Introduction to Lie algebras

N. Perrin

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Part I

Lie algebras

Chapter 1

Algebras

In this chapter we recall very basic facts on general algebra. We state the results without proofs (with the exception of basic properties of derivations) the proof are identical to the similar statement for commutative algebras.

1.1 Basics

Definition 1.1.1 *An* algebra A *over* k *is a vector space over* k *together with a bilinear map* $A \times A \to A$ *denoted* $(x, y) \mapsto xy$. *In symbols we have:*

- $x(y + z) = xy + xz$ *and* $(x + y)z = xz + yz$ *for all* $(x, y, z) \in A^3$,
- $(ax)(by) = (ab)(xy)$ *for all* $(a, b) \in K^2$ *and* $(x, y) \in A^2$.

Remark 1.1.2 Remark that we assume neither the commutativity of the product nor its associativity. Remark also that A has, a priori, no unit.

Proposition & Definition 1.1.3 *Let* A *be an algebra, the vector space* A *together with the multiplication defined by* $(x, y) \mapsto yx$ *is again an algebra called the* opposite algebra *and denoted* A^{op}.

Proposition & Definition 1.1.4 *A subvector space* B *of* A *which is stable for the multiplication has a natural structure of algebra inherited from the algebra stucture of* A. *Such a subvector space* B *is called a* subalgebra *of* A.

Definition 1.1.5 *A subvector space* I *of an algebra* A *is called an* left ideal *(resp.* right ideal *) if* $xy \in I$ *for all* $x \in A$ *and* $y \in I$ *(resp. for all* $x \in I$ *and* $y \in A$*).*

A left and right ideal is called a two-sided ideal.

Proposition 1.1.6 *For* I *a two-sided ideal of an algebra* A, *the quotient has a natural structure of algebra defined by* $\overline{x} \cdot \overline{y} = \overline{xy}$ *where* $\overline{x}$ *is the class of an element* x *of* A *in the quotient* A/I.

Definition 1.1.7 *For* I *a two-sided ideal of an algebra* A, *the algebra* A/I *is called* the quotient algebra of A by I.

Definition 1.1.8 *Let* f *be a linear map between two algebras* A *and* B. *The map* f *is called a* morphism of algebra *if we have the equality* $f(xy) = f(x)f(y)$ *for all* $(x, y) \in A^2$.

Example 1.1.9 The morphism $A \to A/I$ from an algebra A to the quotient algebra is a morphism of algebras.

1.2 Derivations

As derivations are less classical objects, we shall give proofs of their basic properties.

Definition 1.2.1 *An endomorphism D of an algebra A is called a* derivation *of A if the equality $D(xy) = xD(y) + D(x)y$ holds for all $(x, y) \in A^2$.*

Proposition 1.2.2 *The kernel of a derivation of A is a subalgebra of A.*

Proof. Let x and y be in the kernel, we need to prove that xy is again in the kernel. This follows from the definition of a derivation. □

Proposition 1.2.3 *Let D_1 and D_2 be two derivation of an algebra A, then the commutator $[D_1, D_2] = D_1D_2 - D_2D_1$ is again a derivation of A.*

Proof. Let us compute $[D_1, D_2](xy)$ using the fact that D_1 and D_2 are derivations:

$$\begin{aligned}[D_1, D_2](xy) = &\ D_1(xD_2(y) + D_2(x)y) - D_2(xD_1(y) + D_1(x)y) \\ = &\ xD_1D_2(y) + D_1(x)D_2(y) + D_1D_2(x)y + D_2(x)D_1(y)) \\ &-(xD_2D_1(y) + D_2(x)D_1(y) + D_2D_1(x)y + D_1(x)D_2(y)) \\ = &\ [D_1, D_2](x)y + x[D_1, D_2](y)\end{aligned}$$

and the result follows. □

1.3 Product of algebras

Proposition & Definition 1.3.1 *Let A and B be two algebras over k and consider the map $A \times B \to A \times B$ defined by $((x, y), (x', y')) \mapsto (xx', yy')$. This defines a structure of algebra over k on $A \times B$ called the* product algebra of A and B.

Proposition 1.3.2 *The algebras A and B are supplementary two-sided ideals in $A \times B$.*

Proposition 1.3.3 *Let C be an algebra such that A and B are supplementary two sided ideals in C, then C is isomorphic to the product algebra $A \times B$.*

1.4 Restriction and extension of the scalars

Let k_0 be a subfield of k and k_1 be an extension of k. By restriction of the scalar, we may consider any algebra A over k as an algebra over k_0.

Proposition & Definition 1.4.1 *Let A be an algebra over k, the vector A seen as a vector space over k_0 and with its multiplication over k is an algebra over k_0 called the* algebra obtained by restriction of the scalars from k to k_0.

Proposition & Definition 1.4.2 *Let A be an algebra over k, the algebra structure over k_1 on $A \otimes_k k_1$ defined by $(x \otimes a)(y \otimes b) = xy \otimes ab$ is called the* algebra obtained by extension of the scalar from k to k_1.

1.5 Exercices

Exercice 1.5.1 Prove the assertions in Propositions & Definition 1.1.3, 1.1.4, 1.3.1, 1.4.1, 1.4.2 in Propositions 1.1.6, 1.3.2, 1.3.3 and in Example 1.1.9.

Exercice 1.5.2 Prove that all the Definitions and statements in this chapter are valid if we only assume that k is a commutative ring with unit and if we replace vector spaces over k by modules over k. Prove that the notion of restriction of scalars and extension of scalar makes sense for any ring morphisms $k_0 \to k$ and $k \to k_1$ where k_0 and k_1 are commutative rings with unit.

Exercice 1.5.3 Let A be an algebra (non associative *a priori*) and let D be a derivation in A. Prove that the kernel $\ker D$ is a subalgebra of A. Prove that the subset

$$B = \{x \in A \;/\; \exists k \in \mathbb{Z}_{>0},\; D^k(x) = 0\}$$

is a subalgebra of A.

Chapter 2

Lie algebras

2.1 Definition

Definition 2.1.1 *An algebra A is called a* Lie algebra *(and its product will be denoted $(x, y) \mapsto [x, y]$ and called the* Lie bracket*) if the following properties hold:*

- $[x, x] = 0$ *for all* $x \in A$,
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, *for all* $(x, y, z) \in A^3$.

The last identity is called the *Jacobi identity.*

Remark 2.1.2 The first identity implies that the product, the Lie bracket, is antisymmetric: for all $(x, y) \in A^2$, we have $[y, x] = -[x, y]$.

Proposition 2.1.3 *Any subalgebra of a Lie algebra, any quotient algebra of a Lie algebra, any product algebra of Lie algebras are again Lie algebras.*

Proposition 2.1.4 *The opposite algebra $A^{\rm op}$ of a Lie algebra A is again a Lie algebra and the morphism $A^{\rm op} \to A$ defined by $x \mapsto -x$ is an isomorphism.*

Example 2.1.5 Let A be an associative algebra, then the bracket $[x, y] = xy - yx$ defines a Lie algebra structure on A.

Example 2.1.6 Let V be a k vector space of finite dimension, then $\mathrm{End}(V)$, the vector space of endomorphisms of V has a structure of associative algebra and therefore a structure of Lie algebra denoted $\mathfrak{gl}(V)$.

Example 2.1.7 The subspace $\mathfrak{sl}(V)$ of trace free endomorphisms in $\mathfrak{gl}(V)$ is a Lie subalgebra.

Example 2.1.8 Let $(e_i)_{i\in[1,n]}$ be a basis of V and define the complete flags $V_\bullet = (V_i)_{i\in[1,n]}$ and $V'_\bullet = (V'_i)_{i\in[1,n]}$ of V by $V_i = \langle e_1, \cdots, e_i\rangle$ and $V'_i = \langle e_n, \cdots, e_{n+1-i}\rangle$ for all $i \in [1, n]$.

The subspace $\mathfrak{t}(V_\bullet)$ of endomorphisms stabilising all the subspaces of the flag $V_\bullet$ is a Lie subalgebra of $\mathfrak{gl}(V)$ and its intersection $\mathfrak{st}(V_\bullet)$ is a subalgebra of $\mathfrak{sl}(V)$.

The subspace $\mathfrak{n}(V_\bullet)$ of endomorphisms f such that $f(V_i) \subset V_{i-1}$ for all $i \in [1, n]$ with $V_0 = \{0\}$ is a Lie subalgebra of $\mathfrak{st}(V_\bullet)$.

The subspace $\mathfrak{diag}(V_\bullet, V'_\bullet) = \mathfrak{t}(V_\bullet) \cap \mathfrak{t}(V'_\bullet)$ is a Lie subalgebra of $\mathfrak{gl}(V)$ and its intersection with $\mathfrak{sl}(V)$, denoted by $\mathfrak{sdiag}(V_\bullet, V'_\bullet)$, is a Lie subalgebra of $\mathfrak{sl}(V)$.

We have $\mathfrak{gl}(V) = \mathfrak{n}(V'_\bullet) \oplus \mathfrak{diag}(V_\bullet, V'_\bullet) \oplus \mathfrak{n}(V_\bullet,)$ and $\mathfrak{sl}(V) = \mathfrak{n}(V'_\bullet) \oplus \mathfrak{sdiag}(V_\bullet, V'_\bullet) \oplus \mathfrak{n}(V_\bullet,)$.

Example 2.1.9 Let G be a Lie group *i.e.* a variety G which is a group and such that the multiplication and the inverse map are morphisms of varieties. The space DiffOpp(G) of differential operators on G form an associative algebra and we may therefore define a Lie algebra structure on DiffOpp(G). Define Lie(G) to be the vector subspace of Diffopp(G) of left G-invariant vector fields. They form a Lie subalgebra of OppDiff(G). By invariance, the Lie algebra Lie(G) is identified with the tangent space to G at e_G the identity element of G.

For any morphism $f : G \to G'$ of Lie groups, the differential $d_{e_G} f$ of f at e_G is a Lie algebra morphism $\mathrm{Lie}(G) \to \mathrm{Lie}(G')$.

For V a finite dimensional vector space, the Lie algebra Lie(GL(V)) of the linear group GL(V) is identified with $\mathfrak{gl}(V)$. The Lie algebra of the subgroup SL(V) of GL(V) is the subalgebra $\mathfrak{sl}(V)$.

2.2 Adjoint representation

Definition 2.2.1 *Let $\mathfrak{g}$ be a Lie algebra and let $x \in \mathfrak{g}$, the linear map $\mathfrak{g} \to \mathfrak{g}$ defined by $y \mapsto [x, y]$ is called the* adjoint application of x *and denoted* $\mathrm{ad}_{\mathfrak{g}} x$ *or* $\mathrm{ad}\, x$.

Proposition 2.2.2 *Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g}$, then the morphism $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $x \mapsto \mathrm{ad}\, x$ is a Lie algebra morphisms which factors through $\mathfrak{Der}(\mathfrak{g})$ the Lie algebra of derivations of $\mathfrak{g}$.*

Furthermore, for $D \in \mathfrak{Der}(\mathfrak{g})$, we have $[D, \mathrm{ad}\, x] = \mathrm{ad}\,(Dx)$

Proof. The Jacobi identity, $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, gives for all $(x, y, z) \in \mathfrak{g}^3$:

$$\mathrm{ad}\, x(\mathrm{ad}\, y(z)) - \mathrm{ad}\, y(\mathrm{ad}\, x(z)) - \mathrm{ad}\,[x, y](z) = 0$$

proving that the morphisms $x \mapsto \mathrm{ad}\, x$ is a Lie algebra morphism. The Jacobi formula also reads $\mathrm{ad}\, x([y, z]) = [y, \mathrm{ad}\, x(z)] + [\mathrm{ad}\, x(y), z]$ proving the fact that $\mathrm{ad}\, x$ is a derivation.

For D a derivation of $\mathfrak{g}$, we have

$$[D, \mathrm{ad}\, x](y) = D(\mathrm{ad}\, x(y)) - \mathrm{ad}\, x(Dy) = D[x, y] - [x, Dy] = [Dx, y] = \mathrm{ad}\,(Dx)(y)$$

where we used the derivation identity $D[x, y] = [x, Dy] + [Dx, y]$. □

Definition 2.2.3 *The Lie algebra morphism $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $x \mapsto \mathrm{ad}\, x$ is called the* adjoint representation. *It will be denoted by* ad . *The derivations in the image of* ad *in $\mathfrak{Der}(\mathfrak{g})$ are called* inner derivations.

Example 2.2.4 Let G be a Lie group and let G act on itself by conjugation. Any $g \in G$ defined a group morphism $\mathrm{Int}g : G \to G$ by $\mathrm{Int}g(g') = gg'g^{-1}$. The differential of this group morphism at e_G defines a Lie algebra morphism $d_{e_G}\mathrm{Int}g : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$. The map $\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}(G))$ defined by $g \mapsto d_{e_G}(\mathrm{Int}g)$ is a morphism of Lie group and its differential $d_{e_G}\mathrm{Ad} : \mathrm{Lie}(G) \to \mathfrak{gl}(\mathrm{Lie}(G))$ is the adjoint representation ad of Lie(G).

2.3 Ideals

Ideals of Lie algebras are defined as ideals of the underlying algebra.

Remark 2.3.1 The antisymmetry of the Lie bracket implies that any left or right ideal is a two-sided ideal. We shall only say ideal of a Lie algebra. An ideal of a Lie algebra is a subspace stable under the inner derivations.

Definition 2.3.2 *An subspace of a Lie algebra* $\mathfrak{g}$ *stable under any derivation of* $\mathfrak{g}$ *is called a* characteristic ideal*.*

Example 2.3.3 Let G be a Lie group and H a Lie subgroup which is normal. Then the Lie algebra $rmLie(H)$ of H is an ideal of Lie(G) the Lie algebra of G.

Proposition 2.3.4 *Let* $\mathfrak{g}$ *be a Lie algebra, let* $\mathfrak{a}$ *be an ideal (resp. a characteristic ideal) of* $\mathfrak{g}$ *and let* $\mathfrak{b}$ *be a characteristic ideal of* $\mathfrak{a}$*. Then* $\mathfrak{b}$ *is an ideal (resp. a characteristic ideal) of* $\mathfrak{g}$*.*

Proof. Let D be an inner derivation (resp. a general derivation) of $\mathfrak{g}$. Because, $\mathfrak{a}$ is an ideal (resp. a characteristic ideal) of $\mathfrak{g}$, the derivation D stabilises $\mathfrak{a}$ and induces therefore a derivation of $\mathfrak{a}$. As $\mathfrak{b}$ is a characteristic ideal of $\mathfrak{a}$ is it stabilised by D. □

Definition 2.3.5 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *two subvector space of a Lie algebra* $\mathfrak{g}$*, then we denote by* $[\mathfrak{a}, \mathfrak{b}]$ *the subspace generated by the elements* $[x, y]$ *for* $x \in \mathfrak{a}$ *and* $y \in \mathfrak{b}$*.*

Remark that we have $[\mathfrak{a}, \mathfrak{b}] = [\mathfrak{b}, \mathfrak{a}]$*.*

Proposition 2.3.6 *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are ideals (resp. characteristic ideals) of a Lie algebra* $\mathfrak{g}$*, then* $[\mathfrak{a}, \mathfrak{b}]$*, is an ideal (resp. a charateristic ideal) of* $\mathfrak{g}$*.*

Proof. Let D be an inner derivation (resp. a general derivation) of $\mathfrak{g}$ and let $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$, we have $D[x, y] = [x, Dy] + [Dx, y] \in [\mathfrak{a}, \mathfrak{b}]$ and the result follows. □

Example 2.3.7 The Lie algebra $\mathfrak{g}$ is a characteristic ideal of $\mathfrak{g}$. As a consequence of the previous proposition $[\mathfrak{g}, \mathfrak{g}]$ is again a characteristic ideal of $\mathfrak{g}$.

2.4 Abelian Lie agebras

Definition 2.4.1 *Two elements* x *and* y *of a Lie algebra* commute *if* $[x, y] = 0$*.*

A Lie algebra $\mathfrak{g}$ *is said to be* abelian *or* commutative *if any two of its element commute.*

Example 2.4.2 Let A be an associative algebra with Lie algebra structure $[x, y] = xy - yx$. Then A is abelian as a Lie algebra if and only if A is commutative.

Example 2.4.3 The Lie algebras $\mathfrak{diag}(V_\bullet, V'_\bullet)$ and $\mathfrak{sdiag}(V_\bullet, V'_\bullet)$ are abelian.

Example 2.4.4 Let G be a Lie group, if G is abelian, then Lie(G) is an abelian Lie algebra.

2.5 Derived, central ascending and central descending series

Definition 2.5.1 *Let* $\mathfrak{g}$ *be a Lie algebra, the* derived ideal *of* $\mathfrak{g}$ *is the characteristic ideal* $[\mathfrak{g}, \mathfrak{g}]$*. We denote it by* $\mathcal{D}\mathfrak{g}$*.*

Lemma 2.5.2 *Any subspace of* $\mathfrak{g}$ *containing* $\mathcal{D}\mathfrak{g}$ *is an ideal of* $\mathfrak{g}$*.*

Proof. Indeed, the image of any inner derivation is contained in $\mathcal{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. □

Definition 2.5.3 *We call* derived serie *the sequence* $(\mathcal{D}^i\mathfrak{g})_{i\geq 0}$ *of characteristic ideals with* $\mathcal{D}^0\mathfrak{g} = \mathfrak{g}$ *and* $\mathcal{D}^{i+1}\mathfrak{g} = \mathcal{D}(\mathcal{D}^i g) = [\mathcal{D}^i\mathfrak{g}, \mathcal{D}^i\mathfrak{g}]$ *for* $i \geq 0$*.*

Definition 2.5.4 *We call* central descending serie *the sequence* $(\mathcal{C}^i\mathfrak{g})_{i\geq 0}$ *of characteristic ideals with* $\mathcal{C}^0\mathfrak{g} = \mathfrak{g}$ *and* $\mathcal{C}^{i+1}\mathfrak{g} = [\mathfrak{g}, \mathcal{C}^i\mathfrak{g}]$ *for* $i \geq 0$.

Remark 2.5.5 We have the inclusions $\mathcal{D}^i\mathfrak{g} \subset \mathcal{C}^i\mathfrak{g}$ for all $i \geq 0$.

Proposition 2.5.6 *Let* $f : \mathfrak{g} \to \mathfrak{g}'$ *be a Lie algebra morphisms, then we have* $f(\mathcal{D}^i\mathfrak{g}) = \mathcal{D}^i f(\mathfrak{g})$ *and* $f(\mathcal{C}^i\mathfrak{g}) = \mathcal{C}^i f(\mathfrak{g})$. *In particular for* f *surjective, we have* $f(\mathcal{D}^i\mathfrak{g}) = \mathcal{D}^i\mathfrak{g}'$ *and* $f(\mathcal{C}^i\mathfrak{g}) = \mathcal{C}^i\mathfrak{g}'$.

Proof. For two subspaces $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{g}$, we have the equality $f([\mathfrak{a}, \mathfrak{b}]) = [f(\mathfrak{a}), f(\mathfrak{b})]$. The result follows by induction. □

Definition 2.5.7 *Let* P *be a subset of a Lie algebra* $\mathfrak{g}$. *We call* centraliser of P *the set of elements in* $\mathfrak{g}$ *commuting with any element of* P. *We denote this set by* $\mathfrak{z}_{\mathfrak{g}}(P)$ *or* $\mathfrak{z}(P)$.

When P *is a unique vector* x *in* $\mathfrak{g}$ *we denote* $\mathfrak{z}(P)$ *by* $\mathfrak{z}(x)$.

Lemma 2.5.8 *For any subset* P *of* $\mathfrak{g}$, *the centraliser* $\mathfrak{z}(P)$ *is a Lie subalgebra of* $\mathfrak{g}$.

Proof. Indeed, the space $\mathfrak{z}(P)$ is the intersection of the kernels of the maps $\operatorname{ad} x$ for $x \in P$. □

Proposition 2.5.9 *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{a}$ *be an ideal (resp. a characteristic ideal) of* $\mathfrak{g}$, *then* $\mathfrak{z}(\mathfrak{a})$ *is an ideal (res. a characteristic ideal) of* $\mathfrak{g}$.

Proof. Let D be an inner derivation (resp. a general derivation), let $x \in \mathfrak{z}(\mathfrak{a})$ and let $y \in \mathfrak{a}$. We have $[Dx, y] = D[x, y] - [x, Dy]$. But y and Dy are in $\mathfrak{a}$ therefore x commutes with y and Dy and the result follows. □

Definition 2.5.10 *The characteristic ideal* $\mathfrak{z}(\mathfrak{g})$ *is the* center *of the Lie algebra* $\mathfrak{g}$.

Proposition 2.5.11 *The center is the kernel of the adjoint representation.*

Proof. The adjoint representation is defined by $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ with $\operatorname{ad} x(y) = [x, y]$. In particular, an element $x \in \mathfrak{g}$ is in $\mathfrak{z}(\mathfrak{g})$ if and only if $\operatorname{ad} x$ vanishes. □

Lemma 2.5.12 *Let* $\mathfrak{a}$ *be a characteristic ideal of* $\mathfrak{g}$ *and* $\mathfrak{b}$ *an ideal containing* $\mathfrak{a}$. *If* $\mathfrak{b}/\mathfrak{a}$ *is a characteristic ideal of* $\mathfrak{g}/\mathfrak{a}$, *then* $\mathfrak{b}$ *is a characteristic ideal of* $\mathfrak{g}$.

Proof. Let D be a derivation of $\mathfrak{g}$, then it induces a derivation $\overline{D}$ of $\mathfrak{g}/\mathfrak{a}$ by setting $\overline{D}(\overline{x}) = \overline{D(x)}$ where we denote by $\overline{x}$ the class of $x \in \mathfrak{g}$ in $\mathfrak{g}/\mathfrak{a}$. Now for $x \in \mathfrak{b}$, we have $\overline{D(x)} = \overline{D}(\overline{x}) \in \mathfrak{b}/\mathfrak{a}$ because $\mathfrak{b}/\mathfrak{a}$ is a characteristic ideal and therefore $D(x) \in \mathfrak{b}$. □

Definition 2.5.13 *We call* central ascending serie *the sequence* $(\mathcal{C}_i\mathfrak{g})_{i\geq 0}$ *of characteristic ideals with* $\mathcal{C}_0 = 0$ *and* $\mathcal{C}_{i+1}\mathfrak{g}$ *is the inverse image under the canonical projection* $\mathfrak{g} \to \mathfrak{g}/\mathcal{C}_i\mathfrak{g}$ *of the center of* $\mathfrak{g}/\mathcal{C}_i\mathfrak{g}$ *for* $i \geq 0$.

2.6 Extensions

Definition 2.6.1 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two Lie algebras, an* extension of $\mathfrak{b}$ by $\mathfrak{a}$ *is a Lie algebra* $\mathfrak{g}$ *together with an exact sequence of Lie algebra*

$$0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{b} \to 0$$

i.e. *an exact sequence where the morphisms are Lie algebra morphisms.*

Remark 2.6.2 By abuse of notation, we shall identify $\mathfrak{a}$ with its image in $\mathfrak{g}$ and consider $\mathfrak{a}$ as an ideal of $\mathfrak{g}$. The Lie algebra $\mathfrak{b}$ is identified to $\mathfrak{g}/\mathfrak{a}$.

We shall also say that $\mathfrak{g}$ is an extension of $\mathfrak{b}$ by $\mathfrak{a}$.

Definition 2.6.3 *Two extentions*

$$0 \longrightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \mathfrak{b} \longrightarrow 0 \quad \textit{and} \quad 0 \longrightarrow \mathfrak{a} \xrightarrow{i'} \mathfrak{g}' \xrightarrow{p'} \mathfrak{b} \longrightarrow 0$$

are equivalent *if there exists a morphism* $f : \mathfrak{g} \to \mathfrak{g}'$ *such that the diagram*

$$\begin{array}{ccccc} \mathfrak{a} & \xrightarrow{i} & \mathfrak{g} & \xrightarrow{p} & \mathfrak{b} \\ \| & & \downarrow f & & \| \\ \mathfrak{a} & \xrightarrow{i'} & \mathfrak{g}' & \xrightarrow{p'} & \mathfrak{b} \end{array}$$

commutes.

Proposition 2.6.4 *The relation "*the extention is equivalent to*" is an equivalence relation on the set of extension of* $\mathfrak{b}$ *by* $\mathfrak{a}$*:*

Proof. We easily see that this relation is reflexive and transitive. To see that it is symmetric, we only need to check that if $f : \mathfrak{g} \to \mathfrak{g}'$ defines an equivalence between two extensions, then f is bijective.

Indeed, by the commutativity of the right hand square, the subspace $\ker f$ is mapped to 0 in $\mathfrak{b}$ and is therefore contained in $\mathfrak{a}$. By the commutativity of the left hand square, this kernel is 0.

Again by the commutativity of the right hand square, the image of f dominantes $\mathfrak{b}$ and by the commutativity of the left hand square it contains $\mathfrak{a}$. □

Proposition 2.6.5 *Let* $0 \longrightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \mathfrak{b} \longrightarrow 0$ *be an extension of* $\mathfrak{b}$ *by* $\mathfrak{a}$*. There exists a subalgebra* $\mathfrak{c}$ *of* $\mathfrak{g}$ *supplementary to* $\mathfrak{a}$ *if and only if* p *has a section* $s : \mathfrak{b} \to \mathfrak{g}$ *(*i.e. *a Lie algebra morphisms such that* $p \circ s = \mathrm{Id}_{\mathfrak{b}}$*).*

Proof. If $\mathfrak{c}$ exists, then the restriction $p|_{\mathfrak{c}}$ of the projection p to $\mathfrak{c}$ maps $\mathfrak{c}$ isomorphically to $\mathfrak{b}$. We may define s as the composition of the inverse of $p|_{\mathfrak{c}}$ with the inclusion of $\mathfrak{c}$ in $\mathfrak{g}$.

Conversely, if s exists, its image is a Lie subalgebra of $\mathfrak{g}$ supplementary to $\mathfrak{a}$ (if $s(x) = i(y)$, then $x = p(s(x)) = p(i(x)) = 0$). □

Definition 2.6.6 *Let* $0 \longrightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \mathfrak{b} \longrightarrow 0$ *be an extension of* $\mathfrak{b}$ *by* $\mathfrak{a}$*. We shall say that the extension is* not essential *(resp.* trivial*) if there exists a Lie subalgebra (resp. an ideal)* $\mathfrak{c}$ *of* $\mathfrak{g}$ *supplementary to* $\mathfrak{a}$*.*

If $\mathfrak{a}$ *is contained in the center of* $\mathfrak{g}$*, we call the extension* central*.*

Lemma 2.6.7 *The extension* $0 \longrightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \mathfrak{b} \longrightarrow 0$ *is trivial if and only if* $\mathfrak{g}$ *is isomorphic to the product Lie algebra* $\mathfrak{a} \times \mathfrak{b}$*.*

Proof. Apply Propositions 1.3.2 and 1.3.3. □

Lemma 2.6.8 *A non essential and central extension is trivial.*

Proof. Denote by $\mathfrak{g}$ the Lie algebra obtained by extension and by $\mathfrak{a}$ the kernel of the extension. Let $\mathfrak{c}$ be a Lie subalgebra supplementary to $\mathfrak{a}$. We have $[\mathfrak{c}, \mathfrak{g}] = [\mathfrak{c}, \mathfrak{a}] + [\mathfrak{c}, \mathfrak{c}] = [\mathfrak{c}, \mathfrak{c}] \subset \mathfrak{c}$ because $\mathfrak{a}$ is in the center of $\mathfrak{g}$. Therefore $\mathfrak{c}$ is an ideal. □

2.7 Semidirect products

We construct in this subsection all the non essential extension of $\mathfrak{b}$ by $\mathfrak{a}$. Let us fix some notation. The extension will be of the form

$$0 \longrightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \mathfrak{b} \longrightarrow 0$$

and because the extension is non essential, we may fix a Lie subalgebra of $\mathfrak{g}$ which is supplementary to $\mathfrak{a}$. We shall identify this subalgebra with $\mathfrak{b}$ and therefore consider $\mathfrak{g}$ as the vector space $\mathfrak{a} \times \mathfrak{b}$. To determine the Lie algebra structure on $\mathfrak{g}$, we only need to determine its Lie bracket. It is of the following form:

$$\begin{aligned} [(a,b),(a',b')] &= [a+b, a'+b'] \\ &= [a,a'] + [b,b'] + [a,b'] + [b,a'] \\ &= [a,a'] + \operatorname{ad} b(a') - \operatorname{ad} b'(a) + [b,b']. \end{aligned}$$

We may consider $\operatorname{ad} b$ and $\operatorname{ad} b'$ as derivations of $\mathfrak{a}$ (because $\mathfrak{a}$ is an ideal), we therefore have a Lie algebra morphism $\mathfrak{b} \to \mathfrak{Der}(\mathfrak{a})$ given by $b \mapsto \operatorname{ad} b$.

Conversely, given a Lie algebra morphism $f : \mathfrak{b} \to \mathfrak{Der}(\mathfrak{a})$, we may define a Lie bracket on $\mathfrak{a} \times \mathfrak{b}$ by

$$[(a,b),(a',b')] = [a,a'] + f(b)(a') - f(b')(a) + [b,b'].$$

Lemma 2.7.1 *This defines a Lie bracket.*

Proof. It is clearly bilinear and alternate. The Jacobi identify follows from the fact that f is a Lie algebra morphism into the derivations on $\mathfrak{a}$. □

Proposition 2.7.2 *Let* $f : \mathfrak{b} \to \mathfrak{Der}(\mathfrak{a})$ *be a Lie algebra morphism, then the Lie algebra* $\mathfrak{g} = \mathfrak{a} \times \mathfrak{b}$ *defined by the above Lie bracket is an non essential extension of* $\mathfrak{b}$ *by* $\mathfrak{a}$*. Any non essential extension is of that form.*

Proof. We need to prove that $\mathfrak{a}$ is an ideal for this Lie bracket. But we may compute the Lie bracket $[(a,0),(a',b')] = [a,a'] + f(0)(a') - f(b')(a) + [0,b'] = [a,a'] - f(b')(a) \in \mathfrak{a}$ because $f(b')$ is a derivation of $\mathfrak{a}$.

This fact that any non essential extension is of that form follows from the observation that, for a non essential extension, the map $\mathfrak{b} \to \mathfrak{Der}(\alpha)$ is a Lie algebra morphism and the coincidence $\operatorname{ad} b(a') = f(b)(a')$. □

Definition 2.7.3 *For* $f : \mathfrak{b} \to \mathfrak{Der}(\mathfrak{a})$ *a Lie algebra morphism, the Lie algebra structure defined above on* $\mathfrak{a} \times \mathfrak{b}$ *using* f *is called a* semidirect product *of* $\mathfrak{a}$ *and* $\mathfrak{b}$*.*

2.8 Exercices

Exercice 2.8.1 Prove the statements in Proposition 2.1.3, 2.1.4 and in Example 2.1.5, 2.1.6, 2.1.7, 2.1.8, 2.4.3.

Describe in terms of matrices in the base $(e_i)_{i\in[1,n]}$ the Lie subalgebras of $\mathfrak{gl}(V)$ described in Example 2.1.8.

Exercice 2.8.2 Prove that all the Definitions and statements in this chapter are valid if we assume that k is a commutative ring with unit and if we replace vector spaces over k by modules over k.

Exercice 2.8.3 Give a complete classification (*i.e.* a complete list) of Lie algebras of dimension 1, 2 and 3. Describe then as subalgebras of $\mathfrak{gl}(k^n)$.

Exercice 2.8.4 Prove that a Lie algebra $\mathfrak{g}$ is associative if and only if $\mathcal{D}\mathfrak{g} \subset \mathfrak{z}(\mathfrak{g})$.

Exercice 2.8.5 Let V be a vector space and W a codimension 1 subvector space. Prove that $\mathfrak{gl}(W)$ can be realised as a Lie subalgebra of $\mathfrak{sl}(V)$.

Exercice 2.8.6 For V a vector space of finite dimension over k, prove that $\mathfrak{z}(\mathfrak{gl}(V)) = k\mathrm{Id}_V$.

Exercice 2.8.7 Let $\mathfrak{g}$ be a Lie algebra of dimension n over k and assume that $\mathfrak{z}(\mathfrak{g})$, the center of $\mathfrak{g}$, is of codimension 1. Prove that $\mathfrak{g}$ is abelian.

Exercice 2.8.8 (ı) For V a vector space of dimension 2, prove that the derived serie, the central descending serie and the central ascending serie of $\mathfrak{gl}(V)$ are described by

- $\mathcal{D}^0\mathfrak{gl}(V) = \mathfrak{gl}(V)$ and $\mathcal{D}^i\mathfrak{gl}(V) = \mathfrak{sl}(V)$ for $i \geq 1$.
- $\mathcal{C}^0\mathfrak{gl}(V) = \mathfrak{gl}(V)$ and $\mathcal{C}^i\mathfrak{gl}(V) = \mathfrak{sl}(V)$ for $i \geq 1$.
- $\mathcal{C}_0\mathfrak{gl}(V) = 0$, $\mathcal{C}_i\mathfrak{gl}(V) = k\mathrm{Id}_V$ for $i \geq 1$.

(ıı) Prove that the same results hold for any finite dimensional vector space V.

Exercice 2.8.9 Let $\mathfrak{g}$ be a Lie algebra over k and let K be a field containing k.

(ı) Prove that $\mathfrak{g} \otimes_k K$ is a Lie algebra for the Lie bracket $[x \otimes a, y \otimes b] = [x, y] \otimes ab$.

(ıı) Prove that if $\mathfrak{a}$ is a Lie subalgebra (resp. an ideal) of $\mathfrak{g}$, then $\mathfrak{a} \otimes_k K$ is a Lie subalgebra (resp. an ideal) of $\mathfrak{g} \otimes_k K$.

(ııı) Prove that we have the equalities

- $[\mathfrak{a} \otimes_k K, \mathfrak{b} \otimes_k K] = [\mathfrak{a}, \mathfrak{b}] \otimes_k K$;
- $\mathcal{D}^i(\mathfrak{g} \otimes_k K) = \mathcal{D}^i\mathfrak{g} \otimes_k K$ and
- $\mathcal{C}^i(\mathfrak{g} \otimes_k K) = \mathcal{C}^i\mathfrak{g} \otimes_k K$

where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in $\mathfrak{g}$.

Exercice 2.8.10 Assume that $\mathrm{char}(k) \neq 2$. Let $\mathfrak{g}$ be a Lie algebra over k and define on the vector space $\mathfrak{G} = \mathfrak{g} \times \mathfrak{g}$ the bracket

$$[(x, x'), (y, y')] = ([x, y] + [x', y'], [x, y'] + [x', y]).$$

(ı) Prove that this defines a Lie bracket on $\mathfrak{G}$.

(ıı) Prove that the morphisms $x \mapsto \frac{1}{2}(x, x)$ and $x \mapsto \frac{1}{2}(x, -x)$ are isomorphisms between $\mathfrak{g}$ and two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathfrak{G}$ with $\mathfrak{G} = \mathfrak{a} \times \mathfrak{b}$ (as a vector space and therefore as a Lie algebra).

(ııı) Assume that $k = \mathbb{R}$ the field of real numbers and prove that $(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}) \times (\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$.

Exercice 2.8.11 Let V be a finite dimensional vector space of dimension n. Recall that $\mathfrak{sl}(V)$ and $\mathfrak{z}(\mathfrak{gl}(V))$ are ideals in $\mathfrak{gl}(V)$. We therefore have extensions

$$0 \to \mathfrak{sl}(V) \to \mathfrak{gl}(V) \to \mathfrak{b} \to 0 \text{ and } 0 \to \mathfrak{z}(\mathfrak{gl}(V)) \to \mathfrak{gl}(V) \to \mathfrak{b}' \to 0.$$

(ı) Prove that for char$k$ prime with n the previous extensions are trivial.

(ıı) If char$k$ divides n, prove that the first extension is non essential and compute for a section of $\mathfrak{b}$ in $\mathfrak{g}$ the Lie algebra morphism $\mathfrak{b} \to \mathfrak{Der}(\mathfrak{sl}(V))$. Prove that the second extension is central but non essential.

Exercice 2.8.12 Finish the computation of the Jacobi formula in the proof of Lemma 2.7.1.

Exercice 2.8.13 Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ a subspace of $\mathfrak{g}$. Define

$$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a}) = \{x \in \mathfrak{g} \ / \ [x,y] \in \mathfrak{a} \ \forall y \in \mathfrak{a}\}.$$

Prove that $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$ is a Lie subalgebra of $\mathfrak{g}$. The algebra $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$ is called the *normaliser* of $\mathfrak{a}$ in $\mathfrak{g}$.

Prove the following equalities

- $\mathfrak{n}_{\mathfrak{gl}(V)}(\mathfrak{t}(V_\bullet)) = \mathfrak{t}(V_\bullet)$,
- $\mathfrak{n}_{\mathfrak{gl}(V)}(\mathfrak{diag}(V_\bullet)) = \mathfrak{diag}(V_\bullet)$
- $\mathfrak{n}_{\mathfrak{gl}(V)}(\mathfrak{n}(V_\bullet)) = \mathfrak{t}(V_\bullet)$

Exercice 2.8.14 Which of the following Lie algebras over $\mathbb{R}$ are isomorphic?

(ı) $\mathbb{R}^3$ with the cross product as Lie bracket;

(ıı) The upper triangular 2×2 matrices;

(ııı) The strict upper triangular 3×3 matrices;

(ıv) The antisymmetric 3×3 matrices.

(v) The traceless 2×2 matrices.

Exercice 2.8.15 Prove that $\mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{C}$ where $\mathbb{R}^3$ has the cross product as Lie bracket and the Lie algebra of traceless 2×2 matrices are isomorphic.

Chapter 3

Envelopping algebra I

In this chapter we shall present the definition and first properties of the universal envelopping algebra. We shall present the Poincaré-Birkhoff-Witt (PBW) theorem in Chapter 15. Here the associative algebras are supposed to have a unit element.

3.1 Definition

The principle of the envelopping algebra is to replace the Lie algebra by an associative algebra but of infinite dimension.

Definition 3.1.1 *Let $\mathfrak{g}$ be a Lie algebra and A be an associative algebra with unit, a linear map $f : \mathfrak{g} \to A$ is called an* α-map *if we have $f([x, y]) = f(x)f(y) - f(y)f(x)$ for all $(x, y) \in \mathfrak{g}^2$.*

The envolopping algebra of $\mathfrak{g}$ is an associative algebra $U(\mathfrak{g})$ with an α-map from $\mathfrak{g}$ to $U(\mathfrak{g})$ such that any α-map from $\mathfrak{g}$ to an associative algebra factor through $U(\mathfrak{g})$. Let us first construct the envelopping algebra $U(\mathfrak{g})$.

Definition 3.1.2 *Let $\mathfrak{g}$ be a Lie algebra and denote by $T(\mathfrak{g})$ its tensor algebra*[1]*. Denote by J the two-sided ideal of $T(\mathfrak{g})$ generated by the tensors $x \otimes y - y \otimes x - [x, y]$ for $(x, y) \in \mathfrak{g}^2$. The* universal envelopping algebra *of $\mathfrak{g}$, denoted by $U(\mathfrak{g})$, is the quotient $T(\mathfrak{g})/J$. The composition $\mathfrak{g} \to T(\mathfrak{g}) \to T(\mathfrak{g})/J$ is denoted by $f_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$ and is called the* canonical map from $\mathfrak{g}$ to $U(\mathfrak{g})$.

If we denote by T_+ the two-sided ideal of $T(\mathfrak{g})$ formed by the tensor without degree 0 term and by $T_0 = k1$ the degree 0 part in $T(\mathfrak{g})$, then we have a direct sum $T(\mathfrak{g}) = T_0 \oplus T_+$.

Because $J \subset T_+$, the image of T_0 in $U(\mathfrak{g})$ is non zero. We denote it by U_0 and if U_+ is the image of T_+ in $U(\mathfrak{g})$, which is a two-sided ideal in $U(\mathfrak{g})$, we have $U(\mathfrak{g}) = U_0 \oplus U_+$. In particular $U(\mathfrak{g})$ is a non trivial associative algebra with unit.

Remark 3.1.3 Because T_+ is generated by the image of $\mathfrak{g}$ in $T(\mathfrak{g})$, the two-sided ideal U_+ is generated by the image of $\mathfrak{g}$ in $U(\mathfrak{g})$. Therefore $U(\mathfrak{g})$ is generated by $f_{\mathfrak{g}}(\mathfrak{g})$ and the unit 1.

Lemma 3.1.4 *The map $f_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$ is an α-map.*

Proof. For any x and y in $\mathfrak{g}$, the elements $x \otimes y - y \otimes x$ and $[x, y]$ are equal in $T(\mathfrak{g})$ modulo J, therefore $f_{\mathfrak{g}}(x)f_{\mathfrak{g}}(y) - f_{\mathfrak{g}}(y)f_{\mathfrak{g}}(x) = f_{\mathfrak{g}}([x, y])$ and the result follows. □

[1] The tensor algebra is the direct sum $T(\mathfrak{g}) = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$

Proposition 3.1.5 *Universal property of the envelopping algebra.*

Let A be an associative algebra and $g : \mathfrak{g} \to A$ an α-map, then there exists a unique morphism $g_U : U(\mathfrak{g}) \to A$ such that $g = g_U \circ f_{\mathfrak{g}}$.

Proof. The unicity comes from the fact that $f_{\mathfrak{g}}$ and 1 generate $U(\mathfrak{g})$. For the existence, let $g_T : T(\mathfrak{g}) \to A$ be the unique morphisme such that $g_T|_{\mathfrak{g}} = g$. Because g is an α-map, we have for any x and y in $\mathfrak{g}$:

$$g_T(x \otimes y - y \otimes x - [x, y]) = g(x)g(y) - g(y)g(x) - g([x, y]) = 0.$$

Therefore $J \subset \ker g_T$ and g_T factors through the quotient $T(\mathfrak{g})/J = U(\mathfrak{g})$. This defined g_U. □

Remark 3.1.6 Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two Lie algebras and $\varpi : \mathfrak{g} \to \mathfrak{g}'$ a Lie algebra morphism. Then the composition $f_{\mathfrak{g}'} \circ \varphi : \mathfrak{g} \to U(\mathfrak{g}')$ is an α-map. Therefore we have a morphism $\widetilde{\varphi} : U(\mathfrak{g}) \to U(\mathfrak{g}')$ such that the diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi} & \mathfrak{g}' \\ {\scriptstyle f_{\mathfrak{g}}}\downarrow & & \downarrow{\scriptstyle f_{\mathfrak{g}'}} \\ U(\mathfrak{g}) & \xrightarrow{\widetilde{\varphi}} & U(\mathfrak{g}') \end{array}$$

commutes. If we have another Lie algebra morphism $\varphi' : \mathfrak{g}' \to \mathfrak{g}''$, then $\widetilde{\varphi' \circ \varphi} = \widetilde{\varphi}' \circ \widetilde{\varphi}$.

3.2 Envelopping algebra of a product

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie algebras and set $\mathfrak{g} = \mathfrak{a} \times \mathfrak{b}$. The injections $\imath : \mathfrak{a} \to \mathfrak{g}$ and $\jmath : \mathfrak{b} \to \mathfrak{g}$ induce canonical morphisms $\widetilde{\imath} : U(\mathfrak{a}) \to U(\mathfrak{g})$ and $\widetilde{\jmath} : U(\mathfrak{b}) \to U(\mathfrak{g})$.

Lemma 3.2.1 *The images of $\widetilde{\imath}$ and $\widetilde{\jmath}$ commute in $U(\mathfrak{g})$. The application $\widetilde{\imath} \otimes \widetilde{\jmath} : U(\mathfrak{a}) \otimes U(\mathfrak{b}) \to U(\mathfrak{g})$ is therefore an algebra morphism.*

Proof. We have in $U(\mathfrak{g})$ the relation $\imath(x) \otimes \jmath(y) - \jmath(y) \otimes \imath(x) - [\imath(x), \jmath(y)] = 0$. But in $\mathfrak{g}$ we have the relation $[\imath(x), \jmath(y)] = 0$, therefore $\imath(x) \otimes \jmath(y) = \jmath(y) \otimes \imath(x)$.

Now we compute

$$\begin{aligned} \widetilde{\imath} \otimes \widetilde{\jmath}(x \otimes y \cdot z \otimes t) &= \widetilde{\imath} \otimes \widetilde{\jmath}(xz \otimes yt) \\ &= \widetilde{\imath}(xz)\widetilde{\jmath}(yt) \\ &= \widetilde{\imath}(x)\widetilde{\imath}(z)\widetilde{\jmath}(y)\widetilde{\jmath}(t) \\ &= \widetilde{\imath}(x)\widetilde{\jmath}(y)\widetilde{\imath}(z)\widetilde{\jmath}(t) \\ &= \widetilde{\imath} \otimes \widetilde{\jmath}(x \otimes y) \cdot \widetilde{\imath} \otimes \widetilde{\jmath}(z \otimes t) \end{aligned}$$

And the result follows. □

Proposition 3.2.2 *The morphism $\widetilde{\imath} \otimes \widetilde{\jmath} : U(\mathfrak{a}) \otimes U(\mathfrak{b}) \to U(\mathfrak{g})$ is an isomorphism.*

Proof. Consider the map $g : \mathfrak{g} \to U(\mathfrak{a}) \otimes U(\mathfrak{b})$ defined by $g(x, y) = f_{\mathfrak{a}}(x) \otimes 1 + 1 \otimes f_{\mathfrak{b}}(y)$ for $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$. We have the equalities:

$$\begin{aligned} g(x, y)g(z, t) - g(z, t)g(x, y) - g([(x, y), (z, t)]) = \; & (f_{\mathfrak{a}}(x) \otimes 1 + 1 \otimes f_{\mathfrak{b}}(y))(f_{\mathfrak{a}}(z) \otimes 1 + 1 \otimes f_{\mathfrak{b}}(t)) \\ & -(f_{\mathfrak{a}}(z) \otimes 1 + 1 \otimes f_{\mathfrak{b}}(t))(f_{\mathfrak{a}}(x) \otimes 1 + 1 \otimes f_{\mathfrak{b}}(y)) \\ & -(f_{\mathfrak{a}}([x, z]) \otimes 1 + 1 \otimes f_{\mathfrak{b}}([y, t])) \\ = \; & (f_{\mathfrak{a}}(x)f_{\mathfrak{a}}(z) - f_{\mathfrak{a}}(z)f_{\mathfrak{a}}(x) - f_{\mathfrak{a}}([x, z])) \otimes 1 \\ & +1 \otimes (f_{\mathfrak{b}}(y)f_{\mathfrak{b}}(t) - f_{\mathfrak{b}}(t)f_{\mathfrak{b}}(y) - f_{\mathfrak{b}}([y, t])) \\ = \; & 0. \end{aligned}$$

Thus g is an α-map and induces a morphism $g_U : U(\mathfrak{g}) \to U(\mathfrak{a}) \otimes U(\mathfrak{b})$ with $g = g_U \circ f_{\mathfrak{g}}$. We compute

$$\begin{aligned} g_U \circ \widetilde{\imath} \otimes \widetilde{\jmath} \circ g(x,y) &= g_U \circ \widetilde{\imath} \otimes \widetilde{\jmath}(f_{\mathfrak{a}}(x) \otimes 1 + 1 \otimes f_{\mathfrak{b}}(y)) \\ &= g_U(\widetilde{\imath}(f_{\mathfrak{a}}(x)) + \widetilde{\jmath}(f_{\mathfrak{b}}(y)) \\ &= g_U(f_{\mathfrak{g}}(\imath(x) + f_{\mathfrak{g}}(\jmath(y)) \\ &= g_U(f_{\mathfrak{g}}(\imath(x) + \jmath(y)) \\ &= g(x,y). \end{aligned}$$

And because the image of g contains $f_{\mathfrak{a}}(\mathfrak{a}) \otimes 1$ and $1 \otimes f_{\mathfrak{b}}(\mathfrak{b})$, it generates $U(\mathfrak{a}) \otimes U(\mathfrak{b})$ as an algebra. This implies that $g_U \circ \widetilde{\imath} \otimes \widetilde{\jmath} = \mathrm{Id}_{U(\mathfrak{a}) \otimes U(\mathfrak{b})}$. We also compute

$$\begin{aligned} \widetilde{\imath} \otimes \widetilde{\jmath} \circ g_U \circ f_{\mathfrak{g}}(x,y) &= \widetilde{\imath} \otimes \widetilde{\jmath}(g(x,y)) \\ &= \widetilde{\imath} \otimes \widetilde{\jmath}(f_{\mathfrak{a}}(x) \otimes 1 + 1 \otimes f_{\mathfrak{b}}(y)) \\ &= \widetilde{\imath}(f_{\mathfrak{a}}(x)) + \widetilde{\jmath}(f_{\mathfrak{b}}(y)) \\ &= f_{\mathfrak{g}}(\imath(x) + \jmath(y)) \\ &= f_{\mathfrak{g}}(x,y). \end{aligned}$$

And because the image of $f_{\mathfrak{g}}$ generates $U(\mathfrak{g})$ as an algebra, this implies that $\widetilde{\imath} \otimes \widetilde{\jmath} \circ g_U = \mathrm{Id}_{U(\mathfrak{g})}$. □

3.3 Envelopping algebra of the opposite Lie algebra

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^{\mathrm{op}}$ its opposite Lie algebra. Recall that the Lie bracket in $\mathfrak{g}^{\mathrm{op}}$ is defined by $[x,y]_{\mathrm{op}} = [y,x]$. Let us denote by $U(\mathfrak{g})^{\mathrm{op}}$ the associative algebra opposite to $U(\mathfrak{g})$. Its product is defined by $x \cdot_{\mathrm{op}} y = yx$ where yx is the product in $U(\mathfrak{g})$. As vector spaces $\mathfrak{g}$ and $\mathfrak{g}^{\mathrm{op}}$ are isomorphic as well as $U(\mathfrak{g})$ and $U(\mathfrak{g})^{\mathrm{op}}$. We therefore have a linear map $f : \mathfrak{g}^{\mathrm{op}} \to U(\mathfrak{g})^{\mathrm{op}}$ which is equal to $f_{\mathfrak{g}}$ on vector spaces.

Lemma 3.3.1 *The map $f : \mathfrak{g}^{\mathrm{op}} \to U(\mathfrak{g})^{\mathrm{op}}$ is an α-map.*

Proof. We have $f(x) \cdot_{\mathrm{op}} f(y) - f(y) \cdot_{\mathrm{op}} f(x) - f([x,y]_{\mathrm{op}}) = f_{\mathfrak{g}}(y)f_{\mathfrak{g}}(x) - f_{\mathfrak{g}}(x)f_{\mathfrak{g}}(y) - f_{\mathfrak{g}}([y,x])$ and the result follows because $f_{\mathfrak{g}}$ is an α-map. □

Proposition 3.3.2 *The induced map $f_U : U(\mathfrak{g}^{\mathrm{op}}) \to U(\mathfrak{g})^{\mathrm{op}}$ is an isomorphism.*

Proof. Indeed, the previous lemma shows that there is an algebra morphism $U((\mathfrak{g}^{\mathrm{op}})^{\mathrm{op}}) \to U(\mathfrak{g}^{\mathrm{op}})^{\mathrm{op}}$. But $(\mathfrak{g}^{\mathrm{op}})^{\mathrm{op}} = \mathfrak{g}$ as a Lie algebras therefore we have an algebra morphism $U(\mathfrak{g}) \to U(\mathfrak{g}^{\mathrm{op}})^{\mathrm{op}}$. The same vector space morphism induces an algebra morphism $g_U : U(\mathfrak{g})^{\mathrm{op}} \to (U(\mathfrak{g}^{\mathrm{op}})^{\mathrm{op}})^{\mathrm{op}} = U(\mathfrak{g}^{\mathrm{op}})$. By definition, we have $f_U \circ g_U \circ f_{\mathfrak{g}^{\mathrm{op}}} = f_{\mathfrak{g}^{\mathrm{op}}}$ and $g_U \circ f_U \circ f_{\mathfrak{g}^{\mathrm{op}}} = f_{\mathfrak{g}}$ and as the images of $f_{\mathfrak{g}^{\mathrm{op}}}$ and of $f_{\mathfrak{g}}$ generate $U(\mathfrak{g}^{\mathrm{op}})$ and $U(\mathfrak{g})$, the result follows. □

Recall that the map $\mathrm{op} : \mathfrak{g} \to \mathfrak{g}^{\mathrm{op}}$ defined by $x \mapsto -x$ is an isomorphism of Lie algebras. It induces an isomorphism $\mathrm{op}_U : U(\mathfrak{g}) \to U(\mathfrak{g}^{\mathrm{op}}) = U(\mathfrak{g})^{\mathrm{op}}$ or an antiautomorphism of U. We have the following multiplicative formula for $(x_i)_{i \in [1,n]}$ elements in $\mathfrak{g}$:

$$\begin{aligned} \mathrm{op}_U(f_{\mathfrak{g}}(x_1) \cdots f_{\mathfrak{g}}(x_n)) &= \mathrm{op}_U(f_{\mathfrak{g}}(x_n)) \cdots \mathrm{op}_U(f_{\mathfrak{g}}(x_1)) \\ &= f_{\mathfrak{g}}(\mathrm{op}(x_n)) \cdots f_{\mathfrak{g}}(\mathrm{op}(x_1)) \\ &= f_{\mathfrak{g}}(-x_n) \cdots f_{\mathfrak{g}}(-x_1) \\ &= (-1)^n f_{\mathfrak{g}}(x_n) \cdots f_{\mathfrak{g}}(x_1). \end{aligned}$$

3.4 Exercices

Exercice 3.4.1 Prove the assertions in Remark 3.1.6.

Chapter 4

Representations

In this chapter we give the very first definition and properties of representations of a Lie algebra. We shall study in more details the representations of semisimple Lie algebras later on in the text.

4.1 Definition

Definition 4.1.1 *Let $\mathfrak{g}$ be a Lie algebra and V be a vector space. A* representation *of $\mathfrak{g}$ in V is a Lie algebra morphism $\mathfrak{g} \to \mathfrak{gl}(V)$.*

Definition 4.1.2 *A injective representation $\mathfrak{g} \to \mathfrak{gl}(V)$ is called* faithful. *The dimension of V is called the* dimension of the representation.

Example 4.1.3 The adjoint representation $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a representation.

Example 4.1.4 Let G be a Lie group and $G \to GL(V)$ be a representation of G, the differential of this map at identity is a representation of Lie(G) in V.

Remark 4.1.5 A representation of $\mathfrak{g}$ is a linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ such that for x and y in $\mathfrak{g}$ and for v in V, we have $\rho([x, y])(v) = \rho(x)\rho(y)(v) - \rho(y)\rho(x)(v)$.

In particular this map is an α-map and induces an algebra morphism $\rho_U : U(\mathfrak{g}) \to \mathfrak{gl}(V)$ such that $\rho = \rho_U \circ f_{\mathfrak{g}}$. This means that V is a $U(\mathfrak{g})$-module with the action given by $x \cdot v = \rho_U(x)(v)$ for $x \in U(\mathfrak{g})$ and $v \in V$.

Conversely, if V is an $U(\mathfrak{g})$-module whose multiplication is determined by an algebra morphism $\rho_U : U(\mathfrak{g}) \to \mathfrak{gl}(V)$, then by composition with $f_{\mathfrak{g}}$ we obtain a representation of $\mathfrak{g}$ in V.

There is therefore a one to one correspondence between representations of the Lie algebra $\mathfrak{g}$ and the $U(\mathfrak{g})$-modules.

We transpose the usual notions like, *isomorphism* , *direct sums* from $U(\mathfrak{g})$-modules to representations of $\mathfrak{g}$. Let us state more precisely some of these definitions.

Definition 4.1.6 *(ı) A representation of $\mathfrak{g}$ in V is called* simple *if V is a simple $U(\mathfrak{g})$-module* i.e. *if the is no non trivial submodule.*

(ıı) A representation V is called reducible *if there is a decomposition $V = V_1 \oplus V_2$ where V_i are subrepresentations of V for $i \in \{1, 2\}$. If the representation is not reducible then we call it* irreducible.

(ııı) A representation of $\mathfrak{g}$ in V is called semisimple *or* completely reducible *if V is a semisimple $U(\mathfrak{g})$-module* i.e. *if it is isomorphic to a direct sum of simple modules.*

(ıv) A representation $\mathfrak{g}$ in W is a subrepresentation *of V if W is an $U(\mathfrak{g})$-submodule of V. A representation $\mathfrak{g}$ in W is a* quotient representation *of V if W is an $U(\mathfrak{g})$ quotient module of V.*

Definition 4.1.7 *For V a representation of the Lie algebra $\mathfrak{g}$ and for x in $\mathfrak{g}$, we shall denote by x_V the endomorphism of V induced by x.*

Lemma 4.1.8 *Let V be a representation of $\mathfrak{g}$ and $v \in V$, then the subset $\mathfrak{g}_v = \{x \in \mathfrak{g} \ / \ x_V \cdot v = 0\}$ of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$.*

Proof. It is a subspace and for x and y in $\mathfrak{g}_v$, we have $[x,y]_V \cdot v = x_V \cdot (y_V \cdot v) - y_V \cdot (x_V \cdot v) = 0$. □

4.2 Tensor product of representations

Taking tensor products is a natural operation on representations. Indeed, let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras and let V_i for $i \in \{1,2\}$ be a representation of $\mathfrak{g}_i$. By the last remark, the vector space V_i for each $i \in \{1,2\}$ is an $U(\mathfrak{g}_i)$-module and therefore $V_1 \otimes V_2$ is an $U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$-module. But we have seen in Proposition 3.2.2 that $U(\mathfrak{g}_1 \times \mathfrak{g}_2) = U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$ therefore $V_1 \otimes V_2$ is a $\mathfrak{g}_1 \times \mathfrak{g}_2$-representation whose action is given by:

$$\begin{aligned}(x_1,x_2)_V \cdot (v_1 \otimes v_2) &= (f_{\mathfrak{g}_1}(x_1) \otimes 1 + 1 \otimes f_{\mathfrak{g}_2}(x_2)) \cdot v_1 \otimes v_2 \\ &= (x_1)_V \cdot v_1 \otimes v_2 + v_1 \otimes (x_2)_V \cdot v_2.\end{aligned}$$

Definition 4.2.1 *If $\mathfrak{g} = \mathfrak{g}_1 = \mathfrak{g}_2$, and composing with the inclusion $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ of Lie algebras given by $x \mapsto (x,x)$ we obtain, for V_1 and V_2 two representations of $\mathfrak{g}$ a representation of $\mathfrak{g}$ in $V_1 \otimes V_2$ called the* tensor product representation. *The action is given by*

$$x_V \cdot (v_1 \otimes v_2) = x_V \cdot v_1 \otimes v_2 + v_1 \otimes x_V \cdot v_2.$$

By induction we get

Proposition 4.2.2 *Let $(V_i)_{i \in [1,n]}$ be representations of the Lie algebra $\mathfrak{g}$, then the tensor product $V = \otimes_{i=1}^n V_i$ is a representation of $\mathfrak{g}$ with action given by the following formula:*

$$x_V \cdot (v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^n v_1 \otimes \cdots \otimes x_{V_i} \cdot v_i \otimes \cdots \otimes v_n.$$

4.3 Representations in the space of morphisms

As in the previous section, let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras and let V_i for $i \in \{1,2\}$ be a representation of $\mathfrak{g}_i$. The vector space V_i for each $i \in \{1,2\}$ is an $U(\mathfrak{g}_i)$-module and therefore $\mathrm{Hom}(V_1,V_2)$ is an $U(\mathfrak{g}_1)^{\mathrm{op}} \otimes U(\mathfrak{g}_2)$-module. But we have seen in Proposition 3.3.2 that $U(\mathfrak{g}_1)^{\mathrm{op}} = U(\mathfrak{g}_1^{\mathrm{op}})$ therefore $\mathrm{Hom}(V_1,V_2)$ is a $\mathfrak{g}_1^{\mathrm{op}} \times \mathfrak{g}_2$-representation whose action is given, for $\phi \in \mathrm{Hom}(V_1,V_2)$ and $v_1 \in V_1$, by:

$$\begin{aligned}((x_1,x_2)_V \cdot \phi)(v_1) &= ((f_{\mathfrak{g}_1^{\mathrm{op}}}(x_1) \otimes 1 + 1 \otimes f_{\mathfrak{g}_2}(x_2)) \cdot \phi)(v_1) \\ &= \phi((x_1)_{V_1} \cdot v_1) + (x_2)_{V_2} \cdot \phi(v_1) \\ &= ((x_2)_{V_2} \circ \phi + \phi \circ (x_1)_{V_1})(v_1).\end{aligned}$$

Definition 4.3.1 *If $\mathfrak{g} = \mathfrak{g}_1 = \mathfrak{g}_2$, and composing with the inclusion $\mathfrak{g} \to \mathfrak{g}^{\mathrm{op}} \times \mathfrak{g}$ of Lie algebras given by $x \mapsto (-x,x)$ we obtain, for V_1 and V_2 two representations of $\mathfrak{g}$ a representation of $\mathfrak{g}$ in $\mathrm{Hom}(V_1,V_2)$. The action is given by*

$$x_V \cdot \phi = x_{V_2} \circ \phi - \phi \circ x_{V_1}.$$

Combining with Proposition 4.2.2 we get the

Proposition 4.3.2 *Let $(V_i)_{i\in[1,n+1]}$ be representations of the Lie algebra $\mathfrak{g}$, then the space of multilinear maps $\mathrm{Hom}(\otimes_{i=1}^n V_i, V_{n+1})$ is a representation of $\mathfrak{g}$ with action given by the following formula:*

$$(x_V\cdot\phi)(v_1\otimes\cdots\otimes v_n)=-\sum_{i=1}^n\phi(v_1\otimes\cdots\otimes x_{V_i}\cdot v_i\otimes\cdots\otimes v_n)+(x_{V_{n+1}})\cdot\phi(v_1\otimes\cdots\otimes v_n).$$

Definition 4.3.3 *Let V be a representation of $\mathfrak{g}$, then $V^\vee=\mathrm{Hom}(V,k)$ is called the* dual representation *of V.*

Let V be a representation of $\mathfrak{g}$ and consider the representation of $\mathfrak{g}$ in $B=\mathrm{Hom}(V\otimes V,k)$ the space of bilinear forms. Let $b\in\mathrm{Hom}(V\otimes V,k)$, then, by Lemma 4.1.8, the set $\mathfrak{g}_b=\{x\in\mathfrak{g}\ /\ x_B\cdot b=0\}$ is a Lie subalgebra of $\mathfrak{g}$. The condition $x_B\cdot b=0$ translates here in

$$b(x_V\cdot v,v')+b(v,x_V\cdot v')=0 \text{ for all } v \text{ and } v'.$$

Example 4.3.4 Let $V=k^n$, $\mathfrak{g}=\mathfrak{gl}(V)$ acting on V by the identity map $\mathfrak{g}=\mathfrak{gl}(V)\to\mathfrak{gl}(V)$. We identify $\mathfrak{g}$ with the vector space of $n\times n$-matrices. Choose $b((x_i)_{i\in[1,n]},(y_i)_{i\in[1,n]})=\sum_{i=1}^n x_iy_i$ the standard symmetric bilinear form, then we have

$$\mathfrak{g}_b=\{A\in\mathfrak{gl}(V)\ /\ A \text{ is antisymmetric}\} \text{ is a Lie algebra.}$$

When $k=\mathbb{R}$, then this is the Lie algebra of the orthogonal group $\mathrm{O}(n,\mathbb{R})$.

Example 4.3.5 Let $V=k^{2n}$, $\mathfrak{g}=\mathfrak{gl}(V)$ acting on V by the identity map $\mathfrak{g}=\mathfrak{gl}(V)\to\mathfrak{gl}(V)$. We identify $\mathfrak{g}$ with the vector space of $2n\times 2n$-matrices. Choose the following standard antisymmetric bilinear form $b((x_i)_{i\in[1,2n]},(y_i)_{i\in[1,2n]})=\sum_{i=1}^n x_iy_{2n+1-i}-\sum_{i=1}^n x_{2n+1-i}y_i$, then we have

$$\mathfrak{g}_b=\left\{\left(\begin{array}{cc}A&B\\C&D\end{array}\right)\in\mathfrak{gl}(V) \text{ with } A,\ B,\ C \text{ and } D\ n\times n\text{-matrices} \ /\ D=-{}^tA,\ B={}^tB \text{ and } C={}^tC\right\}$$

is a Lie algebra. When $k=\mathbb{R}$, then this is the Lie algebra of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$.

4.4 Invariants

Definition 4.4.1 *Let V be a representation of $\mathfrak{g}$, an element $v\in V$ is called* invariant *if $\mathfrak{g}_v=\mathfrak{g}$* i.e. *$x_V\cdot v=0$ for all $x\in\mathfrak{g}$. The set of invariants of $\mathfrak{g}$ in V is denoted by $V^{\mathfrak{g}}$.*

Example 4.4.2 Let G be a Lie group and V be a representation of G *i.e.* we have a Lie group morphism $G\to\mathrm{GL}(V)$. Then the differential at the identity defines a representation $\mathrm{Lie}(G)\to\mathfrak{gl}(V)$ and if $v\in V$ is G-invariant, then v is an invariant for $\mathrm{Lie}(G)$. Indeed, we have for ϵ small and any $x\in\mathrm{Lie}(G)$ the equality $v=(1+\epsilon x)\cdot v=v+\epsilon(x_V\cdot v)$ and the result follows.

Example 4.4.3 Let V and W be representations of $\mathfrak{g}$ and let $\phi\in\mathrm{Hom}(V,W)$. Then ϕ is invariant if and only if $x_V\circ\phi=\phi\circ x_W$ or equivalentely the morphism ϕ is a Lie algebra morphism. In symbols:

$$\mathrm{Hom}(V,W)^{\mathfrak{g}}=\mathrm{Hom}_{\mathfrak{g}}(V,W).$$

Example 4.4.4 Let V be a representation of $\mathfrak{g}$. There is always an invariant in $\mathrm{Hom}(V,V)$, namely Id_V (by the previous example). In particular, because $V^\vee \otimes V$ is isomorphic to $\mathrm{Hom}(V,V)$ has a representation of $\mathfrak{g}$ (Exercice!) is has an invariant element say $c_V^\otimes$. If $(v_i)_{i\in[1,n]}$ is a basis of V and if $(v_i^\vee)_{i\in[1,n]}$ is the dual basis in $V^\vee$, then

$$c_V^\otimes = \sum_{i=1}^n v_i^\vee \otimes v_i.$$

Remark that the element $c_V^\otimes$ does not depend on the choice of the basis.

Example 4.4.5 Let V be a representation of $\mathfrak{g}$ and let $b \in \mathrm{Hom}(V \times V, k)$ a bilinear form. Then b is an element in $\mathrm{Hom}(V \otimes V, k) = \mathrm{Hom}(V, V^\vee)$. Then b is invariant if and only if the corresponding map $b : V \to V^\vee$ is a morphism of representations. In symbols:

$$\mathrm{Hom}(V \times V, k)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, V^\vee).$$

In particular, if V is finite dimensional and b is non degenerate and invariant, then $b : V \to V^\vee$ is an isomorphism of representations. We therefore have an invariant $c_V \in V \otimes V$ corresponding to $c_V^\otimes \in V^\vee \otimes V$. If $(v_i)_{i\in[1,n]}$ is a basis of V and if $(v'_i)_{i\in[1,n]}$ is the dual basis for b defined by $b(v_i, v'_j) = \delta_{i,j}$, then

$$c_V = \sum_{i=1}^n v'_i \otimes v_i.$$

Remark that the element c_V does not depend on the choice of the basis.

Proposition 4.4.6 *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an ideal in $\mathfrak{g}$. Let V be a representation of $\mathfrak{g}$ and consider it as a representation of $\mathfrak{a}$. Then $V^{\mathfrak{a}}$ is a representation of $\mathfrak{g}$.*

Proof. The subset $V^{\mathfrak{a}}$ is a subspace of V. Furthermore, for $x \in \mathfrak{g}$, $y \in \mathfrak{a}$ and $v \in V^{\mathfrak{a}}$, we have $y_V \cdot (x_V \cdot v) = [y,x]_V \cdot v + x_V \cdot (y_V \cdot v) = 0$ because $\mathfrak{a}$ is an ideal in $\mathfrak{g}$. □

4.5 Invariant bilinear forms

Consider the special representation $V = \mathrm{Hom}(\mathfrak{g} \times \mathfrak{g}, k)$ induced by the adjoint representation and the trivial representation.

Definition 4.5.1 *A bilinear form b on $\mathfrak{g}$ is called* invariant *if it is invariant as an element of the representation V.*

Remark 4.5.2 A bilibear form b is invariant if and only if $b(x_{\mathfrak{g}} \cdot y, z) + b(y, x_{\mathfrak{g}} \cdot z) = 0$ *i.e.* iff

$$b([x,y],z) + b(x,[y,z]) = 0$$

for all x, y and z in $\mathfrak{g}$.

Definition 4.5.3 *A bilinear form b on $\mathfrak{g}$ is called* fully invariant *if for any derivation D in $\mathfrak{Der}(\mathfrak{g})$, we have $b(Dx,y) + b(x,Dy) = 0$ for all x and y in $\mathfrak{g}$.*

Proposition 4.5.4 *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an ideal of $\mathfrak{g}$. Let b be an invariant symmetric bilinear form on $\mathfrak{g}$.*

($\imath$) The orthogonal $\mathfrak{b}$ of $\mathfrak{a}$ for b is an ideal of $\mathfrak{g}$.

($\imath\imath$) If $\mathfrak{a}$ is a characteristic ideal and b is fully invariant, then $\mathfrak{b}$ is also a characteristic ideal.

($\imath\imath\imath$) If b is non degenerate, then $\mathfrak{a} \cap \mathfrak{b}$ is commutative.

Proof. For *($\imath$)* and *($\imath\imath$)*, let D be an inner (resp. any derivation of $\mathfrak{g}$). Let x be in $\mathfrak{b}$ and $y \in \mathfrak{a}$. We have $b(Dx, y) = -b(x, Dy) = 0$ and the result follows.

($\imath\imath\imath$) Let x and y in $\mathfrak{a} \cap \mathfrak{b}$. We have $b([x, y], z) = b(x, [y, z]) = 0$ because $x \in \mathfrak{a}$ and $[y, z] \in \mathfrak{b}$ are orthogonal for b. This is true for any z thus $[x, y] = 0$ because b is non degenerate. □

Definition 4.5.5 *Let V be a finite dimensional representation of $\mathfrak{g}$. The* bilinear form *associated to the representation V is the symmetric bilinear form defined by*

$$(x, y) \mapsto \mathrm{Tr}(x_V y_V).$$

If V is the adjoint representation, the the associated bilinear form is called the Killing form. *We denote it by $\kappa_{\mathfrak{g}}$.*

Proposition 4.5.6 *Let V be a finite dimensional representation of $\mathfrak{g}$, then the associated bilinear form is invariant.*

Proof. Let x, y and z in $\mathfrak{g}$. We compute

$$\mathrm{Tr}([x, y]_V z_V) = \mathrm{Tr}(x_V y_V z_V) - \mathrm{Tr}(y_V x_V z_V) = \mathrm{Tr}(x_V y_V z_V) - \mathrm{Tr}(x_V z_V y_V) = \mathrm{Tr}(x_V [y, z]_V)$$

and the result follows. □

Proposition 4.5.7 *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $\mathfrak{h}$ an ideal of $\mathfrak{g}$, then if $\kappa_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$ and $\kappa_{\mathfrak{h}}$ the Killing form of $\mathfrak{h}$ we have $\kappa_{\mathfrak{h}} = \kappa_{\mathfrak{g}}|_{\mathfrak{h}}$.*

Proof. Let x and y be elements in $\mathfrak{h}$, we want to compute the trace of $\mathrm{ad}\, x \mathrm{ad}\, y$ as an endomorphism of $\mathfrak{h}$ and of $\mathfrak{g}$. Let call u the corresponding endomorphism of $\mathfrak{g}$. As $\mathfrak{h}$ is an ideal, the image of u is $\mathfrak{h}$ and u induces an endomorphism $u_{\mathfrak{h}}$ of $\mathfrak{h}$ and $u_{\mathfrak{g}/\mathfrak{h}}$ of $\mathfrak{g}/\mathfrak{h}$. This last endomorphism vasnishes and therefore $\mathrm{Tr} u = \mathrm{Tr} u_{\mathfrak{h}}$. □

Proposition 4.5.8 *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, then the Killing form $\kappa_{\mathfrak{g}}$ is fully invariant.*

Proof. Let D be a derivation of $\mathfrak{g}$ and x and y elements of $\mathfrak{g}$. We start with the following

Lemma 4.5.9 *Let D be a derivation of $\mathfrak{g}$, then there exists a Lie algebra $\mathfrak{g}' = \mathfrak{g} \oplus k x_0$ such that for $x \in \mathfrak{g}$, we have $Dx = [x_0, x]$ and such that $\mathfrak{g}$ is an ideal in $\mathfrak{g}'$.*

Proof. Indeed, the derivation D gives a map $k x_0 \to \mathfrak{Der}(\mathfrak{g})$ defined by $\lambda x_0 \mapsto \lambda D$. As we have already seen, we may then define $\mathfrak{g}'$ the semidirect product of $k x_0$ and $\mathfrak{g}$ which is an extention of $k x_0$ by $\mathfrak{g}$ therefore $\mathfrak{g}$ is an ideal in $\mathfrak{g}'$. The Lie bracket $[x_0, x]$ is by definition Dx. □

By the previous proposition and the above lemma, we have, $\kappa_{\mathfrak{g}}(Dx, y) = \kappa_{\mathfrak{g}'}(Dx, y) = \kappa_{\mathfrak{g}'}([x_0, x], y)$ but the Killing form $\kappa_{\mathfrak{g}'}$ being invariant, we have $\kappa_{\mathfrak{g}'}([x_0, x], y) = -\kappa_{\mathfrak{g}'}(x, [x_0, y]) = -\kappa_{\mathfrak{g}}(x, Dy)$ and the result follows. □

4.6 Casimir element

In this section we construct a very useful element in the envelopping algebra $U(\mathfrak{g})$. Let us first remark that the envelopping algebra $U(\mathfrak{g})$ is a representation of the Lie algebra $\mathfrak{g}$. Indeed, the adjoint representation gives a representation of $\mathfrak{g}$ in itself and by looking at the tensor product and direct sum representation, we get that $T(\mathfrak{g})$, the tensor algebra, is a representation of $\mathfrak{g}$, the action being given by

$$x_{T(\mathfrak{g})} \cdot (x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n} (x_1 \otimes \cdots \otimes [x, x_i] \otimes \cdots \otimes x_n).$$

Lemma 4.6.1 *The ideal J generated by the elements of the form $x \otimes y - y \otimes x - [x, y]$ is a subrepresentation of $T(\mathfrak{g})$.*

Proof. Let x, y and z in $\mathfrak{g}$, we need to prove that $x_{T(\mathfrak{g})}$ maps $y \otimes z - z \otimes y - [y, z]$ to an element of J. We compute

$$\begin{aligned} x_{T(\mathfrak{g})} \cdot (y \otimes z - z \otimes y - [y,z]) = & \ [x,y] \otimes z + y \otimes [x,z] - [x,z] \otimes y - z \otimes [x,y] - [x,[y,z]] \\ = & \ [x,y] \otimes z + y \otimes [x,z] - [x,z] \otimes y - z \otimes [x,y] - [z,[x,y]] - [y,[x,z]] \\ = & \ ([x,y] \otimes z - z \otimes [x,y] - [[x,y],z]) \\ & + (y \otimes [x,z] - [x,z] \otimes y - [y,[x,z]]) \end{aligned}$$

and the terms in the last sum are in J, the result follows. □

Corollary 4.6.2 *The above action of $\mathfrak{g}$ on $T(\mathfrak{g})$ induces an action on $U(\mathfrak{g})$ which is therefore a representation of $\mathfrak{g}$.*

Proposition 4.6.3 *Let $\mathfrak{g}$ be a Lie algebra, let $\mathfrak{h}$ be an ideal of finite dimension n and let b be an invariant bilinear form on $\mathfrak{g}$ whose restriction to $\mathfrak{h}$ is non degenerate. Let $(h_i)_{i \in [1,n]}$ and $(h'_i)_{i \in [1,n]}$ be basis of $\mathfrak{h}$ such that $b(h_i, h'_j) = \delta_{i,j}$, then the element in $U(\mathfrak{g})$ defined by*

$$c = \sum_{i=1}^{n} h_i h'_i$$

is invariant, lies in the center of $U(\mathfrak{g})$ and is independent of the choice of the basis.

Proof. We have already seen in Example 4.4.5 that the element

$$c_{\mathfrak{h}} = \sum_{i=1}^{n} h_i \otimes h'_i \in \mathfrak{h} \otimes \mathfrak{h}$$

does not depend on the basis and is invariant. The above element c is the image of $c_{\mathfrak{h}}$ in $U(\mathfrak{g})$ and therefore is invariant and does not depend on the choice of the base. The element c lies in the center by the next result. □

Lemma 4.6.4 *Let c be an invariant element in $U(\mathfrak{g})$, then c lies in the center of $U(\mathfrak{g})$.*

Proof. First assume that c is the image of a pure tensor *i.e.* $c = x_1 \cdots x_n \in T(\mathfrak{g})$ with the x_i in $\mathfrak{g}$. Then we have

$$\begin{aligned} x_{T(\mathfrak{g})} \cdot c &= \textstyle\sum_{i=1}^n x_1 \cdots [x, x_i] \cdots x_n \\ &= \textstyle\sum_{i=1}^n x_1 \cdots x_{i-1} x x_i \cdots x_n - \sum_{i=1}^n x_1 \cdots x_i x x_{i+1} \cdots x_n \\ &= \textstyle\sum_{i=1}^n x_1 \cdots x_{i-1} x x_i \cdots x_n - \sum_{i=2}^{n+1} x_1 \cdots x_{i-1} x x_i \cdots x_n \\ &= x x_1 \cdots x_n - x_1 \cdots x_n x. \end{aligned}$$

Therefore if c is invariant we have $xc = cx$ for all $x \in \mathfrak{g}$. By linearity, the same is true a general invariant element c. Now the result follows because $\mathfrak{g}$ generates $U(\mathfrak{g})$. □

Definition 4.6.5 *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ an ideal of $\mathfrak{g}$. Let V be a finite dimensional representation of $\mathfrak{g}$ such that the bilinear form $b_V^{\mathfrak{h}}$ on $\mathfrak{h}$ associated to V is non degenerate. The element $c_V^{\mathfrak{h}} \in Z(U(\mathfrak{g}))$ is called the* Casimir element *of $\mathfrak{g}$ associated to $\mathfrak{h}$ and V.*

Proposition 4.6.6 *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ an ideal of $\mathfrak{g}$ of dimension n. Let V be a finite dimensional representation of $\mathfrak{g}$ such that the bilinear form $b_V^{\mathfrak{h}}$ on $\mathfrak{h}$ associated to V is non degenerate. Let $c = c_V^{\mathfrak{h}}$ be the Casimir element of $\mathfrak{g}$ associated to $\mathfrak{h}$ and V.*

(ı) We have $\mathrm{Tr}(c) = n$.

(ıı) If V is simple and n prime to $\mathrm{char}k$, then c is an automorphism of V.

Proof. *(ı)* By definition, for basis $(h_i)_{i \in [1,n]}$ and $(h'_i)_{i \in [1,n]}$ be basis of $\mathfrak{h}$ such that $b_V^{\mathfrak{h}}(h_i, h'_j) = \delta_{i,j}$, we have the equality $c = \sum_{i=1}^n h_i h'_i$. We get

$$\mathrm{Tr}(c) = \sum_{i=1}^n \mathrm{Tr}((h_i)_V (h'_i)_V) = \sum_{i=1}^n b_V^{\mathfrak{h}}(h_i, h'_i) = n.$$

(ıı) If n and $\mathrm{char}k$ are coprime, then $\mathrm{Tr}(c)$ does not vanish and c is not the zero map. But c is in the center of $U(\mathfrak{g})$ and therefore commutes with any x_V for $x \in \mathfrak{g}$. In particular, $\ker c$ is a $\mathfrak{g}$-invariant subspace of V. The representation V being simple, we have $\ker c = V$ or $\ker c = 0$. The first equality would imply $c = 0$, therefore c is injective and an automorphism. □

4.7 Exercices

Exercice 4.7.1 Let V be a representation of a Lie algebra $\mathfrak{g}$. Prove that the subspaces of symmetric and antisymmetric tensors S^nV and Λ^nV and subrepresentations of the tensor product representation $T^n(\mathfrak{g})$.

Exercice 4.7.2 Let $\mathfrak{g}_i$ be a Lie algebra and V_i be a representation for $i \in \{1, 2\}$. Verify, without using envelopping algebras, that the map $\mathfrak{g}_1 \times \mathfrak{g}_2 \to \mathfrak{gl}(V_1 \otimes V_2)$ given by $(x_1, x_2)_V(v_1 \otimes v_2) = (x_1)_V(v_1) \otimes v_2 + v_1 \otimes (x_2)_V(v_2)$ defines a Lie algebra representation of $\mathfrak{g}_1 \times \mathfrak{g}_2$ in $V_1 \otimes V_2$.

Exercice 4.7.3 Prove that $x_{V^\vee} = -{}^t x_V$.

Exercice 4.7.4 Prove that V is simple if and only if $V^\vee$ is simple.

Exercice 4.7.5 Let $\mathfrak{g}_i$ be a Lie algebra and V_i be a representation for $i \in \{1, 2\}$. Prove that the natural isomorphisms

$$V_1^\vee \otimes V_2 \simeq \mathrm{Hom}(V_1, V_2) \text{ and } \mathrm{Hom}(V_1, V_2^\vee) \simeq (V_1 \otimes V_2)^\vee$$

defined by $\phi \otimes v_2 \mapsto (v_1 \mapsto \phi(v_1)v_2)$ and $\phi \mapsto ((v_1 \otimes v_2) \mapsto \phi(v_1)(v_2))$ are isomorphisms of Lie algebras.

Exercice 4.7.6 For V of dimension 2, prove that $\mathfrak{sl}(V)$ is a simple Lie algebra.

Exercice 4.7.7 Let $\mathfrak{g}$ be a Lie algebra over $k = \mathbb{C}$ and let $\mathfrak{g}_0$ be the Lie algebra over $\mathbb{R}$ obtained by considering $\mathfrak{g}$ as an $\mathbb{R}$-vector space (we have $\dim_{\mathbb{R}} \mathfrak{g}_0 = 2 \dim_{\mathbb{C}} \mathfrak{g}$) *i.e.* by restriction of the scalars. Prove the equality

$$\kappa_{\mathfrak{g}_0} = 2\Re(\kappa_{\mathfrak{g}}),$$

where $\Re(z)$ is the real part of $z \in \mathbb{C}$.

Exercice 4.7.8 Let $\mathfrak{g}$ be a Lie algebra, then the multilinear form

$$(x_i)_{i \in [1,n]} \mapsto \mathrm{Tr}(\mathrm{ad}\, x_1 \circ \cdots \circ x_n)$$

is invariant.

Exercice 4.7.9 Let $\mathfrak{g}$ be a Lie algebra, let V and W be two representations of $\mathfrak{g}$ and let $f : V \to W$ be a morphisms of representations, prove the inclusion $f(V^{\mathfrak{g}}) \subset W^{\mathfrak{g}}$.

If furthermore, V and W are semisimple representations, prove the equality $f(V^{\mathfrak{g}}) = W^{\mathfrak{g}}$ (Hint: prove first that the subspace $V^{\mathfrak{g}}$ is a subrepresentation and has a unique supplementary representation generated by the $x_V \cdot v$ for $x \in \mathfrak{g}$ and $v \in V$).

Exercice 4.7.10 (i) Let $\mathfrak{g} = \mathfrak{sl}(V)$ with V a vector space of dimension 2. Compute the Casimir element associated to the adjoint representation in terms of the canonical base

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(ii) Let $\mathfrak{g} = \mathfrak{sl}(V)$ with V a vector space of dimension 3. Compute the Casimir element associated to the representation $\mathfrak{sl}(V) \to \mathfrak{gl}(V)$.

Exercice 4.7.11 Let V be a finite dimensional vector space and let $V_\bullet$ be a complete flag. Let $\mathfrak{g} = \mathfrak{t}(V_\bullet)$ and consider the representation $\mathfrak{g} \to \mathfrak{gl}(V)$ given by the inclusion.

(i) Prove that the only subrepresentations of V are the vector spaces V_k of the flag $V_\bullet$.

(ii) Prove that the V_k are irreducible but not simple (except for V_1).

Exercice 4.7.12 Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$.

(i) Let V be of dimension one and $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. Prove that $\varphi(\mathcal{D}\mathfrak{g}) = 0$.

(ii) Prove that any representation V of $\mathfrak{g}/\mathcal{D}\mathfrak{g}$ induces a representation of $\mathfrak{g}$ in V on which $\mathcal{D}\mathfrak{g}$ acts trivially.

(iii) Prove that if $\mathfrak{g} \neq \mathcal{D}\mathfrak{g}$, then $\mathfrak{g}$ has infinitely many one dimensional non isomorphic representations while if $\mathfrak{g} = \mathcal{D}\mathfrak{g}$, then the only one dimensional representation of $\mathfrak{g}$ is the trivial one.

Hint: any linear map $\mathfrak{g}/\mathcal{D}\mathfrak{g} \to \mathfrak{gl}(V)$ with $\dim V = 1$ is a representation.

Chapter 5

Nilpotent Lie algebras

5.1 Definition

Definition 5.1.1 *A Lie algebra is called* nilpotent *if there exists a decreasing finite sequence* $(\mathfrak{g}_i)_{i\in[0,k]}$ *of ideals such that* $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_k = 0$ *and* $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ *for all* $i \in [0, k-1]$.

Proposition 5.1.2 *Let* $\mathfrak{g}$ *be a Lie algebra, the following conditions are equivalent:*

(ı) the Lie algebra $\mathfrak{g}$ *is nilpotent;*

(ıı) we have $\mathcal{C}^k\mathfrak{g} = 0$ *for* k *large enough;*

(ııı) we have $\mathcal{C}_k\mathfrak{g} = \mathfrak{g}$ *for* k *large enough;*

(ıv) there exists an integer k *such that* $\operatorname{ad} x_1 \circ \cdots \circ \operatorname{ad} x_k = 0$ *for any sequence* $(x_i)_{i\in[1,k]}$ *of elements in* $\mathfrak{g}$*;*

(v) there exists a decreasing sequence of ideals $(\mathfrak{g}_i)_{i\in[0,n]}$ *with* $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_n = 0$ *and such that* $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ *and* $\dim g_i/g_{i+1} = 1$ *for all* $i \in [0, n-1]$.

Proof. We start with the equivalence of the first three conditions. If *(ıı)* or *(ııı)* holds, then the sequence $(\mathcal{C}^i\mathfrak{g})_{i\in[1,k]}$ or $(\mathcal{C}_{k-i}\mathfrak{g})_{i\in[1,k]}$ satisfy the conditions of the definition and $\mathfrak{g}$ is nilpotent.

Conversely, if the exists a sequence of ideals $(\mathfrak{g}_i)_{i\in[0,k]}$ as in the definition, we prove by induction that $\mathcal{C}^i\mathfrak{g} \subset \mathfrak{g}_i$ and $\mathcal{C}_i\mathfrak{g} \supset \mathfrak{g}_{k-i}$. This is true for $i = 0$. Assume that $\mathcal{C}^i\mathfrak{g} \subset \mathfrak{g}_i$ and $\mathcal{C}_i\mathfrak{g} \supset \mathfrak{g}_{k-i}$, then we have the inclusions $\mathcal{C}^{i+1}\mathfrak{g} = [\mathfrak{g}, \mathcal{C}^i\mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ and $[\mathfrak{g}/\mathcal{C}_i, (\mathfrak{g}_{k-(i+1)} + \mathcal{C}_i\mathfrak{g})/\mathcal{C}_i\mathfrak{g}] \subset (\mathfrak{g}_{k-i} + \mathcal{C}_i\mathfrak{g})/\mathcal{C}_i\mathfrak{g} = 0$. The last inclusion implies that $(\mathfrak{g}_{k-(i+1)} + \mathcal{C}_i\mathfrak{g})/\mathcal{C}_i\mathfrak{g}$ is in the center of $\mathfrak{g}/\mathcal{C}_i\mathfrak{g}$ and therefore $\mathfrak{g}_{k-(i+1)} \subset \mathcal{C}_{i+1}\mathfrak{g}$. We get $\mathcal{C}^k\mathfrak{g} = 0$ and $\mathcal{C}_k\mathfrak{g} = \mathfrak{g}$.

Now *(ıı)* and *(ıv)* are equivalent. Indeed, the ideal $\mathcal{C}^k\mathfrak{g}$ is composed of the linear combinations of elements of the form $[x_1, [x_2, [\cdots[x_k, y]\cdots]]] = \operatorname{ad} x_1 \circ \cdots \circ \operatorname{ad} x_k(y)$ with $x_i \in \mathfrak{g}$ for all i and $y \in \mathfrak{g}$.

Finally *(ı)* and *(v)* are equivalent. Indeed the last condition imply the first. Conversely, assume that $(\mathfrak{g}_i)_{i\in[0,k]}$ is a sequence of ideals as in the definition of a nilpotent Lie algebra. Then let us complete the sequence $(\mathfrak{g}_i)_{i\in[0,k]}$ to a sequence $(\mathfrak{g}'_i)_{i\in[0,n]}$ with $n = \dim\mathfrak{g}$, $\dim\mathfrak{g}'_i = n-i$, $\mathfrak{g}_{i+1} \subset \mathfrak{g}_i$ and $\mathfrak{g}'_{n-\dim\mathfrak{g}_j} = \mathfrak{g}_j$. We only need to prove that $[\mathfrak{g}, \mathfrak{g}'_i] \subset \mathfrak{g}'_{i+1}$. But, for $i \in [0, n]$, we define $i_s = \max\{j \ / \ \mathfrak{g}'_i \subset \mathfrak{g}_j\}$. We have $\mathfrak{g}_{i_s+1} \subset \mathfrak{g}'_{i+1} \subset \mathfrak{g}'_i \subset \mathfrak{g}_{i_s}$. Therefore $[\mathfrak{g}, \mathfrak{g}'_i] \subset [\mathfrak{g}, \mathfrak{g}_{i_s}] \subset \mathfrak{g}_{i_s+1} \subset \mathfrak{g}'_{i+1}$. □

Corollary 5.1.3 *The center of a non trivial nilpotent Lie algebra is non trivial.*

Proof. Indeed, we must have $\mathcal{C}_1\mathfrak{g} \neq 0$ otherwise there is no k with $\mathcal{C}_k\mathfrak{g} = \mathfrak{g}$. □

Corollary 5.1.4 *The Killing form $\kappa_{\mathfrak{g}}$ vanishes for $\mathfrak{g}$ nilpotent.*

Proof. For any $(x, y) \in \mathfrak{g}^2$, the element $\operatorname{ad} x \circ \operatorname{ad} y$ is nilpotent thus $\kappa_{\mathfrak{g}}(x, y) = \operatorname{Tr}(\operatorname{ad} x \circ \operatorname{ad} y) = 0$. □

Proposition 5.1.5 *Any subalgebra, any quotient algebra, any central extension a Lie subalgebra is again a Lie subalgebra. A finite product of nilpotent Lie algebras is again a nilpotent Lie algebra.*

Proof. Let $\mathfrak{g}$ be a nilpotent Lie algebra.

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, then $\mathcal{C}^k\mathfrak{h} \subset \mathcal{C}^k\mathfrak{g}$ and the result follows for $\mathfrak{h}$.

Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$ and let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ be the projection. We proved in Proposition 2.5.6 that $\pi(\mathcal{C}^k\mathfrak{g}) = \mathcal{C}^k(\mathfrak{g}/\mathfrak{a})$ and the result follows for $\mathfrak{g}/\mathfrak{a}$.

Let $0 \to \mathfrak{a} \to \mathfrak{g}' \xrightarrow{p} \mathfrak{g} \to 0$ be a central extension, then $p(\mathcal{C}^k\mathfrak{g}') = \mathcal{C}^k\mathfrak{g}$. Therefore if $\mathcal{C}^k\mathfrak{g} = 0$, then $\mathcal{C}^k\mathfrak{g}' \subset \mathfrak{a}$ and $\mathcal{C}^{k+1}\mathfrak{g}' = 0$ because $\mathfrak{a} \subset \mathfrak{z}(\mathfrak{g}')$.

The last assertion follows from the condition *(iv)* in the previous proposition. □

5.2 Engel's Theorem

Theorem 5.2.1 *Let V be a vector space and $\mathfrak{g}$ be a finite dimensional Lie subalgebra of $\mathfrak{gl}(V)$, such that for all x is nilpotent for all $x \in \mathfrak{g}$, then there is a $v \in V$ with $x(v) = 0$ for all $x \in \mathfrak{g}$.*

Proof. We proceed by induction on $n = \dim \mathfrak{g}$. For $n = 0$, this is clear. We shall need a

Lemma 5.2.2 *Let V be a vector space and $x \in \mathfrak{gl}(V)$ nilpotent, then element f of $\mathfrak{gl}(\mathfrak{gl}(V))$ defined by $y \mapsto [x, y]$ is nilpotent.*

Proof. Indeed, we can compute that $f^m(y)$ is a linear combinaison of terms of the form $x^i y x^{m-i}$ and the result follows. □

Now let $\mathfrak{h}$ be a strict subalgebra of $\mathfrak{g}$. We define a map $\sigma : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ sending $x \in \mathfrak{h}$ to the map $\sigma(x)$ defined by $\overline{y} \mapsto \overline{[x, y]}$ where $\overline{y}$ is the class of $y \in \mathfrak{g}$ in the quotient $\mathfrak{g}/\mathfrak{h}$. By the previous lemma, the map $x \mapsto [x, y]$ is nilpotent so $\sigma(x)$ is nilpotent. Therefore $\sigma(\mathfrak{h})$ satisfies the conditions of the Theorem and $\dim \sigma(\mathfrak{h}) < n$. By induction, there exists $\overline{y}$ a non trivial vector in $\mathfrak{g}/\mathfrak{h}$ with $\sigma(x)(y) = 0$ for all $x \in \mathfrak{h}$. Therefore, there is a y not in $\mathfrak{h}$ with $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}$. This imples that $\mathfrak{h}$ is an ideal in the subalgebra $\mathfrak{h} \oplus ky$ of $\mathfrak{g}$.

By induction starting with $\mathfrak{h} = 0$, we get a codimension 1 ideal $\mathfrak{h}$ in $\mathfrak{g}$. The result is true for $\mathfrak{h}$ therefore, the subspace W of all $v \in V$ such that $x(v) = 0$ for all $x \in \mathfrak{h}$ is non trivial. Let $y \in \mathfrak{g}$ with $y \notin \mathfrak{h}$, then y stabilises W. Indeed, for $v \in W$, we have $x(y(v)) = y(x(v)) + [x, y](v) = 0$ because $[x, y]$ and x are in $\mathfrak{h}$. Now y is nilpotent on W therefore there exists v non trivial in W with $y(v) = 0$. The vector v does the job. □

Corollary 5.2.3 *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\operatorname{ad} x$ is nilpotent for all $x \in \mathfrak{g}$.*

Proof. By Proposition 5.1.2, if $\mathfrak{g}$ is nilpotent then $\operatorname{ad} x$ is nilpotent for all $x \in \mathfrak{g}$. Conversely, if $\operatorname{ad} x$ is nilpotent for all x, then the image of the adjoint representation in $\mathfrak{gl}(\mathfrak{g})$ satisfies the conditions of Engel's Theorem. Therefore, there is a non trivial $x \in \mathfrak{g}$ such that $[x, y] = \operatorname{ad} x(y) = 0$ for all $y \in \mathfrak{g}$. Therefore the center of $\mathfrak{g}$ is non trivial. Now the Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ satisfies the same hypothesis and we conclude by induction that $\mathcal{C}_k\mathfrak{g} = \mathfrak{g}$ for k large enough. □

Corollary 5.2.4 *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an ideal of $\mathfrak{g}$. Assume that $\mathfrak{g}/\mathfrak{a}$ is nilpotent and that for all $x \in \mathfrak{g}$, the restriction $\operatorname{ad} x|_{\mathfrak{a}}$ is nilpotent, then $\mathfrak{g}$ is nilpotent.*

Proof. Let $x \in \mathfrak{g}$, we prove that $\operatorname{ad} x$ is nilpotent. Indeed, it is nilpotent on $\mathfrak{a}$ and on $\mathfrak{g}/\mathfrak{a}$ (there are k and k' such that $\operatorname{ad}^k x(\mathfrak{g}) \subset \mathfrak{a}$ and $\operatorname{ad}^{k'} x(\mathfrak{a}) = 0$ therefore $\operatorname{ad}^{k+k'} x(\mathfrak{g}) = 0$). □

Corollary 5.2.5 *Let V be a vector space and $\mathfrak{g}$ a Lie subalgebra of $\mathfrak{gl}(V)$ such that all the elements $x \in \mathfrak{g}$ are nilpotent endomorphisms of V, then $\mathfrak{g}$ is nilpotent.*

Proof. Indeed by Lemma 5.2.2, for any $x \in \mathfrak{g}$, the element $\operatorname{ad} x$ is nilpotent. We conclude by applying Corollary 5.2.3 □

Example 5.2.6 For V a vector space and $V_\bullet$ a complete flag, the Lie algebra $\mathfrak{n}(V_\bullet)$ is nilpotent.

5.3 Maximal nilpotent ideal

Definition 5.3.1 *An ideal $\mathfrak{a}$ in $\mathfrak{g}$ is called nilpotent if it is nilpotent as a Lie algebra.*

Lemma 5.3.2 *An ideal $\mathfrak{a}$ of $\mathfrak{g}$ is nilpotent if and only if for all $x \in \mathfrak{a}$, we have that $\operatorname{ad}_{\mathfrak{g}} x$ is nilpotent.*

Proof. The condition is sufficent (we only need that $\operatorname{ad}_{\mathfrak{a}} x$ is nilpotent). Conversely, if $\mathfrak{a}$ is nilpotent, then $\operatorname{ad}_{\mathfrak{a}} x$ is nilpotent and $\operatorname{ad}_{\mathfrak{g}} x(\mathfrak{g}) \subset \mathfrak{a}$ and the result follows. □

We shall need the following general result on representations.

Lemma 5.3.3 *Let V be a finite dimensional representation of the Lie algebra $\mathfrak{g}$, then there exists an increasing sequence $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ of subrepresentations of V such that V_i/V_{i-1} is simple for all $i \in [1, n]$.*

Proof. By induction on the dimension of V, we only need to prove that there exists a subrepresentation W of V such that V/W is simple. We also prove this by induction on $\dim V$. Indeed, if V is simple, we are done. Otherwise, there exists a non trivial subrepresentation V' of V and we apply our induction hypothesis on V/V'. We get W/V' a subrepresentation of V/V' (image of the subspace W in V) such that $(V/V')/(W/V')$ is simple. But W is a subrepresentation of V and $V/W \simeq (V/V')/(W/V')$ is simple. □

Lemma 5.3.4 *Let V be a simple representation of $\mathfrak{g}$ and $\mathfrak{a}$ an ideal such that for all $x \in \mathfrak{a}$, the element x_V is nilpotent. Then for all $x \in \mathfrak{a}$, we have $x_V = 0$.*

Proof. By Proposition 4.4.6, the subspace $V^{\mathfrak{a}} = \{v \in V \ / \ x_V \cdot v = 0 \text{ for all } x \in \mathfrak{a}\}$ is a subrepresentation of V. Furthermore, by Engel's Theorem (Theorem 5.2.1), this space is non trivial. Because V is simple we have $V = V^{\mathfrak{a}}$. □

Lemma 5.3.5 *The sum of any two nilpotent ideals is again a nilpotent ideal.*

Proof. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two nilpotent ideals and $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$. We need to prove that if $\mathrm{ad}_{\mathfrak{g}}(x+y)$ is nilpotent. For this consider the sequence of subrepresentations $\mathfrak{g}_0 = 0 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ of the adjoint representation given by Lemma 5.3.3. Because $\mathrm{ad}_{\mathfrak{g}}x$ and $\mathrm{ad}_{\mathfrak{g}}y$ are nilpotent, for any $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$, we have that $x_{\mathfrak{g}_i/\mathfrak{g}_{i-1}}$ and $y_{\mathfrak{g}_i/\mathfrak{g}_{i-1}}$ are nilpotent for all $i \in [1,n]$. By Lemma 5.3.4 and because $\mathfrak{g}_i/\mathfrak{g}_{i-1}$ is simple, we have the equalities that $x_{\mathfrak{g}_i/\mathfrak{g}_{i-1}} = 0$ and $y_{\mathfrak{g}_i/\mathfrak{g}_{i-1}} = 0$ for all $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ and for all $i \in [1,n]$. In particular $(x+y)_{\mathfrak{g}_i/\mathfrak{g}_{i-1}} = 0$ for all $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ and for all $i \in [1,n]$. We have $\mathrm{ad}_{\mathfrak{g}}(x+y)(\mathfrak{g}_i) \subset \mathfrak{g}_{i-1}$ for all $i \in [1,n]$ and $\mathrm{ad}_{\mathfrak{g}}(x+y)$ is nilpotent. □

Corollary 5.3.6 *There exists a maximal nilpotent ideal $\mathfrak{n}_{\mathfrak{g}}$ in any finite dimensional Lie algebra $\mathfrak{g}$.*

Remark 5.3.7 The quotient $\mathfrak{g}/\mathfrak{n}_{\mathfrak{g}}$ may have nilpotent ideals.

5.4 Exercices

Exercice 5.4.1 Let $\mathfrak{g}$ be a nilpotent Lie algebra and let p (resp. q) be the smallest integer such that $\mathcal{C}^p\mathfrak{g} = 0$ (resp. $\mathcal{C}_q\mathfrak{g} = \mathfrak{g}$). Prove that $p = q$ and that $\mathcal{C}_i\mathfrak{g} \supset \mathcal{C}^{p-i}\mathfrak{g}$.

Exercice 5.4.2 Prove that a Lie algebra $\mathfrak{g}$ is nilpotent if and only if all its two dimensional Lie subalgebras are abelian.

Hint: reduce to the case where k is algebraically closed.

Exercice 5.4.3 Let $\mathfrak{g}$ be the semidirect product of a one-dimensional Lie algebra $\mathfrak{b}$ and an abelian ideal $\mathfrak{a}$. Let $x \in \mathfrak{b}$ non trivial and $u = \mathrm{ad}_{\mathfrak{g}}x|_{\mathfrak{a}}$.

(i) Prove that $\mathfrak{g}$ is nilpotent if and only if u is nilpotent.

(ii) Prove that the Killing form $\kappa_{\mathfrak{g}}$ vanishes if and only if $\mathrm{Tr}(u^2) = 0$.

(iii) Give an example of a non nilpotent Lie algebra with vanishing Killing form.

(iv) Give an example of a nilpotent Lie algebra such that the inclusion $\mathcal{C}_i\mathfrak{g} \supset \mathcal{C}^{p-i}\mathfrak{g}$ of the previous exercice is strict.

Exercice 5.4.4 Assume $\mathrm{char}k = 2$, prove that for V of dimension 2 over k the Lie algebra $\mathfrak{sl}(V)$ is nilpotent.

Exercice 5.4.5 Let $\mathfrak{g}$ be a nilpotent Lie algebra, let $\mathfrak{z}$ be its center and let $\mathfrak{a}$ be a nonzero ideal of $\mathfrak{g}$. Prove that $\mathfrak{z} \cap \mathfrak{a} \neq 0$.

Exercice 5.4.6 Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an ideal contained in $\mathfrak{z}(\mathfrak{g})$. Prove that $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}/\mathfrak{a}$ is nilpotent.

Deduce that nilpotent Lie algebras are the Lie algebras obtained from abelian Lie algebras by performing central extensions.

Exercice 5.4.7 Describe all nilpotent Lie algebras of dimension at most 3.

Exercice 5.4.8 Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two Lie algebras and $\mathfrak{n}_{\mathfrak{g}}$ and $\mathfrak{n}_{\mathfrak{g}'}$ their maximal nilpotent ideals, prove that $\mathfrak{n}_{\mathfrak{g}\times\mathfrak{g}'} = \mathfrak{n}_{\mathfrak{g}} \times \mathfrak{n}_{\mathfrak{g}'}$.

Exercice 5.4.9 Give an example of a Lie algebra $\mathfrak{g}$ where the quotient $\mathfrak{g}/\mathfrak{n}_{\mathfrak{g}}$ has nilpotent ideals.

Chapter 6

Semisimple and nilpotent elements

In this chapter, all vector spaces are finite dimensional and we assume that k is a perfect field (for example char$k = 0$).

6.1 Semisimple endomorphisms

Recall the basic definition.

Definition 6.1.1 *We call semisimple any endomorphism which is diagonalisable in an extension of the base field k. Equivalently, the minimal polynomial is separable.*

Lemma 6.1.2 *If $x \in \mathrm{End}(V)$ is semismple and if $x(W) \subset W$ for W a subspace of V, then $x|_W \in \mathrm{End}(W)$ is semisimple.*

Proof. Let P_x be the minimal polynomial of x. We may compute $P_x(x|_W)(w) = P_x(x)(w) = 0$ therefore P_x kills $x|_W$ and the minimal polynomial $P_{x|_W}$ of $x|_W$ divides P_x. As P_x is separable, so is $P_{x|_W}$. □

Lemma 6.1.3 *Any two commuting semisimple endomorphisms x and y of V can be simultaneously diagonalised in an extension of k. In particular their sum is again diagonalisable in that extension, in other words $x + y$ is semisimple.*

Proof. We may assume that k is algebraically closed and that x and y are diagonalisable. Now let $V = \oplus V_\lambda(x)$ be the eigenspaces decomposition for x where λ is the eigenvalue. Then for $v \in V_\lambda(x)$, we have $x(y(v)) = y(x(v)) = y(\lambda(v)) = \lambda y(v)$ therefore $V_\lambda(x)$ is stabilised by y and by the previous lemma $y|_{V_\lambda(x)}$ is diagonalisable. The result follows. □

6.2 Semisimple and nilpotent decomposition

Theorem 6.2.1 *Assume that k is perfect (for example char$k = 0$). Let V be a finite dimensional vector space over k and let $x \in \mathrm{End}(V)$.*

(ı) There exists a unique decomposition $x = x_s + x_n$ in $\mathrm{End}(V)$ such that x_s is semisimple, x_n is nilpotent and x_s and x_n commute.

(ıı) There exists polynomial P and Q in $k[T]$ such that $x_s = P(x)$ and $x_n = Q(x)$. In particular x_s and x_n commute with any endomorphism commuting with x.

(ııı) If $U \subset W \subset V$ are subspaces such that $x(W) \subset U$, then x_s and x_n also map W in U.

Proof. We start to prove the result when k contains all the roots of the characteristic polynomial χ_x of x. Let us write

$$\chi_x(T) = \prod_{i=1}^{n}(T - \lambda_i)^{a_i}$$

where the λ_i are pairwise distinct. The space $V^{\lambda_i}(x) = \ker((x - \lambda_i \mathrm{Id}_V)^{a_i})$ is invariant under x and we have $V = \oplus V^{\lambda_i}(x)$. Let P be a polynomial such that $P \equiv \lambda_i \pmod{(T - \lambda_i)^{a_i}}$ for all i and $P \equiv 0 \pmod{T}$. This is possible by the Chinese Remainder Theorem and because the $(T - \lambda_i)^{a_i}$ are coprime (if $\lambda_i = 0$ for some i, then the last condition is satisfied).

Now put $x_s = P(x)$, then the restriction of x_s to $V^{\lambda_i}(x)$ is simply the multiplication by a_i therefore x_s is semisimple on V. Now put $Q(T) = T - P(T)$ and $x_n = Q(x)$. But the only eigenvalues of x and x_s on $V^{\lambda_i}(x)$ is λ_i therefore x_n has only 0 as eigenvalue and x_n is nilpotent.

Let us prove that the decomposition is unique. Let $x = s + n$ be another decomposition. Write $s - x_s = x_n - n$. All these endomorphism commute, therefore $s - x_s$ is semisimple and $x_n - n$ is nilpotent. Thus both vanish.

We now need to prove the general case. To simplify notation, we will consider matrices instead of endomorphisms therefore, we fix a basis for V and denote by X the matrix of x in that basis.

Let K be the field of decomposition for χ_X the characteristic polynomial of X. We consider the matrix X as a matrix with coefficients in K. By what we already proved, we know that there exists polynomials P and Q in $K[T]$ such that $X = P(X)+Q(X)$ with $P(X)$ semisimple and $Q(X)$ nilpotent. We may replace P and Q by their rest in the Euclidian division by P_X minimal polynomial of X.In particular P and Q have degree smaller than the degree of P_X.

Now let σ be an automorphism of K leaving k fixed. For any matrix $M = (m_{i,j})$ we may define the matrix $M^\sigma = (\sigma(m_{i,j}))$. We may also define, for $R(T) = \sum a_i T^i$, the element $R^\sigma(T) = \sum \sigma(a_i) T^i$. We have

$$(P(X))^\sigma = P^\sigma(X^\sigma),\ (Q(X))^\sigma = Q^\sigma(X^\sigma) \text{ and } X^\sigma = P^\sigma(X^\sigma) + Q^\sigma(X^\sigma).$$

But X has coefficients in k therefore $X^\sigma = X$ and we get $X = P^\sigma(X) + Q^\sigma(X)$. But because $P(X)$ (resp. (QX)) is semisimple (resp. nilpotent), the same is true for $(P(X))^\sigma$ (resp. $(Q(X))^\sigma$). Indeed, it is clear for the nilpotent case. If (e_i) is a basis of V_K and (λ_i) are scalars in K such that $P(X)(e_i) = \lambda_i e_i$, then we have $(P(X))^\sigma(\sigma(e_i)) = \sigma(P(X)(e_i)) = \sigma(\lambda_i e_i) = \sigma(\lambda_i)\sigma(e_i)$ and $(\sigma(e_i))$ is an eigenbasis for $(P(X))^\sigma$. By unicity, we have $P^\sigma(X) = P(X)$ and $Q^\sigma(X) = Q(X)$. By minimality of the degree, we get $P = P^\sigma$ and $Q = Q^\sigma$. Now because the field is perfect, the extension K/k is a Galois extension and P and Q have to be in $k[T]$.

This proves *(ı)* and *(ıı)*, for *(ııı)* we only need to remark that P and Q can be choosen without constant terms. This was the case for k containing the roots of P_X. In the general case, if X is invertible then P_X has non zero constant term therefore, because $P_X(X) = 0$, we have that Id is a polynomial in X without constant term and the result follows. If X is not invertible, then $\ker X$ is non trivial and $Q(X)$ stabilises $\ker(X)$. But $Q(X)$ is nilpotent therefore there exists $v \in \ker(X)$ such that $Q(X)(v) = 0$. Write $Q(T) = \sum a_i T^i$, we have $Q(X)(v) = \sum a_i X^i(v) = a_0$ because $v \in \ker X$. Therefore $a_0 = 0$ and the result follows. ☐

Definition 6.2.2 *The elements x_s (resp. x_n) is called the* semisimple part *of $x \in \mathrm{End}(V)$ (resp.* nilpotent part *The decomposition $x = x_s + x_n$ is called the* Jordan-Chevalley decomposition.

Lemma 6.2.3 *Let $x \in \mathrm{End}(V)$ and denote by $\mathrm{ad} : \mathrm{End}(V) \to \mathfrak{gl}(\mathrm{End}(V))$ the adjoint representation of $\mathrm{End}(V)$ defined by $\mathrm{ad}\, x(y) = xy - yx$. Then we have the formulas:*

$$\mathrm{ad}\,(x_s) = (\mathrm{ad}\, x)_s \text{ and } \mathrm{ad}\,(x_n) = (\mathrm{ad}\, x)_n.$$

Proof. We have $x = x_s + x_n$ therefore $\operatorname{ad} x = \operatorname{ad} x_s + \operatorname{ad} x_n$. Furthermore, because $[x_s, x_n] = 0$, we get $[\operatorname{ad} x_s, \operatorname{ad} x_n] = 0$. We are left to prove, by unicity that $\operatorname{ad} x_s$ is semisimple and $\operatorname{ad} x_n$ is nilpotent. Lemma 5.2.2 gives that $\operatorname{ad} x_n$ is nilpotent.

To prove that $\operatorname{ad} x_s$ is semisimple, we may assume that k is algebraically closed. Choose a basis (e_i) where x_s is diagonalisable with $x_s(e_i) = \lambda_i$. Then if $(E_{i,j})$ is the canonical basis for $\dim V \times \dim V$ matrices, we have $\operatorname{ad} x_s(E_{i,j}) = (\lambda_i - \lambda_j)E_{i,j}$ and $\operatorname{ad} x_s$ is also semisimple. □

6.3 Exercice

Exercice 6.3.1 Take $k = \mathbb{F}_2(T)$ and consider the endomorphism of k^2 given by the matrix

$$\begin{pmatrix} 0 & T \\ 1 & 0 \end{pmatrix}.$$

Prove that the semisimple and nilpotent parts of that endomorphism are not in $\mathrm{End}(k^2)$ but in $\mathrm{End}(K^2)$ with $K = k[\sqrt{T}]$.

Exercice 6.3.2 Prove that if x and y commute, then $(x+y)_s = x_s + y_s$ and $(x+y)_n = x_n + y + n$. Prove that this is flase in general, look for example at the following matrices:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Chapter 7

Solvable Lie algebras

From now on we assume that char$k = 0$ and that the dimension of all Lie algebras is finite.

7.1 Definition

Definition 7.1.1 *A Lie algebra is called* solvable *if we have $\mathcal{D}^k\mathfrak{g} = 0$ for k large enough;*

Proposition 7.1.2 *Let $\mathfrak{g}$ be a Lie algebra, the following conditions are equivalent:*

($\imath$) the Lie algebra $\mathfrak{g}$ is solvable;

($\imath\imath$) there exists a decreasing finite sequence $(\mathfrak{g}_i)_{i\in[0,k]}$ of ideals such that $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_k = 0$ and $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian for all $i \in [0, k-1]$;

($\imath\imath\imath$) there exists a decreasing finite sequence $(\mathfrak{g}_i)_{i\in[0,k]}$ of subalgebras with $\mathfrak{g}_{i+1}$ ideal in $\mathfrak{g}_i$ such that $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_k = 0$ and $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian for all $i \in [0, k-1]$;

($\imath v$) there exists a decreasing sequence of subalgebras $(\mathfrak{g}_i)_{i\in[0,n]}$ with $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_n = 0$ and such that $\mathfrak{g}_{i+1}$ is an codimension 1 ideal in $\mathfrak{g}_i$.

Proof. We easily have that *($\imath$)* implies *($\imath\imath$)* which implies *($\imath\imath\imath$)* implying *($\imath v$)*. For the first implication, put $\mathfrak{g}_i = \mathcal{D}^i\mathfrak{g}$. For the last one, any complete flag in $\mathfrak{g}$ containing the $(\mathfrak{g}_i)_{i\in[0,k]}$ will do.

For the fact that *($\imath$)* implies *($\imath$)*, we use the next proposition: any extension of solvable Lie algebra is again solvable and induction on the dimension. In dimension1, the Lie algebra is abelian and therefore solvable. We assume that $\mathfrak{g}_1$ is solvable but then $\mathfrak{g}$ is the extension $0 \to \mathfrak{g}_1 \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_1 \to 0$ and because $\mathfrak{g}_1$ and $\mathfrak{g}/\mathfrak{g}_1$ (this last one is abelian) are solvable, so is $\mathfrak{g}$. □

Proposition 7.1.3 *Any subalgebra and any quotient algebra of a solvable Lie algebra is solvable. Any extension of solvable Lie algebras is solvable.*

Proof. Let $\mathfrak{g}$ be a solvable Lie algebra. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, then $\mathcal{D}^k\mathfrak{h} \subset \mathcal{D}^k\mathfrak{g}$ and the result follows for $\mathfrak{h}$. Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$ and let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ be the projection. We proved in Proposition 2.5.6 that $\pi(\mathcal{D}^k\mathfrak{g}) = \mathcal{D}^k(\mathfrak{g}/\mathfrak{a})$ and the result follows for $\mathfrak{g}/\mathfrak{a}$.

Let $0 \to \mathfrak{a} \to \mathfrak{g}' \xrightarrow{p} \mathfrak{g} \to 0$ be an extension, then $p(\mathcal{D}^k\mathfrak{g}') = \mathcal{D}^k\mathfrak{g}$. Therefore if $\mathcal{D}^k\mathfrak{g} = 0$ and $\mathcal{D}^l\mathfrak{a} = 0$, then $\mathcal{D}^k\mathfrak{g}' \subset \mathfrak{a}$ and $\mathcal{D}^{k+l}\mathfrak{g}' = \mathcal{D}^l(D^k\mathfrak{g}') \subset \mathcal{D}^l\mathfrak{a} = 0$. □

Corollary 7.1.4 *A Lie algebra is solvable if and only if it can be obtained by successive extensions of abelian Lie algebras.*

Example 7.1.5 The Lie algebra $\mathfrak{t}(V_\bullet)$ is solvable but not nilpotent.

7.2 Radical

Lemma 7.2.1 *Let $\mathfrak{g}$ be a Lie algebra, then there exists a maximal solvable ideal in $\mathfrak{g}$.*

Proof. Indeed, let $\mathfrak{a}$ and $\mathfrak{b}$ be two solvable ideals, then $\mathfrak{a}+\mathfrak{b}$ is again an ideal and we have $(\mathfrak{a}+\mathfrak{b})/\mathfrak{b} \simeq \mathfrak{a}/(\mathfrak{a}\cap\mathfrak{b})$ therefore $(\mathfrak{a}+\mathfrak{b})/\mathfrak{b}$ is solvable as quotient of $\mathfrak{a}$. But $\mathfrak{b}$ being solvable, then $\mathfrak{a}+\mathfrak{b}$ is an extension of solvable Lie algebras and is therefore solvable. The sum of all solvable ideals is again a solvable ideal and is maximal. □

Definition 7.2.2 *We denote by $\mathfrak{r}(\mathfrak{g})$ or simply $\mathfrak{r}$ the maximal solvable ideal of $\mathfrak{g}$ and we call it the* radical ideal *of $\mathfrak{g}$.*

Proposition 7.2.3 *The radical $\mathfrak{g}$ is the smallest ideal such that $\mathfrak{g}/\mathfrak{r}$ has trivial radical.*

Proof. Let $\mathfrak{a}$ be a solvable ideal in $\mathfrak{g}$, let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ be the quotient map, then the map $\mathfrak{b} \mapsto \pi^{-1}(\mathfrak{b})$ is a bijection between solvable ideals in $\mathfrak{g}/\mathfrak{a}$ and solvable ideal in $\mathfrak{g}$ containing $\mathfrak{a}$.

Indeed, if $\mathfrak{b}$ is solvable, then its inverse image $\pi^{-1}(\mathfrak{b})$ in $\mathfrak{g}$ is an extension of $\mathfrak{b}$ by $\mathfrak{a}$ and is therefore a solvable ideal conversely, if $\mathfrak{c}$ is a solvable ideal in $\mathfrak{g}$ containing $\mathfrak{a}$, then $\pi(\mathfrak{c}) = \mathfrak{c}/\mathfrak{a}$ is again a solvable ideal in $\mathfrak{g}/\mathfrak{a}$. □

7.3 Lie's Theorem

In our first result we need to assume more on the field k, namely we need it to be algebraically closed since we want to have eigenvalues.

Theorem 7.3.1 *Assume that k is algebraically closed.*

Let V be a non trivial finite dimensional vector space and let $\mathfrak{g}$ be a solvable Lie subalgebra of $\mathfrak{gl}(V)$, then there exists a common eigenvector for all the elements in $\mathfrak{g}$.

Proof. We proceed by induction on $\dim\mathfrak{g}$, the case $\dim\mathfrak{g} = 0$ being trivial. We first choose $\mathfrak{a}$ an codimension 1 ideal. This is possible because $\mathcal{D}\mathfrak{g}$ is a strict ideal in $\mathfrak{g}$ and $\mathfrak{g}/\mathcal{D}\mathfrak{g}$ is abelian. Therefore, any codimension 1 subspace in $\mathfrak{g}/\mathcal{D}\mathfrak{g}$ is an ideal in $\mathfrak{g}/\mathcal{D}\mathfrak{g}$ and its inverse image in $\mathfrak{g}$ is a codimension 1 ideal in $\mathfrak{g}$.

Then, the ideal $\mathfrak{a}$ being solvable of dimension smaller, there is a common eigenvector $v \in V$ for all elements in $\mathfrak{a}$. Therefore, for any $x \in \mathfrak{a}$, we have $x(v) = \lambda(x)v$. It is easy to check that λ is a linear form on $\mathfrak{a}$. Now consider the subspace

$$W = \{w \in V \ / \ x(w) = \lambda(x)w \text{ for all } x \in \mathfrak{a}\}.$$

The subspace is non trivial.

Let us prove that $\mathfrak{g}$ stabilises W *i.e.* $y(W) \subset W$ for any $y \in \mathfrak{g}$. Let $w \in W$ and $x \in \mathfrak{a}$, we may compute $x(y(w)) = [x, y](w) + y(x(w)) = \lambda([x, y])w + \lambda(x)y(w)$. We therefore have to prove that $\lambda([x, y]) = 0$. For this we use the following

Lemma 7.3.2 *Let U be the vector space spanned by $(y^k(w))_{k\geq 0}$. Then for any $t \in \mathfrak{a}$, we have $\mathrm{Tr}(t|_U) = \lambda(t) \cdot \dim U$.*

Proof. Let us prove by induction that, for any $t \in \mathfrak{a}$ and for $i \geq 0$, there exists scalars $a_{i,j}(t) \in k$ such that

$$t(y^i(w)) = \lambda(t)y^i(w) + \sum_{j<i} a_{i,j}(t)y^j(w).$$

For $i = 0$ we have $t(w) = \lambda(t)w$, for $i = 1$, we have already seen the formula $t(y(w)) = \lambda([t, y])w + \lambda(t)y(w)$. By induction, we have

$$\begin{aligned} t(y^{i+1}(w)) &= [t, y](y^i(w)) + y(t(y^i(w))) \\ &= \lambda([t, y])y^i(w) + \textstyle\sum_{j<i} a_{i,j}([t, y])y^j(w) + y(\lambda(t)y^i(w) + \sum_{j<i} a_{i,j}(t)y^j(w)) \\ &= \lambda(t)y^{i+1}(w) + \lambda([t, y])y^i(w) + \textstyle\sum_{j<i} a_{i,j}([t, y])y^j(w) + \sum_{j<i} a_{i,j}(t)y^{j+1}(w) \end{aligned}$$

giving the induction. The familly $(y^i(w))_{i\geq 0}$ is a generating familly. We can extract a basis of it and the matrix of $t|_U$ in that basis is upper triangular with $\lambda(t)$ on the diagonal. The result follows. □

Now $0 = \mathrm{Tr}([x, y]|_U) = \dim U \cdot \lambda([x, y])$ and because $\mathrm{char} k = 0$ and U is non trivial (the space U contains w) we get $\lambda(x, y) = 0$. Therefore W is stable under $\mathfrak{g}$.

Take z in $\mathfrak{g}$ not in $\mathfrak{a}$, then z acts on W and because k is algebraically closed, it has an eigenvector $w \in W$. Now w is a common eigenvector for $\mathfrak{g}$. □

For the next result we also need k to be algebraically closed.

Corollary 7.3.3 *Assume that k is algebraically closed.*

Let V be a finite dimensional vector space and let $\mathfrak{g}$ be a solvable Lie subalgebra of $\mathfrak{gl}(V)$, then there exists a flag $V_\bullet$ of V such that $\mathfrak{g} \subset \mathfrak{t}(V_\bullet)$.

Proof. We proceed by induction on the dimension of V, if $\dim V = 0$, the result follows. By the previous Theorem, there is an common eigenvector v. Therefore V/kv is again a representation of $\mathfrak{g}$. The image of $\mathfrak{g}$ in $\mathfrak{gl}(V/kv)$ is solvable and by induction, there is a complete flag in V/kv stabilised by $\mathfrak{g}$. Its inverse image in V is also stabilised and the result follows. □

In the last result of this subsection, we do not need k to be algebraically closed.

Corollary 7.3.4 *A Lie algebra $\mathfrak{g}$ is solvable if and only if $\mathfrak{D}\mathfrak{g}$ is nilpotent.*

Proof. If $\mathfrak{D}\mathfrak{g}$ is nilpotent, then $\mathfrak{g}$ is the extension of $\mathfrak{g}/\mathfrak{D}\mathfrak{g}$, which is abelian therefore solvable, by $\mathfrak{D}\mathfrak{g}$ which is nilpotent therefore solvable. The result follows.

Conversely, consider the adjoint representation, its image is $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ and we have the equality $\mathfrak{D}(\mathfrak{g}/\mathfrak{z}(\mathfrak{g})) = \mathfrak{D}\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. Therefore $\mathfrak{D}\mathfrak{g}$ is a central extension of $\mathfrak{D}\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ by $\mathfrak{z}(\mathfrak{g})$ and it is enough to prove that $\mathfrak{D}\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is nilpotent. We may therefore assume that $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ (with $V = \mathfrak{g}$ for example).

Take $\overline{k}$ an algebraic closure of k. For the Lie algebra $\overline{\mathfrak{g}} = \mathfrak{g} \otimes_k \overline{k}$, the previous result gives $\overline{\mathfrak{g}} \subset \mathfrak{t}(\overline{V}_\bullet)$ for some complete flag $\overline{V}_\bullet$ of $\overline{V} = V \otimes_k \overline{k}$. Therefore we have $\mathfrak{D}\overline{\mathfrak{g}} \subset \mathfrak{n}(\overline{V}_\bullet)$ which is nilpotent. Any element in $\mathfrak{D}\mathfrak{g}$ is contained in $\mathfrak{D}\overline{\mathfrak{g}}$ is therefore nilpotent *i.e.*, by Corollary 5.2.5, the Lie algebra $\mathfrak{D}\mathfrak{g}$ is nilpotent. □

7.4 Cartan's criterion

We prove a characterisation with the Killing form of solvable Lie algebras.

Lemma 7.4.1 *Let V be a finite dimensional vector space and U and W two subspaces of $\mathfrak{gl}(V)$. Define the subset T of $\mathfrak{gl}(V)$ by*

$$T = \{t \in \mathfrak{gl}(V) \;/\; [t, U] \subset W\}.$$

Let $x \in T$ such that for all $t \in T$, we have $\mathrm{Tr}(xt) = 0$, then x is nilpotent.

Proof. We may assume that k is algebraically closed. Let $x = x_s + x_n$ be the Chevalley-Jordan decomposition of x. We want to prove that all the eigenvalues (λ_i) of x_s vanish. Let us denote by (e_i) the associated basis of eigenvectors.

Let M be the subvector space, over $\mathbb{Q}$, of k generated by the λ_i and let $f : \mathbb{Q}(\lambda_i) \to \mathbb{Q}$ be a linear form. We define t in $\mathfrak{gl}(V)$ by $t(e_i) = f(\lambda_i)e_i$. For $(E_{i,j})$ the canonical basis of $\mathfrak{gl}(V)$ associated to (e_i), we have

$$\begin{aligned}(\mathrm{ad}\, x_s)(E_{i,j}) &= (\lambda_i - \lambda_j)E_{i,j}\\ (\mathrm{ad}\, t)(E_{i,j}) &= (f(\lambda_i) - f(\lambda_j))E_{i,j} = f(\lambda_i - \lambda_j)E_{i,j}.\end{aligned}$$

Let $P \in \mathbb{Q}(T)$ be a polynomial such that $P(\lambda_i - \lambda_j) = f(\lambda_i - \lambda_j)$ and $P(0) = 0$. We have the equality $\mathrm{ad}\, t(E_{i,j}) = f(\lambda_i - \lambda_j)E_{i,j} = P(\lambda_i - \lambda_j)E_{i,j} = P(\mathrm{ad}\, x_s)(E_{i,j})$ therefore $\mathrm{ad}\, t = P(\mathrm{ad}\, x_s)$ and $\mathrm{ad}\, t(U) \subset W$. We thus have $t \in T$. We get $0 = \mathrm{Tr}(xt) = \sum \lambda_i f(\lambda_i)$. Applying f which is $\mathbb{Q}$-linear, we get $\sum f(\lambda_i)^2 = 0$ thus, because $f(\lambda_i) \in \mathbb{Q}$, we have $f(\lambda_i) = 0$ for all i and all linear form f. This implies that $\lambda_i = 0$ for all i. □

Theorem 7.4.2 *Let $\mathfrak{g}$ be a Lie algebra and V be a finite dimensional representation of $\mathfrak{g}$ and b the associated bilinear form. Then the image of $\mathfrak{g}$ in $\mathfrak{gl}(V)$ is solvable if and only if $\mathcal{D}\mathfrak{g} \subset \mathfrak{g}^\perp$ where $\perp$ is taken with respect to b.*

Proof. We may of course replace $\mathfrak{g}$ by its image and assume that the representation is injective.

Let $\overline{k}$ be an algebraic closure of k, let $\overline{\mathfrak{g}} = \mathfrak{g} \otimes_k \overline{k}$ and $\overline{V} = V \otimes_k \overline{k}$. By Corollary 7.3.3, there exists a complete flag $\overline{V}_\bullet$ of $\overline{V}$ such that $\overline{\mathfrak{g}} \subset \mathfrak{t}(\overline{V}_\bullet)$ and we have $\mathcal{D}\overline{\mathfrak{g}} \subset \mathfrak{n}(\overline{V}_\bullet)$. In particular for $x \in \mathfrak{g}$ and $y \in \mathcal{D}\mathfrak{g}$, the product xy lies in $\mathfrak{n}(\overline{V}_\bullet)$ and therefore $\mathrm{Tr}(xy) = 0$ *i.e.* $\mathcal{D}\mathfrak{g} \subset \mathfrak{g}^\perp$.

Conversely, assume that $\mathcal{D}\mathfrak{g} \subset \mathfrak{g}^\perp$. Consider the set T defined as in the previous lemma with $U = \mathfrak{g}$ and $W = \mathcal{D}\mathfrak{g}$. Let $t \in T$ and $x \in \mathcal{D}\mathfrak{g}$, we can write $x = \sum_i [y_i, z_i]$ with y_i and z_i in $\mathfrak{g}$. We compute

$$\mathrm{Tr}(tx) = \sum_i \mathrm{Tr}(t[y_i, z_i]) = \sum_i \mathrm{Tr}([t, y_i]z_i) = 0$$

because $[t, y_i] \in \mathcal{D}\mathfrak{g}$ and by hypothesis $\mathcal{D}\mathfrak{g} \subset \mathfrak{g}^\perp$. Thus x is nilpotent. By Corollary 5.2.5, the Lie algebra $\mathcal{D}\mathfrak{g}$ is nilpotent and by Corollary 7.3.4 the Lie algebra $\mathfrak{g}$ is solvable. □

Corollary 7.4.3 *Let $\mathfrak{g}$ be a Lie algebra such that for the Killing form $\kappa_\mathfrak{g}$ we have $\mathcal{D}\mathfrak{g} \subset \mathfrak{g}^\perp$, then $\mathfrak{g}$ is solvable.*

Proof. Consider the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. The Killing form is the associated bilinear form and by the previous result we know that $\mathrm{ad}\,(\mathfrak{g})$ is solvable. But the kernel of ad is $\mathfrak{z}(\mathfrak{g})$, which is abelian and thus solvable therefore $\mathfrak{g}$ is solvable. □

7.5 Exercices

Exercice 7.5.1 Prove the assertion in Example 7.1.5.

Exercice 7.5.2 Prove that if $\mathfrak{g}$ and $\mathfrak{g}'$ are Lie algebras, then $\mathfrak{r}(\mathfrak{g} \times \mathfrak{g}') = \mathfrak{r}(\mathfrak{g}) \times \mathfrak{r}(\mathfrak{g}')$.

Exercice 7.5.3 Let $\mathfrak{g}$ be a Lie algebra. Prove that $\mathfrak{r}(\mathfrak{g})$ is contained in any maximal solvable Lie subalgebra of $\mathfrak{g}$.

Exercice 7.5.4 Consider the dimension 3 Lie algebra $\mathfrak{g}$ defined over $\mathbb{R}$ in the basis (x, y, z) by the relations $[x, y] = z$, $[x, z] = -y$ and $[y, z] = 0$

(i) Prove that $\mathfrak{g}$ is solvable.

(ii) Prove that there is no decreasing sequence of ideals of dimensions 3, 2, 1, 0.

Exercice 7.5.5 Prove that in the only non commutative two dimensional Lie algebra $\mathfrak{g}$, there is a decreasing sequence of ideals of dimensions 2, 1, 0. In particular $\mathfrak{g}$ is solvable. Prove that it is not nlpotent.

Exercice 7.5.6 Assume $\mathrm{char} k = 2$

(i) Consider the $p \times p$ matrices

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

check that $[x, y] = x$ and that the Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(k^2)$ generated by x and y is solvable.

(ii) Prove that x and y have no common eigenvector giving a counterexample to Lie's Theorem in positive cxharacteristic.

(iii) Consider $\mathfrak{g}' = \mathfrak{g} \oplus k^2$ and define on $\mathfrak{g}'$ a Lie bracket by $[(f, v), (g, w)] = [f, g] + f(w) - g(v)$. Prove that $\mathfrak{g}'$ is solvable.

(iv) Prove that $\mathcal{D}\mathfrak{g}'$ is $kx \oplus k^2$ and is not nilpotent.

Exercice 7.5.7 Assume $\mathrm{char} k = p$ and consider the $p \times p$ matrices

$$x = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & 2 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & 0 & \ddots & 0 & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & p-3 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & p-2 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & p-1 \end{pmatrix}$$

prove again that the Lie algebra generated by x and y gives a counter example to Lie's Theorem.

Exercice 7.5.8 Let $\mathfrak{g}$ be a Lie algebra and define $\mathcal{D}^\infty\mathfrak{g} = \cap_i \mathcal{D}^i\mathfrak{g}$. Prove that $\mathfrak{g}/\mathcal{D}^\infty\mathfrak{g}$ is solvable.

Chapter 8

Semisimple Lie algebras

8.1 Definition

Definition 8.1.1 *A Lie algebra* $\mathfrak{g}$ *is called* semisimple *if the only commutative ideal of* $\mathfrak{g}$ *is* $\{0\}$.

Example 8.1.2 The 0-dimensional Lie algebra is semisimple.

Example 8.1.3 If $\mathfrak{g}$ and $\mathfrak{g}'$ are semisimple, the $\mathfrak{g} \times \mathfrak{g}'$ are also semisimple.

Fact 8.1.4 *For* $\mathfrak{g}$ *a semisimple Lie algebra, we have* $\mathfrak{z}(\mathfrak{g}) = 0$*, therefore the adjoint representation* $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is injective.*

8.2 First caracterisation of semisimple Lie algebras

Theorem 8.2.1 *Let* $\mathfrak{g}$ *be a Lie algebra, the following are equivalent:*
(ı) the Lie algebra is semisimple;
(ıı) the radical $\mathfrak{r}(\mathfrak{g})$ *vanishes;*
(ııı) the Killing form $\kappa_{\mathfrak{g}}$ *is non degenerate.*

Proof. Assume that $\mathfrak{g}$ is semisimple and let $\mathfrak{r}$ be its radical. Assume that $\mathfrak{r} \neq 0$ and let r be the biggest integer such that $\mathcal{D}^r\mathfrak{r} \neq 0$ (such an integer exists because $\mathfrak{r}$ is solvable). Then $\mathcal{D}^r\mathfrak{r}$ is abelian but $\mathcal{D}^r\mathfrak{r}$ being characteristic in the ideal $\mathfrak{r}$, it is an ideal of $\mathfrak{g}$, a contradiction.

Assume that $\mathfrak{r}(\mathfrak{g}) = 0$ and consider $\mathfrak{g}^\perp$ the orthogonal of $\mathfrak{g}$ for the Killing form. This is an ideal of $\mathfrak{g}$ by Proposition 4.5.4 (and even a characteristic ideal of $\mathfrak{g}$). Furthermore, the Killing form on $\mathfrak{g}^\perp$ is the restriction of $\kappa_{\mathfrak{g}}$ and therefore vanishes. In particular, we have $\mathcal{D}(\mathfrak{g}^\perp) \subset \mathfrak{g}^\perp = (\mathfrak{g}^\perp)^\perp$. By Theorem 7.4.2, the Lie algebra $\mathfrak{g}^\perp$ is solvable and contained in $\mathfrak{r}(\mathfrak{g})$.

Finally, assume that the Killing form is non degenerate and let $\mathfrak{a}$ be an abelian ideal in $\mathfrak{g}$. Let $x \in \mathfrak{g}$ and $y \in \mathfrak{a}$, we want to compute $\mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y)$. But $\mathrm{ad}\, x$ maps $\mathfrak{g}$ to $\mathfrak{g}$ and $\mathfrak{a}$ to $\mathfrak{a}$ while $\mathrm{ad}\, y$ maps $\mathfrak{g}$ to $\mathfrak{a}$ and $\mathfrak{a}$ to 0. Therefore $\mathrm{ad}\, x \circ \mathrm{ad}\, y$ maps $\mathfrak{g}$ to $\mathfrak{a}$ and $(\mathrm{ad}\, x \circ \mathrm{ad}\, y)^2$ maps $\mathfrak{g}$ to 0. Therefore $\kappa_{\mathfrak{g}}(x, y) = 0$ and $\mathfrak{a} \subset \mathfrak{g}^\perp$ and the result follows. □

Corollary 8.2.2 *For* $\mathfrak{g}$ *semisimple, we have* $\mathcal{D}\mathfrak{g} = \mathfrak{g}$.

Proof. Let $x \in (\mathcal{D}\mathfrak{g})^\perp$ and let y and z in $\mathfrak{g}$. We have the equalities $\kappa_{\mathfrak{g}}([x, y], z) = \kappa_{\mathfrak{g}}(x, [y, z]) = 0$ therefore $[x, y]$ is orthogonal to $\mathfrak{g}$ and because the Killing form is non degenerate $[x, y] = 0$. Therefore $x \in \mathfrak{z}(\mathfrak{g})$ but $\mathfrak{z}(\mathfrak{g}) = 0$ thus $x = 0$. Therefore $(\mathcal{D}\mathfrak{g})^\perp = 0$ and $\mathcal{D}\mathfrak{g} = \mathfrak{g}$. □

Corollary 8.2.3 *Let $\mathfrak{g} \to \mathfrak{gl}(V)$ be a representation with $\mathfrak{g}$ semisimple, then it factors through $\mathfrak{sl}(V)$.*

Proof. Indeed, for $x \in \mathfrak{g}$, by the previous Corollary we may write $x = \sum_i [y_i, z_i]$. Therefore we have $\mathrm{Tr}(x_V) = \mathrm{Tr}(\sum_i [y_i, z_i]_V) = \mathrm{Tr}([(y_i)_V, (z_i)_V]) = 0$ □

Proposition 8.2.4 *Let $\mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful representation with $\mathfrak{g}$ semisimple. Then the quadratic form b induced by this representation is non degenerate.*

Proof. Let $\mathfrak{a} = \mathfrak{g}^{\perp_b}$ be the orthogonal of $\mathfrak{g}$ for b. It is an ideal in $\mathfrak{g}$. The representation induces a faithful representation $\mathfrak{a} \to \mathfrak{gl}(V)$ such that the induced form vanishes. Therefore, we have $\mathfrak{a}^{\perp_b} = \mathfrak{a} \supset \mathcal{D}\mathfrak{a}$. By Theorem 7.4.2, the Lie algebra $\mathfrak{a}$ is solvable thus trivial because $\mathfrak{g}$ is semisimple. □

Corollary 8.2.5 *(ı) Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ a semisimple Lie subalgebra of $\mathfrak{g}$. Then $\mathfrak{a}^\perp$ the orthogonal for the Killing form is a supplementary for $\mathfrak{a}$ and we have $[\mathfrak{a}, \mathfrak{a}^\perp] \subset \mathfrak{a}^\perp$.*

(ıı) If furthermore $\mathfrak{a}$ is an ideal in $\mathfrak{g}$, then $\mathfrak{a}^\perp$ is also an ideal and $\mathfrak{g} = \mathfrak{a} \times \mathfrak{a}^\perp$. Furthermore $\mathfrak{a}^\perp = \mathfrak{z}_\mathfrak{g}(\mathfrak{a})$.

Proof. *(ı)* Consider the representation $\mathfrak{a} \to \mathfrak{gl}(\mathfrak{g})$ defined by $x \mapsto \mathrm{ad}_\mathfrak{g}(x)$. This representation is faithful: if x is in the kernel, then $0 = \mathrm{ad}_\mathfrak{g}(x)(y) = [x, y]$ for all $y \in \mathfrak{g}$ in particular $x \in \mathfrak{z}(\mathfrak{a}) = 0$. By the above Proposition, the associated bilinear form, which is $\kappa_\mathfrak{g}|_\mathfrak{a}$ is non degenerate and its kernel is $\mathfrak{a} \cap \mathfrak{a}^\perp$. It is trivial because $\kappa_\mathfrak{g}|_\mathfrak{a}$ is non degenerate.

For x in $\mathfrak{a}^\perp$ and y and z in $\mathfrak{a}$ we have $\mathrm{Tr}([x, y], z) = \mathrm{Tr}(x, [y, z]) = 0$ because $[y, z] \in \mathfrak{a}$. The desired inclusion follows.

(ıı) If $\mathfrak{a}$ is an ideal then $\mathfrak{a}^\perp$ is also an ideal and we have for x and y in $\mathfrak{g}$ unique writings $x = x_1 + x_2$ and $y = y_1 + y_2$ where x_1 and y_1 are in $\mathfrak{a}$ while x_2 and y_2 are in $\mathfrak{a}^\perp$. We get $[x, y] = [x_1, y_1] + [x_2, y_2] + [x_1, y_2] + [x_2, y_1]$. But the last two terms are in $\mathfrak{a} \cap \mathfrak{a}^\perp$ therefore vanish thus $\mathfrak{g} = \mathfrak{a} \times \mathfrak{a}^\perp$. Finally, because $\mathfrak{a}^\perp$ commutes with $\mathfrak{a}$ and $\mathfrak{z}(\mathfrak{a}) = 0$, the last statement follows. □

Corollary 8.2.6 *Any extension of a semisimple Lie algebra by a semisimple Lie algebra is trivial and semisimple.*

Proof. Indeed, let $0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{b} \to 0$ be such an extension. By the previous result, we have $\mathfrak{g} = \mathfrak{a} \times \mathfrak{b}$ which is semisimple. □

Corollary 8.2.7 *Let $\mathfrak{g}$ be a semisimple Lie algebra, then any derivation of $\mathfrak{g}$ is an inner derivation.*

Proof. Indeed, the Lie algebra $\mathrm{ad}\,\mathfrak{g}$ image of $\mathfrak{g}$ under the adjoint representation is isomorphic to $\mathfrak{g}$ and therefore semisimple. But it is also an ideal in $\mathfrak{Der}(\mathfrak{g})$ the Lie algebra of derivations of $\mathfrak{g}$ and we have

$$\mathfrak{Der}(\mathfrak{g}) = \mathrm{ad}\,\mathfrak{g} \times \mathfrak{z}_{\mathfrak{Der}(\mathfrak{g})}(\mathrm{ad}\,\mathfrak{g}).$$

Let us compute $\mathfrak{z}_{\mathfrak{Der}}(\mathrm{ad}\,\mathfrak{g})$ the centraliser of $\mathfrak{g}$ in the Lie algebra of derivations. Let $D \in \mathfrak{z}_{\mathfrak{Der}}(\mathrm{ad}\,\mathfrak{g})$. We have $[D, \mathrm{ad}\,x] = 0$ for all $x \in \mathfrak{g}$, but $[D, \mathrm{ad}\,x] = \mathrm{ad}\,D(x)$ therefore $D = 0$ and $\mathfrak{z}_{\mathfrak{Der}}(\mathrm{ad}\,\mathfrak{g}) = 0$. □

8.3 Semisimplicity of representations

In this section, we prove the following result of H. Weyl

Theorem 8.3.1 *Any finite dimensional representation of a semisimple Lie algebra is semisimple.*

Proof. To prove this result we need several lemmas

Lemma 8.3.2 *Let $\mathfrak{g}$ be a semisimple Lie algebra, then the adjoint representation is semisimple.*

Proof. Indeed, the $\mathfrak{a}$ be a subrepresentation of $\mathfrak{g}$ (*i.e.* $\mathfrak{a}$ is an ideal of $\mathfrak{g}$). Then $\mathfrak{a}^\perp$ is again an ideal of $\mathfrak{g}$ and $\mathfrak{a} \cap \mathfrak{a}^\perp$ is an abelian ideal therefore trivial. We have $\mathfrak{g} = \mathfrak{a} \times \mathfrak{a}^\perp$ and the result follows. □

Lemma 8.3.3 *Let $\mathfrak{g}$ be a Lie algebra. The following are equivalent:*

(ı) All the finite dimensional representations of $\mathfrak{g}$ are semisimple.

(ıı) Given a finite dimensional representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ and a subspace W of codimension 1 in V with $\rho(x)(V) \subset W$ for all $x \in \mathfrak{g}$, there exist a supplementary line L for W stable under $\mathfrak{g}$.

Proof. The first condition implies the second one. Conversely, assume that the second condition holds and let $\sigma : \mathfrak{g} \to \mathfrak{gl}(E)$ be a finite dimensional representation and F a subrepresentation of E. Consider the induced representation $\tau : \mathfrak{g} \to \mathrm{Hom}(E, F)$. Recall that $x_{\mathrm{Hom}(E,F)} \cdot \phi = x_F \circ \phi - \phi \circ x_E$. Let V resp. W be the subspace of maps $\phi \in \mathrm{Hom}(E, F)$ such that the restriction $\phi|_F$ is a scalar multiple of Id_F (respectively $\phi|_F = 0$).

The spaces V and W are subrepresentations of $\mathrm{Hom}(E, F)$. Indeed, if $\phi|_F = \lambda \mathrm{Id}_F$, then we have $(x_{\mathrm{Hom}(E,F)} \cdot \phi)|_F = x_F \circ \phi|_F - \phi|_F \circ x_F = x_F \circ \lambda \mathrm{Id}_F - \lambda \mathrm{Id}_F \circ x_F = 0$. Let us denote by $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ the induced representation. Then we are in position to apply our hypothesis, therefore there exists a line $L = k\phi$ in V supplementary to W and stable under $\mathfrak{g}$ *i.e.* there exists a $\phi \in \mathrm{Hom}(E, F)$ with $\phi|_F = \lambda \mathrm{Id}_F$ with $\lambda \neq 0$ and $\rho(x)(\phi) \subset L \cap W = 0$. Therefore, we have for all $x \in \mathfrak{g}$ the equality $x_E \circ \phi = \phi \circ x_E$. Let G be the kernel of ϕ, it is supplementary to F. Because of the last commutation relation G is a subrepresentation of E. □

Lemma 8.3.4 *Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite dimensional representation and let W be subspace of codimension 1 in V with $\rho(x)(V) \subset W$ for all $x \in \mathfrak{g}$. Then there exist a supplementary line L for W stable under $\mathfrak{g}$.*

Proof. Let $\sigma : \mathfrak{g} \to \mathfrak{gl}(W)$ be the restriction of ρ to W. If $\sigma = 0$, then for all x and y in $\mathfrak{g}$, we have $\rho(x)\rho(y) = 0$ therefore $\rho([x, y]) = 0$. We get $\rho(\mathfrak{g}) = \rho(\mathfrak{D}\mathfrak{g}) = 0$ and the result is true. We may thus assume $\sigma \neq 0$.

Let us first assume that W is simple for the representation σ. Let $\mathfrak{a}$ be the kernel of σ which is an ideal of $\mathfrak{g}$. By Lemma 8.3.2, there is an ideal $\mathfrak{h}$ supplementary to $\mathfrak{a}$. The restriction of σ to $\mathfrak{h}$ is faithful. By Proposition 8.2.4, the associated quadratic form b is non degenerate. We may therefore define the Casimir element $c \in U(\mathfrak{g})$ associated to $\mathfrak{h}$ and W (see Definition 4.6.5). By Proposition 4.6.6, the element c_W is an automorphism of W. But we have $\rho(c)(V) \subset W$ therefore $\ker \rho(c)$ Is a line L supplementary to W and because c lies in the center of $U(\mathfrak{g})$, this line is $\mathfrak{g}$-invariant.

In general, we proceed by induction on the dimension of V. Let $\tau : \mathfrak{g} \to \mathfrak{gl}(U)$ be a minimal non trivial $\mathfrak{g}$-stable subspace of W. Then U is simple. For the quotient representation V/U, we have by induction hypothesis a $\mathfrak{g}$-stable line Z/U supplementary to W/U. Its inverse image Z is $\mathfrak{g}$-stable. The trace $W \cap Z$ of W in Z is U and we have $\tau(x)(Z) \subset U$ for all $x \in \mathfrak{g}$. As U is simple the above argument yields a $\mathfrak{g}$-stable line L supplementary to U in Z. The line L is supplementary to W in V. □

The Theorem follows from the combinaison of Lemma 8.3.3 and Lemma 8.3.4. □

Corollary 8.3.5 *A Lie algebra $\mathfrak{g}$ is semisimple if and only if all its finite dimensional representations are semisimple.*

Proof. The previous theorem implies that if $\mathfrak{g}$ is semisimple, then any finite dimensional representation is semisimple. Conversely, suppose that $\mathfrak{g}$ is not semisimple, we need to find a non semisimple representtion of $\mathfrak{g}$. If the adjoint representation is not semisimple, we are done. Otherwise, let $\mathfrak{a}$ be an abelian ideal in $\mathfrak{g}$ and $\mathfrak{b}$ a supplementary ideal. Let $x \in \mathfrak{a}$ non zero and $\mathfrak{h}$ a supplementary for kx in $\mathfrak{a}$. We have as Lie algebra the isomorphisms $\mathfrak{g} \simeq \mathfrak{a} \times \mathfrak{b} \simeq kx \times \mathfrak{h} \times \mathfrak{b}$. In particular, any representation of kx induces a representation of $\mathfrak{g}$. But the representation of kx in k^2 given by

$$x_{k^2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not semisimple and the same holds for $\mathfrak{g}$. □

8.4 Simple Lie algebras

We recall the definition of simple algebras, for Lie algebras, we want to avoid the 1-dimensional Lie algebra to be simple therefore we slighly modify the definition.

Definition 8.4.1 *An algebra A is called* simple *if there is no non trivial ideal (or equivalently A is simple as left A-module).*

Definition 8.4.2 *An Lie algebra is called* simple *if it is not abelian and has no non trivial ideal.*

Example 8.4.3 A simple Lie algebra is semisimple.

Proposition 8.4.4 *A Lie algebra $\mathfrak{g}$ is semisimple if and only if it is a product of simple Lie algebras.*

Proof. A product of simple Lie lagebras is semisimple. Conversely, let $\mathfrak{g}$ be a semisimple Lie algebra. The adjoint representation is completely reducible therefore $\mathfrak{g}$ is a direct sum of simple representations $\mathfrak{g}_i$. The $\mathfrak{g}_i$ are ideals of $\mathfrak{g}$ and therefore $\mathfrak{g}$ is the product of the $\mathfrak{g}_i$.

Let us prove that the $\mathfrak{g}_i$ are simple. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}_i$, it is also an ideal of $\mathfrak{g}$ because $\mathfrak{g}$ is the product of the $\mathfrak{g}_i$. But $\mathfrak{g}_i$ is simple as a $\mathfrak{g}$-representation therefore $\mathfrak{a} = 0$ or $\mathfrak{a} = \mathfrak{g}_i$. □

Corollary 8.4.5 *A semisimple Lie algebra $\mathfrak{g}$ is the product of its simple ideals $(\mathfrak{g}_i)_{i \in [1,n]}$. Any ideal is the product of some of the $\mathfrak{g}_i$.*

Proof. By the last proposition, $\mathfrak{g}$ is the product of simple Lie algebras $\mathfrak{g}_i$. with $\mathfrak{g}_i$ a simple ideal of $\mathfrak{g}$. Let us remark that $\mathfrak{z}_\mathfrak{g}(\mathfrak{g}_i) = \oplus_{j \neq i} \mathfrak{g}_j$. Indeed we easily have $\oplus_{j \neq i} \mathfrak{g}_j \subset \mathfrak{z}_\mathfrak{g}(\mathfrak{g}_i)$. But conversely if $x = (x_1, \cdots, x_n) \in \mathfrak{z}_\mathfrak{g}(\mathfrak{g}_i)$, then $[x_i, \mathfrak{g}_i] = 0$ and therefore $x_i = 0$ as $\mathfrak{g}_i$ is simple.

Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$, then as $\mathfrak{g}_i$ is simple, we have the alternative $\mathfrak{g}_i \subset \mathfrak{a}$ or $\mathfrak{a} \cap \mathfrak{a}_i = 0$. If $\mathfrak{g}_i \not\subset \mathfrak{a}$ then $[\mathfrak{a}, \mathfrak{g}_i] \subset \mathfrak{a} \cap \mathfrak{g}_i = 0$, thus $\mathfrak{a} \subset \mathfrak{z}_\mathfrak{g}(\mathfrak{g}_i)$. Therefore the ideal $\mathfrak{a}$ is the product of the $\mathfrak{g}_i$ contained in it. The simple ideals are the $\mathfrak{g}_i$. □

Definition 8.4.6 *The simple ideals of a semisimple Lie algebra $\mathfrak{g}$ are the* simple components *of $\mathfrak{g}$.*

Corollary 8.4.7 *Any ideal and any quotient of a semisimple Lie algebra is again semisimple.*

Corollary 8.4.8 *Let $f : \mathfrak{g} \to \mathfrak{g}'$ be a surjective morphism of Lie algebras then $f(\mathfrak{r}(\mathfrak{g})) = \mathfrak{r}(\mathfrak{g}')$.*

Proof. The image $f(\mathfrak{r}(\mathfrak{g}))$ is a solvable ideal of $\mathfrak{g}'$ thus $f(\mathfrak{r}(\mathfrak{g})) \subset \mathfrak{r}(\mathfrak{g}')$. The Lie algebra $\mathfrak{g}/\mathfrak{r}(\mathfrak{g})$ is semisimple and $\mathfrak{g}'/\mathfrak{r}(\mathfrak{g}')$ is a quotient therefore semisimple and we get the reverse inclusion. □

8.5 Jordan-Chevalley decomposition

Proposition 8.5.1 *Let V be a finite dimensional vector space and let $\mathfrak{g}$ be a semisimple Lie subalgebra of $\mathfrak{gl}(V)$. Then for any element $x \in \mathfrak{g}$, its semisimple part x_s and its nilpotent part x_n are in $\mathfrak{g}$.*

Proof. Let us first remark that we can assume k to be algebraically closed. Indeed, the Jordan-Chevalley decomposition of $x \in \mathfrak{gl}(V)$ is the restriction of the Jordan-Chevalley of $x \otimes 1 \in \mathfrak{gl}(V \otimes_k K)$. Furthermore, the Killing form of $\mathfrak{g} \otimes_k K$ is given by $\kappa_{\mathfrak{g}\otimes_k K}(x \otimes \lambda, y \otimes \mu) = \kappa_{\mathfrak{g}}(x, y) \otimes \lambda\mu$ and therefore $\kappa_{\mathfrak{g}\otimes_k K}$ is non-degenerate if and only if $\kappa_{\mathfrak{g}}$ is non-degenerate.

For any sub-$\mathfrak{g}$-representation W of V, let us denote by $\mathfrak{g}_W$ the Lie subalgebra of $\mathfrak{sl}(W)$ of elements stabilising W. Because $\mathfrak{g}$ is semisimple, we have $\mathfrak{g} \subset \mathfrak{sl}(W)$ therefore $\mathfrak{g} \subset \mathfrak{g}_W$.

Let $x \in \mathfrak{g}$ and let $x = x_s + x_n$ be its Jordan-Chevalley decomposition with x_s and x_n in $\mathfrak{gl}(V)$ and let W be any sub-$\mathfrak{g}$-representation of V. We have $x_s(W) \subset W$ and $x_n(W) \subset W$ (see Theorem 6.2.1). Furthermore, as x_n is nilpotent, we have $\mathrm{Tr}((x_n)|_W) = 0$ and we get $\mathrm{Tr}((x_s)|_W) = \mathrm{Tr}(x|_W) - \mathrm{Tr}((x_n)|_W) = 0$. Therefore x_s and x_n are also in $\mathfrak{g}_W$.

Let $\mathfrak{n}_{\mathfrak{gl}(V)}(\mathfrak{g})$ be the normaliser of $\mathfrak{g}$ in $\mathfrak{gl}(V)$ *i.e.* the ideal of all elements $y \in \mathfrak{gl}(V)$ such that $[y, x] \in \mathfrak{g}$ for any $x \in \mathfrak{g}$. Let us define the Lie algebra

$$\mathfrak{g}^* = \mathfrak{n}_{\mathfrak{gl}(V)}(\mathfrak{g}) \cap \bigcap_W \mathfrak{g}_W$$

where W runs in the set of subrepresentations of V. By the above arguments, we have $\mathfrak{g} \subset \mathfrak{g}^*$ and x_s and x_n are in $\mathfrak{g}^*$. We therefore only need to prove that $\mathfrak{g} = \mathfrak{g}^*$. The Lie algebra $\mathfrak{g}$ is an ideal in $\mathfrak{g}^*$ and is semisimple. By Corollary 8.2.5 we have $\mathfrak{g}^* = \mathfrak{g} \times \mathfrak{z}_{\mathfrak{g}^*}(\mathfrak{g})$. Let $y \in \mathfrak{z}_{\mathfrak{g}^*}(\mathfrak{g})$ and let W be a minimal subset of V stable under $\mathfrak{g}$. Let λ be an eigenvalue for y in W. The eigenspace $W_\lambda(y) = \ker(y|_W - \lambda \mathrm{Id}_W)$ is $\mathfrak{g}$-stable and non trivial. But W is simple thus $W_\lambda(y|_W) = W$ *i.e.* $y|_W = \lambda \mathrm{Id}_W$. But $\mathrm{Tr}(y|_W) = 0$ therefore (recall that $\mathrm{char} k = 0$), we have $\lambda = 0$ thus $y|_W = 0$. As V is semisimple, it is a direct sum of simple subrepresentations thus $y = 0$ and the result follows. □

Corollary 8.5.2 *Let V be a finite dimensional vector space and $\mathfrak{g}$ a semisimple Lie subalgebra of $\mathfrak{gl}(V)$. Then an element $x \in \mathfrak{g}$ is semisimple (resp. nilpotent) if and only if* $\mathrm{ad}\, x$ *is.*

Proof. Let $x = x_s + x_n$ be the Jordan-Chevalley decomposition of x. Then $\mathrm{ad}\, x = \mathrm{ad}\, x_s + \mathrm{ad}\, x_n$ is the Jordan-Chevalley decomposition of $\mathrm{ad}\, x$ by Lemma 6.2.3. If x is semisimple (resp. nilpotent), then $x_n = 0$ (resp. $x_s = 0$) and we get that $\mathrm{ad}\, x_n = 0$ (resp. $\mathrm{ad}\, x_s = 0$) thus $\mathrm{ad}\, x$ is semisimple (resp. nilpotent). Conversely, if $\mathrm{ad}\, x$ is semisimple (resp. nilpotent), then $\mathrm{ad}\, x_n = 0$ (resp. $\mathrm{ad}\, x_s = 0$) and because $\mathfrak{g}$ is semisimple, the adjoint representation is faithful therefore we get that $x_n = 0$ (resp. $x_s = 0$) thus x is semisimple (resp. nilpotent). □

Definition 8.5.3 *Let $\mathfrak{g}$ be a semisimple Lie algebra. An element $x \in \mathfrak{g}$ is called* semisimple *(resp.* nilpotent*) if for any finite dimensional representation V of $\mathfrak{g}$, the element x_V is semisimple (resp. nilpotent).*

Proposition 8.5.4 *Let $f : \mathfrak{g} \to \mathfrak{g}'$ be a morphism between two semisimple Lie algebras, then if $x \in \mathfrak{g}$ is semisimple (resp. nilpotent), then so is $f(x)$.*

If furthermore f is surjective, then any semisimple (resp. nilpotent) element in $\mathfrak{g}'$ is the image of a semisimple (resp. nilpotent) element in $\mathfrak{g}$.

Proof. Let V be a representation of $\mathfrak{g}'$, then it is also a representation of $\mathfrak{g}$ and $f(x)_V = x_V$. The first part of the proposition follows.

If f is surjective, we have an exact sequence $0 \to \ker f \to \mathfrak{g} \to \mathfrak{g}' \to 0$ and by Corollary 8.2.6, the extension is trivial thus there exists $f' : \mathfrak{g}' \to \mathfrak{g}$ a morphism such that $f \circ f' = \mathrm{Id}_{\mathfrak{g}'}$. Then for $x' \in \mathfrak{g}'$, we have $x' = f(f'(x'))$ and $f'(x')$ is semisimple (resp. nilpotent) when x' is. □

Theorem 8.5.5 *Let $\mathfrak{g}$ be a semisimple Lie algebra.*

(ı) An element x is semisimple (resp. nilpotent) if and only if there exists a faithful finite dimensional representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ such that $\rho(x)$ is semisimple (resp. nilpotent).

(ıı) Any element $x \in \mathfrak{g}$ can be written as the sum $x = x_s + x_n$ where $x_s \in \mathfrak{g}$ is semisimple, $x_n \in \mathfrak{g}$ is nilpotent and $[x_s, x_n] = 0$.

Proof. *(ı)* If x is semisimple (resp. nilpotent), the $\mathrm{ad}\, x$ is and as the adjoint representation is faithful, the result follows.

Conversely, if there is a faithful representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ as above and assume that $\rho(x)$ is semisimple (resp. nilpotent). Remark that $\mathrm{ad}\, x$ is semisimple (resp. nilpotent) by Corollary 8.5.2.

Let $\sigma : \mathfrak{g} \to \mathfrak{gl}(W)$ be any finite dimensional representation of $\mathfrak{g}$, let $\mathfrak{a}$ be the kernel of σ and let $\mathfrak{b}$ be a supplementary ideal of $\mathfrak{a}$ which is also semisimple. Denote by $\pi : \mathfrak{g} \to \mathfrak{b}$ be the projection, we have $\mathrm{ad}_{\mathfrak{b}}\pi(x) = (\mathrm{ad}\, x)|_{\mathfrak{b}}$ therefore $\mathrm{ad}_{\mathfrak{b}}\pi(x)$ is semisimple (resp. nilpotent). But the restriction of σ to $\mathfrak{b}$ is faithful therefore by Corollary 8.5.2 the element $\sigma(\pi(x))$ is semisimple (resp. nilpotent) and the result follows because $\sigma(x) = \sigma(\pi(x))$.

(ıı) Take the Jordan-Chevalley decomposition in the adjoint representation. □

8.6 Examples of semisimple Lie algebras

We shall first state a very general result without proof and see how easy it becomes to prove that classical Lie algebra are semisimple. We shall give a proof for $\mathfrak{sl}(V)$ of the semisimplicity the other cases can be done in the same way as Exercice.

Theorem* 8.6.1 *Let $\mathfrak{g}$ be a Lie algebra, the following are equivalent:*

(ı) $\mathcal{D}\mathfrak{g}$ is semisimple;

(ıı) $\mathfrak{g}$ is a product of a semisimple and a commutative algebra;

(ııı) $\mathfrak{g}$ has a finite dimensional representation whose associated bilinear form is non degenerate;

(ıv) $\mathfrak{g}$ has a faithful semisimple finite dimensional representation;

(v) $\mathfrak{r}(\mathfrak{g})$ is the center of $\mathfrak{g}$.

Definition 8.6.2 *A Lie algebra satisfying one of the above properties is called* reductive.

Corollary* 8.6.3 *A reductive Lie algebra is semisimple if and only if its center is trivial.*

Proposition 8.6.4 *Let V be a finite dimensional vector space, then $\mathfrak{gl}(V)$ is reductive and $\mathfrak{sl}(V)$ is semisimple.*

Proof. The representation of $\mathfrak{gl}(V)$ (resp. $\mathfrak{sl}(V)$) in V is simple therefore $\mathfrak{gl}(V)$ and $\mathfrak{sl}(V)$ are reductive. Furthermore, because $\mathfrak{z}(\mathfrak{sl}(V))$ is trivial the last assertion follows.

We now give a proof without using Theorem* 8.6.1. Let $(e_i)_{i\in[1,n]}$ be a basis for V and let us define $E_{i,j} \in \mathfrak{gl}(V)$ by $E_{i,j}(e_k) = \delta_{j,k}e_i$. We have $E_{i,j} \in \mathfrak{sl}(V)$ for $i \neq j$ and $H_i = E_{i,i} - E_{i+1,i+1} \in \mathfrak{sl}(V)$ for all $i \leq n-1$. These elements are clearly linearly independent and generate a (n^2-1)-dimensional subspace in $\mathfrak{sl}(V)$. As $\mathfrak{sl}(V)$ is defined by the unique equation $\mathrm{Tr}(x) = 0$, it is of codimension 1 in $\mathfrak{gl}(V)$ and therefore the above elements form a basis for $\mathfrak{sl}(V)$. We shall call it the *standard basis* for $\mathfrak{sl}(V)$. To prove the result, we only need to prove that $\kappa_{\mathfrak{sl}(V)}$ is non-degenerate.

One way to conclude is by brute force computing the following

$$\begin{array}{lll} \kappa_{\mathfrak{sl}(V)}(E_{i,j}, E_{k,l}) & = 2n \cdot \delta_{i,l}\delta_{j,k} & \text{for } i \neq j \text{ and } k \neq l, \\ \kappa_{\mathfrak{sl}(V)}(E_{i,j}, H_k) & = 0 & \text{for } i \neq j \text{ and } k \in [1, n-1], \\ \kappa_{\mathfrak{sl}(V)}(H_i, H_j) & = 2n \cdot \left\{ \begin{array}{ll} -\delta_{i,j+1} - \delta_{i+1,j} & \text{for } i \neq j \text{ in } [1, n-1] \\ 2 & \text{for } i = j \text{ in } [1, n-1]. \end{array} \right. \end{array}$$

On the subspace spanned by the $E_{i,j}$ with $i \neq j$, the quadratic form is non degenerate and the dual basis is $E_{j,i}$. On the space spanned by the H_i for $i \in [1, n-1]$, which is orthogonal to the previous one, the Killing form has the following matrix

$$2n \cdot \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

whose determinant is $2^n n^{n+1}$ (by induction, if Δ_n is the determinant of the matrix on the right, then developping with respect to the first line yields the equation $\Delta_n = 2\Delta_{n-1} - \Delta_{n-2}$ and we get $\Delta_n = n$ and the result follows). Therefore the Killing form is non degenerate and $\mathfrak{sl}(V)$ is semisimple.

Another way to proceed, which will be more useful for other classical Lie algebras is to consider the action of the element $E = \sum_i t_i E_{i,i}$ where $\sum_i t_i = 0$. We have the formula $[E, E_{i,j}] = (t_i - t_j)E_{i,j}$. Because the Killing form is invariant, we get

$$(t_j - t_i)\kappa_{\mathfrak{sl}(V)}(E_{i,j}, E_{k,l}) = \kappa_{\mathfrak{sl}(V)}([E_{i,j}, E], E_{k,l}) = \kappa_{\mathfrak{sl}(V)}(E_{i,j}, [E, E_{k,l}]) = (t_k - t_l)\kappa_{\mathfrak{sl}(V)}(E_{i,j}, E_{k,l})$$

and we deduce that $\kappa_{\mathfrak{sl}(V)}(E_{i,j}, E_{k,l}) = 0$ for $(i, j) \neq (l, k)$. We are therefore reduced to prove that the resctriction of the Killing form to $\mathfrak{h} = \oplus_i H_i$ is non degenerate and we proceed as above. □

Proposition 8.6.5 *Let b be a symmetric (resp. antisymmetric) non degenerate bilinear form on a finite dimensional vector space V, then the subspace*

$$\mathfrak{gl}_b(V) = \{x \in \mathfrak{gl}(V) \ / \ b(x(v), v') + b(v, x(v')) = 0 \text{ for all } v \text{ and } v' \text{ in } V\}$$

is a reductive subalgebra of $\mathfrak{gl}(V)$ and even of $\mathfrak{sl}(V)$. It is semissimple except for $\dim V = 2$ and b symmetric.

Definition 8.6.6 *When the non degenerate bilinear form b is symmetric, we write $\mathfrak{so}(V, b) = \mathfrak{gl}_b(V)$ and when b is antisymmetric, we write $\mathfrak{sp}(V, b) = \mathfrak{gl}_b(V)$.*

Proof. The bilinear form b induces an isomorphism $\varphi : V \to V^\vee$ defined by $\varphi(v)(v') = b(v, v')$. For any $x \in \mathfrak{gl}(V)$, let us define $x^* \in \mathrm{End}(V)$ by $x^* = \varphi^{-1} \circ {}^t x \circ \varphi$. This is the *adjoint* of x with respect to b. By definition, we have the equality

$$b(x^*(v'), v) = b(\varphi^{-1} \circ {}^t x \circ \varphi(v'), v) = {}^t x \circ \varphi(v')(v) = \varphi(v')(x(v)) = b(v', x(v))$$

for all v and v' in V. Because b is non degenerate, this condition determines x^*. Remark that the map $x \mapsto x^*$ is linear, is involutive: $(x^*)^* = x$ and antimultiplicative: $(xy)^* = y^*x^*$.

The condition $x \in \mathfrak{gl}_b(V)$ translates into $b(-x(v'), v) = b(v', x(v)$ for all v and v' in V therefore x is in $\mathfrak{gl}_b(V)$ if and only if $x^* = -x$ or $x + x^* = 0$. By the above properties of the adjoint, we get that $\mathfrak{gl}_b(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$.

Lemma 8.6.7 *We have* $\mathrm{Tr}(x) = \mathrm{Tr}(x^*)$.

Proof. We put $\epsilon = 1$ if b is symmetric and $\epsilon = -1$ if b is antisymmetric. Let (e_i) be a basis for V and $(e_i^\vee)$ be the dual basis in $V^\vee$ (by definition $e_i^\vee(e_j) = \delta_{i,j}$). Recall that we have by definition $\mathrm{Tr}(x^*) = \sum_i e_i^\vee(x^*(e_i))$ (this is independent of the choice of the basis (e_i)).

Let $e'_i = \varphi^{-1}(e_i^\vee)$, the family (e'_i) is again a basis of V and we have $e_i^\vee(v) = b(e'_i, v)$. Let us prove that $(\epsilon\varphi(e_i))$ is the dual basis of (e'_i), in symbols $(e'_i)^\vee = \epsilon\varphi(e_i)$. Indeed, we have

$$\epsilon\varphi(e_i)(e'_j) = \epsilon b(e_i, e'_j) = \epsilon^2 b(e'_j, e_i) = e_j^\vee(e_i) = \delta_{i,j}.$$

Now we compute:

$$\begin{aligned} \mathrm{Tr}(x^*) &= \textstyle\sum_i e_i^\vee(x^*(e_i)) = \sum_i b(e'_i, x^*(e_i)) \\ &= \epsilon \textstyle\sum_i b(x^*(e_i), e'_i) = \epsilon \sum_i b(e_i, x(e'_i)) \\ &= \textstyle\sum_i \epsilon\varphi(e_i)(x(e'_i)) = \sum_i (e'_i)^\vee(x(e'_i)) \\ &= \mathrm{Tr}(x) \end{aligned}$$

□

The lemma implies that $x \in \mathfrak{sl}(V)$. Indeed we have $\mathrm{Tr}(x) = \mathrm{Tr}(x^*) = -\mathrm{Tr}(x)$.

Let us now prove that the bilinear form induced by the representation V is non degenerate. This form is non degenerate on $\mathfrak{sl}(V)$ by Proposition 8.2.4 and because $\mathfrak{sl}(V)$ is semisimple. Let $x \in \mathfrak{gl}_b(V)$ be in the kernel of that form *i.e.* $\mathrm{Tr}(xy) = 0$ for all $y \in \mathfrak{gl}_b(V)$. Remark that for any $y \in \mathfrak{sl}(V)$ we may construct elements in $\mathfrak{gl}_b(V)$ by taking $y - y^*$. Indeed, we have $(y - y^*)^* = y^* - (y^*)^* = -(y - y^*)$. For $y \in \mathfrak{sl}(V)$ we therefore have $\mathrm{Tr}(x(y - y^*)) = 0$ thus $\mathrm{Tr}(xy) = \mathrm{Tr}(xy^*)$. We get the equality

$$\mathrm{Tr}(xy) = \mathrm{Tr}(xy^*) = \mathrm{Tr}((xy^*)^*) = \mathrm{Tr}(yx^*) = -\mathrm{Tr}(yx) = -\mathrm{Tr}(xy)$$

therefore $\mathrm{Tr}(xy) = 0$ for all $y \in \mathfrak{sl}(V)$ and this form in $\mathfrak{sl}(V)$ is non degenerate, we get $x = 0$. Therefore V induces a non degenerate form and by Theorem* 8.6.1 the Lie algebra $\mathfrak{gl}_b(V)$ is reductive.

We are reduced to prove that the center is trivial. For this, we can assume that k is algebraically closed (we have $\mathfrak{z}(\mathfrak{g} \otimes_k K) = \mathfrak{z}(\mathfrak{g}) \otimes_k K$).

For b symmetric, we may assume that there is a basis (e_i) such that $b(e_i, e_j) = \delta_{i,j}$. In this basis the matrices of elements in $\mathfrak{gl}_b(V)$ are antisymmetric. Let A be an antisymmetric matrix commuting with any other antisymmetric matrix. In particular $A = (a_{i_0,j_0})$ commutes with $E_{i,j} - E_{j,i}$ leading to the equations $a_{j,i_0} = 0$ for $i \neq j_0$, $a_{i_0,j} = 0$ for $i \neq j_0$. For $\dim V > 2$, for i_0 and j_0 fixed, there exist indices (i, j) with $j = j_0$, $i \neq i_0$ and $i \neq j$ therefore $A = 0$.

For b antisymmetric, we may assume that $\dim V = 2n$ and that there is a basis (e_i) such that $b(e_i, e_{2n+1-j}) = \delta_{i,j}$. In this basis the matrices of elements in $\mathfrak{gl}_b(V)$ are of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $D = -{}^tA$ and B and C are symmetric. Assume that this element lies in the center of the Lie algebra. Commuting with matrices of the form

$$\begin{pmatrix} X & 0 \\ 0 & -{}^tX \end{pmatrix} \text{ and } \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \text{ with } Y \text{ symmetric}$$

gives $AX = XA$ therefore $A = \lambda \mathrm{Id}$ and $CY = YC = 0$ and $AY = DY$. This implies that $C = 0$ and $\lambda = 0$. The same argument shows that $B = 0$ and the result follows. $\square$

8.7 Exercices

Exercice 8.7.1 (i) Prove that there is no semisimple Lie algebras of dimension 1 or 2.

(ii) For char$k > 2$, prove that $\mathfrak{sl}(V)$ with $\dim V = 2$ is a semisimple Lie algebra of dimension 3.

(iii) Without assumption on char$k$, consider a three dimensional vector space $\mathfrak{g}$ with basis (x, y, z). Define a product on $\mathfrak{g}$ by $[x, y] = z$, $[x, z] = -y$ and $[y, z] = x$.

Prove that this defines a semisimple and even a simple Lie algebra structure on $\mathfrak{g}$.

Exercice 8.7.2 Let char$k$ be arbitrary. Prove that if $\kappa_{\mathfrak{g}}$ is non degenerate, then $\mathfrak{g}$ is semisimple. For which values of char$k$ is $\mathfrak{sl}(V)$ semisimple for V a finite dimensional vector space over k.

Exercice 8.7.3 Let b be a non degenerate symmetric or antisymmetric bilinear form on a vector space V. In this exercice, we prove without using Theorem* 8.6.1 that $\mathfrak{gl}_b(V)$ is semisimple (except for $\dim V = 2$ and b symmetric).

(i) By considering the Killing form, prove that we may assume that k is algebraically closed.

(ii) Symmetric case. Let $n = \dim V$ and consider a basis (e_i) of V such that $b(e_i, e_j) = \delta_{i,n+1-j}$.

(a) Prove that, with notation as in the proof of Proposition 8.6.4, the elements $A_{i,j} = E_{i,j} - E_{n+1-i,n+1-j}$ are in $\mathfrak{gl}_b(V)$ and form a basis for $i, j \leq n/2$ with $(i, j) \neq (n/2, n/2)$.

(b) Consider the elements of the form $E = \sum_i t_i E_{i,i}$ lying in $\mathfrak{gl}_b(V)$. Using the fact that the Killing form is invariant, prove that the restriction of the Killing form on the span of the $A_{i,j}$ for $i \neq j$ is non degenerate.

(c) Prove that the Killing form on the span of the $A_{i,i}$ is non degenerate and conclude.

(iii) Antisymmetric case. Let $2n = \dim V$ and consider a basis (e_i) of V such that $b(e_i, e_j) = \delta_{i,2n+1-j}$.

(a) Prove that, with notation as in the proof of Proposition 8.6.4, the elements $A_{i,j} = E_{i,j} - E_{2n+1-i,2n+1-j}$ for $i, j \in [1, n]$, $B_{i,j} = E_{i,j} + E_{n+1-i,3n+1-j}$ for $i \in [1, n]$ and $j \in [n + 1, 2n]$ and $C_{i,j} = E_{i,j} + E_{3n+1-i,n+1-j}$ for $i \in [n + 1, 2n]$ and $j \in [1, n]$are in $\mathfrak{gl}_b(V)$ and form a basis.

(b) Consider the elements of the form $E = \sum_i t_i E_{i,i}$ lying in $\mathfrak{gl}_b(V)$. Using the fact that the Killing form is invariant, prove that the restriction of the Killing form on the span of the $A_{i,j}$ for $i \neq j$ and $B_{i,j}$ and $C_{i,j}$ is non degenerate.

(c) Prove that the Killing form on the span of the $A_{i,i}$ is non degenerate and conclude.

Exercice 8.7.4 Let b be a non degenerate symmetric or antisymmetric bilinear form on a vector space V, prove that the Lie algebras $\mathfrak{sl}(V)$, $\mathfrak{so}(V, b)$ and $\mathfrak{sp}(V, b)$ are simple.

Exercice 8.7.5 Let $\mathfrak{g}$ be a Lie algebra and construct a sequence $(\mathfrak{a}_i)$ of ideals in $\mathfrak{g}$ by setting $\mathfrak{a}_0 = 0$ and taking for $\mathfrak{a}_{i+1}/\mathfrak{a}_i$ a maximal commutative ideal in $\mathfrak{g}/\mathfrak{a}_i$.

Let p be the smallest integer such that $a_p = a_{p+1} = \cdots$, prove that we have $\mathfrak{r}(\mathfrak{g}) = \mathfrak{a}_p$.

Exercice 8.7.6 Prove that the simple ideals of a semisimple Lie algebra are characteristic ideals.

Exercice 8.7.7 Let $\mathfrak{g}$ be a Lie algebra. An ideal $\mathfrak{a}$ is called *minimal* if $\mathfrak{a} \neq 0$ and for any ideal $\mathfrak{b}$ of $\mathfrak{g}$ contained in $\mathfrak{a}$, we have $\mathfrak{b} = 0$ or $\mathfrak{b} = \mathfrak{a}$.

(i) Prove that any simple ideal of $\mathfrak{g}$ is minimal.

(ii) Let $\mathfrak{a}$ be a minimal ideal, we have the alternative

- $\mathfrak{a} \subset \mathfrak{r}(\mathfrak{g})$ and then $\mathfrak{a}$ is abelian or,
- $\mathfrak{a} \cap \mathfrak{r}(\mathfrak{g}) = 0$ and $\mathfrak{a}$ is simple.

Exercice 8.7.8 Prove that a Lie algebra $\mathfrak{g}$ is reductive if and only if $\mathfrak{z}(\mathfrak{g}) = \mathfrak{n}_{\mathfrak{g}}$ (recall that $\mathfrak{n}_{\mathfrak{g}}$ is the maximal nilpotent ideal). Hint: if the condition is satisfied, then prove that $\mathcal{D}(\mathfrak{r}(\mathfrak{g})) \subset \mathfrak{z}(\mathfrak{g})$.

Exercice 8.7.9 Let $\mathfrak{g}$ be a simple Lie algebra and b an invariant bilinear form.
(i) Prove that b is either trivial or non degenerate.
(ii) Prove that if k is algebraically closed, then b is a scalar multiple of the Killing form.

Part II

Classification of complex semisimple Lie algebras

Chapter 9

Cartan subalgebras

9.1 definition

Definition 9.1.1 *Let* $\mathfrak{g}$ *be a Lie algebra. A subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ *is called a* Cartan subalgebra *if it satisfies the following two conditions:*

- $\mathfrak{h}$ *is nilpotent;*
- $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Example 9.1.2 Let V be a finite dimensional space and (e_i) a basis. Define the complete flags $V_\bullet$ and $V'_\bullet$ associated to (e_i) as in Example 2.1.8. Then $\mathfrak{diag}(V_\bullet, V'_\bullet)$ is a Cartan subalgebra of $\mathfrak{sl}(V)$.

9.2 Regular elements

Definition 9.2.1 *Let* $\mathfrak{g}$ *be a Lie algebra and* $x \in \mathfrak{g}$*, we denote by* $P_x(T)$ *the characteristic polynomial of* $\operatorname{ad} x$*. Let* n *be the dimension of* $\mathfrak{g}$*, we define the functions* $a_i(x)$ *by the following identity:*

$$P_x(T) = \sum_{i=1}^{n} a_i(x) T^i.$$

Definition 9.2.2 *The* rank *of* $\mathfrak{g}$ *is the smallest integer* i *such that* a_i *is not identically zero. We denote it by* $\operatorname{rk}(\mathfrak{g})$.

Remark 9.2.3 We have the inequalities $1 \leq \operatorname{rk}(\mathfrak{g}) \leq \dim \mathfrak{g}$.

Indeed, the rank of $\mathfrak{g}$ is at most $\mathfrak{n} = \dim \mathfrak{g}$. This is the case if and only if $\mathfrak{g}$ is nilpotent. For the other inequality, we have that x is in the kernel of $\operatorname{ad} x$ therefore $a_0(x) = 0$ for all $x \in \mathfrak{g}$.

Definition 9.2.4 *An element* $x \in \mathfrak{g}$ *is called regular if* $a_{\operatorname{rk}(\mathfrak{g})}(x) \neq 0$.

Proposition 9.2.5 *Let* $\mathfrak{g}$ *be a Lie algebra, the set* $\mathfrak{g}_r$ *of regular elements is a connected, dense open subset of* $\mathfrak{g}$.

Proof. Let us denote with F the closed subset of $\mathfrak{g}$ defined by the vanishing of the function $a_{\operatorname{rk}(\mathfrak{g})}$. We have $\mathfrak{g}_r = \mathfrak{g} \setminus F$ therefore $\mathfrak{g}_r$ is open. Because $a_{\operatorname{rk}(\mathfrak{g})}$ is a polynomial function in x (take basis and check this) non vanishing on $\mathfrak{g}$ (by definition of the rank), the closed subset F has an empty interior, therefore $\mathfrak{g}_r$ is dense. Furthermore, for x and y in $\mathfrak{g}_r$, take L to be the line through x and y. Then L meets F in a finite number of points. Therefore as L is a 2-dimensional real vector space there is a path connecting x and y in L not meeting $F \cap L$ therefore $\mathfrak{g}_r$ is connected.

9.3 Cartan subalgebras associated with regular elements

Definition 9.3.1 *Let $x \in \mathfrak{g}$ and $\lambda \in k$, we denote by $\mathfrak{g}_x^\lambda$ the* caracteristic subspace *of x associated to λ that is the subspace defined by*

$$\mathfrak{g}_x^\lambda = \{y \in \mathfrak{g} \ / \ (\mathrm{ad}\, x - \lambda \mathrm{Id}_\mathfrak{g})^n(y) = 0 \text{ for some } n\}.$$

Lemma 9.3.2 *Let $n \geq 0$. Let $x, y, z \in \mathfrak{g}$ and $\lambda, \mu \in k$, we have the formula*

$$(\mathrm{ad}\, x - (\lambda+\mu)\mathrm{Id}_\mathfrak{g})^n[y,z] = \sum_{i=0}^n \binom{n}{i} \left[(\mathrm{ad}\, x - \lambda\mathrm{Id}_\mathfrak{g})^i y, (\mathrm{ad}\, x - \mu\mathrm{Id}_\mathfrak{g})^{n-i} z\right].$$

Proof. By induction on n. For $n = 0$ there is nothing to prove. Assume that the formula holds for n. We compute

$$\begin{aligned}
(\mathrm{ad}\, x - (\lambda+\mu)\mathrm{Id}_\mathfrak{g})^{n+1}[y,z] = & \ (\mathrm{ad}\, x - (\lambda+\mu)\mathrm{Id}_\mathfrak{g}) \left(\sum_{i=0}^n \binom{n}{i} \left[(\mathrm{ad}\, x - \lambda\mathrm{Id}_\mathfrak{g})^i y, (\mathrm{ad}\, x - \mu\mathrm{Id}_\mathfrak{g})^{n-i} z\right] \right) \\
= & \ \mathrm{ad}\, x \left(\sum_{i=0}^n \binom{n}{i} \left[(\mathrm{ad}\, x - \lambda\mathrm{Id}_\mathfrak{g})^i y, (\mathrm{ad}\, x - \mu\mathrm{Id}_\mathfrak{g})^{n-i} z\right] \right) \\
& -\lambda \sum_{i=0}^n \binom{n}{i} \left[(\mathrm{ad}\, x - \lambda\mathrm{Id}_\mathfrak{g})^i y, (\mathrm{ad}\, x - \mu\mathrm{Id}_\mathfrak{g})^{n-i} z\right] \\
& -\mu \sum_{i=0}^n \binom{n}{i} \left[(\mathrm{ad}\, x - \lambda\mathrm{Id}_\mathfrak{g})^i y, (\mathrm{ad}\, x - \mu\mathrm{Id}_\mathfrak{g})^{n-i} z\right] \\
= & \sum_{i=0}^n \binom{n}{i} \left[(\mathrm{ad}\, x - \lambda\mathrm{Id}_\mathfrak{g})^{i+1}(x), (\mathrm{ad}\, x - \mu\mathrm{Id}_\mathfrak{g})^{n-i}(y)\right] \\
& + \sum_{i=0}^n \binom{n}{i} \left[(\mathrm{ad}\, x - \lambda\mathrm{Id}_\mathfrak{g})^{i}(x), (\mathrm{ad}\, x - \mu\mathrm{Id}_\mathfrak{g})^{n+1-i}(y)\right] \\
= & \sum_{i=0}^n \left(\binom{n}{i-1} + \binom{n}{i} \right) \left[(\mathrm{ad}\, x - \lambda\mathrm{Id}_\mathfrak{g})^{i}(x), (\mathrm{ad}\, x - \mu\mathrm{Id}_\mathfrak{g})^{n+1-i}(y)\right]
\end{aligned}$$

and the result follows by Pascal's formula. □

Proposition 9.3.3 *Let $x \in \mathfrak{g}$, then we have*
(i) $\mathfrak{g}$ is the direct sum of the $\mathfrak{g}_x^\lambda$,
(ii) we have $[\mathfrak{g}_x^\lambda, \mathfrak{g}_x^\mu] \subset \mathfrak{g}_x^{\lambda+\mu}$.
(iii) In particular $\mathfrak{g}_x^0$ is a Lie subalgebra of $\mathfrak{g}$.

Proof. (i) This is a classical statement of linear algebra.
(ii) This is a direct consequence of the above Lemma.
(iii) By *(ii)*, we have that $\mathfrak{g}_x^0$ is stable under the Lie bracket. □

Lemma 9.3.4 *If $x \in \mathfrak{g}$, we have $\dim \mathfrak{g}_x^0 \geq \mathrm{rk}(\mathfrak{g})$. If x is regular, then $\dim \mathfrak{g}_x^0 = \mathrm{rk}(\mathfrak{g})$.*

Proof. The characteristic polynomial P_x of $\mathrm{ad}\, x$ on $\mathfrak{g}$ is the product of the characteristic polynomial P_x^λ of the restrictions of $\mathrm{ad}\, x$ on $\mathfrak{g}_x^\lambda$. We have $P_x^0(T) = T^{\dim \mathfrak{g}_x^0}$ and for $\lambda \neq 0$ we have $P_x^\lambda(0) \neq 0$. Therefore we have $P_x(T) = T^{\dim \mathfrak{g}_x^0} Q(T)$ with $Q(0) \neq 0$. In particular $a_k(x) = 0$ for $k \leq \dim \mathfrak{g}_x^0$ and $a_{\dim \mathfrak{g}_x^0}(x) \neq 0$. This implies the inequality.

For the equality, recall that by definition, if x is regular, we have $a_{\mathrm{rk}(\mathfrak{g})}(x) \neq 0$ and $a_k(x) = 0$ for all $k \leq \mathrm{rk}(\mathfrak{g})$. The result follows. □

Theorem 9.3.5 *Let x be a regular element in the Lie algebra $\mathfrak{g}$, then $\mathfrak{g}_x^0$ is a Cartan subalgebra of $\mathfrak{g}$. Its dimension is* $\mathrm{rk}(\mathfrak{g})$.

Proof. Let us first show that $\mathfrak{g}_x^0$ is nilpotent. We only need to prove that for any $y \in \mathfrak{g}_x^0$, the adjoint endomorphism $(\mathrm{ad}\, y)|_{\mathfrak{g}_x^0}$ is nilpotent. Let us denote by A_y the restriction $(\mathrm{ad}\, y)|_{\mathfrak{g}_x^0}$ and by B_y the endomorphism of the quotient $\mathfrak{g}/\mathfrak{g}_x^0$ induced by $\mathrm{ad}\, y$. We consider the following sets:

$$U = \{y \in \mathfrak{g}_x^0 \ / \ Y_y \text{ is not nilpotent}\} \qquad V = \{y \in \mathfrak{g}_x^0 \ / \ Z_y \text{ is invertible}\}.$$

These sets are open in $\mathfrak{g}_x^0$ (given by the non vanishing of at least one coefficient of the characteristic polynomial of Y_y for the first one and by the non vanishing of the determinant for the second one). Remark that V is non empty since it contains x (indeed the eigenvalues of x on $\mathfrak{g}/\mathfrak{g}_x^0$ are different from 0 by definition).

We want to prove that U is empty. If not, then its complement (as well as the complement of V are closed subsets defined by the vanishing of a non trivial polynomial. Therefore U (and V) are dense open subsets and so is the intersection $U \cap V$. We may thus pick $y \in U \cap V$. But we see that $\mathfrak{g}_y^0$ is strictly contained in $\mathfrak{g}_x^0$ giving $\dim \mathfrak{g}_y^0 < \dim \mathfrak{g}_x^0$. But by the above Lemma we have $\dim \mathfrak{g}_x^x = \mathrm{rk}(\mathfrak{g})$ a contradiction to the inequality in the Lemma.

Let us prove that $\mathfrak{g}_x^0$ is self normalised. Let $z \in \mathfrak{n}(\mathfrak{g}_x^0)$, then $\mathrm{ad}\, z(\mathfrak{g}_x^0) \subset \mathfrak{g}_x^0$. Thus we have $\mathrm{ad}\, x(z) = -\mathrm{ad}\, z(x) \in \mathfrak{g}_x^0$ and thus there exists an integer m such that $(\mathrm{ad}\, x)^m(\mathrm{ad}\, x(z)) = 0$. Therefore we have $z \in \mathfrak{g}_x^0$. ☐

9.4 Conjugacy of Cartan subalgebras

Definition 9.4.1 *We denote by* $\mathrm{Aut}(\mathfrak{g})$ *the subgroup of Lie algebra automorphisms in* $\mathrm{GL}(\mathfrak{g})$. *In symbols:*

$$\mathrm{Aut}(\mathfrak{g}) = \{f \in \mathrm{GL}(\mathfrak{g}) \ / \ f([x,y]) = [f(x), f(y)] \textit{ for all } x \textit{ and } y \textit{ in } \mathfrak{g}\}.$$

Proposition 9.4.2 *Let $x \in \mathfrak{g}$, then* $\exp \mathrm{ad}\, x$ *is in* $\mathrm{Aut}(\mathfrak{g})$.

Proof. The element $\exp \mathrm{ad}\, x$ is in $\mathrm{GL}(\mathfrak{g})$, its inverse is $\exp(-\mathrm{ad}\, x)$. Now we compute for y and z in $\mathfrak{g}$ and using Lemma 9.3.2:

$$\begin{aligned}
(\exp \mathrm{ad}\, x)[y,z] &= \sum_{n=0}^{+\infty} \frac{1}{n!} (\mathrm{ad}\, x)^n [y,z] \\
&= \sum_{n=0}^{+\infty} \frac{1}{n!} \sum_{i=0}^{n} [(\mathrm{ad}\, x)^i(y), (\mathrm{ad}\, x)^{n-i}(z)] \\
&= \sum_{n=0}^{+\infty} \sum_{i=0}^{n} \frac{1}{i!(n-i)!} (\mathrm{ad}\, x)^i(y) (\mathrm{ad}\, x)^{n-i}(z) - \sum_{n=0}^{+\infty} \sum_{i=0}^{n} \frac{1}{i!(n-i)!} (\mathrm{ad}\, x)^{n-i}(z) (\mathrm{ad}\, x)^i(y) \\
&= (\exp \mathrm{ad}\, x)(y)(\exp \mathrm{ad}\, x)(z) - (\exp \mathrm{ad}\, x)(z)(\exp \mathrm{ad}\, x)(y).
\end{aligned}$$

The result follows. ☐

Definition 9.4.3 *Let $\mathfrak{g}$ be a Lie algebra, we call* inner automorphism group *and denote by G the subgroup of* $\mathrm{Aut}(\mathfrak{g})$ *generated by the elements* $\exp \mathrm{ad}\, x$ *for $x \in \mathfrak{g}$.*

Theorem 9.4.4 *The group G acts transitively on the set of Cartan subalgebras of $\mathfrak{g}$.*

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For $x \in \mathfrak{h}$ we denote by Y_x (resp. Z_x) the restriction of $\operatorname{ad} x$ to $\mathfrak{h}$ (resp. the morphism induced by $\operatorname{ad} x$ on $\mathfrak{g}/\mathfrak{h}$).

Lemma 9.4.5 *The set $V = \{x \in \mathfrak{h} \ / \ Z_x$ is invertible$\}$ is open and non empty.*

Proof. This set is open because given by the non vanishing of a determinant. Let us prove that it is non empty. We consider $\mathfrak{g}/\mathfrak{h}$ as a representation of $\mathfrak{h}$. The action of $x \in \mathfrak{h}$ is given by Z_x. Because $\mathfrak{h}$ is nilpotent, it is solvable and therefore we may apply Lie's Theorem to $\mathfrak{g}/\mathfrak{h}$. We get a complete flag $0 = V_0 \subset \cdots \subset V_n = \mathfrak{g}/\mathfrak{h}$ stable under $\mathfrak{h}$. On the quotients V_{i+1}/V_i, the Lie algebra $\mathfrak{h}$ acts by scalar multiplication *i.e.* there is a linear form $\alpha_i : \mathfrak{h} \to \mathbb{C}$ such that $Z_x(v) \in \alpha_i(x)v + V_i$ for any $v \in V_{i+1}$. In particular, for $x \in \mathfrak{h}$, the eigenvalues of Z_x on $\mathfrak{g}/\mathfrak{h}$ are the $(\alpha_i(x))_{i\in[1,n]}$. To prove the result, we only need to prove that none of the linear forms α_i are trivial. Indeed, in that case the set V is the complement of the union of the hyperplanes $\{x \in \mathfrak{h} \ / \ \alpha_i(x) = 0\}$ and therefore non empty.

Let us assume that one of the α_i is trivial. Let k be the smallest interger such that $\alpha_k = 0$. We have that Z_x is invertible on V_{k-1} but not on V_k. The kernel K of Z_x on V_k is therefore one dimensional and supplementary to V_{k-1}. It is also the nilspace for Z_x *i.e.* the set of z such that $Z_x^m(z) = 0$ for some m. Let us prove that for $y \in \mathfrak{h}$ and $z \in K$, we have $Z_y(z) = 0$. Indeed, first remark that we have by the Jacobi formula the equality $Z_x(Z_y(z)) = (\operatorname{ad} x)(\operatorname{ad} y(z)) = (\operatorname{ad} y)(\operatorname{ad} x(z)) + (\operatorname{ad}(\operatorname{ad} x(y)))(z) = Z_{\operatorname{ad} x(y)}(z)$. By induction we may compute

$$Z_x^m(Z_y(z)) = (\operatorname{ad} x)^m(\operatorname{ad} y(z)) = Z_{(\operatorname{ad} x)^m(y)}(z).$$

But because $\mathfrak{h}$ is nilpotent, we have $(\operatorname{ad} x)^m(y) = 0$ for some m. Therefore we have $Z_x^m(Z_y(z)) = 0$ and $Z_y(z) \in K$. But Z_y maps V_k to V_{k-1} (because $\alpha_k = 0$) thus $Z_y(z) \in K \cap V_{k-1} = 0$.

Now let $z \in \mathfrak{g}$ such that its image $\overline{z}$ in $\mathfrak{g}/\mathfrak{h}$ lies in K. For any $y \in \mathfrak{h}$, we have $Z_y(\overline{z}) = 0$ therefore $\operatorname{ad} y(z) \in \mathfrak{h}$ or equivalentely $\operatorname{ad} z(y) \in \mathfrak{h}$. We get that $z \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ but $z \notin \mathfrak{h}$. A contradiction with the definition of Cartan subalgebras. ☐

Lemma 9.4.6 *Let $W = G \cdot V$ be the union of all translates of V by elements of G. The set W is open in $\mathfrak{g}$.*

Proof. Let $x \in V$, we only need to prove that W contains a neighbourhood of x in $\mathfrak{g}$. For this we consider the map $\mu : G \times V \to \mathfrak{g}$ defined by $\mu(g, y) = g \cdot y$ and whose image is W. We want to compute the image I of $T_{(e,x)}(G \times V)$ by the differential $d_{(e,x)}\mu$. This image contains the image of $\mathfrak{h} = T_x V$ under the differential of the inclusion $\mathfrak{h} \to \mathfrak{g}$ (this is the map $\mu(e, \cdot)$). Furthermore, we have for $y \in \mathfrak{g}$ the curve $t \mapsto \exp(t \operatorname{ad} y)(x) = x + t[y, x] + O(t^2)$. The image of the tangent vector of that curve is in I. In particular $[y, x]$ is in I. Therefore $\operatorname{ad} x(\mathfrak{g}) \subset I$. But for $x \in V$, we have Z_x invertible therefore the projection of $\operatorname{ad} x(\mathfrak{g})$ to $\mathfrak{g}/\mathfrak{h}$ is sujective and $I = \mathfrak{g}$.

By the Implicit function Theorem, the map μ is locally submersive at (e, x) therefore its image contains an neighbourhood of x. ☐

Lemma 9.4.7 *There is a regular element x such that $\mathfrak{h} = \mathfrak{g}_x^0$.*

Proof. Recall that the set of regular elements is dense in $\mathfrak{g}$. Because W is open and non empty, there exists y regular and in W. Let us write $y = g \cdot x$ for $g \in G$ and $x \in V$. The element x is regular. Indeed, we have $\operatorname{ad} y(z) = (\operatorname{ad} g \cdot x)(z) = g \cdot (\operatorname{ad} x(g^{-1} \cdot z))$ and $\operatorname{ad} y$ and $\operatorname{ad} x$ are conjugate. We thus have a regular element $x \in \mathfrak{h}$. Because $\mathfrak{h}$ is nilpotent, we have that Y_x is nilpotent, therefore $\mathfrak{h} \subset \mathfrak{g}_x^0$ but Z_x is invertible therefore $\mathfrak{g}_x^0 = \mathfrak{h}$. ☐

We are left to prove that for any two regular elements x and y, the algebras $\mathfrak{g}_x^0$ and $\mathfrak{g}_y^0$ are conjugate under G. Let us define the equivalence relation

$$R(x,y) \text{ holds} \Leftrightarrow \ \mathfrak{g}_x^0 \text{ and } \mathfrak{g}_y^0 \text{ are conjugate under } G.$$

Lemma 9.4.8 *The equivalence classes for R are open in $\mathfrak{g}_r$.*

Proof. Let $x \in \mathfrak{g}_r$, we need to prove that there exists a small neighbouhood U of x such that all elements $y \in U$ are such that $R(x,y)$ holds. Let $\mathfrak{h} = \mathfrak{g}_x^0$ which is a Cartan subalgebra. We have that V (as defined above) is open and non empty and that W is also open. We have $x \in V \cap \mathfrak{g}_r \subset W \cap \mathfrak{g}_r$. Thus there is a neighbourhood U of x contained in $W \cap \mathfrak{g}_r$. Let y in U. It is regular and of the form $g \cdot z$ with $z \in V$. Therefore z is regular, $z \in \mathfrak{g}_x^0$ and $Z_z$invertible. This implies (same proof as in the last lemma that $\mathfrak{g}_z^0 = \mathfrak{g}_x^0$ and therefore $\mathfrak{g}_y^0$ is conjugated to $\mathfrak{g}_z^0 = \mathfrak{g}_x^0$. We have that $R(x,y)$ holds. □

To finish the proof, recall that $\mathfrak{g}_r$ is connected so there is only one equivalence class for R. □

9.5 The semisimple case

For $\mathfrak{g}$ a semisimple Lie algebra, the Cartan subalgebras take a simpler form.

Theorem 9.5.1 *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. The following propositions hold:*
(ı) The restriction of the Killing form $\kappa_{\mathfrak{g}}$ to $\mathfrak{h}$ is non degenerate;
(ıı) $\mathfrak{h}$ is abelian;
(ııı) $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$;
(ıv) every element in $\mathfrak{h}$ is semisimple.

Proof. *(ı)* There is a regular element $x \in \mathfrak{g}$ such that $\mathfrak{h} = \mathfrak{g}_x^0$. Let us consider the direct sum decomposition $\mathfrak{g} = \oplus_\lambda \mathfrak{g}_x^\lambda$. For $y \in \mathfrak{g}_x^\lambda$ and $z \in \mathfrak{g}_x^\mu$ we may compute by induction on $n+m$ with $(\mathrm{ad}\, x - \lambda \mathrm{Id})^n(y) = 0$ and $(\mathrm{ad}\, x - \mu \mathrm{Id})^m(z) = 0$ that

$$\lambda \kappa_{\mathfrak{g}}(y,z) = \kappa_{\mathfrak{g}}([x,y],z) = -\kappa_{\mathfrak{g}}(y,[x,z]) = -\mu \kappa_{\mathfrak{g}}(y,z)$$

thus $\kappa_{\mathfrak{g}}(y,z) = 0$ unless $\lambda + \mu = 0$. Therefore $\mathfrak{g}_x^0$ is orthogonal to all the $\mathfrak{g}_x^\lambda$ for $\lambda \neq 0$. Because $\kappa_{\mathfrak{g}}$ is non degenerate, this implies that its resctriction to $\mathfrak{h} = \mathfrak{g}_x^0$ is also non degenerate.

(ıı) Consider the representation $\mathrm{ad} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$. It is injective as the restriction of the adjoint representation of $\mathfrak{g}$ (which is semisimple). Its image $\mathrm{ad}\,(\mathfrak{h})$ is solvable (as the quotient of a solvable — even nilpotent— Lie algebra). Therefore by Cartan's criterion (Theorem 7.4.2) we have $[\mathrm{ad}\,(\mathfrak{h}), \mathrm{ad}\,(\mathfrak{h})] \subset \mathrm{ad}\,(\mathfrak{h})^\perp$ where the orthogonal is taken with respect to the Killing form $\kappa_{\mathfrak{g}}$. But $\kappa_{\mathfrak{g}}|_{\mathfrak{h}}$ is non degenerate by *(ı)* thus $[\mathrm{ad}\,(\mathfrak{h}), \mathrm{ad}\,(\mathfrak{h})] = 0$ and $\mathrm{ad}\,(\mathfrak{h})$ is abelian, the result follows because the representation is faithful.

(ııı) We have $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ thus $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) \subset \mathfrak{h}$. But as $\mathfrak{h}$ is abelian, the inclusion $\mathfrak{h} \subset \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$ holds.

(ıv) Let $x \in \mathfrak{h}$ and write $x = x_s + x_n$ is Jordan-Chevalley decomposition. Recall that because $\mathfrak{g}$ is semisimple, this is well defined. Now because $x \in \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$, the elements x_s and x_n (as polynomials with non constant terms in x) are also in $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. Now let $y \in \mathfrak{h}$, we want to compute $\kappa_{\mathfrak{h}}(x_n, y) = \mathrm{Tr}(\mathrm{ad}\, y \circ \mathrm{ad}\, x_n)$. But $\mathrm{ad}\, x_n$ is nilpotent and $\mathrm{ad}\, x_n$ commutes with $\mathrm{ad}\, y$ thus $\mathrm{ad}\, y \circ \mathrm{ad}\, x_n$ is nilpotent. We get $\kappa_{\mathfrak{h}}(x_n, y) = 0$ for all $y \in \mathfrak{h}$. Because $\kappa_{\mathfrak{h}}$ is non degenerate we deduce that $x_n = 0$. □

Corollary 9.5.2 *Let $\mathfrak{g}$ be semisimple, then a Cartan subalgebra is a maximal commutative algebra.*

Proof. This follows from *(iii)* of the previous theorem. □

Corollary 9.5.3 *Every regular element in a semisimple Lie algebra is semisimple.*

Proof. For x regular $x \in \mathfrak{g}_x^0$ and $\mathfrak{g}_x^0$ is a Cartan subalgebra. Therefore x is semisimple. □

9.6 Exercices

Exercice 9.6.1 Prove the assertion in Example 9.1.2.

Exercice 9.6.2 Let $\mathfrak{g}$ be 2-dimensional Lie algebra, give an example of a maximal abelian subalgebra which is not a Cartan subalgebra.

Exercice 9.6.3 Let $\mathfrak{g}$ be a semisimple Lie algebra. A subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ s called *toral* if all its elements are semisimple.

(i) Prove that there exists toral subalgebras in semisimple Lie algebras.

(ii) Prove that a toral subalgebra $\mathfrak{h}$ is abelian. Hint: for $x \in \mathfrak{h}$, it suffices to prove that the only eigenvalue of $\operatorname{ad} x$ is 0. Suppose this is not true and let $y \in \mathfrak{h}$ with $\operatorname{ad} x(y) = ay$ with $a \neq 0$. Because $\operatorname{ad} y$ is semisimple the element $\operatorname{ad} y(x)$ is decomposed as sum of eigenvectors with non trivial eigenvalues for $\operatorname{ad} y$ but is also an eigenvector associated to 0 for $\operatorname{ad} y$. This gives a contradiction.

(iii) Prove that there is a direct sum decomposition $\mathfrak{g} = \oplus \mathfrak{g}_{\mathfrak{h}}^{\lambda}$ where $\lambda \in \mathfrak{h}^{\vee}$ and

$$\mathfrak{g}_{\mathfrak{h}}^{\lambda} = \{x \in \mathfrak{g} \ / \ [y, x] = \lambda(y)x \text{ for all } y \in \mathfrak{h}\}.$$

(iv) Prove that $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_{\mathfrak{h}}^{0}$ and that the restriction of the Killing form on $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$ is non degenerate.

(v) Prove that $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$ contains the semisimple and nilpotent parts of its elements.

(vi) Prove that all semisimple elements of $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$ lie in $\mathfrak{h}$.

(vii) Prove that the restriction of the Killing form $\kappa_{\mathfrak{g}}$ to $\mathfrak{h}$ is non degenerate.

(viii) Prove that $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$ is nilpotent.

(ix) Prove that $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$ is abelian (Hint: use Cartan's criterion for $\operatorname{ad} : \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) \to \mathfrak{gl}(\mathfrak{g})$)

(x) Deduce that $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

(xi) Prove that $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

(xii) Conclude that the Cartan subalgebras are the maximal toral subalgebras.

Exercice 9.6.4 Let V be a finite dimensional vector space and b a non degenerate symmetric (resp. symplectic) form on V. Describe Cartan subalgebras for $\mathfrak{so}(V, b)$ (resp. $\mathfrak{sp}(V, b)$).

Chapter 10

The Lie algebra $\mathfrak{sl}_2$

10.1 Definition, standard basis and simplicity

Definition 10.1.1 *The Lie algebra $\mathfrak{sl}(k^2)$ will be denoted by $\mathfrak{sl}_2$ and identified with the Lie algebra of 2×2 matrices. We define the following three elements X, Y and H in $\mathfrak{sl}_2$ and call (X, H, Y) the* canonical basis *for $\mathfrak{sl}_2$*

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and } \ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Fact 10.1.2 *The Lie algebra $\mathfrak{sl}_2$ is of dimension 3 and (X, H, Y) is a k-vector space basis for $\mathfrak{sl}_2$. We have the formulas $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$.*

Corollary 10.1.3 *The Lie algebra $\mathfrak{sl}_2$ is simple and in particular semisimple.*

Proof. Indeed, let $\mathfrak{a}$ be a non trivial ideal in $\mathfrak{sl}_2$ and let $aX + bH + cY$ be a non zero element in $\mathfrak{a}$. If $a = c = 0$, we have $H \in \mathfrak{a}$ therefore $2X = [H, X]$ and $2Y = [H, Y]$ are in $\mathfrak{a}$. This implies that $\mathfrak{a} = \mathfrak{sl}_2$. We may therefore assume that $a \neq 0$ or $c \neq 0$ and by symmetry, we may assume that $a \neq 0$. We then have $[Y, [Y, aX + bH + cY]] = [Y, aH + 2bY] = aY$, therefore $Y \in \mathfrak{a}$. Applying $\operatorname{ad} X$ we get that H is in $\mathfrak{a}$ and conclude as before. ☐

Corollary 10.1.4 *The endomorphism $\operatorname{ad} H$ has three eigenvalues 2, 0, -2 in particular H is semisimple. The Lie algebra $\mathfrak{h} = kH$ is a Cartan subalgebra.*

Proof. The first part is obvious from the above Fact. The Lie algebra $\mathfrak{h}$ is abelian and therefore nilpotent and if $aX + bH + cY$ is in $\mathfrak{n}_{\mathfrak{sl}_2}(\mathfrak{h})$, we have $[H, aX + bH + cY] \in \mathfrak{h}$ *i.e.* $2aX - 2cY \in \mathfrak{h}$ thus $a = c = 0$. Thus we have $\mathfrak{n}_{\mathfrak{sl}_2}(\mathfrak{h}) = \mathfrak{h}$. ☐

Definition 10.1.5 *We denote by $\mathfrak{n}$ the subalgebra generated by X and by $\mathfrak{b}$ the subalgebra generated by X and H. We call $\mathfrak{b}$ the* canonical Borel subalgebra of $\mathfrak{sl}_2$.

Corollary 10.1.6 *The subalgebra $\mathfrak{n}$ is nilpotent and the subalgebra $\mathfrak{b}$ is solvable.*

Proof. The algebra $\mathfrak{n}$ being 1-dimensional it is nilpotent and even abelian. We have $\mathcal{D}\mathfrak{b} = \mathfrak{n}$ which is nilpotent therefore $\mathfrak{b}$ is solvable. ☐ a=b

10.2 Representations, weights and primitive elements

Let V be a representation of $\mathfrak{sl}_2$. We denote by V^λ the eigenspace for H on V associated to the eigenvalue λ.

Definition 10.2.1 *An element $\lambda \in k$ with V^λ non trivial is called a* weight *of V. An element in V^λ is said to have weight λ.*

Proposition 10.2.2 *($\imath$) The sum $\sum_\lambda V^\lambda$ is a direct sum.*
($\imath\imath$) If $v \in V$ has weight λ, then $X \cdot v$ (resp. $Y \cdot v$) has weight $\lambda + 2$ (resp. $\lambda - 2$).

Proof. *($\imath$)* This is simply the fact that the eigenspaces for different eigenvalues are in direct sum.
($\imath\imath$) Let v such that $H(v) = \lambda v$. We have $H(X(v)) = [H, X](v) + X(H(v)) = 2X(v) + \lambda X(v)$ and $H(Y(v)) = [H, Y](v) + Y(H(v)) = -2Y(v) + \lambda Y(v)$. □

Remark 10.2.3 If V is finite dimensional, because $\operatorname{ad} H$ is semisimple, the element H is semisimple on V. In particular, V can be decomposed as $V = \bigoplus_\lambda V^\lambda$. This is not true for V infinite dimensional.

Definition 10.2.4 *A non zero vector v in V is called* primitive *of weight λ if $v \in V^\lambda$ and $X(v) = 0$.*

Lemma 10.2.5 *A non zero vector $v \in V$ is primitive if and only if it is stable under the canonical Borel subalgebra $\mathfrak{b}$.*

Proof. If v is primitive, the result is clear. Conversely, assume that $\mathfrak{b}(v) \subset kv$ *i.e.* we have $X(v) = av$ and $H(v) = bv$. We only need to prove that $a = 0$. But we have the equalities of vectors in V $2av = 2X(v) = [H, X](v) = HX(v) - XH(v) = aH(v) - bX(v) = abv - abv = 0$. We get $a = 0$. □

Proposition 10.2.6 *Every non-zero finite dimensional representation of $\mathfrak{sl}_2$ has a primitive element.*

Proof. By Lie's Theorem 7.3.1 and because $\mathfrak{b}$ is solvable, there is, in the representation a common eigenvector v to all elements in $\mathfrak{b}$. Therefore v is stabilised by $\mathfrak{b}$ and is primitive by the previous Lemma. □

10.3 The subrepresentation generated by a primitive element

Let us look at the action of the canonical basis of $\mathfrak{sl}_2$ on the vectors obtained from a promitive element.

Proposition 10.3.1 *Let V be a representation of $\mathfrak{sl}_2$ and let v be a primitive element of weight λ in V. Let $v_n = \frac{1}{n!}Y^n(v)$ and $v_{-1} = 0$, then, for all $n \geq 0$, we have the formulas:*

$$\begin{aligned} H(v_n) &= (\lambda - 2n)v_n \\ Y(v_n) &= (n+1)v_{n+1} \\ X(v_n) &= (\lambda - n + 1)v_{n-1}. \end{aligned}$$

Proof. The first formula is a consequence, by easy induction, of Proposition 10.2.2. The second formula follows from the definition of v_n and v_{n+1}. We prove the last formula by induction on n. For $n = 0$ the formula is true by definition of a primitive element. Assume that the formula holds for n. We compute:

$$\begin{aligned}(n+1)X(v_{n+1}) &= XY(v_n) = YX(v_n) + [X,Y](v_n)\\ &= Y((\lambda - n + 1)v_{n-1}) + H(v_n)\\ &= (\lambda - n + 1)nv_n + (\lambda - 2n)v_n\\ &= (n+1)(\lambda - n)v_n\end{aligned}$$

and the result follows by dividing by $n + 1$. □

With the notation as in the above proposition.

Corollary 10.3.2 *We have the following alternative:*
(ı) the elements $(v_n)_{n\geq 0}$ are linearly independent;
(ıı) λ is an integer $m \geq 0$, the elements $(v_n)_{n\in[0,m]}$ are linearly independent and $v_n = 0$ for $n > m$.

Proof. Remark that because the weights of v_n is $\lambda - 2n$, all the weights of the vectors $(v_n)_{n\geq 0}$ are distinct therefore these elements are linearly independent as soon as they do not vanish. Let m be the greatest integer such that $v_m \neq 0$ and $v_{m+1} = 0$. We clearly get by definition of v_n that $v_n = 0$ for $n > m$. Furthermore by definition of m, we have $v_n \neq 0$ for $n \in [0, m]$. Therefore we have that the elements $(v_n)_{n\in[0,m]}$ are linearly independent. But now we have $0 = X(v_{m+1}) = (\lambda - m)v_m$ and because $v_m \neq 0$, we get $\lambda = m$. □

With the notation as in the above proposition.

Corollary 10.3.3 *If V is finite dimensional, then we are in the second case of the above corollary and the subspace W generated by $(v_n)_{n\in[0,m]}$ is stable under $\mathfrak{sl}_2$ and is an irreducible representation.*

Proof. We are in the second case otherwise we would have an infinite family of linearly independent vectors. Furthermore for any $n \in [0, m]$, we have $X(v_n) = (\lambda - n + 1)v_{n-1} \in W$ (because $v_{-1} = 0$), $H(v_n) = (\lambda - 2n)v_n \in W$ and $Y(v_n) = (n + 1)v_{n+1} \in W$ (because $v_{m+1} = 0$).

Let us prove that W is irreducible. Let U be a submodule and let $u \in U$ be a non zero vector. Write $u = \sum_n u_n v_n$ and let k be the smallest integer such that $u_k \neq 0$. We have $Y^{m-k}(u) = \frac{m!}{k!}u_k v_m$ therefore $v_m \in U$ and by succesive application of X we get that $v_n \in U$ for all n. □

10.4 Structure of finite dimensional representations

We shall see that the modules generated by primitve vectors are the only irreducible representations of $\mathfrak{sl}_2$. Let us first construct an irreducible $\mathfrak{sl}_2$ representation on any $m+1$ dimensional vector space.

Proposition 10.4.1 *Let W_m be an $m+1$ dimensional vector space with a fixed basis $(v_n)_{n\in[0,m]}$. Let us define endomorphisms X, Y and H on W_m by the formulas*

$$\begin{aligned}H(v_n) &= (m - 2n)v_n\\ Y(v_n) &= (n + 1)v_{n+1}\\ X(v_n) &= (m - n + 1)v_{n-1}.\end{aligned}$$

for all $n \in [0, m]$ with the convention $v_{-1} = v_{m+1} = 0$, then W_m is a $\mathfrak{sl}_2$-representation.

Proof. We need to check the commutation relations $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$. This follows easily from the definitions. □

Theorem 10.4.2 *(ı) The representations W_m are irreducible for all $m \geq 0$.*

(ıı) Any irreducible finite dimensional representation V of $\mathfrak{sl}_2$ is isomorphic to W_m with $m + 1 = \dim V$.

Proof. (ı) In W_m, the element v_0 is by definition primitive of weight m. Therefore, W_m contains as an irreducible subrepresentation the span of the $(v_n)_{n\in[0,m]}$ which is W_m itself.

(ıı) Let V be irreducible and let v be a primitive element of weight λ. Then by Corollary 10.3.2 the weight λ is an integer m and by Corollary 10.3.2 the space V contains an irreducible subrepresentation isomorphic to W_m. The result follows. □

Corollary 10.4.3 *Any finite dimensional representation V of $\mathfrak{sl}_2$ is a direct sum of representations W_m.*

Proof. This is a direct consequence of Weyl's Theorem 8.3.1 and the previous statement. □

Recall that for V a $\mathfrak{sl}_2$-representation, V^λ denotes the eigenspace associated to the eigenvalue λ for the action of H.

Corollary 10.4.4 *Let V be a finite dimensional representation of $\mathfrak{sl}_2$.*

(ı) Then the endomorphism induced by H is diagonalisable, its eigenvalues are integers and if n is an eigenvalue, then so are the integers $(n - 2k)_{k\in[0,n]}$.

(ıı) For any integer $n \geq 0$, the linear maps $Y^n : V^n \to V^{-n}$ and $X^n : V^{-n} \to V^n$ are isomorphisms. In particular $\dim V^n = \dim V^{-n}$.

Proof. By the previous Corollary, we only need to prove these results on the irreducible representations W_m. But all these results are trivial in this case. □

Example 10.4.5 Let X be a compact Kähler manifold of complex dimension n (say for example a compact projective variety). Then Hodge theory defines endomorphisms L and Λ on $H^*(X, \mathbb{C})$. Set $X = L$ and $Y = \Lambda$ and $H(v) = (n - p)v$ for $v \in H^p(X, \mathbb{C})$. Then one can prove that this defines a $\mathfrak{sl}_2$-representation structure on $H^*(X, \mathbb{C})$. Then Corollary 10.4.4 (ıı) for $V = H^*(X, \mathbb{C})$ is called the Hard Lefschetz Theorem. Of course the difficulty here is to construct the endomorphisms L and Λ and prove that they satisfy the correct commuting relations.

10.5 Exercices

Exercice 10.5.1 Give a very simple proof (without using Lie's Theorem) of the fact that in any non trivial finite dimensional representation of $\mathfrak{sl}_2$, there is a primitive element.

Exercice 10.5.2 Let V be a finite dimensional representation V of $\mathfrak{sl}_2$.
(ı) Prove that V is determined by the eigenvalues of H on V.
(ıı) Prove that if V is the direct sum of n simple modules, we have the formula:

$$n = \dim V^0 + \dim V^1$$

where $V^i = \{v \in V \ / \ H(v) = iv\}$.

Exercice 10.5.3 Let V be a 2-dimensional vector space over $\mathbb{C}$. Describe the representations W_1 and W_2 in terms of V.

Exercice 10.5.4 Consider the embedding $\iota : \mathfrak{sl}_2 \to \mathfrak{sl}_3$ defined by

$$\iota \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This defines a structure of $\mathfrak{sl}_2$ representation on $\mathfrak{sl}_3$ by $x \cdot y = [\iota(x), y]$ for $x \in \mathfrak{sl}_2$ and $y \in \mathfrak{sl}_3$. Prove that as $\mathfrak{sl}_2$ representation, we have $\mathfrak{sl}_3 \simeq W_0 \oplus W_1 \oplus W_1 \oplus W_1 \oplus W_2$.

Exercice 10.5.5 Let $\mathbb{C}[x, y]$ be the space of polynomials in the variables x and y. For $P(x, y) \in \mathbb{C}[x, y]$ and $A \in \mathfrak{sl}_2$, define

$$(A \cdot P)(x, y) = \left(\frac{\partial P}{\partial x}(x, y), \frac{\partial P}{\partial y}(x, y) \right) A \begin{pmatrix} x \\ y \end{pmatrix}.$$

(ı) Prove that this gives to $\mathbb{C}[x, y]$ a structure of $\mathfrak{sl}_2$ representation.
(ıı) Let us denote by $\mathbb{C}[x, y]_m$ the subspace in $\mathbb{C}[x, y]$ of homogeneous polynomials of degree m.
(a) Prove that $\mathbb{C}[x, y]_m$ is stable under $\mathfrak{sl}_2$.
(b) Prove that $\mathbb{C}[x, y]_m$ is isomorphic to W_m.
(c) Deduce that $W_m = S^m W_1$.

Exercice 10.5.6 (ı) With the same notation as in the previous exercice but replacing $\mathbb{C}[x, y]$ by $k[x, y]$ with $\mathrm{char} k = p > 0$, prove that $k[x, y]$ is again a $\mathfrak{sl}_2(k)$ representation and that $k[x, y]_m$ are subrepresentations.
(ıı) Prove that $k[x, y]_p$ is not irreducible. Give the irreducible submodules in W_p.

Chapter 11

Root systems

This chapter is independent of the previous chapters. We define and study the root systems and give a classification of all root systems. In this chapter, all vector spaces will be considered over $\mathbb{R}$ the field of real numbers. All vector spaces will be finite dimensional.

11.1 Definition

Definition 11.1.1 *Let V be a finite dimensional vector space and $\alpha \in V$ a nonzero vector.* A symmetry with vector α *is an automorphism s of V such that*

- *$s(\alpha) = -\alpha$*
- *the set $H = \{\beta \in V \ / \ s(\beta) = \beta\}$ of fixed elements is an hyperplane of V.*

We have the following fact where we use the notation $\langle f, v\rangle = f(v)$ for $f \in V^\vee$ and $v \in V$.

Fact 11.1.2 *Let s be a symmetry with vector α.*
($\imath$) The space $H = \{\beta \in V \ / \ s(\beta) = \beta\}$ is a complement for $\mathbb{R}\alpha$ in V.
($\imath\imath$) The element s has order 2.
($\imath\imath\imath$) There is a unique element $\alpha^\vee \in V^\vee$ such that $\langle \alpha^\vee, H\rangle = 0$ and $\langle \alpha^\vee, \alpha\rangle = 2$. We have

$$s(v) = v - \langle \alpha^\vee, v\rangle\alpha,$$

($\imath v$) If α and $\alpha^\vee$ are elements in V and $V^\vee$ such that $\langle \alpha^\vee, \alpha\rangle = 2$, then the map s defined by $s(v) = v - \langle \alpha^\vee, v\rangle\alpha$ is a symmetry with vector α.

Proof. *($\imath$)* The space H is of codimension 1 and does not contain α, the result follows.
($\imath\imath$) The order of s on H is 1 and 2 on $\mathbb{R}\alpha$, the result follows from *($\imath$)*.
($\imath\imath\imath$) The uniqueness is clear because H and $\mathbb{R}\alpha$ are in direct sum. Furthermore, for $v \in V$, write $v = h + \lambda\alpha$, we have $s(v) = h - \lambda\alpha$ and $v - \langle \alpha^\vee, v\rangle\alpha = h + \lambda\alpha - 2\lambda\alpha = h - \lambda\alpha$ and the formula follows.
($\imath v$) Let H be the kernel of $\alpha^\vee$. It is an hyperplane and for $h \in H$, we have $s(h) = h$. Furthermore we have the equality $s(\alpha) = -\alpha$ proving the result. □

Lemma 11.1.3 *Let α be a nonzero element in V and R be a finite subset of V which spans V. Then there is at most one symmetry with vector α leaving R invariant.*

Proof. Let s and s' be two such symmetries and let $u = s \circ s'$.

On the one hand, we have $u(\alpha) = \alpha$ and u induces the identity on the quotient $V/\mathbb{R}\alpha$. This proves that the eigenvalues of u are all equal to 1.

On the other hand, we have $u(R) = R$ therefore u induces a permutation of R and therefore, there exists an integer n such that $(u|_R)^n = \mathrm{Id}_R$. But because R spans V we get $u^n = \mathrm{Id}_V$. In particular u is semisimple. As is has only 1 as eigenvalue we get $u = \mathrm{Id}_V$ and $s = s'$. □

Definition 11.1.4 *subset R of a vector space V is called a* root system in V *if the following conditions are satisfied:*

(1) R is finite, spans V and does not contain 0;

(2) for each $\alpha \in R$, there exists a symmetry s_α with vector α leaving R invariant;

*(3) for each α and β in R, the vector $s_\alpha(\beta) - \beta$ is an integer multiple of α (*i.e. $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$*).*

Remark that the symmetry s_α is unique by the above Lemma. The dimension of V is called the *rank of the root system* and the elements α in R are called the *roots* of the root system. The element $\alpha^\vee$ is called the *dual root* or *inverse root* of α.

Remark 11.1.5 By (2), we have for α in R that $-\alpha = s_\alpha(\alpha) \in R$.

Definition 11.1.6 *A root system is called* reduced *if the intersection of R with $\mathbb{R}\alpha$ for $\alpha \in R$ is the set $\{\alpha, -\alpha\}$.*

Fact 11.1.7 *Let R be a non reduced root system and $\alpha \in R$ such that $R \cap \mathbb{R}\alpha$ contains more than the two roots $\{-\alpha, \alpha\}$, then we have*

$$R \cap \mathbb{R}\alpha = \{-2\alpha, -\alpha, \alpha, 2\alpha\}, \text{ or } R \cap \mathbb{R}\alpha = \{-\alpha, -\frac{1}{2}\alpha, \frac{1}{2}\alpha, \alpha\}.$$

Proof. By taking for α the root with biggest coefficient in $R \cap \mathbb{R}_+\alpha$, we may assume that any other root $\beta \in \mathbb{R} \cap \mathbb{R}_+\alpha$ is of the form $t\alpha$ with $0 < t < 1$. Applying point (3) of the definition, we have $s_\alpha(\beta) - \beta = -t\alpha - t\alpha = -2t\alpha \in \mathbb{Z}\alpha$. Therefore we have $t = \frac{1}{2}$ and the result follows. □

Example 11.1.8 (i) The only reduced root system of rank 1 is $\{-\alpha, \alpha\}$ and is called of type A_1.

(ii) The only nonreduced root system of rank 1 is $\{-2\alpha, -\alpha, \alpha, 2\alpha\}$.

(iii) The following subsets in $\mathbb{R}^2$ are root systems:

- $\{-\beta, -\alpha, \alpha, \beta\}$ with $\alpha = (1, 0)$ and $\beta = (0, 1)$. Root system of type $A_1 \times A_1$.
- $\{-\alpha - \beta, -\beta, -\alpha, \alpha, \beta, \alpha + \beta\}$ with $\alpha = (1, 0)$ and $\beta = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$. Root system of type A_2.
- $\{-2\alpha - \beta, -\alpha - \beta, -\beta, -\alpha, \alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$ with $\alpha = (1, 0)$ and $\beta = (-1, 1)$. Root system of type $B_2 = C_2$.
- $\{-3\alpha - 2\beta, -3\alpha - \beta, -2\alpha - \beta, -\alpha - \beta, -\beta, -\alpha, \alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}$ with $\alpha = (1, 0)$ and $\beta = (-\frac{3}{2}, \frac{\sqrt{3}}{2})$. Root system of type G_2.

11.2 Weyl group

Definition 11.2.1 *Let R be a root system in the $\mathbb{R}$-vector space V.*

(ı) The Weyl group *of R is the subgroup of $\mathrm{GL}(V)$ generated by the symmetrie s_α for $\alpha \in R$, we denote it by $W(R)$.*

(ıı) The group of automorphisms *of R is the subgroup of all elements in $\mathrm{GL}(V)$ preserving R, we denote it by $\mathrm{Aut}(R)$.*

Fact 11.2.2 *The Weyl group $W(R)$ is a normal subgroup of $\mathrm{Aut}(R)$ and both are finite.*

Proof. The two groups are contained in the group of permutation of R (the map $\mathrm{Aut}(R) \to \mathfrak{S}(R)$ is injective because R generates V). Furthermore, if $u \in \mathrm{Aut}(R)$, then $us_\alpha u^{-1} = s_{u(\alpha)}$ for all $\alpha \in R$ therefore $W(R)$ is normal in $\mathrm{Aut}(R)$. □

Example 11.2.3 (ı) When R is a reduced root system of rank 2 as in the previous example, then the Weyl group is isomorphic to the dihedral group of order $2n$ (which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}/n\mathbb{Z}$) with $n = 2$ for type $A_1 \times A_1$, $n = 3$ for A_2, $n = 4$ for B_2 and $n = 6$ for G_2.

(ıı) For the automorphism group $\mathrm{Aut}(R)$ of the previous example, we have $|\mathrm{Aut}(R) : W(R)| = 2$ for type $A_1 \times A_1$ and A_2 and $\mathrm{Aut}(R) = W(R)$ in the last two cases.

11.3 Invariant bilinear form

Definition 11.3.1 *Let B be a bilinear form on V, an element $u \in \mathrm{GL}(V)$ preserves B if for all v and v' in V, we have $B(u(v), u(v')) = B(v, v')$. The subgroup of all elements preserving B is called the* orthogonal group *associated to B and denoted by $\mathrm{O}(V, B)$.*

Definition 11.3.2 *A bilinear form B on V is called invariant under a subgroup $G \subset \mathrm{GL}(V)$ if we have $G \subset \mathrm{O}(V, B)$.*

Proposition 11.3.3 *Let R be a root system, there exists a positive definite symmetric bilinear form $(\ , \ )$ on V which is invariant under the Weyl group $W(R)$.*

Proof. Let B be any positive definite bilinear form on V. We define the following symmetric bilinear form:

$$(v, v') = \frac{1}{|W(R)|} \sum_{u \in W(R)} B(u(v), u(v')).$$

We have that $(\ , \ )$ is positive definite and invariant under the Weyl group $W(R)$. □

From now on, we fix a positive definte $W(R)$-invariant bilinear form $(\ , \ )$ on V which has therefore the structure of an Euclidean space. We dnote by $\mathrm{O}(V)$ the group $\mathrm{O}(V, (\ , \ ))$. We have the inclusion $W(R) \subset \mathrm{O}(V)$.

Fact 11.3.4 *For $\alpha \in R$ and $v \in V$ we have the formula*

$$s_\alpha(v) = v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha.$$

Equivalentely, identifying V with $V^\vee$ using $(\ , \ )$ we have the equality

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

Proof. Let $v \in V$ such that $s_\alpha(v) = v$, we have $(v, \alpha) = (s_\alpha(v), s_\alpha(\alpha)) = (-\alpha, v)$ therefore $(v, \alpha) = 0$. For an element $v \in V$, we write $v = \lambda\alpha + v_0$ with $v_0 \in \alpha^\perp$ (the orthogonal being taken for the form $(\ , \ )$). We have $s_\alpha(v) = -\lambda\alpha + v_0$ and $v - 2\frac{(\alpha,v)}{(\alpha,\alpha)}\alpha = v - 2\lambda\alpha = s_\alpha(v)$.

This means that for $v \in V$, we have $\langle \alpha^\vee, v\rangle = 2\frac{(\alpha,v)}{(\alpha,\alpha)} = \left(\frac{2\alpha}{(\alpha,\alpha)}, v\right)$ giving the second assertion. □

11.4 Dual root system

Proposition 11.4.1 *Let R be a root system and denote by $R^\vee$ the set of all dual roots $\alpha^\vee$ for $\alpha \in R$.*
(ı) The set $R^\vee$ in $V^\vee$ is a root system.
(ıı) We have $(\alpha^\vee)^\vee = \alpha$ and $(R^\vee)^\vee = R$.

Proof. Let us check the three axioms of roots systems on $R^\vee$.

(1) The set $R^\vee$ is in bijection with R and is therefore finite. Furthermore, for $\alpha \in R$, the symmetry s_α is not the identity (because $s_\alpha(\alpha) = -\alpha$) therefore $\alpha^\vee$ is not trivial. Finally, the set $R^\vee$ spans $V^\vee$. Indeed, if not there would be a non-zero vector $v \in V$ such that $\alpha^\vee(v) = 0$ for all $\alpha \in R$. This would imply $v \in R^\perp$ but R spans V thus we have $R^\perp = V^\perp = 0$ a contradiction.

(2) Let us define $s_{\alpha^\vee}(f) = f - \langle f, v\rangle\alpha^\vee$. This is a symmetry in $V^\vee$ and $s_{\alpha^\vee}(\alpha^\vee) = -\alpha^\vee$ therefore it has vector $\alpha^\vee$. We have $s_{\alpha^\vee} = {}^t s_\alpha$. Indeed, by definition of the transpose, for any $f \in V^\vee$ and $v \in V$, we have ${}^t s_\alpha(f)(v) = f \circ s_\alpha(v) = f(v - \langle\alpha^\vee, v\rangle\alpha) = f(v) - \langle\alpha^\vee, v\rangle f(\alpha) = (f - \langle f, \alpha\rangle\alpha^\vee)(v) = s_{\alpha^\vee}(f)(v)$. In particular, for $\alpha, \beta \in R$ and $v \in V$, we have

$$\langle s_{\alpha^\vee}(\beta^\vee), v\rangle = \langle \beta^\vee, s_\alpha(v)\rangle = \frac{2(\beta, s_\alpha(v))}{(\beta,\beta)} = \frac{2(s_\alpha(\beta), v)}{(s_\alpha(\beta), s_\alpha(\beta))} = \langle s_\alpha(\beta)^\vee, v\rangle$$

therefore $s_{\alpha^\vee}(\beta^\vee) = s_\alpha(\beta)^\vee$ and $s_{\alpha^\vee}$ preserves $R^\vee$.

We see from this point that $(\alpha^\vee)^\vee = \alpha$ and that $(R^\vee)^\vee = R$.

(3) This condition is equivalent to the condition $\langle\alpha^\vee, \beta\rangle \in \mathbb{Z}$ for all α and β in R. But we have $\langle\alpha^\vee, \beta\rangle = \langle\alpha^\vee, (\beta^\vee)^\vee\rangle$ and point (3) follows. □

Definition 11.4.2 *The root system $R^\vee$ is called the* dual root system.

Proposition 11.4.3 *The Weyl group $W(R^\vee)$ is isomorphic to the Weyl group $W(R)$.*

More precisely, $\Phi : \mathrm{GL}(V) \to \mathrm{GL}(V^\vee)$ be the isomorphism defined by $\Phi(u) = {}^t u^{-1}$. Then Φ restricts to an isomorphism from $W(R)$ to $W(R^\vee)$.

Proof. We have seen that ${}^t s_\alpha = s_{\alpha^\vee}$ therefore $\Phi(s_\alpha) = s_{\alpha^\vee}$. Furthermore, because Φ is involutive we get the result. □

11.5 Relative position of two roots

Let $R \subset V$ be a root system. From now on we fix a positive definite bilinear form $(\ , \ )$ on V which is invariant under the Weyl group $W = W(R)$. This defines an Euclidian structure on V. We want in this section to describe all possibilities for the relative position of two roots.

Let α and β be two roots and let us denote by $|\alpha|$ the length of α (*i.e.* $|\alpha| = \sqrt{(\alpha,\alpha)}$). Let us denote by ϕ the angle between the lines geberated by α and β. Remark that if α and β are colinear, we already know they relative position.

Proposition 11.5.1 *If α and β are non colinear, then there are 7 possibilites (up to transposition of α and β):*

- $\langle\alpha^\vee,\beta\rangle = 0$, $\langle\beta^\vee,\alpha\rangle = 0$ *and* $\phi = \pi/2$;
- $\langle\alpha^\vee,\beta\rangle = 1$, $\langle\beta^\vee,\alpha\rangle = 1$, $\phi = \pi/3$ *and* $|\beta| = |\alpha|$;
- $\langle\alpha^\vee,\beta\rangle = -1$, $\langle\beta^\vee,\alpha\rangle = -1$, $\phi = 2\pi/3$ *and* $|\beta| = |\alpha|$;
- $\langle\alpha^\vee,\beta\rangle = 2$, $\langle\beta^\vee,\alpha\rangle = 1$, $\phi = \pi/4$ *and* $|\beta| = \sqrt{2}|\alpha|$;
- $\langle\alpha^\vee,\beta\rangle = -2$, $\langle\beta^\vee,\alpha\rangle = -1$, $\phi = 3\pi/4$ *and* $|\beta| = \sqrt{2}|\alpha|$;
- $\langle\alpha^\vee,\beta\rangle = 3$, $\langle\beta^\vee,\alpha\rangle = 1$, $\phi = \pi/6$ *and* $|\beta| = \sqrt{3}|\alpha|$;
- $\langle\alpha^\vee,\beta\rangle = -3$, $\langle\beta^\vee,\alpha\rangle = -1$, $\phi = 5\pi/6$ *and* $|\beta| = \sqrt{3}|\alpha|$.

Proof. We habe the equality $(\alpha,\beta) = |\alpha||\beta|\cos\phi$. In particular, we get

$$2\frac{|\beta|}{|\alpha|}\cos\phi = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} = \langle\alpha^\vee,\beta\rangle \in \mathbb{Z}$$

and we deduce that $4\cos^2\phi = \langle\alpha^\vee,\beta\rangle\langle\beta^\vee,\alpha\rangle \in \mathbb{Z}$. We choose α and β such that $|\alpha| \leq |\beta|$. This implies that $\langle\alpha^\vee,\beta\rangle \geq \langle\beta^\vee,\alpha\rangle$.

The previous relation gives in particular that we have $4\cos^2\phi = 0;\ 1;\ 2;\ 3;\ 4$ and the corresponding values for $\langle\alpha^\vee,\beta\rangle$ and $\langle\beta^\vee,\alpha\rangle$ are $0,0;\ 1,1$ or $-1,-1;\ 2,1$ or $-2,-1;\ 3,1$ or $-3,-1;\ 4,1$ or $-4,-1$. Remark that in the case $\cos^2\phi = 1$, then $\phi = 0$ or π and α and β are colinear so this case does not happend.

In the other cases we have $\cos\phi = 0;\ \pm1/2;\ \pm\sqrt{2}/2;\ \pm\sqrt{3}/2$ and therefore $\phi = \pi/2;\ \pi/3$ or $2\pi/3$; $\pi/4$ or $3\pi/4$; $\pi/6$ or $5\pi/6$ and the result follows. □

Proposition 11.5.2 *Let α and β be non colinear roots. If $\langle\alpha^\vee,\beta\rangle > 0$, then $\alpha - \beta$ is a root.*

Proof. From the previous proposition, we get that $\langle\alpha^\vee,\beta\rangle = 1$ of $\langle\beta^\vee,\alpha\rangle = 1$. In the first case, we have $s_\alpha(\beta) = \beta - \langle\alpha^\vee,\beta\rangle\alpha = \beta - \alpha$ therefore $\alpha - \beta = -s_\alpha(\beta) \in R$. In the second case, we have $\alpha - \beta = \alpha - \langle\beta^\vee,\alpha\rangle\beta = s_\beta(\alpha) \in R$. □

11.6 System of simple roots

Definition 11.6.1 *A subset S of R is called a* system of simple roots *or a* base of R *if the following two conditions are satisfied:*

- *S is a basis for V;*
- *each root $\beta \in R$ can be written as a linear combination of roots in S as $\beta = \sum_{\alpha\in S} a_\alpha\alpha$ with the alternative*
  - *$a_\alpha \geq 0$ for all $\alpha \in S$ or*
  - *$a_\alpha \leq 0$ for all $\alpha \in S$.*

Example 11.6.2 With the notation as in Example 11.1.8, the set $S = \{\alpha, \beta\}$ is a base for the different root systems.

Theorem 11.6.3 *There exists a base.*

Proof. We will prove a more precise statement. Let $t \in V^\vee$ be an element such that $\langle t, \alpha\rangle \neq 0$ for all $\alpha \in R$. This is possible because R is a finite set and $\mathbb{R}$ an infinite field. Define the set

$$R_t^+ = \{\alpha \in R \ / \ \langle t, \alpha\rangle > 0\}.$$

We have $R = R_t^+ \cup (-R_t^+)$.

Definition 11.6.4 *An element $\alpha \in R_t^+$ is called* decomposable *if there exists two root β and γ in R_t^+ such that $\alpha = \beta + \gamma$. If α is not decomposable, it is called* indecomposable. *We denote by S_t the set of indecomposable elements in R_t^+.*

We will prove that S_t is a base for the root system and that any base for the root system is of the form S_t.

Lemma 11.6.5 *Any element $\alpha \in R_t^+$ is a linear combination with non-negative integer coefficients of elements in S_t.*

Proof. Let I be the set of all elements $\alpha \in R_t^+$ for which the above property is not satisfied. If I were non empty, then there exists an element $\alpha \in I$ with $\langle t, \alpha\rangle$ minimal. But α is not in S_t (otherwise α can be written as a linear combination with non-negative integer coefficients of elements in S_t), therefore it is reducible and $\alpha = \beta + \gamma$ with $\beta, \gamma \in R_t^+$. We have $\beta \in I$ or $\gamma \in I$ (otherwise $\alpha \in I$). Say we have $\beta \in I$, then we have $\langle t, \beta\rangle = \langle t, \alpha\rangle - \langle t, \gamma\rangle < \langle t, \alpha\rangle$ a contradiction to the minimality of $\langle t, \alpha\rangle$. □

Lemma 11.6.6 *We have $(\alpha, \beta) \leq 0$ for all α and β in S_t.*

Proof. If $(\alpha, \beta) > 0$, then by Proposition 11.5.2 (remark that α and β cannot be colinear otherwise one of them would not be in S_t) we know that $\gamma = \alpha - \beta$ is a root. If $\gamma \in R_t^+$, then we have $\alpha = \beta + \gamma$ a contradiction to $\alpha \in S_t$. If $\gamma \in (-R_t^+)$, then $-\gamma \in R_t^+$ and we have $\beta = \alpha + (-\gamma)$ a contradiction to $\beta \in S_t$ □

Lemma 11.6.7 *Let $t \in V^\vee$ and $A \subset V$ such that*

- $\langle t, \alpha\rangle > 0$ *for $\alpha \in A$;*
- $(\alpha, \beta) \leq 0$ *for $\alpha, \beta \in A$,*

then the elements in A are linearly independent.

Proof. Let us consider a relation $\sum_{\alpha \in A} a_\alpha \alpha = 0$. Let B be the subset of elements $\alpha \in A$ with $a_\alpha \geq 0$ and C be the subset of elements $\alpha \in A$ with $a_\alpha \leq 0$. Define $b_\alpha = \alpha_\alpha$ for $\alpha \in B$ and $c_\alpha = -\alpha_\alpha$ for $\alpha \in C$, we have b_α and c_α non negative. We obtain a relation

$$\sum_{\alpha \in B} b_\alpha \alpha = \sum_{\alpha \in C} c_\alpha \alpha.$$

We have the inequality

$$\left(\sum_{\alpha\in B} b_\alpha\alpha, \sum_{\alpha\in B} b_\alpha\alpha\right) = \left(\sum_{\alpha\in B} b_\alpha\alpha, \sum_{\alpha\in C} c_\alpha\alpha\right) = \sum_{\alpha\in B}\sum_{\beta\in C} b_\alpha c_\beta(\alpha,\beta) \leq 0$$

therefore $\sum_{\alpha\in B} b_\alpha\alpha = 0$. Applying $\langle t, \ \rangle$, we get

$$0 = \langle t, \sum_{\alpha\in B} b_\alpha\alpha\rangle = \sum_{\alpha\in B} b_\alpha\langle t,\alpha\rangle$$

and because $\langle t,\alpha\rangle > 0$ for all $\alpha \in A$ and $b_\alpha \geq 0$, we have $b_\alpha = 0$ for all $\alpha \in B$. The symmetric argument gives that $c_\alpha = 0$ for all $\alpha \in C$. ☐

The above three lemmas prove that S_t is a base for R. Let us now prove that any base S for R is of the form S_t for some $t \in V^\vee$. Indeed, let $t \in V^\vee$ such that $\langle t,\alpha\rangle > 0$ for all $\alpha \in S$. This is possible because S is a base of V. Let us denote by R^+ the set of all roots $\alpha \in R$ such that α can be written as a linear combination of elements in S with non negative coefficients. We have $R^+ \subset R_t^+$ and $(-R^+) \subset (-R_t^+)$ therefore we have equalities in both inclusions. Let $\alpha \in S$, then α is indecomposable in R_t^+. Indeed, if $\alpha = \beta + \gamma$ with $\beta, \gamma \in R_t^+$, then β and γ are linear combination of elements in S with non negative coefficients, therefore so is α. But the only combinaison for α is $\alpha = \alpha$, therefore $\{\beta,\gamma\} = \{\alpha, 0\}$ a contradiction. We thus have $S \subset S_t$ but they have the same number of elements are they are basis for V and we have $S = S_t$. ☐

From now on we fix a base S of the root system R.

Definition 11.6.8 *We denote by R^+ the set of roots $\alpha \in R$ which can be written as a linear combination of elements in S with non negative coefficients. The elements in R^+ are called* positive roots*. The elements in $R^- = (-R^+)$ are called the* negative roots.

Proposition 11.6.9 *Any positive α root can be written as $\alpha = \sum_{i=1}^k \alpha_i$ with $\alpha_i \in S$ such that for all $j \in [1,k]$ the sum $\sum_{i=1}^j \alpha_i$ is a root.*

Proof. Let $t \in V^\vee$ be the element defined by $\langle t,\beta\rangle = 1$ for all $\beta \in S$. We proceed by induction on $\langle t,\alpha\rangle$. For $\langle t,\alpha\rangle = 1$, then $\alpha \in S$ and the result follows. Assume that the result holds for all root $\beta \in R^+$ such that $\langle t,\beta\rangle < \langle \alpha, t\rangle$. We may assume that $\alpha \notin S$.

Lemma 11.6.10 *There exists a simple root β such that $(\alpha,\beta) > 0$.*

Proof. Otherwise, by Lemma 11.6.7, the set $S \cup \{\alpha\}$ would be linearly independent, a contradiction to the fact that $\alpha \notin S$ and that S is a base. ☐

There is therefore a simple root $\alpha_k \in S$ such that $(\alpha,\alpha_k) > 0$. But this implies that $\alpha - \alpha_k$ is a root and $\langle t, \alpha - \alpha_k\rangle = \langle t,\alpha\rangle - 1 > 0$ therefore $\alpha - \alpha_k \in R^+$ and we apply the induction hypothesis on $\alpha - \alpha_k$ to conclude the proof. ☐

Proposition 11.6.11 *If R is reduced, then for $\alpha \in S$, the symmetry s_α associated with α leaves $R \setminus \{\alpha\}$ invariant.*

Proof. Let $\beta \in R\setminus\{\alpha\}$ and write $\beta = \sum_{\gamma\in S} a_\gamma\gamma$ with $a_\gamma \geq 0$. There is a simple root γ_0 different from α such that $a_{\gamma_0} > 0$. Now we have

$$s_\alpha(\beta) = \sum_{\gamma\in R} a_\gamma s_\alpha(\gamma) = \sum_{\gamma\in R} a_\gamma\gamma - \left(\sum_{\gamma\in R} a_\gamma\langle\alpha^\vee,\gamma\rangle\right)\alpha.$$

Therefore, the coefficient of γ_0 is again $a_{\gamma_0} > 0$ for the root $s_\alpha(\beta)$ which has to be positive and different from α. ☐

Corollary 11.6.12 *If R is reduced and let ρ be half the sum of the positive roots. We have the equality $s_\alpha(\rho) = \rho - \alpha$ for all $\alpha \in S$.*

Proof. We have by definition the following formula for ρ and we define ρ_α by the right hand side formula:

$$\rho = \frac{1}{2} \sum_{\beta \in R^+} \beta \text{ and } \rho_\alpha = \frac{1}{2} \sum_{\beta \in R^+ \setminus \{\alpha\}} \beta.$$

Therefore $\rho = \rho_\alpha + \alpha/2$. By the above proposition, we have $s_\alpha(\rho_\alpha) = \rho_\alpha$, therefore we have the equality $s_\alpha(\rho) = \rho_\alpha - \alpha/2 = \rho - \alpha$. ☐

Proposition 11.6.13 *If R is reduced, then the set $S^\vee = \{\alpha^\vee \ / \ \alpha \in S\}$ is a root system for the dual root system $R^\vee$.*

Proof. Recall that we fixed an invariant bilinear form $(\ , \ )$ on V identifying V with $V^\vee$ via the map $\Phi : V \to V^\vee$ defined by $u \mapsto (v \mapsto (u, v))$. We therefore also have an invariant bilinear form on $V^\vee$ defined by $((\varphi, \psi)) = (\Phi^{-1}(\varphi), \Phi^{-1}(\psi))$. Remark that under this identification, we have

Fact 11.6.14

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)} \Phi(\alpha).$$

Proof. Indeed, we have seen that the following formula $\langle \alpha^\vee, v \rangle = \frac{2}{(\alpha,\alpha)}(\alpha, v)$ holds for any $v \in V$, the result follows. ☐

In particular, we know that the set of roots $S^\vee$ is a basis of the vector space $V^\vee$ because non-zero multiple of its elements are the image of the basis S by Φ. In symbols,

$$S^\vee = (\alpha^\vee)_{\alpha \in S} = \left(\frac{2}{(\alpha, \alpha)} \Phi(\alpha) \right)_{\alpha \in S}.$$

Furthermore, we know that S is a basis for the roots system therefore any root $\beta \in R$ can be writen as a sum

$$\beta = \sum_{\alpha \in S} a_\alpha \alpha$$

where the coefficients a_α are integers all non negative or non positive at the same time. We get

$$\beta^\vee = \frac{2}{(\beta, \beta)} \Phi(\beta) = \frac{2}{(\beta, \beta)} \Phi \left(\sum_{\alpha \in S} a_\alpha \alpha \right) = \frac{2}{(\beta, \beta)} \sum_{\alpha \in S} a_\alpha \Phi(\alpha) = \sum_{\alpha \in S} a_\alpha \frac{(\alpha, \alpha)}{(\beta, \beta)} \alpha^\vee$$

therefore the coefficients are all non negative or non positive at the same time. We however *a priori* do not know if these coefficients are integers.

To prove this, fix $t \in V^\vee$ such that $\langle t, \alpha \rangle = 1$ for $\alpha \in S$. We have $S = \{\alpha \in R \ / \ \langle t, \alpha \rangle = 1\}$ and for $\beta \in R$, we have $\langle t, \beta \rangle \in \mathbb{Z}$. Let $T = \Phi^{-1}(t) \in V$. We have the equality

$$\langle t, \alpha \rangle = (T, \alpha) = \frac{(\alpha, \alpha)}{2} \langle \alpha^\vee, T \rangle$$

therefore we have the equality $(R^\vee)^+_T = \{\alpha^\vee \in R^\vee \ / \ \alpha \in R^+_t\}$. The set $S^\vee_T$ of indecomposable elements in $(R^\vee)^+_T$ is a basis for $R^\vee$. By definition, $S^\vee$ is contained in $(R^\vee)^+_T$. By what we proved above, the cone generated by $S^\vee$ is equal to the cone generated by $(R^\vee)^+_T$. Therefore, the extremal rays of these cones, which are the half line generated by the elements in $S^\vee$ or in $S^\vee_T$ are the same. The elements in $S^\vee$ and in $S^\vee_T$ are therefore proportional and because R is reduced, the result follows. □

The next result describes how the Weyl group acts on the set of basis.

Theorem 11.6.15 *Assume that R is reduced, let W be its Weyl group and fix S a basis of R.*
(ı) For any $t \in V^\vee$, there exists an element $w \in W$ such that $\langle w(t), \alpha\rangle \geq 0$ for all $\alpha \in S$.
(ıı) If S' is another basis of R, there exists $w \in W$ such that $w(S') = S$.
(ııı) For each $\beta \in R$, there exists $w \in W$ such that $w(\beta) \in S$.
(ıv) The group W is generated by the symmetries s_α for $\alpha \in S$.

Proof. Let us define W_S to be the subgroup of W generated by the symmetries s_α for $\alpha \in S$. We first prove *(ı)*, *(ıı)* and *(ııı)* and then prove that $W = W_S$.

(ı) for W_S. We proceed by induction on the number $n(t)$ of simple roots α such that $\langle t, \alpha\rangle < 0$. If $n(t) = 0$, then just take $w = \mathrm{Id}$. Assume that the result is true for $n(t) = n$ and let t such that $n(t) = n + 1$. Let α be a simple root with $\langle t, \alpha\rangle < 0$. Then we consider $s_\alpha(t)$. We have $\langle s_\alpha(t), \alpha\rangle = \langle t, s_\alpha(\alpha)\rangle = -\langle t, \alpha\rangle > 0$ and for $\beta \in S \setminus \{\alpha\}$, we have $\langle s_\alpha(t), \beta\rangle = \langle t, s_\alpha(\beta)\rangle$. But on the set $S \setminus \{\alpha\}$, the linear form $\langle t, \ \rangle$ takes n times a negative value and because $s_\alpha(S \setminus \{\alpha\}) = S \setminus \{\alpha\}$ the same is true for $s_\alpha(t)$. Therefore we have $n(s_\alpha(t)) = n$ and the result follows by induction.

(ıı) If S' is a basis, there exists an element $t \in V^\vee$ such that $S' = S_t$. Let $w \in W_S$ given by *(ı)* such that $\langle w(t), \alpha\rangle \geq 0$ for $\alpha \in S$. The choice of t also implies that $\langle t, \beta\rangle \neq 0$ for $\beta \in R$ therefore $\langle w(t), \alpha\rangle > 0$ for $\alpha \in S$. The set of positive roots R^+ is $R^+_{w(t)}$ therefore $S = S_{w(t)}$. This gives $S = S_{w(t)} = w(S_t) = w(S')$.

(ııı) Let t be a linear form such that $\langle t, \alpha\rangle \neq$ for all $\alpha \in R$ and such that

$$|\langle t, \beta\rangle| = \min\{|\langle t, \alpha\rangle| \ / \ \alpha \in R\}$$

and $\langle t, \beta\rangle > 0$. We prove that $\beta \in S_t$ and the result will follow from *(ıı)*. Assume we have $\beta = \alpha + \gamma$ with α and γ in R^+_t. We have $\langle t, \beta\rangle = \langle t, \alpha\rangle + \langle t, \gamma\rangle$ therefore $0 < \langle t, \alpha\rangle = \langle t, \beta\rangle - \langle t, \gamma\rangle < \langle t, \beta\rangle$ a contradition to the minimality.

(ıv) We need to prove that $s_\beta \in W_S$ for β any root. By *(ııı)*, there exists $w \in W_S$ such that $w(\beta) = \alpha \in S$, therefore we have $ws_\beta w^{-1} = s_\alpha \in W_S$ ands the result follows. □

Definition 11.6.16 *The set C_S of elements t in $V^\vee$ such that $\langle t, \alpha\rangle > 0$ for all $\alpha \in S$ is called the* Weyl chamber *associated to S.*

Remark 11.6.17 The Weyl chambers are the connected components of

$$V^\vee \setminus \bigcup_{\alpha \in R} \alpha^\perp$$

where $\alpha^\perp = \{t \in V^\vee \ / \ \langle t, \alpha\rangle = 0\}$. The previous result proves that the Weyl group W act transitively on the Weyl chambers.

We shall prove later (if time permits) the following:

Theorem* 11.6.18 *The group W acts simply transitively on the set of Weyl chambers.*

Remark 11.6.19 One can be more precise and prove that the Weyl group is generated by the symmetries s_α with $\alpha \in S$ and that the only relations among the $(s_\alpha)_{\alpha\in S}$ are the relations

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$$

with $m(\alpha, \beta) = 2, 3, 4$ or 6 if the angle between α and β is $\pi/2$, $2\pi/3$, $3\pi/4$ or $5\pi/6$.

11.7 The Cartan matrix

Definition 11.7.1 *The* Cartan matrix *for the root system associated to the base S is by definition the matrix $(\langle\beta^\vee, \alpha\rangle)_{\alpha,\beta\in S}$.*

Example 11.7.2 Consider the rank two root systems as in Example 11.1.8 and choose a base S as in Example 11.6.2. The Cartan matrix for type A_2, $B_2 = C_2$ or G_2 are the following matrices:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

Proposition 11.7.3 *Let R and R' be two reduced root systems in two vector spaces V and V'. Let S and S' be basis for R and R' and let $\phi : S \to S'$ be a bijection such that $\langle\phi(\alpha)^\vee, \phi(\beta)\rangle = \langle\alpha^\vee, \beta\rangle$. Then there exists a unique isomoprhism $f : V \to V'$ such that $f|_S = \phi$ and realising a bijection from R to R'.*

Proof. Because S and S' are basis of V and V', there exists a unique isomorphism $f : V \to V'$ such that $f|_S = \phi$. Let us prove that f maps bijectively R into R'. For this we prove the comuutation relation: $s_{\phi(\alpha)} \circ f = f \circ s_\alpha$. To prove this we only need to check on a basis, for example on S. We have for $\beta \in S$:

$$s_{\phi(\alpha)}(f(\beta)) = s_{\phi(\alpha)}(\phi(\beta)) = \phi(\beta) - \langle\phi(\alpha)^\vee, \phi(\beta)\rangle\phi(\alpha) = f(\beta) - \langle\alpha^\vee, \beta\rangle f(\alpha) = f(s_\alpha(\beta))$$

and the commutation relation holds. Therefore if W resp. W' is the Weyl group of R resp. R', then we have $W' = fWf^{-1}$, therefore, because $W(S) = R$ and $W'(S') = R'$ we get $f(R) = R'$. □

Corollary 11.7.4 *A reduced root system is determined by its Cartan matrix.*

Definition 11.7.5 *The subgroup* $\mathrm{Out}(R)$ *of the group* $\mathrm{Aut}(S)$ *of bijections of S defined by*

$$\mathrm{Out}(R) = \{\sigma \in \mathrm{Aut}(S) \ / \ \langle\sigma(\alpha)^\vee, \sigma(\beta)\rangle = \langle\alpha^\vee, \beta\rangle \textit{ for all } \alpha \textit{ and } \beta \textit{ in } S\}$$

is called the group of outer automorphism of the root system.

Corollary 11.7.6 *The group* $\mathrm{Out}(R)$ *is the subgroup of* $\mathrm{Aut}(R)$ *leaving S invariant.*

Proposition* 11.7.7 *The group* $\mathrm{Aut}(R)$ *is the semidirect product of* $\mathrm{Out}(R)$ *and W .*

Proof. First remark that W is normal in $\mathrm{Aut}(R)$. Indeed, if $f \in \mathrm{Aut}(R)$ and if $\alpha \in R$ is a root, then $fs_\alpha f^{-1} = s_{f(\alpha)}$. Let w be in the intersection $W \cap \mathrm{Out}(R)$, then w maps the Weyl chamber defined by C to itself therefore $w = 1$ by Theorem* 11.6.18. Furthermore, for $f \in \mathrm{Aut}(R)$, we have that $f(S)$ is a base for R therefore, there exists $w \in W$ such that $w(f(S)) = S$ thus $w \circ f \in \mathrm{Out}(R)$ and the result follows. □

Corollary* 11.7.8 *The group* $\mathrm{Out}(R)$ *is isomorphic to the quotient* $\mathrm{Aut}(R)/W$.

11.8 The Coxeter graph

Definition 11.8.1 *A* Coxeter graph *is a finite graph such that the vertices are linked by 0, 1, 2 or 3 edges.*

Definition 11.8.2 *Let R be a root system and let S be a base of R. The* Coxeter graph of R with respect to S *is the graph whose vertices are the elements in S and such that two vertices α and β in S are linked by $\langle \alpha^\vee, \beta\rangle\langle \beta^\vee, \alpha\rangle$ vertices.*

Remark 11.8.3 Recall that we have $\langle \alpha^\vee, \beta\rangle\langle \beta^\vee, \alpha\rangle$ is an integer with value $4\cos^2\phi$ where ϕ is the angle between α and β so that, because α and β are not colinear, the above definition defines a Coxeter graph.

Lemma 11.8.4 *If S and S' are two basis of R, then the Coxeter graph of R associated to S and S' are isomorphic.*

Proof. We know that there is an element $w \in W$ such that $w(S) = S'$. This element w induces a bijection from S to S'.

Fact 11.8.5 *For w in W, $f \in V^\vee$ and $v \in V$, we have $\langle w(f), w(v)\rangle = \langle f, v\rangle$.*

Proof. We only need to check that $\langle s_\alpha(f), v\rangle = \langle f, s_\alpha(v)\rangle$ for any root α. But we have

$$\begin{aligned}\langle s_\alpha(f), v\rangle &= \langle f - \langle f,\alpha\rangle\alpha^\vee, v - \langle\alpha^\vee, v\rangle\alpha\rangle\\ &= \langle f, v\rangle - \langle f,\alpha\rangle\langle\alpha^\vee, v\rangle - \langle f,\alpha\rangle\langle\alpha^\vee, v\rangle + \langle f,\alpha\rangle\langle\alpha^\vee, v\rangle\langle\alpha^\vee,\alpha\rangle\\ &= \langle f, v\rangle - \langle f,\alpha\rangle\langle\alpha^\vee, v\rangle - \langle f,\alpha\rangle\langle\alpha^\vee, v\rangle + 2\langle f,\alpha\rangle\langle\alpha^\vee, v\rangle\\ &= \langle f, v\rangle.\end{aligned}$$

The result follows. □

Wew therefore have $\langle w(\alpha)^\vee, w(\beta)\rangle\langle w(\beta)^\vee, w(\alpha)\rangle = \langle\alpha^\vee,\beta\rangle\langle\beta^\vee,\alpha\rangle$, giving that the two Coxeter graph are isomorphic. □

Example 11.8.6 The Coxeter graphs of roots systems of type A_1, $A_1 \times A_1$, A_2, $B_2 = C_2$ and G_2 are the following:

A_1 $\quad A_1 \times A_1$ $\quad A_2$ $\quad B_2 = C_2$ $\quad G_2$

11.9 Irreducible root systems

Proposition 11.9.1 *Fort $i \in \{1, 2\}$, let R_i a root system in a vector space V_i. Let V be the direct sum of V_1 and V_2 and identify R_i as subsets of V. Then the union $R = R_1 \cup R_2$ is a root system in V.*

Proof. *(1)* We have that R is finite, spans V and does not contain 0.

(2) For α an element in R then α is in R_1 or in R_2. Say it is in R_1, then there is a symmetry s_α on V_1 and we extend this symmetry on V by the identity on V_2. This defines a symmetry on V mapping R_1 to R_1 because s_α does on V_1 and R_2 on R_2 because it is the identity on V_2.

(3) Let α and β be elements in R. If both are in R_1 or in R_2, then $s_\alpha(\beta) - \beta$ is an integer multiple of α because R_1 and R_2 are root systems. If $\alpha \in R_1$ and $\beta \in R_2$, then $s_\alpha(\beta) = \beta$ and $s_\alpha(\beta) - \beta = 0$ is an integer multiple of α. □

Definition 11.9.2 *A root system R in V is called* reductible *if there exists a non trivial direct sum $V = V_1 \oplus V_2$ with $R_1 = V_1 \cap R$ and $R_2 = V_2 \cap R$ root systems in V_1 and V_2. A root system is called* irreducible *if it is not reductible.*

Proposition 11.9.3 *Let R be a root system in V and suppose that V is a direct sum of V_1 and V_2 such that R is contained in $V_1 \cup V_2$. Let $R_i = V \cap R$ for $i \in \{1, 2\}$.*

($\imath$) The spaces V_1 and V_2 are orthogonal for any invariant bilinear form.

($\imath\imath$) For $i \in \{1, 2\}$, the subset R_i is a root system in V_i.

Proof. *($\imath$)* Let $\alpha \in R_1$ and $\beta \in R_2$. We have that $s_\alpha(\beta) = \beta - \langle\alpha^\vee, \beta\rangle\alpha$ is a root therefore in V_1 or in V_2. But β in is V_2 and non zero therefore $s_\alpha(\beta) \in V_2$ which implies that $\langle\alpha^\vee, \beta\rangle\alpha \in V_1 \cap V_2 = 0$ thus $\langle\alpha^\vee, \beta\rangle = 0$ and for any invariant form we have $(\alpha, \beta) = 0$. Remark that R_1 and R_2 span V_1 and V_2 (because R spans V), the result follows.

($\imath\imath$) We know that R_i is finite and does not contain 0 (because R is finite and does not contain 0) and because R spans V, the R_i respectively span V_i. Let $\alpha \in R_1$, then α is a root in R therefore s_α is a bijection of $R_1 \cup R_2$. But because $\langle\alpha^\vee, \beta\rangle = 0$ for $\beta \in R_2$, we have that s_α is the identity on R_2 and therefore maps R_1 on itself. The same is true for $\alpha \in R_2$. Finally, if α and β are in R_1, then $s_\alpha(\beta) - \beta$ is an integer multiple of α because this is true for R. □

Corollary 11.9.4 *A root system R in V is reductible if and only if there exists a non trivial decomposition $V = V_1 \oplus V_2$ such that R is contained in $V_1 \cup V_2$.*

Proposition 11.9.5 *A root system is irreducible if and only if its Coxeter graph is non empty and connected.*

Proof. If R is reductible, write $R = R_1 \cup R_2$ and we take $S = S_1 \cup S_2$ where S_i is a basis of R_i for $i \in \{1, 2\}$. This is easily seen to be a basis of R. By the above proposition, we see that the Coxeter graph is not connected: the graphs with vertices in S_1 and S_2 are disconnected.

Conversely, if the Coxeter graph of a root system R in V is not connected, then let $S = S_1 \cup S_2$ be a decomposition of a basis S such that for $\alpha \in S_1$ and $\beta \in S_2$, we have $\langle\alpha, \beta\rangle = 0$. Let V_1 be the span of S_1 and let V_2 be the span of S_2. These spaces are non trivial and we need to prove that R is contained in $V_1 \cup V_2$. But for $\alpha \in R$, there exists $w \in W$ such that $w(\alpha) \in S$, therefore $w(\alpha)$ is is S_1 or in S_2. Let us assume it is in S_1. We prove that α is in V_1. Indeed, the group W is spanned by reflections s_β for $\beta \in S$. We have $s_\beta(V_1) \subset V_1$ and $s_\beta(V_2) \subset V_2$ (because s_β maps elements in S_1 resp. S_2 to linear combinaison of elements in S_1 resp. S_2). Therefore $\alpha = w^{-1}(w(\alpha))$ is in V_1. □

11.10 Exercices

Exercice 11.10.1 Prove that the only two dimensional reduced roots systems are those of type A_2, $B_2 = C_2$ and G_2.

Exercice 11.10.2 Complete the root system B_2 to obtain a nonreduced root system. Can one do the same with root systems of type A_2 and G_2?

Exercice 11.10.3 Let R be a non reduced root system and S a base of R. Let S_1 (resp. S_2) be the subset of S of roots α such that 2α is not a root (resp. such that 2α is a root). Prove that the set

$$S^\vee = \{\alpha^\vee \ / \ \alpha \in S_1\} \cup \{\alpha^\vee/2 \ / \ \alpha \in S_2\}$$

is a base for $R^\vee$.

Chapter 12

Classification of Connected Coxeter graphs

In this chapter we give the classification of a special class of Coxeter graphs. In this chapter again, the base field is the fiels of real numbers $\mathbb{R}$.

We first define, for a Coxeter graph, a vector space V with a symmetric bilinear form b. Let C be a Coxeter graph and denote by C_0 and C_1 the set of vertices and the set of edges in C. Let V_C be a vector space of of dimension the number of elements in C_0 (*i.e.* the number of vertices in C) and let us fix a base $(e_v)_{v\in C_0}$. We define a quadratic form b_C on V_C by linearity and by

$$b_C(e_v, e_v) = 1 \text{ and } b_C(e_v, e_{v'}) = -\cos\frac{\pi}{m+2} \text{ for } v \neq v'$$

where m is the number of edges between v and v' (recall that $m \in \{0, 1, 2, 3\}$). We shall denote $b_c(e_v, e_{v'})$ by $q_{v,v'}$ and the number m of edges between v and v' by $m(v, v')$.

Definition 12.0.4 *The vector space V_C together with the bilinear form b_C is called the* geometric representation *of C.*

Definition 12.0.5 *A Coxeter graph is called* finite *or* non degenerate *if the bilinear form b_C of the geometric representation is positive definite.*

12.1 Contraction of an edge

Let C be a finite Coxeter graph and let e be an edge of C *i.e.* $e \in C_1$ and let v and v' be the two vertices related by e.

Lemma 12.1.1 *Let C be a finite connected Coxeter graph, then C has no loop.*

Proof. Indeed, if there were a loop, then there would be a sequence of vertices $(e_i)_{i\in[1,n]}$ such that there is an edge between e_{i-1} and e_i for $i \in [1, n]$ and between e_0 and e_n. Let $x = e_0 + \cdot + e_n$. We have $B_C(x, x) = n + 1 + \sum_{i\neq j} q_{i,j}$ where $q_{i,j} = b_C(e_i, e_j)$. But this last term is $-\cos(\pi/(m + 2)) = 0, -1/2, -\sqrt{2}/2, -\sqrt{3}/2$ for $m = 0, 1, 2, 3$ thus for $m \neq 0$ we have $-\cos(\pi/(m+2)) \leq -1/2$. This gives

$$B_C(x, x) \leq 2(n + 1)(-1/2) = 0$$

which implies $x = 0$ because b_C is positive definite, a contradiction. □

Definition 12.1.2 *For C a finite Coxeter graph, we define the* graph obtained by contraction of e *denoted by $C(e)$, as follows:*

- *the set of vertices $C(e)_0$ is the set $(C_0 \setminus \{v, v'\}) \cup \{e\}$;*
- *the edges between two points in $C_0 \setminus \{v, v'\}$ is the same as in C;*
- *the edges between a point v'' in $C_0 \setminus \{v, v'\}$ and e is the union of the edges from v'' to v and v'.*

Remark that in the last case, there can not be edges from v and v' to v'' at the same time otherwise there would be a loop (v, v', v'') in C.

Proposition 12.1.3 *If C is a finite Coxeter graph and if e is an edge between two vertices v and v' such that $m(v, v') = 1$ (*i.e. *e is the only edge between v and v'), then $C(e)$ is again a finite Coxeter graph.*

Proof. For any two vertices i and j of $C(e)$, we denote by $q'_{i,j}$ the value of $B_{C(e)}(e_i, e_j)$. Let i and j be two vertices of $C(e)$ different from e, we have $q'_{i,j} = q_{i,j}$. We also have for i a vertex of $C(e)$ different from e the equality $q'_{i,e} = q_{i,v} + q_{i,v'}$ (one of the two vanishes). Write $x = \sum_{i \in C(e)_0, i \neq e} x_i e_i + x_e e_e \in V_{C(e)}$ and define $y = \sum_{i \in C(e)_0, i \neq e} x_i e_i + x_e e_v + x_e e_{v'} \in V_C$. We have

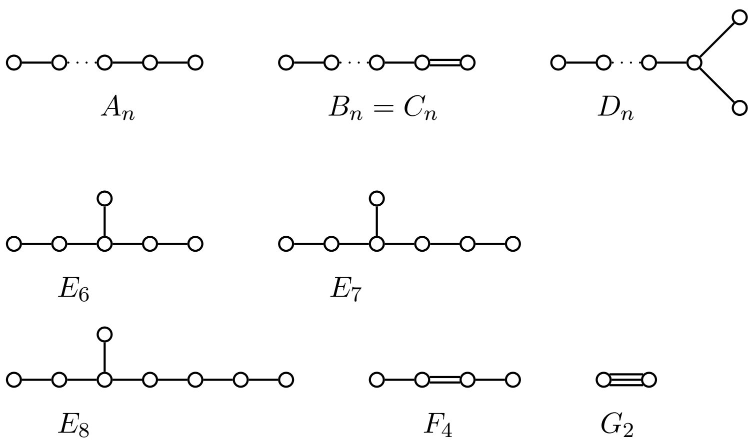
$$b_{C(e)}(x, x) = \sum_{i,j \neq e} q_{i,j} x_i x_j + 2 \sum_{i \neq e} x_e x_i (q_{i,v} + q_{i,v'}) + x_e^2 \text{ while}$$

$$b_C(y, y) = \sum_{i,j \neq e} q_{i,j} x_i x_j + 2 \sum_{i \neq e} x_e x_i q_{i,v} + 2 \sum_{i \neq e} x_e x_i q_{i,v'} + x_e^2 (2 + 2 q_{v,v'}).$$

Because $q_{v,v'} = -1/2$ we get that these two values are equal thus $b_{C(e)}(x, x)$ is non negative and if it vanishes, we have $y = 0$ *i.e.* $x = 0$. □

12.2 Classification

Theorem 12.2.1 *The connected finite Coxeter graphs are the following (the index indicates the number of vertices):*



Proof. We prove several Lemmas which imply the result. The first lemma controls the local structure of the graph at one special vertex.

Lemma 12.2.2 *Let v be a vertex in C, then $\sum_{v'\neq v} q_{v,v'}^2 < 1$.*

Proof. For v' and v'' distinct and distinct from v with $q_{v,v'} \neq 0$ and $q_{v,v''} \neq 0$, we have $q_{v',v''} = 0$ (otherwise there is a loop in C). Therefore if C_v is the set of vertices v' with $q_{v,v'} \neq 0$ and if $x = -e_v + \sum_{v'\in C_v} e_{v'}$, we have

$$0 < b_C(x,x) = \sum_{v',v''\in C_v} q_{v,v'}q_{v,v''}q_{v',v''} - 2\sum_{v'\in C_v} q_{v,v'}^2 + 1 = 1 - \sum_{v'\in C_v} q_{v,v'}^2$$

and the result follows. □

Corollary 12.2.3 *($\imath$) If a vertex v of C is connected to three different vertices v_1, v_2 and v_3, then $m(v,v_i) = 1$ for $i \in [1,3]$ and v is not connected to any other vertex.*

($\imath\imath$) There is at most one double edge starting form a vertex v.

*($\imath\imath\imath$) If there is a triple edge between v and v', then C has only two vertices (*i.e. *is of type G_2).*

Proof. Let us apply the previous Lemma to the vertex v, where we denote by $(v_i)_{i\in[1,n]}$ the vertices connected to v. Assume there are n_k vertices v_i with $m(v,v_i) = k$ for $k \in [1,3]$. If $m(v,v_i) = k$, we have $q_{v,v_i}^2 \geq k/4$ thus

$$1 > \sum_{i=1}^{n} q_{v,v_i}^2 = \frac{n_1 + 2n_2 + 3n_3}{4}.$$

This gives $n_1 + 2n_2 + 3n_3 \leq 3$ thus

- $n_1 \leq 3$ and if $n_1 = 3$, then $n_2 = n_3 = 0$, proving *($\imath$)*;
- $n_2 \leq 1$ proving *($\imath\imath$)*;
- $n_3 \leq 1$ and if $n_3 = 1$, then $n_1 = n_2 = 0$.

To finish the proof of *($\imath\imath\imath$)*, remark that by the above v and v' are only connected to each other. Because C is connected this implies that C has only two vertices. □

The next lemma controls the number of special vertices in the graph.

Lemma 12.2.4 *We have the alternative:*

- *the graph C has one ramification point with exactly 3 simple edges and all the edges of the graph are simple;*
- *the graph has no ramification point and at most one double edge.*

Proof. We proceed by induction on the number n of vertices in the graph. If $n = 1$, 2 or 3 then the result follows by the previous Corollary (in the case $n = 3$). Assume the result holds for any finite Coxeter graph with n vertices. Let C be a finite Coxeter graph with $n + 1$ vertices.

If C has a ramification point v, then by the previous Corollary, we know that this point is related to exactly 3 vertices v_1, v_2 and v_3 by simple edges. If none of these 3 vertices is related to another vertex, then $n + 1 = 4$ and the result follows. If at least one of these vertices, say v_1, is related to another vertex, let e be an edge linking v_1 to v. We know that $C(e)$ is again a finite Coxeter graph and it has a ramification at e. Therefore by induction $C(e)$ has only simple edges and so has C.

If C has no ramification point by a double edge between v and v'. If C has only two vertices, we are done. If not, there must be another edge e starting from v or v', say from v, and this edge is simple by the previous Corollary. Therefore $C(e)$ is again a finite Coxeter graph and has a double edge. By induction it has no other double edge and so is C. □

We are therefore left with two types of graphs: chains with at most one double edge of a graph with only simple edges and a unique ramification points. We thus need to rule out few more cases.

Lemma 12.2.5 *Assume that the is a chain $(v_i)_{i\in[1,n]}$ of vertices in C with only simple edges between v_i and v_{i+1} for $i \in [1, n-1]$. Put $x = \sum_{i=1}^{n} i e_{v_i}$, then we have*

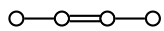
$$b_C(x, x) = \frac{n(n+1)}{2}.$$

Proof. We compute

$$b_C(x, x) = \sum_{i,j} ij b_C(e_{v_i}, e_{v_j}) = \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n-1} i(i+1) = n^2 - \sum_{i=1}^{n-1} i$$

and the result follows. □

Lemma 12.2.6 *Assume that C has no ramification and that there is a double edge whose end vertices are connected to other vertices. Then C is the following graph:*



Proof. By Lemma 12.2.4, the graph C is a chain. Let us denote by v_1 and w_1 the vertices of the double edge. There are chains $(v_i)_{i\in[1,n]}$ and $(w_j)_{j\in[1,m]}$ of vertices in C with only simple edges between v_i and v_{i+1} (and between w_j and w_{j+1}) for $i \in [1, n-1]$ and $j \in [1, m-1]$. Put $x = \sum_{i=1}^{n}(n+1-i)e_{v_i}$ and $y = \sum_{j=1}^{m}(m+1-j)e_{w_j}$, we have $b_C(x, x) = n(n+1)/2$ and $b_C(y, y) = m(m+1)/2$. By Cauchy-Schwartz we get $|b_C(x, y)^2| < b_C(x, x)b_C(y, y)$ *i.e.*

$$\frac{1}{2}n^2m^2 < \frac{n(n+1)}{2}\frac{m(m+1)}{2}$$

which gives $(n-1)(m-1) \leq 1$. By hypothesis we have $n, m \geq 2$ thus we have equality. □

If C has a ramification point r, then C is the union of r and three chains $(u_k)_{k\in[1,l]}$, $(v_i)_{i\in[1,n]}$ and $(w_j)_{j\in[1,m]}$ with only simple edges between u_k and k_{k+1}, between v_i and v_{i+1} and between w_j and w_{j+1}) for $k \in [1, l-1]$, $i \in [1, n-1]$ and $j \in [1, m-1]$ and with edges linking r to u_1, v_1 and w_1.

Lemma 12.2.7 *The only possible values of (l, m, n) with $l \leq m \leq n$ are $(1, 2, 2)$, $(1, 2, 3)$, $(1, 2, 4)$, $(1, 1, n)$.*

Proof. As in the previous lemma we define the elements $x = \sum_{k=1}^{l}(l+1-k)e_{u_k}$, $y = \sum_{i=1}^{n}(n+1-i)e_{v_i}$ and $z = \sum_{j=1}^{m}(m+1-j)e_{w_j}$. We have $b_C(x, x) = l(l+1)/2$, $b_C(y, y) = n(n+1)/2$, $b_C(z, z) = m(m+1)/2$ and $b_C(x, e_r) = -l/2$, $b_C(y, e_r) = -n/2$ and $b_C(z, e_r) = -m/2$. Let F be the span of x, y and z and let us denote by $||v||$ the non negative square root of $b_C(v,v)$. The elements $x/||x||$, $y/||y||$ and $z/||z||$ form an orthonormal basis of F therefore the distance of e_r to F is given by

$$b_C(e_r, e_r) - b_C(e_r, x/||x||)^2 - b_C(e_r, y/||y||)^2 - b_C(e_r, z/||z||)^2$$

and is positive. We get the inequality

$$1 - \frac{l}{2(l+1)} - \frac{m}{2(m+1)} - \frac{n}{2(n+1)} > 0$$

and thus

$$\frac{1}{l+1} + \frac{1}{m+1} + \frac{1}{n+1} > 1$$

Because $l \leq m \leq n$ we get $3 > l+1$ thus $l \leq 1$ *i.e.* $l = 1$. Pluging this value in the inequality we get $4 > m+1$ *i.e.* $m \leq 2$. For $m = 2$ we get $6 > n+1$ thus $n \leq 4$. □

We are left with the graphs given in the theorem. To finish the proof one needs to check that the associated bilinear form b_C is indeed positive definite for these graphs. We shall do this when explicitely constructing the root systems. □

12.3 Dynkin diagrams and classification of root systems

Let us go back to root systems. We shall assume in the sequel that our root systems are reduced and irreducible.

We have seen that to any root system R with a base S, there is a Coxeter graph associated to the situation and that the graph does not depend on the choice of the base S. However, the Coxeter graph does not determine the Cartan matrix: a Cartan matrix and its transpose (which is the Cartan matrix of the dual root system) have the same Coxeter graph. This comes from the fact that the Coxeter graph only determines the angle between two roots but not which of the roots is the longest and which is the shortest. This problem is solved by looking at Dynkin diagrams.

To define the Dynkin diagrams, we fix an positive definite form (,) invariant under the Weyl group.

Definition 12.3.1 *Let R be a root system and $C(R)$ be its Coxeter graph. The* Dynkin diagram *of R is the Coxeter graph $C(R)$ together with, for each vertex v_α corresponding to a simple root α, the length (α, α) attached to the vertex v_α.*

Proposition 12.3.2 *The Dynkin diagram determines the Cartan matrix (and therefore the root system).*

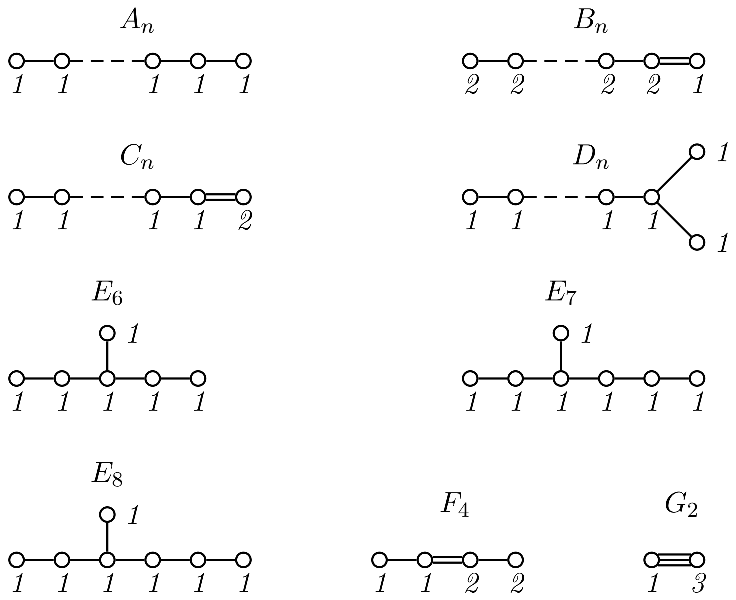
Proof. Let us describe the Cartan matrix in terms of the Dynkin diagram. Let α and β be two simple roots.

- If $\beta = \alpha$, then $\langle \beta^\vee, \alpha \rangle = 2$.
- If we have $(\beta, \beta) \geq (\alpha, \alpha)$, then $\langle \beta^\vee, \alpha \rangle = -1$.
- If we have $(\beta, \beta) < (\alpha, \alpha)$, then $\langle \beta^\vee, \alpha \rangle = -$number of edges connecting v_α and v_β.

In particular knowing the relative length gives the Cartan matrix. □

Recall that a root system is irreducible if and only if its Coxeter graph, or equivalently its Dynkin diagram, is connected. The following result therefore gives a complete classification of irreducible and reduced root systems.

Theorem 12.3.3 *The connected Dynkin diagrams are the following*



Proof. Let us look at the vector space V in which the root system R lives. We have a natural basis given by the simple roots $\alpha \in S$ and we define a new basis by $e_\alpha = \alpha/|\alpha|$. With this base we have

$$(e_\alpha, e_\beta) = \frac{(\alpha, \beta)}{|\alpha| \cdot |\beta|} = \cos\phi$$

with ϕ the angle between α and β and we get the bilinear form defined by a Coxeter graph. In particular, we know that this form in positive definite therefore the Coxeter graph is finite and connected (this is equivalent to R irreducible). The only possible graphs are therefore those given in the statement. We shall now construct the corresponding root systems.

In $\mathbb{R}^n$ we denote by $(e_i)_{i\in[1,n]}$ the canonical basis and we denote by L_n the lattice define by

$$L_n = \bigoplus_{i=1}^{n} \mathbb{Z} e_i.$$

In $\mathbb{R}^n$ we shall also consider the canonical positive definite bilinear form defined by $(e_i, e_j) = \delta_{i,j}$.

Type A_n. We set $V = (e_1 + \cdots + e_{n+1})^\perp \subset \mathbb{R}^{n+1}$, and define $R \subset V$ by

$$R = \{\alpha \in L_{n+1} \cap V \ / \ (\alpha, \alpha) = 2\}.$$

This set does not contain 0 and is finite (intersection of a compact and a discrete subset). The symmetry s_α for $\alpha \in R$ is defined $s_\alpha(v) = v - (\alpha, v)\alpha$. The elements in R are the vectors $e_i - e_j$ for $i \neq j$. One easily checks that this is a root system and that a basis is given by $(e_i - e_{i+1})_{i\in[1,n]}$. This gives a Dynkin diagram of type A_n. The Weyl group is the group $\mathfrak{S}_{n+1}$ acting by permutation on the elements of the canonical basis in $\mathbb{R}^{n+1}$.

Type B_n. We set $V = \mathbb{R}^n$, and define $R \subset V$ by

$$R = \{\alpha \in L_n \cap V \ / \ (\alpha, \alpha) = 1 \text{ or } (\alpha, \alpha) = 2\}.$$

This set does not contain 0 and is finite (intersection of a compact and a discrete subset). The elements in R are the vectors $\pm e_i$ for all i and $\pm e_i \pm e_j$ for $i \neq j$. One easily checks that this is a root system and that a basis is given by $((e_i - e_{i+1})_{i \in [1,n-1]}, e_n)$. This gives a Dynkin diagram of type B_n. The Weyl group is the group $\mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ where $\mathfrak{S}_n$ acts by permutation on the elements of the canonical basis in $\mathbb{R}^n$ and $(\mathbb{Z}/2\mathbb{Z})^n$ acts by sign change on each coordinate in the canonical basis in $\mathbb{R}^n$.

Type C_n. This is simply the dual root system of B_n. We can realise it as follows: we set $V = \mathbb{R}^n$ take R to be the union of all elements of the form $\pm 2e_i$ for all i and $\pm e_i \pm e_j$ for $i \neq j$. One easily checks that this is a root system and that a basis is given by $((e_i - e_{i+1})_{i \in [1,n-1]}, 2e_n)$. This gives a Dynkin diagram of type C_n. The Weyl group is the group $\mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ as in type B_n and acts in the same way: $\mathfrak{S}_n$ acts by permutation on the elements of the canonical basis in $\mathbb{R}^n$ and $(\mathbb{Z}/2\mathbb{Z})^n$ acts by sign change on each coordinate in the canonical basis in $\mathbb{R}^n$.

Type D_n. We set $V = \mathbb{R}^n$, and define $R \subset V$ by

$$R = \{\alpha \in L_n \cap V \ / \ (\alpha, \alpha) = 2\}.$$

This set does not contain 0 and is finite (intersection of a compact and a discrete subset). The elements in R are the vectors $\pm e_i \pm e_j$ for $i \neq j$. One easily checks that this is a root system and that a basis is given by $((e_i - e_{i+1})_{i \in [1,n-1]}, e_{n-1} + e_n)$. This gives a Dynkin diagram of type D_n. The Weyl group is the group $\mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$ where $\mathfrak{S}_n$ acts by permutation on the elements of the canonical basis in $\mathbb{R}^n$ and $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ acts by sign change on each coordinate in the canonical basis in $\mathbb{R}^n$ such that the product of the sign changes is 1 (*i.e.* it acts on an even number of coordinates).

Type G_2. This root system was described in Example 11.1.8. Its Weyl group is $\mathbb{Z}/6\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. It can also be described an follows: let ζ be a primitive third root of 1 and let $K = \mathbb{Q}(\zeta)$. There is a norm defined on K by $N_K(a + b\zeta) = (a + b\zeta)(a + b\bar{\zeta}) = a^2 + b^2 - ab$. Let $\mathcal{O} \subset K$ be the set of intergers in K *i.e.* $\mathcal{O} = \mathbb{Z}[\zeta]$, then

$$R = \{\alpha \in \mathcal{O} \ / \ N_K(\alpha) = 1 \text{ or } N_K(\alpha) = 3\}.$$

A basis is given by $(1, \zeta - 1)$.

Type F_4. We set $V = \mathbb{R}^4$ and let L'_4 be the lattice generated by L_4 and the vector $\frac{1}{2}(e_1+e_2+e_3+e_4)$. We define $R \subset V$ by

$$R = \{\alpha \in L'_4 \cap V \ / \ (\alpha, \alpha) = 1 \text{ or } (\alpha, \alpha) = 2\}.$$

This set does not contain 0 and is finite (intersection of a compact and a discrete subset). The elements in R are the vectors $\pm e_i$ for all i, $\pm e_i \pm e_j$ for $i \neq j$ and $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$. One easily checks that this is a root system and that a basis is given by $(e_2 - e_3, e_3 - e_4, e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4))$. This gives a Dynkin diagram of type F_4. The Weyl group is of order $2^7 3^2$.

Type E_8. We set $V = \mathbb{R}^8$, let L'_8 be the lattice generated by L_8 and the vector $\frac{1}{2}(e_1 + \cdots + e_8)$ and define L''_8 to be the sublattice of L'_8 of elements $\sum_i a_i e_i$ with $\sum_i a_i$ even. We define $R \subset V$ by

$$R = \{\alpha \in L''_8 \cap V \ / \ (\alpha, \alpha) = 2\}.$$

This set does not contain 0 and is finite (intersection of a compact and a discrete subset). The elements in R are the vectors

$$\pm e_i \pm e_j \text{ for } i \neq j \text{ and } \frac{1}{2}\sum_{i=1}^{8}(-1)^{m(i)}e_i \text{ with } \sum_{i=1}^{8} m(i) \text{ even.}$$

One easily checks that this is a root system and that a basis is given by

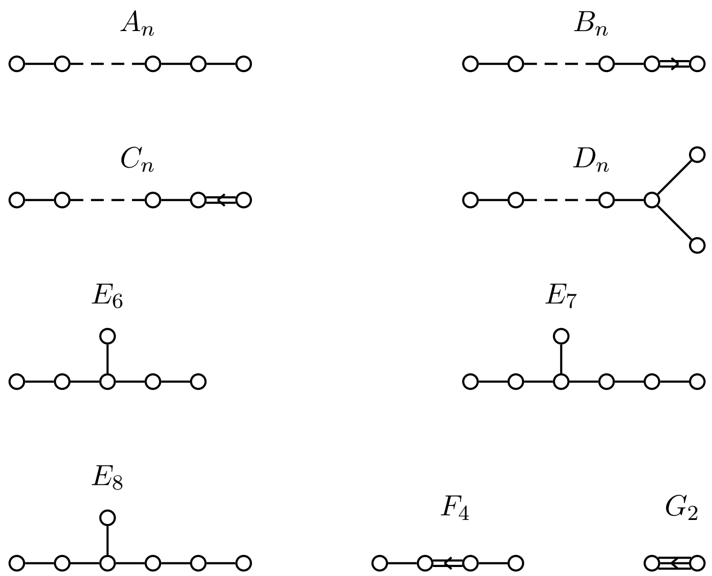
$$\left(\frac{1}{2}(e_1 + e_8 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7), e_1 + e_2, e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4, e_6 - e_5, e_7 - e_6\right).$$

This gives a Dynkin diagram of type E_8. The Weyl group is of order $2^{14}3^5 5^2 7$.

Type E_7. Take the intersection of the root system of E_8 with the subspace spanned by the vectors $(e_1, e_2, e_3, e_4, e_5, e_6, e_7)$.

Type E_6. Take the intersection of the root system of E_8 with the subspace spanned by the vectors $(e_1, e_2, e_3, e_4, e_5, e_6)$. □

Remark 12.3.4 As there are only two different value for the length of roots, instead of drawing all the values of the length, one usually indicates which root is longer than the other by drawing an arrow goning from the longest root to the shortest one. Here are the Dynkin diagrams one obtain in this way:



12.4 Exercices

Exercice 12.4.1 Prove that the sets described below define root systems and basis.

Type A_n. We set $V = (e_1 + \cdots + e_{n+1})^\perp \subset \mathbb{R}^{n+1}$, and define $R = \{\alpha \in L_{n+1} \cap V \ / \ (\alpha, \alpha) = 2\}$. Prove that this is a root system and that a basis is given by $(e_i - e_{i+1})_{i\in[1,n]}$. Verify that the associated Dynkin diagram is of type A_n.

Type B_n. We set $V = \mathbb{R}^n$, and $R = \{\alpha \in L_n \cap V \ / \ (\alpha, \alpha) = 1 \text{ or } (\alpha, \alpha) = 2\}$. Prove that this is a root system and that a basis is given by $((e_i - e_{i+1})_{i\in[1,n-1]}, e_n)$. Verify that the associated Dynkin diagram is of type B_n.

Type C_n. Consider the dual root system of the previous one. Prove that (identifying V with its dual using the form $(\ , \ )$) that it is given by the union of all elements of the form $\pm 2e_i$ for all i and $\pm e_i \pm e_j$ for $i \neq j$. Prove that a basis is given by $((e_i - e_{i+1})_{i\in[1,n-1]}, 2e_n)$. Verify that the associated Dynkin diagram is of type C_n.

Type D_n. We set $V = \mathbb{R}^n$, and define $R = \{\alpha \in L_n \cap V \ / \ (\alpha, \alpha) = 2\}$. Prove that this is a root system and that a basis is given by $((e_i - e_{i+1})_{i\in[1,n-1]}$. Verify that the associated Dynkin diagram is of type D_n.

Type G_2. Let ζ be a primitive third root of 1 and let $K = \mathbb{Q}(\zeta)$. There is a norm defined on K by $N_K(a + b\zeta) = (a + b\zeta)(a + b\bar{\zeta}) = a^2 + b^2 - ab$. Let $\mathcal{O} \subset K$ be the set of intergers in K *i.e.* $\mathcal{O} = \mathbb{Z}[\zeta]$, then define $R = \{\alpha \in \mathcal{O} \ / \ N_K(\alpha) = 1 \text{ or } N_K(\alpha) = 3\}$. Prove that this is a root system, that a basis is given by $(1, \zeta - 1)$ and that the associated Dynkin diagram is of type G_2.

Type F_4. We set $V = \mathbb{R}^4$ and let L'_4 be the lattice generated by L_4 and the vector $\frac{1}{2}(e_1+e_2+e_3+e_4)$. We define $R = \{\alpha \in L'_4 \cap V \ / \ (\alpha, \alpha) = 1 \text{ or } (\alpha, \alpha) = 2\}$. Prove that this is a root system and that a basis is given by $(e_2 - e_3, e_3 - e_4, e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4))$. Prove that the associated Dynkin diagram is of type F_4.

Type E_8. We set $V = \mathbb{R}^8$, let L'_8 be the lattice generated by L_8 and the vector $\frac{1}{2}(e_1 + \cdots + e_8)$ and define L''_8 to be the sublattice of L'_8 of elements $\sum_i a_i e_i$ with $\sum_i a_i$ even. Define $R = \{\alpha \in L''_8 \cap V \ / \ (\alpha, \alpha) = 2\}$. Prove that this is a root system and that a basis is given by $(\frac{1}{2}(e_1 + e_8 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7), e_1 + e_2, e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4, e_6 - e_5, e_7 - e_6)$. Prove that the asociated Dynkin diagram is of type E_8.

Type E_7. Take the intersection of the root system of E_8 with the subspace spanned by the vectors $(e_1, e_2, e_3, e_4, e_5, e_6, e_7)$. Prove that this is a root system and that the basis of E_8 restricts to a basis. Prove that the associated Dynkin diagram is of type E_7.

Type E_6. Take the intersection of the root system of E_8 with the subspace spanned by the vectors $(e_1, e_2, e_3, e_4, e_5, e_6)$. Prove that this is a root system and that the basis of E_8 restricts to a basis. Prove that the associated Dynkin diagram is of type E_6.

Exercice 12.4.2 Let R bs a root system and S be a basis of R. Let S' be a subset of S and let R' the the subset of R of all roots which are linear combination of elements in S'.

Prove that R' is a root system and that S' is a base for R'.

Exercice 12.4.3 Prove that the Weyl group of a root system of type A_n is isomorphic to $\mathfrak{S}_{n+1}$ and that it acts transitively on the roots.

Exercice 12.4.4 (i) Let R be an irreducible root system and S be a basis. Let $\alpha \in S$, prove that any simple root of the same length as α is in the orbit of α under the action of the Weyl group.

Hint: consider the subsystem generated by the roots of the same length as α (see Exercice 12.4.2) and the use Exercice 12.4.3.

(ii) Let $\beta = \sum_{\gamma \in S} b_\gamma \gamma$ be a positive root. Prove by induction on $\sum_\gamma b_\gamma$ that β is in the orbit under the Weyl group of some simple root.

(iii) Prove that the orbits of the Weyl group on R are the sets of roots with the same length.

Exercice 12.4.5 Montrer que les racines de meme longueur forment un sous-systeme et decrire les sous systemes ainsi obtenus.

Chapter 13

Classification of complex semisimple Lie algebras

13.1 Decomposition of the Lie algebra

Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$. We know that $\mathfrak{h}$ is abelian and self centralising. Furthermore all its elements are semisimple and because they commute they are diagonalisable simultaneously. Let us fix such a diagonalising base $(e_i)_{i\in[1,n]}$ with $n = \dim \mathfrak{g}$. For any $x \in \mathfrak{h}$, we thus have

$$\mathrm{ad}\,(x)(e_i) = [x, e_i] = \lambda_i(x)e_i$$

where $\lambda_i(x)$ is the eigenvalue of x associated to the eigenvector e_i.

Fact 13.1.1 *The map $\lambda_i : \mathfrak{h} \to k$ defined by $x \mapsto \lambda_i(x)$ is a linear form.*

Proof. Let u in k and x, y in $\mathfrak{h}$ we compute $\lambda_i(x + uy)e_i = [x + uy, e_i] = [x, e_i] + u[y, e_i] = (\lambda_i(x) + u\lambda_i(y))e_i$ and the result follows. □

Corollary 13.1.2 *We have a decomposition*

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in\mathfrak{h}^\vee\setminus\{0\}} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \ / \ \forall y \in \mathfrak{h}\ \mathrm{ad}\,(y)(x) = \alpha(y)x\}$.

Definition 13.1.3 *A linear form $\alpha \in \mathfrak{h}^\vee$ such that $\mathfrak{g}_\alpha$ is not trivial is called a* root of the Lie algebra. *We denote by R the set of all roots of $\mathfrak{g}$, this is a (finite) subset of $\mathfrak{h}^\vee$.*

For $\alpha \in R$, elements in $\mathfrak{g}_\alpha$ are said to have weight α *and elements in $\mathfrak{h}$ are said to have weight 0.*

Proposition 13.1.4 *Let $(\ , \ )$ be an invariant non degenerate bilinear form on $\mathfrak{g}$ (for example the Killing form).*

(ı) The subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal for $(\ , \ )$ except if $\alpha + \beta = 0$.

(ıı) The subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are dual with respect to $(\ , \ )$ and its restriction to $\mathfrak{h}$ is non degenerate.

(ııı) If $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ and $h \in \mathfrak{h}$, then we have

$$(h, [x, y]) = \alpha(h)(x, y).$$

(iv) Let $\alpha \in R$ and let h_α be the element in $\mathfrak{h}$ corresponding to α under the isomorphism $\mathfrak{h} \simeq \mathfrak{h}^\vee$ defined by $(\, , \,)$. We have

$$[x, y] = (x, y)h_\alpha$$

for $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$.

Proof. (i) Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. We have for all $h \in \mathfrak{h}$ the formula:

$$\alpha(h)(x, y) = ([h, x], y) = -(x, [h, y]) = -\beta(h)(x, y)$$

therefore $(\alpha(h) - \beta(h))(x, y) = 0$ and if $\alpha + \beta \neq 0$, there exists an element $h \in \mathfrak{h}$ with $\alpha(h) - \beta(h) \neq 0$ giving $(x, y) = 0$.

(ii) We have a direct sum of orthogonal spaces for the bilinear form $(\, , \,)$ as follows:

$$\mathfrak{g} = \mathfrak{h} \overset{\perp}{\bigoplus_{\alpha \in R}} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$$

and because the bilinear form is non degenerate, it has to be non degenerate on each spaces and the result follows.

(iii) By invariance of $(\, , \,)$ we have $(h, [x, y]) = ([h, x], y) = \alpha(h)(x, y)$.

(iv) For any element h in $\mathfrak{h}$ we compute:

$$([x, y], h) = \alpha(h)(x, y) = (h_\alpha, h)(x, y) = ((x, y)h_\alpha, h).$$

Remark that $[x, y] \in \mathfrak{h}$ (indeed $[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha(h)[x, y] - \alpha(h)[x, y] = 0$ thus $[x, y]$ has weight 0 and is in $\mathfrak{h}$). Thus because $(\, , \,)$ is non degenerate on $\mathfrak{h}$ we get the result. □

13.2 Structure theorem for complex semisimple Lie algebras

Theorem 13.2.1 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra.*

(i) The set R is a root system in $\mathfrak{h}^\vee$ and is reduced.

(ii) For α a root, the space $\mathfrak{g}_\alpha$ is one dimensional as well as the subspace $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ of $\mathfrak{h}$.

(iii) There is a unique element $H_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(H_\alpha) = 2$, this is the inverse root of α.

(iv) Let α be a root, for each non zero element $X_\alpha \in \mathfrak{g}_\alpha$, there is a unique element $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, Y_\alpha] = H_\alpha$.

(v) We have the formulas $[H_\alpha, X_\alpha] = 2X_\alpha$ and $[H_\alpha, Y_\alpha] = 2Y_\alpha$ thus $\mathfrak{s}_\alpha = \mathfrak{g}_{-\alpha} \oplus \mathfrak{h}_\alpha \oplus \mathfrak{g}_\alpha$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$.

(vi) For α and β two roots such that $\alpha + \beta \neq 0$, we have the equality $\mathfrak{g}_{\alpha+\beta} = [\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$.

Proof. We will not prove this result linearly but in several steps. We fix a non degenerate invariant bilinear form $(\, , \,)$. Let us first prove the following very easy fact (that we already partially used in the previous proposition).

• We have the inclusion $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

We have for $x \in \alpha_\alpha$, $y \in \mathfrak{g}_\beta$ and $h \in \mathfrak{h}$ the equalities $[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha(h)[x, y] + \beta(h)[x, y]$ and the result follows.

• R spans $\mathfrak{h}^\vee$.

For this we only need to prove that if $h \in \mathfrak{h}$ is such that $\alpha(h) = 0$ for all $\alpha \in R$, then $h = 0$. But for such an element h, we have $[h, \mathfrak{g}] = [h, \mathfrak{h}] + \sum_\alpha [h, \mathfrak{g}_\alpha] = \sum_\alpha \alpha(h)\mathfrak{g}_\alpha = 0$. Thus h lies in the center of $\mathfrak{g}$ and because $\mathfrak{g}$ is semisimple, the result follows.

• We have $\dim \mathfrak{h}_\alpha = 1$.

This follows from the previous proposition: all the elements in this spaces are colinear to h_α.

• There exists an element H_α in $\mathfrak{h}_\alpha$ with $\alpha(H_\alpha) = 2$.

In view of the previous point, we only need to prove that α is non trivial on $\mathfrak{h}_\alpha$. If it were not the case, then $\alpha(\mathfrak{h}_\alpha) = 0$. Because $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are dual to each other for $(\ , \ )$, there exist elements $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ such that $(x, y) \neq 0$ thus $z = [x, y] = (x, y)h_\alpha \neq 0$. We have $\alpha(z) = 0$. We thus have the formulas $[x, y] = z$, $[z, x] = [z, y] = 0$, thus the span $\mathfrak{s}$ of x, y and z is a subalgebra of $\mathfrak{g}$.

This subalgebra $\mathfrak{s}$ is nilpotent and therefore solvable. Thus in the representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, the image stabilises a fixed flag. Furthermore, the elements z lie in $\mathfrak{D}\mathfrak{s}$ therefore its image is nilpotent. But z lies in $\mathfrak{h}$ thus is semisimple and so is its image under the adjoint representation. Therefore $\mathrm{ad}\,(z) = 0$ thus $z = 0$ a contradiction.

• Let α be a root, for each non zero element $X_\alpha \in \mathfrak{g}_\alpha$, there is an element $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, Y_\alpha] = H_\alpha$.

For $y \in \mathfrak{g}_{-\alpha}$, we have $[X_\alpha, y] = (X_\alpha, y)h_\alpha$ and $h_\alpha = cH_\alpha$ for some $c \in k^\times$. Let $y \in \mathfrak{g}_{-\alpha}$ such that $(X_\alpha, y) \neq 0$ (this is possible because $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are dual for $(\ , \ )$). We can therefore set $Y_\alpha = y/(c(X_\alpha, y))$.

• We have the formulas $[H_\alpha, X_\alpha] = 2X_\alpha$ and $[H_\alpha, Y_\alpha] = 2Y_\alpha$ thus $\mathfrak{s}_\alpha = \mathfrak{g}_{-\alpha} \oplus \mathfrak{h}_\alpha \oplus \mathfrak{g}_\alpha$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$.

We compute $[H_\alpha, X_\alpha] = \alpha(H_\alpha)X_\alpha = 2X_\alpha$ and $[H_\alpha, Y_\alpha] = -\alpha(H_\alpha)Y_\alpha = -2Y_\alpha$. We therefore have an identification with the canonical basis of $\mathfrak{sl}_2$.

We shall in the sequel consider $\mathfrak{g}$ as an $\mathfrak{sl}_2 = \mathfrak{s}_\alpha$ representation thanks to the adjoint representation.

• We have $\dim \mathfrak{g}_\alpha = 1$ for $\alpha \in R$, the element Y_α is unique and we have $\mathfrak{s}_\alpha = \mathfrak{g}_{-\alpha} \oplus \mathfrak{h}_\alpha \oplus \mathfrak{g}_\alpha$.

If $\dim \mathfrak{g}_\alpha > 1$, then the same is true for $\mathfrak{g}_{-\alpha}$ because it is dual to $\mathfrak{g}_\alpha$. Therefore there exists a non zero element $y \in \mathfrak{g}_{-\alpha}$ such that $(X_\alpha, y) = 0$. We therefore have $[X_\alpha, y] = (X_\alpha, y)h_\alpha = 0$ by the previous proposition. But we have $[H_\alpha, y] = -\alpha(H_\alpha)y = -2y$ thus y is a primitive element of negative weight. This is not possible. The last two assertions follow from the first one.

• Let α and β be two roots, then $\beta(H_\alpha)$ is an integer and $\beta - \beta(H_\alpha)\alpha$ is a root.

Let $x \in \mathfrak{g}_\beta$ be a non trivial vector. We have $[H_\alpha, x] = \beta(H_\alpha)x$ therefore $\beta(H_\alpha)$ is a weight for the representation of $\mathfrak{s}_\alpha$ in $\mathfrak{g}$. We get that $\beta(H_\alpha)$ is an integer. If $\beta(H_\alpha) \geq 0$, we set $y = Y_\alpha^{\beta(H_\alpha)}(x)$ and if $\beta(H_\alpha) \leq 0$, we set $y = X_\alpha^{-\beta(H_\alpha)}(x)$. We know that $y \neq 0$ but we have that y has weight $\beta - \beta(H_\alpha)\alpha$ which has therefore to be a root.

• R is a root system.

We already know that R is finite does not contain 0 and spans $\mathfrak{h}^\vee$. Let us define, for $\alpha \in R$, the endomorphism $s_\alpha : \mathfrak{h}^\vee \to \mathfrak{h}^\vee$ by $s_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha$. Because $\alpha(H_\alpha) = 2$, this is a reflection. Furthermore by the previous points, this reflection maps R to itself and $s_\alpha(\beta) - \beta$ is an integer multiple of α.

• The root system R is reduced.

Let $\alpha \in R$ and let $x \in \mathfrak{g}_{2\alpha}$. We want to show that $x = 0$. Let us remark that we already know (because R is a root system) that $\mathfrak{g}_{3\alpha} = 0$. We have $[H_\alpha, x] = 2\alpha(H_\alpha)x = 4x$. On the other hand, we have $[H_\alpha, x] = [[X_\alpha, Y_\alpha], x] = [[X_\alpha, x], Y_\alpha] + [X_\alpha, [Y_\alpha, x]] = [X_\alpha, [Y_\alpha, x]]$ because $[X_\alpha, x] \in \mathfrak{g}_{3\alpha} = 0$. But $[Y_\alpha, x] \in \mathfrak{g}_\alpha$ is colinear to X_α thus $4x = [H_\alpha, x] = 0$ and the result follows.

• Let α and β be non proportional roots. Let p and q the greatest integers such that $\beta - p\alpha$ and $\beta + q\alpha$ are roots. Let

$$E = \bigoplus_{k=-p}^{q} \mathfrak{g}_{\beta + k\alpha}.$$

Then E is an irreducible representation of $\mathfrak{s}_\alpha$ of dimension $p+q+1$ and for $k \in [-p, q-1]$, the map

$$\mathrm{ad}\,(X_\alpha) : \mathfrak{g}_{\beta+k\alpha} \to \mathfrak{g}_{\beta+(k+1)\alpha}$$

is an isomorphism. We have $\beta(H_\alpha) = p - q$.

This is clearly a subrepresentation of $\mathfrak{g}$. The weights of H_α on E are given by $(\beta + k\alpha)(H_\alpha) = \beta(H_\alpha) + 2k$ for those k such that $\beta + k\alpha$ is a root. Because all these weights have multiplicity one, this in particular implies that E is irreducible. The assertions on the dimension and the map $\mathrm{ad}\,(X_\alpha)$ follow from the theory of representation of $\mathfrak{sl}_2$. Furthermore, we have that $\beta(H_\alpha) - 2p$ is the lowest weight of E while $\beta(H_\alpha) + 2q$ is its highest weight. We thus have $\beta(H_\alpha) - 2p = -(\beta(H_\alpha) + 2q)$ giving the last assertion.

• For $\alpha \in R$, $\beta \in R$ with $\alpha + \beta \in R$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

With the notation of the previous point we have $q \geq 1$ and $p \geq 0$, thus for $k = 0$ we get that $\mathrm{ad}\,(X_\alpha) : \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$ is an isomorphism and in particular $\mathfrak{g}_{\alpha+\beta} \subset [\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$. The converse inclusion was proved as first point. □

Corollary 13.2.2 *The root system R does not depend on the choice of $\mathfrak{h}$ but only on the semisimple Lie algebra $\mathfrak{g}$.*

Proof. Indeed, let $\mathfrak{h}$ and $\mathfrak{h}'$ be two Cartan subalgebras and let R and R' be the corresponding root systems. We know that there exists an automorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra such that the restriction of ϕ to $\mathfrak{h}$ is an isomorphisms $\phi|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}'$. Let us denote by ${}^t\phi^{-1} : \mathfrak{h}^\vee \to (\mathfrak{h}')^\vee$ the transpose of its inverse. For $\alpha \in R$, $x \in \mathfrak{g}_\alpha$ and $h \in \mathfrak{h}$, we have

$$[\phi(h), \phi(x)] = \phi([h, x]) = \alpha(h)\phi(x)$$

therefore $\phi(x)$ is an eigenvector for $h' = \phi(h) \in \mathfrak{h}'$ with eigenvalue $\alpha(h) = \alpha(\phi^{-1}(h')) =^t \phi^{-1}(\alpha)(h')$. Therefore ${}^t\phi^{-1}(\alpha)$ is a root. Thus ${}^t\phi^{-1}$ is an isomorphism of root systems from R to R' (its inverse is ${}^t\phi$). □

Corollary 13.2.3 *Let $\mathfrak{g}$ be a semisimple Lie algebra and R be the root system associated to $\mathfrak{g}$ defined in the previous theorem. Then $\mathfrak{g}$ is simple if and only if R is irreducible.*

Proof. If $\mathfrak{g}$ is not simple, then $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ with $\mathfrak{g}_i$ semisimple Lie algebras. We easily get that if $\mathfrak{h}_i$ is a Cartan subalgebra for $\mathfrak{g}_i$, then $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ is a Cartan subalgebra for $\mathfrak{g}$ and that the root system for $\mathfrak{h}$ in $\mathfrak{g}$ is the union of the root systems for $\mathfrak{h}_1$ in $\mathfrak{g}_1$ and $\mathfrak{h}_2$ in $\mathfrak{g}_2$.

Conversely, if R is reducible, then it is the orthogonal union of two root systems R_1 and R_2 and if we set $\mathfrak{h}_i$ to be the span in $\mathfrak{h}$ of the elements H_α for $\alpha \in R_i$ and if we define

$$\mathfrak{g}_i = \mathfrak{h}_i \oplus \bigoplus_{\alpha \in R_i} \mathfrak{g}_\alpha$$

then we have $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$. □

13.3 Free Lie algebras, Lie algebras defined by generators and relations

Recall that for V a vector space, the tensor algebra $T(V)$ is the free associative algebra generated by V. Let us denote by i the natural inclusion $V \to T(V)$. Recall also that we have the following universal property characterising free associative algebras.

Proposition 13.3.1 *For any associative algebra A and any linear map $f : V \to A$, there exists a unique associative algebra morphism $F : T(V) \to A$ such that $f = F \circ i$.*

Let us now, in the same spirit, define Free Lie algebras.

Definition 13.3.2 *Let V be a vector space and let $T(V)$ be its tensor algebra viewed as a Lie algebra under the bracket $[a, b] = a \otimes b - b \otimes a$. The* free Lie algebra generated by V, *denoted $F(V)$, is the Lie subalgebra generated by the subspace V in $T(V)$. We denote by j the embedding $V \to F(V)$.*

If $(e_i)_{i\in[1,m]}$ is a base of V, we call $F(V)$ the free Lie algebra generated by the $(e_i)_{i\in[1,m]}$.

The following is the characteristic property of free Lie algebras.

Proposition 13.3.3 *For any Lie algebra $\mathfrak{g}$ and any linear map $f : V \to \mathfrak{g}$, there exists a unique Lie algebra morphism $F : F(V) \to \mathfrak{g}$ such that $f = F \circ j$.*

Proof. Let $T(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$. We have a linear map $V \to \mathfrak{g} \to T(\mathfrak{g})$ therefore by the universal property of $T(V)$, we get a morphism of associative algebras $T(V) \to T(\mathfrak{g})$. We may then compose this map with the quotient $T(\mathfrak{g}) \to U(\mathfrak{g})$ to the universal envelopping algebra.

Let us now consider the restrition of this map to $F(V)$ and denote it by ϕ. We claim that the image of ϕ is contained in the image of $\mathfrak{g}$ in $U(\mathfrak{g})$. Indeed, the Lie algebra $F(V)$ is generated by the elements in V but if v and v' are in V, we have $\phi([v, v']) = \phi(v \otimes v' - v' \otimes v) = \phi(v) \otimes \phi(v') - \phi(v') \otimes \phi(v) = [\phi(v), \phi(v')]$. As f maps V to $\mathfrak{g}$, the elements $\phi(v)$ and $\phi(v')$ as well as the element $[\phi(v), \phi(v')]$ live in the image of $\mathfrak{g}$ in $U(\mathfrak{g})$.

To conclude, we need to invoque Poincaré-Birkhoff-Witt Theorem 15.0.7 and the fact that the map $f_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$ is injective therefore the Lie algebra morphism $\phi : F(V) \to f_{\mathfrak{g}}(\mathfrak{g}) \subset U(\mathfrak{g})$ lifts to a Lie algebra morphism $F(V) \to \mathfrak{g}$. □

Definition 13.3.4 *Let E be a finite set, let V be the vector space k^E and consider $F(V)$ the free Lie algebra generated by E. Let R be a subset of $F(V)$,* the Lie algebra defined by the generators E and the relations R *is the quotient of $F(V)$ by the ideal generated by R.*

13.4 Serre's presentation

In this section we give a description of a semisimple Lie algebra by generators and relations. Let us denote by $\mathfrak{n}$ the sum of the $\mathfrak{g}_\alpha$ for $\alpha \in R_+$ and $\mathfrak{n}_-$ the sum of the $\mathfrak{g}_\alpha$ for $\alpha \in R_-$. Remark that because H_α is the dual root of $\alpha \in R$, we have $\alpha(H_\beta) = \langle \beta^\vee, \alpha \rangle$.

Theorem 13.4.1 *(ı) The subspaces $\mathfrak{n}$ resp. $\mathfrak{n}_-$ are subalgebras in $\mathfrak{g}$ and are generated by the X_α resp. Y_α with $\alpha \in S$.*

(ıı) The Lie algebra $\mathfrak{g}$ is generated by the elements X_α, Y_α and H_α for $\alpha \in S$.

(ııı) We have the relations (called Weyl relations)

- $[X_\alpha, Y_\alpha] = H_\alpha$ *for* $\alpha \in S$,
- $[X_\alpha, Y_\beta] = 0$ *for* $\alpha, \beta \in S$ *with* $\alpha \neq \beta$,
- $[H_\alpha, X_\beta] = \langle \alpha^\vee, \beta \rangle X_\alpha$ *for* $\alpha, \beta \in S$,
- $[H_\alpha, Y_\beta] = -\langle \alpha^\vee, \beta \rangle Y_\alpha$ *for* $\alpha, \beta \in S$,

and the relations (called Serre relations)

- $\mathrm{ad}\,(X_\alpha)^{1-\langle\alpha^\vee,\beta\rangle}(X_\beta) = 0$ *for* $\alpha, \beta \in S$ *with* $\alpha \neq \beta$,
- $\mathrm{ad}\,(Y_\alpha)^{1-\langle\alpha^\vee,\beta\rangle}(Y_\beta) = 0$ *for* $\alpha, \beta \in S$ *with* $\alpha \neq \beta$.

Proof. (i) Let us prove that $\mathfrak{n}$ is generated by the elements X_α for $\alpha \in S$. For this we only need to prove that for $\beta \in R_+$, the element X_β is in the subalgebra generated by the $(X_\alpha)_{\alpha\in S}$. From Proposition 11.6.9, the root β can be written as $\beta = \sum_{i=1}^k \alpha_i$ with $\alpha_i \in S$ such that for all $j \in [1, k]$ the sum $\sum_{i=1}^j \alpha_i$ is a root. By Theorem 13.2.1 and induction, we get that the element

$$[X_{\alpha_k}, \cdots, [X_2, X_1]]$$

is non trivial in $\mathfrak{g}_\beta$ therefore a multiple of it is equal to X_β.

By the same argument we get the result for $\mathfrak{n}_-$.

(ii) Because the H_α for $\alpha \in S$ generate $\mathfrak{h}$ (they form a base for the dual root system $R^\vee$) we get the result for $\mathfrak{g}$.

(iii) In the Weyl relations, only the relations $[X_\alpha, Y_\beta] = 0$ for $\alpha, \beta \in S$ with $\alpha \neq \beta$, are not already known from Theorem 13.2.1. But the weight of $[X_\alpha, Y_\beta] = 0$ is $\alpha - \beta$ which is not a root (because root are linear combinations with constant signs of simple roots).

To prove Serre relations we proceed in the same way. The weight of $\mathrm{ad}\,(X_\alpha)^{1-\langle\alpha^\vee,\beta\rangle}(X_\beta) = 0$ is equal to $\beta + \alpha - \langle\alpha^\vee, \beta\rangle\alpha = \alpha + s_\alpha(\beta) = s_\alpha(\beta - \alpha)$. Because $\beta - \alpha$ is not a root, the same is true for $s_\alpha(\beta - \alpha)$ and the result follows. The same method works for the second type of Serre relations. □

We want to prove that the above relations define a semisimple Lie algebra. For R a reduced root system and S a base for R, we first study the Lie algebra $\widehat{\mathfrak{g}}$ generated by the elements H_α, Y_α and H_α for $\alpha \in S$ subject only to the Weyl relations.

Proposition 13.4.2 *Let* $\widehat{\mathfrak{g}}$ *be the Lie algebra generated by elements* H_α, Y_α *and* H_α *for* $\alpha \in S$ *subject to the relations:*

- $[X_\alpha, Y_\alpha] = H_\alpha$ *for* $\alpha \in S$,
- $[X_\alpha, Y_\beta] = 0$ *for* $\alpha, \beta \in S$ *with* $\alpha \neq \beta$,
- $[H_\alpha, X_\beta] = \langle\alpha^\vee, \beta\rangle X_\alpha$ *for* $\alpha, \beta \in S$,
- $[H_\alpha, Y_\beta] = -\langle\alpha^\vee, \beta\rangle Y_\alpha$ *for* $\alpha, \beta \in S$,

(i) The elements H_α *for* $\alpha \in S$ *are linearly independent in* $\widehat{\mathfrak{g}}$. *Their span is denoted by* $\widehat{\mathfrak{h}}$.

(ii) Denote by $\widehat{\mathfrak{n}}$ *resp.* $\widehat{\mathfrak{n}}_-$ *the Lie subalgebras of* $\widehat{\mathfrak{g}}$ *generated by the* X_α *resp.* Y_α, *then we have the decomposition*

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}$$

and $\widehat{\mathfrak{n}}$ *resp.* $\widehat{\mathfrak{n}}_-$ *is isomorphic to the free Lie algebra generated by the* X_α *resp.* Y_α.

Proof. Remark that in general, the Lie algebra $\widehat{\mathfrak{g}}$ is of infinite dimension. To study this Lie algebra, we define a representation of $\widehat{\mathfrak{g}}$ in a vector space V. To do this we only need to define a linear map $\widehat{\mathfrak{g}} \to \mathfrak{gl}(V)$ such that the images of the elements H_α, X_α and Y_α satisfy the Weyl relations.

Let W be a vector space of dimension $|S|$ and fix a base $(e_\alpha)_{\alpha\in S}$ in W. Let $V = T(W)$ be the tensor algebra over W, a base is given by the elements $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}$ for any sequence $(\alpha_1, \cdots, \alpha_k)$

of simple roots $\alpha_i \in S$. Let us define elements in $\mathfrak{gl}(V)$ as follows (the elements $\widehat{X}_\alpha$ is defined by induction):

$$\begin{cases} \widehat{H}_\alpha(1) = 0 \\ \widehat{H}_\alpha(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}) = -(\langle \alpha^\vee, \alpha_1 \rangle + \cdots + \langle \alpha^\vee, \alpha_k \rangle) e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} \end{cases}$$

$$\begin{cases} \widehat{Y}_\alpha(1) = e_\alpha \\ \widehat{Y}_\alpha(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}) = e_\alpha \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} \end{cases}$$

$$\begin{cases} \widehat{X}_\alpha(1) = 0 \\ \widehat{X}_\alpha(e_\beta) = 0 \text{ for } \beta \in S \\ \widehat{X}_\alpha(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}) = e_{\alpha_1} \otimes X_\alpha(e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_k}) - \delta_{\alpha,\alpha_1}(\langle \alpha^\vee, \alpha_2 \rangle + \cdots + \langle \alpha^\vee, \alpha_k \rangle) e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_k} \end{cases}$$

Lemma 13.4.3 *The map $\widehat{\mathfrak{g}} \to \mathfrak{gl}(V)$ defined by $X_\alpha \mapsto \widehat{X}_\alpha$, $Y_\alpha \mapsto \widehat{Y}_\alpha$ and $H_\alpha \mapsto \widehat{H}_\alpha$ is a Lie algebra morphism.*

Proof. We only need to check the Weyl relations. Remark that the elements $\widehat{H}_\alpha$ act diagonally in the base $(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k})$, therefore we have $[\widehat{H}_\alpha, \widehat{H}_\beta] = 0$.

We also have the equalities $[\widehat{H}_\alpha, \widehat{Y}_\beta](1) = -\langle \alpha^\vee, \beta \rangle e_\beta = -\langle \alpha^\vee, \beta \rangle \widehat{Y}_\beta(1)$ and together with the equalities $[\widehat{H}_\alpha, \widehat{Y}_\beta](e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}) = -\langle \alpha^\vee, \beta \rangle e_\beta \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} = -\langle \alpha^\vee, \beta \rangle \widehat{Y}_\beta(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k})$ we get $[\widehat{H}_\alpha, \widehat{Y}_\beta] = -\langle \alpha^\vee, \beta \rangle \widehat{Y}_\beta$.

We compute $[\widehat{X}_\alpha, \widehat{Y}_\beta](1) = 0 = \delta_{\alpha,\beta} \widehat{H}_\alpha(1)$, $[\widehat{X}_\alpha, \widehat{Y}_\beta](e_\gamma) = -\delta_{\alpha,\beta} \langle \alpha^\vee, \gamma \rangle e_\gamma = \delta_{\alpha,\beta} \widehat{H}_\alpha(e_\gamma)$ and the equalities $[\widehat{X}_\alpha, \widehat{Y}_\beta](e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}) = e_\beta \otimes \widehat{X}_\alpha(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}) - \delta_{\alpha,\beta}(\langle \alpha^\vee, \alpha_1 \rangle + \cdots + \langle \alpha^\vee, \alpha_k \rangle) e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} - e_\beta \otimes \widehat{X}_\alpha(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}) = \delta_{\alpha,\beta} \widehat{H}_\alpha(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k})$ we get $[\widehat{X}_\alpha, \widehat{Y}_\beta] = \delta_{\alpha,\beta} \widehat{H}_\alpha$.

We compute $[\widehat{H}_\alpha, \widehat{X}_\beta](1) = 0 = \langle \alpha^\vee, \beta \rangle \widehat{X}_\beta(1)$, $[\widehat{H}_\alpha, \widehat{X}_\beta](e_\gamma) = 0 = \langle \alpha^\vee, \beta \rangle \widehat{X}_\beta(e_\gamma)$. For the general case, we first need to prove the following formula

$$\widehat{H}_\alpha(\widehat{X}_\beta(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k})) = (\langle \alpha^\vee, \beta \rangle - (\langle \alpha^\vee, \alpha_1 \rangle + \cdots + \langle \alpha^\vee, \alpha_k \rangle)) \widehat{X}_\beta(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}).$$

We prove this formula by induction on k. For $k = 0$ or $k = 1$ we get the trivial equality $0 = 0$. Assume the equality is true and compute

$$\begin{aligned} \widehat{H}_\alpha(\widehat{X}_\beta(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} \otimes e_{\alpha_{k+1}})) = \; & \widehat{H}_\alpha(e_{\alpha_1} \otimes \widehat{X}_\beta(e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_{k+1}})) \\ & -\delta_{\beta,\alpha_1}(\langle \alpha^\vee, \alpha_2 \rangle + \cdots + \langle \alpha^\vee, \alpha_{k+1} \rangle) e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_{k+1}}. \end{aligned}$$

By induction $\widehat{X}_\beta(e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_{k+1}})$ is an eigenvector for $\widehat{H}_\alpha$ therefore it can be writen as a linear combinaison of elements $e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_r}$ with the same eigenvalue *i.e.* $-(\langle \alpha^\vee, \gamma_1 \rangle + \cdots + \langle \alpha^\vee, \gamma_r \rangle)) = (\langle \alpha^\vee, \beta \rangle - (\langle \alpha^\vee, \alpha_2 \rangle + \cdots + \langle \alpha^\vee, \alpha_k \rangle))$. We get the equality

$$\widehat{H}_\alpha(e_{\alpha_1} \otimes \widehat{X}_\beta(e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_{k+1}})) = (\langle \alpha^\vee, \beta \rangle - (\langle \alpha^\vee, \alpha_1 \rangle + \cdots + \langle \alpha^\vee, \alpha_{k+1} \rangle)) e_{\alpha_1} \otimes \widehat{X}_\beta(e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_{k+1}}).$$

Discussing on the cases $\beta = \alpha_1$ and $\beta \neq \alpha_1$, we get the desired equality. We may now compute the equalities $[\widehat{H}_\alpha, \widehat{X}_\beta](e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}) = (\langle \alpha^\vee, \beta \rangle - (\langle \alpha^\vee, \alpha_1 \rangle + \cdots + \langle \alpha^\vee, \alpha_k \rangle)) \widehat{X}_\beta(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}) + (\langle \alpha^\vee, \alpha_1 \rangle + \cdots + \langle \alpha^\vee, \alpha_k \rangle) \widehat{X}_\beta(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}) = \langle \alpha^\vee, \beta \rangle \widehat{X}_\beta(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k})$ thus $[\widehat{H}_\alpha, \widehat{X}_\beta] = \langle \alpha^\vee, \beta \rangle \widehat{X}_\beta$.
□

(ı) From this representation we easily deduce that all the H_α are linearly independent. Indeed, we only have to show that the $\widehat{H}_\alpha$ are linearly independent. But we have $\sum_i a_i \widehat{H}_{\alpha_i}(e_\beta) = -\sum_i a_i \langle \alpha_i^\vee, \beta \rangle e_\beta$

thus $\sum_i a_i \widehat{H}_{\alpha_i} = 0$ if and only if $\sum_i a_i \langle \alpha_i^\vee, \beta \rangle = 0$ for all $\beta \in S$. This gives

$$\left(\sum_i \frac{2a_i \alpha_i}{(\alpha_i, \alpha_i)}, \beta \right) = 0$$

thus $a_i = 0$ for all i because $(\ ,\ )$ is non degenerate and the α_i are linearly independent.

(ıı) Let us first prove that the elements X_α, Y_α and H_α are linearly independent (here α may vary in S). For α fixed, the span of X_α, Y_α and H_α is a quotient of $\mathfrak{sl}_2$ (map X, Y and H to X_α, Y_α and H_α). This map is a Lie algebra morhism and is not trivial (because $H_\alpha \neq 0$). As $\mathfrak{sl}_2$ is simple it has to be injective. Thus X_α, Y_α and H_α are linearly independent and in particular do not vanish. Now let a general element $H = \sum_\beta \lambda_\beta H_\beta$ act by the adjoint action. We see that the elements X_α, Y_α and H_α are eigenvectors with eigenvalue $\sum_\beta \lambda_\beta \langle \beta^\vee, \alpha \rangle$, $-\sum_\beta \lambda_\beta \langle \beta^\vee, \alpha \rangle$ and 0. Thus these elements are linearly independent.

Let us now compute the action of $\widehat{\mathfrak{g}}$ on elements of the form $[X_{\alpha_1}, [X_{\alpha_2}, \cdots, [X_{\alpha_{k-1}}, X_{\alpha_k}]]]$ and $[Y_{\alpha_1}, [Y_{\alpha_2}, \cdots, [Y_{\alpha_{k-1}}, Y_{\alpha_k}]]]$. By induction and using Jacobi identity, we get

$$[H_\alpha, [X_{\alpha_1}, [X_{\alpha_2}, \cdots, [X_{\alpha_{k-1}}, X_{\alpha_k}]]]] = \langle \alpha^\vee, \alpha_1 + \cdots + \alpha_k \rangle [X_{\alpha_1}, [X_{\alpha_2}, \cdots, [X_{\alpha_{k-1}}, X_{\alpha_k}]]]$$
$$[H_\alpha, [Y_{\alpha_1}, [Y_{\alpha_2}, \cdots, [Y_{\alpha_{k-1}}, Y_{\alpha_k}]]]] = -\langle \alpha^\vee, \alpha_1 + \cdots + \alpha_k \rangle [Y_{\alpha_1}, [Y_{\alpha_2}, \cdots, [Y_{\alpha_{k-1}}, Y_{\alpha_k}]]].$$

We also have, by easy induction the following inclusions for $k \geq 2$:

$$[Y_\alpha, [X_{\alpha_1}, [X_{\alpha_2}, \cdots, [X_{\alpha_{k-1}}, X_{\alpha_k}]]]] \in \widehat{\mathfrak{n}}$$
$$[X_\alpha, [Y_{\alpha_1}, [Y_{\alpha_2}, \cdots, [Y_{\alpha_{k-1}}, Y_{\alpha_k}]]]] \in \widehat{\mathfrak{n}}_-.$$

We therefore see that the Lie algebras $\widehat{\mathfrak{n}}$ resp. $\widehat{\mathfrak{n}}_-$ are generated by the elements of the form $[X_{\alpha_1}, [X_{\alpha_2}, \cdots, [X_{\alpha_{k-1}}, X_{\alpha_k}]]]$ resp. $[Y_{\alpha_1}, [Y_{\alpha_2}, \cdots, [Y_{\alpha_{k-1}}, Y_{\alpha_k}]]]$. These elements are eigenvectors for the abelian algebra $\widehat{\mathfrak{h}}$ with eigenvalue $\langle \alpha^\vee, \alpha_1 + \cdots + \alpha_k \rangle$ resp. $-\langle \alpha^\vee, \alpha_1 + \cdots + \alpha_k \rangle$ therefore the sum $\widehat{\mathfrak{n}}_- + \widehat{\mathfrak{h}} + \widehat{\mathfrak{n}}$ is direct. This sum is a subalgebra by the above inclusion and contain generators of $\widehat{\mathfrak{g}}$ thus we have $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}$.

Finally, consider the free Lie algebra $F(Y)$ over the elements Y_α. We have a natural morphism $F(Y) \to F \to \widehat{\mathfrak{g}}$ where F is the free Lie algebra generated by the X_α, the Y_α and the H_α. The first map is injective therefore we only need to prove that the intersection of $F(X)$ (seen as a subalgebra in F) and the ideal spanned by the Weyl relations is trivial. For this we again use the repesentation in $\mathfrak{gl}(V)$. Recall that $F(Y)$ is the Lie subalgebra of $T(Y)$ (where here $Y = k^S$ with base the elements Y_α for $\alpha \in S$). We have a natural representation of $T(Y)$ (as an associative algebra) in V defined by:

$$Y_\alpha \cdot e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} = e_\alpha \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}.$$

This representation restricts to a representation of Lie algebra $T(Y) \to \mathfrak{gl}(V)$ and to a representation $F(Y) \to \mathfrak{gl}(V)$. Furthermore, this representation is the representation obtained by the composition $F(Y) \to F \to \widehat{\mathfrak{g}} \to \mathfrak{gl}(V)$. But the map $T(Y) \to \mathfrak{gl}(V)$ is injective (remark that $T(Y) \simeq V$ and that this representation is simply the left multiplication which is injective because $T(Y)$ is a domain) therefore the same is true for the composition and the result follows. □

Remark 13.4.4 For λ a linear form on $\widehat{\mathfrak{h}}$, let us denote by $\widehat{\mathfrak{g}}_\lambda$ the eigenspace associated to λ *i.e.*:

$$\widehat{\mathfrak{g}}_\lambda = \{x \in \widehat{\mathfrak{g}} \ / \ [h, x] = \lambda(h)x \text{ for all } h \in \widehat{\mathfrak{h}}\}.$$

The previous proof shows that the only linear form λ for which $\widehat{\mathfrak{g}}_\lambda$ is non trivial are such that $\lambda = \sum_{\alpha \in S} a_\alpha \alpha$ with the a_α integers all non negative at the same time (denoted by $\lambda \succ 0$) or all non positive at the same time (denoted by $\lambda \prec 0$)

Theorem 13.4.5 *Let R be a reduced root system with base S and let $\mathfrak{g}$ be the Lie algebra defined by generators and relations as in the previous theorem, then $\mathfrak{g}$ is a semisimple Lie algebra, the Lie subalgebra $\mathfrak{h}$ generated by the H_α for $\alpha \in S$ is a Cartan subalgebra and the root system of $\mathfrak{g}$ is R.*

Proof. Let $\theta_{\alpha,\beta}$ and $\theta^-_{\alpha,\beta}$ be the image in $\widehat{\mathfrak{g}}$ of the elements $\mathrm{ad}\,(X_\alpha)^{1-\langle\alpha^\vee,\beta\rangle}(X_\beta)$ and $\mathrm{ad}\,(Y_\alpha)^{1-\langle\alpha^\vee,\beta\rangle}(Y_\beta)$ for $\alpha,\beta\in S$. Denote by $\mathfrak{u}$ resp. $\mathfrak{u}_-$ the ideal of $\widehat{\mathfrak{n}}$ and $\widehat{\mathfrak{n}}_-$ generated by the $(\theta_{\alpha,\beta})_{\alpha,\beta\in S}$ resp. $(\theta^-_{\alpha,\beta})_{\alpha,\beta\in S}$. Let $\mathfrak{r}=\mathfrak{u}\oplus\mathfrak{u}_-$.

Lemma 13.4.6 *The spaces $\mathfrak{u}$ and $\mathfrak{u}_-$ are ideals of $\widehat{\mathfrak{g}}$.*

Proof. Consider the ideal $\mathfrak{u}_{\alpha,\beta}$ of $\widehat{\mathfrak{g}}$ generated by the element $\theta_{\alpha,\beta}$. If $U(\widehat{\mathfrak{g}})$ is the envelopping algebra of $\widehat{\mathfrak{g}}$, then $\mathfrak{u}_{\alpha,\beta}$ is generated as vector space by the elements $U\cdot\theta_{\alpha,\beta}$ for $U\in U(\widehat{\mathfrak{g}})$. We have the following fact (which is a special case of Poicaré-Birkhoff-Witt theorem 15.0.7).

Fact 13.4.7 *Any element $U\in U(\widehat{\mathfrak{g}})$ is a linear combinaison of elements of the form XYH where X, Y and H are respectively image in $U(\widehat{\mathfrak{g}})$ of tensors of elements in $\widehat{\mathfrak{n}}$, $\widehat{\mathfrak{n}}_-$ and $\widehat{\mathfrak{h}}$.*

Proof. Any element in $U(\widehat{\mathfrak{g}})$ is the image (under the map $T(\widehat{\mathfrak{g}})\to U(\widehat{\mathfrak{g}})$) of a linear combinaison of pure tensors of elements X_α, Y_α and H_α. Let us write such a pure tensor in the following form $v=v_1\otimes\cdots\otimes v_n$. We prove that $v=X_{\alpha_1}\otimes\cdots\otimes X_{\alpha_p}\otimes Y_{\beta_1}\otimes\cdots\otimes Y_{\beta_q}\otimes H_{\gamma_1}\otimes\cdots\otimes H_{\gamma_r}$ for simple roots α_i, β_j and γ_k.

We first prove that we can move all the X_α's to the right. We proceed by induction on $l(v)=n+m_v$ where

$$m_v=\max\{i\ /\ v_i=X\alpha \text{ for some } \alpha\in S \text{ and } v_{i+1}=H_\beta \text{ or } Y_\beta \text{ for some } \beta\in S\}.$$

We thus have $v_{m_v}\otimes v_{m_v+1}=X_\alpha\otimes Y_\beta$ or $X_\alpha\otimes H_\beta$. But in $U(\widehat{\mathfrak{g}})$ we have the relation $x\otimes y=y\otimes x+[x,y]$ thus we have $v_{m_v}\otimes v_{m_v+1}=X_\alpha\otimes Y_\beta=Y_\beta\otimes X_\alpha+[X_\alpha,Y_\beta]$ or $v_{m_v}\otimes v_{m_v+1}=X_\alpha\otimes H_\beta=H_\beta\otimes X_\alpha+[X_\alpha,H_\beta]$. We have $[X_\alpha,Y_\beta]=\delta_{\alpha,\beta}H_\alpha$ and $[X_\alpha,H_\beta]=-\beta(\alpha)X_\alpha$ thus replacing $v_{m_v}\otimes v_{m_v+1}$ by $v_{m_v+1}\otimes v_{m_v}+[v_{m_v},v_{m_v+1}]$ in v gives a new expression for v as linear combinaison of two elements v' and v'' with smaller $l(v')$ and $l(v'')$. We conclude by induction.

The same arguments with the Y_α's and H_β's give the result. □

So we get that the element $U\cdot\theta_{\alpha,\beta}=XYH\cdot\theta_{\alpha,\beta}$. But $H\cdot\theta_{\alpha,\beta}$ is a multiple of $\theta_{\alpha,\beta}$. We prove the following

Fact 13.4.8 *We have $Y_\gamma\cdot\theta_{\alpha,\beta}=0$ for any root γ.*

Proof. We have $Y_\gamma\cdot\theta_{\alpha,\beta}=\mathrm{ad}\,(Y_\gamma)\circ\mathrm{ad}\,(X_\alpha)^{1-\langle\alpha^\vee,\beta\rangle}(X_\beta)$.

If $\gamma\neq\alpha$, then Y_γ and X_α commute therefore we have $Y_\gamma\cdot\theta_{\alpha,\beta}=\mathrm{ad}\,(X_\alpha)^{1-\langle\alpha^\vee,\beta\rangle}([Y_\gamma,X_\beta])$. If $\gamma\neq\beta$ we get the result because $[Y_\gamma,X_\beta]=0$ in that case. If $\gamma=\beta$, then $[Y_\gamma,X_\beta]=H_\beta$ and $Y_\gamma\cdot\theta_{\alpha,\beta}=\mathrm{ad}\,(X_\alpha)^{1-\langle\alpha^\vee,\beta\rangle}(H_\beta)=-\langle\beta^\vee,\alpha\rangle\mathrm{ad}\,(X_\alpha)^{\langle\alpha^\vee,\beta\rangle}(X_\alpha)$. But because α and β are distinct simple roots, we have $\langle\alpha^\vee,\beta\rangle\leq 0$ and $\langle\beta^\vee,\alpha\rangle\leq 0$ with equality simultaneously. If both vanish, the result follows, if not then $\mathrm{ad}\,(X_\alpha)^{\langle\alpha^\vee,\beta\rangle}(X_\alpha)$ vanishes.

If $\gamma=\alpha$, by an easy induction, we get for $k\geq 0$:

$$\mathrm{ad}\,(Y_\alpha)\circ\mathrm{ad}\,(X_\alpha)^{k+1}(X_\beta)=-(k+1)(k+\langle\alpha^\vee,\beta\rangle)\mathrm{ad}\,(X_\alpha)^k(X_\beta).$$

For $k=-\langle\alpha^\vee,\beta\rangle\geq 0$ we get the result. □

From this fact we deduce that the element $Y\cdot\theta_{\alpha,\beta}$ is a multiple of $\theta_{\alpha,\beta}$ (it is non zero if $Y=1$ in $U(\mathfrak{g})$). Because $\theta_{\alpha,\beta}$ is a weight vector, $H\cdot\theta_{\alpha,\beta}$ is also a multiple of $\theta_{\alpha,\beta}$ thus $XYH\cdot\theta_{\alpha,\beta}$ is a multiple of $X\cdot\theta_{\alpha,\beta}$. This implies that $\mathfrak{u}_{\alpha,\beta}$ is contained in $\mathfrak{u}$. We thus have $\mathfrak{u}=\sum_{\alpha,\beta}\mathfrak{u}_{\alpha,\beta}$ thus $\mathfrak{u}$ is an ideal in $\widehat{\mathfrak{g}}$ because the $\mathfrak{u}_{\alpha,\beta}$'s are. The same proof gives the result for $\mathfrak{u}_-$. □

As a consequence of the previous lemma, we see that the ideal $\mathfrak{r} = \mathfrak{u}_- \oplus \mathfrak{u}$ is the ideal generated by the $\theta_{\alpha,\beta}$ and the $\theta^-_{\alpha,\beta}$ in $\widehat{\mathfrak{g}}$. We also know that $\mathfrak{u} \subset \widehat{\mathfrak{n}}$ and $\mathfrak{u}_- \subset \widehat{\mathfrak{n}}_-$ thus we have the decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$$

where $\mathfrak{n} = \widehat{\mathfrak{n}}/\mathfrak{u}$ and $\mathfrak{n}_- = \widehat{\mathfrak{n}}_-/\mathfrak{u}_-$. We shall now need the following definition:

Definition 13.4.9 *Let V be a vector space (possibly of infinite dimension, otherwise the definition is the one of nilpotent elements) and let $u \in \mathrm{End}(V)$. The endomorphism u is called* locally nilpotent *if for any element $v \in V$, there exists an integer n with $u^n(v) = 0$.*

Example 13.4.10 (ı) If V is finite dimensional, then u is locally nilpotent if and only if u is nilpotent.

(ıı) Let $V = k[x]$ the vector space of polynomial in the variable x and let $u = \partial/\partial x$. Then u is locally nilpotent but not nilpotent.

If u is a locally nilpotent element, the endomorphism $\exp(u)$ is well defined because for $v \in V$ fixed, $u^n(v)$ vanishes for large n.

Lemma 13.4.11 *The endomorphisms* $\mathrm{ad}\,(X_\alpha)$ *and* $\mathrm{ad}\,(Y_\alpha)$ *are locally nilpotent on* $\mathfrak{g}$.

Proof. Remark that since we do not know that $\mathfrak{g}$ is finite dimensional yet, it makes sense to ask for local nilpotence and not simply nilpotence.

Let $\mathfrak{g}(\alpha) = \{x \in \mathfrak{g} \ / \ \mathrm{ad}\,(X_\alpha)^k(x) = 0 \text{ for some } k \geq 0\}$. We prove that $\mathfrak{g}(\alpha)$ is a Lie subalgebra of $\mathfrak{g}$. Indeed, if x and y are in $\mathfrak{g}(\alpha)$, we have $\mathrm{ad}\,(X_\alpha)^k(x) = 0$ and $\mathrm{ad}\,(X_\alpha)^l(y) = 0$. But because $\mathrm{ad}\,(X_\alpha)$ is a derivation, we have by Lemma 9.3.2 with $\lambda = \mu = 0$ the equality

$$\mathrm{ad}\,(X_\alpha)^n([x,y]) = \sum_{i=0}^{n} \binom{n}{i} \left[\mathrm{ad}\,(X_\alpha)^i(x), \mathrm{ad}\,(X_\alpha)^{n-i}(y)\right].$$

The result follows.

We have $X_\alpha \in \mathfrak{g}(\alpha)$. By the Serre relations, we know that $X_\beta \in \mathfrak{g}(\alpha)$ and by the Weyl relations, H_β and Y_β are in $\mathfrak{g}(\alpha)$. Therefore $\mathfrak{g}(\alpha) = \mathfrak{g}$. The same method gives the result for Y_α. □

We now introduce for the Lie algebra $\mathfrak{g}$ the same notation as we did for semisimple Lie algebras with roots: for λ a linear form on $\mathfrak{h}$, we define $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \ / \ [h,x] = \lambda(h)x \text{ for all } h \in \mathfrak{h}\}$. The forms λ with $\mathfrak{g}_\lambda \neq 0$ are called the weights of $\mathfrak{g}$ and a element in $\mathfrak{g}_\lambda$ is said to have weight λ. By Remark 13.4.4 and Proposition 13.4.2 we know that $\widehat{\mathfrak{g}}$ is a direct sum of weight spaces (the sum is direct because the spaces are eigenspaces for different eigenvalues) with weights $\lambda \prec 0$ or $\lambda \succ 0$. By quotienting by $\mathfrak{r}$, the same is true for $\mathfrak{g}$. Furthermore, we have

$$\mathfrak{h} = \mathfrak{g}_0, \ \mathfrak{n} = \bigoplus_{\lambda \succ 0} \mathfrak{g}_\lambda \text{ and } \mathfrak{n}_- = \bigoplus_{\lambda \prec 0} \mathfrak{g}_\lambda$$

Lemma 13.4.12 *If $\lambda = w(\mu)$ for $w \in W$, then* $\dim \mathfrak{g}_\lambda = \dim \mathfrak{g}_\mu$.

Proof. It is enough to prove this for $w = s_\alpha$ where α is a simpe root because W is generated by reflection with respect to simple roots. Thus we assume $\lambda = s_\alpha(\mu)$. Let us define the automorphism $\sigma_\alpha : \mathfrak{g} \to \mathfrak{g}$ by

$$\sigma_\alpha = \exp(\mathrm{ad}\,(X_\alpha)) \exp(-\mathrm{ad}\,(Y_\alpha)) \exp(\mathrm{ad}\,(X_\alpha)).$$

For $x \in \mathfrak{g}_\lambda$, we check that $\sigma_\alpha(x) \in \mathfrak{g}_\mu$. We first compute, for $h \in \mathfrak{h}$, the formula (the first equality comes from the fact that the exponential of nilpotent element is a Lie algebra morphism):

$$[\sigma_\alpha(h), \sigma_\alpha(x)] = \sigma_\alpha([h,x]) = \sigma_\alpha(\lambda(h)x) = \lambda(h)\sigma_\alpha(x).$$

We now compute $\sigma_\alpha(h)$ for $h \in \mathfrak{h}$.

Fact 13.4.13 *We have the equality* $\sigma_\alpha(h) = h - \alpha(h)H_\alpha$.

Proof. Indeed, we first compute $\exp(\mathrm{ad}\,(X_\alpha))(H) = H - \alpha(H)X_\alpha$, then we get $\exp(-\mathrm{ad}\,(Y_\alpha))(H - \alpha(H)X_\alpha) = H - \alpha(H)(X_\alpha + H_\alpha)$ and finally the formula follows. □

Therefore, we see that for $h' = \sigma_\alpha(h)$ we have $h = \sigma_\alpha(h')$ and for any $h \in \mathfrak{h}$ (or any $h' \in \mathfrak{h}$) we have the equality

$$[h', \sigma_\alpha(x)] = \lambda(h)\sigma_\alpha(x) = \lambda(h' - \alpha(h')H_\alpha)\sigma_\alpha(x) = (\lambda - \lambda(H_\alpha)\alpha)(h')\sigma_\alpha(x) = \mu(h')\sigma_\alpha(x)$$

proving that $\sigma_\alpha(x) \in \mathfrak{g}_\mu$. Applying the inverse of σ_α we map $\mathfrak{g}_\mu$ to $\mathfrak{g}_\lambda$ therefore σ_α realises an isomorphism from $\mathfrak{g}_\lambda$ onto $\mathfrak{g}_\mu$ and the result follows. □

Lemma 13.4.14 *We have* $\dim \mathfrak{g}_\alpha = 1$ *and* $\dim \mathfrak{g}_{m\alpha} = 0$ *for* $\alpha \in S$ *and* $m \notin \{-1, 0, 1\}$. *For* $\beta \in R$, *we have* $\dim \mathfrak{g}_\beta = 1$ *and* $\dim \mathfrak{g}_{m\beta} = 0$ *for* $m \notin \{-1, 0, 1\}$.

Proof. We have seen that $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}$ and that $\widehat{\mathfrak{n}}$ is generated by the X_s's for $\alpha \in S$. Therefore we have $\widehat{\mathfrak{g}}_\alpha = kX_\alpha$ for $\alpha \in S$ and $\widehat{\mathfrak{g}}_{m_\alpha} = 0$ for $\alpha \in S$ and $m \notin \{-1, 0, 1\}$. As $X_\alpha \notin \mathfrak{r}$, the result follows. The result for $\beta \in R$ comes from the simple root case and the previous lemma. □

Lemma 13.4.15 *Let* λ *be real linear combinaison of simple roots in* $\mathfrak{h}^\vee$ *such that* λ *is not colinear to any root. Then there exists an element* $w \in W$ *such that* $w(\lambda) = \sum_{\alpha\in S} t_\alpha \alpha$ *with some* $t_\alpha > 0$ *and some* $t_\alpha < 0$.

Proof. Let $\mathfrak{h}^\vee_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}$ the real spans of the $\alpha \in S$ and H_α for $\alpha \in S$. Let L (resp. L_α) be the hyperplane orthogonal to λ (resp. to α) in $\mathfrak{h}_\mathbb{R}$. The hyperplane L is distinct form all the hyperplanes H_α. Let $H \in L$ such that $\alpha(H) \neq 0$ for all $\alpha \in S$. We know that there exists $w \in W$ such that $\alpha(w(H)) > 0$ for all $\alpha \in S$. We therefore get, writing $w(\lambda) = \sum_{\alpha\in S} t_\alpha\alpha$, the equality

$$0 = \lambda(H) = (\lambda)(w^{-1}w(H)) = (w(\lambda))(w(H)) = \sum_{\alpha\in S} t_\alpha \alpha(w(H)).$$

Therefore the result follows because all the t_α do not vanish. □

Lemma 13.4.16 *If* λ *is not a root and is not zero then* $\mathfrak{g}_\lambda = 0$.

Proof. We know that the weights of $\widehat{\mathfrak{g}}$ are integral linear combinaison of simple roots with constant sign coefficients. In particular, if λ is a multiple of a root, then it is an integer multiple of a root and by Lemma 13.4.14 we have $\mathfrak{g}_\lambda = 0$.

If λ is not a multiple of a root, then we know by the previous lemma that there exists $w \in W$ with $w(\lambda)$ not a linear combinaison with constant sign coefficients of the simple roots. In particular $\widehat{\mathfrak{g}}_{w(\lambda)} = 0$ thus $\mathfrak{g}_{w(\lambda)} = 0$ and we get $\mathfrak{g}_\lambda = 0$. □

The Lie algebra $\mathfrak{g}$ is therefore finite dimensional and its dimension is $|S| + |R|$. Indeed, we have the decomposition into weight spaces which is direct

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha.$$

For $\alpha \in S$, we have that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional equal to the span of H_α. The Lie algebra $\mathfrak{s}_\alpha$ generated by $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$ and H_α is isomorphic to $\mathfrak{sl}_2$. Applying elements of the form σ_α, we see that for any root $\beta \in R$, there is an element $H_\beta \in \mathfrak{h}$ such that the Lie algebra $\mathfrak{s}_\beta$ generated by $\mathfrak{g}_\beta$, $\mathfrak{g}_{-\beta}$ and H_β is isomorphic to $\mathfrak{sl}_2$.

Lemma 13.4.17 *The Lie algebra* $\mathfrak{g}$ *is semisimple.*

Proof. Let $\mathfrak{a}$ be an abelian ideal in $\mathfrak{g}$. Because it is stable under $\mathfrak{h}$, it has to be a direct sum of weight spaces for $\mathfrak{h}$ thus we have

$$\mathfrak{a} = \mathfrak{a} \cap \mathfrak{h} \bigoplus_{\alpha \in R} \mathfrak{a} \cap \mathfrak{g}_\alpha.$$

But $\mathfrak{a} \cap \mathfrak{s}_\alpha$ is an abelian ideal in $\mathfrak{s}_\alpha$ thus $\mathfrak{a} \cap \mathfrak{s}_\alpha = 0$ therefore $\mathfrak{a} \cap \mathfrak{g}_\alpha = 0$ and $\mathfrak{a} \subset \mathfrak{h}$. Let $h \in \mathfrak{a} \cap \mathfrak{h}$, we have $[h, X_\alpha] = \alpha(h)X_\alpha \in \mathfrak{a}$ thus $\alpha(h) = 0$ for all root α thus $h = 0$ and $\mathfrak{a} = 0$. □

To finish we prove that $\mathfrak{h}$ is a Cartan subalgebra and that the root system is R. We have that $\mathfrak{h}$ is abelian therefore nilpotent. Furthermore, if $x \in \mathfrak{n}_\mathfrak{g}(\mathfrak{h})$, then $x = h + \sum_{\alpha \in R} a_\alpha X_\alpha$ where $h \in \mathfrak{h}$. We get for all $h' \in \mathfrak{h}$ the inclusion

$$[h', x] = \sum_{\alpha \in R} a_\alpha \alpha(h') X_\alpha \in \mathfrak{h}$$

thus for all $\alpha \in R$ and all $h' \in \mathfrak{h}$, the equalities $a_\alpha \alpha(h') = 0$ giving $a_\alpha = 0$ for all $\alpha \in R$ and $x \in \mathfrak{h}$. The fact that R is the root system is clear. □

Corollary 13.4.18 *For any reduced root system* R*, there exists a semisimple Lie algebra with root system* R*.*

Corollary 13.4.19 *Let* $\mathfrak{g}$ *be a semisimple Lie algebra, it is equal to the Lie algebra generated by* X_α*,* Y_α *and* H_α *for* $\alpha \in S$ *with relations the Weyl and Serre relations.*

Proof. Let $\mathfrak{g}'$ be the Lie algebra with the above presentation with generators X'_α, Y'_α and H'_α for $\alpha \in S$. It is semisimple with root system R. By Theorem 13.4.1, the relations over the elements X_α, Y_α and H_α for $\alpha \in S$ are satisfied in $\mathfrak{g}$ therefore the map $\mathfrak{g}' \to \mathfrak{g}$ sending X'_α to X_α, Y'_α to Y_α and H'_α to H_α is a Lie algebra morphism. It is surjective by *loc. cit.*. Furthermore, we have in both case by Theorem 13.2.1 the equalities $\dim \mathfrak{g}' = |S| + |R| = \dim \mathfrak{g}$ and the result follows. □

Corollary 13.4.20 *Two Lie algebras are isomorphic if and only if they have the same root system.*

Proof. If the two Lie algebra are isomorphic they have the same root system (all the Cartan subalgebra are conjugated therefore the root system does not depend on the choice of a Cartan subalgebra).

Conversely, if two Lie algebra have the same root system, they have the same presentation and are therefore isomorphic by the previous corollary. □

Corollary 13.4.21 *The simple Lie algera are in one to one correspondence with connected Dynkin diagrams.*

Corollary 13.4.22 *Let* $\mathfrak{g}$ *be a semisimple Lie algebra, there exists a Lie algebra involution* ι *mapping* X_α *to* $-Y_\alpha$*,* Y_α *to* $-X_\alpha$ *and* H_α *to* $-H_\alpha$*.*

Proof. This is true because the elements $-Y_\alpha$, $-X_\alpha$ and $-H_\alpha$ satisfy the same relations as X_α, Y_α and H_α. □

13.5 Exercices

Exercice 13.5.1 Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra in $\mathfrak{g}$. Let R be the associated root system in $\mathfrak{h}^\vee$.

(i) Prove that $\dim \mathfrak{g} = \dim \mathfrak{h} + |R|$ where $|R|$ is even.

(ii) Prove that there are no semisimple Lie algebra of dimension 4,5 and 7.

(iii) Prove that any 3-dimensional semisimpple Lie algebra is isomorphic to $\mathfrak{sl}_2$.

Chapter 14

Representations of semisimple Lie algebras

In this chapter we study a special type of representations of semisimple Lie algberas: the so called highest weight representations. In particular every finite dimensional representation is an highest weight representation.

We fix in all the chapter a semisimple Lie algebras $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$. We denote by R the associated root system and we fix a base S of R. We denote by R_+ (resp. R_- the set of positive (resp. negative) roots and we fix for any positive root $\alpha \in R_+$ elements $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, Y_{-\alpha}] = H_\alpha$. We define the subalgebras

$$\mathfrak{n} = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha,\ \mathfrak{n}_- = \bigoplus_{\alpha \in R_-} \mathfrak{g}_\alpha \text{ and } \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}.$$

14.1 Weights

Definition 14.1.1 *Let V be a representation of $\mathfrak{g}$ (not necessarily finite dimensional) and let $\lambda \in \mathfrak{h}^\vee$ be a linear form on $\mathfrak{h}$. An element $v \in V$ is said of* weight λ *if for all $h \in \mathfrak{h}$, we have $h \cdot v = \lambda(h)v$. The set of all elements of weight λ is a vector space V_λ called a* weight space of V *. If V_λ is non trivial then λ is calles a*weight of the representation *. The dimension* $\dim V_\lambda$ *is called the* multiplicity of the weight λ in V.

Proposition 14.1.2 *Let V be a representation of $\mathfrak{g}$.*

(ı) We have $\mathfrak{g}_\alpha \cdot V_\lambda = V_{\lambda+\alpha}$ for $\lambda \in \mathfrak{h}^\vee$ and $\alpha \in R$.

(ıı) The sum $V' = \sum_\lambda V_\lambda$ is direct and is a subrepresentation of V.

Proof. *(ı)* Let $x \in \mathfrak{g}_\alpha$, $h \in \mathfrak{h}$ and $v \in V_\lambda$, we have the equalities

$$h \cdot (x \cdot v) = [h, x] \cdot v + x \cdot (h \cdot v) = \alpha(h) x \cdot v + \lambda(h) x \cdot v$$

and the result follows.

(ıı) The fact that the sum is direct comes from the classical fact that eigenspaces are in direct sum. Furthermore, from *(ı)*, we get that V' is a subrepresentation. □

14.2 Primitive elements

Definition 14.2.1 *Let V be a representation of $\mathfrak{g}$ and $\lambda \in \mathfrak{h}^\vee$. A vector $v \in V$ is called a* promitive element of weight λ *if $h \cdot v = \lambda(h)v$ for all $h \in \mathfrak{h}$ and $x \cdot v = 0$ for all $x \in \mathfrak{n}$.*

Remark 14.2.2 The last condition defining a primitive element is equivalent to the following two conditions:

- $X_\alpha \cdot v = 0$ for all $\alpha \in R_+$,
- $X_\alpha \cdot v = 0$ for all $\alpha \in S$.

The primitive elements are also the eigenvalues of the Borel subalgebra $\mathfrak{b}$.

Proposition 14.2.3 *Let V be a representation of $\mathfrak{g}$ and let $v \in V$ be a primitive element of weight λ. Let E be the submodule generated by v in V.*

($\imath$) If $\beta_1, \cdots, \beta_n$ are the positive roots, then E is spanned by the elements $Y_{\beta_1}^{k_1} \cdots Y_{\beta_n}^{k_n} \cdot v$ with $k_i \in \mathbb{Z}_{\geq 0}$

($\imath\imath$) The weight of E have finite multiplicity and are of the form

$$\lambda - \sum_{\alpha \in S} l_\alpha \alpha \textit{ with } l_\alpha \in \mathbb{Z}_{\geq 0}.$$

($\imath\imath\imath$) The weight λ has multiplicity 1 in E.

($\imath v$) The representation E is irreducible.

Proof. *($\imath$)* The subrepresentation E generated by v is the subspace of all elements of the form $U \cdot v$ for $U \in U(\mathfrak{g})$. Recall from Fact 13.4.7 that any element $U \in U(\mathfrak{g})$ can be written in the form $U = YXH$ (we exchange the role of positive and negative roots here) with $X \in U(\mathfrak{n})$, $Y \in U(\mathfrak{n}_-)$ and $H \in U(\mathfrak{h})$. By the definition of primitive elements, we have that $H \cdot v$ and $X \cdot v$ are multiples of v (for $X \cdot v$, this multiple is 0 except for $X = 1$). We therefore get $U \cdot v = Y \cdot v$. But any element of $U(\mathfrak{n}_-)$ can be written in the form $Y_{\beta_1}^{k_1} \cdots Y_{\beta_n}^{k_n}$, the result follows (for this last result, use the relations $Y_{\beta_i} \otimes Y_{\beta_j} = Y_{\beta_j} \otimes Y_{\beta_i} + [Y_{\beta_i}, Y_{\beta_j}]$ for $i > j$).

($\imath\imath$) This is direct consequence of *($\imath$)* and the previous proposition.

($\imath\imath\imath$) This comes from the fact that $\lambda - \sum_{\alpha \in S} l_\alpha \alpha = \lambda$ if and only if $l_\alpha = 0$ for all $\alpha \in S$.

($\imath v$) Assume that $E = E_1 \oplus E_2$ where E_i is a subrepresentation of E. In particular, we may consider the weight spaces $(E_i)_\lambda$.

Fact 14.2.4 *We have $E_\lambda = (E_1)_\lambda \oplus (E_2)_\lambda$.*

Proof. We easily have that the sum is direct and the inclusion of the right hand side in the left hand side. Let $v \in E_\lambda$, we have $v = v_1 + v_2$ with $v_i \in E_i$. For $h \in \mathfrak{h}$, we get

$$\lambda(h)(v_1 + v_2) = \lambda(h)v = h \cdot v = h \cdot v_1 + h \cdot v_2$$

but $h \cdot v_i \in E_i$ thus $h \cdot v_i = \lambda(h)v_i$ and the result follows. □

In our situation, because E_λ is one dimensional, we get that one of the two $(E_i)_\lambda$ vanishes and the other is E_λ. Let us say $(E_1)_\lambda = E_\lambda$, because E is generated by $v \in E_\lambda$ we get $E = E_1$ and $E_2 = 0$. □

14.3 Highest weight representations

Definition 14.3.1 *An simple representation of $\mathfrak{g}$ with a primitive vector of weight λ is called an* highest weight representation of weight λ*.*

Theorem 14.3.2 *Let V be an highest weight representation of weight λ.*

(ı) There is a unique primitive vector modulo scalar multiplication, its weight is λ.

(ıı) The weights of V have finite multiplicity and are of the form

$$\lambda - \sum_{\alpha\in S} l_\alpha \alpha \text{ with } l_\alpha \in \mathbb{Z}_{\geq 0}.$$

(ııı) The weight λ has multiplicity 1.

(ıv) Two highest weight representations V_1 and V_2 of highest weights λ_1 and λ_2 are isomorphic if and only if $\lambda_1 = \lambda_2$.

Proof. *(ı)* The submodule E generated by a primitive vector v in V is non trivial (because it contains v) and therefore it is equal to V because V is simple. This implies that the weight of v is λ by the previous proposition. The result now follows from point *(ııı)* of the previous proposition.

(ıı) Follows from point *(ıı)* of the previous proposition.

(ııı) Follows from point *(ııı)* of the previous proposition.

(ıv) Let V_i for $i \in \{1, 2\}$ be two highest weight representations of highest weight λ_i and let v_i be primitive vectors. If $V_1 \simeq V_2$, we have $\lambda_1 = \lambda_2$ by *(ı)*. Conversely, assume that $\lambda_1 = \lambda_2$ and define $V = V_1 \oplus V_2$, the vector $v = v_1 + v_2$ is a primitive vector for V of weight $\lambda = \lambda_1 = \lambda_2$. Let E be the subrepresentation generated by v in V. The projection $V \to V_i$ induces a morphism of representations $E \to V_i$ mapping v to v_i. Because v_i spans V_i as representation, this implies that this map $E \to V_i$ is surjective. Its kernel is $E\cap V_{3-i}$ and is a submodule of V_{3-i}. It does not contain v_1 because $\dim E_\lambda = 1$ thus it is a proper submodule of V_{3-i}. But V_{3-i} being simple we get $E \cap V_{3-i} = 0$ and the projection $E \to V_i$ is an isomorphism. Therefore $V_1 \simeq E \simeq V_2$. □

Remark 14.3.3 There are simple modules with no primitive elements, these are infinite dimensional.

Theorem 14.3.4 *For each linear form $\lambda \in \mathfrak{h}^\vee$, there is an highest weight representation of highest weight λ.*

Proof. We first construct a representation of $\mathfrak{g}$ with a primitive element of weight λ. Let $L(\lambda)$ be a one dimensional vector space and define a representation of $\mathfrak{b}$ on $L(\lambda)$ as follows: if $v \in L(\lambda)$ is any vector we define

$$X_\alpha \cdot v = 0 \text{ and } H_\alpha \cdot v = \lambda(H_\alpha)v.$$

This is indeed a representation because these elements satisfy the relations between the generators X_α and H_α for $\alpha \in S$ in $\mathfrak{b}$. Therefore $L(\lambda)$ is a $U(\mathfrak{b})$-module, but because of the inclusion $\mathfrak{b} \subset \mathfrak{g}$ we have a morphism $U(\mathfrak{b}) \to U(\mathfrak{g})$. We can therfore consider

$$V(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} L(\lambda).$$

This module is generated by the element $w = 1 \otimes v$. We have $X_\alpha \cdot w = 1 \otimes X_\alpha \cdot w = 0$ and $H_\alpha \cdot w = 1 \otimes H_\alpha \cdot v = \lambda(H_\alpha)w$. Therefore w is an primitive vector of highest weight λ as soon as it is non trivial. The non triviality comes from Poincaré-Birkhoff-Witt Theorem 15.0.7: the algebra $U(\mathfrak{g})$ is a free $U(\mathfrak{b})$-module. We also see, using Fact 13.4.7 (and exchanging the role of the X_α's and of the

Y_α's), that $V(\lambda)$ is generated as vector space by the elements of the form $YHX \cdot w$ therefore of the form $Y_{\alpha_1}^{k_1} \cdots Y_{\alpha_m}^{k_m} \cdot w$. Its weight with respect to $\mathfrak{h}$ is $\lambda - \sum_i k_i \alpha_i$ and we have

$$V(\lambda) = \bigoplus_{\mu \in \mathfrak{h}^\vee} V(\lambda)_\mu.$$

We now want to construct a simple module out of $V(\lambda)$. Let V' be any proper submodule of $V(\lambda)$ and consider the weight space V'_λ. We have $V'_\lambda \subset V(\lambda)_\lambda$. But the last one is one dimensional thus either $V'_\lambda = V(\lambda)_\lambda$ or $V'_\lambda = 0$. In the first case, because w generates $V(\lambda)$ we get $V' = V(\lambda)$ a contradiction, thus for any proper submodule V' we have $V'_\lambda = 0$. Because V' is stable under $\mathfrak{h}$, we also have by restriction of the decomposition of $V(\lambda)$ a decomposition

$$V' = \bigoplus_{\mu \in \mathfrak{h}^\vee} V'_\mu$$

and we get the inclusion

$$V' \subset \bigoplus_{\mu \neq \lambda} V(\lambda)_\mu = W.$$

Thus, the sum N of all proper submodules is contained in W and is again proper. It is the maximal proper submodule and the quotient $V(\lambda)/N$ is a highest weight module of highest weight λ. □

14.4 Finite dimensional representations

Proposition 14.4.1 *Let V be a finite dimensional representation of $\mathfrak{g}$. The following properties hold.*
(ı) We have the equality $V = \oplus_\lambda V_\lambda$.
(ıı) If λ is a weight of V, then $\lambda(H_\alpha) \in \mathbb{Z}$ for all $\alpha \in R$.
(ııı) If V is not trivial, then v contains a primitive element.
(ıv) If V is generated by a primitive element, then V is simple.

Proof. *(ı)* All elements of $\mathfrak{h}$ act as diagonalisable endomorphisms and are therefore simultaneously diagonalisable because $\mathfrak{h}$ is abelian.

(ıı) If $\mathfrak{g}$ acts, then the subalgebra $\mathfrak{s}_\alpha$ which is isomorphic to $\mathfrak{sl}_2$ also acts. The weights of H_α are therefore integers and these weights are the scalars $\lambda(H_\alpha)$.

(ııı) This follows from Lie's theorem: because $\mathfrak{b}$ is solvable it has a non zero eigenvector which is therefore primitive.

(ıv) By Weyl's Theorem, the representation V is completely reducible. But we know that V, being the representation generated by a primitive element, is irreductible, therefore it is simple. □

Corollary 14.4.2 *Every finite dimensional representation has a highest weight.*

Theorem 14.4.3 *Let $\lambda \in \mathfrak{h}^\vee$ and let $V(\lambda)$ be a simple representation of $\mathfrak{g}$ with highest weight λ. Then $V(\lambda)$ is finite dimensional if and only if $\lambda(H_\alpha) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in R_+$.*

Proof. If $V(\lambda)$ is finite dimensional, then $\lambda(H_\alpha)$ is the weight of a primitive elements in the finite dimensional $\mathfrak{sl}_2 \simeq \mathfrak{s}_\alpha$-representation V. Therefore we have $\lambda(H_\alpha) \in \mathbb{Z}_{\geq 0}$.

Conversely, let v be a primitive element of weight λ and let $m_\alpha = \lambda(H_\alpha)$ for α a simple root. We have $m_\alpha \geq 0$ and put $v_\alpha = Y_\alpha^{m_\alpha+1}(v)$. We have for $\beta \neq \alpha$ a simple root the equality

$$X_\beta(v_\alpha) = Y_\alpha^{m_\alpha+1}(X_\beta(v)) = 0$$

and also $X_\alpha(v_\alpha) = 0$ by our formula in Proposition 10.3.1 on $\mathfrak{sl}_2$-representations (here v is a primitive vector of weight m_α for $V(\lambda)$ seen have a $\mathfrak{sl}_2 = \mathfrak{s}_\alpha$-representation). Therefore v_α is a primitive element of weight $\lambda - (m_\alpha + 1)\alpha$ this is not possible because there are only primitive vectors of weight λ in $V(\lambda)$. Thus we have $v_\alpha = 0$.

Denote by F_α the subrepresentation of $V(\lambda)$ as a $\mathfrak{s}_\alpha$-representation generated by v. It is finite dimensional and spanned by the elements $Y_\alpha^k(v)$ for $k \in [0, m_\alpha]$. Let T_α be the set of all finite dimensional sub-$\mathfrak{s}_\alpha$-representations of $V(\lambda)$.

Fact 14.4.4 *Let $E \in T_\alpha$, we have $\mathfrak{g} \cdot E \in T_\alpha$.*

Proof. Because $\mathfrak{g}$ and E are finite dimensional, the same is true for $\mathfrak{g} \cdot E$. We have to prove that $\mathfrak{g} \cdot E$ is a representation of $\mathfrak{s}_\alpha$ but this is clear since $\mathfrak{s}_\alpha \cdot \mathfrak{g} \cdot E \subset \mathfrak{g} \cdot E$. ☐

Let $E_\alpha = \sum_{E \in T_\alpha} E$.

Fact 14.4.5 *The space E_α is a representation of $\mathfrak{g}$.*

Proof. Indeed, let $x \in E \in T_\alpha$, we have $\mathfrak{g} \cdot x \in \mathfrak{g} \cdot E \in T_\alpha$ thus $x \in E_\alpha$. As any element in E_α is a linear combination of elements in $E \in T_\alpha$ for some E, the result follows. ☐

We therefore have a subrepresentation E_α of $V(\lambda)$ which is non trivial because it contains v. Therefore, as $V(\lambda)$ is simple, we have $E_\alpha = V(\lambda)$ and $V(\lambda)$ is a sum of finite dimensional $\mathfrak{s}_\alpha$-representations.

Let P_λ be the set of weights of $V(\lambda)$. It is enough to prove that P_λ is finite since all weight spaces are finite dimensional. Recall also that P_λ is contained in the set of linear forms of the form

$$\lambda - \sum_{\alpha \in S} k_\alpha \alpha$$

with $k_\alpha \in \mathbb{Z}_{\geq 0}$. We therefore only need to bound P_λ to get the result. For this we use the action of the Weyl group.

Fact 14.4.6 *The set P_λ is invariant under the action of the Weyl group.*

Proof. Let μ be a weight and let v' be a vector of weight μ. We only need to prove that P_λ is stable under the action of simple reflections. Let α be a simple root, we know that v' is contained in some finite dimensional subspace F_α which is stable under $\mathfrak{s}_\alpha$. We therefore have $m = \mu(H_\alpha) \in \mathbb{Z}$ and we can look at

$$x = Y_\alpha^m(v') \text{ for } m \geq 0 \text{ and } x = X_\alpha^{-m}(v') \text{ for } m \leq 0.$$

By our study on $\mathfrak{sl}_2$-representations, we know that x is non trivial and its weight is $\mu - m\alpha = s_\alpha(\mu)$. The result follows. ☐

We apply this result to the element w of the Weyl group sending the base S to its opposite $-S$. We have that for any weight μ, the linear form $w(\mu)$ is a weight of $V(\lambda)$ thus can be written in the form

$$w(\mu) = \lambda - \sum_{\alpha \in S} k_\alpha \alpha$$

with $k_\alpha \in \mathbb{Z}_{\geq 0}$ and applying w^{-1} we get $\mu = w^{-1}(\lambda) - \sum_{\alpha \in S} k_\alpha w^{-1}(\alpha)$ but because $w(S) = -S$ and because $\mu \in P_\lambda$, we have

$$\mu = w^{-1}(\lambda) + \sum_{\alpha \in S} k_{w(\alpha)} \alpha \text{ and } \mu = \lambda - \sum_{\alpha \in S} l_\alpha \alpha$$

with $k_\alpha \in \mathbb{Z}_{\geq 0}$ and with $l_\alpha \in \mathbb{Z}_{\geq 0}$. Writing $\lambda - w^{-1}(\lambda) = \sum_\alpha r_\alpha \alpha$ we get $l_\alpha + k_{w(\alpha)} = r_\alpha$ therefore for any weight $\mu \in P_\lambda$, we have the bound $l_\alpha \leq r_\alpha$ thus P_λ is bounded and therefore finite. ☐

Proposition 14.4.7 *Let V be a finite dimensional representation of $\mathfrak{g}$ and let $P(V)$ be the set of weights of V. Then W acts on $P(V)$ and if λ and μ are in the same orbit then* $\dim V_\lambda = \dim V_\mu$.

Proof. The first part of the proposition was proved in the proof of the previous theorem. One easily check that for any simple root (ad even any root) α, the element X_α acts nilpotently on V. Indeed, if x has weight λ, then $X_\alpha^k \cdot x$ has weight $\lambda + k\alpha$. Because the set of weights is finite we must have $X_\alpha^k \cdot x = 0$ for k large enough. The same is true for the action of Y_α. We may therefore define the action of $\sigma_\alpha = \exp(X_\alpha)\exp(-Y_\alpha)\exp(X_\alpha)$ on V. This action is bijective. But because σ_α acts on $\mathfrak{h}^\vee$ as s_α we see that $\sigma_\alpha(V_\lambda) = V_{s_\alpha(\lambda)}$, these spaces therefore have the same dimension. The result follows because the Weyl group is generated by the simple reflections. □

14.5 Application to the Weyl group

Definition 14.5.1 *The* fundamental weights $(\varpi_\alpha)_{\alpha\in S}$ *form by definition the dual base in $\mathfrak{h}^\vee$ to the base $(H_\alpha)_{\alpha\in S}$ of $\mathfrak{h}$.*

Remark 14.5.2 Let $V(\lambda)$ be the highest weight module of highest weight λ. Then $V(\lambda)$ is finite dimensional if and only if λ is a linear combination of fundamental weights with non negative integer coefficients. We denote by P the set of fundamental weights.

Proposition 14.5.3 *The Weyl group acts simply transitively on the set of bases of the root system.*

Proof. We already know that the Weyl group acts transitively so we only need to prove that if $w \in W$ is such that $w(S) = S$, then $w = 1$. We first remark that if $w(S) = S$, then the elements w acting on $\mathfrak{h}$ satisfies $w(S^\vee) = S^\vee$ *i.e.* the element w permutes the dual simple roots. But the fundamental weights forming the dual base to $S^\vee$ we get the equality $w(P) = P$.

On the other hand, for $\varpi \in P$, let $V(\varpi)$ be the simple highest weight module with highest weight w. This module is finite dimensional therefore $w(\varpi)$ is a weight of $V(\varpi)$ and thus $\varpi - w(\varpi)$ is a linear combination of simple roots with non negative integer coefficients. Now we have

$$\sum_{\varpi\in P}(\varpi - w(\varpi)) = \sum_{\varpi\in P}\varpi - \sum_{\varpi\in P}\varpi = 0.$$

This is possible only for $\varpi - w(\varpi) = 0$ for any $\varpi \in P$. Therefore w acts trivially on P and because P is a base $w = 1$. □

14.6 Characters

Definition 14.6.1

(ı) Let $\mathcal{P}$ be the subgroup of $\mathfrak{h}^\vee$ of linear forms with integer values on the base $(H_\alpha)_{\alpha\in S}$. The group $\mathcal{P}$ is a free abelian group generated by the fundamental weights $(\varpi_\alpha)_{\alpha\in S}$. The group $\mathcal{P}$ is called the weight lattice.

(ıı) We denote by $\mathcal{Q}$ the $\mathbb{Z}$-submodule of $\mathfrak{h}^\vee$ generated by the simple roots. The group $\mathcal{Q}$ is called the root lattice. *We have the inclusion $\mathcal{Q} \subset \mathcal{P}$.*

(ııı) We denote by $\mathcal{A}$ the algebra $\mathbb{Z}[\mathcal{P}]$ which is the group algebra of the group $\mathcal{P}$ with coefficients in $\mathbb{Z}$. It has a $\mathbb{Z}$-basis given by $(e(\lambda))_{\lambda\in\mathcal{P}}$ and multiplication defined by

$$e(\lambda)e(\mu) = e(\lambda + \mu).$$

Remark that there is a natural action of the Weyl group W on the algebra $\mathcal{A}$ defined by $w(e(\lambda)) = e(w(\lambda))$.

Definition 14.6.2 *Let V be a finite dimensional representation of $\mathfrak{g}$. We define the character of V to be the element*

$$\mathrm{Ch}(V) = \sum_{\lambda\in\mathcal{P}} \dim V_\lambda e(\lambda)$$

of the algebra $\mathcal{A}$.

Proposition 14.6.3 *($\imath$) Let V be a finite dimensional representation of $\mathfrak{g}$, then the character* $\mathrm{Ch}(V)$ *is invariant under the action of the Weyl group* i.e. $\mathrm{Ch}(V) \in \mathcal{A}^W$.

($\imath\imath$) We have the formulas

$$\mathrm{Ch}(V \oplus V') = \mathrm{Ch}(V) + \mathrm{Ch}(V'),$$
$$\mathrm{Ch}(V \otimes V') = \mathrm{Ch}(V)\mathrm{Ch}(V').$$

($\imath\imath\imath$) Two finite dimensional representations are isomorphic if and only if their character coincide.

Proof. *($\imath$)* This is a consequence of Proposition 14.4.7.

($\imath\imath$) We have the simple formulas $(V \oplus V')_\lambda = V_\lambda \oplus V_\lambda$ and

$$(V \otimes V')_\lambda = \bigoplus_{\mu+\nu=\lambda} V_\mu \otimes V'_\nu.$$

These formulas easily imply the statement.

($\imath\imath\imath$) If V and V' are isomorphic then we have $\mathrm{Ch}(V) = \mathrm{Ch}(V')$. Conversely we proceed by induction on $\dim V$. If $\dim V = 0$, then $\mathrm{Ch}(V') = \mathrm{Ch})V) = 0$ thus $V' = 0$. Otherwise, let λ be a weight in V which is maximal *i.e.* $\lambda + \alpha$ is not a weight of V for any simple root α. Then λ is also a maximal weight for V'. Let E and E' be the submodules in V and V' generated by a primitive elements of weight λ. The modules are simple highest weight modules with the same weight therefore $E \simeq E'$. Furthermore, because V and V' are finite dimensional, we have by Weyl's Theorem the existence of subrepresentations W and W' of V and V' such that $V = E \oplus W$ and $V' = E' \oplus W'$. But we have $\mathrm{Ch}(V) = \mathrm{Ch}(V')$ and $\mathrm{Ch}(E) = \mathrm{Ch}(E')$ therefore $\mathrm{Ch}(W) = \mathrm{Ch}(W')$. By induction we get $W \simeq W'$ and $V \simeq V'$. □

We also have the following result we shall not prove.

Theorem* 14.6.4 *Let T_α be the character of the simple module of higest weight ϖ_α, then the elements $(T_\alpha)_{\alpha\in S}$ are algebraically independent in $\mathcal{A}$ and generate the algebra $\mathcal{A}^W$.*

Corollary* 14.6.5 *The map* Ch *between isomophism classes of representations with sum $\oplus$ and product $\otimes$ to $\mathcal{A}^W$ is an isomorphism.*

Proof. The injectivity comes from the above proposition while the surjectivity comes from the above theorem. □

Example 14.6.6 In the case of $\mathfrak{sl}_{n+1}$, let $(e_i)_{i\in[1,n=1]}$ be a base of $\mathbb{R}^{n+1}$ such that the roots are described by the elements $\alpha_{i,j} = e_i^* - e_j^*$ for $i \neq j$. The fundamental weights are given by

$$\varpi_i = \frac{1}{n+1}(\sum_{k=1}^{i}(n+1-i)e_k^* - \sum_{k=i+1}^{n} ie_k^*).$$

The corresponding representations are the vector spaces $\Lambda^i\mathbb{C}^{n+1}$. The character T_i as a function on the basis vectors is writen as

$$T_i = \sigma_i(e_1^*, \cdots, e_{n+1}^*)$$

where σ_i is the i-th symmetric function and $(e_i^*)_{i\in[1,n+1]}$ is the dual base to $(e_i)_{i\in[1,n+1]}$. The ring $\mathcal{A}^W$ is therefore the ring of symmetric polynomials in $n+1$ variables.

14.7 Weyl's character formula

In this section we state without proof the formula of H. Weyl computing the character of any finite dimensional simple representation.

Definition 14.7.1 *Sign representation. We define the group morphism $\varepsilon : W \to \{\pm 1\}$ by the equality $\varepsilon(w) = \det(w)$ where w is seen as an element in $\mathrm{GL}(\mathfrak{h}^\vee)$.*

Remark 14.7.2 We have $\varepsilon(w) = 1$ if w can be written as a product of an even number of simple reflection and -1 otherwise. If we define the length of w to be $\ell(w) = \min\{n \;/\; w = s_{\alpha_1}\cdots s_{\alpha_n} \text{ with } \alpha_i \in S\}$, then $\varepsilon(w) = (-1)^{\ell(w)}$.

Definition 14.7.3 *Recall the definition of ρ by*

$$\rho = \frac{1}{2}\sum_{\alpha\in R_+}\alpha.$$

Recall also that we have $\langle\alpha^\vee, \rho\rangle = 1$ for $\alpha \in S$. This means $\rho(H_\alpha) = 1$ for $\alpha \in S$ therefore $\rho \in \mathcal{P}$.

(ı) We define the element $D \in \mathbb{Z}[\frac{1}{2}\mathcal{P}]$ by

$$D = \prod_{\alpha\in R_+}(e(\alpha/2) - e(-\alpha/2)).$$

Proposition* 14.7.4 *We have the equality in $\mathbb{Z}[\frac{1}{2}\mathcal{P}]$:*

$$D = \sum_{w\in W}\varepsilon(w)e(w(\rho)).$$

Theorem* 14.7.5 (Weyl's character formula) *Let V be a finite dimensional simple representation of $\mathfrak{g}$ of highest weight λ, then we have the equality*

$$D\cdot \mathrm{Ch}(V) = \sum_{w\in W}\varepsilon(w)e(w(\lambda+\rho)).$$

In other words we have the formula

$$\mathrm{Ch}(V) = \frac{\displaystyle\sum_{w\in W}\varepsilon(w)e(w(\lambda+\rho))}{\displaystyle\sum_{w\in W}\varepsilon(w)e(w(\rho))}.$$

Corollary* 14.7.6 *With the notation as in the above theorem, we have the equality*

$$\dim V = \prod_{\alpha\in R_+}\frac{\langle\alpha^\vee, \lambda+\rho\rangle}{\langle\alpha^\vee, \rho\rangle} = \prod_{\alpha\in R_+}\frac{(\alpha, \lambda+\rho)}{(\alpha, \rho)}$$

Chapter 15

Envelopping algebra II

In this chapter, we state the Poincaré-Birkhoff-Witt Theorem. We do have have time in the lectures to prove this statement. We refer to [3, Chapitre I, paragraphe 2, numero 7] for a proof.

Theorem* 15.0.7 (Poincaré-Birkhoff-Witt) *Let* $\mathfrak{g}$ *be a Lie algebra and let* $(e_i)_{i\in[1,n]}$ *be a base of* $\mathfrak{g}$*. Then the monomials* $e_1^{k_1} \otimes \cdots \otimes e_n^{k_n}$ *form a base of the universal envelopping algebra* $U(\mathfrak{g})$ *of* $\mathfrak{g}$*.*

Corollary* 15.0.8 *Assume that we have a decomposition* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$*, then we have an algebra isomorphism* $U(\mathfrak{g}) \simeq U(\mathfrak{n}) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_-)$*.*

Corollary* 15.0.9 *Let* $\mathfrak{b} \subset \mathfrak{g}$ *be an inclusion of Lie algebras, then* $U(\mathfrak{g})$ *is a free* $U(\mathfrak{b})$*-module.*

Corollary* 15.0.10 *The morphism* $f_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$ *is injective.*

Chapter 16

Groups

In this chapter we give some unproved information on the classification of complex Lie groups corresponding to the semisimple Lie algebras.

Definition 16.0.11 *(ı) A complex Lie group G is a complex variety with a group structure such that the maps $\mu : G \times G \to G$ defined by $\mu(g, g') = gg'$ and $i : G \to G$ defined by $i(g) = g^{-1}$ are holomorphic.*

(ıı) A complex Lie subgroup of a complex Lie group is a subgroup H of G such that the inclusion map $H \to G$ is a morphism.

Example 16.0.12 Consider the groups $G = (\mathbb{C}^*)^2$ and $H = \mathbb{C}^*$. Define a map $\phi : H \to G$ by $\phi(z) = (z, z^\alpha)$ for $\alpha \notin 2\pi\mathbb{Q}$. Then ϕ is a morphism and bijective onto its image. However the image of ϕ is not closed (it is dense in G) therefore ϕ is not homeomorphic to its image.

We denote by $\mathfrak{g}$ the tangent space of G at the unit element $e \in G$. Let us define for $g \in G$ the map $\mathrm{Int}_g : G \to G$ by $\mathrm{Int}_g(g') = gg'g^{-1}$. This map is a group morphism and its differential at e is denoted $\mathrm{Ad}(g) : d_e\mathrm{Int_g} : \mathfrak{g} \to \mathfrak{g}$. It is an automorphism of $\mathfrak{g}$, we therefore have a natural map

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$

The differential at e of this map is the map

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}).$$

Proposition* 16.0.13 *Define the map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by $[x, y] = \mathrm{ad}\,(x)(y)$. This defines a Lie algebra structure on $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is called the Lie algebra of the group G.*

Theorem* 16.0.14 *Let G be a complex Lie group, there is a bijection between the set of complex Lie subgroups $\phi : H \to G$ of G and the set of Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g}$, the map being defined by taking for $\mathfrak{h}$ the image by $d_e\phi$ of the Lie algebra of the subgroup H.*

Remark 16.0.15 Not every Lie subgroup of G is a closed subgroup. Therefore not every subalgebra of $\mathfrak{g}$ is the Lie subalgebra of a closed subgroup. For example, for the group $G = (\mathbb{C}^*)^2$, then the Lie algebra of G is $\mathfrak{g} = \mathbb{C}^2$ and if $\mathbb{C}e_i$ are the Lie subalgebras of the subgroups given by the two factors of the products $G = (\mathbb{C}^*)^2$, then the only one dimensional subalgebras in $\mathfrak{g}$ that come from closed subgroups of G are the subalgebras $ae_1 + be_2$ with $(a, b) \in \mathbb{Z}^2$.

Definition 16.0.16 *A complex Lie group is called* semisimple *if its Lie algebra $\mathfrak{g}$ is semisimple.*

Definition 16.0.17 *Let G be a complex semisimple Lie group and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ the Lie algebra of G. Then the complex Lie subgroup H of G corresponding to $\mathfrak{h}$ is called a* Cartan subgroup of G. *The conjuguates of H are called the* Cartan subgroups of G.

Theorem* 16.0.18 *(ı) The group H is a closed subgroup of G.*
(ıı) The group H is isomorphic to a product $(\mathbb{C}^*)^n$.

Corollary* 16.0.19 *The group H is a closed subgroup of the group $\mathrm{GL}_n(\mathbb{C})$ and the Lie algebra $\mathfrak{h}$ a subalgebra of $\mathfrak{gl}_n(\mathbb{C})$.*

Proof. Take the diagonal matrices. □

Definition 16.0.20 *Define the* exponential map $\exp : \mathfrak{h} \to H$ *by taking the restriction of the exponential $\mathfrak{gl}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ on matrices.*

Let R be the root system in $\mathfrak{h}^\vee$ and let $R^\vee$ be the dual root system which lives in $\mathfrak{h}$. Let us denote by $\mathcal{Q}^\vee$ and by $\mathcal{P}^\vee$ the root lattice and weight lattice of the dual root system $R^\vee$.

Theorem* 16.0.21 *The exponential map $\exp : \mathfrak{h} \to H$ is surjective and its kernel $\Gamma(G)$ satisfies the inclusions:*

$$\mathcal{Q}^\vee \subset \Gamma(G) \subset \mathcal{P}^\vee.$$

In particular H is isomorphic to $\mathfrak{h}/\Gamma(G)$ and therefore $\pi_1(H) = \Gamma(G)$.

Theorem* 16.0.22
(ı) exp *defines an isomorphism from $\mathcal{P}^\vee/\Gamma(G)$ onto $Z(G)$ the center of G.*
(ıı) The map $\pi_1(H) \to \pi_1(G)$ is sujective and induces an isomorphism from $\Gamma(G)/\mathcal{Q}^\vee$ onto $\pi_1(G)$.
(ııı) Let $\mathfrak{g}$ be a semisimple Lie algebra and Γ be a subgroup of $\mathfrak{h}$ such that $\mathcal{Q}^\vee \subset \Gamma \subset \mathcal{P}^\vee$. Then there is a unique, up to unique isomorphism complex Lie group G with Lie algebra $\mathfrak{g}$ and with $\Gamma(G) = \Gamma$.

Definition 16.0.23 *Let $\mathfrak{g}$ be a semisimpe Lie algebra.*
(ı) There is a unique simple complex Lie group with Lie algebra $\mathfrak{g}$. This group is called the adjoint group *associated to $\mathfrak{g}$. We have $\pi_1(G) = \mathcal{P}^\vee/\mathcal{Q}^\vee$.*
(ıı) There is a unique simply connected complex Lie group with Lie algebra $\mathfrak{g}$. This group is called the semisimple group *associated to $\mathfrak{g}$. We have $Z(G) = \mathcal{P}^\vee/\mathcal{Q}^\vee$.*

Remark 16.0.24 (ı) The map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a representation of any complex Lie group in its Lie algebra called the adjoint representation. . If G is a complex semisimple Lie group associated to a semisimple Lie algebra $\mathfrak{g}$, then the adjoint group associated to $\mathfrak{g}$ is $\mathrm{Ad}(G)$ the image of G in the adjoint representation.
(ıı) If G is a complex semisimple Lie group associated to a semisimple Lie algebra $\mathfrak{g}$, then the simply connected group associated to $\mathfrak{g}$ is the universal covering $\widetilde{G}$ of G.

Remark 16.0.25 This proves that the fundamental group of G is always abelian.

Example 16.0.26 If $\mathfrak{g}$ is the Lie algebra $\mathfrak{sl}_2$, then we easily compute the equality

$$\mathcal{P}^\vee/\mathcal{Q}^\vee = \mathbb{Z}/2\mathbb{Z}.$$

There are therefore exacly two semisimple Lie groups whose Lie algebra is $\mathfrak{sl}_2$, the adjoint group and the simply connected group. The adjoint group is $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{PSL}_2(\mathbb{C})$ *i.e.* the quotient of $\mathrm{GL}_2(\mathbb{C})$ by homotheties. The simply connected group is $\mathrm{SL}_2(\mathbb{C})$. There is a natural 2 to 1 Galois cover

$$\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$$

given by dividing by the center of $\mathrm{SL}_2(\mathbb{C})$.

Proposition* 16.0.27 *The group $\mathcal{P}^\vee/\mathcal{Q}^\vee$ is described in the following array for the simple Lie algebras:*

Type	$\mathcal{P}^\vee/\mathcal{Q}^\vee$
A_n	$\mathbb{Z}/(n+1)\mathbb{Z}$
B_n	$\mathbb{Z}/2\mathbb{Z}$
C_n	$\mathbb{Z}/2\mathbb{Z}$
$D_n,\ n$ *odd*	$\mathbb{Z}/4\mathbb{Z}$
$D_n,\ n$ *even*	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
E_6	$\mathbb{Z}/2\mathbb{Z}$
E_7	$\mathbb{Z}/2\mathbb{Z}$
E_8	0
F_4	0
G_2	0

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