div}(f_Y).$$

Proof. Let δ be a globally generated line bundle. Let f_Y be a B-eigenfunction such that $w(f_Y) = -l_Y$. The divisor $\mathrm{Div}(f_Y) + \delta$ is then effective B-stable and its support does not contain Y.

Conversely, if δ' is a B-stable divisor, linearly equivalent to δ whose support does not contain Y, then we may write $\delta' = \delta + \mathrm{Div}(f)$ and f is B-stable and its weight χ has to satisfy

$$\langle\rho(\nu_D), \chi\rangle + \langle l_Y, \chi\rangle = 0$$

for any $D \in D_Y(X)$. We get $\chi|_{\mathcal{C}^\vee_Y(X)} = -l_Y|_{\mathcal{C}^\vee_Y(X)}$ and since Y is closed and X proper we have $\chi = -l_Y$. Thus f is a multiple of f_Y and $\mathrm{Div}(f) = \mathrm{Div}(f_Y)$, the result follows. ☐

Let us prove the result for ample line bundles. If δ is ample, then $n\delta$ separates closed orbits for n large enough. By the previous lemma, this implies that $l_Y \neq l_Z$ for Y and Z two distinct closed orbits. Furthermore, the restriction of $\delta + \mathrm{Div}(f_Y)$ to $X_{Y,G}$, which is a simple spherical variety, has to be ample. This gives $\langle\rho(\nu_D), -l_Y\rangle + n_D > 0$ for all $D \in \mathcal{F}^c(X)$ which is the second desired condition.

Conversely, assume that δ satisfies the assumptions of the corollary. Let Y be a closed G-orbit and let η_Y be the canonical section of $\mathcal{O}_{X_{Y,G}}(\delta + \mathrm{Div}(f_Y))$ on $X_{Y,G}$. Since the function l is strictly convex and since $\langle\rho(\nu_D), l_Y\rangle < n_D$ for $D \in \mathcal{F}^c(X)$ (which gives that the coefficient m_D of $\delta + \mathrm{Div}(f_Z)$ is $m_D = -\langle\rho(\nu_D), l_Y\rangle + n_D > 0$), the ampleness criterion for simple spherical varieties gives that δ is ample on $X_{Y,G}$. The union of all simple spherical varieties $X_{Y,G}$ for Y a closed orbit is X proving the result. ☐

Corollary 11.2.13 *Let X be a spherical G-variety.*

(ı) Assume that X is proper, then X is projective if and only if there exists a piecewise linear function on $\mathcal{C}^\vee(X)$ which strictly convex.

(ıı) The variety X is affine is and only if X is simple and there exists $\chi \in \mathbb{X}(G/H)$ with

$$\chi \in \mathcal{C}^\vee(X)^\perp,\ \chi|_{\mathcal{V}(G/H)} \geq 0 \text{ and } \langle\rho(\nu_D), \chi\rangle < 0 \text{ for all } D \in \mathcal{F}^c(X).$$

(ııı) The homogeneous space G/H is affine if and only if $\rho(D(G/H))$ does not contain 0 and spans a convex cone which meets $\mathcal{V}(G/H)$ only in 0.

Proof. (ı) If X is projective then there exists an ample line bundle δ and the piece-wise linear function associated to δ is strictly convex by the previous result. If such a function l exist, then choose δ as in the previous corollary with the n_D very large. Then δ is ample.

(ıı) Assume that X is affine.

Lemma 11.2.14 *The variety X is simple.*

Proof. Let Y be a closed orbit and let Z be a closed G-stable subset disjoint from Y. Since they are disjoint, we have the equality $k[X] = I_Y + I_Z$. In particular we get a surjective map $I_Z \to k[Y]$. The constant functions on Y are G-invariant therefore $k[Y]^G$ is non trivial and since I_Z is also G-stable and we have a surjection $I_Z \to k[Y]$, we get a non trivial element in I_Z^G. But X has a dense G-orbit thus the G-invariant functions on X are constant and $I_Z = k[X]$ thus Z is empty. In particular Y is the unique closed orbit. □

We may thus assume that X is simple with closed orbit Y. This variety is affine if and only if there exists a projective embedding $\bar{X}$ with $X \subset \bar{X}$ and δ ample with support equal to $\bar{X} \setminus X$.

Assume X is affine and let $l \in PL(\bar{X})$ be the corresponding function. Then l is strictly convex and vanishes on $\mathcal{C}_Y^\vee(X)$ (since X is in the complement of the support of δ). Thus l is positive on the cones of $\bar{X}$ which are not cones of X. In particular, since $\bar{X}$ is convex, the function l is positive on $\mathcal{C}^\vee(X) \setminus \mathcal{C}_Y^\vee(X) \supset \mathcal{V}(G/H) \setminus \mathcal{C}_Y^\vee(X)$. In particular l is non negative on $\mathcal{V}(G/H)$. If $D \in \mathcal{F}^c(\bar{X})$, then D is a B-stable non G-stable divisor thus D meets G/H and thus X non trivially. In particular the multiplicity of the divisor δ on D is 0. Because δ is ample, this multiplicity is strictly bigger $\langle \rho(\nu_D), l \rangle$ proving the last condition for l. Now let V_X be the vector space spanned by $\mathcal{C}_Y^\vee(X)$ and let Σ_X the cone spanned by V_X and $\mathcal{F}^c(X)$. The function l is non positive on Σ_X and negative on $\Sigma_X \setminus V_X$. In particular we get that the intersection of $\mathcal{V}(G/H)$ with Σ_X is contained in V_X. This implies the assertion.

Conversely, let X be a simple spherical variety with closed orbit Y. Denote by $\mathcal{C}_0$ the cone spanned by $\mathcal{C}_Y^\vee(X)$ and by $\rho(\nu_D)_{D \in D(X)}$. Assume that such a χ exists. Let us consider the algebra $k[X]$ of regular functions on X. A function $f \in k(G/H)^{(B)}$ is in $k[X]$ is and only if its weight $w(f)$ is in $\mathcal{C}_0^\vee$ (since X is normal, it is regular if and only if it is regular on all the divisors giving the result). In particular, the weights of $k[X]^{(B)} = k[X]^U$ are in a cone therefore $k[X]^{(B)}$ is finitely generated.

Lemma 11.2.15 *If $k[X]^{(B)}$ is finitely generated, so is $k[X]$.*

Proof. Let $f_1, \cdots, f_n$ be generators of $k[X]^{(B)}$. Let $\lambda_1, \cdots, \lambda_n$ be their weights. We choose a basis (h_{k,λ_i}) of the simple modules $V(\lambda_i)$ and claim that this generates $k[X]$. Indeed, let $f \in k[X]$. Decomposing f in terms of the simple representations of G appearing in $k[X]$ we may assume that f is in a simple G-module $V(\lambda)$. Because the f_i generate $k[X]^{(B)}$, there is a multiplication map

$$V(\lambda_1)^{\otimes r_1} \otimes \cdots \otimes V(\lambda_n)^{\otimes r_n} \to V(\lambda)$$

which contains the highest weight vector of $V(\lambda)$. This map is a morphism of G-modules and the target is simple therefore the map is surjective proving the lemma. □

We can therefore define $X' = \mathrm{Spec}(k[X])$. This is an affine spherical variety embedding of G/H' (the algebra $k[X]$ is normal and multiplicity free). Note that the function χ restricted to $\mathcal{C}_0$ is non negative and is positive outside $\mathcal{C}_Y^\vee(X)$. In particular if this cone contains a line then this line in contained in $\mathcal{C}_Y^\vee(X)$ which is strictly convex a contradiction. Thus $\mathcal{C}_0$ does not contain lines. In particular $\mathcal{C}_0^\vee = \{w(f) \ / \ f \in k[X]^{(B)}\} = \{w(f) \ / \ f \in k[X']^{(B)}\}$ is of maximal dimension in $\mathbb{X}(G/H)$. This implies that the inclusion $\mathbb{X}(G/H') \subset \mathbb{X}(G/H)$ is an equality. This implies that $\rho(\mathcal{F}_\varphi)$ (where $\varphi : G/H \to G/H'$ is the map induced by $X \to X'$) is contained in $\mathcal{C}_\varphi = \ker(\mathbb{X}^\vee(G/H) \to \mathbb{X}^\vee(G/H')) = \{0\}$. But the colors $\mathcal{F}(X)$ of X are not mapped to 0 by ρ by definition of colored cones and the elements $D \in \mathcal{F}^c(X)$ are not such that $\rho(\nu_D) = 0$ since $\chi(\rho(\nu_D)) > 0$. Thus $\mathcal{F}_\varphi = \emptyset$ and $G/H = G/H'$.

We also have a morphism $X \to X'$ therefore we have inclusions $\mathcal{C}^\vee(X) \subset \mathcal{C}^\vee(X')$ and $\mathcal{F}(X) \subset \mathcal{F}(X')$. Furthermore, we have the equalities $\mathcal{C}_0^\vee = \{w(f) \ / \ f \in k[X]^{(B)}\} = \{w(f) \ / \ f \in k[X']^{(B)}\} =$

$\{w(f) \ / \ \langle \rho(\nu_D), w(f)\rangle \geq 0, \text{ for } D \in D(X')\}$ which implies that $\mathcal{C}^\vee(X') \cup \rho(D(G/H'))$ spans $\mathcal{C}_0$. In particular we have $\mathcal{C}^\vee(X') \subset \mathcal{C}_0$.

But the existence of χ implies that $\mathcal{C}^\vee(X)$ is the maximal face of $\mathcal{C}_0$ satisfying condition $(CC2)$ *i.e.* having an interior which non trivially intersects $\mathcal{V}(G/H)$. In particular we must have $\mathcal{C}(X') \subset \mathcal{C}(X)$ and thus equality. Furthermore the existence of χ implies that if $D \in D(G/H)$ is such that $\rho(\nu_D) \in \mathcal{C}_Y^\vee(X)$, then $\chi(\rho(\nu_D)) = 0$ and D has to be a color of X *i.e.* $D \in \mathcal{F}(X)$. Now if $D \in \mathcal{F}(X')$, then $\rho(\nu_D) \in \mathcal{C}(X') = \mathcal{C}(X)$ thus $D \in \mathcal{F}(X)$. The colored cones of X and X' are the same therefore $X = X'$.

($\imath\imath\imath$) Apply the previous result to the trivial embedding with $Y = X = G/H$. Note that we have Then $\mathcal{C}_Y^\vee(X) = 0$ and $\mathcal{F}_Y(X) = \emptyset$ thus $\mathcal{F}^c(X) = D(G/H)$. □

Chapter 12

Canonical divisor for spherical varieties

In this chapter we assume that char(k) the characteristic of the base field k is 0.

12.1 A simplification step

As any spherical variety X is normal, we can define a canonical divisor on X by extending the canonical divisor of the smooth locus. In symbols

$$\omega_X = i_* \omega_{X^{\mathrm{sm}}}$$

where $i : X^{\mathrm{sm}} \to X$ is the embedding of the smooth locus of X in X. To compute this divisor we shall even restrict our selves to an open subset of X with more properties.

Let us fix the following notation. We denote by $\mathcal{V}(X)$ respectively by $D(G/H)$ the set of B-stable divisors which are G-stable, respectively which are not G-stable. We denote by X' the union of the dense orbit and the G-stable divisors, in symbols

$$X' = G/H \cup \bigcup_{D \in \mathcal{V}(X)} D.$$

Then the complement of X' in X is of codimension at least 2 and any B-stable divisor $D \in D(G/H)$ meets X' non trivially. Furthermore, the divisor $D \cap X'$ of X' for $D \in D(G/H)$ does not contain any G-orbit of X' therefore X' is toroidal. To compute ω_X we may thus compute $\omega_{X'}$ and set

$$\omega_X = j_* \omega_{X'}$$

with $j : X' \to X$ the inclusion. We are therefore left to compute the canonical divisor of a smooth toroidal spherical variety.

12.2 local structure of toroidal varieties

To study toroidal varieties we shall need a specific local structure result. Let X be a spherical variety and let Δ_X be the union of the B-stable non G-stable divisors of X.

Fact 12.2.1 *The divisor Δ_X is the closure in X of $G/H \setminus BH/H$.*

Proof. We have seen that the complement of $BH/H = X_{G/H,B}$ is the union of all B-stable divisors. The G-stable divisors are exactely those not meeting G/H concluding the proof. □

Using this fact we may write the following formula

$$\Delta_X = \bigcup_{D \in D(G/H)} \overline{D}$$

where $\overline{D}$ is the closure of D in X. Let us denote by P_X the stabiliser of Δ_X. Note that the previous fact implies that P_X is also the stabiliser of BH/H (and in particular a birational invariant of X).

Proposition 12.2.2 *Let X be a spherical variety. The following conditions are equivalent.*

($\imath$) The variety X is toroidal.

($\imath\imath$) There exists a Levi subgroup L of P_X depending on G/H (and not on X) and a closed subvariety Z of $X \setminus \Delta_X$ stable under L such that the map

$$R_u(P_X) \times Z \to X \setminus \Delta_X$$

is an isomorphism. The group $D(L)$ acts trivially on Z which is therefore a toric variety for a quotient of $L/D(L)$. Furthermore any G-orbit meets Z along a unique L-orbit.

Proof. Assume that X is toroidal. We want to apply Theorem 10.1.3. The divisor Δ_X is Cartier and globally generated (take the trivial function on all the cones; another proof would be that the non Cartier and non globally generated locus is G-stable and contained in Δ_X which contains no closed G-orbit thus these loci are empty). Let η be the canonical section of the line bundle $\mathcal{O}_X(\Delta_X)$. This section is an element of $H^0(X, \mathcal{O}_X(\Delta_X))$. We may assume, replacing G by a covering G' that Δ_X is G-linearised thus G acts on $H^0(X, \mathcal{O}_X(\Delta_X))$. Let $V^\vee$ be the G-submodule spanned by η. We have a G-equivariant morphism (this is indeed a morphism since Δ_X is globally generated):

$$\varphi : X \to \mathbb{P}(V)$$

defined by $x \mapsto [\sigma \mapsto \sigma(x)]$ for $\sigma \in V^\vee \subset H^0(X, \mathcal{O}_X(\Delta_X))$. By definition of η, we have $X \setminus \Delta_X \subset \mathbb{P}(V)_\eta$. Note also that since P_X stabilises Δ_X the function η is a P_X-eigenfunction therefore P_X also stabilises $[\eta]$ or $\mathbb{P}(V)_\eta$. The map $X \setminus \Delta_X \to \mathbb{P}(V)_\eta$ is therefore P_X-equivariant.

Since V is simple, the intersection of all translates gH_η of the vanising divisor H_η of η is empty. Therefore if S is a closed orbit in $\mathbb{P}(V)$, it will not be contained in H_η (otherwise it would be contained in the intersection of the translates which is empty). Choose s in the dense P-orbit in S and B a Borel subgroup such that Bs is dense in S. We may apply Theorem 10.1.3 to get a closed L-stable subvariety S' of $\mathbb{P}(V)_\eta$ such that the morphism $R_u(P_X) \times S' \to \mathbb{P}(V)_\eta$ is a P-equivariant isomorphism. In particular S' meets the image of $X \setminus \Delta_X$. Let $Z = \varphi^{-1}(S')$. This is a closed L-stable subvariety of $X \setminus \Delta_X$ and we have a Cartesian diagram

$$\begin{tikzcd} R_u(P_X) \times Z \arrow[r] \arrow[d, "\mathrm{Id}\times\varphi"'] & X \setminus \Delta_X \arrow[d, "\varphi"'] \\ R_u(P_X) \times S' \arrow[r] & \mathbb{P}(V)_\eta. \end{tikzcd}$$

This proves that the top map is an isomorphism.

We have finitely many B-orbits in $X \setminus \Delta_X$ since X is spherical thus B also has finitely many orbits in $R_u(P_X) \times Z$. Recall that $P = R_u(P_X)L$ thus $B = R_u(P_X)(L \cap B)$ and that $L \cap B$ is a Borel subgroup of L. Recall also that the action of $P = R_u(P_X)L$ on $R_u(P_X) \times Z$ is given by $ul \cdot (u', x) = (ulu'l^{-1}, l \cdot x)$ thus $B \cap L$ must have finitely many orbits in Z. Since $X \setminus \Delta_X$ is normal (recall that X is spherical thus normal) the variety Z is also normal thus Z is L-spherical.

The former Fact gives that the intersection $G/H \cap (X \setminus \Delta_X)$ is BH/B. Therefore the intersection $G/H \cap Z$ is equal to $BH/H \cap Z$ and is therefore a $B \cap L$-orbit. But it is also a L-orbit. By Lemma 10.2.2 we get that $D(L)$ acts trivially on Z which has to be a toric variety under the action of a quotient of $L/D(L)$.

Let Y be a G-orbit in X. Then Y is not contained in Δ_X therefore $Y \cap X \setminus \Delta_X$ is dense in Y. Therefore $R_u(P_X)(Z \cap Y)$ is also dense in Y. But $Z \cap Y$ is the closure of an L-orbit. The variety Z is toric for some torus T_Z. Let Z' be the above orbit. The structure Theorem of spherical varieties applied to toric varieties gives a $T_{Z'}$-variety $S_{Z'}$ and an isomorphism

$$T \times^{T_{Z'}} S_{Z'} \to Z.$$

The orbit Z' therefore corresponds to a $T_{Z'}$-fixed point s in $S_{Z'}$. Consider the cone $\mathcal{C}_s^\vee(S_{Z'})$ associated to s and choose a basis $(\rho(\nu_D))$ (over $\mathbb{Q}$ of this cone) given by $L_{Z'}$-stable divisors D. The affine charts give that s is the intersection of these divisors therefore Z' is the intersection of $D_1, \cdots D_r$ divisors of Z with $r = \mathrm{codim}_Z(Z')$. Then each divisor $X_i = \overline{R_u(P_X)D_i}$ is irreducible B-stable and does not meet G/H (this is true since D_i does not meets the dense L-orbit of Z). This implies that X_i is G-stable. Now consider $X' = \overline{R_u(P_X)Z'}$. It is a subvariety of codimension r in X which is contained in the intersection of the X_i. Since Δ_X contains no closed G-orbit the intersection of the X_i has a dense open subset given by $(\bigcup_i X_i) \cap X \setminus \Delta_X$. The variety X' has to be an irreducible components of the intersection of the X_i and is thus G-stable. We get $X' = \overline{Y}$ and $Y \cap (X \setminus \Delta_X) = R_u(P_X)Z'$ thus $Y \cap Z = Z'$ proving the result.

Conversely, any G-orbit of X meets Z and thus is not contained in Δ_X and therefore is not contained in any B-stable bu tnot G-stable divisor. $\square$

Remark 12.2.3 Note that the G-orbit of the open subset $X \setminus \Delta_X$ is X, in symbols: $G(X \setminus \Delta_X) = X$.

12.3 Toric varieties

We start with the case of smooth toric varieties. We will even compute the tangent bundle in this case. We start with the following lemma.

Lemma 12.3.1 *Let X be a toric variety, then X is smooth if and only if it is locally factorial* i.e. *if and only if any cone of the fan of X is spanned by a basis of the monoid $\mathbb{X}(\mathbb{G}/H)$.*

Proof. If X is smooth then it is locally factorial. Conversely, if it is locally factorial, then its cones $\mathcal{C}_Y^\vee(X)$ are saturated and spanned by a basis of the monoid. This means that we have the equality $k[X_{Y,B}] = k[f_1, \cdots, f_n]$ where $(w(f_i))$ is a basis of the weight monoid. In particular the elements (f_i) must be algebraically independent (otherwise a linear combinaison of their weight would vanish). We thus have $X_{Y,B} \simeq \mathbb{A}^n$. Since the subspaces $X_{Y,B}$ are equal to the subspaces $X_{Y,G}$ which form an open covering the result follows. $\square$

Corollary 12.3.2 *If X is toroidal, then X is smooth if and only if it is locally factorial* i.e. *if and only if the elements $(\rho(\nu_D))_{D \in D(X)}$ form a basis of $\mathbb{X}(G/H)^\vee$.*

Proof. By the structure Theorem of toroidal varieties, the open subset $X \setminus \Delta_X$ is isomorphic to $R_u(P_X) \times Z$ with Z a toric variety. Therefore the singularities of $X \setminus \Delta_X$ are those of $R_u(P_X) \times Z$. This open subset is smooth if and only if it is locally factorial. But since the complement of $X \setminus \Delta_X$ is the union of the B-stable non G-stable divisors, the G-orbit of $X \setminus \Delta_X$ is X which is smooth if and only if it is locally factorial. $\square$

Definition 12.3.3 *Recall that locally on an affine open subset $U = \mathrm{Spec}(A)$ of a variety X, the tangent sheaf T_X is defined as the derivations $\mathrm{Der}_k(A, A)$.*

If Z is a closed subset of X, we define the logarithmic tangent sheaf $T_X(-\log(Z))$ as the subsheaf defined locally by the derivation $\partial \in \mathrm{Der}_k(A, A)$ such that $\partial(I) \subset I$ where I is the ideal of $Z \cap U$ in U.

Example 12.3.4 Let $X = \mathbb{A}^n$ so that $X = \mathrm{Spec}(k[x_1, \cdots, x_n])$. The tangent sheaf is free of rank n defined by

$$T_X = \bigoplus_{i=1}^{n} k[x_1, \cdots, x_n] \frac{\partial}{\partial x_i}.$$

Let $Z = D_1 \cup \cdots \cup D_n$ with D_i the divisor defined by the equation $x_i = 0$. The ideal I_Z is $(x_1 \cdots x_n)$ and the logarithmic tangent sheaf for Z is

$$T_X(-\mathrm{logZ}) = \bigoplus_{i=1}^{n} k[x_1, \cdots, x_n] x_i \frac{\partial}{\partial x_i}.$$

Indeed, any element of the tangent sheaf can be written in the form

$$\partial = \sum_{i=1}^{n} P_i(x_1, \cdots, x_n) \frac{\partial}{\partial x_i}$$

and $\partial(x_1 \cdots x_n) \subset I_Z$. This implies that P_i is a multiple of x_i.

Fact 12.3.5 *Let X be a smooth G-variety and let $\mathfrak{g}$ be the Lie algebra of G. There is a G-equivariant morphism*

$$\mathrm{Act} : \mathfrak{g} \otimes \mathcal{O}_X \to T_X$$

defined by $(\eta, x) \mapsto (d_e f_x(\eta), x)$ where $f_x : G \to X$ is the orbit morphism defined by $f_x(g) = gx$.

Proof. We only need to prove that this morpism is G-equivariant. The action of G on its Lie algebra is given by the adjoint action *i.e.* by $g \cdot \eta = \mathrm{Ad}\,(g)(\eta)$ where $Ad(g) = d_e\mathrm{Int}(g)$ and $\mathrm{Int}(g) : G \to G$ is defined by $\mathrm{Int}(g)(h) = ghg^{-1}$. We compute

$$\mathrm{Act}(g \cdot (\eta, x)) = \mathrm{Act}(\mathrm{Ad}\,(g)(\eta), gx) = (d_e f_{gx}(d_e\mathrm{Int}(g)(\eta)), gx).$$

But the composition $f_{gx} \circ \mathrm{Int}(h) = ghx = \tau_g \circ f_x(h)$ where $\tau_g : X \to X$ is defined by $\tau_g(x) = gx$. We get the equality

$$\mathrm{Act}(g \cdot (\eta, x)) = (d_x\tau_g(d_e f_x(\eta)), gx)$$

and since the action of G on T_X is given by $(g, (\eta, x)) \mapsto (d_x\tau_g(\eta), gx)$ the result follows. □

Theorem 12.3.6 *Let X be a smooth toric variety under the action of the torus T whose Lie algebra is $\mathfrak{t}$. Let ∂X be the union of T-stable divisors in X.*

Then the action map induces an isomorphism

$$\mathfrak{t} \otimes \mathcal{O}_X \simeq T_X(-\log(\partial X)).$$

Proof. We only have to check this on the affine covering $X_{Y,B} = X_{Y,G}$ with $G = B = T$ and Y any closed orbit. Since the variety X is smooth, the affine open subsets $X_{Y,B}$ are isomorphic to $\mathbb{A}^n$ with algebra isomorphic to $k[x_1, \cdots, x_n]$. On these affine subspaces the torus T acts on x_i by scalar multiplication therefore the T-stable divisors are given by the union of the coordinate hyperplanes, its ideal is $(x_1 \cdots x_n)$. The Lie algebra of the torus is isomorphic to

$$\mathfrak{t} = \bigoplus_{i=1}^{n} k\eta_i$$

and since the element $(t_1, \cdots, t_n)$ of the torus acts on $(x_1, \cdots, x_n)$ by $(t_1 x_1, \cdots, t_n x_n)$ we have

$$\mathrm{Act}((\eta_1, \cdots, \eta_n), (x_1, \cdots, x_n)) = (\eta_1 x_1, \cdots, \eta_n x_n), (x_1, \cdots, x_n)).$$

The result the follows by the previous example (we have a surjective morphism between vector bundles of the same rank thus it is an isomorphism). □

Corollary 12.3.7 *The canonical divisor of a smooth toric variety X is given by $\omega_X = \mathcal{O}_X(-\partial X)$.*

Proof. Note that we have a G-equivariant morphism $\mathfrak{g} \otimes \mathcal{O}_X \to T_X$ which gives by taking the highest wedge product a G-equivariant morphism $\mathcal{O}_X \to \omega_X^{-1}$. To compute its image, we may again do this on the affine covering $X_{Y,B}$. The algebra of this variety is given by $A = k[x_1, \cdots, x_n]$ and vector bundle $T_X(-\log(\partial Z))$ has a basis given by the $(x_i \partial/\partial x_i)$ Taking maximal exterior power gives the above map which is given in coordinated by

$$(x_1 \cdots x_n) \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}.$$

In particular this section vanishes exactly on the restriction of the boundary ∂X to $X_{Y,B}$. The section $\mathcal{O}_X \to \omega_X^{-1}$ has therefore $\omega_X^{-1}(-\partial X)$ as image thus is isomorphic on that image, the result follows. □

12.4 The canonical divisor of a spherical variety

In this section we compute a canonical divisor for a spherical variety X. Let us denote by ∂X the union of G-stable divisors of X that is to say the union of the divisors D with $D \in \mathcal{V}(X)$.

Theorem 12.4.1 *There exists a canonical divisor K_X of X such that*

$$-K_X = \partial X + \sum_{D \in D(G/H)} a_D \overline{D}$$

with a_D a non negative integer.

Remark 12.4.2 It can be proved, see for example [BrIn94] that the coefficients a_D are unique and positive.

Proof. As already explained, we may assume that X is smooth and toroidal. Consider the action map $\mathrm{Act} : \mathfrak{g} \otimes \mathcal{O}_X \to T_X$. This is a G-equivariant morphism and we consider its restriction to the open subset $X \setminus \Delta_X$ which is isomorphic to $R_u(P_X) \times Z$ with Z a toric variety for a quotient T of $L/D(L)$ where L is a Levi subgroup of P_X. We have an isomorphism $T_{X \setminus \Delta_X} = T_{R_u(P)} \times T_Z$ and by the description of the tangent space for toric varieties, we have an isomorphism $\mathfrak{t} \otimes \mathcal{O}_Z \simeq T_Z(_{\log}(\partial Z))$. But because there is a correspondence between orbits in X and in Z, we have an isomorphism $T_{X \setminus \Delta_X}(-\log(\partial X)) \simeq (\mathfrak{r}_u(P) \oplus \mathfrak{t}) \otimes \mathcal{O}_{X \setminus \Delta_X}$. Therefore the action map

$$\mathfrak{p}_X \otimes \mathcal{O}_{X \setminus \Delta_X} \to T_{X \setminus \Delta_X}$$

on $X \setminus \Delta_X$ induced by the action of P_X has $T_{X\setminus\Delta_X}(-\log(\partial X))$ for image. We have a commutative diagram

$$\begin{tikzcd} \mathfrak{g}\otimes\mathcal{O}_{X\setminus\Delta_X} \arrow[r, "\mathrm{Act}_G"] & T_{X\setminus\Delta_X} \\ \mathfrak{p}_X\otimes\mathcal{O}_{X\setminus\Delta_X} \arrow[u] \arrow[r, "\mathrm{Act}_{P_X}"] & T_{X\setminus\Delta_X} \arrow[u, equal] \end{tikzcd}$$

where the first line is the restrition of the action map from X to $X \setminus \Delta_X$ and the right vertical line is the inclusion of Lie algebras. In particular, the image Act_G contains $T_{X\setminus\Delta_X}(-\log(\partial X))$. Since this last sheaf is the restriction of $T_X(-\log(\partial X))$ to $X \setminus \Delta_X$, the image I of the action map $\mathfrak{g}\otimes\mathcal{O}_X \to T_X$ contains $T_X(-\log(\partial X))$.

Let us prove that it is equal to $T_X(-\log(\partial X))$. For this we only need to consdier the restriction of Act_G at a divisor D of the boundary. Let $x_D \in D\cap(X \setminus \Delta_X)$ such that $\overline{G\cdot x_D} = D$. Let $H_D = \mathrm{stab}(x_D)$. The P_X-orbit of x_D is dense in D thus $P_X H_D$ is dense in G thus $\mathfrak{g} = \mathfrak{p} + \mathfrak{h}_D$ where $\mathfrak{h}_D$ is the Lie algebra of H_D. Now since H_D acts trivially on x_D, we get that the action map

$$\mathfrak{h}_D \otimes \mathcal{O}_X|_{x_D} \to T_X|_{x_D}$$

vanishes. Therefore the action map $\mathfrak{g}\otimes\mathcal{O}_X|_{x_D} \to T_X|_{x_D}$ factors through the action map induced by the P_X action $\mathfrak{p}_X\otimes\mathcal{O}_X|_{x_D} \to T_X|_{x_D}$. Since this is true for any x_D with $\overline{G\cdot x_D} = D$, we get that the action map $\mathfrak{g}\otimes\mathcal{O}_X|_D \to T_X|_D$ factors through the action map induced by the P_X action $\mathfrak{p}_X\otimes\mathcal{O}_X|_D \to T_X|_D$. This implies the equality $I = T_X(-\log(\partial X))$.

Taking the maximal exterior power we get a surjective morphism

$$\Lambda^{\dim X}\mathfrak{g}\otimes\mathcal{O}_X \to \omega_X^{-1}(-\partial X).$$

The fact that the maximal exterior product of $T_X(-\log(\partial X))$ is $\omega_X^{-1}(-\partial X)$ comes from the same result on toric varieties, the G-action and the restriction to $X \setminus \Delta_X \simeq R_u(P_X)\times Z$.

The result now follows: pick a non trivial global section of $\omega_X^{-1}(-\partial X)$ which is a B-eigensection. The correspondig divisor is of the form

$$-K_X - \partial X = \sum_{D\in D(G/H)} a_D\overline{D} + \sum_{D\in\mathcal{V}(X)} b_D D$$

with a_D and b_D non negative. Its restriction to $X \setminus \Delta_X$ is trivial thus $b_D = 0$ for all $D \in \mathcal{V}(X)$. The result follows. $\square$

Chapter 13

Horospherical varieties

In this chapter we study more in details the case of horospherical varieties. Recall that G/H is called horospherical if H does contain a maximal unipotent subgroup U of G. Any embedding of a horospherical homogeneous variety is called horospherical. Recall that this is equivalent to the equality $\mathcal{V}(G/H) = \mathbb{X}(G/H)$.

13.1 Homogeneous horospherical varieties

We first classify all homogeneous horospherical varieties.

Proposition 13.1.1 *Let H be a closed subgroup in G such that G/H is horospherical. Then $N_G(H)$ is a parabolic subgroup and we have an exact sequence*

$$1 \to H \to N_G(H) \xrightarrow{\chi} \mathbb{G}_m^k \to 1$$

where χ is a product of k characters.

Conversely, any subgroup H obtained this way is horospherical.

Proof. By Chevalley's Theorem, there exists V a G-representation and L a line in V such that $H = \mathrm{Stab}_G(L)$. Let U be a maximal unipotent subgroup contained in H and let $B = TU$ be a Borel subgroup with T a maximal torus. Let us decompose V in simple representations. Then we have

$$V = \bigoplus_{\lambda \in \hat{G}} V(\lambda)^{m(\lambda)}.$$

Since U is contained in H we have $L \subset V^U = \oplus_{\lambda \in \hat{G}} (kv_\lambda)^{m(\lambda)}$ with v_λ an highest weight vector of weight λ. Let Λ be the minimal subset of $\hat{G}$ such that $L \subset V^U = \oplus_{\lambda \in \Lambda} (kv_\lambda)^{m(\lambda)}$. Since L is of dimension 1, there exists a subspace $V' = \oplus_{\lambda \in \Lambda} V(\lambda)$ such that the projection from V to V' maps L to L' with the property that $\mathrm{stab}_G(L') = H$ (simply pick V' containing L). We may therefore assume that L is spanned by v with

$$v = \sum_{\lambda \in \Lambda} a_\lambda v_\lambda$$

and $a_\lambda \neq 0$ for all λ.

Let us denote by P the following parabolic subgroup

$$P = \bigcap_{\lambda \in \Lambda} P(\lambda)$$

where $P(\lambda)$ is the stabiliser of kv_λ in $V(\lambda)$. We have the equalities

$$\begin{aligned} H &= \{g \in G \ / \ \exists \lambda_0(g) \in \mathbb{G}_m \ g \cdot v = \lambda_0(g)v\} \\ &= \{g \in P \ / \ \forall \lambda, \mu \in \Lambda, \ \lambda(g) = \mu(g)\}. \end{aligned}$$

Fixing $\mu \in \Lambda$ we get

$$H = P \cap \bigcap_{\lambda \in \Lambda} \ker(\lambda - \mu) \subset P.$$

We finish the proof by the equality $P = N_G(H)$. We have the equality $R_u(H) = R_u(P)$ and since $P = N_G(R_u(P))$ we get the inclusion $N_G(H) \subset N_G(R_u(H)) = N_G(R_u(P)) = P$. Let $p \in P$ and $h \in H$, then php^{-1} is in P since $H \subset P$ and $(\lambda - \mu)(php^{-1}) = (\lambda - \mu)(h) = 0$ this $php^{-1} \in H$ and $P \subset N_G(H)$. □

13.2 Colored fans

We are now in position to describe the weight lattice $\mathbb{X}(G/H)$ and its dual $\mathbb{X}^\vee(G/H)$. Let H be a horospherical subgoup and let $P = N_G(H)$.

Definition 13.2.1 *(ı) Let us denote by I the set of simple roots defining P* i.e. *the set of simple roots α such that $U_{-\alpha} \not\subset P$.*

(ıı) Define M as the set of characters χ of P such that $\chi|_M$ is trivial, in symbols:

$$M = \{\chi \in X^*(P) \ / \ \chi(h) = 1, \ \forall h \in H\}.$$

We have an inclusion $M \subset X^*(T)$.

(ııı) Define N as the dual lattice of M, i.e. *$N = \hom_{\mathbb{Z}}(M, \mathbb{Z})$.*

Fact 13.2.2 *We have the equalities $M = \mathbb{X}(G/H)$ and $N = \mathbb{X}^\vee(G/H)$.*

Proof. The fibration $\pi : G/H \to G/P$ is locally trivial over the dense affine open subset $R_u(P^-)P/P$. Indeed we have the two inverse morphisms $p : \pi^{-1}(R_u(P^-)P/P) \to R_u(P^-)P/P \times P/H$ and $q : R_u(P^-)P/P \times P/H \to \pi^{-1}(R_u(P^-)P/P)$ defined by $p(y) = (\pi(y), u^{-1}y)$ where $\pi(y) = uP/P$ with $u \in R_u(P^-)$ and $q(uP/P, x) = ux$.

Therefore G/H has an open affine subset of the form $\Omega = R_u(P^-) \times P/H$ and the Borel B^- acts with a dense orbit in this variety. The field $k(G/H)$ is the same as $k(\Omega)$ and the weights of B^- in Ω are the weights of T on P/H therefore these weights are exactly M. □

Fact 13.2.3 *The set $D(G/H)$ is in bijection with I.*

Proof. Let D be a B^--stable divisor of G/H. Assume that its image by $\pi : G/H \to G/P$ is dominant. Then its image contains the dense U^- orbit thus D induces a B-stable divisor in $\pi^{-1}(R_u(P^-)P/P) = R_u(P^-)P/P \times P/H$. But this variety is a B-orbit. A contradiction.

Thus the image of D by $\pi : G/H \to G/P$ is a B-stable divisor of G/P. This is the closure of a B-orbit of codimension 1 on G/P. But the B-orbits in G/P are of the form BwP/P with $w \in W$. Furthermore, remark that since W_P the subgroup of the Weyl group spanned by the simple reflections s_α with $\alpha \notin I$ is contained in P, we may only consider elements $w \in W/W_P$. The divisors are given by elements of maximal length minus 1. They are of the form $w_0 s_\alpha$ with $\alpha \in I$ (see for example [Hum75]). □

Definition 13.2.4 *For each element $\alpha \in I$, we will denote by D_α the corresponding divisor in G/H.*

Example 13.2.5 Let Y be a toric variety for the torus P/H. The group P acts by left multiplication on Y via the map $P \times Y \to P/H \times Y \to Y$. We may therefore construct the contracted product

$$X = G \times^P Y.$$

This contains $G/H = G \times^P P/H$ as dense orbit and is therefore an embedding of the horospherical homogeneous variety G/H.

If Y_i is a P/H stable subvariety in Y, then $G \times^P Y_i$ is a G-stable subvariety in X. In particular, the G-stable divisors of X are of the form $G \times^P Y_i$ with Y_i any P/H stable divisor on Y.

Note that we have a G-equivariant morphism $\pi : X \to G/P$. Any G-orbit is mapped surjectively onto G/P therefore it is not contained in any B-stable divisor. The variety X is thus toroidal.

We prove that any toroidal horospherical variety is of the above form.

Fact 13.2.6 *Any toroidal horospherical embedding of G/H is of the form $G \times^P Y$ with Y a toric variety for the torus P/H.*

Proof. Indeed, consider the fan $\mathbb{F}(X)$ in N. This fan defines a toric variety for the torus P/H (which is such that $\mathbb{X}(P/H) = M$ and $\mathbb{X}^\vee(P/H) = N$). Consider the horospherical variety $G \times^P Y$. Its fan is obviously the fan $\mathbb{F}(X)$ thus $X = G \times^P Y$. □

Proposition 13.2.7 *Let X be a horospherical variety, then there exists a canonical class of the form*

$$K_X = -\partial X - \sum_{\alpha \in I} a_\alpha \overline{D}_\alpha$$

with $a_\alpha = \langle 2\rho^P, \alpha^\vee \rangle$ and $2\rho^P = \sum_{\beta \in R^+ \setminus R_P} \beta$.

Proof. As for the case of spherical varieties, we can assume that X is smooth and toroidal. Therefore X is of the form $X = G \times^P Y$ with Y a toric variety for P/H. Consider the morphism $\pi : X \to G/P$. Since the canonical bundle $K_Y = -\partial Y$ is P/H-invariant and thus P-invariant, the relative canonical bundle for π, denoted by K_π, is given by $K_\pi = -\partial X$. A canonical divisor is therefore given by $K_X = K_\pi + \pi^* K_{G/P}$ and the formula follows from the case of homogeneous spaces.

For homogeneous spaces, way to compute a canonical class is as follows. First note that the canonical line bundle is defined by its weight for P and that there will be a unique B-stable canonical divisor (because homogeneous spaces are spherical and therefore multiplicity free for example). But we have an open subset which is B-stable given by $R_u(P^-)P/P$ and on that affine space a canonical B-stable divisor class given by $dx_1 \wedge \cdots \wedge dx_n$ where x_i are the coordinates in $R_u(P^-)$. The T-weight of that class is given by $-\sum_{\beta \in R^+ \setminus R_P} \beta = 2\rho^P$. The result follows. □

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