PROJECTED GROMOV-WITTEN VARIETIES IN COMINUSCULE SPACES

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Abstract. A projected Gromov-Witten variety is the union of all rational curves of fixed degree that meet two opposite Schubert varieties in a homogeneous space $X = G/P$. When X is cominuscule we prove that the map from a related Gromov-Witten variety is cohomologically trivial. This implies that all (3 point, genus zero) K-theoretic Gromov-Witten invariants of X are determined by projected Gromov-Witten varieties, which extends an earlier result of Knutson, Lam, and Speyer, and provides an alternative version of the 'quantum equals classical' theorem. Our proof uses that any projected Gromov-Witten variety in a cominuscule space is also a projected Richardson variety.

1. Introduction

Let $X = G/P$ be a homogeneous space defined by a simple complex Lie group G and a parabolic subgroup P. Given an effective degree $d \in H_2(X;\mathbb{Z})$, the Kontsevich moduli space $M_d = \overline{\mathcal{M}}_{0,3}(X,d)$ parametrizes the set of 3-pointed stable maps to X of degree d and genus zero. Let $\mathrm{ev}_i : M_d \to X$ be the i-th evaluation map. Given two opposite Schubert varieties X_u and X^v in X, let $M_d(X_u, X^v) = \mathrm{ev}_1^{-1}(X_u) \cap \mathrm{ev}_2^{-1}(X^v) \subset M_d$ denote the *Gromov-Witten variety* of stable maps that send the first marked point to X_u and the second marked point to X^v. It was proved in [7] that $M_d(X_u, X^v)$ is either empty or unirational with rational singularities. The image $\Gamma_d(X_u, X^v) = \mathrm{ev}_3(M_d(X_u, X^v)) \subset X$ is called a *projected Gromov-Witten variety*. This variety $\Gamma_d(X_u, X^v)$ is the union of all connected rational curves of degree d in X that meet both of the Schubert varieties X_u and X^v.

Gromov-Witten varieties are closely related to the (ordinary and K-theoretic) Gromov-Witten invariants. These invariants are defined by

$$I_d([\mathcal{O}_{X_u}], [\mathcal{O}_{X^v}], \sigma) = \chi_{M_d}(\mathrm{ev}_1^*[\mathcal{O}_{X_u}] \cdot \mathrm{ev}_2^*[\mathcal{O}_{X^v}] \cdot \mathrm{ev}_3^*(\sigma))\,,$$

where $\sigma \in K(X)$ is any K-theory class and $\chi_{M_d} : K(M_d) \to \mathbb{Z}$ is the sheaf Euler characteristic map. It follows from Sierra's sheaf-theoretic version of Kleiman's transversality theorem [27] that $\mathrm{ev}_1^*[\mathcal{O}_{X_u}] \cdot \mathrm{ev}_2^*[\mathcal{O}_{X^v}] = [\mathcal{O}_{M_d(X_u,X^v)}] \in K(M_d)$, see

Date: May 4, 2017.

2000 *Mathematics Subject Classification.* Primary 14N35; Secondary 19E08, 14N15, 14M15, 14M20, 14M22.

The first author was supported in part by NSF grant DMS-1205351.

The third author was supported in part by NSA Awards H98230-13-1-0208 and H98320-16-1-0013 and a Simons Collaboration Grant.

The fourth author was supported by a public grant as part of the Investissement d'avenir project, reference ANR-11-LABX-0056-LMH, LabEx LMH..

[10, §4]. In particular, any 2-point Gromov-Witten invariant of X is determined by $I_d([\mathcal{O}_{X_u}], [\mathcal{O}_{X^v}], 1) = \chi_{M_d}([\mathcal{O}_{M_d(X_u,X^v)}])$, and so is equal to zero or one by the geometric properties of Gromov-Witten varieties proved in [7]. More generally, the projection formula implies that the 3-point Gromov-Witten invariants of X are given by

$$I_d([\mathcal{O}_{X_u}], [\mathcal{O}_{X^v}], \sigma) = \chi_X((\mathrm{ev}_3)_*[\mathcal{O}_{M_d(X_u,X^v)}] \cdot \sigma)\,.$$

It is therefore natural to study the cohomological properties of the restricted evaluation map

$$\mathrm{ev}_3 : M_d(X_u, X^v) \to \Gamma_d(X_u, X^v)\,. \tag{1}$$

In this paper we study the projected Gromov-Witten varieties when X is a cominuscule variety, i.e. a Grassmannian of type A, a Lagrangian Grassmannian, a maximal orthogonal Grassmannian, a quadric hypersurface, or one of two exceptional varieties called the Cayley plane and the Freudenthal variety. Our main result states that, when X is cominuscule, the restricted evaluation map (1) is *cohomologically trivial*, which means that the pushforward of the structure sheaf of $M_d(X_u, X^v)$ is the structure sheaf of $\Gamma_d(X_u, X^v)$ and the higher direct images of the first sheaf are zero. This result implies that the (small) quantum cohomology ring $\mathrm{QH}(X)$ and the quantum K-theory ring $\mathrm{QK}(X)$ are determined by the projected Gromov-Witten varieties in X. More precisely, the K-theoretic (3-point genus zero) Gromov-Witten invariants of X satisfy the identity

$$I_d([\mathcal{O}_{X_u}], [\mathcal{O}_{X^v}], \sigma) = \chi_X([\mathcal{O}_{\Gamma_d(X_u,X^v)}] \cdot \sigma)\,. \tag{2}$$

Knutson, Lam, and Speyer have earlier proved this identity for cohomological Gromov-Witten invariants of Grassmannians of type A [20]. We apply our identity to compute the square of a point in the quantum K-theory ring of any cominuscule variety.

The identity (2) can be interpreted as an alternative version of the 'quantum equals classical' theorem for cominuscule Gromov-Witten invariants, which was proved in various generalities in the papers [9, 12, 10, 13]. It has the advantage of being completely uniform. In particular, it avoids a special case of the original version that concerns the degree $d = 3$ for the Cayley plane E_6/P_6 [13]. In addition it reveals that certain (equivariant) K-theoretic Gromov-Witten invariants have alternating signs (see Corollary 4.3 and Remark 4.4), generalizing results of Buch [5], Brion [4], and Anderson, Griffeth and Miller [1].

If $d = 0$, the Gromov-Witten variety $M_d(X_u, X^v)$ is an intersection $X_u \cap X^v$ of opposite Schubert varieties in X, also called a *Richardson variety*. Let $B \subset P$ be a Borel subgroup, set $F = G/B$, and let $\rho : F \to X$ be the projection. If $R \subset F$ is any Richardson variety, then the image $\rho(R) \subset X$ is called a *projected Richardson variety*. Projected Richardson varieties have been studied by Lusztig [24] and Rietsch [26] in the context of total positivity. For arbitrary homogeneous spaces $X = G/P$ it was proved by Billey and Coskun [2] and by Knutson, Lam, and Speyer [21] that any projected Richardson variety $\rho(R)$ is Cohen-Macaulay with rational singularities, and the restricted map $\rho : R \to \rho(R)$ is cohomologically trivial. He and Lam have recently related the K-theory classes of projected Richardson varieties to the K-homology of the affine Grassmannian [18].

The results of this paper show that the restricted evaluation map ev_3 and the map ρ have similar properties when X is a cominuscule variety. We already mentioned cohomological triviality. In addition, if X is cominuscule, then any projected Gromov-Witten variety in X is also a projected Richardson variety. This result is derived from the main construction used to prove the K-theoretic version of the 'quantum equals classical' theorem in [10, 13]. In type A this analysis shows that any variety $\Gamma_d(X_u, X^v)$ is the image of a Richardson variety in a three-step flag manifold obtained from the Richardson variety of kernel-span pairs of rational curves passing through X_u and X^v; see §5, and also [6, 9, 20]. In most cases we can derive the cohomological triviality of the map $\mathrm{ev}_3 : M_d(X_u, X^v) \to \Gamma_d(X_u, X^v)$ from the cohomological triviality of a projection $\rho : R \to \rho(R) = \Gamma_d(X_u, X^v)$ of a Richardson variety R. However, a separate argument is needed when X is the Cayley plane E_6/P_6 and $d = 3$, as in this case the analogue of the three-step flag manifold constructed in [13] fails to be a homogeneous space. In this case we obtain our result by proving that the general fibers of the restricted evaluation map are rationally connected.

This paper is organized as follows. In Section 2 we briefly discuss cohomologically trivial maps and state some useful results. Section 3 explains our notation for Schubert varieties and proves two results about intersections of Schubert varieties in special position. In Section 4 we define projected Gromov-Witten varieties and state our main result as well as some consequences. Section 5 recalls the 'quantum equals classical' theorem for cominuscule varieties and uses it to prove our main result whenever X is not the Cayley plane or $d \neq 3$. This section also explains how to compute the dimension of a projected Gromov-Witten variety and gives an example of a projected Richardson variety in a cominuscule space that is not a projected Gromov-Witten variety. Finally, Section 6 proves the main theorem for the Cayley plane, and Section 7 gives the formula for the square of a point in the quantum K-theory ring.

Acknowledgment. We thank Thomas Lam for useful discussions regarding the dimension of a projected Richardson variety. We also thank the referee for several helpful comments.

2. Cohomologically trivial maps

In this section we state some facts about cohomologically trivial maps which are required in later sections.

Definition 2.1. A morphism $f : X \to Y$ of schemes is *cohomologically trivial* if we have $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^i f_*\mathcal{O}_X = 0$ for $i > 0$.

Notice that if $f : X \to Y$ is proper and cohomologically trivial, then we have $f_*[\mathcal{O}_X] = [\mathcal{O}_Y] \in K_\circ(Y)$ in the Grothendieck group of coherent sheaves on Y. All the cohomologically trivial maps encountered in this paper are also proper. It would be interesting to know if cohomological triviality implies properness.

An irreducible complex variety X has *rational singularities* if there exists a cohomologically trivial resolution of singularities $\pi : \widetilde{X} \to X$, i.e. $\widetilde{X}$ is a nonsingular variety and π is a proper birational and cohomologically trivial morphism. If X has rational singularities, then X is normal, and all resolutions of singularities of X are cohomologically trivial.

An irreducible variety X is *rationally connected* if a general pair of points $(x, y) \in X \times X$ can be joined by a rational curve, i.e. both x and y belong to the image of some morphism $\mathbb{P}^1 \to X$. The following result provides a sufficient condition for cohomological triviality. It was proved in [10, Thm. 3.1] as an application of a theorem of Kollár [22], see also [7, Prop. 5.2].

Proposition 2.2. *Let $f : X \to Y$ be a surjective morphism between complex projective varieties with rational singularities. Assume that the general fibers of f are rationally connected. Then f is cohomologically trivial.*

To establish that a variety is rationally connected, the following result from [17] is invaluable.

Proposition 2.3 (Graber, Harris, Starr)**.** *Let $f : X \to Y$ be any dominant morphism of complete irreducible complex varieties. If Y and the general fibers of f are rationally connected, then X is rationally connected.*

We need the following statement about cohomological triviality of compositions, which will be applied in both directions.

Lemma 2.4. *Let $f : X \to Y$ be a cohomologically trivial morphism, and let $g : Y \to Z$ be any morphism of schemes. Then g is cohomologically trivial if and only if gf is cohomologically trivial.*

Proof. The Grothendieck spectral sequence shows that $R^i g_* \mathcal{O}_Y = R^i(gf)_* \mathcal{O}_X$ for all $i \geq 0$. $\square$

3. Intersections of Schubert varieties

In this section we fix our notation for Schubert varieties and state some results. Let $X = G/P$ be a homogeneous space defined by a reductive complex linear algebraic group G and a parabolic subgroup P. An irreducible closed subvariety $\Omega \subset X$ is called a *Schubert variety* if there exists a Borel subgroup $B \subset G$ such that Ω is B-stable, i.e. $B.\Omega = \Omega$.

Let $Q \subset P$ be a parabolic subgroup contained in P and consider the projection $\pi : G/Q \to G/P$. We need the following result.

Proposition 3.1. *Let Ω be any Schubert variety in G/Q. Then for any $x \in G/P$, the fiber $\pi^{-1}(x)$ is a homogeneous space for a conjugate of a Levi subgroup of P. Furthermore, the intersection $\Omega \cap \pi^{-1}(x)$ is a Schubert variety in $\pi^{-1}(x)$ for all points x in a dense open subset of $\pi(\Omega)$.*

Proof. We may assume that $\Omega = \overline{B.Q}$ and $\pi(\Omega) = \overline{B.P}$ for some Borel subgroup $B \subset G$. Since the dense open orbit $B.P \subset \pi(\Omega)$ is a principal homogeneous space for a subgroup of B, it follows that the restricted map $\pi : \Omega \to \pi(\Omega)$ can be trivialized over this orbit, i.e. there is an isomorphism $\pi^{-1}(B.P) \cap \Omega \cong B.P \times F$ with $F = \pi^{-1}(1.P) \cap \Omega$, such that the map $\pi^{-1}(B.P) \cap \Omega \to B.P$ is the projection to the first factor (see also [7, Prop. 2.3]). It follows that F is irreducible, and it suffices to show that F is a Schubert variety in $\pi^{-1}(1.P) = P/Q$. Choose a maximal torus T in G such that $T \subset B \cap Q$, and let $P = LU$ be the Levi decomposition of P with respect to T, i.e. L is a (reductive) Levi subgroup containing T and U is the unipotent radical. Since $U \subset Q$ it follows that $\pi^{-1}(1.P) = L/(L \cap Q)$ is a homogeneous space for L. By using the root space decomposition of $\mathrm{Lie}(G)$ it follows that $B' = B \cap L$ is a Borel subgroup of L. The proposition now follows because F is a B'-stable closed irreducible subvariety of $\pi^{-1}(1.P)$. $\square$

From now on we will fix a Borel subgroup B and a maximal torus T such that $T \subset B \subset P \subset G$. Let $W = N_G(T)/T$ be the Weyl group of G, let $W_P = N_P(T)/T \subset W$ be the Weyl group of P, and let $W^P \subset W$ be the subset of minimal representatives for the cosets in W/W_P. Each element $u \in W$ defines a Schubert variety $X_u = \overline{Bu.P}$ and an opposite Schubert variety $X^u = \overline{B^-u.P}$ in X. Here $B^- \subset G$ is the Borel subgroup opposite to B. For $u \in W^P$ we have $\dim(X_u) = \operatorname{codim}(X^u, X) = \ell(u)$, where $\ell(u)$ denotes the length of u. Any non-empty intersection of the form $X_u \cap X^v$ with $u, v \in W$ is called a *Richardson variety*. Richardson varieties are known to be rational [25], and they have rational singularities [4].

Proposition 3.2. *Let $u, v \in W^P$ be such that $X_u \cap X^v \neq \emptyset$. Then $X_u \cap g.X^v$ is connected for all $g \in G$.*

Proof. We follow Brion's proof of [4, Lemma 2]. Let G act on $G \times X^v$ by left multiplication on the first factor. Then the multiplication map $m : G \times X^v \to X$ defined by $m(g, x) = g.x$ is G-equivariant. It follows that m is a locally trivial fibration, see [7, Prop. 2.3]. We deduce that $Z = m^{-1}(X_u) = X_u \times_X (G \times X^v)$ is an irreducible variety. Let $\pi : Z \to G$ be the projection. For $g \in G$ we then have $\pi^{-1}(g) = X_u \cap g.X^v$. Since this is a translate of a Richardson variety for all elements g in a dense open subset of G, it follows that the general fibers of π are connected. By Zariski's Main Theorem and Stein Factorization, this implies that all fibers of π are connected, as required. $\square$

Set $F = G/B$ and let $\rho : F \to X$ be the projection. If R is any Richardson variety in F, then the image $\rho(R)$ is called a *projected Richardson variety*. We need the following result which was proved in the papers [2, 21].

Proposition 3.3 ([2, 21])**.** *Let $R \subset G/B$ be a Richardson variety and let $\rho(R) \subset G/P$ be the corresponding projected Richardson variety.*

(a) *The variety $\rho(R)$ is Cohen-Macaulay and has rational singularities.*

(b) *The restricted map $\rho : R \to \rho(R)$ is cohomologically trivial.*

4. Projected Gromov-Witten varieties

Let Φ be the root system of (G, T), with positive roots Φ^+ and simple roots $\Delta \subset \Phi^+$. In the rest of this paper we will assume that $X = G/P$ is a *cominuscule variety*. This means that P is a maximal parabolic subgroup of G corresponding to a simple root $\gamma \in \Delta$ such that $s_\gamma \notin W_P$. In addition γ is a cominuscule simple root, i.e. when the highest root in Φ^+ is written as a linear combination of simple roots, the coefficient of γ is one. The collection of cominuscule varieties consists of Grassmannians $\operatorname{Gr}(m, N)$ of type A, Lagrangian Grassmannians $\operatorname{LG}(m, 2m)$, maximal orthogonal Grassmannians $\operatorname{OG}(m, 2m)$, quadric hypersurfaces Q^m, and two exceptional varieties called the Cayley plane E_6/P_6 and the Freudenthal variety E_7/P_7.

We will identify the homology group $H_2(X;\mathbb{Z})$ with the integers $\mathbb{Z}$, so that the generator $[X_{s_\gamma}] \in H_2(X;\mathbb{Z})$ corresponds to $1 \in \mathbb{Z}$. Given a non-negative degree $d \in \mathbb{Z}$ and a positive integer n, the Kontsevich moduli space $\overline{\mathcal{M}}_{0,n}(X, d)$ parametrizes the isomorphism classes of n-pointed stable (genus zero) maps $f : C \to X$ for which $f_*[C] = d$, and comes with an evaluation map $\mathrm{ev} = (\mathrm{ev}_1, \dots, \mathrm{ev}_n) : \overline{\mathcal{M}}_{0,n}(X,d) \to X^n := X \times \cdots \times X$. A detailed construction of this space can be found in the survey [14]. For the special case $n = 3$ we write simply $M_d = \overline{\mathcal{M}}_{0,3}(X,d)$. Given two closed subvarieties Ω_1 and Ω_2 in X, define the *Gromov-Witten variety* $M_d(\Omega_1,\Omega_2) = \mathrm{ev}_1^{-1}(\Omega_1) \cap \mathrm{ev}_2^{-1}(\Omega_2) \subset M_d$. The corresponding *projected Gromov-Witten variety* is defined by $\Gamma_d(\Omega_1,\Omega_2) = \mathrm{ev}_3(M_d(\Omega_1,\Omega_2)) \subset X$. It was proved in [7] that any Gromov-Witten variety $M_d(X_u, X^v)$ defined by two opposite Schubert varieties is either empty or unirational with rational singularities. Our main result is the following theorem.

Theorem 4.1. *Let X be a cominuscule variety and let X_u and X^v be opposite Schubert varieties in X such that $\Gamma_d(X_u, X^v) \neq \emptyset$.*

(a) *The projected Gromov-Witten variety $\Gamma_d(X_u, X^v)$ is a projected Richardson variety. In particular, it is Cohen-Macaulay and has rational singularities.*

(b) *The restricted map $\mathrm{ev}_3 : M_d(X_u, X^v) \to \Gamma_d(X_u, X^v)$ is cohomologically trivial.*

Let $K(X)$ denote the Grothendieck ring of algebraic vector bundles on X. We use the notation $\mathcal{O}_u = [\mathcal{O}_{X_u}]$ and $\mathcal{O}^u = [\mathcal{O}_{X^u}]$ for the classes in $K(X)$ defined by the structure sheaves of Schubert varieties. The set $\{\mathcal{O}_u \mid u \in W^P\} = \{\mathcal{O}^u \mid u \in W^P\}$ is a $\mathbb{Z}$-basis for $K(X)$. Let $\mathcal{O}_w^\vee$ denote the basis element dual to $\mathcal{O}^w$, in the sense that $\chi_X(\mathcal{O}^u \cdot \mathcal{O}_w^\vee) = \delta_{u,w}$ for all $u, w \in W^P$. Here $\chi_X : K(X) \to \mathbb{Z}$ denotes the sheaf Euler characteristic defined by $\chi_X(\mathcal{F}) = \sum_{k \geq 0} (-1)^k \dim H^i(X;\mathcal{F})$. Given three classes $\sigma_1, \sigma_2, \sigma_3 \in K(X)$, define a K-theoretic Gromov-Witten invariant by

$$I_d(\sigma_1,\sigma_2,\sigma_3) = \chi_{M_d}(\mathrm{ev}_1^*(\sigma_1) \cdot \mathrm{ev}_2^*(\sigma_2) \cdot \mathrm{ev}_3^*(\sigma_3))\,.$$

Theorem 4.1 has the following consequence.

Corollary 4.2. *In the ring $K(X)$ we have*

$$[\mathcal{O}_{\Gamma_d(X_u,X^v)}] = (\mathrm{ev}_3)_*[\mathcal{O}_{M_d(X_u,X^v)}] = \sum_{w \in W^P} I_d(\mathcal{O}_u, \mathcal{O}^v, \mathcal{O}_w^\vee)\, \mathcal{O}^w\,.$$

We note that the second equality in this corollary is clear from the definition of K-theoretic Gromov-Witten varieties. In fact, we have $[\mathcal{O}_{M_d(X_u,X^v)}] = \mathrm{ev}_1^*(\mathcal{O}_u) \cdot \mathrm{ev}_2^*(\mathcal{O}^v)$ in $K(M_d)$ by [27, Thm. 2.2] (see [10, §4.1]), from which we deduce that

$$\chi_X((\mathrm{ev}_3)_*[\mathcal{O}_{M_d(X_u,X^v)}] \cdot \mathcal{O}_w^\vee) = \chi_{M_d}([\mathcal{O}_{M_d(X_u,X^v)}] \cdot \mathrm{ev}_3^*(\mathcal{O}_w^\vee)) = I_d(\mathcal{O}_u, \mathcal{O}^v, \mathcal{O}_w^\vee)\,,$$

as required.

A theorem of Brion [4] states that, if a closed irreducible subvariety of a homogeneous space has rational singularities, then the expansion of its Grothendieck class in the basis of Schubert structure sheaves has alternating signs. This combined with Corollary 4.2 has the following consequence.

Corollary 4.3. *The K-theoretic Gromov-Witten invariants $I_d(\mathcal{O}_u, \mathcal{O}^v, \mathcal{O}_w^\vee)$ have alternating signs in the sense that*

$$(-1)^{\ell(w) - \operatorname{codim} \Gamma_d(X_u,X^v)}\, I_d(\mathcal{O}_u, \mathcal{O}^v, \mathcal{O}_w^\vee) \geq 0\,.$$

We will explain how to compute the codimension of $\Gamma_d(X_u, X^v)$ in Remark 5.3.

Remark 4.4. Since all relevant maps and classes are T-equivariant, Corollary 4.2 holds more generally for the class of $\Gamma_d(X_u, X^v)$ in the Grothendieck ring $K_T(X)$ of T-equivariant vector bundles on X, with the same proof. To be precise, let

$\mathcal{O}_u = [\mathcal{O}_{X_u}]$ and $\mathcal{O}^v = [\mathcal{O}_{X^v}]$ denote T-equivariant Schubert classes in $K_T(X)$, define $\mathcal{O}_w^\vee \in K_T(X)$ by $\chi_X^T(\mathcal{O}^u \cdot \mathcal{O}_w^\vee) = \delta_{u,w}$ where $\chi_X^T : K_T(X) \to K_T(\mathrm{pt})$ is the pushforward map along the structure morphism $X \to \{\mathrm{pt}\}$, and define T-equivariant K-theoretic Gromov-Witten invariants of X by

$$I_d^T(\mathcal{O}_u, \mathcal{O}^v, \mathcal{O}_w^\vee) = \chi_{M_d}^T(\mathrm{ev}_1^*(\mathcal{O}_u) \cdot \mathrm{ev}_2^*(\mathcal{O}^v) \cdot \mathrm{ev}_3^*(\mathcal{O}_w^\vee)) \ \in K_T(\mathrm{pt})\,.$$

Then the equivariant Grothendieck class of $\Gamma_d(X_u, X^v)$ is given by

$$[\mathcal{O}_{\Gamma_d(X_u,X^v)}] = \sum_{w \in W^P} I_d^T(\mathcal{O}_u, \mathcal{O}^v, \mathcal{O}_w^\vee)\, \mathcal{O}^w \ \in K_T(X)\,.$$

Furthermore, a generalization of Brion's theorem by Anderson, Griffeth, and Miller [1, Cor. 5.1] implies that the equivariant Gromov-Witten invariants of X satisfy the positivity property

$$(-1)^{\ell(w) - \operatorname{codim} \Gamma_d(X_u, X^v)}\, I_d^T(\mathcal{O}_u, \mathcal{O}^v, \mathcal{O}_w^\vee) \ \in \ \mathbb{N}\big[\,[\mathbb{C}_\beta] - 1 : \beta \in \Delta\,\big]\,.$$

In other words, up to a sign the invariant $I_d^T(\mathcal{O}_u, \mathcal{O}^v, \mathcal{O}_w^\vee)$ can be written as a polynomial with non-negative integer coefficients in the classes $[\mathbb{C}_\beta]-1 \in K_T(\mathrm{pt})$ defined by the simple roots $\beta \in \Delta$. Here $\mathbb{C}_\beta$ denotes the one-dimensional representation of T defined by $t.z = \beta(t)z$ for $t \in T$ and $z \in \mathbb{C}$.

5. The quantum equals classical theorem

In order to prove Theorem 4.1 we need the 'quantum equals classical' theorem for Gromov-Witten invariants of cominuscule varieties, which can be found in various generalities in the papers [9, 12, 10, 13]. We will say that a non-negative degree $d \in H_2(X)$ is *well behaved* if X is not the Cayley plane E_6/P_6 or $d \neq 3$. We will explain the theorem only for Gromov-Witten invariants of well behaved degrees. The correct statement for Gromov-Witten invariants of the Cayley plane of degree 3 can be found in [13].

For any non-negative degree d and $n \in \mathbb{N}$ we set $\mathcal{Z}_{d,n} = \mathrm{ev}(\overline{\mathcal{M}}_{0,n}(X,d)) \subset X^n$. Given two points $x, y \in X$, we let $d(x,y)$ denote the smallest degree of a rational curve containing x and y [28]. Equivalently, $d(x,y)$ is the minimal degree d for which $(x,y) \in \mathcal{Z}_{d,2}$. For $n \in \mathbb{N}$ we also let $d_X(n)$ be the smallest degree for which any collection of n points in X is contained in a connected rational curve of degree $d_X(n)$, i.e. $d_X(n)$ is minimal with the property that $\mathcal{Z}_{d_X(n),n} = X^n$. The numbers $d_X(2)$ and $d_X(3)$ are given in the following table (see [12, Prop. 18], [13, Prop. 3.4], and [7, §4]).

X	$\dim(X)$	$d_X(2)$	$d_X(3)$
$\mathrm{Gr}(m, m+k)$	mk	$\min(m,k)$	$\min(2m, 2k, \max(m,k))$
$\mathrm{LG}(m, 2m)$	$\frac{m(m+1)}{2}$	m	m
$\mathrm{OG}(m, 2m)$	$\frac{m(m-1)}{2}$	$\lfloor \frac{m}{2} \rfloor$	$\lceil \frac{m}{2} \rceil$
Q^m	m	2	2
E_6/P_6	18	2	4
E_7/P_7	27	3	3

Fix a well behaved degree $d \in H_2(X)$. The main ingredient in the 'quantum equals classical' theorem is a homogeneous space Y_d that parametrizes a family of subvarieties in X. If $X = \mathrm{Gr}(m, N) = \{V \subset \mathbb{C}^N \mid \dim(V) = m\}$ is a Grassmann variety of type A, then Y_d is the two-step partial flag variety defined by

$$Y_d = \mathrm{Fl}(a, b; N) = \{(A, B) \mid A \subset B \subset \mathbb{C}^N,\ \dim(A) = a\,,\ \dim(B) = b\}$$

where $a = \max(m - d, 0)$ and $b = \min(m + d, N)$. For each point $\omega = (A, B) \in Y_d$ we set $X_\omega = \mathrm{Gr}(m - a, B/A) = \{V \in X \mid A \subset V \subset B\}$. The idea is that the kernel A and span B [6] of a general rational curve $C \subset X$ of degree d form a point $\omega = (A, B) \in Y_d$ such that $C \subset X_\omega$.

If X is not a Grassmannian of type A, then we have either $d \leq d_X(2)$ or $d \geq d_X(3)$. In these cases we use the set-theoretic definition

$$Y_d = \{\Gamma_d(x, y) \subset X \mid x, y \in X \text{ and } d(x, y) = \min(d, d_X(2))\}\,.$$

For each element $\omega \in Y_d$ we let X_ω denote the corresponding subvariety of X of the form $\Gamma_d(x, y)$. It follows from [12, Prop. 18] that Y_d can be identified with a projective homogeneous space for G, and that X_ω is a non-singular Schubert variety in X for each point $\omega \in Y_d$. Notice that for $d \geq d_X(3)$, the variety Y_d is a single point and $X_\omega = X$ for $\omega \in Y_d$.

Define the incidence variety $Z_d = \{(\omega, x) \in Y_d \times X \mid x \in X_\omega\}$, and let $p : Z_d \to X$ and $q : Z_d \to Y_d$ be the two projections. For $X = \mathrm{Gr}(m, N)$ we have $Z_d = \mathrm{Fl}(a, m, b; N)$ where a and b are defined as above. Otherwise it follows from [12, Prop. 18] that the stabilizer of X_ω in G acts transitively on X_ω, which implies that Z_d is a projective homogeneous space for G [1]. The following result holds more generally for the T-equivariant K-theoretic Gromov-Witten invariants of X.

Theorem 5.1 ([9, 12, 10, 13])**.** *Let X be a cominuscule variety and $d \in H_2(X)$ a well behaved degree. Given classes $\sigma_1, \sigma_2, \sigma_3 \in K(X)$, the corresponding K-theoretic Gromov-Witten invariant of X of degree d is given by*

$$I_d(\sigma_1, \sigma_2, \sigma_3) = \chi_{Y_d}(q_* p^*(\sigma_1) \cdot q_* p^*(\sigma_2) \cdot q_* p^*(\sigma_3))\,.$$

Define the variety $\mathrm{B}\ell_d = \{(\omega, f) \in Y_d \times M_d \mid \mathrm{Im}(f) \subset X_\omega\}$ and let $\pi : \mathrm{B}\ell_d \to M_d$ be the projection. This map is birational by [10, Lemma 4.6] and [12, Prop. 19]. We also define $Z_d^{(3)} = \{(\omega, x_1, x_2, x_3) \in Y_d \times X^3 \mid x_i \in X_\omega \text{ for } 1 \leq i \leq 3\}$, and let $\phi : \mathrm{B}\ell_d \to Z_d^{(3)}$ be the morphism defined by $\phi(\omega, f) = (\omega, \mathrm{ev}(f))$. It follows from [10, Cor. 2.2] and [13, Thm. 0.2] that the general fibers of ϕ are rational. For $1 \leq i \leq 3$ we also define $e_i : Z_d^{(3)} \to Z_d$ by $e_i(\omega, x_1, x_2, x_3) = (\omega, x_i)$. We then have the following commutative diagram from [10].

$$\begin{tikzcd}
\mathrm{B}\ell_d \arrow[rr, "\pi"] \arrow[d, "\phi"] & & M_d \arrow[d, "\mathrm{ev}_i"] \\
Z_d^{(3)} \arrow[r, "e_i"] & Z_d \arrow[r, "p"] \arrow[d, "q"] & X \\
& Y_d &
\end{tikzcd} \tag{3}$$

For $u, v \in W^P$ we define a Richardson variety in Z_d by

$$Z_d(X_u, X^v) = q^{-1}(q(p^{-1}(X_u)) \cap q(p^{-1}(X^v)))\,.$$

[1]The variety Z_d is called I_d in [12], while Y_d is called F_d, and X_ω is called Y_d.

Since both maps π and ϕ are surjective, it follows from the definitions that the projected Gromov-Witten variety for u and v is given by

$$\Gamma_d(X_u, X^v) = p(Z_d(X_u, X^v))\,. \tag{4}$$

In particular, the projected Gromov-Witten variety $\Gamma_d(X_u, X^v)$ is also a projected Richardson variety. This proves Theorem 4.1(a). We pause here to give a direct proof of Corollary 4.2 from Theorem 5.1, without invoking Theorem 4.1(b).

Direct proof of Corollary 4.2. Assume first that d is a well behaved degree. For $u, v, w \in W^P$ we then obtain from Proposition 3.3 and Theorem 5.1 that

$$\begin{aligned}\chi_X([\mathcal{O}_{\Gamma_d(X_u,X^v)}] \cdot \mathcal{O}_w^\vee) &= \chi_X(p_*[\mathcal{O}_{Z_d(X_u,X^v)}] \cdot \mathcal{O}_w^\vee) = \chi_{Z_d}([\mathcal{O}_{Z_d(X_u,X^v)}] \cdot p^*(\mathcal{O}_w^\vee)) \\ &= \chi_{Y_d}(q_*p^*(\mathcal{O}_u) \cdot q_*p^*(\mathcal{O}^v) \cdot q_*p^*(\mathcal{O}_w^\vee)) = I_d(\mathcal{O}_u, \mathcal{O}^v, \mathcal{O}_w^\vee)\,,\end{aligned}$$

as required.

Assume next that $X = E_6/P_6$ is the Cayley plane and $d = 3$. It then follows from [7, Cor. 4.6] that $\Gamma_3(1.P, w_0.P)$ is a translate of the Schubert divisor X^{s_γ} in X, where $w_0 \in W$ is the longest element. This in turn implies that, if either X_u or X^v has positive dimension, then $\Gamma_3(X_u, X^v) = X$, see Lemma 6.2. It follows that

$$[\mathcal{O}_{\Gamma_3(X_u,X^v)}] = \begin{cases} \mathcal{O}^{s_\gamma} & \text{if } \dim(X_u) = \dim(X^v) = 0\,; \\ 1 & \text{otherwise.} \end{cases}$$

This is compatible with the quantum equals classical theorem for the Cayley plane proved in [13], which implies that $I_3(\mathcal{O}_u, \mathcal{O}^v, \mathcal{O}_w^\vee)$ is equal to δ_{w,s_γ} when $\dim(X_u) = \dim(X^v) = 0$, and equal to $\delta_{w,1}$ otherwise. □

Let $u, v \in W^P$ and define the varieties $\mathrm{B}\ell_d(X_u, X^v) = \pi^{-1}(M_d(X_u, X^v))$ and $Z_d^{(3)}(X_u, X^v) = (pe_1)^{-1}(X_u) \cap (pe_2)^{-1}(X^v)$. Then (3) restricts to the following commutative diagram. It follows from [7, Prop. 3.2 and Thm. 2.5] that all varieties in this diagram are irreducible and have rational singularities.

$$\begin{tikzcd} \mathrm{B}\ell_d(X_u, X^v) \arrow[rr, "\pi"] \arrow[d, "\phi"] & & M_d(X_u, X^v) \arrow[d, "\mathrm{ev}_3"] \\ Z_d^{(3)}(X_u, X^v) \arrow[r, "e_3"] & Z_d(X_u, X^v) \arrow[r, "p"] & \Gamma_d(X_u, X^v) \end{tikzcd} \tag{5}$$

It follows from Kleiman's transversality theorem [19] that the restricted maps π and ϕ of (5) have the same properties as the corresponding maps of the diagram (3), i.e. π is birational and the general fibers of ϕ are rational. Proposition 2.2 therefore implies that π and ϕ are cohomologically trivial, and Proposition 3.3 shows that p is cohomologically trivial. Theorem 4.1(b) therefore follows from Lemma 2.4 and Proposition 2.2 together with the following statement.

Proposition 5.2. *The general fibers of the restricted map $e_3 : Z_d^{(3)}(X_u, X^v) \to Z_d(X_u, X^v)$ are rational.*

Proof. For any $\omega \in Y_d$ the variety $X_\omega \cong q^{-1}(\omega) \subset Z_d$ is a homogeneous space. Furthermore, for any Schubert variety $\Omega \subset X$ we have $\Omega \cap X_\omega \cong p^{-1}(\Omega) \cap q^{-1}(\omega)$. It therefore follows from Proposition 3.1 that $\Omega \cap X_\omega$ is a Schubert variety for all points ω in a dense open subset of $q(p^{-1}(\Omega)) \subset Y_d$. Since the fiber of $e_3 : Z_d^{(3)} \to Z_d$

over an arbitrary point $(\omega, x_3) \in Z_d$ is given by $e_3^{-1}(\omega, x_3) \cong X_\omega \times X_\omega$, we obtain $Z_d^{(3)}(X_u, X^v) \cap e_3^{-1}(\omega, x_3) \cong (X_u \cap X_\omega) \times (X^v \cap X_\omega)$. This proves the proposition. □

Remark 5.3. The codimension of $\Gamma_d(X_u, X^v)$ can be computed as follows. We assume that d is a well behaved degree and $u, v \in W^P$. Let $Q \subset G$ be a parabolic subgroup containing B such that $Y_d = G/Q$. Let $\overline{u}$ be the maximal element in the coset uw_PW_Q where w_P is the longest element in W_P, and let $\overline{v}$ be the minimal element in vW_Q. Set $F = G/B$ and let $\rho : F \to X$ be the projection. It follows from (4) that $\Gamma_d(X_u, X^v) = \rho(F_{\overline{u}} \cap F^{\overline{v}})$, so we have $\Gamma_d(X_u, X^v) \neq \emptyset$ if and only if $\overline{v} \leq \overline{u}$ in the Bruhat order on W. The *Hecke product* on W is the unique associative monoid product such that $s_i \cdot w$ is equal to $s_i w$ if $\ell(s_i w) > \ell(w)$ and equal to w otherwise (see [11, §3] for details and references). Choose $a \in W_P$ such that $\overline{v}a$ is the maximal element in $\overline{v}W_P$. It follows from [21, Prop. 3.3] that $\rho(F_{\overline{u}} \cap F^{\overline{v}}) = \rho(F_{\overline{u}\cdot a} \cap F^{\overline{v}a})$ where $\overline{u} \cdot a$ is the Hecke product of $\overline{u}$ and a. Since ρ maps $F^{\overline{v}a}$ birationally onto its image in X, we deduce that $\Gamma_d(X_u, X^v)$ is birational to $F_{\overline{u}\cdot a} \cap F^{\overline{v}a}$. This implies that $\dim \Gamma_d(X_u, X^v) = \ell(\overline{u} \cdot a) - \ell(\overline{v}a)$.

Example 5.4. Let $G = \mathrm{GL}(6)$ and set $F = \mathrm{Fl}(6) = G/B$ where B is the Borel subgroup of upper triangular matrices. The Weyl group of G is the symmetric group S_6, where each permutation $w \in S_6$ is identified with the permutation matrix whose entry in position (i, j) is equal to 1 whenever $i = w(j)$. Let $\mathcal{O}^w = [\mathcal{O}_{F^w}] \in K(F)$ be the Grothendieck class of the opposite Schubert variety $F^w = \overline{B^- w.B} \subset F$. Consider the Richardson variety $R = F_u \cap F^v$ in F defined by the permutations $u = 642153$ and $v = 132546$ in S_6. A computation with Grothendieck polynomials [23] then gives

$$[\mathcal{O}_R] = \mathcal{O}^{w_0 u} \cdot \mathcal{O}^v = \mathcal{O}^{154623} + 2\mathcal{O}^{245613} + \mathcal{O}^{253614} - \mathcal{O}^{345612} - 3\mathcal{O}^{254613} + \mathcal{O}^{354612}$$

in $K(F)$ where $w_0 = 654321$ is the longest permutation.

Now let P be the parabolic subgroup such that $B \subset P \subset G$ and $X = G/P = \mathrm{Gr}(2, 6)$ is the Grassmann variety of 2-planes in $\mathbb{C}^6$. Let $\rho : F \to X$ be the projection. The set W^P consists of permutations w for which $w(j) < w(j+1)$ for $j \neq 2$, and each such permutation w can be identified with the partition $(w(2)-2\,,\, w(1)-1)$. With this notation we obtain from Proposition 3.3 that the class of the projected Richardson variety $\rho(R)$ is given by

$$[\mathcal{O}_{\rho(R)}] = \rho_*([\mathcal{O}_R]) = \mathcal{O}^{(3,0)} + 2\mathcal{O}^{(2,1)} - 2\mathcal{O}^{(3,1)} - \mathcal{O}^{(2,2)} + \mathcal{O}^{(3,2)}$$

in $K(X)$. The Grothendieck classes of all projected Gromov-Witten varieties $\Gamma_d(X_\lambda, X^\mu)$ in X can be computed using Corollary 4.2 combined with Theorem 5.1 or the Pieri rule for $\mathrm{QK}(X)$ obtained in [10, Thm. 5.4]. It turns out that the class $[\mathcal{O}_{\rho(R)}]$ is not the class of any projected Gromov-Witten variety in X. Therefore not all projected Richardson varieties in X are projected Gromov-Witten varieties.

6. The Cayley plane

The 'quantum equals classical' theorem proved in [13] for Gromov-Witten invariants of degree 3 of the Cayley plane E_6/P_6 involves a variety Z_3 that is not a homogeneous space. It is therefore not possible to use Proposition 3.3 to prove that the map p of the diagram (5) is cohomologically trivial. In this section we give a different proof of Theorem 4.1(b) for the Cayley plane when $d = 3$.

Let X be a cominuscule variety and fix a degree $d \geq 0$. Set $\mathcal{Z}_d = \mathcal{Z}_{d,3} = \mathrm{ev}(M_d) \subset X^3$. Given three subvarieties Ω_1, Ω_2, Ω_3 of X we define the varieties

$M_d(\Omega_1, \Omega_2, \Omega_3) = \mathrm{ev}^{-1}(\Omega_1 \times \Omega_2 \times \Omega_3) \subset M_d$, $\mathcal{Z}_d(\Omega_1, \Omega_2, \Omega_3) = \mathcal{Z}_d \cap (\Omega_1 \times \Omega_2 \times \Omega_3)$, and $\mathcal{Z}_d(\Omega_1, \Omega_2) = \mathcal{Z}_d \cap (\Omega_1 \times \Omega_2 \times X)$. It was proved in [10, 13] that the general fibers of the map $\mathrm{ev} : M_d \to \mathcal{Z}_d$ are rational. Let $U \subset \mathcal{Z}_d$ be a dense open subset such that $M_d(x, y, z) = \mathrm{ev}^{-1}(x, y, z)$ is rational for all points $(x, y, z) \in U$. Let G act diagonally on X^3 and on $\mathcal{Z}_d$. Since the evaluation map is equivariant for this action, we may assume that U is G-stable.

Lemma 6.1. *Let $u, v \in W^P$ satisfy $\Gamma_d(X_u, X^v) \neq \emptyset$. Then $\mathcal{Z}_d(X_u, X^v) \cap U \neq \emptyset$.*

Proof. Set $d_0 = \min(d, d_X(2))$. The assumption implies that we may choose $(x_0, y_0) \in X_u \times X^v$ such that $d(x_0, y_0) = d_0$. Furthermore, since $\mathcal{Z}_d$ contains a dense open subset of points (x, y, z) for which $d(x, y) = d_0$, we may choose $(x, y, z) \in U$ with $d(x, y) = d_0$. Finally, since G acts transitively on the set of pairs of points of distance d_0 by [12, Prop. 18], we have $(x_0, y_0) = g.(x, y)$ for some $g \in G$. It follows that $(x_0, y_0, g.z) = g.(x, y, z) \in \mathcal{Z}_d(X_u, X^v) \cap U$, as required. □

Now let $X = E_6/P_6$ be the Cayley plane. We need the following fact.

Lemma 6.2. *Let X be the Cayley plane and let Ω_1 and Ω_2 be irreducible closed subvarieties of X. If both of these varieties are single points and $d(\Omega_1, \Omega_2) = 2$, then $\Gamma_3(\Omega_1, \Omega_2)$ is a translate of the Schubert divisor X^{s_γ}. In all other cases we have $\Gamma_3(\Omega_1, \Omega_2) = X$.*

Proof. If Ω_1 and Ω_2 are single points, then the lemma follows from [7, Cor. 4.6]. Assume that Ω_2 has positive dimension and let $x \in \Omega_1$ be any point. It is enough to show that $\Gamma_3(x, \Omega_2) = X$. If $z \in X$ is any point, then $\Gamma_3(x, z)$ contains a divisor, and this implies that $\Gamma_3(x, z) \cap \Omega_2 \neq \emptyset$. It follows that $z \in \Gamma_3(x, \Omega_2)$. □

Theorem 4.1(b) for the Cayley plane and $d = 3$ is a consequence of the following result together with Proposition 2.2.

Proposition 6.3. *Let X be the Cayley plane and let $u, v \in W^P$. Then the general fibers of the evaluation map $\mathrm{ev}_3 : M_3(X_u, X^v) \to \Gamma_3(X_u, X^v)$ are rationally connected.*

Proof. For convenience we let $x_0 = 1.P$ denote the B-fixed point and $y_0 = w_0.P$ the B^--fixed point in X. The evaluation map $\mathrm{ev}_3 : M_3(X_u, X^v) \to \Gamma_3(X_u, X^v)$ can be factored as

$$M_3(X_u, X^v) \xrightarrow{\mathrm{ev}} \mathcal{Z}_3(X_u, X^v) \xrightarrow{p_3} \Gamma_3(X_u, X^v)$$

where p_3 is the restriction of the third projection $X^3 \to X$. If we have $\dim(X_u) = \dim(X^v) = 0$, then $X_u = \{x_0\}$, $X^v = \{y_0\}$, and the map p_3 is an isomorphism. The proposition therefore follows from Lemma 6.1.

Assume now that $\dim(X^v) \geq 1$. Choose $z_0 \in X$ such that $d(x_0, z_0) = 2$ and $\Gamma_2(x_0, z_0)$ is a B-stable Schubert variety. It then follows from [7, Cor. 4.6] that $\Gamma_3(x_0, z_0)$ is the B-stable Schubert divisor. Since $G.(x_0, z_0)$ is a dense open subset of X^2 and B^-B is a dense open subset of G, it follows that $O = B^-B.(\{x_0\} \times X \times \{z_0\})$ is a dense open subset of X^3. We claim that $O \cap U \cap \mathcal{Z}_3(X_u, X^v) \neq \emptyset$. To see this, note at first that $\mathcal{Z}_3(X_u, X^v) = \mathrm{ev}(M_3(X_u, X^v))$ is irreducible, so $U \cap \mathcal{Z}_3(X_u, X^v)$ is a dense open subset by Lemma 6.1. In addition, if $y' \in \Gamma_3(x_0, z_0) \cap X^v$ is any point, then $(x_0, y', z_0) \in O \cap \mathcal{Z}_3(X_u, X^v)$, so $O \cap \mathcal{Z}_3(X_u, X^v)$ is also a dense open subset of $\mathcal{Z}_3(X_u, X^v)$. This proves the claim. Since $p_3 : \mathcal{Z}_3(X_u, X^v) \to X$ is surjective, it follows from Kleiman's transversality theorem [19] that there exists a dense open subset $V \subset X$ such that, for each $z \in V$ the fiber $p_3^{-1}(z) = \mathcal{Z}_3(X_u, X^v, z)$ is locally irreducible and each component of this fiber meets $O \cap U$.

Fix a point $z \in V$ and consider the first projection $p_1 : \mathcal{Z}_3(X_u, X^v, z) \to X_u$. For any point $x \in X_u$ we have $p_1^{-1}(x) = \mathcal{Z}_3(x, X^v, z) \cong \Gamma_3(x, z) \cap X^v$. Since this variety is connected by Lemma 6.2 and Proposition 3.2, we deduce that $\mathcal{Z}_3(X_u, X^v, z)$ is connected and hence irreducible. By definition of V we have $\mathcal{Z}_3(X_u, X^v, z) \cap O \neq \emptyset$. If (x, y, z) is any point in this intersection, then we may write $(x, z) = b'b.(x_0, z_0)$ where $b' \in B^-$ and $b \in B$, after which we obtain $p_1^{-1}(x) \cong b'b.\Gamma_3(x_0, z_0) \cap X^v = b'.(\Gamma_3(x_0, z_0) \cap X^v)$. This shows that the general fibers of the map $p_1 : \mathcal{Z}_3(X_u, X^v, z) \to X_u$ are translates of Richardson varieties, so it follows from Proposition 2.3 that $\mathcal{Z}_3(X_u, X^v, z)$ is rationally connected. The definition of V also implies that $\mathcal{Z}_3(X_u, X^v, z) \cap U \neq \emptyset$. Therefore the general fibers of $\mathrm{ev} : M_3(X_u, X^v, z) \to \mathcal{Z}_3(X_u, X^v, z)$ are rationally connected, so a second application of Proposition 2.3 shows that $M_3(X_u, X^v, z)$ is rationally connected for all $z \in V$. This finishes the proof. □

7. The square of a point

In this section we apply our results to compute the square of a point in the (small) quantum K-theory ring $\mathrm{QK}(X)$ of any cominuscule variety X. The ring $\mathrm{QK}(X)$ is a formal deformation of the Grothendieck ring $K(X)$, which as a group is defined by $\mathrm{QK}(X) = K(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\![q]\!]$. The product $\mathcal{O}^u \star \mathcal{O}^v$ of two Schubert structure sheaves in $\mathrm{QK}(X)$ is given by

$$\mathcal{O}^u \star \mathcal{O}^v = \sum_{w, d \geq 0} N_{u,v}^{w,d} \, q^d \, \mathcal{O}^w$$

where the sum is over all $w \in W^P$ and non-negative degrees d. The structure constants $N_{u,v}^{w,d}$ are defined by the recursive identity

$$N_{u,v}^{w,d} \;=\; I_d(\mathcal{O}^u, \mathcal{O}^v, \mathcal{O}_w^\vee) - \sum_{\kappa, e > 0} N_{u,v}^{\kappa, d-e} \cdot I_e(\mathcal{O}^\kappa, \mathcal{O}_w^\vee)\,,$$

where the sum is over all $\kappa \in W^P$ and non-zero degrees e with $0 < e \leq d$. The degree zero constants $N_{u,v}^{w,0}$ are the structure constants of the Grothendieck ring $K(X)$. The quantum K-theory ring $\mathrm{QK}(X)$ was defined by Givental [15] and was further studied in connection with the finite difference Toda lattice [16, 3]. While the product $\mathcal{O}^u \star \mathcal{O}^v$ is defined as a power series in q and might *a priori* contain infinitely many non-zero terms, it was proved in [7] that $N_{u,v}^{w,d} = 0$ whenever X is cominuscule and $d > d_X(2)$. A finiteness result for the quantum K-theory of a larger class of homogeneous spaces was obtained in [8].

Corollary 7.1. *Let X be a cominuscule variety and let $\mathcal{O}_{\mathrm{pt}} \in K(X)$ denote the class of a point. Choose $\kappa \in W^P$ such that $\Gamma_{d_X(2)}(1.P, w_0.P)$ is a translate of X^κ. Then we have $\mathcal{O}_{\mathrm{pt}} \star \mathcal{O}_{\mathrm{pt}} = q^{d_X(2)}\, \mathcal{O}^\kappa$ in $\mathrm{QK}(X)$.*

Proof. Recall that $\Gamma_{d_X(2)}(1.P, w_0.P)$ is a Schubert variety in X by [12, Prop. 18]. Let $N_{\mathrm{pt},\mathrm{pt}}^{w,d}$ denote the structure constants defining the product $\mathcal{O}_{\mathrm{pt}} \star \mathcal{O}_{\mathrm{pt}}$. It follows from Corollary 4.2 that $I_d(\mathcal{O}_{\mathrm{pt}}, \mathcal{O}_{\mathrm{pt}}, \mathcal{O}_w^\vee) = \chi_X([\mathcal{O}_{\Gamma_d(1.P, w_0.P)}] \cdot \mathcal{O}_w^\vee)$ for all degrees d and $w \in W^P$. Since $\Gamma_d(1.P, w_0.P) = \emptyset$ for $d < d_X(2)$, it follows by induction on d that $N_{\mathrm{pt},\mathrm{pt}}^{w,d} = 0$ for $d < d_X(2)$. This in turn implies that

$N^{w,d_X(2)}_{\mathrm{pt},\mathrm{pt}} = \chi_X([\mathcal{O}_{\Gamma_{d_X(2)}(1.P,w_0.P)}] \cdot \mathcal{O}^\vee_w) = \chi_X(\mathcal{O}^\kappa \cdot \mathcal{O}^\vee_w) = \delta_{\kappa,w}$. The corollary follows from this together with the fact [7] that $N^{w,d}_{\mathrm{pt},\mathrm{pt}} = 0$ for $d > d_X(2)$. □

References

[1] D. Anderson, S. Griffeth, and E. Miller, *Positivity and Kleiman transversality in equivariant K-theory of homogeneous spaces*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 1, 57–84. MR 2735076

[2] S. Billey and I. Coskun, *Singularities of generalized Richardson varieties*, Comm. Algebra **40** (2012), no. 4, 1466–1495. MR 2912998

[3] A. Braverman and M. Finkelberg, *Finite difference quantum Toda lattice via equivariant K-theory*, Transform. Groups **10** (2005), no. 3-4, 363–386. MR 2183117 (2007d:17019)

[4] M. Brion, *Positivity in the Grothendieck group of complex flag varieties*, J. Algebra **258** (2002), no. 1, 137–159, Special issue in celebration of Claudio Procesi's 60th birthday. MR 1958901 (2003m:14017)

[5] A. S. Buch, *A Littlewood-Richardson rule for the K-theory of Grassmannians*, Acta Math. **189** (2002), no. 1, 37–78. MR 1946917

[6] ———, *Quantum cohomology of Grassmannians*, Compositio Math. **137** (2003), no. 2, 227–235. MR 1985005

[7] A. S. Buch, P.-E. Chaput, L. Mihalcea, and N. Perrin, *Finiteness of cominuscule quantum K-theory*, Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), no. 3, 477–494 (2013). MR 3099983

[8] ———, *Rational connectedness implies finiteness of quantum K-theory*, Asian J. Math. **20** (2016), no. 1, 117–122. MR 3460760

[9] A. S. Buch, A. Kresch, and H. Tamvakis, *Gromov-Witten invariants on Grassmannians*, J. Amer. Math. Soc. **16** (2003), no. 4, 901–915. MR 1992829

[10] A. S. Buch and L. Mihalcea, *Quantum K-theory of Grassmannians*, Duke Math. J. **156** (2011), no. 3, 501–538. MR 2772069

[11] ———, *Curve neighborhoods of Schubert varieties*, J. Differential Geom. **99** (2015), no. 2, 255–283. MR 3302040

[12] P.-E. Chaput, L. Manivel, and N. Perrin, *Quantum cohomology of minuscule homogeneous spaces*, Transform. Groups **13** (2008), no. 1, 47–89. MR 2421317 (2009e:14095)

[13] P.-E. Chaput and N. Perrin, *Rationality of some Gromov-Witten varieties and application to quantum K-theory*, Commun. Contemp. Math. **13** (2011), no. 1, 67–90. MR 2772579 (2012k:14077)

[14] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 45–96. MR 1492534 (98m:14025)

[15] A. Givental, *On the WDVV equation in quantum K-theory*, Michigan Math. J. **48** (2000), 295–304, Dedicated to William Fulton on the occasion of his 60th birthday. MR 1786492 (2001m:14078)

[16] A. Givental and Y.-P. Lee, *Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups*, Invent. Math. **151** (2003), no. 1, 193–219. MR 1943747 (2004g:14063)

[17] T. Graber, J. Harris, and J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67 (electronic). MR 1937199 (2003m:14081)

[18] X. He and T. Lam, *Projected Richardson varieties and affine Schubert varieties*, arXiv:1106.2586.

[19] S. L. Kleiman, *The transversality of a general translate*, Compositio Math. **28** (1974), 287–297. MR 0360616 (50 #13063)

[20] A. Knutson, T. Lam, and D. Speyer, *Positroid varieties: juggling and geometry*, Compos. Math. **149** (2013), no. 10, 1710–1752. MR 3123307

[21] ———, *Projections of Richardson varieties*, J. Reine Angew. Math. **687** (2014), 133–157. MR 3176610

[22] J. Kollár, *Higher direct images of dualizing sheaves. I*, Ann. of Math. (2) **123** (1986), no. 1, 11–42. MR 825838 (87c:14038)

[23] A. Lascoux and M.-P. Schützenberger, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 11, 629–633. MR 686357 (84b:14030)

[24] G. Lusztig, *Total positivity in reductive groups*, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 531–568. MR 1327548 (96m:20071)

[25] R. W. Richardson, *Intersections of double cosets in algebraic groups*, Indag. Math. (N.S.) **3** (1992), no. 1, 69–77. MR 1157520 (93b:20081)

[26] K. Rietsch, *Closure relations for totally nonnegative cells in G/P*, Math. Res. Lett. **13** (2006), no. 5-6, 775–786. MR 2280774 (2007j:14073)

[27] S. J. Sierra, *A general homological Kleiman-Bertini theorem*, Algebra Number Theory **3** (2009), no. 5, 597–609. MR 2578891 (2011d:14088)

[28] F. L. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs, vol. 127, American Mathematical Society, Providence, RI, 1993, Translated from the Russian manuscript by the author. MR 1234494 (94i:14053)

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