# Introduction to Kac-Moody groups and Lie algebras

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# Introduction

This text consists of lecture notes for lectures given in Bonn University.

The main inspiration for these notes was the book of V. Kac [Ka90]. I stayed very close from this book at least for the begining of the lecture: definition and first properties of Kac-Moody Lie algebras. In particular I have choosen to keep the definition of $\mathfrak{g}(A)$ the Kac-Moody Lie algebra associated to a generalised Cartan matrix $A$ as the quotient of the Lie algebra $\widetilde{\mathfrak{g}}(A)$ with the obvious commuting relations but the quotient of the maximal ideal with trivial intersection with the Cartan subalgebra instead taking the definition using Serre relations. This has many advantages when one wants to identity a Lie algebra as a special Kac-Moody Lie algebra in particular in the explicit constructions of finite dimensional simple Lie algebras (Chapter 13) or of twisted and untwisted affine Lie algebras (Chapter 12 and 14).

The second source of inspiration was the book of S. Kumar [Ku02]. In particular we took his point of view to define the Weyl group of a Kac-Moody Lie algebra and also to some extend in the presentation of the invariant bilinear form on a Kac-Moody Lie algebra as well as in the presentation of the representation theory of Kac-Moody Lie algebras and the celebrated character formula.

Finally the books [Hu90] and [Bo54] where my main sources for the treatment of Coxeter groups in general even if the link with Weyl group of Kac-Moody Lie algebras is inspired from [Ku02].

As a conclusion, I would like to thank the HCM and the University of Bonn for giving me the opportunity to give these lectures. I also thank the students for their patience when I tried to explain the flavour of some tedious computations that may appear in proofs in the subject. I would finally like to thank Manfred Lehn whose lectures on affine Lie algebras in WS2001-2002 in Cologne was my first and inspirating introduction to the subject.

Let me briefly review the content of these lecture notes: in the first part I give a quick introduction to semisimple Lie algebras and semisimple groups. This part should serve as a guide to what we want to obtain for Kac-Moody Lie algebras (and Kac-Moody groups that should be treated in a forthcoming third part). The main references are J.E. Humpreys book [Hu72] for Chapter 1 and T.A. Springer's book [Sp98] for Chapter 2.

In a third chapter, we recall some basic facts and definitions on algebras and Lie algebras like free Lie algebras, enveloping algebras an the Poincaré-Birkhoff-Witt Theorem.

We then enter the subject with the defintion and first properties of Kac-Moody Lie algebras in Chapter 4. In Chapter 5 we define the Weyl group of a Kac-Moody Lie algebra and associated geometric representation. In the next Chapter, we review the theory of Coxeter group and end with a proof of the fact that the Weyl group of a Kac-Moody Lie algebra is a cristallographic Coxeter group.

At this point we start to study some particular generalised Cartan matrices especially, in Chapter 7, the so called symmetrisable Cartan matrices $A$ for which the Kac-Moody Lie algebra $\mathfrak{g}(A)$ has an invariant bilinear form. These are the Kac-Moody Lie algebra for which we prove the character formula. In Chapter 8, we give the classification of generalised Cartan matrices in finite, affine and general type. We also give the possible connected Dynkin diagrams in finite and affine case. In Chapter 9, we give a first description of the root system of Kac-Moody Lie algebras. In particular a new phenomenon occurs here: not all roots are in the orbit of simple roots, these are the imaginary roots.

We then start with the representation theory of Kac-Moody Lie algebras in Chapter 10. We define Verma modules and intergrable highest weight. In Chapter 11, we define the Casimir operator which exists only if the algebra is symmetrisable. We use this operator to prove the character formula for symmetrisable Kac-Moody Lie algebras.

We then start to realise explicitly the Kac-Moody Lie algebras of finte and affine type. We start in Chapter 12 with untwisted affine Lie algebras. Then in Chapter 13 we come back to the finite case and explicitly realise simply laced and non simply laced simple Lie algebras. We use this construction in Chapter 14 to construct twisted affine Lie algebras. The explicit construction in particular induce description of the root system and Weyl groups of the affine Lie algebras.

# Contents

# Part I

# Quick review of semisimple Lie algebras and semisimple groups

# Chapter 1

# Semisimple Lie algebras

## 1.1 Semisimple Lie algebras and Killing form

**Definition 1.1.1** A bf Lie algebra $\mathfrak{g}$ is a vector space over a field $k$ together with an alternate bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity $[x,[y,z]] + [[x,z],y] + [z,[x,y]] = 0$.

**Example 1.1.2** The basic examples of Lie algebras are $\mathfrak{gl}(V) = \mathrm{End}(V)$ the set of endomorphisms of a fixed vector space $V$ with Lie bracket $[f,g] = f \circ g - g \circ f$. The Lie algebra $\mathfrak{sl}(V)$ is defined for $V$ of finite dimension by $\mathfrak{sl}(V) = \{f \in \mathfrak{gl}(V) \ / \ \mathrm{Tr}(f) = 0\}$.

To a Lie algebra one defines its **adjoint representation** $\mathrm{ad}\ : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) = \mathrm{End}(\mathfrak{g})$ defined by $\mathrm{ad}\,(x)(y) = [x,y]$. This is a Lie algebra morphism thanks to Jacobi identity.

**Definition 1.1.3** More generally, a **Lie algebra representation** of $\mathfrak{g}$ is a vector space $V$ together with a Lie algebra map $\mathfrak{g} \to \mathfrak{gl}(V)$.

When $V$ is a finite dimensional representation of $\mathfrak{g}$, one define an invariant quadratic form on $\mathfrak{g}$ by $(x,y) = \mathrm{Tr}_V(\varphi(x)\varphi(y))$ where $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ defines the representation. Here invariant means that we have $(x,[y,z]) = ([x,y],z)$. In particular, if $\mathfrak{g}$ is finite dimensional, the adjoint representation leads to the **Killing form** $\kappa(x,y)$ on $\mathfrak{g}$.

**Definition 1.1.4** (ı) An **ideal** $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is a subvectorspace such that $[\mathfrak{g},\mathfrak{a}] \subset \mathfrak{a}$.
Remark that if $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then the quotient $\mathfrak{g}/\mathfrak{a}$ is again a Lie algebra.
(ıı) A Lie algebra $\mathfrak{g}$ is **abelian** if $[\mathfrak{g},\mathfrak{g}] = 0$.
(ııı) A Lie algebra $\mathfrak{g}$ is **simple** if $2 \leq \dim \mathfrak{g} < +\infty$ and $\mathfrak{g}$ as no non trivial ideal.
(ıv) A Lie algebra $\mathfrak{g}$ is **semisimple** if $\mathfrak{g}$ is a direct sum of finitely many simple Lie algebras.

**Example 1.1.5** The Lie algebra $\mathfrak{sl}(V)$ is simple for any finite dimensional vector space $V$.

**Proposition 1.1.6** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The following are equivalent:*
*(ı) The Lie algebra $\mathfrak{g}$ is semisimple.*
*(ıı) The Killing form is non degenerate.*
*(ııı) There is no non trivial abelian ideal.*

For Kac-Moody Lie algebras one can not define the Killing form because the Lie algebra will be of infinite dimension so that the trace is not defined. However, we will look for equivariant non degenerate quadratic forms. This will not always exist but in a large class of Kac-Moody Lie algebras called **symmetrisable** it will be the case.

## 1.2 Cartan subalgebras, roots and Weyl group

I will now give a brief account of the classification theory of semisimple Lie algebras. A very important tool in this classification is the existence of Cartan subalgebras.

**Definition 1.2.1** A Lie subalgebra $\mathfrak{h}$ of a semisimple Lie algebra$\mathfrak{g}$ is said to be Cartan if $\mathfrak{h}$ is abelian and $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$,where $N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \ / \ \forall h \in \mathfrak{h}, \ [x,h] \in \mathfrak{h}\}$.

**Proposition 1.2.2** *There always exists a Cartan subalgebra and all Cartan subalgebras are conjugated by $Int(\mathfrak{g})$.*

**Example 1.2.3** A Cartan Lie subalgebra of $\mathfrak{sl}(V)$ is given by the choice of a basis of $V$ by taking $\mathfrak{h}$ to be all endomorphisms that are diagonalised in that basis.

The important point on Cartan Lie algebras is the following result.

**Proposition 1.2.4** *If $\mathfrak{g}$ is semisimple then for all $h \in \mathfrak{h}$ the endomorphism $\operatorname{ad} h : \mathfrak{g} \to \mathfrak{g}$ is semisimple.*

In particular there is a simultaneous decomposition of the vector space $\mathfrak{g}$ into eigenspaces with respect to $\mathfrak{h}$. Set $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \ / \ \forall h \in \mathfrak{h}, \ [h,x] = \alpha(h)x\}$, in particular, we have $\mathfrak{g}_0 = \mathfrak{h}$ and the decomposition:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

**Definition 1.2.5** Define the set $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \ / \ \dim \mathfrak{g}_\alpha \neq 0\}$. An element of this set will be called a root of the Lie algebra $\mathfrak{g}$. The integer $\dim \mathfrak{h} = l$ is called the rank of $\mathfrak{g}$.

**Example 1.2.6** In the case of $\mathfrak{sl}(V)$, assume that a Cartan algebra is fixed (i.e. a basis of $V$ is fixed). We may identify $\mathfrak{sl}(V)$ with the $n \times n$ matrices of vanishing trace 9where $n = \dim(V)$. Consider the linear formes $\epsilon_i$ for $i \in [1,n]$ on $\mathfrak{h}$ for given by $\text{Diag}(a_k)_{k\in[1,n]} \mapsto a_i$. Then we have

$$\Delta = \{\epsilon_i - \epsilon_j, \ \text{for } i \neq j\}.$$

For $\alpha = \epsilon_i - \epsilon_j$, the vector space $\mathfrak{g}_\alpha$ is one dimensional generated by the matrix $E_{i,j} = (\delta_{(k,l);(i,j)})_{k,l\in[1,n]}$ where $\delta$ is the Kronecker symbol. You can easily see the decomposition of $\mathfrak{g}$ on the matrices.

**Proposition 1.2.7** *The Killing form $\kappa$ is non degenerate on $\mathfrak{h}$ and we have for all $\alpha \in \Phi$, the equality* $\dim \mathfrak{g}_\alpha = 1$.

Because the Killing form $\kappa$ is non degenerate on $\mathfrak{h}$ it defines an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$ and in particular the Killing from induces a non degenerate bilinear form on $\mathfrak{h}^*$. Denote by $( \ , \ )$ this form. Then for any $\alpha \in \Delta$, we have $(\alpha,\alpha) \neq 0$ and we may define the reflection $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$s_\alpha(u) = u - 2\frac{(u,\alpha)}{(\alpha,\alpha)}\alpha.$$

These reflections preserve the bilinear form $( \ , \ )$.

**Definition 1.2.8** The Weyl group of $\mathfrak{g}$ is the subgroup of $SO(\mathfrak{h}^*, ( \ , \ ))$ generated by these reflections. Remark that because any two Cartan algebras are conjugate, this does only depend on $\mathfrak{g}$.

**Proposition 1.2.9** *The Weyl group is finite.*

The set of roots $\Delta$ satisfy the following properties:

**Theorem 1.2.10** *(ı) The set $\Delta$ spans $\mathfrak{h}^*$ and $0 \notin \Delta$.*

*(ıı) If $\sigma \in \Delta$, then $-\alpha \in \Delta$ but no other scalar multiple of $\alpha$ is a root.*

*(ııı) The set of roots $\Delta$ is stable under $W$.*

*(ıv) For $u \in \mathfrak{h}^*$, define $\langle\langle \alpha, u \rangle\rangle = 2\frac{(u,\alpha)}{(\alpha,\alpha)}$, then for all $(\alpha, \beta) \in \Delta^2$ we have $\langle\langle \alpha, \beta \rangle\rangle \in \mathbb{Z}$.*

**Definition 1.2.11** A subset $\Delta$ in an Euclidian vector space satisfying the conclusion of the previous theorem is called a root system. The associated reflection group $W$ is called the Weyl group of the root system.

**Example 1.2.12** In the case $\mathfrak{sl}(V)$ with $V$ of dimension $n$, the Weyl group will be $\mathfrak{S}_n$ acting on the $\epsilon_i$ by permutation.

The Cartan Lie algebras will be the starting point of the consruction of Kac-Moody Lie algebras so they will be given for free. The decomposition will follow and the definition of roots will be similar. The Weyl group will exist but an important difference is that it will not be finite any more. It will however still be a Coxeter group, a natural generalisation of the Weyl groups of semisimple Lie algebras.

## 1.3 Cartan matrices and Dynkin diagrams

### 1.3.1 Simple roots

**Definition 1.3.1** A subset $\Pi$ of $\Delta$ is called a base if it is a basis of $\mathfrak{h}^*$ and if each root $\beta$ can be written as $\beta = \sum k_\alpha \alpha$ with integral coefficients $k_\alpha$ all nonnegative or all non positive. The roots in $\Pi$ are called simple.

It is not completely obvious that such a base exists. Let $H_\alpha$ the hyperplane orthogonal to $\alpha$ (with respect to $( \; , \; )$). An element $\gamma$ in $\mathfrak{h}^* \setminus \cup_\alpha H_\alpha$ is called regular. The connected components of the former set are called Weyl chambers. Let $\Delta_+(\gamma) = \{\alpha \in \Delta \; / \; (\alpha, \gamma) > 0\}$. An element $\alpha \in \Delta_+(\gamma)$ is called indecomposable if it is not the sum of two elements in $\Delta_+(\gamma)$.

**Theorem 1.3.2** *The set $\Pi(\gamma)$ of indecomposable roots in $\Delta(\gamma)$ is a base of $\Delta$.*

**Theorem 1.3.3** *Let $\Pi$ be a basis of $\Delta$.*

*(ı) If $\gamma$ is regular, then there exist $w \in W$ such that $(w(\gamma), \alpha) > 0$ for all $\alpha \in \Pi$ (in other words $W$ acts transitively on the Weyl Chambers). In this case $\Pi = \Pi(w(\gamma))$*

*(ıı) If $\Pi'$ is another basis, then there exists $w \in W$ such that $w(\Pi') = w$ (so $W$ acts transitively on the bases).*

*(ııı) If $\alpha \in \Delta$, then there exists $w \in W$ such that $w(\alpha) \in \Pi$.*

*(ıv) The group $W$ is generated by the $s_\alpha$ with $\alpha \in \Pi$.*

*(v) If $w(\Pi) = \Pi$ and $w \in W$, then $w = 1$ (so $W$ acts simply transitively on the basis and the chambers).*

### 1.3.2 Cartan matrices

**Definition 1.3.4** Let $\Pi$ be a basis of the root system. Fix an ordering on $\Pi$, then the Cartan matrix is the matrix $A = (\langle\langle \alpha_i, \alpha_j \rangle\rangle)_{i,j \in [1,n]}$.

**Remark 1.3.5** Let $\alpha \in \Delta$, and define the coroots $\alpha^\vee \in \mathfrak{h}$ by $\langle \alpha^\vee, \beta \rangle = \langle\langle \alpha, \beta \rangle\rangle$ for all $\beta \in \Delta$ (this is well defined because these conditions when $\beta \in \Pi$ imply the others). With these notation we have $A = (\langle \alpha_i^\vee, \alpha_j \rangle)_{i,j \in [1,n]}$.

**Proposition 1.3.6** *Modulo conjugation by a permutation matrix, the Cartan matrix only depend on* $\mathfrak{g}$. *Furthermore, the Cartan matrix determines* $\Delta$ *up to isomorphism.*

**Theorem 1.3.7** *The Cartan matrix* $A = (a_{i,j})_{i,j \in [1,n]}$ *satisfies the following properties*
*(ı)* $a_{i,i} = 2$.
*(ıı)* $a_{i,j}$ *for* $i \neq j$ *are non positive integers.*
*(ııı)* $a_{i,j} = 0 \ \Leftrightarrow \ a_{j,i} = 0$.
*(ıv)* $\det(A) > 0$.
*Furthermore, any matrix with these properties is a Cartan matrix for a semisimple Lie algebra* $\mathfrak{g}$.

**Example 1.3.8** Let us give the Cartan matrix of type $A_n$:

$$A_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

**Proposition 1.3.9** *The coroots define a root system in* $\mathfrak{h}$ *called the* **dual root system**. *The associated Cartan matrix is the transpose of the Cartan matrix.*

**Example 1.3.10** For more Cartan matrices, see Proposition 8.2.7

### 1.3.3 Serre's presentation

Let $\Pi$ be a root base and $\Pi^\vee$ the corresponding set of coroots (it is a base for the dual root system $\Delta^\vee$ formed by the coroots). Fix generators $e_\alpha$ (resp. $f_\alpha$) of $\mathfrak{g}_\alpha$ (resp. of $\mathfrak{g}_{-\alpha}$) for all $\alpha \in \Pi$ such that $[e_\alpha, f_\alpha] = \alpha^\vee$.

**Theorem 1.3.11 (Serre)** *The Lie algebra* $\mathfrak{g}$ *is the quotient of the free Lie algebra*[1], *generated by the* $e_\alpha$, $f_\alpha$ *and* $\alpha^\vee$ *for* $\alpha \in \Pi$, *by the ideal generated by the following relations:*
*(ı)* $[\alpha^\vee, \beta^\vee] = 0$ *for all* $\alpha$, $\beta$ *in* $\Pi$.
*(ıı)* $[e_\alpha, f_\beta] = \delta_{\alpha,\beta}\alpha^\vee$ *for all* $\alpha$, $\beta$ *in* $\Pi$.
*(ııı)* $[\beta^\vee, e_\alpha] = \langle \alpha^\vee, \beta \rangle e_\alpha$ *and* $[\beta^\vee, f_\alpha] = \langle \alpha^\vee, \beta \rangle f_\alpha$ *for all* $\alpha$, $\beta$ *in* $\Pi$.
*(ıv)* $(\mathrm{ad}\, e_\alpha)^{1-\langle \alpha^\vee, \beta \rangle}(e_\beta) = 0$ *for all* $\alpha \neq \beta$ *in* $\Pi$.
*(v)* $(\mathrm{ad}\, f_\alpha)^{1-\langle \alpha^\vee, \beta \rangle}(f_\beta) = 0$ *for all* $\alpha \neq \beta$ *in* $\Pi$.

**Example 1.3.12** For the only rank one Cartan matrix we get a Lie algebra of dimension 3 with 3 generators $e$, $f$ and $h$ such that $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. This is $\mathfrak{sl}_2$.

[1] see chapter 3 for more on free Lie algebras.

### 1.3.4 Dynkin diagrams

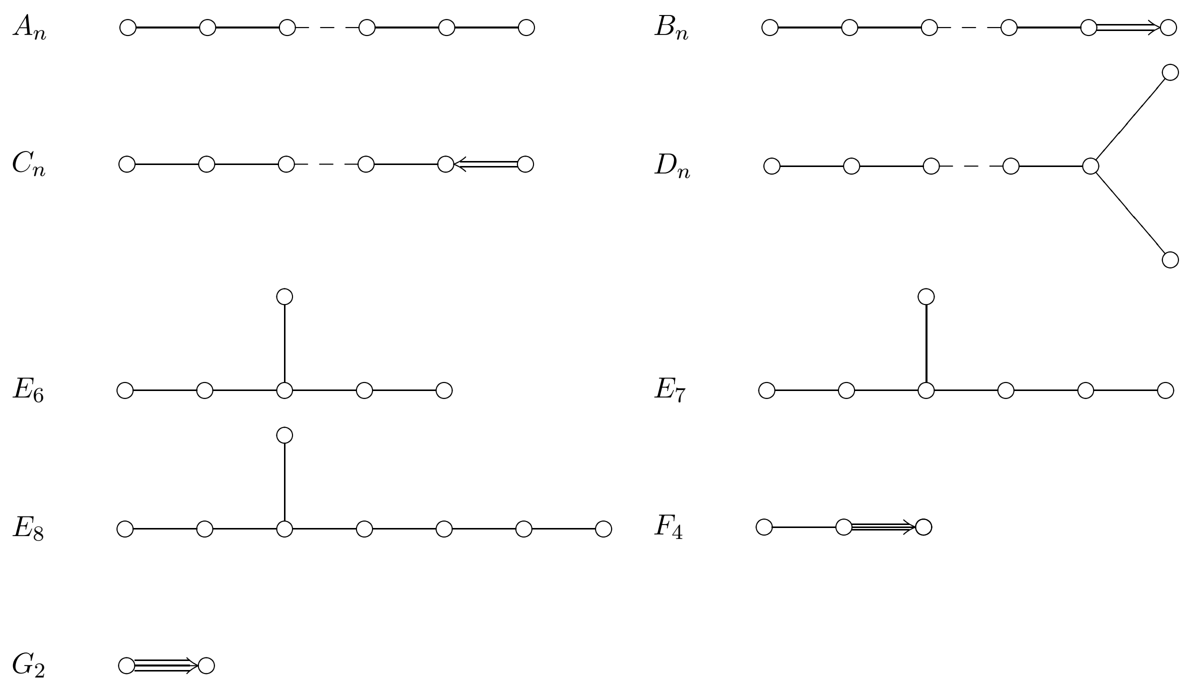
Let $\Pi$ be a basis of $\Delta$.

**Proposition 1.3.13** *For $\alpha$ and $\beta$ two distinct positive roots (i.e. linear combination of elements in $\Pi$ with non negative coefficients) then $\langle\alpha,\beta\rangle\langle\langle\beta,\alpha\rangle\rangle = \langle\langle\alpha,\beta\rangle\rangle\langle\langle\beta,\alpha\rangle\rangle = 0$, 1, 2 or 3.*

**Definition 1.3.14** The Dynkin diagram is the graph having $l = |\Pi| = \dim\mathfrak{h}$ vertices indexed by $\Pi$ and the vertex $\alpha$ is jointed to the vertex $\beta$ by $\langle\alpha,\beta\rangle\langle\langle\beta,\alpha\rangle\rangle$ edges with an arrow going from the vertex indexed by the longest root to the vertex indexed by the shortest one (with respect to ( , )) and no arrow if the two roots have the same length.

**Proposition 1.3.15** *The connected components of the Dynkin diagram are in one to one correspondence with the semisimple factors of $\mathfrak{g}$.*

**Theorem 1.3.16** *The connected Dynkin diagrams are the following:*




*For each of them, there exist a corresponding indecomposable semisimple Lie algebra.*

Cartan matrices (in fact generalised Cartan matrices where the condition on the determinant is removed) will really be the starting point of the construction of Kac-Moody Lie algebras so that Cartan matrices and simple root will also show up in Kac-Moody theory. For the Serre equations, this could be a way to define Kac-Moody Lie algebras (see for example the book of Kumar [Ku02] where this point of view is taken). At least we will see that the Serre relations are satisfied by Kac-Moody Lie algebras and that for symmetrisable ones, they are defined by these equations like in the semisimple theory.

## 1.4 Representation theory of semisimple Lie algebras

We now want to describe the representations of semisimple Lie algebras. In fact we will focus on finite dimensional representation. For Kac-Moody Lie algebras, this will be different because one of the first representations we want to study is the adjoint representation. So if the Lie algebra is infinite dimensional, we will need to admit more representations that only finite dimensional ones. This will lead to some technicalities on finite or infinite sums...

The starting point is the following result:

**Theorem 1.4.1 (Weyl)** *Let $\mathfrak{g}$ be a semisimple Lie algebra, then any finite dimensional representation is a direct sum of irreducible representations.*

In particular we only need to describe the irreducible representations of $\mathfrak{g}$. Let $V$ be a finite dimensional representation of $\mathfrak{g}$.

### 1.4.1 Weights of a representation

Because any element $h \in \mathfrak{h}$ is semisimple on the adjoint representation, this implies that is has to be semisimple on any representation. In particular, we get a decomposition of $V$ with respect to the action of $\mathfrak{h}$ as follows:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

where $V_\lambda = \{v \in V \ / \ h(v) = \langle \lambda, h \rangle v, \ \forall h \in \mathfrak{h}\}$.

**Definition 1.4.2** An element $\lambda \in \mathfrak{h}^*$ such that $V_\lambda \neq 0$ is a weight of $V$. The set $P(V)$ is the set of weights of $V$. The multiplicity $\text{mult}\lambda$ of $\lambda \in P(V)$ is the dimension of $V_\lambda$.

**Example 1.4.3** (ı) For the adjoint representation, the weights are the roots plus the zero weight. All the roots have (as weights) multiplicity one and the zero weight has dimension $\text{rk}(\mathfrak{g})$ the rank of $\mathfrak{g}$ i.e. the dimension of the Cartan subalgebra.

(ıı) For $\mathfrak{sl}(V)$, then $V$ can be seen as a representation by the map $\text{Id} : \mathfrak{sl}(V) \to \mathfrak{sl}(V)$. This is the standard representation. The weights of this representation are given by $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$. In the basis given by $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = \epsilon_2 - \epsilon_3$ the weights are of the form $\frac{1}{3}(2\alpha_1 + \alpha_2)$, $\frac{1}{3}(\alpha_2 - \alpha_1)$ and $\frac{1}{3}(-\alpha_1 - 2\alpha_2)$

If you look at the representation $\Lambda^2 V$, then it is isomorphic to $V^*$ and its weights are the opposite of the weights of $V$. It is easy to compute these weights because they are the sum of two different weights of $V$. Remark that if we try to compute the weights of $\mathfrak{sl}_3 = V \otimes V^*/\mathbb{C}$ by this way we get back the roots.

### 1.4.2 Weight lattice

**Definition 1.4.4** Consider the root system $\Delta$ in $\mathfrak{h}$ and define **the root lattice** to be the lattice $Q$ generated by the roots. The **weight lattice** is the dual lattice $P$ for the form $\langle\langle \ , \ \rangle\rangle$ that is

$$P = \{\lambda \in \mathfrak{h}^* \ / \ \langle\langle \lambda, \alpha \rangle\rangle \in \mathbb{Z} \text{ for all } \alpha \in Q\}.$$

Remark that the root lattice is always contained in the weight lattice.

**Example 1.4.5** For $\mathfrak{sl}_3$ it is not hard to compute that the weight lattice if generated by the weights $\epsilon_1$ and $-\epsilon_3$ that is to say by the weights of $V$ and $\Lambda^2 V$. In particular the weight lattice is stricly bigger that the root lattice. On can check that $P/Q = \mathbb{Z}/3\mathbb{Z}$.

**Proposition 1.4.6** *The quotient $P/Q$ is always finite and we have $|P/Q| = \det(A)$. It is called* **the fundamental group** *of the situation.*

**Proposition 1.4.7** *(ı) Let $V$ be a finite dimensional representation of $\mathfrak{g}$, then the set $P(V)$ is contained in the weight lattice.*

*(ıı) Let $w \in W$, we have an isomorphism $V_{w(\lambda)} \simeq V_\lambda$. In particular the set $P(V)$ is $W$-invariant and $\lambda$ and $w(\lambda)$ have the same multiplicity.*

### 1.4.3 Dominant weights

**Definition 1.4.8** Chose a basis $\Pi$ of $\Delta$ or equivalently a Weyl chamber of equivalently a set of positive roots.

(ı) Then there is an induced partial order on weights of $V$ defined for $\lambda$ and $\mu$ by $\lambda \preccurlyeq \mu$ in $P(V)$ if $\mu$ is obtained from $\lambda$ by adding positive roots.

(ıı) We may also define a **unipotent subalgebra** $\mathfrak{u}$ by

$$\mathfrak{u} = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha.$$

**Proposition 1.4.9** *Let $V$ be a representation of $\mathfrak{g}$, then we have the following property:*

$$\mathfrak{g}_\alpha(V_\lambda) \subset V_{\lambda+\alpha}.$$

**Definition 1.4.10** (ı) Let $V$ be a representation of $\mathfrak{g}$, we say that $V$ has a highest weight $\lambda \in P(V)$ if the unipotent Lie subalbegra $\mathfrak{u}$ acts trivially on $V_\lambda$.

**Proposition 1.4.11** *(ı) Any finite dimensional representation $V$ has a highest weight.*

*(ıı) If $\lambda$ is a highest weight of $V$, then any weight $\mu$ of $V$ satisfies $\mu \preccurlyeq \lambda$.*

*(ııı) If $V$ is irreducible, then the highest weight is unique.*

**Definition 1.4.12** The set $P^+$ of **dominant weights** is defined by (it is the principal Weyl chamber associated to $\Pi$):

$$P_+ = \{\lambda \in P \ / \ \langle\langle \alpha, \lambda \rangle\rangle > 0 \ \forall \alpha \in \Delta_+\}.$$

**Proposition 1.4.13** *Any highest weight is a dominant weight.*

**Theorem 1.4.14** *There is a canonical bijection $\lambda \mapsto V(\lambda)$ between $P_+$ and the set of finite dimensional irreducible representations of $\mathfrak{g}$ modulo isomorphism.*

### 1.4.4 Verma modules

In this section, we give a more explicit description of the bijection in Theorem 1.4.14. Let $\mathfrak{b}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and the $e_\alpha$ for $\alpha \in \Pi$. Let $\lambda \in P_+$ and let $\mathfrak{b}$ act on $\mathbb{C}$ by $h(x) = \langle \lambda, h \rangle x$ and $e_\alpha(x) = 0$ for $x \in \mathbb{C}$ and $\alpha \in \Pi$. Consider the following tensor product:

$$M(\lambda) = \mathfrak{u}(\mathfrak{g}) \otimes_{\mathfrak{u}(\mathfrak{b})} \mathbb{C}.$$

This gives us the representation $M(\lambda)$ of $\mathfrak{g}$. This representation is called the Verma module. We have the following Theorem.

**Theorem 1.4.15** *We have*

*(ı) $\lambda$ is the unique highest weight of $M(\lambda)$.*

*(ıı) There exist a unique maximal subrepresentation $M'(\lambda)$ of $M(\lambda)$.*

*(ııı) The quotient $V(\lambda) = M(\lambda)/M'(\lambda)$ is irreducible of highest weight $\lambda$.*

*(ıv) $V(\lambda) \simeq V(\lambda') \ \Leftrightarrow \ \lambda = \lambda'$.*

*(v) $\dim V(\lambda) < \infty \ \Leftrightarrow \ \lambda \in P_+$.*

*(vı) Any finite dimensional irreducible representation of $\mathfrak{g}$ is isomorphic to some $V(\lambda)$ for $\lambda \in P_+$.*

### 1.4.5 Character formula

The purpose of this formula is to describe the multiplicity $m_\lambda(\mu) = \dim(V(\lambda)_\mu)$ of any weight $\mu$ of $V(\lambda)$.

**Definition 1.4.16** (ı) Let $f : P \to \mathbb{Z}$ be a function. We define the support of $f$ denoted by $\mathrm{Supp}(f)$ to be the set $\{\lambda \in P \ / \ f(\lambda) \neq 0\}$.

(ıı) Define the following subset $\mathcal{H}$ of the set $\mathcal{F}(P, \mathbb{Z})$ from $P$ to $\mathbb{Z}$:

$$\mathcal{H} = \{f \in \mathcal{F}(P, \mathbb{Z}) \ / \ \exists\ S \subset P \text{ a finite subset, such that } \forall\ \lambda \in \mathrm{Supp}(f),\ \lambda \preccurlyeq s \text{ for some } s \in S\}$$

We define a ring structure on $\mathcal{H}$ by $(f+g)(\lambda) = f(\lambda) + g(\lambda)$ and

$$(fg)(\lambda) = \sum_{\mu \in P} f(\mu) g(\lambda - \mu).$$

This is well defined because of our hypothesis on $\mathcal{H}$. For any $\lambda \in P$, we define a function $e^\lambda$ by $e^\lambda(\mu) = \delta_{\lambda,\mu}$.

**Proposition 1.4.17** *We have*

*(ı)* $e^\lambda \in \mathcal{H}$.

*(ıı)* $e^\lambda e^\mu = e^{\lambda+\mu}$.

*(ııı)* $e^0$ *is a 1 for* $\mathcal{H}$.

**Definition 1.4.18** Let $V$ be an $\mathfrak{h}$-diagonalisable $\mathfrak{g}$-representation. Then we define $\mathrm{Ch}_V(\lambda) = \dim V_\lambda$. It is called the character of $V$.

**Proposition 1.4.19** *Let* $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$ *(the Weyl vector), then*

$$\mathrm{Ch}_{M(\lambda)} = \frac{e^{\lambda+\rho}}{\prod_{\alpha \in \Delta_+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})}.$$

**Theorem 1.4.20 (Weyl character formula)** *The character of the irreducible representation* $V(\lambda)$ *is given by:*

$$\mathrm{Ch}_{V(\lambda)} \left( \sum_{w \in W} \mathrm{sgn}(w) e^{w(\rho)} \right) = \sum_{w \in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)}.$$

# Chapter 2

# Semisimple groups

## 2.1 Algebraic groups

### 2.1.1 First properties

Here when I use the word variety, you can think either to an algebraic variety or to a complex variety.

**Definition 2.1.1** An algebraic group $G$ is a variety and a group such that the multiplication map $\mu : G \times G \to G$ and the inverse map $G \to G$ sending an element to its inverse are morphisms.

From now on we will assume that our group is generically reduced. In particular it has a smooth point and by action of the group it is smooth. In particular if an algebraic group is connected then it is irreducible. There is a natural notion of tangent space $T_x X$ at a point $x$ of an algebraic variety $X$.

**Definition 2.1.2** We define the Lie algebra of $G$ to be $\mathfrak{g} = T_e G$.

We define the adjoint action as follows. Consider for $g \in G$ the automorphism $G \to G$ defined by $g' \mapsto g g' g^{-1}$. Then we may differentiate this map to get $\mathrm{Ad}\,(g) : \mathfrak{g} \to \mathfrak{g}$. The map

$$\mathrm{Ad}\, : G \to \mathrm{Aut}(\mathfrak{g})$$

is called the adjoint action of $G$ on $\mathfrak{g}$. We may differentiate the map Ad to get a map $\mathrm{ad}\, : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.

**Proposition 2.1.3** *The Lie algebra $\mathfrak{g}$ of $G$ is endowed with a structure of Lie algebra defined by $[x, y] = \mathrm{ad}\,(x)(y)$.*

In characteristic 0, there are very strong links between the group and its Lie algebra. It is also true in positive characteristic but some caution has to be taken. For example a non abelian group can have an abelian Lie algebra.

**Theorem 2.1.4** *The Kernel of the adjoint action* Ad *is the center $Z(G)$ of $G$. The center of* ad *is the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$.*

We see that the adjoint action encaptures all the group except his center. We will see that this has a very nice presentation is the semisimple case. We can go back from the Lie algebra to the group. Indeed, we have the

**Proposition 2.1.5** *There is a unique map* $\exp : \mathfrak{g} \to G$ *taking* $0$ *to* $e$ *and whose differential* $\mathfrak{g} = T_0\mathfrak{g} \to T_e G = \mathfrak{g}$ *is the identity. This map is called the exponential map.*

### 2.1.2 Semisimple groups

**Definition 2.1.6** A algebraic group is semisimple if it is connected and has no non trivial closed connected commutative normal subgroup.

**Theorem 2.1.7** *Assume that we are in characteristic zero. Let $G$ be a connected algebraic group. The correspondence $H \to \mathfrak{h}$ is bijective and preserves the inclusion between the set of closed connected subgroups $H$ of $G$ and the set of their Lie algebras regarded as subalgebras of $\mathfrak{g}$. In this correspondence, normal subgroups are sent to ideals of $\mathfrak{g}$.*

This leads to the equivalent definition of semisimple groups

**Theorem 2.1.8** *In char. 0, a connected algebraic group $G$ is semisimple if and only if its Lie algebra $\mathfrak{g}$ is semisimple.*

So that we will be able to use the classification of semisimple Lie algebras for the study of semisimple Lie groups.

**Theorem 2.1.9** *Let $G$ be a semisimple group, then $\mathrm{Ad}\,(G) = (\mathrm{Aut}\mathfrak{g})^\circ$ the connected component of identity of $\mathrm{Aut}\mathfrak{g}$.*

**Theorem 2.1.10** *Let $G$ be a semisimple group with Lie algebra $\mathfrak{g}$, then*
*(ı) there is a unique simple group $G^{\mathrm{ad}}$ with the same Lie algebra. This group is the adjoint group $\mathrm{Ad}\,(G)$.*
*(ıı) There is a unique simply connected semisimple Lie group $\widetilde{G}$ with the same Lie algebra and $Z(\widetilde{G}) = \widetilde{G}/\mathrm{Ad}\,(G)$ is finite.*
*(ııı) The group $G$ is lies between $Ad(G)$ and $\widetilde{G}$. More precisely $G$ is a quotient of $\widetilde{G}$ whose kernel is a subgroup of $Z(\widetilde{G})$.*

We will describe more precisely the center of $G$ in the following.

## 2.2 Some subgroups of $G$

### 2.2.1 Maximal torus and root systems

**Definition 2.2.1** An algebraic group $T$ is a **Torus** if it is isomorphic to the group of diagonal matrices in $GL_n$ for some $n$.

**Definition 2.2.2** On a field $k$, a character of an algebraic group $G$ is a group morphism $G \to k^\times$. The set of all characters is a group denoted $X(G)$.

**Proposition 2.2.3** *Let $T$ be a torus, then the group $X(T)$ is a free abelian group of rank $\dim T$. More over all elements of a torus are semisimple.*

**Definition 2.2.4** Let $G$ be an algebraic group and $T$ a torus in $G$. Then $T$ acts by the adjoint representation on $\mathfrak{g}$ and this action induces a decomposition

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in X(T)} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \ / \ Ad(t)(x) = \alpha(t)x \text{ for all } t \in T\}$. The set $\Delta(G,T)$ of characters such that $\mathfrak{g}_\alpha$ is non zero is called the set of roots of $G$ with respect to $T$.

**Theorem 2.2.5** *(ı) Let $G$ be a semisimple algebraic group, then all maximal torus are conjugated. The dimension of such maximal torus is called the rank og $G$.*

*(ıı) Let $T$ be a maximal torus, the group character $X(T)$ does not depend on $T$ and we have a natural isomorphism of $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathfrak{h}^*$ where $\mathfrak{h}$ is the Lie algebra of $T$.*

*(ııı) Let $T$ be a maximal torus, then the group $N_G(T)/T$ is idependent of $T$ and is finite. It is called the Weyl group of $G$.*

*(ııı) The set $\Delta(G,T)$ in $\mathfrak{h}^*$ does not depend on $T$ and is a root system. This root system is the root system of $\mathfrak{g}$ as a Lie algebra. The Weyl group of $G$ and of $\mathfrak{g}$ are isomorphic.*

**Theorem 2.2.6** *(ı) The lattice $Q$ (the root lattice) generated by the set of roots $\Delta$ in $X(T)$ (the weigth lattice) for any maximal torus in $G$ semisimple if of maximal rank and*

$$|X(T)/Q| = |W| = \det(A)$$

*where $A$ is the associated Cartan matrix.*

*(ıı) The center $Z(G)$ of $G$ is isomorphic to the dual of the quotient $X(T)/Q$.*

Here we see the problem of dual groups appearing: when $A$ is a Cartan matrix, then $A^t$ is also a Cartan matrix and the associated group is the dual group. The dual of the quotient $X(T)/Q$ is the same quotient but for the dual group. These king of dualities will be more accurate for Kac-Moody Lie algebras and Kac-Moody groups.

### 2.2.2 Borel subgroups

**Definition 2.2.7** A **Borel subgroup** of an algebraic group $G$ is a maximal closed connected solvable subgroup of $G$.

**Theorem 2.2.8** *Let $B$ be a Borel subgroup of $G$, then the quotient $G/B$ is a projective variety.*

**Theorem 2.2.9** *(ı) Any maximal torus $T$ is contained in a Borel subgroup.*

*(ıı) All Borel subgroups are conjugated and even all pairs $T \subset B$ of a maximal torus contained in a Borel subgroup are conjugated.*

*(ııı) Let $T$ be a maximal torus contained in a Borel $B$, then $T$ acts on $\mathfrak{b}$ the Lie algebra of $B$ by the adjoint representation and we have the decomposition*

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(B)} \mathfrak{g}_\alpha$$

*where $Delta(B)$ is a set of positive roots in $\Delta$ the root system of $G$. In particular $B$ defines a basis of $\Delta$ i.e. a set of simple roots.*

*(ıv) Conversely, any set of positive roots is obtained from a Borel subgroup containing $T$.*

## 2.3 Characters and line bundles

We will now review more on the geometry of the variety $G/B$ for $B$ a Borel subgroup of a semisimple algebraic group $G$. Because all Borel subgroups are conjugated, this variety is isomorphic to the variety $\mathfrak{B}$ of all Borel subgroups in $G$.

Let $\chi \in X(T)$ be a character of a maximal torus contained in $B$ a Borel subgroup. Then $T$ acts on $\mathbb{C}$ via the map $T \to \mathbb{C}^*$ and this induces an action of $B$ on $\mathbb{C}$ because of the structure of $B$: there is an exact sequence

$$1 \to U \to B \to T \to 1$$

where $U$ is an unipotent normal subgroup of $B$. We extend the action of $T$ to $B$ by letting $U$ acts trivially.

**Definition 2.3.1** We may define the line bundle $\mathcal{L}_\chi$ associated to $\chi$ as the quotient of the product $G \times \mathbb{C}$ under the action of $B$ on the right on $G$ and on the left on $\mathbb{C}$. This variety together with the first projection map to $G/B$ has a structure of line bundle on $G/B$.

**Theorem 2.3.2** *The map $X(T) \to \mathrm{Pic}(G)$ defined by $\chi \mapsto \mathcal{L}_\chi$ is an isomorphism of abelian groups.*

Recall that $B$ defines in $\Delta \subset X(T) \otimes_{\mathbb{Z}} \mathbb{R} = \mathfrak{h}^*$ a set $\Delta(B)$ of positive roots. In particular, it defines a cones

$$C = \{x \in \mathfrak{h}^* \ / \ (x, \alpha) \geq 0 \text{ for all } \alpha \in \Delta(B)\}$$

where ( , ) is the Killing form. Let us define the hyperplanes $H_\alpha = \{x \in \mathfrak{h}^* \ / \ (x, \alpha) = 0\}$.

**Theorem 2.3.3** *The Weyl group $W$ acts simply transitively on the connected components of the set $\mathfrak{h}^* \setminus \bigcup_{\alpha \in \Delta} H_\alpha$ and $C$ is a fundamental domain for this action.*

**Theorem 2.3.4** *Let $\chi$ be a domiant weigth (i.e. in $C$, the dominant chamber), then we have the following vanishing $H^i(G/B, \mathcal{L}_\chi) = 0$ for all $i > 0$. Moreover $H^0(G/B, \mathcal{L}_\chi)$ is non zero and is the irreducible representation of $G$ of highest weight $\chi$.*

**Theorem 2.3.5** *More generaly in characteristic zero, let us consider the following action $w * \chi = w(\lambda + \rho) - \rho$ where $\rho$ is half the sum of all positive roots. Then for $\chi$,*

*(ı) either the orbit $W * \chi$ does not meet $C$ and in this case all the cohomology group $H^i(G/B, \mathcal{L}_\chi)$ vanish,*

*(ıı) or there exist a unique $w \in W$ such that $w * \chi \in C$. In that case $H^i(G/B, \mathcal{L}_\chi) = 0$ for $i \neq \ell(w)$ the length of $w$ and $H^i(G/B, \mathcal{L}_\chi)$ is the representation with highest weight $w * \chi$.*

# Part II

# Kac-Moody Lie algebras

# Chapter 3

# Some facts on associative algebras

## 3.1 Free algebras

**Definition 3.1.1** Let $V$ be a vector space, then $T(V)$, the tensor algebra, is the free associative algebra generated by $V$. Let us denote by $i$ the natural inclusion $V \to T(V)$.

The following is the characteristic property of free associative algebras.

**Proposition 3.1.2** *For any associative algebra $A$ and any linear map $f : V \to A$, there exists a unique associative algebra morphism $F : T(V) \to A$ such that $f = F \circ i$.*

**Definition 3.1.3** Let $V$ be a vector space and let $T(V)$ be its tensor algebra viewed as a Lie algebra under the bracket $[a, b] = a \otimes b - b \otimes a$. The free Lie algebra generated by $V$, denoted $F(V)$, is by definition the Lie subalgebra generated by the subspace $V$ in $T(V)$. Let us denote by $j$ the embedding $V \to F(V)$.

The following is the characteristic property of free Lie algebras.

**Proposition 3.1.4** *For any Lie algebra $A$ and any linear map $f : V \to A$, there exists a unique Lie algebra morphism $F : F(V) \to A$ such that $f = F \circ j$.*

## 3.2 Enveloping algebras

Let us recall basic facts on enveloping algebras. For more details and proofs see [Hu72].

**Definition 3.2.1** Let $\mathfrak{g}$ be a Lie algebra and consider the associative $\mathbb{C}$-algebra $U(\mathfrak{g})$ defined as the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal generated by elements of the form $x \otimes y - [x, y]$ for any $x$ and $y$ in $\mathfrak{g}$. Denote by $\tau$ the map from $\mathfrak{g}$ to $U(\mathfrak{g})$.

Remark that $U(\mathfrak{g})$ is generated, as an algebra, by the image of $\mathfrak{g}$. The enveloping algebra has the following universal property:

**Proposition 3.2.2** *For any associative $\mathbb{C}$-algebra $A$, and any linear map $f : \mathfrak{g} \to A$ such that $f([x, y]) = f(x)f(y) - f(y)f(x)$, there exists a unique $\mathbb{C}$-algebra morphism $F : U(\mathfrak{g}) \to A$ such that $f = F \circ \tau$.*

**Corollary 3.2.3** *Let $\mathfrak{g}$ be a Lie algebra and assume $A$ is a free associative $\mathbb{C}$-algebra with a linear map $f : \mathfrak{g} \to A$ such that $f([x, y]) = f(x)f(y) - f(y)f(x)$ and the image of $f$ contains a $\mathbb{C}$-algebra basis of $A$. Then $A$ is the enveloping algebra of $\mathfrak{g}$.*

**Proof :** Let $B$ be a $\mathbb{C}$-algebra and let $g : \mathfrak{g} \to B$ be a linear map such that $g([x, y]) = g(x)g(y) - g(y)g(x)$. Let $(f(e_i))$ be a basis of $A$ contained in the image of $f$and let us define the map $G : A \to B$ by $G(f(e_i)) = g(e_i)$. Because $A$ is free and $(f(e_i))$ is a basis, this map is well defined and we have $g = G \circ f$. □

**Theorem 3.2.4 (Poincaré-Birkhoff-Witt)** *Let $(e_1, \cdots, e_n)$ be a basis of $\mathfrak{g}$. Then the monomials $x_1^{a_1} \cdots x_n^{a_n}$ (the product being taken in this order) for $(a_1, \cdots, a_n) \in \mathbb{N}^n$, form a basis of $U(\mathfrak{g})$ as a $\mathbb{C}$-vector space.*

**Corollary 3.2.5** *($\imath$) The map $\tau : \mathfrak{g} \to U(\mathfrak{g})$ is injective.*

*($\imath\imath$) If we have the decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, then $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$.*

**Remark 3.2.6** Any $\mathfrak{g}$-representation will be a $U(\mathfrak{g})$-module and the converse is also true. In particular, if $\mathfrak{b} \subset \mathfrak{g}$ is a subalgebra and $V$ is a representation of $\mathfrak{b}$, we get an induced representation of $\mathfrak{g}$ by considering the $U(\mathfrak{g})$-module obtained by $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V$ (the enveloping algebra of $\mathfrak{g}$ is a module over $U(\mathfrak{b})$).

# Chapter 4

# Kac-Moody Lie algebras

## 4.1 Lie algebras associated to a complex square matrix

### 4.1.1 Realization of a matrix

Let $A$ be a complex $n \times n$ matrix. In this section, we define for any complex square matrix $A$ of size $n$ two Lie algebras $\widetilde{\mathfrak{g}}(A)$ and $\mathfrak{g}(A)$ and study their first properties. The rank of $A$ will be denoted by $\ell$.

**Definition 4.1.1** A **realization** of a matrix $A$ is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where $\mathfrak{h}$ is a complex vector space, $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ and $\Pi^\vee = \{\alpha_1^\vee, \cdots, \alpha_n^\vee\}$ are indexed subsets in $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively satisfying the following conditions:

(R1) $\Pi$ and $\Pi^\vee$ are linearly independent in $\mathfrak{h}^*$ and $\mathfrak{h}$;

(R2) $\langle \alpha_i^\vee, \alpha_j \rangle = a_{i,j}$ where $\langle \ , \ \rangle$ denotes the natural pairing between a vector space and its dual;

(R3) $\dim \mathfrak{h} = n + \mathrm{Corank}(A)$.

**Proposition 4.1.2** *Let $A$ be a complex square matrix of size $\ell$.*

*(ı) Let $\mathfrak{h}$ be a complex vector space satisfying conditions (R1) and (R2) of the preceding definition, then* $\dim \mathfrak{h} \geq n + \mathrm{Corank}(A)$.

*(ıı) There exist a unique up to isomorphism (non unique isomorphism if* $\det(A) = 0$*) realization of $A$.*

**Proof :** (ı) Consider $\mathfrak{t} \subset \mathfrak{h}$ the orthogonal to $(\alpha_1, \cdots, \alpha_n)$. This subspace is of codimension $n$ and the image of the subspace spanned by $(\alpha_1^\vee, \cdots, \alpha_n^\vee)$ in $\mathfrak{h}/\mathfrak{t}$ is of dimension $\mathrm{Corank}(A)$. This implies that $\dim \mathfrak{h} \geq n + \mathrm{Corank}(A)$.

(ıı) We may assume that $A$ has the following form

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where $A_1$ is non degenerate of rank $\ell$. We may consider the following square matrix of size $2n - \ell$:

$$B = \begin{pmatrix} A_1 & A_2 & 0 \\ A_3 & A_4 & I_{n-\ell} \\ 0 & I_{n-\ell} & 0 \end{pmatrix}.$$

We have $\det(B) = \det(A_1) \neq 0$. Set $\mathfrak{h} = \mathbb{C}^{2n-\ell} = \mathbb{C}^{n+\mathrm{Corank}(A)}$ and take $\alpha_1, \cdots, \alpha_n$ the first $n$ linear coordinates and $\alpha_1^\vee, \cdots, \alpha_n^\vee$ the first $n$ rows of the matrix. This yields a realization of $A$.

Conversely, let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of $A$. Complete $\Pi^\vee$ to a basis of $\mathfrak{h}$ by adding elements $\alpha_{n+1}^\vee, \cdots, \alpha_{2n-\ell}^\vee$. Define elements $\alpha_{n+1}, \cdots, \alpha_{2n-\ell}$ in $\mathfrak{h}^*$ such that

$$C = (\langle \alpha_i^\vee, \alpha_j \rangle)_{i,j\in[1,2n-\ell]} = \begin{pmatrix} A_1 & A_2 & 0 \\ A_3 & A_4 & I_{n-\ell} \\ X_1 & X_2 & 0 \end{pmatrix}$$

for some matrices $X_1$ and $X_2$ and with $A_1$ still non degenerate. We prove that $C$ is non degenerate. For this, we may assume that $A_3 = 0$ and $A_4 = 0$. But then $(\mathrm{Vect}(\alpha_{\ell+1}^\vee, \cdots, \alpha_n^\vee))^\perp = \mathrm{Vect}(\alpha_1, \cdots, \alpha_n)$ and $(\mathrm{Vect}(\alpha_1^\vee, \cdots, \alpha_\ell^\vee, \alpha_{n+1}^\vee, \cdots, \alpha_{2n-\ell}^\vee))^\perp = \mathrm{Vect}(\alpha_{n+1}, \cdots, \alpha_{2n-\ell})$. This implies that $(\alpha_1, \cdots, \alpha_{2n-\ell})$ is a basis and the matrix $C$ is non degenerate.

Now changing $\alpha_{n+1}^\vee, \cdots, \alpha_{2n-\ell}^\vee$ we may assume that $X_1 = 0$ in $C$. Then $\det(X_2) \neq 0$ and by a new change, the matrix $C$ becomes equal to $B$ and we are done. □

**Remark 4.1.3** (i) If $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of a matrix $A$, then $(\mathfrak{h}^*, \Pi^\vee, \Pi)$ is a realization of $A^t$.

(ii) If $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ and $(\mathfrak{h}_2, \Pi_2, \Pi_2^\vee)$ are respectively realizations of the matrices $A_1$ and $A_2$, then $(\mathfrak{h}_1 \oplus \mathfrak{h}_2, \Pi_1 \times \{0\} \cup \{0\} \times \Pi_2, \Pi_1^\vee \times \{0\} \cup \{0\} \times \Pi_2^\vee)$ is a realization of the matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

Such a matrix is called a **direct sum matrix**.

**Definition 4.1.4** (i) A matrix $A$ and its realization is said **decomposable** if after reordering the indices, $A$ decomposes into a non trivial direct sum. Any matrix can be decomposed as a direct sum of indecomposable matrices.

(ii) We define the **root lattice** $Q \subset \mathfrak{h}^*$ to be the $\mathbb{Z}$-module generated by $\Pi$ (this set is called the **root basis** and its elements the **simple roots**) and the **coroot lattice** $Q^\vee \subset \mathfrak{h}$ to be the $\mathbb{Z}$-module generated by $\Pi^\vee$ (this set is called the **coroot basis** and its elements the **simple coroots**). Denote by $Q_+$ the monoid generated in $Q$ by the simple roots (i.e. $Q_+ = \sum_i \mathbb{Z}_+ \alpha_i$).

(iii) For an element $\alpha \in Q$ with $\alpha = \sum_i k_i \alpha_i$, denote by $\mathrm{ht}\alpha = \sum_i k_i$ the **height** of $\alpha$. We introduce a partial ordering on $Q$ by setting $\alpha \geq \beta$ if $\alpha - \beta \in Q_+$.

### 4.1.2 The Lie algebra $\widetilde{\mathfrak{g}}(A)$

The uniqueness of a realization allows the following definition.

**Definition 4.1.5** Let $A = (a_{i,j})_{i,j\in[1,n]}$ be a complex matrix and $(\mathfrak{h}, \Pi, \Pi^\vee)$ a realization of $A$. The Lie algebra $\widetilde{\mathfrak{g}}(A)$ has $(e_i)_{i\in[1,n]}$, $(f_i)_{i\in[1,n]}$ and $\mathfrak{h}$ for generators and the following relations:

- $[e_i, f_j] = \delta_{i,j} \alpha_i^\vee$
- $[h, h'] = 0$
- $[h, e_i] = \langle \alpha_i, h \rangle e_i$
- $[h, f_i] = -\langle \alpha_i, h \rangle f_i$.

Denote by $\widetilde{\mathfrak{n}}_+$ and $\widetilde{\mathfrak{n}}_-$ the Lie subalgebras of $\widetilde{\mathfrak{g}}(A)$ generated by $(e_i)_{i\in[1,n]}$ and $(f_i)_{i\in[1,n]}$ respectively.

We have the

**Theorem 4.1.6** *The Lie algebra* $\widetilde{\mathfrak{g}}(A)$ *satisfies the following properties:*

*(ı)* $\widetilde{\mathfrak{g}}(A) = \widetilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \widetilde{\mathfrak{n}}_+$

*(ıı)* $\widetilde{\mathfrak{n}}_+$ *(resp.* $\widetilde{\mathfrak{n}}_-$*) is freely generated by* $(e_i)_{i\in[1,n]}$ *(resp.* $(f_i)_{i\in[1,n]}$*).*

*(ııı) The map* $e_i \mapsto -f_i$*,* $f_i \mapsto -e_i$*,* $h \mapsto -h$ *can be uniquely extended to an involution* $\omega$ *of the Lie algebra* $\widetilde{\mathfrak{g}}(A)$*.*

*(ıv) There is a root space decomposition with respect to* $\mathfrak{h}$*-eigenvalues:*

$$\widetilde{\mathfrak{g}}(A) = \left(\bigoplus_{\alpha\in Q_+,\ \alpha\neq 0} \widetilde{\mathfrak{g}}_{-\alpha}\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in Q_+,\ \alpha\neq 0} \widetilde{\mathfrak{g}}_{\alpha}\right),$$

*where* $\widetilde{\mathfrak{g}}_\alpha(A) = \{x \in \widetilde{\mathfrak{g}}(A)\ /\ [h, x] = \langle\alpha, h\rangle x,\ \forall h \in \mathfrak{h}\}$*,* $\dim \widetilde{\mathfrak{g}}_\alpha < \infty$ *and* $\widetilde{\mathfrak{g}}_\alpha \subset \mathfrak{n}_\pm$ *for* $\pm\alpha \in Q_+$*.*

*(v) There exist a unique maximal ideal* $\mathfrak{r}$ *in* $\widetilde{\mathfrak{g}}(A)$ *among the ideals intersecting* $\mathfrak{h}$ *trivially. Furthermore we have the following sum of ideals:*

$$\mathfrak{r} = (\mathfrak{r} \cap \widetilde{\mathfrak{n}}_-) \oplus (\mathfrak{r} \cap \widetilde{\mathfrak{n}}_+).$$

**Proof :** We will first define a representation of the Lie algebra $\widetilde{\mathfrak{g}}(A)$. Let $V$ be a vector space of dimension $n$ with basis $v_1, \cdots, v_n$. Let $\alpha$ be any element in $\mathfrak{h}^*$, we define an action of $\widetilde{\mathfrak{g}}(A)$ on the tensor algebra $T(V)$ by:

- $f_i(a) = v_i \otimes a$ for $a \in T(V)$,
- $h(1) = \langle\alpha, h\rangle 1$, and inductively on $s$, $h(v_j \otimes a) = -\langle\alpha_j, h\rangle v_j \otimes a + v_j \otimes h(a)$ for $a \in T^{s-1}(V)$,
- $e_i(1) = 0$, and inductively on $s$, $e_i(v_j \otimes a) = \delta_{i,j}\alpha_i^\vee(a) + v_j \otimes e_i(a)$ for $a \in T^{s-1}(V)$.

Let us first check that this indeed defines a representation of the Lie algebra $\widetilde{\mathfrak{g}}(A)$. We need to check that the defining relations of $\widetilde{\mathfrak{g}}(A)$ are satisfied. We have:

$$(e_i f_j - f_j e_i)(a) = e_i(v_j \otimes a) - v_j \otimes e_i(a) = \delta_{i,j}\alpha_i^\vee(a) + v_j \otimes e_i(a) - v_j \otimes e_i(a) = \delta_{i,j}\alpha_i^\vee(a).$$

Since $\mathfrak{h}$ acts diagonally, the relations $[h, h']$ are satisfied. For the third relation, we proceed by induction on $s$. Let $a \in T^{s-1}(V)$, we have:

$$(he_i - e_ih)(a \otimes v_j) = h(\delta_{i,j}\alpha_i^\vee(a) + v_j \otimes e_i(a)) - e_i(-\langle\alpha_j, h\rangle v_j \otimes a + v_j \otimes h(a))$$

$$= \delta_{i,j}h\alpha_i^\vee(a) - \langle\alpha_j, h\rangle v_j \otimes e_i(a) + v_j \otimes he_i(a) + \langle\alpha_j, h\rangle\delta_{i,j}\alpha_i^\vee(a) + \langle\alpha_j, h\rangle v_j \otimes e_i(a) - \delta_{i,j}\alpha_i^\vee h(a) - v_j \otimes e_ih(a)$$

$$= v_j \otimes [h, e_i](a) + \langle\alpha_j, h\rangle\delta_{i,j}\alpha_i^\vee(a) = \langle\alpha_i, h\rangle v_j \otimes e_i(a) + \langle\alpha_i, h\rangle\delta_{i,j}\alpha_i^\vee(a) = \langle\alpha_i, h\rangle e_i(a \otimes v_j).$$

And we also have for the last relation:

$$(hf_i - f_ih)(a) = h(v_i \otimes a) - v_i \otimes h(a) = -\langle\alpha_i, h\rangle v_i \otimes a = -\langle\alpha_i, h\rangle f_i(a).$$

Let us also consider the map $\widetilde{\mathfrak{n}}_- \to T(V)$ sending $f_i$ to $v_i = f_i(1)$ or more generally $n_- \in \widetilde{\mathfrak{n}}_-$ to $n_-(1)$. We prove that this is the embedding of $\widetilde{\mathfrak{n}}_-$ in its enveloping algebra $U(\widetilde{\mathfrak{n}}_-)$. Indeed, because of what we just proved, there is a surjective map $U(\widetilde{\mathfrak{n}}_-) \to T(V)$. But because $T(V)$ is free, corollary 3.2.3 implies that $T(V)$ is the enveloping algebra of $\widetilde{\mathfrak{n}}_-$ with $n_- \mapsto n_-(1)$ its natural embedding.

Let us now prove the Theorem. An easy induction proves that we have $\widetilde{\mathfrak{g}}(A) = \widetilde{\mathfrak{n}}_- + \mathfrak{h} + \widetilde{\mathfrak{n}}_+$ (simply remark that the subspace $\widetilde{\mathfrak{n}}_- + \mathfrak{h} + \widetilde{\mathfrak{n}}_+$ is stable by the adjoint action of $\widetilde{\mathfrak{g}}(A)$ and contains the generators of $\widetilde{\mathfrak{g}}(A)$). Assume there is a relation $n_- + h + n_+ = 0$ with $n_- \in \widetilde{\mathfrak{n}}_-$, $h \in \mathfrak{h}$ and $n_+ \in \widetilde{\mathfrak{n}}_+$. We have $n_-(1) + h(1) + n_+(1) = 0$. But $\widetilde{\mathfrak{n}}_+$ being generated by the $(e_i)_{i\in[1,n]}$, we have $n_+(1) = 0$ and

$n_-(1) + h(1) = 0$ or $n_-(1) = \langle\alpha, h\rangle 1$. This is true for all $\alpha \in \mathfrak{h}^*$ so that $h = 0$ and $n_-(1) = 0$. Because the map $n_- \mapsto n_-(1)$ is an embedding we get $n_- = 0$ and the part (ɪ) of the theorem.

For point (ɪɪ), we have seen that $U(\widetilde{\mathfrak{n}}_-) = T(V)$ so that $\widetilde{\mathfrak{n}}_-$ is the Lie subalgebra of $T(V)$ generated by the $(f_i)_{i\in[1,n]}$ which is by definition a free Lie algebra. For $\widetilde{\mathfrak{n}}_+$ we use point (ɪɪɪ) which is trivial because the involution respects the defining relations of $\widetilde{\mathfrak{g}}(A)$.

Using Jacobi identity and the defining relations of $\widetilde{\mathfrak{g}}(A)$, it is easy to see that any composition of $\mathrm{ad}\,(e_i)$'s applied to some $e_j$ is an eigenvector for $\mathfrak{h}$ with eigenvalue in $Q_+$. This proves, together with the symmetry in (ɪɪɪ), the decompositions

$$\mathfrak{n}_\pm = \bigoplus_{\alpha\in Q_+,\ \alpha\neq 0} \widetilde{\mathfrak{g}}_\pm.$$

Furthermore, because $\mathfrak{n}_+$ is generated by the $(e_i)_{i\in[1,n]}$, we have the inequality $\dim \widetilde{\mathfrak{g}}_\alpha \leq n^{|\mathrm{ht}\alpha|}$, proving (ɪv). Remark that $\mathfrak{h} = \widetilde{\mathfrak{g}}_0$.

For the last assertion, let us prove the following useful Lemma:

**Lemma 4.1.7** *Let $\mathfrak{h}$ be a commutative Lie algebra and $V$ be a diagonalisable $\mathfrak{h}$-module meaning that we have a decomposition*

$$V = \bigoplus_{\alpha\in\mathfrak{h}^*} V_\alpha, \quad \textit{with } V_\alpha = \{v \in V \ / \ h(v) = \langle\alpha, h\rangle v\}.$$

*Then any submodule $U$ of $V$ can be decomposed as*

$$U = \bigoplus_{\alpha\in\mathfrak{h}^*} U \cap V_\alpha.$$

**Proof :** Let $u \in U$, it is decomposed as $u = \sum_{i=1}^m u_{\alpha_i}$ with $u_{\alpha_i} \in V_{\alpha_i}$. We want to prove that $u_{\alpha_i} \in U$ for any $i$. But let $h \in \mathfrak{h}$ such that all the $\langle\alpha_i, h\rangle$ are distinct. We have $h^k(u) = \sum_{i=1}^m \langle\alpha_i, h\rangle^k u_{\alpha_i} \in U$ because $U$ is an $\mathfrak{h}$-module. This is an independent system of equations proving the result. □

Now let $\mathfrak{u}$ be any ideal of $\widetilde{\mathfrak{g}}(A)$. We have the decomposition

$$\mathfrak{u} = \bigoplus_{\alpha\in Q} (\widetilde{\mathfrak{g}}_\alpha \cap \mathfrak{u}),$$

hence a sum of ideals intersecting $\mathfrak{h} = \mathfrak{g}_0$ trivially will intersect $\mathfrak{h}$ trivially. The sum of all these ideals is the desired ideal $\mathfrak{r}$. We have the decomposition

$$\mathfrak{r} = (\widetilde{\mathfrak{n}}_- \cap \mathfrak{r}) \oplus (\widetilde{\mathfrak{n}}_+ \cap \mathfrak{r})$$

as vector spaces thanks to the previous Lemma. But $[f_i, \widetilde{\mathfrak{n}}_+] \subset \mathfrak{n}_+ \oplus \mathfrak{h}$ hence $[f_i, \widetilde{\mathfrak{n}}_+ \cap \mathfrak{r}] \subset (\mathfrak{n}_+ \oplus \mathfrak{h}) \cap \mathfrak{r} \subset \mathfrak{n}_+ \cap \mathfrak{r}$. This implies the inclusion $[\widetilde{\mathfrak{g}}(A), \mathfrak{n}_+ \cap \mathfrak{r}] \subset \mathfrak{n}_+ \cap \mathfrak{r}$ proving that $\mathfrak{n}_+ \cap \mathfrak{r}$ is an ideal. The same method shows that $\mathfrak{n}_- \cap \mathfrak{r}$ is an ideal. □

### 4.1.3 The Lie algebra $\mathfrak{g}(A)$

We are now in position to define the Lie algebra $\mathfrak{g}(A)$ associated to a complex $n \times n$-matrix $A$.

**Definition 4.1.8** Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of $A$ and let $\widetilde{\mathfrak{g}}(A)$ the Lie algebra defined in the previous section.

(ɪ)Let us set $\mathfrak{g}(A) = \widetilde{\mathfrak{g}}(A)/\mathfrak{r}$ where $\mathfrak{r}$ is the ideal defined by Theorem 4.1.6.

(ɪɪ) Remark that by Theorem 4.1.6, the abelian Lie algebra $\mathfrak{h}$ is contained in $\widetilde{\mathfrak{g}}(A)$ and because $\mathfrak{r}$ does not meet $\mathfrak{h}$ is it also contained in $\mathfrak{g}(A)$. The quadruple $(\mathfrak{g}(A), \mathfrak{h}, \Pi, \Pi^\vee)$ is called the $(\mathfrak{g}, \mathfrak{h})$-pair associated to $A$.

Remark that the decomposition of $\widetilde{\mathfrak{g}}(A)$ induces a decomposition, called the root space decomposition

$$\mathfrak{g}(A) = \left( \bigoplus_{\alpha \in Q_+,\ \alpha \neq 0} \mathfrak{g}_{-\alpha} \right) \oplus \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in Q_+,\ \alpha \neq 0} \mathfrak{g}_{\alpha} \right),$$

where $\mathfrak{g}_\alpha(A) = \{x \in \mathfrak{g}(A) \ / \ [h,x] = \langle \alpha, h\rangle x,\ \forall h \in \mathfrak{h}\}$. We have the estimate $\dim \mathfrak{g}_\alpha < n^{|\mathrm{ht}\alpha|}$ and $\mathfrak{g}_0 = \mathfrak{h}$. We also call $\dim \mathfrak{g}_\alpha$ the multiplicity of $\alpha$ and denote it by $\mathrm{mult}(\alpha)$. We denote by $\mathfrak{n}_\pm$ the image of $\widetilde{\mathfrak{n}}_\pm$ in $\mathfrak{g}(A)$. We have the decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

**Definition 4.1.9** An element $\alpha \in Q$ is called a root if $\alpha \neq 0$ and $\mathrm{mult}\alpha \neq 0$. A root $\alpha$ is called positive if $\alpha \in Q_+$ and negative if $-\alpha \in Q_+$. We denote by $\Delta$, $\Delta_+$ and $\Delta_-$ the set of roots, positive roots and negative roots. We have a disjoint union $\Delta = \Delta_+ \cup \Delta_-$.

**Remark 4.1.10** (ı) The space $\mathfrak{g}_\alpha$, for $\alpha \in \Delta_+$ (resp. for $\alpha \in \Delta_-$), is the linear span of the elements of the form $[\cdots[[e_{i_1}, e_{i_2}], e_{i_3}]\cdots, e_{i_k}]$ (resp. $[\cdots[[f_{i_1}, f_{i_2}], f_{i_3}]\cdots, f_{i_k}]$) such that $\alpha_{i_1} + \alpha_{i_2} = \cdots + \alpha_{i_k} = \alpha$ (resp. $-\alpha$). In particular, we have

$$\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i,\ \mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i \text{ and } \mathfrak{g}_{s\alpha_i} = 0 \text{ for } |s| > 1.$$

Indeed, the first fact comes from the inclusion of $\mathfrak{g}_\alpha$ in $\mathfrak{n}_+$ which is generated by the $e_i$'s. The multiplicity of the simple roots follows immediately. The last point follows from the multiplicity of simple roots.

(ıı) The involution defined in Theorem 4.1.6 induces an involution $\omega$ on $\mathfrak{g}(A)$. Because of (ı), we have $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$. In particular $\mathrm{mult}\alpha = \mathrm{mult}(-\alpha)$ and $\omega(\Delta_+) = \Delta_-$.

**Proposition 4.1.11** *Let $\mathfrak{g}'(A) = [\mathfrak{g}(A), \mathfrak{g}(A)]$ be the derived algebra.*

*(ı) We have $\mathfrak{g}(A) = \mathfrak{g}'(A) + \mathfrak{h}$ and $\mathfrak{g}'(A) = \mathfrak{g}(A)$ if and only if $\det(A) \neq 0$.*

*(ıı) Let us denote by $\mathfrak{h}'$ the subspace of $\mathfrak{h}$ spanned by the images of the $\alpha_i$'s (one more time still denoted $\alpha_i$ in $\mathfrak{g}(A)$). We have $\mathfrak{g}'(A) \cap \mathfrak{h} = \mathfrak{h}'$ and $\mathfrak{g}'(A) \cap \mathfrak{g}_\alpha = \mathfrak{g}_\alpha$.*

*(ııı) The image of the $e_i$'s and the $f_i$'s in $\mathfrak{g}(A)$ are, by abuse of notation, denoted by $e_i$ and $f_i$, they generate the subalgebra $\mathfrak{g}'(A)$.*

**Proof :** In view of the defining relations of $\widetilde{\mathfrak{g}}(A)$, it is clear that the $e_i$'s and the $f_i$'s are in $\mathfrak{g}'(A)$. In particular $\mathfrak{n}_- \oplus \mathfrak{n}_+$ is contained in $\mathfrak{g}'(A)$ proving that $\mathfrak{g}(A) = \mathfrak{g}'(A) + \mathfrak{h}$ and that $\mathfrak{g}'(A) \cap \mathfrak{g}_\alpha = \mathfrak{g}_\alpha$ for $\alpha \in Q$ with $\alpha \neq 0$.

It is clear from the relations defining $\widetilde{\mathfrak{g}}(A)$ that $\mathfrak{h}'$ is contained in $\mathfrak{g}'(A) \cap \mathfrak{h}$. For the converse inclusion, it suffices to prove that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}'$ for $\alpha \in Q_+$. We prove it by induction on $\mathrm{ht}\alpha$. It is clear if $\mathrm{ht}\alpha = 1$. Assume $\mathrm{ht}\alpha > 1$ and let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$. Now $x$ is in $\mathfrak{n}_+$ and we may assume that $x = [e_i, z]$ for some $i$ and some $z \in \mathfrak{g}_{\alpha-\alpha_i}$. We compute $[x,y] = [e_i,[z,y]] - [z,[e_i,y]]$ and conclude by induction.

In particular, we have $\mathfrak{g}'(A) = \mathfrak{g}(A)$ if and only if $\mathfrak{h} \subset \mathfrak{g}'(A)$ or if and only if $\mathfrak{h}' = \mathfrak{h}$ which is equivalent to $\det(A) = 0$.

We can now prove (ııı). The $e_i$'s and $f_i$'s generate $\mathfrak{n}_- \oplus \mathfrak{h}' \oplus \mathfrak{n}_+$ which is $\mathfrak{g}'(A)$ by what we already proved. □

**Proposition 4.1.12** *The center $\mathfrak{c}$ of $\mathfrak{g}(A)$ is given by*

$$\mathfrak{c} = \{h \in \mathfrak{h} \ / \ \langle \alpha, h\rangle = 0,\ \forall \alpha \in \Pi\}.$$

*We have the inclusion $\mathfrak{c} \subset \mathfrak{h}'$.*

**Proof :** Let $x \in \mathfrak{c}$, we have $[h, x] = 0$ for all $h \in \mathfrak{h}$ so that $x \in \mathfrak{g}_0 = \mathfrak{h}$. Furthermore, we have $[x, y] = 0$ for any $y \in \mathfrak{g}_\alpha$ which implies that $\langle \alpha, x \rangle = 0$ for all $\alpha \in \Delta$ (we need $y$ to be non zero) which is equivalent to $\langle \alpha, x \rangle = 0$ for all $\alpha \in \Pi$. Conversely, if $x \in \mathfrak{h}$ satisfies the previous condition, then it lies in the center.

For the last condition, we have that $\dim \mathfrak{c} = n - \ell = \mathrm{Corank}(A)$. But $\dim(\mathfrak{c} \cap \mathfrak{h}') = \mathrm{Corank}(A)$ and the conclusion follows. □

## 4.2 Kac-Moody Lie algebras

### 4.2.1 Generalized Cartan matrices

We will now consider a special type of complex matrices: the generalized Cartan matrices. Recall that a Cartan matrix $A = (a_{i,j})_{i,j \in [1,n]}$ has the following characteristic properties:

- $a_{i,i} = 2$;
- $a_{i,j} \leq 0$ for $i \neq j$;
- $a_{i,j} = 0 \Rightarrow a_{j,i} = 0$;
- $\det(A) > 0$.

We only relax the condition on the rank of the matrix $A$.

**Definition 4.2.1** A matrix $A = (a_{i,j})_{i,j \in [1,n]}$ is said to be a generalized Cartan matrix (GCM) if it satisfies the following conditions:

- $a_{i,i} = 2$;
- $a_{i,j} \leq 0$ for $i \neq j$;
- $a_{i,j} = 0 \Rightarrow a_{j,i} = 0$.

**Definition 4.2.2** A Lie algebra $\mathfrak{g}(A)$ associated to a generalised Cartan algebra $A$ is called a **Kac-Moody Lie algebra**.

**Remark 4.2.3** The defining relations of $\widetilde{\mathfrak{g}}(A)$ are the relations in Serre's presentation (see section 1.3) except for the last two relations. We will see that these relations are contained in $\mathfrak{r}$ and that in many occasions these relations generate $\mathfrak{r}$ so that we could have defined $\mathfrak{g}(A)$ as the Lie algebra generated by $(e_i)_{i \in [1,n]}$, $(f_i)_{i \in [1,n]}$ and $\mathfrak{h}$ for generators and the following relations:

- $[e_i, f_j] = \delta_{i,j} \alpha_i^\vee$
- $[h, h'] = 0$
- $[h, e_i] = \langle \alpha_i, h \rangle e_i$
- $[h, f_i] = -\langle \alpha_i, h \rangle f_i$
- $(\mathrm{ad}\, e_i)^{1 - a_{i,j}}(e_j) = 0$ for $i \neq j$
- $(\mathrm{ad}\, f_i)^{1 - a_{i,j}}(f_j) = 0$ for $i \neq j$.

### 4.2.2 Isomorphisms of Kac-Moody Lie algebras

In this section we try to understand in what extend the Lie algebra $\mathfrak{g}(A)$ determines the generalized Cartan matrix $A$. We start with the following proposition which is an easy consequence of Proposition 4.1.2 and Theorem 4.1.6.

**Proposition 4.2.4** *($\imath$) Let $\mathfrak{g}$ be a Lie algebra, let $\mathfrak{h}$ be a commutative subalgebra, let $e_1, \cdots, e_n$, $f_1, \cdots, f_n$ be elements in $\mathfrak{g}$ and let $\Pi^\vee = \{\alpha_1, \cdots, \alpha_n\} \subset \mathfrak{h}$, $\Pi = \{\alpha_1^\vee, \cdots, \alpha_n^\vee\} \subset \mathfrak{h}^*$ be linearly independent sets such that*

- $[e_i, f_j] = \delta_{i,j}\alpha_i^\vee$
- $[h, e_i] = \langle\alpha_i, h\rangle e_i$, $[h, f_i] = -\langle\alpha_i, h\rangle f_i$.

*Suppose that the $e_i$'s, the $f_i$'s and $\mathfrak{h}$ generate $\mathfrak{g}$ and that $\mathfrak{g}$ has a no non zero ideal which intersect $\mathfrak{h}$ trivially. Finally, set $A = (\langle\alpha_i^\vee, \alpha_j\rangle)$ and suppose that $\dim\mathfrak{h} = 2n - \mathrm{Rank}(A)$. Then $(\mathfrak{g}, \mathfrak{h}, \Pi, \Pi^\vee)$ is the $(\mathfrak{g}, \mathfrak{h})$-pair associated to $A$.*

*($\imath\imath$) Given two $n \times n$ complex matrices $A$ and $A'$, there exist an isomorphism of the associated $(\mathfrak{g}, \mathfrak{h})$-pairs if and only if there exist a non-degenerated diagonal matrix $D$ such that $A'$ can be obtained from $DA$ by a permutation of the rows and the same permutation of the columns.*

**Proof :**($\imath$) We have that $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of $A$. Let $\widetilde{\mathfrak{g}}(A)$ the associated Lie algebra already defined. Because of the relations in $\mathfrak{g}$, we have a map $\widetilde{\mathfrak{g}}(A) \to \mathfrak{g}$ (surjective because $\mathfrak{g}$ is generated by the $e_i$'s, the $f_i$'s and $\mathfrak{h}$). Because $\mathfrak{h}$ is a subalgebra in $\mathfrak{g}$, this map has to factorize the map $\widetilde{\mathfrak{g}}(A) \to \mathfrak{g}(A)$. But because $\mathfrak{g}$ has a no non zero ideal which intersect $\mathfrak{h}$ trivially, we have an isomorphism $\mathfrak{g} \to \mathfrak{g}(A)$.

($\imath\imath$) Any isomorphism of $(\mathfrak{g}, \mathfrak{h})$-pair gives the permutation between the simple root. And I can only see this permutation and no matrix $D$. However, if we do only fix simple roots and not simple coroots, we get by rescaling a matrix $D$: the map $e_i \mapsto e_i$, $f_i \mapsto d_i f_i$ and identity on $\mathfrak{h}$ is an isomorphism from $\mathfrak{g}(A)$ to $\mathfrak{g}(DA)$ with $D = \mathrm{diag}(d_i)$. □

One has the following Theorem:

**Theorem 4.2.5 (see Peterson-Kac [PK83])** *Let $\mathfrak{g}$ be a Kac-Moody Lie algebra, then any two maximal diagonalisable subalgebra are conjugate. As a consequence, any two Kac-Moody algebras are isomorphic if and only if their generalized Cartan matrices can be obtained from each other by a reordering of the index set.*

### 4.2.3 Serre relations

In this section we prove that Serre relations are satisfied in a Kac-Moody Lie algebra. Let us recall results on $\mathfrak{sl}_2$ representations.

Recall that $\mathfrak{sl}_2 = \{A \in \mathfrak{gl}_2 \ / \ \mathrm{Tr}(A) = 0\}$. Let us define the following elements in $\mathfrak{sl}_2$:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then we have $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$.

**Proposition 4.2.6** *($\imath$) In $U(\mathfrak{sl}_2)$, the following formulas hold: $[h, f^k] = -2kf^k$, $[h, e^k] = 2ke^k$, $[e, f^k] = -k(k-1)f^{k-1} + kf^{k-1}h$.*

*($\imath\imath$) Let $V$ be a $\mathfrak{sl}_2$-module and $v \in V$ such that $h(v) = av$ for some $a \in \mathbb{C}$, then we have the relation $h(f^j(v)) = (a - 2j)f^j(v)$ and if moreover $e(v) = 0$ then $e(f^j(v)) = j(a - j + 1)f^{j-1}(v)$.*

*(iii) For each integer $k \in \mathbb{N}$, there is a unique irreducible $\mathfrak{sl}_2$ representation of dimension $k+1$ (up to isomorphism) denoted by $S_k$. Moreover, there exist a basis $(v_i)_{i\in[0,k]}$ of $S_k$ such that*

$$h(v_i) = (k-2i)v_i, \quad f(v_i) = v_{i+1} \quad \text{and} \quad e(v_i) = j(k+1-j)v_{j-1}$$

*where $v_{k+1} = v_0 = 0$.*

**Proof :** (i) We proceed by induction on $k$. We have

$$[h, f^k] = h \otimes f^k - f^k \otimes h = f \otimes [h, f^{k-1}] - 2f^k = -2(k-1)f \otimes f^{k-1} - 2f^k = -2kf^k.$$

The second formula comes in the same way. For the last one we have:

$$[e, f^k] = f \otimes [e, f^{k-1}] + h \otimes f^{k-1} = f \otimes [e, f^{k-1}] + [h, f^{k-1}] - f^{k-1} \otimes h$$

and we conclude by induction.

(ii) We compute $h(f^j(v)) = [h, f^j](v) + f^j(h(v)) = -2jf^j(v) + af^j(v)$. If $e(v) = 0$, we get $e(f^j(v)) = [e, f^j](v) + f^j(e(v)) = -i(i-1)f^j(v) + jf^j(h(v)) = j(a-j+1)f^j(v)$.

(iii) Let $V$ be an irreducible representation of dimension $k+1$. Let $u$ be an eigenvector for $h$ say of eigenvalue $a$. If $e^s(u) \neq 0$, then it is an eigenvector of $h$ with eigenvalue $a+2s$ so that $(u, \cdots, e^s(u))$ is linearly independent. In particular there is an $s$ such that $e^s(u) = 0$. Take $s$ minimal with this property and set $v = e^{s-1}(u)$. Then $v$ is an eigenvector of $h$ say of eigenvalue $b$ and $e(v) = 0$. Set $v_j = f^j(v)$. We know that $v_0$ is an eigenvector for $h$ and so are the $v_j$'s as soon as they don't vanish. Furthermore, the non vanishing $v_j$'s having different eigenvalues are linearly independent so there exist a $j$ with $v_j = 0$. Let $l$ be the smallest number such that $v_l \neq 0$. The subspace spanned by the $v_j$'s is easily seen to be an $\mathfrak{sl}_2$-module. It has to be $V$ and $l = k$. The rest of the proposition comes from (ii). □

**Proposition 4.2.7** *Let $\mathfrak{g}(A)$ be a Kac-Moody Lie algebra and let $e_i$ and $f_i$ be its Chevalley generators. Then the Serre relations hold, namely:*

$$(\text{ad}\, e_i)^{1-a_{i,j}}(e_j) = 0 \quad \text{and} \quad (\text{ad}\, f_i)^{1-a_{i,j}}(f_j) = 0, \quad \text{for} \quad i \neq j.$$

**Proof :** Each relation can be deduced from the other thanks to the involution $\omega$. We prove the second one. Remark that $\mathfrak{g}_{(i)} = \mathfrak{g}_{-\alpha_i} \oplus \mathbb{C}\alpha_i^\vee \oplus \mathfrak{g}_{\alpha_i}$ is isomorphic to $\mathfrak{sl}_2$ with elements $e$, $f$ and $h$ corresponding to $e_i$, $f_i$ and $\alpha_i^\vee$. Consider $\mathfrak{g}(A)$ as a $\mathfrak{sl}_2 = g_{(i)}$-module that to the adjoint action. For $j \neq i$, set $v = f_j$, we have $e(v) = 0$ and $h(v) = -a_{i,j}v$. We want to prove the relation $w = f^{1-a_{i,j}}(v) = 0$. Let us prove the following

**Lemma 4.2.8** *Let $x \in \mathfrak{n}_+$ (resp. in $\mathfrak{n}_-$) such that $[f_i, x] = 0$ (resp. $[e_i, x] = 0$) for all $i$, then $x = 0$.*

**Proof :** Consider the subspace $E$ generated by the $e_i$ and set $\mathfrak{i} = \sum_{i,j}(\text{ad}\ E)^i(\text{ad}\ \mathfrak{h})^j(x)$. We claim it is an ideal of $\mathfrak{g}(A)$. Indeed, elements in $\mathfrak{i}$ are linear combination of elements of the form $[e_{i_1}, [\cdots [e_{i_s}, [h_1, [\cdots [h_k, x] \cdots]$ and ad $(h)$ for $h \in \mathfrak{h}$ sends (by easy induction and Jacobi) an element of that form to a linear combination of elements of that form. It it clear for ad $(e_i)$. For ad $(f_i)$, we use Jacobi to end up with linear combinations of elements of that form and elements of the form $[e_{i_1}, [\cdots [e_{i_s}, [h_1, [\cdots [h_k, [f_i, x]] \cdots]$ which vanishes.

Furthermore any such element is in $\mathfrak{n}_+$ because $a$ is and because $\mathfrak{h}$ and $\mathfrak{n}_+$ stabilize $\mathfrak{n}_+$. The ideal $\mathfrak{i}$ does not intersect $\mathfrak{h}$. It has to be trivial and $a = 0$. □

In view of the previous Lemma, we only need to prove that $[e_k, w] = 0$ for all $k$. This is clear for $k \notin \{i, j\}$. Now Lemme 4.2.6 gives

$$[e_i, w] = (1 - a_{i,j})(-a_{i,j} - (1 - a_{i,j}) + 1)w = 0.$$

Finally, we have

$$[e_j, w] = (\mathrm{ad}\, f_i)^{1-a_{i,j}}[e_j, f_j] = (\mathrm{ad}\, f_i)^{1-a_{i,j}}(\alpha_j^\vee) = -a_{i,j}(\mathrm{ad}\, f_i)^{-a_{i,j}}(f_i).$$

This vanishes and the result follows. □

### 4.2.4 Ideals in the Kac-Moody Lie algebras

Let us prove the following Lemma

**Lemma 4.2.9** *Let $A$ be a Cartan matrix. The matrix is indecomposable if and only if the following condition hold: for any pair of indices $(i, j) \in [1, n]^2$, there exists a sequence of indices $i_0 = i, \cdots, i_k = j$ such that $a_{i_0,i_1} \cdots a_{i_{k-1},i_k} \neq 0$.*

**Proof :** Define the equivalence relation on the set of indices $[1, n]$ by setting $i \sim j$ if there exists a sequence of indices $i_0 = i, \cdots, i_k = j$ such that $a_{i_0,i_1} \cdots a_{i_{k-1},i_k} \neq 0$. Now we may reorder the indices so that the first indices are in the same orbit for this equivalence, the next in the same orbit and so on. The matrix $A$ becomes a block matrix with blocks given by the orbits of the equivalence relation. In particular, it is indecomposable if and only if there is an unique orbit for this relation. □

**Proposition 4.2.10** *Let $A$ be an indecomposable Cartan matrix.*

*(ı) The Lie algebra $\mathfrak{g}(A)$ is simple if and only if $\det(A) \neq 0$.*

*(ıı) Any ideal of the Lie algebra $\mathfrak{g}(A)$ is either contained in the center $\mathfrak{c}$ or contains the derived Lie algebra $g'(A)$.*

*(ııı) Assume that there exist no root $\alpha$ such that $\alpha|_{\mathfrak{h}'} = 0$ then $\mathfrak{g}'(A)/\mathfrak{c} = 0$*

**Proof :** (ı) Assume that the Lie algebra is simple. Then its center has to be trivial so that $\mathfrak{c} = \{h \in \mathfrak{h} \ / \ \langle \alpha_i, h \rangle = 0\}$ is trivial. This implies that $0 = \dim \mathfrak{c} = \dim \mathfrak{h} - n$ i.e. $\mathfrak{h} = \mathfrak{h}'$ and thus $\det(A) \neq 0$.

Let $\mathfrak{i}$ be an ideal of $\mathfrak{g}(A)$ not contained in the center.

Assume first that $\mathfrak{i}$ contains a element $h \in \mathfrak{h}$ not in the center. Then there exists an index $i \in [1, n]$ such that $\langle \alpha_i, h \rangle \neq 0$ and we have $[h, e_i] = \langle \alpha_i, h \rangle e_i$ and $e_i$ is in $\mathfrak{i}$. As a consequence, $\alpha_i^\vee = [e_i, f_i] \in \mathfrak{i}$. But by letting $\alpha_i^\vee$ act on the elements $e_j$ and because of the characterisation of indecomposable matrices of the previous Lemma, we get any $\alpha_j^\vee$, any $e_j$ and any $f_j$ is in $\mathfrak{i}$ thus $\mathfrak{i}$ contains $\mathfrak{g}'(A)$.

Let us prove the general case. We know that $\mathfrak{i}$ is decomposed into its eigenspaces $\mathfrak{i} \cap \mathfrak{g}_\alpha$. If for any $\alpha$ we have $\mathfrak{i} \cap \mathfrak{a} = 0$ then $\mathfrak{i} \subset \mathfrak{h}$ and there must be an element $h \in \mathfrak{h} \cap \mathfrak{i}$ not in the center. We get a contradiction thanks to the previous study. We can therefore take $\alpha$ a root minimal for the order defined on roots by $Q_+$ and such that $\mathfrak{i} \cap \mathfrak{g}_\alpha \neq 0$. Let $x$ be an element in that intersection. Because this element is non zero, the Lemma 4.2.8 gives us an index $i$ such that $[f_i, x] \neq 0$. But $[f_i, x]$ is in $\mathfrak{i} \cap \mathfrak{g}_{\alpha - \alpha_i}$ and by minimality of $\alpha$ this implies that $\alpha - \alpha_i = 0$ and $x$ is colinear to $e_i$. We use the previous argument to conclude for the end of (ı) and for (ıı).

(ııı) Consider $\mathfrak{i}'$ an ideal of the quotient and $\mathfrak{i}$ its inverse image in $\mathfrak{g}'(A)$. It is an ideal of $\mathfrak{g}'(A)$. If there exists an element $h \in \mathfrak{i} \cap \mathfrak{h}$ not in the center, then $h \in \mathfrak{h}'$ and because it is not in the center, we may use the previous argument because all the products are taken with elements in $\mathfrak{g}'(A)$.

To prove the general case we consider the decomposition of $\mathfrak{g}(A)$ with respect to the action of $\mathfrak{h}'$. It is given by the subspaces

$$\mathfrak{g}^\beta = \bigoplus_{\alpha \in f^{-1}(\beta)} \mathfrak{g}_\alpha$$

where $\beta \in \mathfrak{h}'^*$ and $f : \mathfrak{h}^* \to \mathfrak{h}'^*$ is the natural projection. The set $\Delta'$ of roots for this decomposition is $f(\Delta)$. Remark that $\mathfrak{g}^0 = \mathfrak{g}_0 = \mathfrak{h}$ because of our hypothesis (the fibre $f^{-1}(0)$ is $\{0\}$). If $\mathfrak{i}'$ is not trivial, then there exists an element $x \in \mathfrak{i}$ not in the center. Let us consider the decomposition of $x$ as the sum $x = n + h$ where $n \in \mathfrak{n}_- \oplus \mathfrak{n}_+$ and $h \in \mathfrak{h}$. Because $\mathfrak{i}$ is $\mathfrak{h}'$-stable, we know that the element $h$ is in $\mathfrak{i} \cap \mathfrak{g}^0 = \mathfrak{i} \cap \mathfrak{h} \subset \mathfrak{i}$. If $h$ is not in the center, we are done. Otherwise, the element $n$ is not in the center. Let us consider the set $X_\mathfrak{i}$ of all the elements $x \in \mathfrak{i}$ such that $x = \sum_{\alpha \in \Delta} x_\alpha$ with $x_\alpha \in \mathfrak{g}_\alpha$ and the set $\Delta_\mathfrak{i}$ of roots $\gamma \in \Delta$ such that there exists an element $x \in X_\mathfrak{i}$ with $x = \sum_\alpha x_\alpha$ and $x_\gamma \neq 0$. We know that $X_\mathfrak{i}$ is not empty so that $\Delta_\mathfrak{i}$ is also not empty and at least one of the sets $\Delta_\mathfrak{i} \cap \Delta_+$ or $\Delta_\mathfrak{i} \cap \Delta_-$ is not empty. Let us assume that $\Delta_\mathfrak{i} \cap \Delta_+$ is not empty (the other case is similar). Let $\gamma$ minimal in $\Delta_\mathfrak{i} \cap \Delta_+$ and let $x \in X_\mathfrak{i}$ such that $x = \sum_\alpha x_\alpha$ with $x_\gamma \neq 0$. The element $x_\gamma$ is a non zero element in $\mathfrak{n}_+$ thus thanks to Lemma 4.2.8 there exists an index $i$ such that $[f_i, x_\gamma] \neq 0$. We have $[f_i, x] \in \mathfrak{i}$ and $[f_i, x]_{\gamma - \alpha_i} = [f_i, x_\gamma] \neq 0$. By minimality, this implies that $\gamma = \alpha_i$ and $x_\gamma$ is a non zero multiple of $e_i$. But then $[f_i, x_\gamma]$ is a non zero multiple of $\alpha_i^\vee$. This element is the component of $[f_i, x]$ in $\mathfrak{h}$ and is not in the center. But as $[f_i, x] \in \mathfrak{i}$ its zero component is also in $\mathfrak{i}$ and we are done. $\square$

# Chapter 5

# Weyl group

In this chapter, we define and study the Weyl group of a Kac-Moody Lie algebra and describe the similarities and differences with the classical situation. Here because the Lie algebra $\mathfrak{g}(A)$ is not finite dimensional, we will need to be more careful with sums. In particular, we will need locally finite and locally nilpotent elements as well as integrable representations. In all the chapter $\mathfrak{g}(A)$ will be a Kac-Moody Lie algebra.

## 5.1 locally finite and nilpotent elements

We start with the

**Definition 5.1.1** (ı) Let $T : V \to V$ be an endomorphism of a complex vector space $V$. It is called locally finite at $v \in V$ if there exists a finite dimensional subspace $W$ of $V$ containing $v$ and stable by $T$. If $T|_W$ is nilpotent, then $T$ is called locally nilpotent at $v$. The endomorphism $T$ is called locally finite (resp. locally nilpotent) if it is locally finite (resp. locally nilpotent) at every $v \in V$.

(ıı) For a locally finite $T : V \to V$, we can define an automorphism $\exp T : V \to V$ by

$$\exp T = \sum_{n\geq 0} \frac{T^n}{n!}$$

and we have the formula $\exp(kT) = (\exp T)^k$. If $T$ is locally finite at $v$, we can define $(\exp T)(v)$.

Let us prove some useful formulas:

**Lemma 5.1.2** *(ı) Let $A$ be an associative algebra, let $D$ be a derivation on $A$, let $x$, $y$ and $z$ be elements in $A$ and let $[x, y] = xy - yx$, then we have the following formulas:*

$$D^k[x,y] = \sum_{i=0}^{k} \binom{k}{i} [D^i x, D^{k-i} y],\ \ x^k y = \sum_{i=0}^{k} \binom{k}{i} ((\operatorname{ad} x)^i y) x^{k-i} \text{ and } (\operatorname{ad} x)^k y = \sum_{i=0}^{k} (-1)^i \binom{k}{i} x^{k-i} y x^i.$$

*(ıı) Let $\mathfrak{g}$ be a Lie algebra and $x$, $y$ and $z$ be elements in $\mathfrak{g}$, then we have:*

$$(\operatorname{ad} x)^k [y,z] = \sum_{i=0}^{k} \binom{k}{i} [(\operatorname{ad} x)^i y, (\operatorname{ad} x)^{k-i} z].$$

**Proof :** (ı) We prove the first formula by induction (it is simply the Leibnitz rule. For $k = 0$ this is true and for $k = 1$ it is the formula $D[x, y] = [x, Dy] + [Dx, y]$. Compute

$$D^{k+1}[x, y] = D\left(\sum_{i=0}^{k}\binom{k}{i}[D^i x, D^{k-i}y]\right)$$

and the result follows from the previous formula. For the second formula, consider the operators $L_x$ and $R_x$ of left and right multiplication by $x$. These operators commute and $\mathrm{ad}\, x = L_x - R_x$ or $L_x = \mathrm{ad}\, x + R_x$ so that the three operators commute. Applying the binomial formula to $L_x = \mathrm{ad}\, x + R_x$ yields the first formula while applying it to $\mathrm{ad}\, x = L_x - R_x$ yields the second.

(ıı) The first formula is true in $U(\mathfrak{g})$ where $\mathrm{ad}\, x$ is a derivation. Applying it to the adjoint representation gives the result. □

**Corollary 5.1.3** *Let $T$ and $S$ be two endomorphisms of $V$ and assume that $T$ is locally finite and that $\{(\mathrm{ad}\, T)^k S \ ;\ k \in \mathbb{N}\}$ spans a finite-dimensional subspace of $\mathrm{End}(V)$, then we have the formula:*

$$(\exp T)S\exp(-T) = \sum_{n\geq 0}\frac{(\mathrm{ad}\, T)^n}{n!}S = (\exp(\mathrm{ad}\, T))(S).$$

**Proof :** The hypothesis on $\mathrm{ad}\, T$ is simply that $\mathrm{ad}\, T$ is locally finite at $S \in \mathrm{End}(V)$. In particular both parts of the equality are well defined. The proof is now a formal computation using the previous Lemma:

$$(\exp T)S\exp(-T) = \sum_{n\geq 0}\frac{T^n S}{n!}\cdot\sum_{n\geq 0}\frac{(-T)^n}{n!} = \sum_{n\geq 0}\sum_{i=0}^{n}(-1)^i\frac{T^{n-i}ST^i}{i!(n-i)!} = \sum_{n\geq 0}\frac{(\mathrm{ad}\, T)^n S}{n!}.$$

□

Let us prove the following Lemmas on finite and nilpotent elements.

**Lemma 5.1.4** *(ı) Let $\mathfrak{s}$ be a Lie algebra and let $\pi : \mathfrak{s} \to \mathfrak{gl}(V)$ be a representation. Assume that $y \in \mathfrak{s}$ is such that $\mathrm{ad}\, y$ is locally finite (resp. locally nilpotent) on $x \in \mathfrak{s}$, then $\pi(\mathrm{ad}\, y)$ is locally finite (resp. locally nilpotent) on $\pi(x) \in \mathfrak{gl}(V)$.*

*(ıı) Let $T \in \mathrm{End}(V)$ be a locally finite (resp. locally nilpotent element) and let $f : V \to W$ be an isomorphism, then $f \circ T \circ f^{-1}$ is locally finite (resp. locally nilpotent).*

**Proof :** (ı) Let $U$ be a finite dimensional subspace of $\mathfrak{s}$ such that $x \in U$ and $\mathrm{ad}\, y$ stabilizes $U$ (and furthermore $(\mathrm{ad}\, y)|_U$ is nilpotent is the locally nilpotent case). Consider the subspace $\pi(U) \subset V$ which contains $\pi(x)$. It is $\pi(\mathrm{ad}\, y)$ stable (and the restriction is nilpotent in the nilpotent case).

(ıı) For this part simply take $f(U)$. □

**Lemma 5.1.5** *(ı) Let $\mathfrak{s}$ be a Lie algebra and let $x \in \mathfrak{s}$. Define*

$$\mathfrak{s}_x = \{y \in \mathfrak{s} \ /\ (\mathrm{ad}\, x)^{n_y}y = 0 \text{ for some } n_y \in \mathbb{N}\}.$$

*Then $\mathfrak{s}_x$ is a Lie subalgebra of $\mathfrak{s}$.*

*(ıı) Let $\pi : \mathfrak{s} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{s}$ and let $x \in \mathfrak{s}$. Define*

$$V_x = \{v \in V \ /\ \pi(x)^{n_v}v = 0 \text{ for some } n_v \in \mathbb{N}\}.$$

*Then $V_x$ is a $\mathfrak{s}_x$-submodule of $V$.*

*(ııı) Let $\pi : \mathfrak{s} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{s}$ such that $\mathfrak{s}$ is generated as a Lie algebra by the set $F_V = \{x \in \mathfrak{s} \ /\ \mathrm{ad}\, x$ is locally finite on $\mathfrak{s}$ and $\pi(x)$ is locally finite on $V\}$. Then*

- *the Lie algebra $\mathfrak{s}$ is spanned by $F_V$ as a $\mathbb{C}$ vector space. In particular, if $\mathfrak{s}$ is generated as a Lie algebra by the set $F$ of its* ad *-locally finite vectors, then $F$ spans $\mathfrak{s}$ as a vector space.*

- *If* $\dim \mathfrak{s} < \infty$, *then any $v \in V$ lies in a finite-dimensional $\mathfrak{s}$-submodule of $V$.*

**Proof :** The points (ı) and (ıı) follow directly from the formulas 2 and 4 of Lemma 5.1.2.

(ııı) Let $x$ and $y$ in $F_V$ and $t \in C$. Because $\mathrm{ad}\, y$ is locally finite, we may consider $\exp(\mathrm{ad}\, y)$ and even $\exp(t{\cdot}\mathrm{ad}\, y)$. This is an endomorphism of $\mathfrak{s}$ so we may apply it to $x$ and get $(\exp(t{\cdot}\mathrm{ad}\, y))(x) \in \mathfrak{s}$. We want to prove that this element is in $F_V$. For this we want to apply Corollary 5.1.3 to $\pi((\exp(t \cdot \mathrm{ad}\, y))(x))$. But because $\pi$ is a Lie algebra morphism, we have $\pi((\mathrm{ad}\, y)^n x) = (\mathrm{ad}\, \pi(y))^n \pi(x)$ and because of Lemma 5.1.4 we have $\pi(\exp(\mathrm{ad}\, y))(x)) = (\exp(\mathrm{ad}\, \pi(y)))(\pi(x))$. But $\pi(y)$ is locally finite and $\mathrm{ad}\, \pi(y)$ is locally finite on $\pi(x)$ so we may apply Corollary 5.1.3 to get

$$\pi(\exp(\mathrm{ad}\, y))(x)) = (\exp(\pi(y)))\pi(x)(\exp(-\pi(y))).$$

Because $\pi(x)$ is locally finite, this proves thanks to Lemma 5.1.4 (ıı) that $\pi(\exp(\mathrm{ad}\, y))(x))$ is locally finite. The same proof shows that ad $(\exp(\mathrm{ad}\, y))(x))$is locally finite and that $\exp(\mathrm{ad}\, y))(x)) \in F_V$. Now we have the formula

$$\lim_{t\to 0} \frac{(\exp(t\mathrm{ad}\, y))x - x}{t} = [y, x].$$

Proving that the linear span of elements in $F_V$ is a Lie subalgebra of $\mathfrak{s}$.

The last result follows from the previous one and Poincaré-Birkhoff-Witt Theorem. □

Let us now prove the following characterizations of locally nilpotent elements.

**Lemma 5.1.6** *(ı) Let $y_1, y_2, \cdots$ be a system of generators of a Lie algebra $\mathfrak{g}$ and let $x \in \mathfrak{g}$ such that $(\mathrm{ad}\, x)^{N_i} y_i = 0$ for some positive integers $N_i$. Then* $\mathrm{ad}\, x$ *is locally nilpotent on $\mathfrak{g}$.*

*(ıı) Let $v_1, v_2, \cdots$ be a system of generators of a $\mathfrak{g}$-module $V$ and let $x \in \mathfrak{g}$ be such that* $\mathrm{ad}\, x$ *is locally nilpotent on $\mathfrak{g}$ and $x^{N_i}(v_i) = 0$ for some positive integer $N_i$. Then $x$ is nilpotent on $V$.*

**Proof :** (ı) If an element is locally nilpotent on a vector space basis, then it is locally nilpotent on the space. This together with formula (ıı) of the Lemma 5.1.2 concludes the proof.

(ıı) To prove this result, we need to prove that a power of $x$ will kill an element of the form $y_1 \cdots y_s(v)$ where $y_i \in \mathfrak{g}$ and $v \in V$. We apply the second formula of Lemma 5.1.2 (ı) to $x^k y_1 \cdots y_k$ in $U(\mathfrak{g})$ and get the result. □

**Corollary 5.1.7** *The elements* $\mathrm{ad}\, e_i$ *and* $\mathrm{ad}\, f_i$ *and locally nilpotent on $\mathfrak{g}(A)$.*

**Proof :** We give two proofs.

We have $\mathrm{ad}\, e_i(f_j) = 0$ and $(\mathrm{ad}\, e_i)^{1-a_{i,j}}(e_j) = 0$ for $i \neq j$. Furthermore, $\mathrm{ad}\, e_i(e_i) = 0$, $(\mathrm{ad}\, e_i)^2(h) = 0$ for any $h \in \mathfrak{h}$. Indeed, we have $(\mathrm{ad}\, e_i)^2(h) = -\langle\alpha_i, h\rangle \mathrm{ad}\, e_i(e_i) = 0$. Finally, we have $(\mathrm{ad}\, e_i)^3(f_i) = 0$ and the result follows from Lemma 5.1.6 (ı). The same proof works for $\mathrm{ad}\, f_i$.

Second proof. Let $x = e_i$ and consider $\mathfrak{g}(A)_x = \{y \in \mathfrak{g}(A) \ / \ (\mathrm{ad}\, x)^{n_y} y = 0$ for some $n_y \in \mathbb{N}\}$ as in Lemma 5.1.5. We know that $\mathfrak{g}(A)_x$ is a Lie subalgebra and because of the defining relations of $\mathfrak{g}(A)$ and of Proposition 4.2.7, we know that $\mathfrak{h}$ and all the $e_j$ and the $f_j$ are in $\mathfrak{g}(A)_x$ so that $\mathfrak{g}(A)_x = \mathfrak{g}(A)$. □

## 5.2 Integrable representations and Weyl group

### 5.2.1 Integrable representations

**Definition 5.2.1** (i) A $\mathfrak{g}(A)$-module $V$ is $\mathfrak{h}$-diagonalisable (sometimes also called a weight module) if there is a decomposition $V = \oplus_{\lambda\in\mathfrak{h}^*} V_\alpha$ where $V_\lambda = \{v \in V \ / \ h(v) = \langle \lambda, h\rangle v \ \forall h \in \mathfrak{h}\}$. The subspace $V_\lambda$ is called a weight space, $\lambda$ is called a weight if $V_\lambda \neq 0$ and $\dim V_\lambda$ is the multiplicity of $\lambda$ denoted by $\mathrm{mult}_V \lambda$.

(ii) An $\mathfrak{h}$-diagonalisable module $V$ over $\mathfrak{g}(A)$ is called integrable if $e_i$ and $f_i$ are locally nilpotent on $V$ for all $i$ in $[1, n]$.

Let us give a Proposition explaining the terminology of integrable representations without proof (this should be proved in chapter 11).

**Proposition 5.2.2** *Let $V$ be an integral representation of $\mathfrak{g}(A)$ and let $g_{(i)}$ be the $\mathfrak{sl}_2$ isomorphic Lie subalgebra of $\mathfrak{g}(A)$ defined by $e_i$, $f_i$ and $\alpha_i^\vee$. Then $V$ decomposes as a direct sum of finite dimensional irreducible $\mathfrak{h}$-invariant modules and in particular the action of $\mathfrak{g}_{(i)}$ can be "integrated" to an action of $SL_2(\mathbb{C})$.*

**Proof :** We only need to apply Lemma 5.1.5. Because $\mathfrak{g}_{(i)}$ is finite dimensional, we know that any element $v \in V$ sits in a $\mathfrak{g}_{(i)}$-stable finite dimensional subspace of $V$. These finite dimensional subspaces are integrable. □

**Proposition 5.2.3** *The adjoint representation of $\mathfrak{g}(A)$ is integrable.*

**Proof :** We already know that the adjoint representation is $\mathfrak{h}$-diagonalisable and the rest follows from Corollary 5.1.7. □

### 5.2.2 Definition of the Weyl group and action on integrable representations

**Definition 5.2.4** (i) For any $i \in [1, n]$, define the reflection $s_i \in \mathrm{Aut}\mathfrak{h}^*$ by $s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee\rangle \alpha_i$, for $\lambda \in \mathfrak{h}^*$. It is the reflection with respect to the hyperplane $\{\lambda \in \mathfrak{h}^* \ / \ \langle \lambda, \alpha_i^\vee\rangle = 0\}$. In particular we have $s_i^2 = 1$.

(ii) Let $W$ the subgroup of $\mathrm{Aut}\mathfrak{h}^*$ generated by the reflections $s_i$ for $i \in [1, n]$. This group is the Weyl group of the Kac-Moody Lie algebra $\mathfrak{g}(A)$ and the reflections $s_i$ are called simple reflections. The faithful representation of $W$ in $\mathfrak{h}^*$ is called the standard representation of $W$.

(iii) Dualizing the representation, we get the congradient representation $W \subset \mathrm{Aut}\mathfrak{h}$, which is explicitly given for any $1 \leq i \leq n$ by $s_i(h) = h - \langle \alpha_i, h\rangle \alpha_i^\vee$, for $h \in \mathfrak{h}$.

(iv) The length of an element $w \in W$ is the smallest $k$ such that we can write $w = s_{i_1} \cdots s_{i_k}$ with the $s_{i_j}$ simple reflections.

**Definition 5.2.5** Let $\pi : \mathfrak{g}(A) \to \mathfrak{gl}(V)$ be an integrable representation of $\mathfrak{g}(A)$. We may define the following element $s_i(\pi) \in \mathrm{End}(V)$ by

$$s_i(\pi) = (\exp \pi(f_i))(exp(-\pi(e_i)))(\exp(\pi(f_i))).$$

**Proposition 5.2.6** *Let $\pi : \mathfrak{gl}(A) \to V$ be an integrable representation. Let $\lambda \in \mathfrak{h}^*$ and $s_i \in W$ a simple reflection.*

*(i) We have the equality $s_i(\pi)(V_\lambda) = V_{s_i(\lambda)}$, in particular $\mathrm{mult}_V \lambda = \mathrm{mult}_V w\lambda$ for all $w \in W$.*

*(ii) For any $v \in V$ and $x \in \mathfrak{g}(A)$, we have $s_i(\pi)(xv) = (s_i(\mathrm{ad}\,)x)(s_i(\pi)v)$. In particular $s_i(\mathrm{ad}\,)$ is a Lie algebra automorphism of $\mathfrak{g}(A)$ (preserves the Lie bracket).*

*(iii) For $v \in V_\lambda$, we have $s_i(\pi)^2(v) = (-1)^{\langle\lambda,\alpha_i^\vee\rangle}v$ and if $v \neq 0$, then $\langle\lambda,\alpha_i^\vee\rangle \in \mathbb{Z}$.*

*(iv)Let $m_{i,j}$ be the order of $s_is_j$ in $W$, then if $m_{i,j} < \infty$, we have*

$$\underbrace{s_i(\pi)s_j(\pi)s_i(\pi)\cdots}_{m_{i,j}\ factors} = \underbrace{s_j(\pi)s_i(\pi)s_j(\pi)\cdots}_{m_{i,j}\ factors}$$

**Proof :** (i) Let us prove the inclusion $s_i(\pi)(V_\lambda) \subset V_{s_i(\lambda)}$. A very similar proof gives the inclusion $s_i(\pi)^{-1}(V_\lambda) \subset V_{s_i(\lambda)}$ and the result follows from the fact that $s_i$ is an involution (the element $s_i(\pi)$ is however not an involution as proved in (iii)).

Let $v \in V_\lambda$ and consider the element $h(s_i(\pi)(v))$ in $V$. If $\langle\alpha_i,h\rangle = 0$, then $h$ and $e_i$ commute in $U(\mathfrak{g}(A))$ and also $h$ and $f_i$ commute in $U(\mathfrak{g}(A))$. In particular, we get $h(s_i(\pi)(v)) = s_i(\pi)(h(v)) = \langle\lambda,h\rangle s_i(\pi)(v)$. But in that case we have $s_i(\lambda) = \lambda - \langle\lambda,\alpha_i^\vee\rangle\alpha_i$ and $\langle s_i(\lambda),h\rangle = \langle\lambda,h\rangle$. We thus have the result for the hyperplane of weights orthogonal to $\alpha_i$. To prove the result we only need to prove it on one element not in that orthogonal (by linearity). For example, it is enough to prove it for $h = \alpha_i^\vee$ that is to say it is enough to prove the relation $\alpha_i^\vee(s_i(\pi)v) = -\langle\lambda,\alpha_i^\vee\rangle s_i(\pi)v$ for $v \in V_\lambda$ or $s_i(\pi)^{-1}(\alpha_i^\vee(s_i(\pi)v)) = -\langle\lambda,\alpha_i^\vee\rangle v$ for $v \in V_\lambda$ or the equality

$$s_i(\pi)^{-1}\pi(\alpha_i^\vee)s_i(\pi) = -\pi(\alpha_i^\vee)$$

of elements in $\mathrm{End}(V)$. Using Corollary 5.1.3, we get the equality

$$s_i(\pi)^{-1}\pi(\alpha_i^\vee)s_i(\pi) = \pi((\exp(-\mathrm{ad}\,f_i))(\exp\mathrm{ad}\,e_i)(\exp(-\mathrm{ad}\,f_i))(\alpha_i^\vee)).$$

But because of the defining relations of $\mathfrak{g}(A)$, we get the equalities $(\exp(-\mathrm{ad}\,f_i))(\alpha_i^\vee) = \alpha_i^\vee - 2f_i$, $(\exp(-\mathrm{ad}\,f_i))(f_i) = f_i$, $(\exp\mathrm{ad}\,e_i)(\alpha_i^\vee) = \alpha_i^\vee - 2e_i$ and $(\exp\mathrm{ad}\,e_i)(f_i) = f_i + \alpha_i^\vee - e_i$. We thus get the formula

$$(\exp(-\mathrm{ad}\,f_i))(\exp\mathrm{ad}\,e_i)(\exp(-\mathrm{ad}\,f_i))(\alpha_i^\vee) = -\alpha_i^\vee.$$

(ii) We compute $s_i(\pi)(xv) = s_i(\pi)\pi(x)s_i(\pi)^{-1}s_i(\pi)(v)$ but the composition $s_i(\pi)\pi(x)s_i(\pi)^{-1}$ is equal to $(\exp\pi(f_i))(\exp\pi(-e_i))(\exp\pi(f_i))x(\exp\pi(-f_i))(\exp\pi(e_i))(\exp\pi(-f_i))$ and applying Corollary 5.1.3 we obtain the equality

$$s_i(\pi)\pi(x)\,s_i(\pi)^{-1} = \pi((\exp\mathrm{ad}\,f_i)(\exp(-\mathrm{ad}\,e_i))(\exp\mathrm{ad}\,f_i)x) = \pi(s_i(\mathrm{ad}\,)(x)).$$

(iii) To prove this result, we may assume that $\mathfrak{g}(A) = \mathfrak{g}_{(i)} \simeq \mathfrak{sl}_2$ and by Lemma 5.1.5 we may assume that $V$ is finite dimensional. By Proposition 4.2.6, we obtain that if $v \neq 0$ then $\langle\lambda,\alpha_i^\vee\rangle \in \mathbb{Z}$. Furthermore, we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{sl}_2 & \xrightarrow{\pi} & \mathrm{End}(V) \\ \downarrow{\scriptstyle \mathrm{Exp}} & & \downarrow{\scriptstyle \exp} \\ SL_2 & \longrightarrow & \mathrm{Aut}(V) \end{array}$$

and it is enough to check the relation $(\mathrm{Exp(f)Exp(-e)Exp(f)})^2 = -1 = \mathrm{Exp}(i\pi h)$ in $SL_2$. This is an easy calculation. Now we compute $\mathrm{Exp}(i\pi h)(v) = \exp(i\pi\langle\lambda,h\rangle)v = (-1)^{\langle\lambda,h\rangle}v$.

(iv) To prove this part, we may assume that $A$ is a $2\times 2$ matrix of the from

$$A = \begin{pmatrix} 2 & a_{i,j} \\ a_{j,i} & 2 \end{pmatrix}.$$

Let us first prove that we may assume that $a = a_{i,j}a_{j,i}$ is an integer in the interval $[0, 3]$. For this we prove that if $a \geq 4$, then the order $m_{i,j}$ of $s_i s_j$ is infinite. We know that both $a_{i,j}$ and $a_{j,i}$ are non positive so $a$ is non negative. Consider the 2-dimensional subspace $U$ of $\mathfrak{h}^*$ generated by $\alpha_i$ and $\alpha_j$. The matrix of $s_i$ (resp. $s_j$) in the basis $\{\alpha_i, \alpha_j\}$ is given by

$$\left(\begin{array}{cc} -1 & -a_{i,j} \\ 0 & 1 \end{array}\right) \ \left(\text{resp. } \left(\begin{array}{cc} 1 & 0 \\ -a_{j,i} & -1 \end{array}\right)\right).$$

The composition $s_i s_j$ is given by the matrix

$$\left(\begin{array}{cc} -1+a & a_{i,j} \\ -a_{j,i} & -1 \end{array}\right).$$

The eigenvalues of this matrix are roots of the polynomial $X^2 + (2 - a)X + 1$ and thus given by $\dfrac{a - 2 \pm \sqrt{a(a-4)}}{2}$. If $a > 5$, then one eigenvalue is bigger than 1 and the composition $s_i s_j$ has to be of infinite order. If $a = 4$, then the eigenvalues are equal to 1. But as the composition is not identity, it has to be of infinite order.

Now we may assume that $a = 0, 1, 2$ or 3. This implies that $\mathfrak{g}(A)$ is of type $A_1 \times A_1$, $A_2$, $B_2$ or $G_2$ (because Serre relations are satisfied and because there is no non trivial ideal meeting $\mathfrak{h}$ trivially in such Lie algebras). By Lemma 5.1.5 we may assume that $V$ is finite dimensional. We have the commutative diagram

$$\begin{array}{ccc} \mathfrak{g}(A) & \xrightarrow{\pi} & \mathrm{End}(V) \\ \downarrow{\scriptstyle \mathrm{Exp}} & & \downarrow{\scriptstyle \exp} \\ G(A) & \longrightarrow & \mathrm{Aut}(V) \end{array}$$

where $G(A)$ is the simply-connected group associated to $\mathfrak{g}(A)$. We need to prove the following relation $S_i S_j S_i \cdots = S_j S_i S_j \cdots$ with $m_{i,j}$ terms and where $S_i = \mathrm{Exp}(f_i)\mathrm{Exp}(-e_i)\mathrm{Exp}(f_i)$ (similar definition for $S_j$). This can be proved case-by-case (see for example [Sp98, Proposition 9.3.2]). □

**Corollary 5.2.7** *(ı) For any simple reflection $s_i$, we have $s_i(\mathrm{ad}\,)|_{\mathfrak{h}} = s_i$ as an automorphism of $\mathfrak{h}$.*

*(ıı) Assume that for some $i$ and $j$ in $[1, n]$ and for $w \in W$ we have $\alpha_j = w(\alpha_i)$, then $w(\alpha_i^\vee) = \alpha_j^\vee$.*

**Proof :** (ı) Take any integrable representation $(V, \pi)$ of $\mathfrak{g}(A)$ and let $v \in V$ of weight $\lambda$ and $x \in \mathfrak{h}$. We have by the previous Proposition:

$$\lambda(x)s_i(\pi)(v) = s_i(\pi)(xv) = (s_i(\mathrm{ad}\,)(x))(s_i(\pi)(v)) = ((s_i\lambda)(s_i(\mathrm{ad}\,)(x)))(s_i(\pi)(v))$$

hence $\lambda(x) = (s_i\lambda)(s_i(\mathrm{ad}\,)(x))$. Replacing $\lambda$ by $s_i\lambda$ we get that $(s_i\lambda)(x) = \lambda(s_i(x)) = \lambda(s_i(\mathrm{ad}\,)(x))$. This is true for any weight $\lambda$ so that $s_i(x) = s_i(\mathrm{ad}\,)(x)$ and we have the result (we shall see in Chapter 10 that the linear span of the weights of integrable representations is $\mathfrak{h}^*$).

We may give a direct proof of this result by simple computation, indeed, we want to compute $s_i(\mathrm{ad}\,)(h) = \exp(\mathrm{ad}\,(f_i))\exp(-\mathrm{ad}\,(e_i))\exp(\mathrm{ad}\,(f_i))(h)$ for $h \in \mathfrak{h}$. But we have the formulas: $\exp(\mathrm{ad}\,(f_i))(h) = h + \langle h, \alpha_i\rangle f_i$, $\exp(-\mathrm{ad}\,(e_i))(h) = h + \langle h, \alpha_i\rangle e_i$, $\exp(-\mathrm{ad}\,(e_i))(f_i) = f_i + \alpha_i^\vee - e_i$ and $\exp(\mathrm{ad}\,(f_i))(f_i) = f_i$. This gives the formula $s_i(\mathrm{ad}\,)(h) = h - \langle h, \alpha_i\rangle\alpha_i^\vee$.

(ıı) We know from (ı) that there exists a Lie algebra automorphism $\widehat{w} : \mathfrak{g}(A) \to \mathfrak{g}(A)$ such that $\widehat{w}|_{\mathfrak{h}} = w$. In particular, we have $[\widehat{w}(e_i), \widehat{w}(f_i)] = \widehat{w}[e_i, f_i] = w(\alpha_i^\vee)$. But because $w(\alpha_i) = \alpha_j$ and by the previous Proposition, we have that $\widehat{w}(e_i) \in \mathfrak{g}_{\alpha_j}$ and $\widehat{w}(f_i) \in \mathfrak{g}_{\alpha_j}$ thus $[\widehat{w}(e_i), \widehat{w}(f_i)] = c\alpha_j^\vee$ for some $c \in \mathbb{C}$. We thus have $w(\alpha_i^\vee) = c\alpha_j^\vee$. Apply $\alpha_j$ to this equality to get $\alpha_j(w(\alpha_i^\vee)) = (w^{-1}(\alpha_j)(\alpha_i(^\vee)) = \alpha_i(\alpha_i^\vee) = 2$ and $\alpha_j(c\alpha_j^\vee) = 2c$ thus $c = 1$ and we are done. □

**Definition 5.2.8** Let $\alpha$ be a root such that $\alpha = w(\alpha_i)$ for some $w$ in the Weyl group and some index $i \in [1, n]$. Such roots will be called real roots (see Chapter 9). We may define the **coroot of** $\alpha$, denoted $\alpha^\vee$ by $w(\alpha_i^\vee)$. The previous Corollary implies that this is well defined.

## 5.3 Using integrable representations to construct groups

In this section we explain how to construct some group associated to the Lie algebra $\mathfrak{g}(A)$. These groups will be studied more in details later on but we can already introduce them by using integrable representations.

**Definition 5.3.1** Let $\mathfrak{g}(A)$ be a Kac-Moody Lie algebra. We know that there exists a $\mathbb{C}$-basis of $\mathfrak{g}(A)$ consisting of locally finite elements (and even of locally nilpotent elements). We define the group $G^*$ to be the free group generated by the set of locally finite elements in $\mathfrak{g}(A)$. For any integrable representation $\pi : \mathfrak{g}(A) \to \mathfrak{gl}(V)$, we define a representation $I(\pi)$ of $G^*$ on $V$ by

$$I(\pi)(x) = \sum_{n \geq 0} \frac{\pi(x)^n}{n!},$$

for $x$ in the generating set of $G$ i.e. locally finite in $\mathfrak{g}(A)$. This exponential exists because of Lemma 5.1.5. Now define $N$ to be the intersection of all $\ker(I(\pi))$ and define $G$ by $G = G^*/N$. This group is called **the group associated to the Lie algebra** $\mathfrak{g}(A)$.

**Proposition 5.3.2** *If $\mathfrak{g}(A)$ is a finite dimensional simple Lie algebra, then $G$ is the semisimple simply connected group associated to $\mathfrak{g}(A)$.*

**Proof :** Let $V$ be an integrable representation. Then thanks to Lemma 5.1.5, we may decompose $V$ into sums of finite dimensional representations and we may construct $G$ using only finite dimensional representations. Now it is clear that for any finite dimensional representation $\pi$, the map $I(\pi) : G^* \to GL(V)$ factors through the simply connected group $G^{sc}$ associated to $\mathfrak{g}(A)$ (see Chapter 2). In particular we may define a map $G^* \to G^{sc}$ whose kernel is contained in all $\ker I(\pi)$. We thus have a map $G \to G^{sc}$. To conclude, it is enough to show that for any element $\exp(x) \in G^{sc}$ there exists a representation on which it acts non trivially. This is true because — for example — $G^{sc}$ is reductive and affine. □

We may construct another group associated to the Lie algebra $\mathfrak{g}(A)$ in the following way. Let $V$ be an integrable representation whose kernel lies in $\mathfrak{h}$ (for example take $V = \mathfrak{g}(A)$ the adjoint representation, whose kernel is the center $\mathfrak{c} \subset \mathfrak{h}'$). This induces a representation of the $\mathfrak{sl}_2$-isomorphic Lie algebra $\mathfrak{g}_{(i)}$ generated by $e_i$ and $f_i$. Furthermore this representation is a sum of finite dimensional representations and thus can be integrated in a representation $\pi_i : SL_2 \to GL(V)$. We may also integrate the action of $\mathfrak{h}$ to an action of an abelian group $T \simeq (\mathbb{C}^*)^{2n-\ell}$ of the same dimension. We define the group $G^\pi$ generated by the images of the $\pi_i$ for $i \in [1, n]$ and by the image of $T$.

**Proposition 5.3.3** *The group $G$ is a central extension of $G^{ad}$.*

**Proof :** This is an easy application of the formula

$$\exp(x)\exp(y)\exp(-x)\exp(-y) = \exp([x, y]).$$

Indeed, this formula shows that the image of $G^*$ in $GL(\mathfrak{g}(A))$ for the adjoint representation is the group $G^{ad}$. Furthermore, the kernel of this representation is the center of $\mathfrak{g}(A)$ an this implies that the kernel of $G \to G^{ad}$ is given by central elements (exponentials of elements in $\mathfrak{c}$): let us start with an element $x \in \mathfrak{g}_\alpha$, then we have $\exp(x)(h) = h + \langle \alpha, h \rangle x$ for any $h \in \mathfrak{h}$. If $\alpha \neq 0$, the endomorphism $\exp(x)$ is never the identity expect if $x = 0$. If $x = x_1 + \cdots + x_k$ with $x_i \in \mathfrak{g}_{\beta_i}$ and $\beta_i \in \Delta \cup \{0\}$, then chose $h \in \mathfrak{h}$ such that $\langle h, \beta_i \rangle = 0$ for $i \neq 1$ and $\langle h, \beta_1 \rangle \neq 0$. Then we have $\exp(x)(h) = \exp(x_1) \cdots \exp(x_k)(h) = \exp(x_1)(h) = h + \langle h, \beta_1 \rangle x_1$ and if $\exp(x) = \mathrm{Id}$ then $x_1 = 0$ and even $x \in \mathfrak{h}$. Now if $h \in \mathfrak{h}$, we have $\exp(h)(e_i) = \exp(\langle h, \alpha_i \rangle) e_i$ and $\exp(h)(e_i) = e_i$ if and only if $\langle h, \alpha_i \rangle \in 2\pi\mathbb{Z}$. This proves the result. $\square$

# Chapter 6

# Coxeter groups

In this chapter we give a quick review of Coxeter groups and prove that the Weyl group of a Kac-Moody Lie algebra is a Coxeter group and even a crystallographic group. We will always assume that the generating set $S$ is finite. The basic references are [Hu90] and [Bo54].

## 6.1 Definition

### 6.1.1 Coxeter systems

**Definition 6.1.1** A **Coxeter system** is a pair $(W, S)$ where $W$ is a group and $S \subset W$ is a set of generators of $W$ satisfying relations of the form $(ss')^{m(s,s')} = 1$ with $m(s, s) = 1$ and $m(s, s') \geq 2$ for $s \neq s'$. If there is no relation between $s$ and $S'$ then $m(s, s') = \infty$.

The number $|S|$ is called the **rank** of the Coxeter system. The group $W$ is called a **Coxeter group**. The elements of $S$ are called the **simple reflections**.

**Definition 6.1.2** To a Coxeter system $(W, S)$ we may associate a graph $\Gamma$ called the **Coxeter graph** as follows: the vertices of $\Gamma$ are in bijection with $S$ and with an edge labeled by $m(s, s')$ between the vertices $s$ and $s'$ whenever $m(s, s') \geq 3$. When $m(s, s') = 3$ we omit the label on the edge. The Coxeter graph determines the Coxeter group.

**Example 6.1.3** *Let* $W = \mathfrak{S}_3$ *the permutation group on the set* $\{1, 2, 3\}$ *and let* $S \in W$ *be the set of transpositions* $\{(1, 2); (2, 3)\}$. *Then* $(W, S)$ *is a Coxeter system whose graph is the following*

$$\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$$

### 6.1.2 Length function

**Definition 6.1.4** Let $(W, S)$ be a Coxeter system, we define a function $\ell : W \to \mathbb{N}$ called the **length function**. If $w \in W$ then we set $\ell(w) = n$ where $n$ is the smallest integer such that $w$ can be written as a product of $n$ elements of $S$. An expression $w = s_1 \cdots s_n$ with $s_i \in S$ and $n = \ell(w)$ is called a **reduced expression**. Note that $\ell(1) = 0$.

**Lemma 6.1.5** *The length satisfies the following properties:*

*(i)* $\ell(w) = \ell(w^{-1})$ *for all* $w \in W$

*(ii)* $\ell(w) = 1$ *if and only if* $w \in S$.

*(iii)* $\ell(w) - \ell(w') \leq \ell(ww') \leq \ell(w) + \ell(w')$ *for all* $(w, w') \in W^2$.

**Proof :** For (ı) use the fact that if $w = s_1 \cdots s_n$ then $w^{-1} = s_n \cdots s_1$. Point (ıı) is clear. For (ııı), if $w = s_1 \cdots s_n$ and $w' = s'_1 \cdots s'_m$ are reduced, then $ww' = s_1 \cdots s_n s'_1 \cdots s'_m$ and the second inequality follows. Forthe first one, we have $\ell(w) = \ell(ww'w'^{-1}) \leq \ell(ww') + \ell(w')$ and the result follows. □

**Proposition 6.1.6** *There is a unique group morphism $\varepsilon : W \to \{\pm 1\}$ sending $s$ to $-1$. Furthermore, we have $\varepsilon(w) = (-1)^{\ell(w)}$ for all $w \in W$.*

**Proof :** To prove that the morphism does exist, we only need to prove that the defining equations of $W$ are satisfies but we have $(\varepsilon(s)\varepsilon(s'))^{m(s,s')} = 1$ for all $s$ and $s'$ in $S$. Now let $w = s_1 \cdots s_{\ell(w)}$ be a reduced expression, we have $\varepsilon(w) = (-1)^{\ell(w)}$. □

**Corollary 6.1.7** *For all $w \in W$ and all $s \in S$, we have $\ell(ws) = \ell(w) \pm 1$ and the same for $\ell(sw)$.*

**Proof :** We already know from the previous Lemma that $\ell(w) - 1 \leq \ell(ws) \leq \ell(w) + 1$. But $\varepsilon(ws) = -\varepsilon(w)$ so that $\ell(ws) - \ell(w) \equiv 1 \bmod 2$. This gives the result.

The same proof works for $sw$. □

## 6.2 Geometric representation

In this section, we give a geometric interpretation of Coxeter groups as groups of hyperplane reflections. Let $(W, S)$ be a Coxeter system and let $V$ be an $\mathbb{R}$-vector space of dimension the rank of the Coxeter group (i.e. $|S|$) and with a fixed basis $(\alpha_s)_{s\in S}$. We define a bilinear form on $V$.

**Definition 6.2.1** (ı) The bilinear form $B$ on $V$ is defined by

$$B(\alpha_s, \alpha_{s'}) = -\cos \frac{\pi}{m(s, s')}$$

with the convention that it equal $-1$ when $m(s, s') = \infty$.

(ıı) Let $s \in W$, we define $\sigma_s : V \to V$ by $\sigma_s(v) = v - 2B(\alpha_s, v)\alpha_s$. We have $\sigma_s(\alpha_s) = -\alpha_s$ and $\sigma_s$ fixes $H_s = \{v \in V \ / \ B(\alpha_s, v) = 0\}$ pointwise.

**Proposition 6.2.2** *(ı) The bilinear form $B$ is invariant under $\sigma_s$*

*(ıı) There is a unique group morphism from $W$ to $O(V, B)$ the subgroup of $GL(V)$ preserving $B$ such that the image of any $s \in S$ is $\sigma_s$.*

**Proof :** (ı) We compute $B(\sigma_s(u), \sigma_s(v)) = B(u, v) - 4B(u, \alpha_s)B(\alpha_s, v) + 4B(u, \alpha_s)B(v, \alpha_s)B(\alpha_s, \alpha_s)$ and the result follows because $B(\alpha_s, \alpha_s) = 1$.

(ıı) Let us first consider the two dimensional subspace $V_{s,s'}$ of $V$ generated by $\alpha_s$ and $\alpha_{s'}$. The restriction of $B$ to $V_{s,s'}$ is positive and non degenerated if and only if $m(s, s') < \infty$. Indeed, for $v = a\alpha_s + b\alpha_{s'}$, we have the formula

$$B(v, v) = \left(a - b\cos\frac{\pi}{m}\right)^2 + \left(b\sin\frac{\pi}{m}\right)^2$$

where $m = m(s, s')$.

Now remark that $\sigma_s$ and $\sigma_{s'}$ leave the space $V_{s,s'}$ stable and we compute the order of the restriction of $\sigma_s\sigma_{s'}$ on this subspace.

In the case $m < \infty$ then $B$ gives $V_{s,s'}$ a structure of Euclidian subspace and $\sigma_s$ and $\sigma_{s'}$ act as orthogonal reflections. The composition is therefore a rotation. Because of the defining relation

$B(\alpha_s, \alpha_{s'}) = -\cos(\pi/m)$, the rays generated by $\alpha_s$ and $\alpha_{s'}$ are separated by an angle of $\pi - \pi/m$ so that the angle of the rotation is $\pi/m$ and the order of $\sigma_s\sigma_{s'}$ restricted to $V_{s,s'}$ is $m$.

But because $B$ is non degenerate on $V_{s,s'}$, we have $V = V_{s,s'} \oplus V_{s,s'}^{\perp}$. But $\sigma_s\sigma_{s'}$ acts trivially on $V_{s,s'}^{\perp}$ so that the order of $\sigma_s\sigma_{s'}$ is $m$.

In the case $m = \infty$, let $u = \alpha_s + \alpha_{s'}$. We have $B(\alpha_s, u) = B(\alpha_{s'}, u) = 0$ so that $\sigma_s(u) = \sigma_{s'}(u) = u$ and $(\sigma_s\sigma_{s'})^n(\alpha_s) = \alpha_s + 2nu$ and $\sigma_s\sigma_{s'}$ is of infinite order.

In all cases we have $\sigma_s\sigma_{s'}$ is of order $m(s, s')$ and there is a unique group morphism $W \to O(V, B)$ sending $s$ ti $\sigma_s$. □

**Corollary 6.2.3** *For any elements $s$ and $s'$ in $S$, the order of $ss'$ is exactly $m(s, s')$. In particular all the elements of $S$ are distinct.*

**Proof :** This follows from the proof of the previous Proposition. Indeed, the image $\sigma_s\sigma_{s'}$ of $ss'$ has order $m(s, s')$. □

### 6.2.1 Root system

We want to study the group $W$ thanks to the geometric representation. In particular, the length will have a nice geometric interpretation. But before doing this, we need to prove that the geometric representation $W \to O(V, B)$ is injective.

**Definition 6.2.4** The **root system** of $W$ associated to the geometric representation $V$ is the set $\Delta$ of all vectors of the form $w(\alpha_s)$ for $w \in W$ and $s \in S$. Any root $\alpha \in \Delta$ can be uniquely written in the form

$$\alpha = \sum_{s \in S} a_s \alpha_s$$

with $a_s \in \mathbb{R}$. We call positive (resp. negative) roots the roots $\alpha$ such that for all $s \in S$ we have $a_s \geq 0$ (resp. $a_s \leq 0$). The set of positive (resp. negative) roots will be denoted $\Delta_+$ (resp. $\Delta_-$).

**Remark 6.2.5** (i) All vectors of $\Delta$ are unit vectors (i.e. $B(\alpha, \alpha) = 1$ for $\alpha \in \Delta$).

(ii) Because $\sigma_s(\alpha_s) = -\alpha_s$, we have $\Delta = -\Delta$.

**Definition 6.2.6** Let $I$ be a subset of $S$. We define the **parabolic subgroup** $W_I$ of $W$ to be the subgroup of $W$ generated by the simple reflections $s \in I$. This group has a length function $\ell_I$ defined by the set $I$ of generators. We have $\ell|_{W_I} \leq \ell_I$ (we will see that there is equality).

**Theorem 6.2.7 (Tits Theorem)** *Let $w \in W$ and $s \in S$. If $\ell(ws) > \ell(w)$, then $w(\alpha_s) > 0$. If $\ell(ws) < \ell(w)$, then $w(\alpha_s) < 0$.*

**Proof :** The second assertion follows from the first one applied to $ws$ in place of $w$.

We proceed by induction on $\ell(w)$. It is clear if $w = 1$. If $\ell(w) > 0$, then there exist $s' \in S$ such that $\ell(ws') < \ell(w)$ (take a reduced expression $w = s_1 \cdots s_n$ and set $s' = s_n$). We then have $\ell(ws') = \ell(w) - 1$. Because $\ell(ws) > \ell(w)$, we have $s \neq s'$.

Let $I = \{s, s'\}$. The idea is to try to translate the problem on a Coxeter group with only two element $s$ and $s'$. Consider the set

$$A = \{v \in W \ / \ v^{-1}w \in W_I \text{ and } \ell(v) + \ell_I(v^{-1}w) = \ell(w)\}.$$

This is a particular choice of elements in the coset $wW_I$. We have $w \in A$. Choose $v \in A$ with minimal length and write $w = vv_I$ with $v_I \in W_I$ and $\ell(w) = \ell(v) + \ell_I(v_I)$. We have $ws' \in A$, indeed

$(ws')^{-1}w = s' \in W_I$ and $\ell(w) = \ell(ws') + 1 = \ell(ws') + \ell_I(s')$. In particular $\ell(v) \leq \ell(ws') = \ell(w) - 1$ and we could apply induction on $v$ if we prove that $\ell(vs) > \ell(v)$.

Assume that $\ell(vs) < \ell(v)$, then we have $\ell(vs) \leq \ell(v) - 1$ and the inequalities

$$\begin{aligned}\ell(w) &\leq \ell(vs) + \ell(sv^{-1}w) \\ &\leq \ell(vs) + \ell_I(sv^{-1}w) \\ &\leq \ell(v) - 1 + \ell_I(v^{-1}w) + 1 \\ &\leq \ell(v) + \ell_I(v^{-1}w) \\ &= \ell(w).\end{aligned}$$

In particular, there is equality in all inequalities and $\ell(w) = \ell(vs) + \ell_I((vs)^{-1}w)$ thus $vs \in A$ but $\ell(vs) < \ell(v)$ contradicts the minimality of $\ell(v)$. We thus have $\ell(vs) > \ell(v)$ and we may apply induction hypothesis to get that $v(\alpha_s) > 0$. The same proof shows that $v(\alpha_{s'}) > 0$.

Now, because of the equality $w = vv_I$, we will be done if we prove that $v_I(\alpha_s) = a\alpha_s + b\alpha_{s'}$ with $a$ and $b$ non negative. Remark that we have $\ell_I(v_Is) \geq \ell_I(v_I)$. Indeed, otherwise we would have $\ell(ws) = \ell(vv_Is) \leq \ell(v) + \ell(v_Is) \leq \ell(v) + \ell_I(v_Is) \leq \ell(v) + \ell_I(v_I) - 1 = \ell(w) - 1$ a contradiction. This implies that a reduced writing of $v_I$ must finish with $s'$. In particular $v_I$ can be written as $v_I = (ss')^a$ or $v_I = s'(ss')^b$. Furthermore, since $(ss')^m = 1$ with $m = m(s, s')$, we may assume that $a < m/2$ and $b \leq [m/2] - 1$.

In the case $m < \infty$, we are on the Euclidian plane with two reflections whose axes form a $\pi/m$ angle. The cone generated by $\alpha_s$ and $\alpha_{s'}$ has an angle of $\pi - \pi/m$. The element $v_I$ is either a rotation of angle $2a\pi/m$ or the composition of $s'$ with a rotation of angle $2b\pi/m$. The image of $\alpha_s$ by such an isometry is always in the cone and we are done.

In the case $m = \infty$, we already did the computation of $(ss')^a(\alpha_s) = \alpha_s + 2au$ where $u = \alpha_s + \alpha_{s'}$. This implies the result for $v_I$ of that form. But now $s'(ss')^b(\alpha_s) = s'(\alpha_s + 2bu) = \alpha_s + (2b+1)u$ and we are done. □

**Corollary 6.2.8** *Any root $\alpha \in \Delta$ is either positive or negative.*

### 6.2.2 Geometric interpretation of length function

We will now describe a geometric way of thinking to the length function. Recall that we denoted by $\Delta$, $\Delta_+$ and $\Delta_-$ the sets of roots, positive roots and negative roots.

**Proposition 6.2.9** *(ı) Let $s \in S$, then $\sigma_s(\alpha_s) = -\alpha_s$ and for any $\alpha \in \Delta_+ \setminus \{\alpha_s\}$ we have $\sigma_s(\alpha) \in \Delta_+ \setminus \{\alpha_s\}$.*

*(ıı) For any $w \in W$ the length $\ell(w)$ is the number of positive roots $\alpha$ such that $w(\alpha)$ is a negative root.*

**Proof :** (ı) We already know that $\sigma_s(\alpha_s) = -\alpha_s$. Let $\alpha$ be a positive root distinct from $\alpha_s$. We have $\alpha = \sum_{u\in S} a_u\alpha_u$ with $a_u > 0$ for some $u \neq s$. But $\sigma_s(\alpha) = (a_s - 2B(\alpha_s, \alpha))\alpha_s + \sum_{u\neq s} a_u\alpha_u$ and has a positive coefficient on some $\alpha_u$. But $\sigma_s(\alpha)$ is a root and thus is positive.

(ıı) Let $n(w)$ be the number of positive roots send to a negative one. We have the equality $n(w) = |\Delta_+ \cap (w^{-1}(\Delta_-))|$. We proceed by induction on $\ell(w)$. The case $\ell(w) = 0$ is trivial and the case $\ell(w) = 1$ is given by (ı). Consider $s \in S$ such that $\ell(ws) = \ell(w) - 1$, then $\ell(ws)$ is $n(ws)$. But the condition $\ell(ws) < \ell(w)$ implies that $w(\alpha_s) < 0$. The set $\Delta_+$ of positive roots is send by $\sigma_s$ to $\Delta_+ \setminus \{\alpha_s\} \cup \{-\alpha_s\}$. In particular $ws(\alpha_s) = -w(\alpha_s) > 0$ and the positive roots send to negative roots by $ws$ are in $\Delta_+ \setminus \{\alpha_s\}$. Because $\sigma_s$ permutes these roots, they are the same number $n(ws) = \ell(w) - 1$ of roots in $\Delta_+ \setminus \{\alpha_s\}$ sent to a negative one by $w$. But $\alpha_s$ is send to a negative roots by $w$ thus $n(w) = n(ws) + 1 = \ell(w)$ □

**Corollary 6.2.10** *If $W$ is infinite, then $\Delta$ is infinite.*

**Proof :** Indeed, let us consider all elements of fixed length $n$. Because $S$ is finite, this is a finite set. This implies that the length of elements in $W$ can be arbitrary large and implies that there are infinitely many roots. □

Let us finish this section with a criterion on the form $B$ for the Coxeter group $W$ to be finite.

**Proposition 6.2.11** *Assume $W$ is irreducible i.e. its Coxeter graph is connected. Let $V^0$ be the subspace of $V$ orthogonal to $V$ with respect to $B$ (the kernel of $B$).*

*(ı) The Coxeter group $W$ acts trivialy on $V^0$ and any subspace of $V$ stable under $W$ is contained in $V^0$.*

*(ıı) If $W$ is finite, then $B$ is positive definite and any $W$-invariant bilinear form is a scalar multiple of $B$. Furthermore if $u \in \mathrm{End}(V)$ commutes with any element of $W$ then $u$ is a homothetie.*

*(ııı) The Coxeter group $W$ is finite if and only if $B$ is positive definite.*

**Proof :** (ı) Let $x \in V^0$, then $\sigma_s(x) = x - 2B(x, \alpha_s)\alpha_s = x$ for all $s \in S$.

If $S = \{s\}$ the second assertion is trivial. Let $V'$ a proper subspace of $V$ stable under $W$. Let $s$ and $s'$ in $S$ with $m(s, s') \geq 3$. Assume that $\alpha_s \in V'$, then $\sigma_{s'}(\alpha_s) = \alpha_s - 2B(\alpha_s, \alpha_{s'})\alpha_{s'} \in V'$ thus $2B(\alpha_s, \alpha_{s'})\alpha_{s'} \in V'$ andbecause $B(\alpha_s, \alpha_{s'}) \neq 0$ we get $\alpha_{s'} \in V'$. Thus if $V'$ contains some $\alpha_s$, it contains all of them and $V' = V$. A contradiction.

Let $s \in S$ and assume that there exists $x \in V'$ with $B(x, \alpha_s) \neq 0$. Then we have

$$\alpha_s = \frac{1}{2B(x, \alpha_s)}(\sigma_s(x) - x) \in V',$$

a contradiction. We get $V' \subset V^0$.

(ıı) If $B$ has a non trivial kernel $V^0$, then this kernel is $W$-stable and because $W$ is finite, it has a stable supplementary (take the kernel of the mean of elements in $W$ of any projection to $V^0$). This is impossible by (ı).

Let $u \in \mathrm{End}(V)$ commuting with any element of $W$. Let $s \in S$ and consider $p_s = \frac{1}{2}(\mathrm{Id}_V - \sigma_s)$ which is a projector on the line $\mathbb{R}\alpha_s$ and commutes with $u$. We have $p_s(u(\alpha_s)) = u(p_s(\alpha_s)) = u(\alpha_s)$ thus $u(\alpha_s) = \lambda\alpha_s$. The subspace $\ker(u - \lambda \mathrm{Id}_V)$ which is non trivial and stable under $W$. It has to be $V$ itself and the result follows.

Because $B$ is non degenerate, any bilinear form $B'$ can be written as

$$B'(x, y) = B(u(x), y)$$

for some $u \in \mathrm{End}(V)$. But if $B'$ is $W$-invariant, this implies that

$$B(u(w(x)), y) = B'(w(x), y) = B'(x, w^{-1}(y)) = B(u(x), w^{-1}(y)) = B(w(u(x)), y)$$

and because $B$ is non degenerate, we get $w(u(x)) = u(w(x))$. This implies $u = \lambda \mathrm{Id}_V$ and $B' = \lambda B$.

Finally we prove that $B$ is symmetric positive: let $B'$ be a positive symmetric form and let $B''$ be the mean of the translate of $B'$ under the group $W$. The form $B''$ is symmetric positive and we have $B'' = \lambda B$ for some $\lambda$. Because $B(\alpha_s, \alpha_s) = 1$ this implies $\lambda > 0$ and $B$ is positive.

(ııı) We proved in (ıı) that $W$ finite implies that $B$ is positive definite. Conversely let us denote by $C$ the set of linear forms $x^*$ on $V$ such that $x^*(\alpha_s) > 0$ for all $s \in S$. In particular we have $x^*(\alpha) > 0$ for any positive root $\alpha$. For $x^* \in C$, the set $U(x^*)$ of elements $g \in GL(V)$ such that $x^* \circ g \in C$ is an open neiboughood of Id. But by Tits Theorem (Theorem 6.2.7) we have $W \cap U(x^*) = \{\mathrm{Id}\}$.

Indeed, if $\ell(w) > 0$, then there exists an element $s \in S$ such that $\ell(ws) < \ell(w)$ which implies that $-\alpha = w(\alpha_s) < 0$ by Tits Theorem and that $x^*w(\alpha_s) = x^*(-\alpha) < 0$ and $w$ is not in the intersection.

The group $W$ is thus discrete in $GL(V)$ but because $B$ is positive definite $O(V, B)$ is compact and this implies that $W$ is finite. □

## 6.3 Exchange conditions

### 6.3.1 Reflections

As we already know from the classical case. The choice of simple roots is not canonical and any root can be chosen simple. Here we consider all the reflections in the Coxeter group $W$. Let $\alpha = w(\alpha_s)$ be a root, we may consider the element $wsw^{-1} \in W$. Its action on $V$ is given by

$$\begin{aligned} wsw^{-1}(v) &= w(w^{-1}(v) - 2B(w^{-1}, \alpha_s)\alpha_s) \\ &= v - 2B(v, \alpha)\alpha. \end{aligned}$$

In particular this element depends only on $\alpha$ and not on $w$ and $s$. We denote it $s_\alpha$ and it acts on $V$ as a reflection. We define the set $T$ of all such reflections $s_\alpha$ by

$$T = \bigcup_{w \in W} wSw^{-1}.$$

**Proposition 6.3.1** *Let $w \in W$ and $\alpha \in \Delta_+$, then $\ell(ws_\alpha) > \ell(w)$ if and only if $w(\alpha) > 0$.*

**Proof :** We prove the implication, the converse will be automatic (because if $\ell(ws_\alpha) < \ell(w)$, the same proof will imply that $w(\alpha) < 0$). We proceed by induction on $\ell(w)$. The case $\ell(w) = 0$ is trivial. If $\ell(w) > 0$, then there exists $s \in S$ such that $\ell(w) = \ell(sw) + 1$. We have $\ell(ws_\alpha) > \ell(w) > \ell(sw)$. This implies that $\ell(sws_\alpha) \geq \ell(ws_\alpha) - 1 > \ell(w) - 1 = \ell(sw)$. The induction hypothesis implies $sw(\alpha) > 0$. Assume that $w(\alpha) < 0$, this implies that $w(\alpha) = \alpha_s$ and in particular $ws_\alpha = sw$ contradicting the length inequalities. □

### 6.3.2 Strong exchange condition

**Theorem 6.3.2 (Strong exchange condition)** *Let $w = s_1 \cdots s_r$ be a not necessary reduced expression with $s_i \in S$. Suppose that a reflection $t \in T$ satisfies $\ell(wt) < \ell(w)$, then there is an index $i$ for which $wt = s_1 \cdots \widehat{s_i} \cdots s_r$. If the expression for $w$ is reduced, then $i$ is unique.*

**Proof :** There exists an $\alpha \in \Delta_+$ such that $t = s_\alpha$. Because of the relation on the length, we have $w(\alpha) < 0$. In particular, there exists an index $i$ for which $s_{i+1} \cdots s_r(\alpha) > 0$ but $s_i \cdots s_r(\alpha) < 0$. This implies that $\alpha_i = s_{i+1} \cdots s_r(\alpha)$ where $\alpha_i$ is the simple root associated to $s_i$. We thus have $s_{i+1} \cdots s_r t s_r \cdots s_{i+1} = s_i$. This gives $wt = s_1 \cdots \widehat{s_i} \cdots s_r$.

Assume the expression is reduced and that there are $i < j$ such that $wt = s_1 \cdots \widehat{s_i} \cdots s_r = s_1 \cdots \widehat{s_j} \cdots s_r$. This gives the relation $s_{i+1} \cdots s_j = s_i \cdots s_{j-1}$ and thus $s_i \cdots s_j = s_{i+1} \cdots s_{j-1}$ a contradiction to the fact that the expression was reduced. □

**Corollary 6.3.3 (Deletion Condition)** *(ı) Let $w = s_1 \cdots s_r$ be a non reduced expression, then there exist indices $i < j$ such that $wt = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_r$.*

*(ıı) Let $w = s_1 \cdots s_r$ be any expression, then a reduced expression of $w$ can be obtained by omitting an even number of $s_i$.*

**Proof :** (ı) By hypothesis, there exists an index $j$ such that $\ell(s_1 \cdots s_j) < l(s_1 \cdots s_{j-1})$. Applying the Strong exchange condition, we get an index $i < j$ such that $s_1 \cdots s_j = s_1 \cdots \widehat{s_i} \cdots s_{j-1}$, the result follows.

(ıı) Apply (ı) as long as the expression is not reduced. □

**Definition 6.3.4** The **Bruhat order** is the order generated on $W$ by $v \leq w$ if there exists a reduced expression $w = s_1 \cdots s_r$ such that $v = s_1 \cdots \widehat{s_i} \cdots s_r$ (or if $\ell(v) < \ell(w)$ and $v = wt$ for some $t \in T$).

## 6.4 Weyl groups of Kac-Moody Lie algebras

### 6.4.1 Equivalent definitions

We now give equivalent conditions for a group $W$ to be a Coxeter group. We will use this result to prove that the Weyl groups of Kac-Moody Lie algebras are Coxeter groups.

Let $W$ be a group generated by a fixed (finite even if it is not necessary) subset $S$ of elements of order 2. We may define the length of any element as in the Coxeter group case.

**Theorem 6.4.1** *The following conditions are equivalent:*

*(ı)* **Coxeter condition***: the group $W$ is the quotient of the free group generated by the set $S$ modulo the relations $s^2 = 1$ for all $S \in S$ and $(st)^{m(s,t)} = 1$ for some integer (eventually infinite) $m(s,t) \geq 2$.*

*(ıı)* **Root system condition***: there exists a representation $V$ of $W$ over $\mathbb{R}$ together with a subset $\Delta$ of $V \setminus \{0\}$ such that*

- *$\Delta = -\Delta$;*
- *$\Delta$ is $W$-invariant;*
- *there exist a subset $\Pi = \{\alpha_s\}_{s \in S}$ of $\Delta$ such that any $\alpha \in \Delta$ is such that exactely one of $\alpha$ or $-\alpha$ belongs to the positive cone generated by $\Pi$. In the first case $\alpha$ is called positive and negative in the other. We denote by $\Delta_+$ and $\Delta_-$ the sets of positive and negative roots;*
- *we have $s\alpha_s \in \Delta_-$ and, for all $\alpha \in \Delta_+ \setminus \{\alpha_s\}$, we have $s\alpha \in \Delta_+$;*
- *For $s$ and $t$ in $S$ and for $w \in W$ such that $w\alpha_s = \alpha_t$, then $wsw^{-1} = t$.*

  *(ııı)* **Strong exchange condition***: let $s \in S$ and $v$ and $w$ in $W$ be such that $\ell(vsv^{-1}w) < \ell(w)$ then for any expression $w = s_1 \cdots s_r$ with $s_k \in S$, we have $vsv^{-1}w = s_1 \cdots \widehat{s_i} \cdots s_r$ for some $i$.*

  *(ıv)* **Exchange condition***: let $s \in S$ and $w \in W$ be such that $\ell(sw) < \ell(w)$ then for any reduced expression $w = s_1 \cdots s_r$ with $s_k \in S$, we have $sw = s_1 \cdots \widehat{s_i} \cdots s_r$ for some $i$.*

**Proof :** We have already proved in our study of Coxeter groups that (ı) implies (ıı). It is evident that (ııı) implies (ıv) and we are left to prove that (ıı) implies (ııı) and that (ıv) implies (ı).

Let us prove the first implication. This will give another proof of the Strong exchange condition for a Coxeter group.

We know that $-s(\alpha_s) > 0$ and $s(-s(\alpha_s)) = -\alpha_s < 0$, this implies by the fourth hypothesis that $s(\alpha_s) - \alpha_s$.

Let us prove that the set $T = \{wsw^{-1} \ / \ s \in S \ w \in W\}$ is in bijection with $\{\alpha > 0 \ / \ \alpha = w(\alpha_s)$ for some $s \in S\}$. For $\alpha > 0$, with $\alpha = w(\alpha_s)$, $s \in S$, we define $t_\alpha = wsw^{-1}$. If $\alpha = w'(a_u)$

with $u \in S$, then we have $w'w^{-1}(\alpha_s) = \alpha_u$ and by hypothesis $w'w^{-1}sww'^{-1} = u$ and in particular $wsw^{-1} = w'uw'^{-1}$ and $t$ does not depend on the choice of $s$ and $w$ but only on $\alpha$. Conversely, let $t \in T$ be such that $t = wsw^{-1}$ and let $\beta_t = w(\alpha_s)$ or $-w(\alpha_s)$ whichever is positive. If $t = w's'w'^{-1}$, then $w^{-1}w's'w'^{-1}w = s$. Set $\alpha = w^{-1}w'(\alpha_{s'})$, then we have $s(\alpha) = w^{-1}w'(-\alpha_{s'}) = -\alpha$. By hypothesis, this implies that $\alpha = \alpha_s$ or $-\alpha_s$. Therefore $\beta_t$ and $w(\alpha) = w'(\alpha_{s'})$ are colinear so that $\beta_t = w'(\alpha_{s'})$ or $-w'(\alpha_{s'})$ whichever is positive. In particular $\beta_t$ does only depend on $t$ and not on $w$ and $s$ such that $t = wsw^{-1}$. These two maps are clearly inverse of each other. Remark that we have $t(\beta_t) = -\beta_t$.

Before proving (iii), we prove that, if $w = s_1 \cdots s_r$ is a non necessary reduced expression and if $t \in T$ is such that $w^{-1}(\beta_t) < 0$, then $tw = s_1 \cdots \widehat{s_i} \cdots s_r$ for some $i$. Indeed, as for Theorem 6.3.2 consider $i$ be such that $s_{i-1} \cdots s_1(\beta_t) > 0$ but $s_i \cdots s_1(\beta_t) < 0$ (this element always exist because $\beta_t > 0$ but $w^{-1}(\beta_t) < 0$. Thus, this implies by hypothesis that $s_{i-1} \cdots s_1(\beta_t) = \alpha_{s_i}$ thus $\beta_t = s_1 \cdots s_{i-1}(\alpha_{s_i})$. And by construction of $\beta_t$ be have $t = s_1 \cdots s_{i-1}s_is_{i-1} \cdots s_1$ thus $tw = s_1 \cdots \widehat{s_i} \cdots s_r$.

In particular, we get that if $w^{-1}(\beta_t) < 0$, then $\ell(wt) < \ell(w)$. Conversely, if $\ell(tw) \leq \ell(w)$, then by remark that the proof of Proposition 6.3.1 applies readily to get that $w^{-1}(\beta_t) < 0$. In particular we proved that $\ell(tw) \leq \ell(w)$ if and inly if $w^{-1}(\beta_t) < 0$ and we are done.

Finally lets prove that (iv) implies (i). This is a bit technical. Consider the group $\widetilde{W}$ which is the quotient of the free group generated by the elements $s \in S$ by the relations $s^2 = 1$ for $s \in S$. For $s \in S$ we denote by $\widetilde{s}$ the corresponding element in $\widetilde{W}$. Let $f : \widetilde{W} \to W$ be the canonical map and denote by $\widetilde{N}$ the normal subgroup generated by the elements $(ss')^{m(s,s')}$ for $s$ and $s'$ in $S$. We want to prove that $\widetilde{N} = \ker f$.

Assume this is not true, then there exists an element $z = \widetilde{s}_1 \cdots \widetilde{s}_k \in \ker f$ such that $z \notin \widetilde{N}$. Assume that $\ell(z) = k$ is minimal for this property. We have $1 = s_1 \cdots s_k$ thus $\ell(s_1 \cdots s_k) = 0$ and $\ell(s_k) = 1$ and there exists an index $i < k$ such that $\ell(s_i \cdots s_k) \leq \ell(s_{i+1} \cdots s_k)$ and $s_{i+1} \cdots s_k$ is reduced. The fact that $\ell(s_1 \cdots s_k) = 0$ implies that $i \geq k/2$. We may now apply Exchange condition to get an index $j$ such that $i < j \leq k$ such that $s_i \cdots s_k = s_{i+1} \cdots \widehat{s_j} \cdots s_k$ i.e. we have $s_i \cdots s_j = s_{i+1} \cdots s_{j-1}$. This implies that $z_0 = \widetilde{s}_i \cdots \widetilde{s}_j \widetilde{s}_{j-1} \cdots \widetilde{s}_{i+1}$ lies in $\ker f$. But we also have that $\ell(z_0) \leq j-i+1+j-i-1 = 2j-2i \leq k$.

If $\ell(z_0) < k$, then by minimality we have that $z_0 \in \widetilde{N}$ and then we get

$$\begin{aligned} z &= \widetilde{s}_1 \cdots \widetilde{s}_k = \widetilde{s}_1 \cdots \widetilde{s}_{i-1} z_0 \widetilde{s}_{i+1} \cdots \widetilde{s}_{j-1} \widetilde{s}_{j+1} \cdots \widetilde{s}_r \\ &= (\widetilde{s}_1 \cdots \widetilde{s}_{i-1} z_0 \widetilde{s}_{i-1} \cdots \widetilde{s}_1) \widetilde{s}_1 \cdots \widetilde{s}_{i-1} \widetilde{s}_{i+1} \cdots \widetilde{s}_{j-1} \widetilde{s}_{j+1} \cdots \widetilde{s}_r \end{aligned} .$$

The two factors on the last line are in $\widetilde{N}$ (because $\widetilde{N}$ is normal and because of the minimality condition) thus $z \in \widetilde{N}$ a contradiction.

We must have $\ell(z_0) = k$ and in particular $i = k/2$ and $j = k$. We thus have the equalities $s_1 \cdots s_k = 1 = s_1 \cdots \widehat{s_i} \cdots \widehat{s_k}$ implying the inclusion $\widetilde{s}_1 \cdots \widehat{\widetilde{s}_i} \cdots \widehat{\widetilde{s}_k} \in \widetilde{N}$ (by minimality). Thus we have $z = \widetilde{s}_1 \cdots \widetilde{s}_{i-1} \widetilde{s}_i \widetilde{s}_{i-1} \cdots \widetilde{s}_1 \widetilde{s}_k \widetilde{s}_k \widetilde{s}_1 \cdots \widehat{\widetilde{s}_i} \cdots \widehat{\widetilde{s}_k} \in \widetilde{s}_1 \cdots \widetilde{s}_{i-1} \widetilde{s}_i \widetilde{s}_{i-1} \cdots \widetilde{s}_1 \widetilde{s}_k \cdot \widetilde{N}$. Let $z_1 = \widetilde{s}_k \widetilde{s}_1 \cdots \widetilde{s}_{i-1} \widetilde{s}_i \widetilde{s}_{i-1} \widetilde{s}_1$, we have $z \in z_1^{-1} \cdot N$ and because $N$ is normal, we get that $z_1 \in z^{-1} \cdot N$. Now we replace $z$ by $z_1$. We have $\ell(z_1) leq 2i = k$ but $z_1 \notin \widetilde{N}$ (otherwise $z$ would be in $\widetilde{N}$) thus $\ell(z_1) = k$. Applying the same method gives us

$$z_2 = \widetilde{s}_1 \widetilde{s}_k \widetilde{s}_1 \cdots \widetilde{s}_{i-2} \widetilde{s}_{i-1} \widetilde{s}_{i-2} \cdots \widetilde{s}_1 \widetilde{s}_k \in z_1^{-1} \cdot N$$

and then a sequence of elements

$$z_{2a} = (\widetilde{s}_1 \widetilde{s}_k)^a \widetilde{s}_1 \cdot \widetilde{s}_{i-2a} \widetilde{s}_{i-2a+1} \widetilde{s}_{i-2a} \cdots \widetilde{s}_1 (\widetilde{s}_k \widetilde{s}_1)^{a-1} \widetilde{s}_k \in z_{2a-1}^{-1} \cdot N$$

$$z_{2a+1} = \widetilde{s}_k (\widetilde{s}_1 \widetilde{s}_k)^a \widetilde{s}_1 \cdot \widetilde{s}_{i-2a-1} \widetilde{s}_{i-2a} \widetilde{s}_{i-2a-1} \cdots \widetilde{s}_1 (\widetilde{s}_k \widetilde{s}_1)^a \in z_{2a}^{-1} \cdot N.$$

In particular, we get that $z_{i-1} = (\widetilde{s}_1 \widetilde{s}_k)^i$ or $(\widetilde{s}_k \widetilde{s}_1)^i$ and $z_{i-1} \in z^{\pm} \cdot \widetilde{N}$. But $z_{i-1} \in \ker f$ thus $m(s_1, s_k)$ divides $i$ and thus $z_{i-1} \in \widetilde{N}$ by definition and thus $\widetilde{N} = \ker f$. □

**Definition 6.4.2** A Coxeter group is called cristallographic if $m(i,j) \in \{2,3,4,6,\infty\}$ for all $i$ and $j$ in $[1,n]$.

**Corollary 6.4.3** *The Weyl group of any Kac-Moody Lie algebra is a Coxeter cristallographic group.*

**Proof :** We prove that the root system condition of the previous Theorem is satisfied. We take $V$ the $\mathbb{R}$-linear span of the $\alpha_i$ for $i \in [1,n]$ in the algebra $\mathfrak{h}^*$ and for $\Delta$ the set of roots of $\mathfrak{g}(A)$. We already know that $\Delta$ is symmetric and by Proposition 5.2.6 that it is $W$-invariant. We also know that any root is either positive or negative. We already know that $s \cdot \alpha_i = -\alpha_i$ and because any root as to be either positive or negative, the same proof as is Proposition 6.2.9 gives the fourth condition.

Let us prove the last condition. Assume that $\alpha_j = w \cdot \alpha_j$ for $i$ and $j$ in $[1,n]$. We need to prove that $ws_iw^{-1} = s_j$. But let $v \in V$ be such that $\langle v, \alpha_j^\vee \rangle = 0$, then we have

$$ws_iw^{-1}(v) = v - \langle w^{-1}(v), \alpha_i^\vee \rangle w(\alpha_i) = v - \langle v, w(\alpha_i^\vee) \rangle w(\alpha_i) = v$$

because $w(\alpha_i^\vee) = \alpha_j^\vee$. On the other hand, we have $ws_iw^{-1}(\alpha_j) = \alpha_j$ so that the equality follows. The group $W$ is a Coxeter group.

We have already seen that if the product $a = a_{i,j}a_{j,i} \geq 4$, then the order of $s_is_j$ is infinite. Furthermore, if $a = 0$, 1, 2, 3 or 4, we have seen that the order of $\sigma_i\sigma_j$ on the vector space generated by $\alpha_i$ and $\alpha_j$ is 2, 3, 4 or 6. Furthermore we have also seen that the order of $\sigma_i\sigma_j$ is equal to that order. □

## 6.5 Dominant chambers and Tits cone

Let $W$ be the Weyl group of a Kac-Moody Lie algebra $\mathfrak{g}(A)$ with Cartan Lie algebra $\mathfrak{h}$.

**Definition 6.5.1** Let us fix a **real form** $\mathfrak{h}_\mathbb{R}$ of $\mathfrak{h}$ (that is to say a real subvector space such that $\mathfrak{h}_\mathbb{R} \otimes \mathbb{C} \simeq \mathfrak{h}$). Assume that $\mathfrak{h}_\mathbb{R}$ satisfies:

- $\{\alpha_1^\vee, \cdots \alpha_n^\vee\} \subset \mathfrak{h}_\mathbb{R}$
- $\alpha_i(\mathfrak{h}_\mathbb{R}) \subset \mathbb{R}$ for any $i \in [1,n]$.

We have that $\mathfrak{h}_\mathbb{R}$ is $W$-stableand define the Dominant Chamber $C_\mathbb{R}$ by

$$C_\mathbb{R} = \{h \in \mathfrak{h}_R \ / \ \langle h, \alpha_i \rangle \in \mathbb{R}_{\geq 0} \text{ for all } i\}.$$

Define the Tits cone $X$ by

$$X = \bigcup_{w \in W} wC_\mathbb{R}.$$

**Theorem 6.5.2** *(ı) For $\lambda \in C_\mathbb{R}$, the isotropy group $W_\lambda = \{w \in W \ / \ w(\lambda) = \lambda\}$ is generated by the simple reflections it contains.*

*(*
*ıı) The dominant chamber $C_\mathbb{R}$ is a fundamental domain for the action of $W$ on $X$ (i.e. any orbit intersects $C_\mathbb{R}$ is exactly one point).*

*(ııı) We have $X = \{\lambda \in \mathfrak{h}_\mathbb{R} \ / \ \langle \lambda, \alpha \rangle < 0$ for a finite number of $\alpha \in \Delta_+\}$. In particular $X$ is a convex cone.*

*(ıv) We have $C_\mathbb{R} = \{h \in \mathfrak{h}_\mathbb{R} \ /$ for every $w \in W, \ h - w(h) = \sum_i c_i\alpha_i^\vee$ where $c_i \geq 0\}$.*

*(v) The following conditions are equivalent:*

- *$|W| < \infty$;*
- *$X = \mathfrak{h}_{\mathbb{R}}$;*
- *$|\Delta| < \infty$;*
- *$|\Delta^\vee| < \infty$ where $\Delta^\vee$ is the root system for the matrix $A^t$.*

*(vi) If $h \in \mathfrak{h}_{\mathbb{R}}$, then $|W_h| < \infty$ if and only if $h$ lies in the interior of $X$ (for the Hausdorff topology).*

**Proof :** We prove (i) and (ii). Let $w \in W$ and $w = s_{i_1} \cdots s_{i_r}$ a reduced expression. Assume that $w(h) = h'$ with $h$ and $h'$ in $C_{\mathbb{R}}$. We have $\langle h, \alpha_{i_r} \rangle \geq 0$ and thus $\langle h', w(\alpha_{i_r}) \rangle \geq 0$. But because the expression is reduced, we have that $w(\alpha_{i_r}) < 0$ thus $\langle h', w(\alpha_{i_r}) \rangle \leq 0$. We deduce that $\langle h', w(\alpha_{i_r}) \rangle = 0$ and $\langle h, a_{i_r} \rangle = 0$. In particular this implies that $s_{i_r}(h) = h$. We conclude (i) and (ii) by induction on the length.

For (iii), set $X'$ be the set defined by the condition in the Theorem. We have $C_{\mathbb{R}} \subset X'$. It is also $W$-invariant (because for any element $w \in W$ only a finite number of roots become negative) so that we have the inclusion $X \subset X'$. Let us prove the reverse inclusion. Take $h \in X'$, and consider $M_h = \{\alpha \in \Delta_+ \ / \ \langle h, \alpha \rangle < 0\}$. This set is finite and if it is empty then $h \in C \subset X$. Otherwise, there exists an $i$ such that $\alpha_i \in M_h$ and applying $s_i$ we get that $|M_{s_i(h)}| < |M_h|$. We conclude by induction.

(iv) Let $h \in C_{\mathbb{R}}$, we proceed by induction on $\ell(w)$ to prove that $h - w(h) = \sum_i c_i \alpha_i^\vee$ with $c_i \geq 0$. Take $w = sv$ with $\ell(w) = \ell(v) + 1$. We have $v^{-1}(\alpha_s) > 0$. But we have

$$w(h) = sv(h) = v(h) - \langle v(h), \alpha_s \rangle \alpha_s^\vee = h - \sum_i c_i \alpha_i^\vee - \langle h, v^{-1}(\alpha_s) \rangle \alpha_s^\vee$$

giving that

$$h - w(h) = \sum_i c_i \alpha_i^\vee + \langle h, v^{-1}(\alpha_s) \rangle \alpha_s^\vee$$

but by induction $c_i \geq 0$ and $\langle h, v^{-1}(\alpha_s) \rangle \geq 0$ because $v^{-1}(\alpha_s) > 0$.

Conversely, we have that $\langle h, \alpha_i \rangle \alpha_i^\vee = h - s_i(h) = \sum_k c_k \alpha_k^\vee$ with $c_k \geq 0$ and we have the result.

(v) Let $h \in \mathfrak{h}_{\mathbb{R}}$ and take $h'$ be a maximal element (for the order defined by positive coroots) in the orbit $W \cdot h$. Such an element exists because $W$ is finite. But thenwe have $s_i(h') = h' - scah'\alpha_i\alpha_i^\vee$ and because of the maximality condition, we get that $\langle h', \alpha_i \rangle \geq 0$ and thus $h' \in C_{\mathbb{R}}$.

Let $h \in \mathfrak{h}_R$ such that $\langle h, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$. We thus have $\langle h, \alpha \rangle < 0$ or $\langle h, -\alpha \rangle < 0$. But $h$ is in $X$ and by the point (iii) we get that $\Delta$ is finite. We already proved in Corollary 6.2.10 that if $\Delta$ is finite then $W$ is finite so that we get the equivalence of the condition except for the dual root system. But remark that the Weyl group $W$ is determined by the generalised Cartan matrix $A$ and that the Weyl group of the transpose matrix is isomorphic to $W$. We get the equivalence for the dual root system.

(v) Consider the set $S'$ of simple reflections in $W_h$. We now by (i) that $S'$ generates $W_h$ and in particular any positive root $\alpha$ such that $s_\alpha$ is in $W_h$ can be written as a linear combination of simple roots in $S'$. These positive roots form the set $\Delta_+^h = \{\alpha \in \Delta_+ \ / \ \langle \alpha, h, = \rangle 0\}$. This is also the root system of the group $W_h$. In particular $W_h$ is finite if and only if $\Delta_+^h$ is finite. But in a small neighborhood of $h$ these only positive roots $\alpha$ on which the sign of the evaluation may change are exactly the roots in $\Delta_+^h$. We conclude by the description of $X$ that this neighborhood is contained in $X$ if and only if $\Delta_+^h$ is finite. $\square$

# Chapter 7

# Invariant bilinear form on $\mathfrak{g}(A)$

## 7.1 Symmetrisable Cartan matrices

We define a special class of Cartan matrices which will include all finite dimensional and affine Kac-Moody Lie algebras. They will have special very usefull properties (in particular the existence of a Casimir operator, see Chapter 11).

**Definition 7.1.1** Let $A = (a_{i,j})$ be an $n \times n$ matrix, it is called symmetrizable if there exists a non degenerated diagonal matrix $D = \mathrm{Diag}(\epsilon_1, \cdots, \epsilon_n)$ and a symmetric matrix $B$ such that $A = DB$.

**Proposition 7.1.2** *(ı) If $A$ is symmetrisable generalised Cartan matrix, then there exists a non degenerate diagonal matrix $D$ with coefficients in $\mathbb{Q}$ such that $D^{-1}A = B$ is symmetric.*

*(ıı) If $A$ is a symmetric indecomposable GCM, then $D$ is unique up to scalar multiple and we may choose all the $\epsilon_i \in \mathbb{Q}$ and positive.*

*(ııı) Assume $A$ to be a symmetric indecomposable GCM, then there is an unique diagonal matrix $D = \mathrm{Diag}(\epsilon_i)$ whose coefficients are in $\mathbb{Z}$ and positive, such that $D^{-1}A$ is symmetric and such that if $D' = \mathrm{Diag}(\epsilon'_i)$ is another matrix with the same properties, then $\epsilon_i \leq \epsilon'_i$. Such a matrix is called the* **minimal** *$D$.*

**Proof :** (ı) Consider the equations for $D$, we have that for all $i$ and $j$ in $[1, n]$, $\epsilon_i^{-1} a_{i,j} = \epsilon_j^{-1} a_{j,i}$. These solutions are homogeneous in $\epsilon_i$. In particular, if $a_{i,j}$ is non vannishing and if the quotient $a_{j,i}/a_{i,j}$ is a rational number, then if the system has a solution, it has a rational solution.

(ıı) If furthermore $A$ is indecomposable, then for any $i$, there exists a sequence $(i_1, \cdots, i_k)$ of integers in $[1, n]$ such that $i_1 = 1$, $i_k = i$, $i_j \neq i_{j+1}$ and $a_{i_j,i_{j+1}} \neq 0$ for $1 \leq j \leq k-1$. This implies that $\epsilon_i$ is determined by $\epsilon_1$ thus the matrix $D$ is unique modulo scalars. Furthermore because all the $a_{i_j,i_{j+1}}$ are non negative, this implies that all the $\epsilon_i$ have the same sign and me may assume they are positive.

(ııı) We have seen that if $A$ is indecomposable then the matrix $D$ is unique up to scalar multiples. The result follows. □

We end this section with a combinatorial characterisation of symmetrisable matrices.

**Proposition 7.1.3** *A matrix $A = (a_{i,j})_{i,j \in [1,n]}$ is symmetrisable if and only if the following two conditions are satisfied*

- *$a_{i,j} = 0$ implies $a_{j,i} = 0$;*
- *for all sequence $i_1 \cdots i_k$ of indexes in $[1, n]$, we have $a_{i_1,i_2} \cdots a_{i_{k-1},i_k} a_{i_k,i_1} = a_{i_2,i_1} \cdots a_{i_k,i_{k-1}} a_{i_1,i_k}$.*

**Proof :** If $A$ is symmetrisable, then there exists a non degenerate diagonal matrix $D$ and there exists a symmetric matrix $B$ such that $A = DB$. We have $a_{i,j} = d_{i,i}b_{i,j}$ and $a_{j,i} = d_{j,j}b_{j,i}$ thus

$$a_{j,i} = \frac{d_{j,j}}{d_{i,i}} a_{i,j}.$$

This proves the first implication.

Conversely, assume that $A$ satisfies the two conditions above. We may assume that $A$ is indecomposable (otherwise we prove the result composentwise). Let $i \in [1, n]$, there exists (see Lemma 4.2.9) a sequence $i_1 = 1, \cdots, i_k = i$ of indices such that $a_{i_1,i_2} \cdots a_{i_{k-1},i_k} \neq 0$. Set $d_{1,1} = 1$ and

$$d_{i,i} = \frac{a_{i_2,i_1} \cdots a_{i_k,i_{k-1}}}{a_{i_1,i_2} \cdots a_{i_{k-1},i_k}}.$$

Remark that because of the second condition, this definition does not depend on the choice of such a sequence of indices and that for any sequence of indices $l_1, \cdots, l_r$ with $l_1 = i$ and $l_r = j$ and with $a_{l_1,l_2} \cdots a_{l_{r-1},l_r} \neq 0$ we have

$$d_{j,j} = \frac{a_{l_2,l_1} \cdots a_{l_r,l_{r-1}}}{a_{l_1,l_2} \cdots a_{l_{r-1},l_r}} d_{i,i}.$$

We may now set $B = D^{-1}A$. We want to prove that $B$ is symmetric. We have

$$b_{i,j} = \frac{a_{i,j}}{d_{i,i}} \quad \text{and} \quad b_{j,i} = \frac{a_{j,i}}{d_{j,j}}$$

and these two are equal by the previous formula. □

## 7.2 Invariant bilinear forms

**Proposition 7.2.1** *(ı) Let $A$ be symmetrizable and indecomposable. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realisation of $A$. There exists a non degenerate symmetric $W$-invariant bilinear form $(\ ,\ )$ on $\mathfrak{h}$.*

*(ıı) The kernel of the resctriction of this form to $\mathfrak{h}' = \sum_i \mathbb{C}\alpha_i^\vee$ is $\mathfrak{c}$.*

*(ııı) The restriction of $(\ ,\ )$ to $\mathfrak{h}'$ is unique up to scalar multiple. Moreover if such a form exists on $\mathfrak{h}'$ then $A$ is symmetrisable.*

**Proof :** (ı) and (ıı). Let $D = \mathrm{Diag}(\epsilon_i)$ be a diagonal matrix such that $D^{-1}A$ is symmetric. Let $\mathfrak{h}''$ be a supplementary to $\mathfrak{h}'$ in $\mathfrak{h}$. We define $(\ ,\ )$ by the following equations:

$$(\alpha_i^\vee, h) = \langle \alpha_i, h \rangle \epsilon_i \text{ for } h \in \mathfrak{h},\ i = 1, \cdots, n;$$

$$(h', h'') = 0, \text{ for } h', h'' \in \mathfrak{h}''.$$

As we have $(\alpha_i^\vee, \alpha_j^\vee) = \langle \alpha_i, \alpha_j^\vee \rangle \epsilon_i = \langle \alpha_j^\vee, \alpha_i \rangle \epsilon_j = (\alpha_j^\vee, \alpha_i^\vee)$ we see that $(\ ,\ )$ is well defined and symmetric.

Let $h \in \mathfrak{h}$ such that $(h, h') = 0$ for all $h' \in \mathfrak{h}$. Then, write $h = \sum_i c_i \alpha_i^\vee + h''$ with $h'' \in \mathfrak{h}''$, then we have $h \in \mathfrak{c} \subset \mathfrak{h}'$. In particular $h'' = 0$ and for any $h' \in \mathfrak{h}$, we have $\langle \sum_i c_i \epsilon_i \alpha_i^\vee, h' \rangle = 0$. This implies that $\sum_i c_i \epsilon_i \alpha_i^\vee = 0$ thus $h = 0$. Furthermore, we proved (ıı) for $(\ ,\ )$.

Let us compute

$$\begin{aligned}(s_i(h), s_i(h')) &= (h - \langle \alpha_i, h \rangle \alpha_i^\vee, h' - \langle \alpha_i, h' \rangle \alpha_i^\vee) \\ &= (h, h') - \langle \alpha_i, h \rangle (\alpha_i^\vee, h') - (h, \langle \alpha_i, h' \rangle \alpha_i^\vee) + \langle \alpha_i, h' \rangle \langle \alpha_i, h' \rangle (\alpha_i^\vee, \alpha_i^\vee) \\ &= (h, h') - \langle \alpha_i, h \rangle \langle \alpha_i, h' \rangle \epsilon_i - \langle \alpha_i, h' \rangle \langle h', \alpha_i \rangle \epsilon_i + 2\langle \alpha_i, h' \rangle \langle \alpha_i, h' \rangle \epsilon_i \\ &= (h, h').\end{aligned}$$

The invariance is proved.

(ııı) Assume that there exists a non zero $W$-invariant bilinear form $((\ ,\ ))$ on $\mathfrak{h}'$. We don't assume it is symmetric nor non degenerate. Let us set $\epsilon_i = ((\alpha_i, \alpha_i^\vee))/2$ and compute

$$2\epsilon_j = ((s_i\alpha_j^\vee, s_i\alpha_j^\vee)) = 2\epsilon_j - \langle \alpha_i, \alpha_j^\vee \rangle ((\alpha_i^\vee, \alpha_j^\vee)) - \langle \alpha_i, \alpha_j^\vee \rangle ((\alpha_j^\vee, \alpha_i^\vee)) + 2\langle \alpha_i, \alpha_j^\vee \rangle^2 \epsilon_i$$

$$((\alpha_i^\vee, \alpha_j^\vee)) = ((s_i\alpha_i^\vee, s_i\alpha_j^\vee)) = 2\epsilon_i \langle \alpha_i, \alpha_j^\vee \rangle - ((\alpha_i^\vee, \alpha_j^\vee)).$$

The second relation gives $((\alpha_i^\vee, \alpha_j^\vee)) = \epsilon_i \langle \alpha_i, \alpha_j^\vee \rangle$ and $((\alpha_j^\vee, \alpha_i^\vee)) = \epsilon_j \langle \alpha_j, \alpha_i^\vee \rangle$. Together with the first one when $\langle \alpha_i, \alpha_j^\vee \rangle \neq 0$, we get $\epsilon_i \langle \alpha_i, \alpha_j^\vee \rangle = \epsilon_j \langle \alpha_j, \alpha_i^\vee \rangle$. This is also true if $\langle \alpha_i, \alpha_j^\vee \rangle = 0$ because in that case $\langle \alpha_j, \alpha_i^\vee \rangle = 0$. This implies because of the indecomposability of $A$ that if $\epsilon_i = 0$ for some $i$, then it is true for all $i$. But in that case the second relation gives $((\alpha_i^\vee, \alpha_j^\vee)) = 0$ for all $i$ and $j$ and $((\ ,\ ))$ would be zero. This implies that all the $\epsilon_i$ are non zero thus $A$ is symmetrisable. To get the unicity, apply the previous to

$$((\ ,\ )) - \frac{((\alpha_1^\vee, \alpha_1^\vee))}{(\alpha_1^\vee, \alpha_1^\vee)}(\ ,\ ).$$

It has to be zero. □

Since the bilinear form $(\ ,\ )$ is non degenerate, it induces an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ defined by $\langle \nu(h), h' \rangle = (h, h')$ for all $h' \in \mathfrak{h}$. We still denote $(\ ,\ )$ the induced bilinear form on $\mathfrak{h}^*$. We have the easy:

**Fact 7.2.2** *We have the following formulas:*

*(ı)* $\nu(\alpha_i^\vee) = \epsilon_i \alpha_i$ *for all* $i \in [1, n]$.

*(ıı)* $(\alpha_i, \alpha_j) = \epsilon_j^{-1} \langle \alpha_i, \alpha_j^\vee \rangle = \epsilon_i^{-1} \langle \alpha_j, \alpha_i^\vee \rangle$ *for all* $i$ *and* $j$ *in* $[1, n]$.

**Remark 7.2.3** We have the inequality $(\alpha_i, \alpha_i) > 0$ and the identity

$$A = \left( \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \right).$$

**Definition 7.2.4** A form satisfying the conclusion of the previous Proposition is called a **normalised form on $\mathfrak{h}$**.

Let $\mathfrak{g}(A)$ be the associated Lie algebra.

**Theorem 7.2.5** *(ı) If $A$ is a indecomposable symmetrisable generalised Cartan matrix, then there exists a bilinear form $(\ ,\ )$ on $\mathfrak{g}(A)$ satisfying*

- *$(\ ,\ )$ is invariant, i.e.* $([x, y], z) = (x, [y, z])$.
- *$(\ ,\ )|_{\mathfrak{h}}$ is a normalised form on $\mathfrak{h}$.*

*(ıı) Moreover, once $(\ ,\ )|_{\mathfrak{h}}$ is fixed, such a form is unique and automatically symmetric. It satisfies*

- $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *unless* $\alpha + \beta = 0$ *for any roots* $\alpha$ *and* $\beta$.
- *The restriction $(\ ,\ )|_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}}$ is non degenerate for $\alpha \in \Delta$ and hence $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are non degenerately paired. The form $(\ ,\ )$ is non degenerate.*
- $[x, y] = (x, y)\nu^{-1}(\alpha)$ *for* $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ *and* $\alpha \in \Delta$.

**Proof :** For $\alpha = \sum_i \alpha_i$ a root we define $|\alpha|$ by $\sum_i k_i$. For any integer $k$ define $\mathfrak{g}_k$ by:

$$\mathfrak{g}_k = \bigoplus_{\alpha\in\Delta\cup\{0\},|\alpha|=k} \mathfrak{g}_\alpha. \qquad \text{Set } \mathfrak{g}(N) = \bigoplus_{k=-N}^{N} \mathfrak{g}_k.$$

We extend the bilinear form $(\ ,\ )$ on $\mathfrak{g}(1)$ by: $(f_j, e_i) = (e_i, f_j) = \delta_{i,j}\epsilon_i$ and $(\mathfrak{g}_{k_1}, \mathfrak{g}_{k_2}) = 0$ for $k_1+k_2 \neq 0$ and $|k_1|$, $|k_2|$ smaller than or equal to 1. We check that this form is invariant with respect to the action of $\mathfrak{g}(1)$. We compute $([e_i, f_j], h) = (\delta_{i,j}\alpha_j^\vee, h) = \delta_{i,j}\epsilon_i\langle\alpha_j, h\rangle = (e_i, [f_j, h])$ this proves the invariance since the other conditions all vanish.

We now extend $(\ ,\ )$ on $\mathfrak{g}(N)$ by induction on $N$ so that $(\mathfrak{g}_{k_1}, \mathfrak{g}_{k_2}) = 0$ if $k_1 + k_2 \neq 0$ and $|k_1|$,$|k_2| \leq N$ and that $([x, y], z) = (x, [y, z])$ for all $x$, $y$, $z$, $[x, y]$ and $[y, z]$ in $\mathfrak{g}(N)$. Assume this form is defined on $\mathfrak{g}(N-1)$. Then we have to define $(x, y)$ and $(y, x)$ for $x \in \mathfrak{g}_N$ and $y \in \mathfrak{g}_{-N}$. We may write $y = \sum_i [u_i, v_i]$ where $u_i$ and $v_i$ are homogeneous of negative degrees and in $\mathfrak{g}(N-1)$ (this is because $y$ lies in $\mathfrak{n}_-$ which is generated by the elements $f_j$). We may define

$$(x, y) = \sum_i ([x, u_i], v_i)$$

which is defined because $[x, u_i]$ lies in $\mathfrak{g}(N-1)$. To prove that this is well defined, we need to prove that it does not depend on the choice of the writing $y = \sum_i [u_i, v_i]$. For this we prove the following relation in $\mathfrak{g}(N-1)$:

$$([[a, b], c], d) = (a, [b, [c, d]])$$

where all the elements $a$, $b$, $c$ and $d$ as well as the brackets $[[a, b], c]$, $[b, [c, d]]$, $[[a, c], b]$, $[a, [b, c]]$, $[a, c]$, $[b, d]$, $[[b, c], d]$ and $[c, [b, d]]$ are in $\mathfrak{g}(N-1)$. Indeed, we have the equalities:

$$\begin{aligned} ([[a, b], c], d) &= ([[a, c], b], d) + ([a, [b, c]], d) \\ &= ([a, c], [b, d]) + (a, [[b, c], d]) \\ &= (a, [c, [b, d]] + [[b, c], d]) \\ &= (a, [b, [c, d]]). \end{aligned}$$

But then if we write $x = \sum_j [s_j, t_j]$ with the $s_j$ and $t_j$ homogeneous of positive degree and in $\mathfrak{g}(N-1)$. Setting $a = s_j$, $b = t_j$, $c = u_i$ and $d = v_i$ we see that the previous conditions are satisfied and we get $([[s_j, t_j], u_i], v_j) = (s_j, [t_j, [u_i, v_i]])$ for all $i$ and $j$ so that we have

$$\sum_i ([x, u_i], v_i) = \sum_j (s_j, [t_j, y]).$$

This implies that this value does not depend on the writing of $x$ nor on the writing of $y$.

For the invariance, we still need to prove that for $x \in \mathfrak{g}_N$, for $y \in \mathfrak{g}_{-N}$ and for all $h$ we have the relations

$$(x, [h, y]) = ([x, h], y) \quad \text{and} \quad ([x, y], h) = (x, [y, h]).$$

We prove it by induction on $N$. We already proved that for $\mathfrak{g}(1)$. Assume this is true for $\mathfrak{g}(N-1)$. Then write $x = \sum_j [s_j, t_j]$ and $y = \sum_i [u_i, v_i]$ we have

$$\begin{aligned} (x, [h, y]) &= \textstyle\sum_i (s_j, [t_j, [h, y]]) \\ &= \textstyle\sum_i (s_j, [h, [t_j, y]]) + \sum_i (s_j, [[t_j, h], y]) \\ &= \textstyle\sum_i ([s_j, h], [t_j, y]) + \sum_i ([s_j, [t_j, h]], y) \\ &= \textstyle\sum_i ([[s_j, h], t_j], y) + \sum_i ([s_j, [t_j, h]], y) \\ &= ([x, h], y) \end{aligned}$$

$$\begin{aligned}(x,[h,y]) &= \textstyle\sum_i(x,[h,[u_i,v_i]])\\ &= \textstyle\sum_i(x,[u_i,[h,v_i]]) + \sum_i(x,[[h,u_i],v_i])\\ &= \textstyle\sum_i([x,u_i],[h,v_i]) + \sum_i([v_i,x],[h,u_i])\\ &= \textstyle\sum_i([v_i,[x,u_i],h) + \sum_i([[v_i,x],u_i],h])\\ &= ([y,x],h)\end{aligned}$$

The third point is proved as follows.

$$0 = (x,[h,y]) + ([h,x],y) = (x,y)\langle\alpha+\beta,h\rangle$$

so that if $\alpha+\beta\neq 0$, we need that $(x,y)=0$. Let us prove the last point. We have for $x\in\mathfrak{g}_\alpha$ and $y\in\mathfrak{g}_{-\alpha}$:

$$([x,y]-(x,y)\nu^{-1}(\alpha),h) = (x,[y,h]) - (x,y)\langle\alpha,h\rangle = 0.$$

This implies that ( , ) is symmetric.

Finally, remark that if the restriction to $\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}$ is degenerate, then because of the third point, the form is degenerate on the full space $\mathfrak{g}(A)$. The kernel has to be an ideal of $\mathfrak{g}(A)$ (because the form is invariant) but the intersection of this ideal with $\mathfrak{h}$ is trivial so that by construction of $\mathfrak{g}(A)$ it has to be trivial, the form is non degenerate and its restriction to $\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}$ is also non degenerate. □

**Remark 7.2.6** If we define the Kac-Moody Lie algebras as the quotient of a free Lie algebra modulo Serre's relations, then the invariant bilinear form still exists but it is not, a priori, necessary non degenerate. This is one of the reasons why we choose this definition of Kac-Moody Lie algebras. In fact in the symmetrisable setting the two definitions coincide so that the form will also be non degenerate.

# Chapter 8

# Classification of Cartan matrices

In this chapter we describe a classification of generalised Cartan matrices. This classification can be compared as the "rough" classification of varieties in terms of Fano varieties ($-K$ ample, discrete moduli space), Calabi-Yau varieties (more generally $K$-trivial varieties, tame moduli space) and varieties of general type ($K$ ample, big moduli space). We will indeed get the case of finite dimensional Lie algebras, of Affine Lie algebras and of Kac-Moody Lie algebras of general type. In the first case, we recover semisimple Lie algebras. The second class is very rich and we shall construct explicitly these algebras in Chapter 12. The last case is more obscure even if special Lie algebras can be studied.

## 8.1 Finite, affine and indefinite case

We will prove the decomposition into these three catgories for a more larger class of matrices than the generalised Cartan matrices. We will deal with square matrices $A = (a_{i,j})$ of size $n$ satisfying the following properties:

- $A$ is indecomposable;
- $a_{i,j} \leq 0$;
- $(a_{i,j} = 0) \Rightarrow (a_{j,i} = 0)$.

Let us recall the following result on systems of inequalities:

**Lemma 8.1.1** *A system of real inequalities $\lambda_i(x_j) = \sum_j u_{i,j}x_j > 0$ for $i \in [1, m]$, $j \in [1, n]$ has a solution if and only if there is no non trivial dependence relation $\sum_i v_i\lambda_i = 0$ with $v_i \leq 0$ for all $i$.*

**Proof :** If there is such a non trivial relation, the system has no solution. We prove the converse by induction on $m$. Assume there is no non trivial such relation. By induction, the system $\lambda_i > 0$ for $i < m$ has a solution. The set of solutions is a cone $C$ containing 0 in its closure. We need to prove that this cone intersects the half space defined by $\lambda_m > 0$.

If it is not the case then we first prove that $\lambda_m$ is in the linear span of the $\lambda_i$ for $i < m$. Indeed, otherwise there would exist an element $x$ such that $\lambda_m(x) = 1$ and $\lambda_i(x) = 0$ for $i < m$. But then there exists a small deformation $y$ of $x$ satisfying $\lambda_m(y) > 0$ and $\lambda_i(y) > 0$ for $i < m$.

Let us consider a relation between $\lambda_m$ and the $\lambda_i$ and write $\sum_{i=1}^m v_i\lambda_i = 0$. We take such a relation of minimal length that is the number of non vanishing $v_i$'s is as small a possible. Let us denote by $I$ the set of indices $i < m$ such that $v_i \neq 0$. Consider the restriction of the cone $C$ to the subspace

$\lambda_m = 0$. If this intersection is non empty, we get as before a solution of our system by perturbing the element in the intersection. If the intersection is empty, this implies that there is a relation

$$\sum_{i\in I}^{m} u_i\lambda_i = 0$$

with all $u_i \leq 0$. But by our minimality condition, the linear forms $\lambda_i$ for $i \in I$ are independent. Thus the restriction of these linear forms on the hyperplane $\lambda_m = 0$ satisfy a unique equation. This equation is given by the restriction of the relation $\sum_{i=1}^m v_i\lambda_i = 0$. Therefore that all the $v_i$ for $i < m$ are non positive. But one of the $v_i$ is positive by hypothesis thus $v_m > 0$ and $v_i \leq 0$ for $i < m$. We are done: the system has a solution has soon as it has a solution for the first $m-1$ inequalities. □

**Definition 8.1.2** A vector $x = (x_i)$ will be called **positive** (resp. **non negative**), this will be denoted $x > 0$ (resp. $x \geq 0$) if for all $i$ we have $x_i > 0$ (resp. $x_i \geq 0$).

**Corollary 8.1.3** *If $A = (a_{i,j})$ is an $n \times m$ real matrix for which there is no $x \geq 0$, $x \neq 0$ such that $A^t x \geq 0$ (here $A^t$ is the transpose of $A$), then there exists $v > 0$ such that $Av < 0$.*

**Proof :** We look for a vector $v = (v_1, \cdots, v_m)$ such that $v_i > 0$ for all $i$ and $\lambda_j = \sum_k a_{j,k}v_k < 0$ that is $-\lambda_j > 0$. We know from the previous Lemma that this system has a solution if and only if there is non non trivial dependence relation between the vectors $v_i$ and $-\lambda_j$ with non positive coefficients.

Assume we have such a relation, it can be written as

$$\sum_i a_i v_i + \sum_j u_j(-\lambda_j) = 0$$

with $a_i$ and $u_j$ non positive. This leads to the equation $v^t a = v^t A^t u$ where $a = (a_i)$ is non positive and $u = (u_j)$ is non positive. This gives the equation $A^t u = a$ and we get that $-u \geq 0$ with $A^t(-u) = -a \geq 0$ which is impossible. □

**Lemma 8.1.4** *Let $A$ satisfy the above three properties. Then $Au \geq 0$ and $u \geq 0$ imply that either $u > 0$ or $u = 0$.*

**Proof :** Let $u$ a be non zero vector such that $Au \geq 0$ and $u \geq 0$. We have

$$a_{i,i}u_i \geq \sum_j (-a_{i,j})u_j.$$

So if $u_i$ vanishes, then all the $u_j$ such that there is a sequence $i_0 = i, \cdots, i_k = j$ with $a_{i_0,i_1} \cdots a_{i_{k-1},i_k} \neq 0$ vanish. As the matrix is indecomposable and thanks to our hypothesis on the matrix, this implies by Lemma 4.2.9 that $u = 0$. This is not the case so that $u > 0$.

Here is another proof without Lemma 4.2.9. Reorder the indexes such that $u_i = 0$ for $i < s$ and $u_i > 0$ for $i \geq s$. Then we get that $a_{i,j} = 0 = a_{j,i}$ for $i < s$ and $j \geq s$. The matrix is decomposable except if $u > 0$. □

We may now prove the classification of our matrices into three disjoint categories.

**Theorem 8.1.5** *Let $A$ be a real $n \times n$-matrix satisfying the above three conditions. Then we have the following alternative for both $A$ and $A^t$:*

- $\det(A) \neq 0$, *there exists $u > 0$ such that $Au > 0$ and $Av \geq 0$ implies that $v > 0$ or $v = 0$.*

- Corank$(A) = 1$, *there exists* $u > 0$ *such that* $Au = 0$ *and* $Av \geq 0$ *implies that* $Av = 0$.

- *there exists* $u > 0$ *such that* $Au < 0$ *and* ($Av \geq 0$ *and* $v \geq 0$) *implies that* $v = 0$.

*A matrix of the first type is called a* **finite type matrix***, of the second case, a* **affine type matrix** *and of the third one, a* **indefinite type matrix***.*

**Proof :** Remark that the third case is disjoint from the first two because in the first two there is no $u > 0$ such that $Au < 0$ (set $v = -u$ and apply the result on $Av \geq 0$). Furthermore because of the rank condition on the matrix the first two cases are disjoint.

We consider the following convex cone:

$$K_A = \{u \ / \ Au \geq 0\}.$$

But the previous Lemma, this cone intersects the cone $\{u \geq 0\}$ only in $\{0\}$ or in its interior $\{u > 0\}$. We thus have the inclusion $K_A \cap \{u \geq 0\} \subset \{0\} \cup \{u > 0\}$. Assume this intersection is not reduced to the point 0. Then we have the alternative:

- $K_A$ is contained in $\{u > 0\} \cup \{0\}$ (and thus does not contain any linear subspace)

- $K_A = \{u \ / \ Au = 0\}$ is a 1-dimensional line.

Indeed, we assume that $K_A$ meets the cone $\{u > 0\}$ in some point say $u$. But if there is an element $v$ of $K_A$ outside this cone, because of the convexity of $K_A$, we know that the interval $[u, v]$ is contained in $K_A$ and this interval has to meet the boundary of $\{u \geq 0\}$. This is only possible on the 0 element thus $v$ has to be in the half line generated by $-u$ and the whole line through $u$ is contained in $K_A$. We are in the second case. Assume there is an element $w \in K_A$ not in that line. Then because $K_A$ is a convex cone we get that the whole half plane generated by $w$ and the line trough $u$ is contained in $K_A$ but this half plane will meet the boundary of the cone $\{u \geq 0\}$ outside 0 which is impossible. Because $u$ and $-u$ are in $K_A$, this implies that $Au = 0$.

The first case is equivalent to the finite type case. If $Av \geq 0$, then $v \in K_A$ and $v > 0$ or $v = 0$ because of the inclusion of $K_A$ in $\{u > 0\} \cup \{0\}$. Because $K_A$ does not contain any linear subspace, $A$ has to be non degenerate. In particular $A$ is surjective thus there exists a vector $u$ with $Au > 0$. This $u$ satisfies $u > 0$ or $u = 0$. The last case is not possible since otherwise we would have $Au = 0$.

The second case is equivalent to the affine case: we know that there is an element $u > 0$ such that $Au = 0$ and the kernel of $A$ is $K_A$ and of dimension 1 and we get the condition on $Av \geq 0$.

Furthermore, in the finite (resp. affine) case, we see that there is no $v > 0$ such that $Av < 0$. By Corollary 8.1.3, this implies that there is an element $u$ in the cone $\{u \geq 0\}$ different from 0 and in $K_{A^t}$. In particular $A^t$ is again of finite (resp. affine type), we distinguish the two cases thanks to the rank of the matrix.

In the last case, we know that there is no $u \geq 0$ with $u \neq 0$ such that $A^t u \geq 0$. Lemma 8.1.3 tells us that there exists an element $v > 0$ such that $Av < 0$. If $Aw \geq 0$ and $w \geq 0$ we know that $w = 0$.
□

**Corollary 8.1.6** *Let* $A$ *be as in the previous Theorem, then* $A$ *is of finite (resp. affine, resp. indefinite) type if and only if there exists and element* $u > 0$ *such that* $Au > 0$ *(resp.* $Au = 0$*, resp.* $Au < 0$*).*

## 8.2 Finite and affine cases

### 8.2.1 First results

**Definition 8.2.1** A matrix $(a_{i,j})_{i,j\in I}$ with $I \subset [1,n]$ is called **a principal submatrix** of $A = (a_{i,j})_{i,j\in[1,n]}$. The determinant of a principal submatrix is a **principal minor**.

**Lemma 8.2.2** *Let $A$ be indecomposable of finite or affine type. Then any proper principal submatrix of $A$ decomposes into a direct sum of matrices of finite type.*

**Proof :** Let $I \subset [1,n]$ and let $A_I$ be the associated principal submatrix. If $u = (u_i)_{i\in[1,n]}$, denote by $u_I$ the vector $(u_i)_{i\in I}$. We know by hypothesis that there exists a vector $u > 0$ such that $Au \geq 0$. Consider the subproduct $A_I u_I$. Because all the $a_{i,j}$ for $i \neq j$ are non positive, this implies that $(Au)_I \leq A_I u_I$ with equality if and only if for all $i \in I$ and all $j \notin I$ we have $a_{i,j} = 0$. We thus have $u_I > 0$ and $A_I u_I \geq 0$ which implies that $A_I$ is of affine or finite type. If it is of affine type then $A_I u_I \geq 0$ implies $A_I u_I = 0$ and in particular $A_I u_I = (Au)_I$. This means that $I = [1,n]$ or $A$ is decomposable. □

**Lemma 8.2.3** *A symmetric matrix $A$ is of finite (resp. affine) type if and only if $A$ is positive definite (resp. positive semidefinite of corank 1).*

**Proof :** Assume $A$ is positive semidefinite, then if there exists $u > 0$ with $Au < 0$, then $u^t Au < 0$ which is impossible. In particular $A$ is of finite of affine type. The rank condition distinguishes the two cases.

Conversely, if $A$ is of finite or affine type, then there exists an $u > 0$ such that $Au \geq 0$. For $\lambda > 0$, we have $(A + \lambda I)u > 0$ thus $A$ is of finite type and hence non degenerate. Therefore the eigenvalues of $A$ are non negative and the result follows with the rank condition. □

**Lemma 8.2.4** *Let $A = (a_{i,j})_{i,j\in[1,n]}$ be a matrix of finite or affine type such that $a_{i,i} = 2$ and $a_{i,j}a_{j,i} \geq 1$. Then $A$ is symmetrisable. Moreover if there exists a sequence $i_1, i_2, i_3, \cdots, i_k$ of indices with $k \geq 3$ such that $a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_{k-1},i_k}a_{i_k,i_1} \neq 0$, then $A$ is of the form*

$$\begin{pmatrix} 2 & -u_1 & \cdots & 0 & -u_n^{-1} \\ -u_1^{-1} & 2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 2 & -u_{n-1} \\ -u_n & 0 & \cdots & -u_{n-1}^{-1} & 2 \end{pmatrix}$$

*where the $u_i$ are some positive numbers.*

**Proof :** Let us prove that the second part of the Lemma implies the first one. Indeed, if there is no such sequence, then the conditions of Proposition 7.1.3 are satisfies (all products $a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_{k-1},i_k}a_{i_k,i_1}$ vanish) and the matrix is symmetrisable).

Consider such a sequence of indices with $a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_{k-1},i_k}a_{i_k,i_1} \neq 0$. We may reorder the indices and suppose that $i_j = j$ so that we have a principal submatrix of $A$ of the form

$$B = \begin{pmatrix} 2 & -b_1 & \cdots & * & -b_k \\ -b'_1 & 2 & & * & * \\ \vdots & & \ddots & & \vdots \\ * & * & & 2 & -b_{k-1} \\ -b'_k & * & \cdots & -b'_{k-1} & 2 \end{pmatrix}$$

where the $*$ are non positive elements and the $b_i$ and $b'_i$ are positive. We know that this matrix $B$ is of affine or finite type so that there exists a vector $u > 0$ such that $Bu \geq 0$. Let $U$ be the diagonal matrix with the coefficients of $u$ on the diagonal. Replacing $B$ by $U^tBU$, we may assume that $u^t = (1, \cdots, 1)$. The condition $Bu \geq 0$ gives us conditions $\sum_j B_{i,j}u_j \geq 0$ and by sum of all these conditions and the fact that the $*$ in $B$ are non positive we get

$$2k - \sum_{i=1}^{k}(b_i + b'_i) \geq 0.$$

But because $b_ib'_i \geq 1$, we get that $b_i + b'_i \geq 2$ (the roots of $X^2 - (b_i + b'_i)X + b_ib'_i$ are real positive thus the discriminant $(b_i + b'_i)^2 - 4b_ib'_i$ is non negative and $b_i + b'_i \geq 2$). But this, together with the previous inequality, implies that for all $i$, we have $b_i = b'_i = 1$ and all the $*$ in $B$ vanish. But then $\det(B) = 0$ and thus $A = B$ (because non proper principal submatrix is affine. Conjugating with $U$ gives the matrix of the lemma. □

To summarise properties of generalised Cartan matrices (as for classical Cartan matrices), it is usefull to define the associated Dynkin diagram $D(A)$.

**Definition 8.2.5** The Dynkin diagram $D(A)$ of a generalised Cartan matrix $A$ is the graph whose vertices are indexed by the row of the matrix (i.e. by $[1, n]$) and whose edges are described as follows

- if $a_{i,j}a_{j,i} \leq 4$ and $|a_{i,j}| \geq |a_{j,i}|$, the vertices $i$ and $j$ are connected by $|a_{i,j}|$ lines equipped with an arrow pointing toward $i$ if $|a_{i,j}| > 1$;
- if $a_{i,j}a_{j,i} > 4$, the vertices $i$ and $j$ are connected by a line colored by the ordered pair of integers $|a_{i,j}|$, $|a_{j,i}|$.

The matrix $A$ is indecomposable if and only if $D(A)$ is connected and the matrix $A$ is determined by the Dynkin diagram $D(A)$ modulo permutation of the indices. We say that the Dynkin diagramm $D(A)$ is of finite, affine or indefinite type if $A$ is.

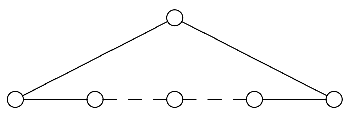
**Proposition 8.2.6** *Let $A$ be any indecomposable generalised Cartan matrix, then we have:*

*(ı) the matrix $A$ is of finite type if and only if all its principal minors are positive.*

*(ıı) The matrix $A$ is of affine type if and only if all its proper principal minors are positive and* $\det(A) = 0$.

*(ııı) If $A$ is of finite or affine type, then any proper subdiagram of $D(A)$ is an union of Dynkin diagrams of finite type.*

*(ıv) If $A$ is of finite type or affine type and if $D(A)$ contains a cycle of length at least 3, then $D(A)$ is the following cycle:*



*(v) The matrix $A$ is of affine type if and only if there exists a vector $\delta > 0$ with $A\delta = 0$. Such a vector is unique up to scalar multiple.*

**Proof :** Let us prove (ı) and (ıı). Let $A_I$ be a principal submatrix of $A$, it is of finite or affine type and is a generalised Cartan matrix. We know from Lemma 8.2.4 that any generalised Cartan matrix of finite or affine type is symmetrisable. Let $D_I$ be a diagonal matrix with positive diagonal coefficients such that $B_I = D_IA_I$ is symmetric. The matrix $B_I$ has the same type as $A_I$ and by Lemma 8.2.3

this implies that $B_I$ is positive and even positive definite if $A_I$ is proper or if $A$ is of finite type. In particular in all these cases we have $\det(B_I) > 0$ and thus $\det(A_I) > 0$.

Conversely, if all principal minors of $A$ are positive, then assume there exists a vector $u > 0$ with $Au < 0$. We get inequalities of the form $\sum_j a_{i,j}u_j < 0$. Now we eliminate the variables $u_i$ for $i > 1$ in the equation $\sum_i a_{1,i}u_i < 0$. For this we remove $a_{1,i}/2$ times the inequality $\sum_j a_{i,j}u_j < 0$. Because $a_{1,i}leq0$ for $i > 1$ we end up with an inequality of the form $\lambda u_1 < 0$. Furthermore, we have $\det(A) = \lambda \det(A')$ where $A'$ is the submatrix of $A$ defined by the indices $[2, n]$. Because all the proper minors are positive, we get that $\lambda > 0$ in the first case and $\lambda = 0$ in the second case. In both case the inequalities $u_1 > 0$ and $\lambda u_1 < 0$ are impossible. The matrix $A$ is thus of finite or affine type and the cases are described by the rank.

(ɪɪɪ) This is a direct consequence of Lemma 8.2.2.

(ɪv) If $D(A)$ contains a cycle of length at least 3, this implies that the condition: *there exists a sequence $i_1, i_2, i_3, \cdots, i_k$ of indices with $k \geq 3$ such that* $a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_{k-1},i_k}a_{i_k,i_1} \neq 0$, of Lemma 8.2.4 is satisfied. The matrix $A$ is thus given by Lemma 8.2.4. Furthermore, in the case of generalised Cartan matrices, if $u_i$ and $u_i^{-1}$ are positive integers, this implies that $u_i = 1$. The associated Dynkin diagram is the desired cycle.

(v) This is a direct consequence of Theorem 8.1.5. □

### 8.2.2 Examples of finite and affine type matrices

In this subsection we give examples of generalised Cartan matrices of finite and affine type and we describe their Dynkin diagrams.

**Proposition 8.2.7**

*(ı) The following matrices are generalised Cartan matrices of finite type. Their associated Dynkin diagrams are given in Table 1.*

Cartan matrices

$$A_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$B_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -2 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$C_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{pmatrix} = B_n^t$$

$$D_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 2 & -1 & 0 & 0 \\ 0 & \ddots & -1 & 2 & -1 & -1 \\ \vdots & \ddots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$E_6 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_7 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$F_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Dynkin diagrams

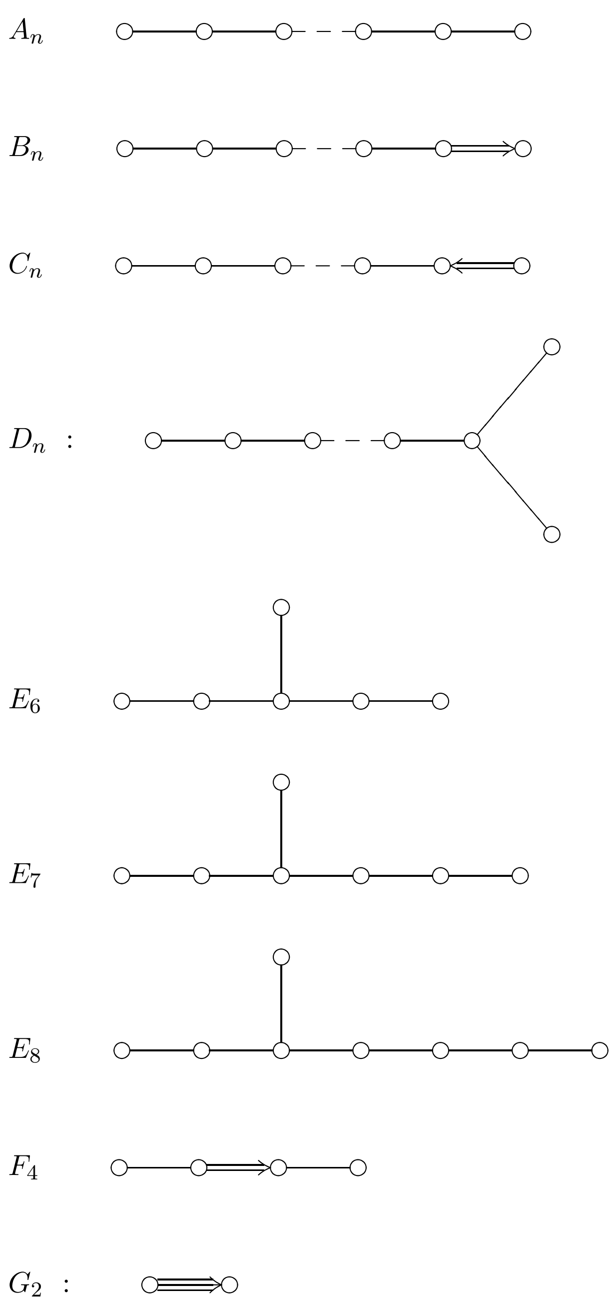



*Table 1.*

*(ii) The following matrices are generalised Cartan matrices of affine type. Their associated Dynkin diagrams are given in Table 2.1, Table 2.2 and Table 2.3. Furthermore the smallest vector $\delta$ with positive integers values such that $A\delta = 0$ is given by $\delta = \sum_i a_i e_i$ where the $a_i$ are the coefficients in the vertices and the $e_i$ are the vector of the canonical basis. The node 0 is represented by a square. We will often forget the tilde in the sequel and denote $A_n^1$ instead of $\widetilde{A}_n^1$ for example. The exponent 4 on the arrow of $\widetilde{A}_2^2$ mens that we have a quadruple arrow.*

CARTAN MATRICES

*Order 1.*

$$\widetilde{A}_1^1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$\widetilde{A}_n^1 = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\widetilde{B}_n^1 = \begin{pmatrix} 2 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 \\ -1 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{pmatrix}$$

$$Ct_n^1 = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -2 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -2 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\widetilde{D}_n^1 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & -1 & 0 & \ddots & \ddots & 0 \\ -1 & -1 & 2 & -1 & \ddots & \ddots & 0 \\ 0 & 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & \ddots & -1 & 2 & -1 & -1 \\ \vdots & \vdots & \ddots & 0 & -1 & 2 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$\widetilde{E}_6^1 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\widetilde{E}_7^1 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\widetilde{E}_8^1 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\widetilde{F}_4^1 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -2 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\widetilde{G}_2^1 = \begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Cartan matrices

*Order 2.*

$$\widetilde{A}_2^2 = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

$$\widetilde{A}_{2n}^2 = \begin{pmatrix} 2 & -2 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -2 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\widetilde{A}^2_{2n-1} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & -1 & 0 & \ddots & \ddots & 0 \\ -1 & -1 & 2 & -1 & \ddots & \ddots & 0 \\ 0 & 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & \ddots & -1 & 2 & -1 & 0 \\ \vdots & \vdots & \ddots & 0 & -1 & 2 & -2 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\widetilde{D}^2_{n+1} = \begin{pmatrix} 2 & -2 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{pmatrix}$$

$$\widetilde{E}^2_6 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

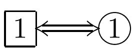
Cartan matrices

*Order 3.*

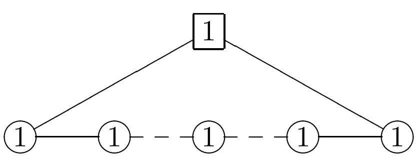
$$\widetilde{D}^3_4 = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Dynkin diagrams

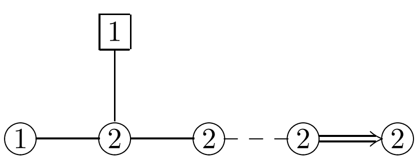
$\widetilde{A}^1_1$



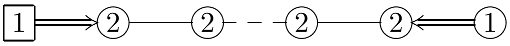

$\widetilde{A}_n$



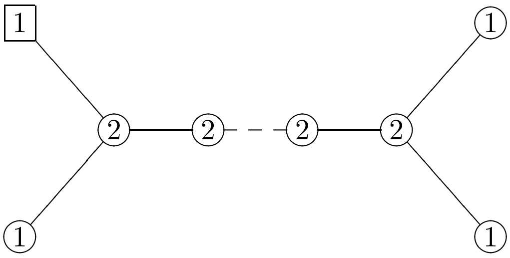

$\widetilde{B}^1_n$



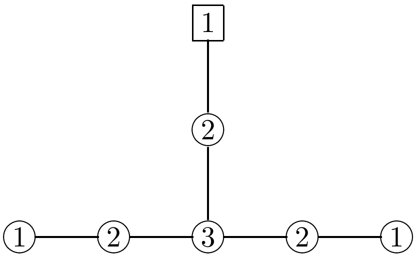
$\widetilde{C}_n^1$



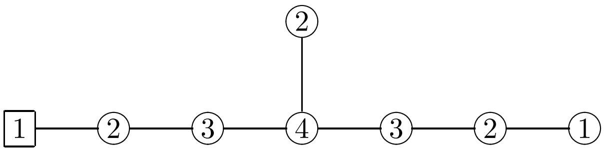

$\widetilde{D}_n^1$



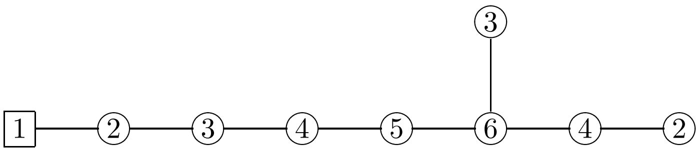

$\widetilde{E}_6^1$



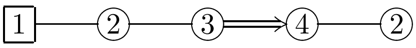

$\widetilde{E}_7^1$




$\widetilde{E}_8^1$




$\widetilde{F}_4^1$




$\widetilde{G}_2^1$

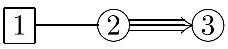
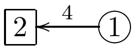
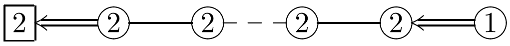



*Table 2.1*

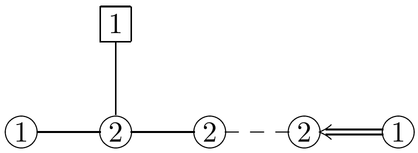
$\widetilde{A}_2^2$



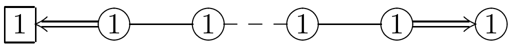

$\widetilde{A}_{2n}^2$




$\widetilde{A}_{2n-1}^2$




$\widetilde{D}_{n+1}^2$




$\widetilde{F}_4^2$

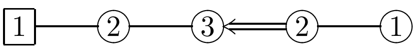



*Table 2.2*

$\widetilde{D}_4^3$

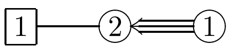



*Table 2.3*

**Remark 8.2.8** (i) One may ask: where do these matrices come from. For the finite type case, these are the Cartan matrices of simple Lie algebras.We will see in Chapter 12 that there is a natural construction associating to any Lie algebra $\mathfrak{g}$ a Lie algebra $\mathfrak{t}$ which is essentially the loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. This construction with $\mathfrak{g}$ a simple Lie algebra give rise to the affine matrices of order 1 described in the proposition. When taking a finite order automorphism of a simple Lie algebra and performing this construction twisted by the automorphism, we get the order 2 and 3 cases (for more details, see Chapter 12.

(ii) One may remark that, in the finite type case, the matrix is symmetric if and only if the Dynkin diagram has only simple edges. This case is called the **simply laced** case. Remark also that even if the matrices $F_4$ and $G_2$ are not symmetric, their transpose give rise to the same Lie algebra, we only need to reorder the simple roots. However the matrices of type $B_n$ and $C_n$ are exchanged by transposition.

Things are a little more complicated for affine type matrices because the order may change while transposing the matrix but we still have that the transpose of an affine type matrix if of affine type. We get the following equalities:

$$^t\widetilde{B}^1_n = \widetilde{A}^2_{2n-1},\quad ^t\widetilde{C}^1_n = \widetilde{D}^2_{n+1},\quad ^t\widetilde{F}^1_4 = \widetilde{E}^2_6 \text{ and } ^t\widetilde{G}^1_2 = \widetilde{D}^3_4.$$

the other matrices are symmetric or their transpose give rise to the same matrix (and thus the same Lie algebra $\mathfrak{g}(A)$) after reordering the indices.

**Proof :** To prove this proposition, we only need to check that the vector $\delta$ given by

$$\delta = \sum_i a_i e_i$$

satisfies $A\delta = 0$ for all the affine type matrices and then to prove that the finite type matrices (or Dynkin diagrams) are principal proper submatrices of matrix of affine type.

This is done case by case. Remark however that in the order 1 case, the vector $\delta$ is the sum $\theta + \alpha_0$ of the longest root of the finite root system and the simple added root. □

### 8.2.3 Classification of finite and affine type matrices

We may now prove the following classification Theorem:

**Theorem 8.2.9** *All the indecomposable generalised Cartan matrices of finite or affine type are those given in Proposition 8.2.7*

**Proof :** We proceed by induction on the number of vertices of the Dynkin diagram. The only matrix of size $1 \times 1$ being the matrix $A = (2)$ of type $A_1$. It is also clear that the rank 2 finite and affine type indecomposable generalised Cartan matrices are given by matrices of type $A_2$, $B_2$, $C_2$, $G_2$, $A^1_1$ and $A^2_2$. We will also need that the following matrices are not of finite or affine type:

$$\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}.$$

But the vectors $v = (2,5,7)$, $v = (4,5,7)$ and $v = (7,15,7)$ are such that $v > 0$ and $Av < 0$ proving the result.

For the induction, we proceed as follows: a diagram $D$ of finite or affine type with at least three vertices is obtained from a smaller diagram $D'$ of finite type (and at least two vertices) by adding a vertex (by Lemma 8.2.2). Furthermore, the diagram $D$ has to be such that when removing any vertex, we obtain a finite type diagram. Furthermore, thanks to Proposition 8.2.6 we may assume that $D$ has no cycle.

Starting with type $A_n$, we may add a vertex with simple edge to root 1 and root 2 (or the last two roots) to root 3 (or $n-2$) if and only if $n \leq 8$ and also to root 4 if $n \leq 7$. We get diagrams of type $A_{n+1}$, $D_{n+1}$, $E_6$, $E_7$, $\widetilde{E}^1_7$, $E_8$ or $\widetilde{E}^1_9$. We may now add a vertex with double edge to root 1 (or $n$) or to root 2 if $n = 3$. We get diagrams of type $B_{n+1}$, $C_{n+1}$, $\widetilde{B}^1_3$ and $\widetilde{A}^2_5$. We may now add a vertex with triple edge to root 1 (or $n$) if $n = 2$. We get diagrams of type $\widetilde{G}^1_2$ and $\widetilde{D}^3_4$.

Starting with type $B_n$ or $C_n$, we may add a vertex with simple edge to root 1 and root 2, we may also add it to root $n$ if and only if $n \leq 4$. We get diagrams of type $B_{n+1}$, $C_{n+1}$, $\widetilde{B}^1_n$, $\widetilde{A}^2_{2n-1}$, $F_4$, $\widetilde{F}^1_4$ or $\widetilde{E}^2_6$. We may now add a vertex with double edge to root 1. We get diagrams of type $\widetilde{C}^1_n$, $\widetilde{A}^2_{2n}$ and $\widetilde{D}^2_{n+1}$. We may not add a vertex with triple edge.

In type $E$, it is clear that we may only add vertices with simple edges. Furthermore, these vertices can be add only to end vertices of the diagram. We get diagrams of type $\widetilde{E}_6^1$, $E_7$, $\widetilde{E}_7^1$ $E_8$ and $\widetilde{E}_8^1$.

For $F_4$, we may only add simple vertices to the end vertices of the diagram. We get diagrams of type $\widetilde{E}_6^2$ and $\widetilde{F}_4^1$.

Finally, for $G_2$, □

We end with a characterisation of Kac-Moody Lie algebras $\mathfrak{g}(A)$ isomorphic to simple finite dimensional Lie algebras.

**Proposition 8.2.10** *The following conditions are equivalent:*

*(ı) The matrix $A$ is a generalised Cartan matrix of finite type.*

*(ıı) The matrix $A$ is symmetrisable and the bilinear form $(\ ,\ )|_{\mathfrak{h}}$ is positive definite.*

*(ııı) The Weyl group $W$ is finite.*

*(ıv) The root system is finite.*

*(v) The Lie algebra $\mathfrak{g}(A)$ is simple and finite dimensional.*

*(vı) There exists a root $\theta \in \Delta$ such that for any simple root $\alpha_i$, we have $\theta + \alpha_i \not\in \Delta$. Such a root is called* **a highest root***.*

**Proof :** (ı) ⇒ (ıı) We know by Lemma 8.2.4 that the matrix $A$ is symmetrisable (so that $(\ ,\ )|_{\mathfrak{h}}$ is well defined) and we also know that the matrix $B = D^{-1}A$ defining the bilinear form is of finite type (because we have $u > 0$ such that $Au > 0$ thus $Bu = D^{-1}Au > 0$ because $D$ can be chosen positive). Now Lemma 8.2.3 gives that the form is positive definite.

(ıı) ⇒ (ııı) Follows from Proposition 6.2.11 and is an equivalence.

(ııı) ⇒ (ıv) Follows from Theorem 6.5.2 and is an equivalence.

(ıv) ⇒ (v) The bilinear form $(\ ,\ )|_{\mathfrak{h}}$ is positive definite and in particular $A$ is regular thus by Proposition 4.2.10 we get that $\mathfrak{g}(A)$ is simple and because all the weight spaces are of finite dimension, the Lie algebra $\mathfrak{g}(A)$ is of finite dimension.

(v) ⇒ (vı) We take $\theta$ a root of maximal height and the result follows.

(vı) ⇒ (ı) Let $\theta$ be a root such that for all simple root $\alpha_i$ we have $\theta + \alpha_i \not\in \Delta$. Consider the $\mathfrak{g}_{(i)}$-submodule of $\mathfrak{g}(A)$ generated by an element $x$ in $\mathfrak{g}_\theta$. It is of finite dimension and by the $\mathfrak{sl}_2$ theory, we obtain that $x$ is a highest weight vector of non negative weight $\langle \alpha_i, \theta \rangle$. In particular we have $\langle \alpha_i, \theta \rangle \geq 0$ for all $i$. This implies that the matrix $A$ is of affine or finite type. If it was of affine type we would have $\langle \alpha_i, \theta \rangle = 0$ for all $i$. Furthermore, if $\theta$ was negative, the condition on $\theta$ would imply that $x = 0$ thus $\theta > 0$ and because $x$ is non zero, there exists an index $i$ such that $[f_i, x] \neq 0$ i.e. $\theta - \alpha_i \in \Delta$. But $\langle \alpha_i, \theta \rangle = 0$ is the highest weight (and thus the lowest weight) of the $\mathfrak{g}_{(i)}$-module generated by $x$, this is a contradiction. □

# Chapter 9

# Real and imaginary roots

In this chapter, we continue the classification of Cartan matrices by studying the root systems of the associated Kac-Moody Lie algebra. On important new feature in this setting is the appearence of imaginary roots i.e. roots not in the orbit of simple roots under the Weyl group.

## 9.1 Definitions and first properties

### 9.1.1 real roots

**Definition 9.1.1** A root $\alpha \in \Delta$ is called **real** if there exists $w \in W$ such that $w(\alpha)$ is a simple root. We denote by $\Delta^{\rm re}$, $\Delta^{\rm re}_+$ and $\Delta^{\rm re}_-$ the set of real roots, positive reals roots and negative real roots.

Recall the definition of the coroot of a real root defined in Chapter 5 (see Definition 5.2.8): a real root $\alpha$ can be written $\alpha = w(\alpha_i)$ for $w \in W$ and $\alpha_i$ a simple root, the coroot $\alpha^\vee$ is defined by $w(\alpha_i^\vee)$ (this is well defined by 5.2.7). In particular we have a canonical bijection between $\Delta^{\rm re}$ and $\Delta^{\vee {\rm re}}$.

We may also define the reflection $s_\alpha$ with respect to any real root $\alpha$ and acting on $\mathfrak{h}^*$ by:

$$s_\alpha(\lambda) = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha, \quad for \lambda \in \mathfrak{h}^*.$$

This is a reflection because $\langle \alpha^\vee, \alpha \rangle = \langle \alpha_i^\vee, \alpha_i \rangle = 2$ and it lies in the Weyl group $W$ because $s_\alpha = w s_i w^{-1}$ for $w$ and $i$ such that $\alpha = w(\alpha_i)$. The real root satisfy similar properties as roots in finite root systems:

**Proposition 9.1.2** *Let $\alpha$ be a real root in a Kac-Moody algebra $\mathfrak{g}(A)$. Then we have:*

*(ı)* $\text{mult}\alpha = 1$.

*(ıı) For $k \in \mathbb{Z}$, the element $k\alpha$ is a root if and only if $k = \pm 1$.*

*(ııı) Suppose that $A$ is symmetrisable and let ( , ) be an invariant bilinear form on $\mathfrak{g}(A)$ as defined in Theorem 7.2.5. Then we have*

- $(\alpha, \alpha) > 0$
- $\alpha^\vee = 2\nu^{-1}(\alpha)/(\alpha, \alpha)$.

*(ıv) If $\alpha$ and $-\alpha$ are not simple, then there exists an index $i$ such that $|\text{ht}(s_i(\alpha))| < |\text{ht}(\alpha)|$.*

**Proof :** We already proved points (ı), (ıı) and (ııı) for simple roots and these properties are invariant under the action of the Weyl group. The result follows. For the last statement, take $\alpha$ a real positive (for example) root such that for all $i$ we have $|\text{ht}(s_i(\alpha))| \geq |\text{ht}(\alpha)|$. This implies that for all $i$ we have $\langle \alpha_i^\vee, \alpha \rangle \leq 0$ and in particular $-\alpha \in C^\vee$ is in the dominant chamber for the dual root system. This implies that $v(\alpha)-\alpha$ for any $v\in W$ is a non negative linear combination of positive roots. Apply this to $v=w^{-1}$ with $\alpha=w(\alpha_i)$ to get $\alpha_i-\alpha$ as a non negative linear combination of positive roots. We thus have

$$\mathrm{ht}(\alpha_i)\geq\mathrm{ht}(\alpha).$$

This implies that $\alpha$ is simple. □

**Definition 9.1.3** Let $A$ be a symmetrisable matrix and $(\ ,\ )$ an associated invariant bilinear form. A real root $\alpha$ is a **short** root if $|\alpha|^2=(\alpha,\alpha)=\min_i|\alpha_i|^2=(\alpha_i,\alpha_i)$.

Remark that this definition does not depend on the invariant bilinear form choosen.

### 9.1.2 Imaginary roots

**Definition 9.1.4** A root $\alpha$ which is not real is called **imaginary**. We denote by $\Delta^{\mathrm{im}}$, $\Delta^{\mathrm{im}}_+$ and $\Delta^{\mathrm{im}}_-$ the set of imaginary, positive imaginary and negative imaginary roots.

We have the following disjoint unions

$$\Delta=\Delta^{\mathrm{re}}\cup\Delta^{\mathrm{im}},\ \Delta^{\mathrm{re}}=\Delta^{\mathrm{re}}_+\cup\Delta^{\mathrm{re}}_-\ \text{ and }\ \Delta^{\mathrm{im}}=\Delta^{\mathrm{im}}_+\cup\Delta^{\mathrm{im}}_-.$$

**Proposition 9.1.5** *(ı) The set $\Delta^{\mathrm{im}}_+$ is $W$-invariant.*

*(ıı) For $\alpha\in\Delta^{\mathrm{im}}_+$, there exists a unique root $\beta\in-C^\vee$ in the $W$ orbit of $\alpha$.*

*(ııı) If $A$ is symmetrisable and $(\ ,\ )$ is a standard invariant bilinear form, then a root $\alpha$ is imaginary if and only if $(\alpha,\alpha)\leq0$.*

**Proof :** (ı) Remark that this is not the case for $\Delta^{\mathrm{re}}_+$ since for example for any positive root $\alpha$ we have $s_\alpha(\alpha)=-\alpha<0$.

It is clear that $\Delta^{\mathrm{im}}$ is $W$-invariant (because $\Delta$ and $\Delta^{\mathrm{re}}$ are $W$-invariant). We thus need to prove that if $\alpha\in\Delta^{\mathrm{im}}_+$, then $w(\alpha)$ is still positive for all $w\in W$. It suffices to show that $s_i(\alpha)$ is positive for all index $i$ and this is clear because $\alpha$ is different from $\alpha_i$.

(ıı) Let $\alpha\in\Delta^{\mathrm{im}}_+$ and consider the orbit of $\alpha$ under the Weyl group. It is contained in the set of positive root. Pick $\beta$ an element of minimal height is that orbit. We have for all index $i$ that $\langle\alpha_i^\vee,\beta\rangle\leq0$ thus $\beta\in-C^\vee$. Furthermore because $C^\vee$ is a fundamental domain for the action of the Weyl group $\beta$ is unique.

(ııı) Let $\alpha\in\Delta^{\mathrm{im}}_+$. We may assume by (ıı) that $-\alpha\in C^\vee$. Write $\alpha=\sum_i a_i\alpha_i$ with $a_i\geq0$. We have

$$\sum_{i\neq j}a_i\langle\alpha_j^\vee,\alpha_i\rangle+2a_j\leq0\ \ \textit{or}\ \ \sum_{i\neq j}a_i2(\alpha_j,\alpha_i)+2a_j(\alpha_j,\alpha_j)\leq0.$$

Therefore we have the inequality:

$$(\alpha,\alpha)=\sum_i a_i^2(\alpha_i,\alpha_i)+\sum_j\sum_{i\neq j}a_ia_j(\alpha_i,\alpha_j)\leq\sum_i a_i^2(\alpha_i,\alpha_i)-\sum_j a_j^2(\alpha_j,\alpha_j)=0.$$

If $\alpha$ is real we already know that $(\alpha,\alpha)>0$. □

**Corollary 9.1.6** *If $A$ is of finite type, then the Kac-Moody algebra has no imaginary root.*

**Proof :** Indeed, in that case we know that the matrix is always symmetrisable and an imaginary root $\alpha$ satisfy $(\alpha,\alpha)\leq0$. However the bilinear form is positive definite is the finite case, a contradiction. □

**Definition 9.1.7** Let $\alpha \in Q$ and write $\alpha = \sum_i a_i\alpha_i$. We define **the support** of $\alpha$, denoted Supp$\alpha$, to be the full subdiagram of the Dynkin diagram consisting of the vertices $i$ such that $a_i \neq 0$.

**Lemma 9.1.8** *The support of a root is connected.*

**Proof :** We first prove:

**Lemma 9.1.9** *Let $I_1$ and $I_2$ be disjoint subsets of the set of indices $I$ such that $a_{i,j} = 0$ for $i \in I_1$ and $j \in I_2$. Let $\beta_k = \sum_{i \in I_k} a_i(k)\alpha_i$ for $k \in \{1, 2\}$. If $\beta_1 + \beta_2$ is a root of $\mathfrak{g}(A)$, then $\beta_1$ or $\beta_2$ vanishes.*

**Proof :** Let $i \in I_1$ and $j \in I_2$, then $[\alpha_i^\vee, e_j] = 0$, $[\alpha_j^\vee, e_i] = 0$, $[e_i, f_j] = 0$ and $[e_j, f_i] = 0$. Compute for any $k$ the Lie bracket:

$$[f_k, [e_i, e_j]] = [[f_k, e_i], e_j] + [e_i, [f_k, e_j]] = \delta_{k,i}[\alpha_i^\vee, e_j] + \delta_{k,j}[e_i, \alpha_j^\vee] = 0.$$

The same computation gives $[e_k, [f_i, f_j]] = 0$ for all $k$. This implies that $[e_i, e_j] = [f_i, f_j] = 0$. For $k \in \{1, 2\}$, denote by $\mathfrak{g}_k$ the Lie subalgebra of $\mathfrak{g}(A)$ generated by the $e_i$ and $f_i$ for $i \in I_k$. The Lie subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ commute. Furthermore, $\mathfrak{g}_{\beta_1+\beta_2}$ lies in the subalgebra generated by $\mathfrak{g}_1$ and $\mathfrak{g}_2$, this implies that it is either contained in $\mathfrak{g}_1$ or in $\mathfrak{g}_2$. □

This lemma proves that the support of a root is connected. □

We define the following subset of $Q_+$:

$$K = \{\alpha \in Q_+ \setminus \{0\} \ / \ \text{Supp}\alpha \text{ is connected and } \ \langle \alpha, \alpha_i^\vee \rangle \leq 0 \text{ for all } i\}.$$

**Proposition 9.1.10** *We have the inclusion $K \subset \Delta_+^{\text{im}}$.*

**Proof :** Let us first remark that in the finite case, the proposition is easy to prove. Indeed, there are no imaginary roots but the set $K$ is empty: if $\alpha \in Q_+$ is such that $\langle \alpha_i^\vee, \alpha \rangle \leq 0$, then because of the formula

$$\langle \alpha_i^\vee, \alpha \rangle = 2\frac{(\alpha_i, \alpha)}{(\alpha_i, \alpha_i)}$$

we have $(\alpha_i, \alpha) \leq 0$ for all $i$. This implies that $(\alpha, \alpha) \leq 0$ and because the form is positive definite that $\alpha = 0$. We may thus assume that $A$ is not of finite type.

Let $\alpha = \sum_i a_i\alpha_i \in K$ and consider

$$\Omega_\alpha = \{\gamma \in \Delta_+ \ / \ \gamma \leq \alpha\}$$

where the order is defined by $Q_+$. The set $\Omega_\alpha$ is finite and non empty because Supp$\alpha \subset \Omega_\alpha$. Let $\beta = \sum_i b_i\alpha_i$ be an element of maximal height in $\Omega_\alpha$.

**Lemma 9.1.11** *We have* Supp$\beta$ = Supp$\alpha$.

**Proof :** Assume that it is not the case. We thus have a simple root $\alpha_i \in$ Supp$\alpha$ such that $\alpha_i \notin$ Supp$\beta$. Furthermore, we may assume that writing $\beta = \sum_j b_j\alpha_j$ there exists an index $j$ such that $b_j\langle a_i^\vee, \alpha_j \rangle \neq 0$ (this is because Supp$\alpha$ is connected). But $\beta + \alpha_i$ is not a root thus the weight $\langle \alpha_i^\vee, \beta \rangle$ is a maximal weight of a $\mathfrak{g}_{(i)}$-module and we have $\langle a_i^\vee, \beta \rangle \geq 0$. But $\beta = \sum_{j \neq i} b_j\alpha_j$ with $b_j \geq 0$ thus $\langle a_i^\vee, \beta \rangle < 0$ a contradiction. □

Let us first prove that $\alpha \in \Delta_+$. Suppose it is not the case, then $\alpha \neq \beta$ and for all index $i$ such that $b_i < a_i$ we have $\beta + \alpha_i \notin \Delta_+$ (and even not in $\Delta$). If this were true for all index $i$ then $\mathfrak{g}(A)$ would have an highest root and thus be of finite type. This is not the case so that there exists an index $i$ such that$a_i = b_i$. Let $I'$ be the set of such indices and consider $R$ a connected components of the diagram $(\mathrm{Supp}\alpha) \setminus I'$. Because $\beta + \alpha_i \notin \Delta$ for any index $i \in R$, we deduce by the usual $\mathfrak{sl}_2$ argument that $\langle \alpha_i^\vee, \beta \rangle \geq 0$.

On the one hand, let us now set $\beta' = \sum_{i \in R} b_i \alpha_i$. Because $\mathrm{Supp}\alpha$ is connected and $I'$ is not empty, the boundary of $R$ — denoted by $\partial R$ and defined as the set of indices $j \in \mathrm{Supp}\alpha \setminus R$ such that there exists an index $i \in R$ with an edge between $i$ and $j$ — is not empty. We get, for $i \in R$, that

$$\langle \alpha_i^\vee, \beta' \rangle = \langle \alpha_i^\vee, \beta \rangle - \sum_{j \in \partial R} b_j \langle \alpha_i^\vee, \alpha_j \rangle \geq 0.$$

Furthermore, because $\partial R$ is not empty and because for $j \in \partial R$, we have $b_j = a_j > 0$ we get that for some $i \in R$ (in fact for any index $i$ connected to an element of $\partial R$) we have $\langle \alpha_i^\vee, \beta' \rangle > 0$. Therefore, considering the submatrix $A_R$ of $A$ defined by $R$, we have a vector $\beta'$ such that $\beta' \geq 0$, $A_R \beta' \geq 0$ and $A_R \beta' \neq 0$. This implies that the Dynkin diagram of $A_R$ is of finite type.

On the other hand, set $\alpha' = sum_{i \in R}(a_i - b_i)\alpha_i$. *Supp*$\alpha'$ is $R$ and is a connected component of $\mathrm{Supp}(\alpha - \beta)$. This implies the formula

$$\langle \alpha_i^\vee, \alpha' \rangle = \langle \alpha_i^\vee, \alpha - \beta \rangle$$

for $i \in R$. But because $\alpha \in K$, we have for all index $i \in R$ the inequality $\langle \alpha_i^\vee, \alpha \rangle \leq 0$. This leads to $\langle \alpha_i^\vee, \alpha' \rangle \leq$ for all $i \in R$ and because $R$ is of finite type we must have $\alpha' = 0$ thus $R = \emptyset$.

Now we proved that any element in $K$ is a positive root. But the condition defining $K$ are invariant under scalar multiplication. In particular if $\alpha$ is a root contained in $K$, then the same is true for $k\alpha$ for all $k \in \mathbb{N}$. This implies that $\alpha$ is imaginary. □

We can describe the set of imaginary roots thanks to $K$:

**Theorem 9.1.12** *We have the following equality*

$$\Delta_+^{\mathrm{im}} = \bigcup_{w \in W} w(K).$$

**Proof :** Let $\alpha \in \Delta_+^{\mathrm{im}}$. We know that there exists an unique root $\beta$ in its $W$-orbit such that $-\beta \in C^\vee$. Because $\beta$ is a root, its support is connected and we thus have $\beta \in K$, the result follows. □

**Corollary 9.1.13** *If $\alpha$ is a positive imaginary root and $r$ is a rational number such that $r\alpha \in Q$, then $r\alpha$ is again an imaginary root.*

**Proof :** This comes directly from the fact that if $\alpha \in K$ and $r \in \mathbb{Q}$ are such that $r\alpha \in Q$, then $r\alpha \in K$. □

We now describe the imaginary roots according to the classification of Cartan matrices and in particular prove their existence in the non finite cases:

**Theorem 9.1.14** *Let $A$ be an indecomposable generalised Cartan matrix.*

*(ı) If $A$ is finite, then $\Delta_+^{\mathrm{im}}$ is empty.*

*(ıı) If $A$ is of finite type, then $\delta_+^{\mathrm{im}} = \{n\delta \ / \ n \in \mathbb{N}\}$ where $\delta$ is the smallest positive vector with integers values such that $A\delta = 0$.*

*(ııı) If $A$ is of indefinite type, then there exists a positive imaginary root $\alpha = \sum_i k_i \alpha_i$ with $k_i > 0$ and $\langle \alpha_i^\vee, \alpha \rangle < 0$ for all $i \in [1, n]$.*

**Proof :** (i) We prove this in Corollary 9.1.6.

(ii) Remark that another definition of $\delta$ is given by $\delta = \sum_i a_i\alpha_i$ where the $a_i$ are the label of the Dynkin diagram in the tables 2.1, 2.2 and 2.3.

Recall that any affine matrix is symmetrisable. Take $\alpha \in K$, then we have $\langle \alpha_i^\vee, \alpha\rangle \leq 0$ thus $A\alpha \leq$ or $A(-\alpha) \geq 0$. This implies that $\alpha = 0$ because $A$ is of affine type. In particular there exists a non negative rational $r$ such that $\alpha = r\delta$. Because of the definition of $\delta$ we must have $r \in \mathbb{N}$.

(iii) Let $u > 0$ with $Au < 0$. We may assume that $v$ has rational coefficients and thus there exists a vector $\alpha \in Q_+$ proportional to $v$. But then $A\alpha < 0$ thus $\alpha \in K$ and the result follows. $\square$

### 9.1.3 Isotropic roots

Let $A$ be a Kac-Moody Lie algebra and fix an invariant bilinear form $( \ , \ )$.

**Definition 9.1.15** A root $\alpha$ is called **isotropic** if $(\alpha, \alpha) = 0$. This does not depend on the invariant bilinear form chosen.

**Proposition 9.1.16** *A root $\alpha$ is isotropic if and only if it is in the $W$-orbit of an imaginary root $\beta$ such that* Supp$\beta$ *is an affine type subdynkin diagram of the Dynkin diagram of $A$.*

**Proof :** Let $\beta$ be an imaginary root whose support is an affine type subdynkin diagram of the Dynkin diagram of $A$. We thus have $\beta = n\delta$ and it suffices to prove that $(\delta, \delta) = 0$. But $\langle \alpha_i^\vee, \delta\rangle = 2(\alpha_i, \delta)/(\alpha_i, \alpha_i)$ and because $A\delta = 0$, we get $(a_i, \delta) = 0$ for all index $i$ thus $(\delta, \delta) = 0$.

Conversely, let $\alpha$ be an isotropic root. Such a root is imaginary. We may assume that $\alpha$ is positive and by letting $W$ act we may assume that $\alpha \in K$. In particular $\langle \alpha_i^\vee, \alpha\rangle \leq 0$ for all index $i$. Write $\alpha = \sum_i a_i\alpha_i$ and consider the last inequality in the proof of Proposition 9.1.5 (iii). It has to be an equality and in particular we must have $\langle \alpha_i^\vee, \alpha\rangle = 0$ for all index $i$ such that $a_i \neq 0$. In particular, if $A_\alpha$ is the proper submatrix of $A$ defined by Supp$\alpha$ we have $A_\alpha\alpha = 0$ and the result follows. $\square$

# Chapter 10

# The category $\mathcal{O}$

In this chapter we define the category $\mathcal{O}$ of modules over the Kac-Moody Lie algebra and derive its first property.

## 10.1 Definition of the category $\mathcal{O}$

Recall the decompositions on $\mathfrak{g}(A)$ and its enveloping algebra $U(\mathfrak{g}(A))$ given by Poincaré-Birkhoff-Witt:

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \quad \text{and} \quad U(\mathfrak{g}(A)) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+).$$

Recall also that a $\mathfrak{g}(A)$-module $V$ is $\mathfrak{h}$-diagonalisable if it decomposes as a direct sum of weight spaces $V_\lambda$ for $\lambda \in \mathfrak{h}^*$. If $V_\lambda$ is non zero then $V_\lambda$ is called a weight space, any non zero vector of $V_\lambda$ is called a weight vector and $\lambda$ is called a weight of $V$. The set of weight of $V$ is denoted by $P(V)$. We will also need the following notation for $\lambda \in \mathfrak{h}^*$ set

$$D(\lambda) = \{\mu \in \mathfrak{h}^* \ / \ \mu \leq \lambda\}$$

where the order is defined thanks to $Q_+$.

**Definition 10.1.1** The category $\mathcal{O}$ is the full subcategory of the category of $\mathfrak{g}(A)$-modules whose objets are $\mathfrak{g}(A)$-modules $V$ which are $\mathfrak{h}$-diagonalisable with finite dimensional weight spaces and such that there exist elements $(\lambda_i)_{i\in[1,s]}$ such that

$$P(V) \subset \bigcup_{i=1}^{s} D(\lambda_i).$$

Because a submodule of an $\mathfrak{h}$-diagonalisable module is again $\mathfrak{h}$-diagonalisable, we have the

**Fact 10.1.2** *Any submodule and quotient of an object in $\mathcal{O}$ is again in $\mathcal{O}$. Any sum or tensor product of a finite number of objects in $\mathcal{O}$ is in $\mathcal{O}$.*

## 10.2 Highest weight modules

**Definition 10.2.1** A $\mathfrak{g}(A)$-module $V$ is called **a highest weight module with highest weight** $\lambda \in \mathfrak{h}^*$ if there exists a nonzero vector $v_\lambda \in V$ such that $v_\lambda$ generates $V$ as a $\mathfrak{g}(A)$-module and

$$\mathfrak{n}_+(v_\lambda) = 0 \quad \text{and} \quad h(v_\lambda) = \langle \lambda, h \rangle v_\lambda \text{ for all } h \in \mathfrak{h}.$$

The vector $v_\lambda$ is called a **highest weight vector**.

An easy consequence of the decomposition of the enveloping algebra is the following

**Fact 10.2.2** *(ı) For $V$ a highest weight module with highest weight vector $v$, we have $U(\mathfrak{n}_-)v = V$.*

*(ıı) A highest weight module is an object in $\mathcal{O}$. Furthermore, if the highest weight is $\lambda$ with highest weight vector $v_\lambda$, then $P(V) \subset D(\lambda)$ and $V_\lambda = \mathbb{C}v_\lambda$.*

**Proof :** The first point comes from the decomposition of the enveloping algebra. For the second, we have a decomposition into weight spaces coming from the decomposition of $\mathfrak{g}(A)$ and the dimension of every weight space is finite because generated by $v$. Furthermore because $V = U(\mathfrak{n}_-)v_\lambda$ the weight are smaller than $\lambda$ and the only vectors in $V_\lambda$ are proportional to $v_\lambda$. □

**Corollary 10.2.3** *Let $V$ be a highest weight module, then* $\mathrm{End}_{\mathfrak{g}(A)}(V) = \mathbb{C}\mathrm{Id}_V$.

**Proof :** The highest weight vector is send to a vector of same weight thus to multiple of itself and the result follows from the fact that the highest weight vector generates $V$. □

## 10.3 Verma modules

**Definition 10.3.1** A $\mathfrak{g}(A)$-module $M(\lambda)$ with highest weight $\lambda$ is called a Verma module if every $\mathfrak{g}(A)$ module with highest weight $\lambda$ is a quotient of $M(\lambda)$.

**Proposition 10.3.2** *(ı) For every $\lambda \in \mathfrak{h}^*$, there exists a unique up to isomorphism Verma module $M(\lambda)$.*

*(ıı) Viewed as a $U(\mathfrak{n}_-)$-module, $M(\lambda)$ is a free module generated by a highest-weight vector.*

*(ııı) The module $M(\lambda)$ contains a unique proper maximal submodule $M'(\lambda)$.*

**Proof :** (ı) If $M$ and $N$ are two Verma modules for the weight $\lambda$, then by definition, there is a surjective morphism $\varphi : M \to N$. This induces a surjective morphism $M_\lambda \to N_\lambda$ and in particular $\dim M_\lambda \geq \dim N_\lambda$. Reversing the roles of $M$ and $N$ we get that these dimensions are equal and $\varphi$ is an isomorphism.

To prove the existence of such a Verma module, it suffices to consider

$$M(\lambda) = U(\mathfrak{g}(A))/I(\lambda),$$

where $I(\lambda)$ is the left ideal generated by $\mathfrak{n}_+$ and the elements $h - \langle\lambda, h\rangle\mathrm{Id}$ in $\mathfrak{g}(A)$. The left multiplication induces a left $\mathfrak{g}(A)$-module structure on $M(\lambda)$ and by definition of $M(\lambda)$ it is a highest weight module of highest weight $\lambda$ and highest weight vector the image of 1 in $M(\lambda)$.

Furthermore, let $V(\lambda)$ be any highest weight module of highest weight $\lambda$ and let $v$ be a highest weight vector. Consider the $\mathfrak{g}(A)$-module morphism $\mathfrak{g}(A) \to V(\lambda)$ defined by $x \mapsto x \cdot v$. It is surjective and the kernel $I$ of this map is a left ideal in $\mathfrak{g}(A)$ containing $I(\lambda)$. The result follows.

(ıı) This is Poincaré-Birkhoff-Witt and the fact that $U(\mathfrak{g}(A)) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$.

(ııı) The sum of all proper submodules is again a submodule (which is $\mathfrak{h}$-diagonalisable because $M(\lambda)$ is) and proper because the grading by weights is preserved and any proper submodule does not contain $M(\lambda)_\lambda$. □

**Corollary 10.3.3** *The quotient $M(\lambda)/M'(\lambda)$ is the unique irreducible module of highest weight $\lambda$.*

We will prove that the modules $L(\lambda)$ are all the irreducible modules in the category $\mathcal{O}$. Let us introduce the following

**Definition 10.3.4** Let $V$ be a $\mathfrak{g}(A)$-module. A vector $v \in V_\lambda$ is called **primitive** if there exists a submodule $U$ of $V$ such that $v \notin U$ and $\mathfrak{n}_+(v) \in U$.

In that case, the weight $\lambda$ is called **a primitive weight**. In the same way, we define primitive vectors and weights for a $\mathfrak{g}'(A)$-module.

**Remark 10.3.5** A weight vector $v$ such that $\mathfrak{n}_+(v) = 0$ is primitive.

**Proposition 10.3.6** *Let $V$ be a nonzero module from the category $\mathcal{O}$. Then we have:*

*(ı) the module $V$ contains a nonzero weight vector such that $\mathfrak{n}_+(v) = 0$.*

*(ıı) The following are equivalent*

- *$V$ is irreducible;*
- *$V$ is a highest weight module and any primitive vector is a highest weight vector;*
- *$V \simeq L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.*

*(ııı) $V$ is generated, as a $\mathfrak{g}(A)$-module by its primitive vectors.*

**Proof :** (ı) Let $\lambda$ be a maximal weight of $V$. This is possible because the weights in $V$ are bounded by a finite number of weights. Let $v \in V_\lambda$, then $\mathfrak{n}_+(v) = 0$.

(ıı) Let $v$ be a primitive vector with $\mathfrak{n}_+(v) = 0$ as above. Then by irreducibility, the $v$ generates $V$ as a $\mathfrak{g}(A)$-module and $v$ is a highest weight vector of weight $\lambda$. Now if $v'$ is a primitive vector, then me must have $\mathfrak{n}_+(v') = 0$ otherwise the submodule $U$ is the definition of primitive vectors would be a proper non trivial submodule. The vector $v'$ is a highest weight vector thus $v' \in V_\lambda$ but this space is of dimension 1 concluding the proof.

Let $V$ be a highest weight module of weight $\lambda$ whose primitive vectors are all highest weight vectors. We have a surjective map $M(\lambda) \to V$. Let us prove that $V$ is irreducible. Take a non trivial submodule $U$ of $V$. By (ı), the module $U$ contains an highest weight vector $u$. In particular, we have $\mathfrak{n}_+(u) = 0$ in $V$ also so that by hypothesis $u$ is primitive and thus a highest weight vector of $V$. We get $U = V$ and the result follows.

(ııı) Let $V'$ the submodule generated by the primitive vectors in $V$. Assume that $V' \neq V$ and take $\lambda$ a maximal weight such that $V'_\lambda \neq V_\lambda$. Let $v \in V_\lambda$ with $v \notin V'$. We have $\mathfrak{n}_+(v) \in V'$ by construction thus $v$ is a primitive vector of $V$ and $v \in V'$ a contradiction. ☐

**Corollary 10.3.7** *There is a bijection between $\mathfrak{h}^*$ and the irreducible modules in the category $\mathcal{O}$ given by $\lambda \mapsto L(\lambda)$. Furthermore, the module $L(\lambda)$ is defined as an (the) irreducible module containing a vector $v$ such that $\mathfrak{n}_+(v) = 0$ and $h(v) = \langle \lambda, h \rangle v$ for $h \in \mathfrak{h}$.*

## 10.4 Lowest weight modules

Let $L(\lambda)^*$ be the dual of $L(\lambda)$. Define on $L(\lambda)^*$ the congradient action of $\mathfrak{g}(A)$ by $(x \cdot f)(v) = -f(x \cdot v)$. Remark that we have $(x \cdot (y \cdot f))(v) = f(y \cdot (x \cdot v))$. This defines an action because of the following computation:

$$((xy - yx) \cdot f)(v) = f(yx \cdot v) - f(xy \cdot v) = -f([x, y] \cdot v) = ([x, y] \cdot f)(v).$$

We may write the module $L(\lambda)^*$ as the product $\prod_\lambda (L(\lambda)_\lambda)^*$ and define the following subspace of $L(\lambda)^*$:

$$L^*(\lambda) = \bigoplus_\lambda (L(\lambda)_l t)^*.$$

**Lemma 10.4.1** *(ı) For any $\mu \in \mathfrak{h}^*$, we have $(L^*(\lambda))_\mu = (L(\lambda)_{-\mu})^*$.*

*(ıı) The module $L^*(\lambda)$ is irreducible. Furthermore, for $f \in L^*(\lambda)_{-\lambda}$ we have $\mathfrak{n}_-(f) = 0$ and $h(f) = -\langle \lambda, h\rangle f$ for $h \in \mathfrak{h}$.*

**Proof :** (ı) This point comes directly from the action of $h \in \mathfrak{h}$: for $v \in L(\lambda)_\mu$ and $f \in L^*(\lambda)$, we have $(h \cdot f)(v) = -f(h \cdot v) = -\langle \mu, h\rangle f(v)$.

(ıı) The action of $h$ is clear. Let $x \in \mathfrak{n}_-$ and $v \in L(\lambda)$, we have $(x \cdot f)(v) = -f(x \cdot v)$. This vanishes unless $x \cdot v \in L(\lambda)_\lambda$ because $f \in (L^*(\lambda))_{-\lambda} = (L(\lambda)_\lambda)^*$. But $x \cdot v$ is never in that space and we get the vanishing condition.

Let $U$ be a submodule of $L^*(\lambda)$. Consider the subset $U^\perp$ of $L(\lambda)$ of vectors $v$ such that $f(v) = 0$ for all $f \in U$. Let us prove that $U^\perp$ is a $\mathfrak{g}(A)$-submodule of $L(\lambda)$. Indeed, for $x \in \mathfrak{g}(A)$, $v \in U^\perp$ and $f \in U$, we have $f(x \cdot v) = -(x \cdot f)(v) = 0$ because $-(x \cdot f) \in U$. The submodule $U^\perp$ is thus either trivial or equal to $L(\lambda)$ and the same is true for $U$. □

**Definition 10.4.2** The module $L^*(\lambda)$ is called a lowest weight module of lowest weight $-\lambda$. There is a bijection between $\mathfrak{h}^*$ and the set of lowest weight modules given by $\lambda \mapsto L^*(-\lambda)$.

We define a twisted action $\cdot_\omega$ of $\mathfrak{g}(A)$ on $L(\lambda)$ using the involution $\omega$ defined in Theorem 4.1.6 by:

$$x \cdot_\omega v = \omega(x) \cdot v$$

for $x \in \mathfrak{g}(A)$ and $v \in L(\lambda)$. We can define in the save way a twisted action of $\mathfrak{g}(A)$ on $L^*(\lambda)$. For this twisted action $L^*(\lambda)$ is an irreducible highest weight module of highest weight $\lambda$ and thus isomorphic to $L(\lambda)$. Denote by $\psi : L(\lambda) \to L^*(\lambda)$ this isomorphism, it defines a non degenerate bilinear $B(\ ,\ )$ on $L(\lambda)$ by $B(u, v) = \psi(u)(v)$ for $u, v \in L(\lambda)$.

**Lemma 10.4.3** *This bilinear form satisfies the following equation*

$$B(x \cdot u, v) = -B(u, \omega(x)\cdot).$$

**Proof :** We compute:

$$B(x \cdot u, v) = (\psi(x \cdot u))(v) = (x \cdot_\omega \psi(u))(v) = (\omega(x) \cdot \psi(u))(v) = -\psi(u)(\omega(x)\cdot) = -B(u, \omega(x) \cdot v).$$

□

**Definition 10.4.4** A bilinear form $B$ on $L(\lambda)$ satisfying the above equation is called a **contravariant bilinear form**.

**Proposition 10.4.5** *There exists a unique, up to scalar multiple, non degenerate contravariant bilinear form $B$ on $L(\lambda)$. This form is symmetric and $L(\lambda)$ decomposes into a orthogonal direct sum of weight spaces with respect to this form.*

**Proof :** We already proved the existence. Assume we have $B$ and $B'$ two such forms, then they define linear maps $\psi_B : L(\lambda) \to L^*(\lambda)$ and $\psi_{B'} : L(\lambda) \to L^*(\lambda)$ by $(\psi_B(u))(v) = B(u, v)$ and $(\psi_{B'}(u))(v) = B'(u, v)$. These applications are morphisms of $\mathfrak{g}(A)$-modules. In particular the compositions $\psi_B \circ \psi_{B'}^{-1}$ and $\psi_{B'} \circ \psi_B^{-1}$ are in $\mathrm{End}_{\mathfrak{g}(A)}(L(\lambda))$. But this space only contains homotheties and the uniqueness follows.

Let us prove that $B(L(\lambda)_\mu, L(\lambda)_\nu) = 0$ for $\mu \neq \nu$. Indeed, take $h \in \mathfrak{h}$, $u \in L(\lambda)_\mu$ and $v \in L(\lambda)_\nu$, we have $\langle \mu, h\rangle B(u, v) = B(h \cdot u, v) = -B(u, -h \cdot v) = \langle \nu, h\rangle B(u, v)$. The result follows. In particular, because $B$ is non degenerate, we get that $B(v, v) \neq 0$ for $v \in L(\lambda)_\lambda$.

Define $B'(u, v) = B(u, v)$. It is a non degenerate bilinear form and we have $B'(x \cdot u, v) = B(v, x \cdot u) = B(v, \omega(\omega(x)) \cdot u) = B(\omega(x) \cdot v, u) = B'(u, \omega(x) \cdot v)$. This form is thus proportional to $B$. But $B'(v, v) = B(v, v) \neq 0$ thus $B' = B$ and $B$ is symmetric. □

## 10.5 Integrable highest weight modules

We start with the following

**Definition 10.5.1** The **weight lattice** $P$, the **dominant weight lattice** $P_+$ and the **regular dominant weight lattice** $P_{++}$ are defined as follows:

$$\begin{aligned} P = & \quad \{\lambda \in \mathfrak{h}^* \ / \ \forall i, \ \langle \lambda, \alpha_i \rangle \in \mathbb{Z}\} \\ P_+ = & \quad \{\lambda \in P \ / \ \forall i, \ \langle \lambda, \alpha_i \rangle \geq 0\} \\ P_{++} = & \quad \{\lambda \in P \ / \ \forall i, \ \langle \lambda, \alpha_i \rangle > 0\}. \end{aligned}$$

Remark that the lattice $Q$ is contained in $P$.

**Lemma 10.5.2** *The $\mathfrak{g}(A)$-module $L(\lambda)$ is integrable if and only if $\lambda \in P_+$.*

**Proof :** Because of the boundedness condition on the weights of objets in $\mathcal{O}$, it is clear that the elements $e_i$ are nilpotent on $L(\lambda)$. We need only to consider the elements $f_i$.

Consider the action of the $\mathfrak{sl}_2 \simeq \mathfrak{g}_{(i)}$-subalgebra on a highest weight vector $v$. Its weight for $\mathfrak{sl}_2$ is $\langle \alpha_i^\vee, \lambda \rangle$ and we have $e_i(v) = 0$. In particular by Proposition 4.2.6 (ıı) we have $e_i(f_i^N(v)) = N(\langle \alpha_i^\vee, \lambda \rangle - N + 1) f^{N-1}(v)$.

Assume that $L(\lambda)$ is integrable and let $N = \min\{n \in \mathbb{N} \ / \ f^n(v) = 0\}$. Because $v \neq 0$ we have $N \geq 1$. But now the preceding equation leads to $N(\langle \alpha_i^\vee, \lambda \rangle - N + 1) = 0$ thus $\langle \alpha_i^\vee, \lambda \rangle = N - 1 \geq 0$.

Conversely, assume that $\lambda \in P_+$. The previous formula gives us $e_i(f_i^{\langle \alpha_i^\vee, \lambda \rangle + 1}(v)) = 0$. But for any $j \neq i$, we have $e_j(f_i^{\langle \alpha_i^\vee, \lambda \rangle + 1}(v)) = 0$. This implies in particular that $f_i^{\langle \alpha_i^\vee, \lambda \rangle + 1}(v)$ either vanishes or is a primitive vector of $L(\lambda)$. The last case is not possible for weight reasons. The result follows. □

**Corollary 10.5.3** *If $\lambda \in P_+$, then we have* $\mathrm{mult}_{L(\lambda)}\mu = \mathrm{mult}_{L(\lambda)} w(\mu)$ *for any $w \in W$. In particular the set of weights of $L(\lambda)$ is $W$-invariant.*

**Corollary 10.5.4** *If $\lambda \in P_+$ and $\mu \in P(\lambda)$ is a weight of $L(\lambda)$, then there exists $w \in W$ such that $w(\mu) \in C^\vee$.*

**Proof :** Take $w$ such that $\mathrm{ht}(\lambda - w(\mu)$ is minimal. We have for any index $i$ that $\mathrm{ht}(\lambda - s_i w(\mu)) \geq \mathrm{ht}(\lambda - w(\mu))$. This implies that $\langle \alpha_i^\vee, w(\mu) \rangle \geq 0$. □

**Lemma 10.5.5** *Let $L$ be an integrable highest weight module of weight $\lambda$, then $\lambda \in P_+$.*

**Proof :** The same proof as for $L(\lambda)$ works. □

## 10.6 Filtration

In this section we try to decompose any object in $\mathcal{O}$ into a sequence of extensions of irreducible module. However, an object $V$ in $\mathcal{O}$ does not always admit a composition series that is a sequence $V \supset V_1 \subset V_2 \cdots$ such that $V_i/V_{i+1}$ is irreducible. We start with the following

**Lemma 10.6.1** *Let $V$ be an object in $\mathcal{O}$. Assume that for any two primitive weights $\lambda$ and $\mu$ such that $\lambda \geq \mu$ we have $\lambda = \mu$. Then the module $V$ is completely reducible (i.e. it decomposes as a direct sum of irreducible submodules).*

**Proof :** Let $V^0 = \{v \in V \ / \ \mathfrak{n}_+(v) = 0\}$. This subspace in $\mathfrak{h}$ invariant and has thus a weight decomposition $V^0 = \oplus V^0_\lambda$. Furthermore, the weight of $V^0$ correspond to primitive weights of $V$. For $\lambda$ such a weight and $v \in V^0_\lambda$, consider the submodule $V' = U(\mathfrak{g}(A))(v)$. We claim that $V'$ is irreducible. Indeed, let $U$ be a submodule of $V'$. Take $u$ a weight vector in $U$ of weight $\mu$ with $n_+(u) = 0$. This is a primitive vector of $V$ and because $U$ is a submodule of $V'$ we have $\mu \leq \lambda$ thus $\mu = \lambda$. But $V'_\lambda$ is one dimensional and $V'$ is generated by this subspace thus $U = V'$. The submodule $V''$ of $V$ generated by $V^0$ is thus completely reducible and sum of modules of the form $L(\lambda)$ for weights $\lambda$ primitive for $V$. In particular any weight of $V''$ is smaller than a primitive weight of $V$.

If $V''$ is proper in $V$, let $\mu$ be a weight maximal for the property $V_\mu/V''_\mu \neq 0$. Let $v \in V_\mu$ and not in $V''_\mu$. We have by maximality $\mathfrak{n}_+(v) \subset V''$. This implies that $v$ and thus $\mu$ are primitive. But because $v \notin V''$ we have $v \notin V^0$ thus there exists an index $i$ such that $e_i(v) \neq 0$ and $e_i(v) \in V''$. The weight of $e_i(v)$ is thus smaller than a primitive weight $\lambda$ of $V$. Thus $\lambda \geq \mu + \alpha_i > \mu$ but $\lambda$ and $\mu$ are primitive weights for $V$ a contradiction. □

The following lemma will replace the lack of composition series:

**Lemma 10.6.2** *Let $V$ be an object in $\mathcal{O}$ and let $\lambda \in \mathfrak{h}^*$. Then there exists a filtration by a sequence of submodules $0 = V_0 \subset \cdot \subset V_m = V$ and a subset $J \subset [1, m]$ such that:*

- *if $j \in J$, then $V_j/V_{j-1} \simeq L(\lambda_j)$ for some weight $\lambda_j \geq \lambda$;*
- *if $j \notin J$, then $(V_j/V_{j-1})_\mu = 0$ for all $\mu \geq \lambda$.*

**Proof :** Let us define the following integer

$$a(V, \lambda) = \sum_{\mu \geq \lambda} \dim V_\mu.$$

We prove the result by induction on $a(V, \lambda)$. If $a(V, \lambda) = 0$ then take the trivial filtration $0 \subset V$. Otherwise, let $\mu$ be a maximal weight in $V$ with $\mu \geq \lambda$. Take $v$ a weight vector for $\mu$. It is primitive with $n_+(v) = 0$. Let $U$ be the submodule of $V$ generated by $v$. Then $U$ is a highest weight module of weight $\mu$ and there exists a maximal proper submodule $S$ in $U$ (the image of $M'(\mu)$ from the surjection $M(\mu) \to U$). We have $U/S \simeq L(\mu)$ and a filtration

$$0 \subset S \subset U \subset V.$$

We have $a(S, \lambda) < a(U, \lambda) \leq a(V, \lambda)$ and $a(V/U, \lambda) < a(V, \lambda)$ thus by induction we have the required filtration on $S$ and $V/U$ and the result follows. □

**Definition 10.6.3** Let $V$ an object in $\mathcal{O}$ and $\mu \in \mathfrak{h}^*$. Take $\lambda \in \mathfrak{h}^*$ with $\lambda \leq \mu$ and a filtration as in Lemma 10.6.2. We define the **multiplicity of $L(\mu)$ in** $V$, denoted $[V : L(\mu)]$ by:

$$[V : L(\mu)] = \text{the number of times } \mu \text{ appears among the set } \{\lambda_j \ / \ j \in J\}.$$

This is well defined and $\mu$ has non zero multiplicity if and only if it is a primitive weight of $V$.

## 10.7 Character formula for Verma modules

We introduce in this section the formal character of a object in $\mathcal{O}$. For this we define the algebra $\mathcal{E}$ whose elements are series of the form

$$\sum_{\lambda \in \mathfrak{h}^*} c_\lambda e(\lambda)$$

with $c_\lambda \in \mathbb{C}$ and $c_\lambda = 0$ for $\lambda$ outside a finite union set $D(\mu)$ for some $\mu \in \mathfrak{h}$. We define an algebra structure on $\mathcal{E}$ by the natural sum and product by a scalar in $\mathbb{C}$ and by setting

$$\left(\sum_{\lambda\in\mathfrak{h}^*} c_\lambda e(\lambda)\right)\left(\sum_{\lambda\in\mathfrak{h}^*} c_\lambda e(\lambda)\right) = \sum_{\lambda\in\mathfrak{h}^*}\left(\sum_{\mu+\nu=\lambda} c_\mu c_\nu\right) e(\lambda).$$

This is well defined because in the sum $\sum_{\mu+\nu=\lambda} c_\mu c_\nu$, the scalars $c_\mu$ vanish for $\mu$ not smaller than all elements of a finite set $\{\mu_1, \cdots \mu_k\}$ and the scalars $c_\nu$ vanish for $\nu$ not smaller than all elements of a finite set $\{\nu_1, \cdots \nu_l\}$. The sum is thus finite and the coefficient of $e(\lambda)$ vanishes for $\lambda$ not smaller than a finite number of elements in $\mathfrak{h}$.

**Remark 10.7.1** This definition of the product leads to the formula $e(\lambda)e(\mu) = e(\lambda+\mu)$. The identity is $e(0)$.

**Definition 10.7.2** Let $V$ be a object in $\mathcal{O}$, then we define the **formal character** of $V$ and denote it by $\mathrm{Ch}V$ by:

$$\mathrm{Ch}V = \sum_{\lambda\in\mathfrak{h}^*}(\dim V_\lambda)e(\lambda).$$

**Lemma 10.7.3** *If $0 \to V' \to V \to V'' \to 0$ is an exact sequence in $\mathcal{O}$, then we have the formula $\mathrm{Ch}V = \mathrm{Ch}V' + \mathrm{Ch}V''$.*

**Proof :** Clear from the definition. □

**Proposition 10.7.4** *Let $V$ be an object in $\mathcal{O}$, then we have the formula*

$$\mathrm{Ch}V = \sum_{\lambda\in\mathfrak{h}^*}[V : L(\lambda)]\mathrm{Ch}L(\lambda).$$

**Proof :** Both formula are additive for exact sequences. Let $\phi(V)$ be the difference. We get that $\phi(L(\lambda)) = 0$. Furthermore, taking the filtration given by Lemma 10.6.2, we get that there exist some modules $M_j$ for $j \notin J$ and weight $\lambda_j$ for $j \in J$ such that

$$\phi(V) = \sum_{j\notin J}\phi(M_j) + \sum_{j\in J}\phi(L(\lambda_j)) = \sum_{j\notin J}\phi(M_j).$$

But the modules $M_j$ are such that for any $\mu \geq \lambda$ we have $(M_j)_\mu = 0$ thus the coefficient of $e(\lambda)$ in $\phi(V)$ is zero. This inplies that $\phi(V) = 0$. □

**Proposition 10.7.5** *For $\lambda \in \mathfrak{h}^*$, we have the following formula:*

$$\mathrm{Ch}M(\lambda) = e(\lambda)\prod_{\alpha\in\Delta_+}(1 - e(-\alpha))^{-\mathrm{mult}(\alpha)}.$$

**Proof :** We know that the module $M(\lambda)$ is a free $U(\mathfrak{n}_-)$-module. In particular, Poincaré-Birlhoff-Witt theorem tells us that there is a basis in terms of monomials in a basis of $\mathfrak{n}_-$. It easily follows that

$$\mathrm{Ch}M(\lambda) = e(\lambda)\prod_{\alpha\in\Delta_+}(1 + e(-\alpha) + e(-2\alpha) + \cdots)^{\mathrm{mult}\alpha}$$

and the result follows. □

# Chapter 11

# Casimir operator and character formula

In this chapter, we assume that the matrix $A$ is symmetrisable. We also fix an invariant bilinear for $(\ ,\ )$ on $\mathfrak{g}(A)$. The existence of this bilinear form will ensure the existence of the Casimir operator which is an important tool in the character formula. Character formula is true for any Kac-Moody Lie algebra but one requires the construction of Kac-Moody group and the use of geometric arguments to prove it.

## 11.1 Casimir operator

### 11.1.1 Some formulas

Fix a root $\alpha \in \Delta_+$ and choose basis $(e_\alpha^k)$ and $(e_{-\alpha}^k)$ of the spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ such that $(e_\alpha^k, e_{-\alpha}^l) = \delta_{k,l}$ for all $k$ and $l$. If $\alpha = \alpha_i$ is simple then there exists an unique vector $e_{\alpha_i}^k$ and we take $e_{\alpha_i}^k = e_i$. In that case because $(e_i, f_i) = \epsilon_i$ we have $e_{-\alpha_i}^k = \frac{1}{\epsilon_i} f_i$.

**Lemma 11.1.1** *For all $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, we have:*

$$(x, y) = \sum_k (x, e_{-\alpha}^k)(y, e_\alpha^k).$$

**Proof :** Write $x = \sum_k (x, e_{-\alpha}^k) e_\alpha^k$ and $y = \sum_k (y, e_\alpha^k) e_{-\alpha}^k$, the result follows. □

**Lemma 11.1.2** *Let $\alpha$ and $\beta$ in $\Delta$ and let $z \in \mathfrak{g}_{\beta-\alpha}$, then we have in $\mathfrak{g}(A) \otimes \mathfrak{g}(A)$:*

$$\sum_s e_{-\alpha}^s \otimes [z, e_\alpha^s] = \sum_s [e_{-\beta}^s, z] \otimes e_\beta^s.$$

**Proof :** Define a non degenerate bilinear form on $\mathfrak{g}(A) \otimes \mathfrak{g}(A)$ by $(x \otimes y, z \otimes t) = (x, z)(y, t)$. To prove the result, it is enought to prove that the equality holds after taking the bilinear form with $e_\alpha^u \otimes e_{-\beta}^v$ for all $u$ and $v$. This gives

$$\left( \sum_s e_{-\alpha}^s \otimes [z, e_\alpha^s], e_\alpha^u \otimes e_{-\beta}^v \right) = \sum_s \delta_{s,u}([z, e_\alpha^s], e_{-\beta}^v) = (e_\alpha^u, [e_{-\beta}^v, z]) \text{ and}$$

$$\left( \sum_s [e_{-\beta}^s, z] \otimes e_\beta^s, e_\alpha^u \otimes e_{-\beta}^v \right) = \sum_s \delta_{s,v}(e_\alpha^u, [e_{-\beta}^s, z]) = (e_\alpha^u, [e_{-\beta}^v, z]).$$

The result follows. □

**Corollary 11.1.3** *With the notation of the previous lemma we have the formulas:*

$$\sum_s [e^s_{-\alpha}, [z, e^s_\alpha]] = -\sum_s [[z, e^s_{-\beta}], e^s_\beta] \text{ in } \mathfrak{g}(A),$$

$$\sum_s e^s_{-\alpha}[z, e^s_\alpha] = -\sum_s [z, e^s_{-\beta}]e^s_\beta \text{ in } U(\mathfrak{g}(A)),$$

**Proof :** Apply the previous lemma and the maps from $\mathfrak{g}(A) \otimes \mathfrak{g}(A)$ to $\mathfrak{g}(A)$ and $U(\mathfrak{g}(A))$ respectively given by $(x, y) \mapsto [x, y]$ and $(x, y) \mapsto xy$. □

**Remark 11.1.4** Remark that the previous lemma and corollary are still true if one of the element $\alpha$, $\beta$ or $\beta - \alpha$ is not a root.

### 11.1.2 Casimir operator

**Definition 11.1.5** We define a special element $\rho$ in $\mathfrak{h}^*$ as follows: take $\rho$ to be a solution of the equations $\langle \rho, \alpha_i^\vee \rangle = 1$ for all index $i$. In fact $\rho$ is uniquely determined only in the finite type case. In that case it is given by half the sum of the positive roots:

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha.$$

In general we take for $\rho$ any solution of these equations.

**Fact 11.1.6** *It follows from the formula* $\langle \rho, \alpha_i^\vee \rangle = \dfrac{2(\rho, \alpha_i)}{(\alpha_i, \alpha_i)}$ *that* $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$.

Choose as before a basis $(e^s_\alpha)$ of $\mathfrak{g}_\alpha$ and let $(e^s_{-\alpha})$ be its dual basis in $\mathfrak{g}_{-a}$.

**Definition 11.1.7** We define the operator $\Omega_0$ on any object $V$ of $\mathcal{O}$ by

$$\Omega_0 = 2 \sum_{\alpha \in \Delta_+} \sum_s e^s_{-\alpha} e^s_\alpha.$$

Remark that this is well defined since because $V$ is in $\mathcal{O}$, for any vector $v \in V$, there is a finite number of positive root $\alpha$ such that $\mathfrak{g}_\alpha(v) \neq 0$. Choose also $(u_k)$ and $(u^k)$ dual bases of $\mathfrak{h}$.

**Definition 11.1.8** We define the **Casimir operator** $\Omega$ on any object $V$ of $\mathcal{O}$ by

$$\Omega = 2\nu^{-1}(\rho) + \sum_k u_k u^k + \Omega_0.$$

**Lemma 11.1.9** *This definitions of $\Omega_0$ and $\Omega$ do not depend on the choice of the dual basis. Therefore definition of $\Omega$ does only depend on the choice of $\rho$ (and of the invariant bilinear form ( , )).*

**Proof :** We denote $\mathfrak{h} = \mathfrak{g}_0$ and $\alpha$ will be a root or 0. We identify $\mathfrak{g}_{-\alpha}$ with $\mathfrak{g}^*_\alpha$ thanks to the bilinear form. With this identification, the element $\sum_s e^s_{-\alpha} \otimes e^s_\alpha \in \mathfrak{g}_{-\alpha} \otimes \mathfrak{g}_\alpha = \mathfrak{g}^*_\alpha \otimes \mathfrak{g}_\alpha$ correspond to the identity. In particular, it does not depend on the choice of the base. □

**Remark 11.1.10** The operators $\Omega_0$ and $\Omega$ live in a completion $\widehat{U}(\mathfrak{g}(A))$ of the enveloping algebra $U(\mathfrak{g}(A))$. Indeed, let $U_d(\mathfrak{n})$ be the subspace of $U(\mathfrak{n})$ formed by the degree $d$ homogeneous elements (the grading is given by the height) and define

$$\widehat{U}(\mathfrak{g}(A)) = \prod_{d\geq 0} U(\mathfrak{n}_-)\otimes U(\mathfrak{h})\otimes U_d(\mathfrak{n}).$$

There a natural product given by

$$\sum_{d\geq 0} x_d \cdot \sum_{m\geq 0} y_m = \sum_k \sum_{d,m\geq 0} (x_d y_m)_k$$

where for $x_d$ and $y_d$ in $U(\mathfrak{n}_-)\otimes U(\mathfrak{h})\otimes U_d(\mathfrak{n}_+)$ the element $(x_d y_m)_k d$ denotes the component of $x_d y_m$ in the factor $U(\mathfrak{n}_-)\otimes U(\mathfrak{h})\otimes U_k(\mathfrak{n}_+)$.

Let us prove the main result on the Casimir operator:

**Theorem 11.1.11** *The action of the Casimir operator $\Omega$ on any module $V$ of the category $\mathcal{O}$ commutes with the action of $\mathfrak{g}(A)$. In other words $\Omega$ lies in $Z(\widehat{U}(\mathfrak{g}(A)))$ the center of the completed enveloping algebra.*

**Proof :** Since the centraliser $Z(\Omega)$ of $\Omega$ is a subalgebra, it suffices to prove that $\Omega$ commute with the generators i.e. with the elements $e_i$, $f_i$ and $h$ for $h\in\mathfrak{h}$. But the weight of $\Omega_0$ is zero as well as the other components of $\Omega$ proving the commutation with $h$ for all $h\in\mathfrak{h}$.We start with the following:

**Lemma 11.1.12** *For $x\in\mathfrak{g}_\alpha$, we have in $U(\mathfrak{g}(A))$:*

$$\sum_k [u^k u_k, x] = x((\alpha,\alpha)+2\nu^{-1}(\alpha)).$$

**Proof :** We first compute $u^k u_k x = u^k[u_k,x] + u^k x u_k = \langle\alpha,u_k\rangle u^k x + u^k x u_k$ and $x u^k u_k = [x,u^k]u_k + u^k x u_k = -\langle\alpha,u^k\rangle x u_k + u^k x u_k$. In particular we get

$$\sum_k [u^k u_k, x] = \sum_k \langle\alpha,u_k\rangle u^k x + \sum_k \langle\alpha,u^k\rangle x u_k.$$

But $u^k x = [u^k,x] + x u^k = \langle\alpha,u^k\rangle x + x u^k$ thus we have

$$\sum_k [u^k u_k, x] = \sum_k \langle\alpha,u_k\rangle\langle\alpha,u^k\rangle x + x\sum_k (\langle\alpha,u_k\rangle u^k + \langle\alpha,u^k\rangle u_k).$$

But recall the formulas $\lambda = \sum_k \langle\lambda,u^k\rangle\nu(u_k) = \sum_k\langle\lambda,u_k\rangle\nu(u^k)$ giving $(\lambda,\mu)=\sum_k\langle\lambda,u^k\rangle\langle\mu,u_k\rangle$. The result follows. □

We compute the second part of the Casimir operator:

**Lemma 11.1.13** *The following formula holds in in $\widehat{U}(\mathfrak{g}(A))$:*

$$[\Omega_0, e_i] = -2e_i((\alpha_i,\alpha_i)+\nu^{-1}(\alpha_i)).$$

**Proof :** In the following sums, we may regard the elements $\alpha$ as positive roots or elements in $Q_+$. In all the terms of the following equalities, this will be the same because by convention we take $e^s_\alpha = e^s_{-\alpha} = 0$ if $\alpha$ is not a root and because of Remark 11.1.4. Let us now compute

$$\begin{aligned}[\Omega_0, e_i] &= 2\sum_{\alpha\in\Delta_+}\sum_s [e^s_{-\alpha}e^s_\alpha, e_i] = 2\sum_{\alpha\in\Delta_+}\sum_s ([e^s_{-\alpha}, e_i]e^s_\alpha + e^s_{-\alpha}[e^s_\alpha, e_i]) \\ &= 2[e_{-\alpha_i}, e_i]e_i + 2\sum_{\alpha\in\Delta_+\setminus\{\alpha_i\}}\left(\sum_s [e^s_{-\alpha}, e_i]e^s_\alpha + \sum_s [e_i, e^s_{-\alpha-\alpha_i}]e^s_{\alpha+\alpha_i}\right).\end{aligned}$$

The last equality comes from Lemma 11.1.3. Furthermore, the last sum is equal to the same sum but with indices of summation $\alpha \in Q_+$. Furthermore, if $\alpha$ is a root not containing $\alpha_i$ in its support, then $e^s_{-\alpha}$ is a linear combination of brackets $[f_{i_1}, \cdots, [f_{i_{p-1}}, f_{i_p}]]$ with all indices $i_j$ different from $i$. In particular the Lie bracket $[[f_{i_1}, \cdots, [f_{i_{p-1}}, f_{i_p}]], e_i]$ vanishes. This implies that the same roots appear in the two terms of the second expression which necessary vanishes. We end up with

$$[\Omega_0, e_i] = 2[e_{-\alpha_i}, e_i]e_i = -\frac{2}{\epsilon_i}\alpha_i^\vee e_i = -\frac{2}{\epsilon_i}\langle\alpha_i^\vee, \alpha_i\rangle e_i - \frac{2}{\epsilon_i}e_i\alpha_i^\vee = -2(\alpha_i, \alpha_i)e_i - 2e_i\nu^{-1}(\alpha_i).$$

The apparition of $\epsilon_i$ comes from the fact that the dual of $e_i$ is $\frac{1}{\epsilon_i}f_i$ (see the definition of the invariant bilinear form in Theorem 7.2.5). ☐

Now Lemma 11.1.12 gives us

$$\sum_k [u^k u_k, e_i] = e_i((\alpha_i, \alpha_i) + 2\nu^{-1}(\alpha_i)).$$

Putting all these formulas together we get:

$$\begin{aligned}[\Omega, e_i] &= [2\nu^{-1}(\rho), e_i] + (\alpha_i, \alpha_i)e_i + 2e_i\nu^{-1}(\alpha_i) - 2(\alpha_i, \alpha_i)e_i - 2e_i\nu^{-1}(\alpha_i) \\ &= [2\nu^{-1}(\rho), e_i] - (\alpha_i, \alpha_i)e_i.\end{aligned}$$

But $[2\nu^{-1}(\rho), e_i] = 2\langle\alpha_i^\vee, \nu^{-1}(\rho)\rangle e_i = 2(\rho, \alpha_i)e_i = (\alpha_i, \alpha_i)e_i$. We get the desired formula

$$[\Omega, e_i] = 0.$$

The same proof works with $f_i$ and the result follows. ☐

**Corollary 11.1.14** *(ı) For any $\lambda \in \mathfrak{h}^*$, we have $\Omega|_{M(\lambda)} = (|\lambda+\rho|^2 - |\rho|^2)\mathrm{Id}_{M(\lambda)}$.*
*(ıı) In particular, for any subquotient $V$ of $M(\lambda)$ we have $\Omega|_V = (|\lambda+\rho|^2 - |\rho|^2)\mathrm{Id}_V$.*

**Proof :** We will consider, as in Proposition 10.3.2, the Verma module $M(\lambda)$ as the quotient:

$$M(\lambda) = U(\mathfrak{g}(A))/I(\lambda).$$

It is generated by 1 and because of the previous theorem, it suffices to show that $\Omega(1) = (|\lambda+\rho|^2 - |\rho|^2)1$. But we have $\mathfrak{n}_+(1) = 0$ and $h(1) = \langle\lambda, h\rangle 1$ for $h \in \mathfrak{h}$. This give the formula

$$\begin{aligned}\Omega(1) &= \langle 2\nu^{-1}(\rho), \lambda\rangle 1 + \textstyle\sum_k \langle\lambda, u^k\rangle\langle\lambda, u_k\rangle 1 \\ &= ((2\rho, \lambda) + (\lambda, \lambda))1 \\ &= ((\rho+\lambda, \rho+\lambda) - (\rho, \rho))1.\end{aligned}$$

☐

## 11.2 Character formula

Let $\rho$ be any solution of the system $\langle\rho, \alpha_i\rangle = 1$ for all $i$ (such a $\rho$ is unique only is the finite case). We prove in this section the following:

**Theorem 11.2.1** *Let $L$ be an integrable highest weight module. Then we have:*

$$\mathrm{Ch}(L(\lambda)) = \frac{\displaystyle\sum_{w\in W}\epsilon(w)e(w(\lambda+\rho)-\rho)}{\displaystyle\prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{\mathrm{mult}\alpha}}.$$

**Corollary 11.2.2** *Let $L(\lambda)$ be an irreducible and integrable highest weight module. Then we have:*

$$\mathrm{Ch}(L(\lambda)) = \frac{\displaystyle\sum_{w\in W}\epsilon(w)e(w(\lambda+\rho))}{\displaystyle\sum_{w\in W}\epsilon(w)e(w(\rho))}.$$

**Proof :** Apply the theorem to $L(0)$ which is the trivial representation. Its character is the unit and we get the so called **denominator identity**:

$$\sum_{w\in W}\epsilon(w)e(w(\rho)-\rho) = \prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{\mathrm{mult}\alpha}.$$

The result follows. □

To prove Theorem 11.2.1 we need some lemmas. First remark that the Weyl groups acts naturally on the algebra $\mathcal{E}$ by $w(e(\lambda)) = e(w(\lambda))$.

**Lemma 11.2.3** *Let $R$ be the element (the denominator) $\displaystyle\prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{\mathrm{mult}\alpha}$ in $\mathcal{E}$. Then we have*

$$w(e(\rho)R) = \epsilon(w)e(\rho)R.$$

**Proof :** It suffices to prove this for simple reflections. We have

$$\begin{aligned} s_i(e(\rho)R) &= e(\rho-\alpha_i)\textstyle\prod_{\alpha\in\Delta_+}(1-e(s_i(-\alpha)))^{\mathrm{mult}\alpha} \\ &= e(\rho-\alpha_i)(1-e(\alpha_i))^{\mathrm{mult}\alpha_i}\textstyle\prod_{\alpha\in\Delta_+\setminus\{\alpha_i\}}(1-e(s_i(-\alpha)))^{\mathrm{mult}\alpha} \\ &= e(\rho)e(-\alpha_i)(1-e(\alpha_i))\textstyle\prod_{\alpha\in\Delta_+\setminus\{\alpha_i\}}(1-e(-\alpha))^{\mathrm{mult}\alpha} \\ &= -e(\rho)R. \end{aligned}$$

The result follows. □

**Lemma 11.2.4** *Let $V$ be a highest weight module of highest weight $\lambda$, then*

$$\mathrm{Ch}V = \sum_{\mu\leq\lambda,\ |\mu+\rho|^2=|\lambda+\rho|^2} c_\mu \mathrm{Ch}M(\mu)$$

*where $c_\mu \in \mathbb{Z}$ and $c_\lambda = 1$.*

**Proof :** It is sufficient to prove this result for $L(\lambda)$ because of Proposition 10.7.4. Now we consider the equation given by Proposition 10.7.4

$$\mathrm{Ch}M(\mu) = \sum_{\nu\in\mathfrak{h}^*}[M(\mu) : L(\nu)]\mathrm{Ch}L(\nu).$$

But Corollary 11.1.14 tells us that the action of the Casimir element on $M(\mu)$ is $(|\mu+\rho|^2 - |\rho|^2)\mathrm{Id}$ and the same for all its subquotient so in particular on $L(\nu)$ if the multiplicity is non zero. But the action of the Casimir on $L(\nu)$ is $(|\nu+\rho|^2 - |\rho|^2)\mathrm{Id}$ so this implies that $|\mu+\rho|^2 = |\nu+\rho|^2$.

Consider the set $S(\mu) = \{\nu \in \mathfrak{h}^* \ / \ \nu \leq \mu \text{ and } |\nu+\rho|^2 = |\mu+\rho|^2\}$. For any $\mu$ we have an equation

$$\mathrm{Ch}M(\mu) = \sum_{\nu\in S(\mu)}[M(\mu) : L(\nu)]\mathrm{Ch}L(\nu).$$

with $[M(\mu) : L(\mu)] = 1$. This system is triangular and in particular considering this system for $\mu \in S(\lambda)$ we get the result by inverting the system. □

**Lemma 11.2.5** *Let $\lambda \in \mathfrak{h}^*$ be such that $\langle\alpha_i^\vee, \lambda\rangle \geq 0$ for all index $i$. Then for any $\nu \in \mathfrak{h}^*$ such that*

- *$\nu \leq \lambda + \rho$,*
- *$(\nu, \nu) = (\lambda+\rho, \lambda+\rho)$,*
- *$\langle\nu, \alpha_i^\vee\rangle \geq 0$ for all $i$,*

*we have $\nu = \lambda + \rho$.*

**Proof :** Write $\nu = \lambda + \rho - \sum_i a_i\alpha_i$ with $a_i \in \mathbb{Z}_{\geq 0}$. We have

$$\begin{array}{ll}(\nu,\nu) & = (\lambda+\rho,\lambda+\rho) - (\lambda+\rho, \sum_i a_i\alpha_i) - (\nu, \sum_i a_i\alpha_i) \\ (\nu,\nu) & = (\lambda+\rho,\lambda+\rho).\end{array}$$

We thus have $(\lambda+\rho, \sum_i a_i\alpha_i) = -(\nu, \sum_i a_i\alpha_i)$. But the second term is non positive by hypothesis and the first one is non negative (recall that $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i) > 0$. This equality is possible if and only if all the $a_i$ vanish. □

We prove the character formula.

**Proof :** There exist integers $d_\mu$ with $d_\lambda = 1$ such that:

$$\mathrm{Ch}L = \sum_{\mu\in S(\lambda)} d_\mu \mathrm{Ch}M(\mu).$$

By multiplying by $e(\rho)R$ and thanks to Proposition 10.7.5 we get

$$e(\rho)R \cdot \mathrm{Ch}L = \sum_{\mu\in S(\lambda)} d_\mu e(\mu+\rho).$$

But now recall that because $L$ is integrable, its character is $W$-invariant. This together with the $W$-anti-invariance of $e(\rho)R$ gives:

$$d_\mu = \epsilon(w)d_{w(\mu+\rho)-\rho}.$$

Fix $\mu$ with $d_\mu \neq 0$. Then for any $w \in W$, we have $d_{w(\mu+\rho)-\rho} \neq 0$ thus $w(\mu+\rho) - \rho \leq \lambda$. Take $v \in W$ such that $\mathrm{ht}(\lambda - (v(\mu+\rho) - \rho))$ is minimal. Set $\nu = v(\mu+\rho)$.

We have $\langle \nu, \alpha_i^\vee \rangle \geq 0$. Indeed, the condition on the height imply the inequality $\mathrm{ht}(\lambda - s_i v(\mu+\rho)+\rho) \geq \mathrm{ht}(\lambda - v(\mu+\rho)+\rho)$. But this implies that $\langle v(\mu+\rho), \alpha_i^\vee \rangle \geq 0$.

We have $\nu = v(\mu+\rho) \leq \lambda + \rho$ and $(\nu,\nu) = (\mu+\rho, \mu+\rho) = (\lambda+\rho, \lambda+\rho)$ because $\mu \in S(\lambda)$. Now the previous lemma implies that $\nu = \lambda + \rho$. In particular, for any $\mu$ with $d_\mu \neq 0$, we have that there exists a $w \in W$ (here $w = v^{-1}$) such that $\mu = w(\lambda+\rho) - \rho$ and

$$d_\mu = \epsilon(v) d_{v(\mu+\rho)-\rho} = \epsilon(v) d_\lambda = \epsilon(v) = \epsilon(w).$$

We thus have

$$e(\rho) R \cdot \mathrm{Ch}(L) = \sum_{w \in W} \epsilon(w) e(w(\lambda+\rho))$$

$$R \cdot \mathrm{Ch}(L) = \sum_{w \in W} \epsilon(w) e(w(\lambda+\rho) - \rho)$$

and the result follows. □

**Corollary 11.2.6** *An integrable highest weight module $L$ is irreducible.*

**Proof :** Indeed, the module $L$ and its irreducible quotient have the same character. □

# Chapter 12

# Untwisted affine Lie algebras

In this chapter, we present an explicit construction of untwisted affine Lie algebras using the existence of simple Lie algebras. We will construct explicitly simple finite dimensional Lie algberas in Chapter 13 and twisted affine Lie algebras in Chapter 14.

## 12.1 Some results on finite root systems

In this section we prove some results on finite dimensional simple Lie algebras that we shall need in the sequel. Let $\mathfrak{g}$ be a simple Lie algebra and denote by $W$ its Weyl group and by $\Delta$ its roots system. Denote by $(\alpha_i)_{i\in[1,n]}$ the simple roots. Recall from Proposition 8.2.10 that there exists a highest root (for the height) in $\Delta$ denoted by $\theta$.

**Proposition 12.1.1** *The Weyl group $W$ acts on $\Delta$ with as many orbits as there are root length (at most two and one in the simply laced case).*

**Proof :** We first prove this result in type $A_n$. In that case, the root system is described as follows. Let $E = \mathbb{R}^{n+1}$ and $(\varepsilon_i)_{i\in[1,n+1]}$, then the simple roots are given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, the roots are described by $(\varepsilon_i - \varepsilon_j)_{1\leq i\neq j\leq n+1}$. The Weyl group is isomorphic to $\mathfrak{S}_{n+1}$ and acts by permutation on the indices of the $\varepsilon_i$. In particular, because $W$ is 2-transitive on $[1, n+1]$, we get that $W$ acts transitively on $\Delta$.

Now consider the general case and recall that all roots are real i.e. all roots are in the orbit of a simple root. We only need to prove that all simple roots of the same length are in the same orbit. All simples roots of the same length are connected in the Dynkin diagram by a connected subgraph of the Dynkin diagram with only simple edges. Furthermore, the subgroup of $W$ generated by the reflections with respect to simple roots in a branch of type $A$ of the Dynkin diagram, is a Weyl group of type $A$ and all the simple roots corresponding to these vertices are thus in the same orbit. This concludes the proof. $\square$

**Corollary 12.1.2** *Let $\alpha$ be a root such that $\langle\theta, \alpha_i^\vee\rangle \geq 0$ for all $i \in [1, n]$, then in the simply laced case $\alpha = \theta$ and in the non simply laced case there are two such roots, one of them being $\theta$.*

**Proof :** If $\theta$ is the highest root, then for all $i$, we have $s_i(\theta) \leq \theta$ giving the inequalities. In particular $\theta \in C^\vee$ where $C^\vee$ is in the dominant chamber for the dual root system.

Conversely such a root $\alpha$ is in $C^\vee$ and by Theorem 6.5.2 this set is a fundamental domain for the action of $W$. In particular any orbit of $W$ meets $C^\vee$ in exactly one point. In the simply laced case there is a unique orbit of $W$ in $\Delta$ and thus a unique root $\alpha = \theta$ in $C^\vee$. In the non simply laced case, there are two orbits of $W$ in $\Delta$ each of them meeting $C^\vee$ in exactly one root. One of this root have to be $\theta$ because $\theta$ is in $C^\vee$. $\square$

Recall that we described some coefficients $(a_i)_{i\in[0,n]}$ associated to the simple roots for all Dynkin diagrams of affine Kac-Moody Lie algebras. These coefficients are such that $\delta = \sum_i a_i\alpha_i$ is the smallest positive integer element of the kernel of the affine Cartan matrix. For $\mathfrak{g}$ of finite dimension consider the coefficients $a_i$ given by affine type of order 1. By convention, in the simply laced case, we consider all roots as long and short.

**Proposition 12.1.3** *The root $\theta$ is long and the coroot $\theta^\vee$ is short. We have the formula* $\theta = \sum_{i=1}^{n} a_i\alpha_i$.

**Proof :** An easy computation (that we already did to prove that $\delta$ is in the kernel of the Cartan matrix of affine type) prove that if we denote by $\alpha$ the right hand side, we have $\langle\alpha_i^\vee, \alpha\rangle = 0$ for all roots except one (or two in type $A$). This (these) root(s) is (are) the root(s) to which the vertex corresponding to $\alpha_0$ is linked. For this (these) root(s) $\alpha_{att}$, we have $\langle\alpha_{att}, \theta\rangle > 0$ (its value is 1 in all cases except type $A_1$ and type $C_n$ for which its value is 2).

**Lemma 12.1.4** *We have $\alpha \in \Delta$ and $\alpha$ is long.*

**Proof :** We assert that $\alpha = w(\alpha_i)$ where $i = 1$ except in type $C_n$ where $i = n$ and $w$ is equal to $s_{\alpha_{i_1}} \cdots s_{\alpha_{i_r}}$ with

$$i_1, \cdots, i_r = \begin{cases} n, \cdots, 2 \text{ in type } A_n \\ 2, \cdots, n, \cdots, 2 \text{ in type } B_n \\ 1, \cdots n-1 \text{ in type } C_n \\ 2, \cdots, n, \cdots, 2 \text{ in type } D_n \\ 2,4,5,3,4,2,6,5,4,3 \text{ in type } E_6 \\ 1,3,4,5,2,4,6,5,3,4,2,7,6,5,4,3 \text{ in type } E_7 \\ 4,2,3,4,5,1,3,4,6,5,2,4,7,6,5,3,4,2,8,7,6,5,4,3 \text{ in type } E_8 \\ 1,2,3,4,3,2 \text{ in type } F_4 \\ 1,2 \text{ in type } G_2. \end{cases}$$

$\square$

This lemma proves that $\alpha = \theta$ in the simply laced case. In the non simply laced case consider the following element $\beta = \alpha^\vee$ where we abuse notation here because $\beta$ is not in $\Delta$ but in $\Delta^\vee$. However if $\Delta$ is of finite type, the same is true for $\Delta^\vee$ so that this defines a root $\beta$ in $\Delta$.

**Lemma 12.1.5** *We have $\beta \in \Delta \cap C^\vee$ and $\beta$ is short.*

**Proof :** Because $\alpha$ is long and $\beta$ is the coroot of $\alpha$ this implies that $\beta$ is short. Now we have $\beta = \sum_{i=1}^n b_i\alpha_i$ with

$$b_1, \cdots, b_n = \begin{cases} 1, \cdots, 1 \text{ in type } A_n \\ 1 \cdots, 1 \text{ in type } B_n \\ 1,2, \cdots, 2,1 \text{ in type } C_n \\ 1,2, \cdots, 2,1,1 \text{ in type } D_n \\ 1,2,2,3,2,1 \text{ in type } E_6 \\ 2,2,3,4,3,2,1 \text{ in type } E_7 \\ 2,3,4,6,5,4,3,2,1 \text{ in type } E_8 \\ 1,2,3,2 \text{ in type } F_4 \\ 1,2 \text{ in type } G_2. \end{cases}$$

It is now easy to get that $\beta \in C^\vee$ indeed, we have $\langle \beta, \alpha_i^\vee \rangle = 0$ for all $i$ except $\alpha_i = \alpha_{att}^\vee$. In fact we do not need to compute the exact value of $\beta$ to get this result because of the following formula:

$$\langle \alpha_i^\vee, \beta \rangle = 2\frac{(\alpha_i, \beta)}{(\alpha_i, \alpha_i)} \text{ and } \langle \alpha_i, \beta^\vee \rangle = 2\frac{(\alpha_i, \beta)}{(\beta, \beta)}$$

and in particular $\langle \alpha_i^\vee, \beta \rangle$ and $\langle \alpha_i, \beta^\vee \rangle$ have the same sign and even vanish at the same time. □

Now $\alpha$ and $\beta$ are the two elements in $\Delta \cap C^\vee$, but $\alpha \geq \beta$, the result follows. □

**Remark 12.1.6** (i) Remark that $\theta^\vee$ is the highest root for the dual root system $\Delta^\vee$ if and only if $\mathfrak{g}$ is simply laced.

(ii) We have the following more general observation: let $\mathfrak{g}$ be finite dimensional of type $X$ and let $\theta_X$ its highest root. Denote by $X^\vee$ the dual finite dimensional type obtained by reversing the arrows in the Dynkin diagram and by $\mathfrak{g}^\vee$ the associated Lie algebra. Then $\theta_X^\vee$ is a root in $\mathfrak{g}^\vee$ and $\theta_{X^\vee}^\vee$ is a root in $\mathfrak{g}$ (it is the root $\beta$ of the preceding proposition).

Consider $\widehat{\mathfrak{g}}$ the affine Lie algebra of order 1 associated with $\mathfrak{g}$ and denote its type by $\widehat{X}$. Denote by $\widehat{\mathfrak{g}}^\vee$ its dual and by $\widehat{X}^\vee$ its type. Consider the affine Lie algebra $\widehat{\mathfrak{g}^\vee}^\vee$ of type $\widehat{X^\vee}^\vee$. Removing its 0 vertex give back a finite Lie algebra $\mathfrak{g}$ of type $X$. We have the following

**Fact 12.1.7** *The root $\theta$ is $\delta_{\widehat{X}} - \alpha_0$ and the root $\theta^\vee$ is $\delta^\vee_{\widehat{X^\vee}^\vee} - \alpha_0^\vee$. In particular, the root $\beta$ of the previous lemma is $\delta_{\widehat{X^\vee}^\vee} - \alpha_0$.*

Let us now describe more explicitly the invariant bilinear form on $\mathfrak{g}$ and compute the norm of some roots. We denote by $a_0, \cdots, a_n$ the coefficients of the extended Dynkin diagram of order 1 and by $a_0^\vee, \cdots, a_n^\vee$ those of the dual Dynkin diagram obtained by reversing the arrows. Let $A$ be the Cartan matrix associated to $\mathfrak{g}$ and $\widehat{A}$ the Cartan of the affine Lie algebra of order one associated. We first make the following remark

**Proposition 12.1.8** *Let $D$ be the following diagonal matrix:*

$$D = \mathrm{Diag}\left(\frac{a_1}{a_1^\vee}, \cdots, \frac{a_n}{a_n^\vee}\right).$$

*Then $D^{-1}A$ is symmetric.*

**Proof :** This fact is true for any generalised Cartan matrix of affine or finite type. Let $D'$ be a regular diagonal matrix such that $B = D'^{-1}A$ is symmetric. Let $\delta = (a_0, \cdots, a_n)^t$ and $\delta^\vee = (a_0^\vee, \cdots, a_n^\vee)^t$. We have $A\delta = 0$ and $A^t\delta^\vee = 0$. This implies that $D'B\delta = 0$ thus $B\delta = 0$ and $\delta^{\vee t}A = 0$ thus $\delta^{\vee t}D'B = 0$ and $BD'\delta^\vee = 0$. Because the corank of $A$ and also the corank of $B$ is 1 we get that $D'\delta^\vee$ and $\delta$ are colinear. But any multiple $D$ of $D'$ is such that $D^{-1}A$ is symmetric. let us take $D$ such that $D\delta^\vee = \delta$. The result follows. □

A direct application of the definition of the invariant bilinear form leads to the following:

**Corollary 12.1.9** *We have the formulas $(\alpha_i^\vee, \alpha_j^\vee) = a_{j,i} \cdot \frac{a_j}{a_j^\vee}$ and $(\alpha_i, \alpha_j) = a_{i,j} \cdot \frac{a_i^\vee}{a_i}$.*

**Corollary 12.1.10** *We have $(\theta, \theta) = 2\frac{a_0^\vee}{a_0} = 2$ and $(\theta^\vee, \theta^\vee) = 2\frac{a_0}{a_0^\vee} = 2$*

**Proof :** Let us recall that in the affine case of order 1 we have $\delta = \theta + \alpha_0$. Recall also that $\langle \delta, \alpha_i^\vee \rangle$ vanishes if and only if $(\delta, \alpha_i)$ does. In particular we have $(\theta, \theta) = (\alpha_0, \alpha_0)$ and the result follows for $\theta$ (in all cases $a_0^\vee = a_0 = 1$). For $\theta^\vee$ this comes from $\theta^\vee = \delta^\vee_{\widehat{X^\vee}^\vee} - \alpha_0^\vee$. □

**Corollary 12.1.11** *We have $\theta = a_0 \nu(\theta^\vee) = \nu(\theta^\vee)$.*

**Proof :** Let us compute $\langle \theta, \alpha_i^\vee \rangle = -\langle \alpha_0, \alpha_i^\vee \rangle = -a_{i,0}$ and $(\theta^\vee, \alpha_i^\vee) = -(\alpha_0^\vee, \alpha_i^\vee)$ the second scalar product being computed for coroots of $\widehat{g^\vee}^\vee$ and the result follows. □

## 12.2 Untwisted affine Lie algebras

### 12.2.1 Construction of affine Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra and $( \ , \ )$ be an invariant bilinear form on $\mathfrak{g}$ such that $(\theta, \theta) = 2$ where $\theta$ is the highest root of the root system of $\mathfrak{g}$. We will denote by $\mathfrak{O}$ the ring $\mathbb{C}[t, t^{-1}]$ and we define the **loop algebra** by

$$\mathcal{L}(\mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{O}.$$

**Proposition 12.2.1** *The loop algebra is a Lie algebra under the Lie bracket*

$$[x \otimes t^a, y \otimes t^b] = [x, y] \otimes t^{a+b}.$$

**Proof :** We extend the Lie bracket by bilinearity and we need to prove the Jacobi identity but it comes directly from the Jacobi identity on $\mathfrak{g}$. □

**Remark 12.2.2** (ı) The loop algebra may be identified with the Lie algebra of regular rational functions $\mathbb{C}^\times \to \mathfrak{g}$, the element $\sum_a x_a \otimes t^a$ being the map $z \mapsto \sum_a z^a x_a$.

Recall the compact involution $\omega_0$ on $\mathfrak{g}$ (for example on $\mathfrak{sl}_n$ it is given by $A \mapsto -\bar{A}^t$ where $^-$ is the complex conjugation. Its fixed point set form the lie subalgebra $\mathfrak{su}_n$) whose fixed point set is a Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ of compact form. We may define a compact involution on $\mathcal{L}(\mathfrak{g})$ such that the fixed points of this involution form the Lie subalgebra of loops in $\mathfrak{k}$ that is to say maps from $S^1$ to $\mathfrak{k}$. This is defined on $\mathfrak{sl}_n$ by $f(t)A \mapsto -\bar{f}(t^{-1})\bar{A}^t$.

(ıı) We may extend the bilinear form $( \ , \ )$ to an $\mathfrak{O}$-valued bilinear form defined by

$$(x \otimes t^a, x \otimes t^b) = (x, y)t^{a+b}.$$

**Definition 12.2.3** Define the affine Lie algebra $\widehat{\mathfrak{g}}$ by $\widehat{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ with Lie bracket

$$[x \otimes t^a + \lambda c + \mu d, y \otimes t^b + \lambda' c + \mu' d] = [x, y] \otimes t^{a+b} + \mu y \otimes bt^b - \mu' x \otimes at^a + a\delta_{a,-b}(x, y)c.$$

**Proposition 12.2.4** *This defines a Lie algebra structure on $\widehat{\mathfrak{g}}$.*

**Proof :** We first study an intermediate Lie algebra $\widehat{\mathfrak{g}}'$ defined by the subspace $\mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c$. The element $c$ is central in this subalgebra (and even in $\widehat{\mathfrak{g}}$) so that if $\widehat{\mathfrak{g}}'$ is a Lie subalgebra, we get a central extension

$$0 \to \mathbb{C}c \to \widehat{\mathfrak{g}}' \to \mathcal{L}(\mathfrak{g}) \to 0.$$

This is a universal central extension of $\mathcal{L}(\mathfrak{g})$ by $\mathbb{C}$. In the same way considering the subalgebra $\bar{\widehat{\mathfrak{g}}}$ defined by the subspace $\mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}d$ then the extension

$$0 \to \mathbb{C}c \to \widehat{\mathfrak{g}} \to \bar{\widehat{\mathfrak{g}}} \to 0$$

is a central extension.

**Lemma 12.2.5** *Let $\phi(P,Q) = Res(\frac{dP}{dt}Q)$, then $\phi$ is bilinear and we have the following properties*

$$\phi(P,Q) = -\phi(Q,P)$$

$$\phi(PQ,R) + \phi(QR,P) + \phi(RP,Q) = 0.$$

**Proof :** The linearity comes from the linearity of the product, the derivation and the residue operator. The first formula comes from the fact that the residue of an exact element $d(PQ)/dt$ vanishes. The last one from the same argument with $d(PQR)/dt$. □

**Corollary 12.2.6** *The vector space $\widehat{\mathfrak{g}}'$ is a Lie algebra with the preceding Lie bracket.*

**Proof :** We first remark that $\phi(t^a, t^b) = a\delta_{a,-b} = -b\delta_{a,-b}$. In particular the Lie bracket on $\widehat{\mathfrak{g}}'$ takes the following form $[x \otimes t^a + \lambda c, y \otimes t^b + \lambda' c] = t^{a+b}[x,y] + a\delta_{a,-b}(x,y)c$. This generalises to $[x \otimes P + \lambda c, y \otimes Q + \lambda' c] = PQ[x,y] + \phi(P,Q)(x,y)c$. The antisymmetry of the Lie bracket comes from the first formula of the preceding Lemma and Jacobi identity comes from the second one. □

To prove that $\widehat{\mathfrak{g}}$ is a Lie algebra, we only need to prove it for $\bar{\widehat{\mathfrak{g}}}$ because $\mu$ and $\mu'$ do not appear in the coefficient of $c$ in the formula defining the Lie bracket.

**Fact 12.2.7** *Define $d_0$ by $d_0(P) = t\frac{dP}{dt}$, then $d_0$ is a derivation (called the degree derivation).*

The Lie bracket on $\bar{\widehat{\mathfrak{g}}}$ is given by $[x \otimes P + \mu d, y \otimes Q + \mu' d] = PQ[x,y] + \mu d_0(Q) \otimes y - \mu' d_0(P) \otimes x$. The antisymmetry of the Lie bracket is clear, let us compute:

$$\begin{aligned}
[x \otimes P + \mu d, [y \otimes Q + \mu' d, z \otimes R + \mu'' d]] &= [x \otimes P + \mu d, [y,z] \otimes QR + \mu' d_0(R) \otimes z - \mu'' d_0(Q) \otimes y] \\
&= [x,[y,z]]PQR + \mu'[x,z]P d_0(R) - \mu''[x,y]P d_0(Q) \\
&\quad + \mu[y,z] \otimes d_0(QR) + \mu\mu' z \otimes d_0^2(R) - \mu\mu'' y \otimes d_0^2(Q).
\end{aligned}$$

It is now an easy computation (using the fact that $d_0$ is a derivation) to verify the Jacobi identity. □

**Remark 12.2.8** The map $\mathfrak{g} \to \widehat{\mathfrak{g}}$ is a Lie algebra embedding. In particular we will consider $\mathfrak{g}$ as a subalgebra of $\widehat{\mathfrak{g}}$. We define the subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$ by $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$.

An easy check with the definition of the Lie bracket given in Definition 12.2.3 gives the following

**Fact 12.2.9** *The Lie subalgebra $\widehat{\mathfrak{h}}$ is abelian.*

### 12.2.2 The Lie algebra $\widehat{\mathfrak{g}}$ is an affine Kac-Moody algebra

We will see that $\widehat{\mathfrak{h}}$ will be a Cartan subalgebra for $\widehat{\mathfrak{g}}$ and that $\widehat{\mathfrak{g}}$ is the affine algebra of untwisted type obtained from $\mathfrak{g}$. Let us be more precise. Let $X$ be the type of $\mathfrak{g}$ with Cartan matrix $A$ and consider the affine Lie algebra of type $\widetilde{X}^1$ whose Cartan matrix is $\widehat{A}$. Recall that we denote by $\delta$ the element $\delta \in Q$ described by

$$\delta = \sum_i a_i \alpha_i$$

where $a_i$ is the coefficient of the vertex $i$ in Table 2.1. Denote by $a_i^\vee$ the coefficients of the vertices in the Dynkin diagram of the dual affine algebra associated to $\widehat{A}^t$. We may define in the same way $\delta^\vee$ by

$$\delta^\vee = \sum_i a_i^\vee \alpha_i^\vee.$$

**Lemma 12.2.10** *The element $\delta^\vee$ generates the center of $\mathfrak{g}(\widehat{A})$.*

**Proof :** We know from Proposition 4.1.12 that the center is of dimension one. We only need to prove that $\delta^\vee$ is in the center. But by the characterisation of the center in Proposition loc. cit. we need to prove that $\langle \delta^\vee, \alpha_i\rangle = 0$ for all $i$. This is true thanks to Proposition 8.2.7 □

**Remark 12.2.11** Remark that in all cases, we have $a_0 = a_0^\vee = 1$. Let us denote by $\theta$ the following root $\theta = \delta - \alpha_0$. It is a root of the finite system whose matrix is $A$ and Lie algebra $\mathfrak{g}$.

We already proved the following:

**Fact 12.2.12** *The root $\theta$ is the highest root for the root system of $\mathfrak{g}$.*

We have seen that $\mathfrak{h}$ is a subalgebra of $\widehat{\mathfrak{g}}$. We may embed $\mathfrak{h}^*$ as a subalgebra of $\widehat{\mathfrak{g}}^*$ as follows: for $\lambda \in \mathfrak{h}^*$ we define $\lambda(c) = \lambda(d) = 0$. This embeds $\mathfrak{h}^*$ in $\widehat{\mathfrak{g}}^*$. We may also define a special element $\widehat{\delta}$ in $\widehat{\mathfrak{g}}^*$ by $\widehat{\delta}|_{\mathfrak{h}} = 0$, $\widehat{\delta}(c) = 0$ and $\widehat{\delta}(d) = 1$. Let us denote by $\theta^\vee$ the coroot of $\theta$.

The first link between $\widehat{\mathfrak{g}}$ and $\mathfrak{g}(\widehat{A})$ is the following:

**Proposition 12.2.13** *The system $(\widehat{\mathfrak{h}}, \Pi, \Pi^\vee)$ is a realisation of the matrix $\widehat{A}$, where we have the equalities $\Pi = \{\widehat{\delta} - \theta, \alpha_1, \cdots, \alpha_n\}$ and $\Pi^\vee = \{c - \theta^\vee, \alpha_1^\vee, \cdots, \alpha_n^\vee\}$.*

**Proof :** The dimension of $\widehat{\mathfrak{h}}$ is $n + 2 = \text{size}(\widehat{A}) + \text{Corank}(\widehat{A})$ the expected dimension. Furthermore, the system $\Pi$ and the system $\Pi^\vee$ are independent because is it the case of the simple roots in $\mathfrak{h}$ and because of the definition of $c$ and $\widehat{\delta}$. Finally the proposition follows from the table of simple Lie algebras. □

**Proposition 12.2.14** *The Lie algebra $\widehat{\mathfrak{g}}$ decomposes as an $\widehat{\mathfrak{h}}$-module as follows:*

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{h}} \oplus \left(\bigoplus_{a\in\mathbb{Z}\setminus\{0\}} \mathfrak{h}\otimes t^a\right) \oplus \left(\bigoplus_{a\in\mathbb{Z},\ \alpha\in\Delta} \mathfrak{g}_\alpha \otimes t^a\right)$$

*where $\Delta$ is the root system of $\mathfrak{g}$ and if $\alpha \in \Delta$ the associated eigenspace in $\mathfrak{g}$ is $\mathfrak{g}_\alpha$. Moreover $\widehat{\mathfrak{h}}$ acts on $\mathfrak{h} \otimes t^a$ by $a\widehat{\delta}$ and on $\mathfrak{g}_\alpha \otimes t^a$ by $a\widehat{\delta} + \alpha$.*

**Proof :** All the weights are distinct and it suffices to prove that $\widehat{\mathfrak{h}}$ acts indeed with these weights. Let $h + \lambda c + \mu d \in \widehat{\mathfrak{h}}$ and let $h' \in \mathfrak{h}$ and $x \in \mathfrak{g}_\alpha$ we have

$$[h + \lambda c + \mu d, h' \otimes t^a] = \mu a h' \otimes t^a = a\widehat{\delta}(h + \lambda c + \mu d)h' \otimes t^a \text{ and}$$

$$[h + \lambda c + \mu d, x \otimes t^a] = \alpha(h)x \otimes t^a + a\mu x \otimes t^a = (\alpha + a\widehat{\delta})(h + \lambda c + \mu d)x \otimes t^a.$$

The result follows. □

Remark that the weights $\widehat{\delta}$ and $\delta$ coincide in $\widehat{\mathfrak{h}}^*$ seen as a realisation of $\widehat{A}$. We will identify them in the rest of the lecture.

Let us fix some more notation. Let $e_i$ and $f_i$ for $i \in [1, n]$ be Chevalley generators of the Lie algebra $\mathfrak{g}$ such that $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$. Let also $e_\theta$ be in $\mathfrak{g}_\theta$ be such that $(e_\theta, w(e_\theta)) = -1$ where $w$ is the Cartan involution in $\mathfrak{g}$ (in particular $w(e_\theta) \in \mathfrak{g}_{-\theta}$). Define $e_0$ and $f_0$ in $\widehat{\mathfrak{g}}$ by $e_0 = -w(x_0) \otimes t$ and $f_0 = x_0 \otimes t^{-1}$. In the same way let $\widehat{e}_i$ and $\widehat{f}_i$ for $i \in [0, n]$ be Chevalley generators of $\widehat{\mathfrak{g}}$ such that $\widehat{e}_i \in \widehat{\mathfrak{g}}_{\alpha_i}$ and $\widehat{f}_i \in \widehat{\mathfrak{g}}_{-\alpha_i}$.

**Theorem 12.2.15** *There is a unique Lie algebra isomorphism* $\psi : \mathfrak{g}(\widehat{A}) \to \widehat{\mathfrak{g}}$ *such that for all* $i$ *in* $[0, n]$ *we have* $\psi(\widehat{e}_i) = e_i$ *and* $\psi|_{\widehat{\mathfrak{h}}} = \mathrm{Id}_{\widehat{\mathfrak{h}}}$.

**Proof :** If it exists $\psi$ is unique because $\mathfrak{g}(\widehat{A})$ is generated by $\widehat{\mathfrak{h}}$ (recall that $\widehat{\mathfrak{h}}$ is the vector space of a realisation of $\widehat{A}$) and by the Chevalley generators $e_i$ and $f_i$ for $i \in [0, n]$.

To prove that this morphism exists, we start by proving its existence on $\widetilde{\mathfrak{g}}(\widehat{A})$. For this we need to prove the following commuting relations in $\widehat{\mathfrak{g}}$ for all $h$ and $h'$ in $\widehat{\mathfrak{h}}$ and all indices $i$ and $j$ in $[0, n]$:

$$[h, h'] = 0, \quad [h, e_i] = \langle \alpha_i, h\rangle e_i, \quad [h, f_i] = -\langle \alpha_i, h\rangle f_i, \quad \text{and} \quad [e_i, f_j] = \delta_{i,j}\alpha_i^\vee.$$

We already checked the first relation. The second is clear for $i \neq 0$ and $h \in \mathfrak{h}$. But it is also clear for $c$ and $d$ and thus for all $h \in \widehat{\mathfrak{h}}$ and $i \neq 0$. For $e_0$, this comes from the fact that $\alpha_0 = \delta - \theta$ giving the relation for $h \in \mathfrak{h}$ because $[h, e_0] = -[h, w(x_0)] \otimes t = -\langle \theta, h\rangle w(x_0) \otimes t = \langle \alpha_0, h\rangle e_0$ because $\delta$ is in the kernel of $\widehat{A}$. We have $\langle \alpha_0, c\rangle = \langle \delta - \theta, c\rangle = 0$ thus we get $[c, e_0] = 0 = \langle a_0, c\rangle e_0$. We have $\langle \alpha_0, d\rangle = \langle \delta - \theta, d\rangle = 1$ thus we get $[d, e_0] = 0 = \langle a_0, d\rangle e_0$. The same computation give the third commuting relations. For the last one, we already know the relations for $i$ and $j$ in $[1, n]$. For $ij = 0$ but $(i, j) \neq (0, 0)$, the fact that $\theta$ is a highest root imply the relation $[e_i, f_j] = 0$. Finally we have $[e_0, f_0] = -[w(x_0) \otimes t, x_0 \otimes t^{-1}] - (w(x_0), x_0)c = [x_0, w(x_0)] \otimes 1 + c = (w(x_0), x_0)\nu^{-1}(\theta) \otimes 1 + c$ thus $[e_0, f_0] = \nu^{-1}(\theta) \otimes 1 + c$ where $\nu : \mathfrak{h} \to \mathfrak{h}^*$ is the isomorphism described by the invariant form $(\ ,\ )$. It is easy to get that $\nu^{-1}(\theta) = \theta^\vee$ and the result follows.

Let us now prove that there is no non trivial ideal $\mathfrak{i}$ intersecting $\widehat{\mathfrak{h}}$ trivially in $\widehat{\mathfrak{g}}$. If such an $\mathfrak{i}$ exists, then from the weight space decomposition, there exists an element $x \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta \cup \{0\}$ and an integer $a \in \mathbb{Z}$ such that $x \otimes t^a \in \mathfrak{i}$. If $\alpha \neq 0$, then because $\mathfrak{g}$ is finite dimensional the dimension of $\mathfrak{g}_\alpha$ is one and $\mathfrak{g}_\alpha \otimes t^a \subset \mathfrak{i}$. In particular $[\mathfrak{g}_\alpha \otimes t^a, \mathfrak{g}_{-\alpha} \otimes t^{-a}] \neq 0$ and is in $\widehat{\mathfrak{h}} \cap \widehat{\mathfrak{g}}$ a contradiction. If $\alpha = 0$ but $a \neq 0$, then $[x \otimes t^a, \mathfrak{h} \otimes t^{-a}] \neq 0$ and is in $\widehat{\mathfrak{h}} \cap \mathfrak{i}$ a contradition. If $\alpha = 0$ and $a = 0$ then $x \otimes t^a$ is already in $\mathfrak{h} \subset \widehat{\mathfrak{h}}$ and this is not possible. This proves that there is a unique such Lie algebra morphism $\psi$.

Let $\mathfrak{r}$ be the kernel of $\psi$, then because $\psi$ is injective on $\widehat{\mathfrak{h}}$, this kernel has a trivial intersection with $\widehat{\mathfrak{h}}$ and is thus trivial by condtruction of the Kac-Moody algebra $\mathfrak{g}(\widetilde{A})$. Let us prove that $\psi$ is surjective. We know that $\mathfrak{g} \otimes 1$ is contained in the image. Furthermore, $[-w(x_0) \otimes t, \mathfrak{g} \otimes 1] = [-w(x_0), \mathfrak{g}] \otimes t$ is in this image but $[-w(x_0), \mathfrak{g}] = \mathfrak{g}$ because $\mathfrak{g}$ is simple, thus $\mathfrak{g} \otimes t$ is in the image. By induction, because $[x \otimes t, y \otimes t^k] = [x, y] \otimes t^{k+1}$ for $k \geq 0$, it follows by inclusion that $\mathfrak{g} \otimes t^k$ is in the image for $k \geq 0$. The same proof gives that $\mathfrak{g} \otimes t^k$ for $k \leq 0$ is in the image and the result follows. □

**Corollary 12.2.16** *The set of roots* $\widehat{\Delta}$ *of* $\mathfrak{g}(\widehat{A})$ *is given by*

$$\widehat{\Delta} = \{a\delta,\ a \in \mathbb{Z} \setminus \{0\}\} \cup \{\alpha + a\delta,\ \alpha \in \Delta,\ a \in \mathbb{Z}\}.$$

*Moreover, the root multiplicity of* $a\delta$ *is* $n$ *and the root multiplicity of* $\alpha + a\delta$ *is 1.*

### 12.2.3 Affine Weyl group

Let $W$ be the Weyl group of $\mathfrak{g}$. It is a finite group. Consider $Q^\vee$ the coroot lattice i.e. the $\mathbb{Z}$ submodule of $\mathfrak{h}$ generated by the simple coroots $\alpha_i^\vee$ for $i \in [1, n]$. Any coroot $\alpha^\vee$ lies in $Q^\vee$. In particular for any $w \in W$, we have $w(\alpha_i) \in Q$ thus $W$ acts on $Q$.

**Definition 12.2.17** The **affine Weyl group** $\widehat{W}$ is the semidirect product of the Weyl group $W$ by the coroot lattice $Q^\vee$. In symbols:

$$\widehat{W} = W \ltimes Q^\vee.$$

For $h \in Q^\vee$ the corresponding element in $\widehat{W}$ will be denoted $t_h$. It acts on $\mathfrak{h}$ (or $Q^\vee$) by translation. For $\alpha$ a root of $\mathfrak{g}$, we will denote by $\widehat{s}_\alpha$ the reflection in $\widehat{W}$ corresponding to the reflection $s_\alpha \in W$.

To describe the action of the Weyl group on $\widehat{\mathfrak{h}}^*$ we will need the following element $\Lambda \in \widehat{\mathfrak{h}}^*$ defined by $\langle \Lambda, \alpha_i^\vee \rangle = \delta_{0,i}$ and $\langle \Lambda, d \rangle = 0$. The elements $((\alpha_i)_{i\in[0,n]}, \Lambda)$ form a basis of $\widehat{\mathfrak{h}}^*$. As a consequence, the elements $((\alpha_i)_{i\in[1,n]}, \delta, \Lambda)$ form a basis of $\widehat{\mathfrak{h}}^*$ and $\mathbb{C}\delta \oplus \mathbb{C}\Lambda$ is a supplementary of $\mathfrak{h}$ in $\widehat{\mathfrak{h}}$.

**Remark 12.2.18** Consider the Weyl group $W_{\text{aff}}$ of the Kac-Moody Lie algebra $\mathfrak{g}(\widehat{A}) = \widehat{\mathfrak{g}}$ and consider the simple reflections $s_0, \cdots, s_n$. Let us denote by $W'$ the subgroup of $W_{\text{aff}}$ generated by $s_1, \cdots, s_n$. The action of $s_i$ on $\delta$ and $\Lambda$ is trivial thus $W'$ acts trivially on $\mathbb{C}\delta \oplus \mathbb{C}\Lambda$ and stabilises $\mathfrak{h}$. This implies (because $W_{\text{aff}}$ acts faithfully on $\widehat{\mathfrak{h}}^*$) that $W'$ acts faithfully on $\mathfrak{h}$. We can thus identify $W$ with $W'$ acting on $\widehat{\mathfrak{h}}$ by its action on $\mathfrak{h}$.

**Theorem 12.2.19** *Let $W_{\text{aff}}$ the Weyl group associated to the Kac-Moody Lie algebra $\mathfrak{g}(\widehat{A}) = \widehat{\mathfrak{g}}$, then there is a unique isomorphism of groups $\phi : W_{\text{aff}} \to \widehat{W}$ such that $\phi(s_0) = t_{\theta^\vee}\widehat{s}_\theta$ and $\phi(s_i) = \widehat{s}_i$ for $i \in [1, n]$.*

**Proof :** Let $h \in \mathfrak{h}$ and define the following element $T_h \in \text{End}(\widehat{\mathfrak{h}}^*)$:

$$T_h(\lambda) = \lambda + \langle \lambda, c\rangle \nu(h) - (\langle \lambda, h\rangle + \frac{1}{2}(h, h)\langle \lambda, c\rangle)\delta.$$

Remark that for $\lambda$ such that $\langle \lambda, c\rangle = 0$ we get $T_h(\lambda) = \lambda - \langle \lambda, h\rangle\delta$. In particular $T_h(\delta) = \delta$. Compute $T_h \circ T_k$ and remark that $\nu(h), \nu(k) \in \mathfrak{h}$, we get:

$$\begin{aligned} T_h \circ T_k(\lambda) &= T_h(\lambda + \langle \lambda, c\rangle \nu(k) - (\langle \lambda, k\rangle + \tfrac{1}{2}(k, k)\langle \lambda, c\rangle)\delta) \\ &= \lambda + \langle \lambda, c\rangle \nu(h) - (\langle \lambda, h\rangle + \tfrac{1}{2}(h, h)\langle \lambda, c\rangle)\delta \\ &\quad + \langle \lambda, c\rangle(\nu(k) - \langle \nu(k), h\rangle\delta) \\ &\quad - (\langle \lambda, k\rangle + \tfrac{1}{2}(k, k)\langle \lambda, c\rangle)\delta \\ &= \lambda + \langle \lambda, c\rangle(\nu(h) + \nu(k)) - (\langle \lambda, h + k\rangle + \tfrac{1}{2}(h + k, h + k)\langle \lambda, c\rangle)\delta \\ &= T_{h+k}(\lambda). \end{aligned}$$

In particular $T_h \in \text{Aut}(\widehat{\mathfrak{h}}^*)$ (its inverse is $T_{-h}$). This gives an embedding of $\mathfrak{h}$ in $\text{Aut}(\widehat{\mathfrak{h}}^*)$ (it is injective because as a group morphism we need to look at the kernel. It is given by the elements $h \in \mathfrak{h}$ such that $T_h = \text{Id}$. This gives for all $\lambda \in \mathfrak{h}^*$ that $\langle \lambda, h\rangle = 0$ thus $h = 0$.)

Recall that $W_{\text{aff}}$ is a subgroup of $\text{Aut}(\widehat{\mathfrak{h}}^*)$. We assert that $Q^\vee \subset \mathfrak{h} \subset \text{Aut}(\widehat{\mathfrak{h}}^*)$ is contained in $W_{\text{aff}}$. Let us first compute the following

$$\begin{aligned} T_{\theta^\vee} \circ s_\theta(\lambda) &= s_\theta(\lambda) + \langle s_\theta(\lambda), c\rangle \nu(\theta^\vee) - (\langle s_\theta(\lambda), \theta^\vee\rangle + \tfrac{1}{2}(\theta^\vee, \theta^\vee)\langle s_\theta(\lambda), c\rangle)\delta \\ &= \lambda - \langle \lambda, \theta^\vee\rangle\theta + \langle \lambda, c\rangle\theta - (\langle \lambda, \theta^\vee\rangle - 2\langle \lambda, \theta^\vee\rangle + \langle \lambda, c\rangle)\delta \\ &= \lambda - (\langle \lambda, c\rangle - \langle \lambda, \theta^\vee\rangle)(\delta - \theta) \\ &= s_{\alpha_0}(\lambda). \end{aligned}$$

We get in particular that $T_{\theta^\vee} \in W_{\text{aff}}$. Furthermore, for $w \in W$ and $\lambda \in \mathfrak{h}^*$, we have

$$\begin{aligned} wT_hw^{-1}(\lambda) &= w(w^{-1}(\lambda) - \langle w^{-1}(\lambda), h\rangle\delta) \\ &= \lambda - \langle \lambda, w(h)\rangle\delta) \\ &= T_{w(h)}(\lambda). \end{aligned}$$

But $wT_hw^{-1}(\delta) = \delta$ and $wT_hw^{-1}(\Lambda) = w(\Lambda + \nu(h) - \frac{1}{2}(h, h)\delta) = \Lambda + w(\nu(h)) - \frac{1}{2}(h, h)\delta$. We deduce that $wT_hw^{-1}(\Lambda) = T_{w(h)}(\Lambda)$ and $wT_hw^{-1} = T_{w(h)}$. We deduce that $W \cdot T_{\theta^\vee} \in W_{\text{aff}}$. To prove that $Q^\vee \subset W_{\text{aff}}$, it suffices to show that $W \cdot \theta^\vee$ generates $Q^\vee$.

**Lemma 12.2.20** *The subset $W \cdot \theta^\vee$ generates $Q^\vee$.*

**Proof :** Let $L$ be the $\mathbb{Z}$-span of $W \cdot \theta^\vee$. It is $W$-invariant. Recall that we proved that the Weyl group of a finite dimensional simple Lie algebra acts transitively on the roots of the same length. Moreover, we proved that $\theta^\vee$ is a short root. In particular all short roots are in $L$. If $\mathfrak{g}$ is simply laced, we are done. Otherwise, take $\alpha$ a short simple root and take $\beta$ a long simple root with $\langle \beta^\vee, \alpha \rangle \neq 0$ (its value has to be $-1$), then $s_\beta(\alpha) - \alpha = \beta$ is in $L$ and all long roots are in $L$. □

Now $Q^\vee$ is contained in $W_{\text{aff}}$ but this subgroup is normalised by $W$ in $W_{\text{aff}}$. Moreover, because all elements in $Q^\vee$ are of infinite order and because $W$ is finite we have $W \cap Q^\vee = \{1\}$ in $W_{\text{aff}}$. Now in the subgroup generated by $W$ and $Q^\vee$, which is $\widehat{W}$, we have all the reflections $s_i$ for $i \neq 0$ and $s_0$ because of a previous computation. □

**Definition 12.2.21** Let us define a bilinear form on $\widehat{\mathfrak{g}} = \mathfrak{g}(\widehat{A})$ as follows, for $x$ and $y$ in $\mathfrak{g}$ and $P$ and $Q$ in $\mathfrak{O}$:

$$(x \otimes P, y \otimes Q) = (x, y)\text{res}(\frac{PQ}{t}).$$

and by

$$(c, \mathfrak{g} \otimes \mathfrak{O}) = (d, \mathfrak{g} \otimes \mathfrak{O}) = (c, c) = (d, d) = 0, \quad (c, d) = 1.$$

**Proposition 12.2.22** *This bilinear form is invariant on $\widehat{\mathfrak{g}}$.*

**Proof :** Remark that taking $\widehat{D} = \text{Diag}(1, \epsilon_1, \cdots, \epsilon_n)$ were $D = \text{Diag}(\epsilon_1, \cdots, \epsilon_n)$ is a diagonal matrix such that $D^{-1}A$ is symmetric, then $\widehat{D}^{-1}\widehat{A}$ is symmetric.

To prove that the above defined bilinear form is the right one, we need to check that it satisfies on $\widehat{\mathfrak{h}}$ the conclusion of Proposition 7.2.1 and on $\widehat{\mathfrak{g}}$ the conclusion of Theorem 7.2.5. On $\widehat{\mathfrak{h}}$, the form is determined by the choice of a supplementary $\widehat{\mathfrak{h}}''$ (here choose $\mathbb{C}d$) of the space $\widehat{\mathfrak{h}}'$ generated by the simple roots and by the formulas $(\alpha_i^\vee, h) = \langle \alpha_i, h \rangle \epsilon_i$ for $h \in \widehat{\mathfrak{h}}$ and $(h', h'') = 0$ for $h'$ and $h''$ in $\widehat{\mathfrak{h}}''$. These are easily checked. We then need to check that this bilinear form is invariant for the Lie algebra structure i.e. $(X, [Y, Z]) = ([X, Y], Z)$. Let $X = x \otimes t^u + \lambda c + \mu d$, $Y = y \otimes t^v + \lambda' c + \mu'' d$ and $Z = z \otimes t^w + \lambda'' c + \mu'' d$, then

$$\begin{aligned}(X, [Y, Z]) &= (x \otimes t^u + \lambda c + \mu d, [y, z] \otimes t^{v+w} + \mu' wz \otimes t^w - \mu'' vy \otimes t^v + v\delta_{v,-w}(y, z)c) \\ &= (x, [y, z])\delta_{u+v+w,0} + (x, z)\mu' w\delta_{u+w,0} - (x, y)\mu'' v\delta_{u+v,0} + (y, z)\mu v\delta_{v+w,0} \\ &= ([x, y], z)\delta_{u+v+w,0} + (x, z)\mu' w\delta_{u+w,0} + (x, y)\mu'' u\delta_{u+v,0} + (y, z)\mu v\delta_{v+w,0}\end{aligned}$$

and because of the symmetry of the last three terms the same computation gives the result. □

## 12.3 Application: Jacobi triple product formula

In this section, we consider the denominator identity for the affine non twisted Lie algebra of type $A_1^1$. We compute explicitly this equality and reprove the famous triple product formula:

**Theorem 12.3.1** *Let $u$ and $v$ be two formal variables, then we have the following (Jacobi triple product) identity:*

$$\prod_{n\geq 1}(1 - u^n v^n)(1 - u^{n-1} v^n)(1 - u^n v^{n-1}) = \sum_{k\in\mathbb{Z}}(-1)^k u^{\frac{1}{2}k(k+1)} v^{\frac{1}{2}k(k-1)}.$$

**Proof :** We consider the denominator identity:

$$\prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{\mathrm{mult}\alpha}=\sum_{w\in W}\epsilon(w)e(w(\rho)-\rho).$$

We compute this formula for the root system of type $A_1^1$. The roots are described by $\Delta=\{\alpha_1+n\delta\ /\ n\in\mathbb{Z}\}\cup\{n\delta\ /\ n\in\mathbb{Z}\setminus\{0\}\}$ and $\Delta_+=\{\alpha_1+n\delta\ /\ n\geq 0\}\cup\{-\alpha_1+n\delta\ /\ n>0\}\cup\{n\delta\ /\ n>0\}$. In particular, we get, because $\mathrm{mult}(\alpha_1+n\delta)=1$ and $\mathrm{mult}(n\delta)=1$ (here the rank is one), the identity:

$$\prod_{n\geq 0}(1-e(-\alpha_1-n\delta))\prod_{n>0}(1-e(\alpha_1-n\delta))(1-e(-n\delta))=\sum_{w\in W}\epsilon(w)e(w(\rho)-\rho).$$

Let us set $u=e(-\alpha_0)$ and $v=e(-\alpha_1)$. Recall that $\delta=\alpha_0+\theta=\alpha_0+\alpha_1$. We get:

$$\prod_{n\geq 0}(1-u^nv^{n+1})\prod_{n>0}(1-u^nv^n)(1-u^nv^{n-1})=\sum_{w\in W}\epsilon(w)e(w(\rho)-\rho)$$

thus

$$\prod_{n\geq 1}(1-u^{n-1}v^n)(1-u^nv^n)(1-u^nv^{n-1})=\sum_{w\in W}\epsilon(w)e(w(\rho)-\rho).$$

Let us now recall the structure of the Weyl group. The finite Weyl group is $\{\pm 1\}$ and the affine Weyl group $W$ is the semi-direct product with $Q^\vee=\mathbb{Z}\alpha_1^\vee$. In particular an element in the Weyl group is of the form $(s_0s_1)^n$ or of the form $s_1(s_0s_1)^n$ for $n\in\mathbb{Z}$. Recall that we have $(\rho,\alpha_0)=\frac{1}{2}(\alpha_0,\alpha_0)=1=\frac{1}{2}(\alpha_1,\alpha_1)=(\rho,\alpha_1)$. This gives $s_0(\rho)=\rho-(\rho,\alpha_0)\alpha_0=\rho-\alpha_0$ and $s_1(\rho)=\rho-\alpha_1$. We also have $s_0(\alpha_1)=\alpha_1+2\alpha_0$, $s_1(\alpha_0)=\alpha_0+2\alpha_1$, $s_0s_1(\alpha_0)=3\alpha_0+2\alpha_1$ and $s_0s_1(\alpha_1)=-2\alpha_0-\alpha_1$.

**Lemma 12.3.2** *We have the following formulas*

$$\begin{aligned}(s_0s_1)^n(\rho)&=\rho+(-2n^2-n)\alpha_0+(-2n^2+n)\alpha_1.\\ s_1(s_0s_1)^n(\rho)&=\rho+(-2n^2-n)\alpha_0+(-2n^2-3n-1)\alpha_1.\end{aligned}$$

**Proof :** We prove these relations by induction on $n$. The second follows from the first. The first relation is clear for $n=0$. We have $s_0s_1(\rho)=s_0(\rho-\alpha_1)=\rho-\alpha_0-\alpha_1-2\alpha_0=\rho-3\alpha_0-\alpha_1$ and the result holds for $n=1$. Assume this is true for $n$, we get

$$\begin{aligned}(s_0s_1)^{n+1}(\rho)&=s_0s_1(\rho+(-2n^2-n)\alpha_0+(-2n^2+n)\alpha_1)\\ &=\rho-3\alpha_0-\alpha_1+(-2n^2-n)(3\alpha_0+2\alpha_1)+(-2n^2+n)(-2\alpha_0-\alpha_1)\\ &=\rho+(-3-6n^2-3n+4n^2-2n)\alpha_0+(-1-4n^2-2n+2n^2-n)\alpha_1\\ &=\rho+(-2n^2-5n-3)\alpha_0+(-2n^2-3n-1)\alpha_1\\ &=\rho+(-2(n+1)^2-(n+1))\alpha_0+(-2(n+1)^2+(n+1))\alpha_1.\end{aligned}$$

□

In particular we get, replacing $e(-\alpha_0)$ by $u$ and $e(-\alpha_1)$ by $v$:

$$\sum_{w\in W}\epsilon(w)e(w(\rho)-\rho)=\sum_{n\in\mathbb{Z}}u^{2n^2+n}v^{2n^2-n}-\sum_{n\in\mathbb{Z}}u^{2n^2+n}v^{2n^2+3n+1}.$$

We set $m=2n$ in the first sum and $m=-(2n+1)$ in the second. When $n\in\mathbb{Z}$, then $m$ describe all even (resp. odd) terms in the first (resp. second) sum. We get

$$\sum_{w\in W}\epsilon(w)e(w(\rho)-\rho)=\sum_{m\text{ even}}u^{\frac{1}{2}m(m+1)}v^{\frac{1}{2}m(m-1)}-\sum_{m\text{ odd}}u^{\frac{1}{2}m(m+1)}v^{\frac{1}{2}m(m-1)}$$

and the result follows. □

# Chapter 13

# Explicit construction of finite dimensional Lie algebras

In the next chapter, we construct the so called twisted affine Lie algebras. These are obtained via finite order automorphisms of finite dimensional Lie algebras. To illustrate the construction, we start in this chapter with the finite dimensional case. We explicitly construct simply laced finite dimensional simple Lie algebras. We then construct the non simply laced simple Lie algebras thanks to automorphisms of the simply laced ones.

## 13.1 Simply laced case

Let $A$ be a symmetric Cartan matrix of finite type. This matrix corresponds to a Dynkin diagram of simply laced type (no double arrow). Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realisation of $A$ and denote by $Q$ and $Q^\vee$ the coroot and root lattices. The matrix $A$ is symmetric non degenerate and defines a non degenerate quadratic form $(\ ,\ )$ on $Q$ (and $Q^\vee$). We denote by $\Delta$ be the root system associated to $A$ and start with the following:

**Fact 13.1.1** *The root system $\Delta$ is given by*

$$\Delta = \{\alpha \in Q \ / \ (\alpha, \alpha) = 2\}.$$

**Proof :** Let us define the set $\Delta'$ to be $\{\alpha \in Q \ / \ (\alpha, \alpha) = 2\}$. We prove that $\Delta'$ and $\Delta$ coincide. We may easily remark that they both live in $Q$ and that the set of simple roots $\Pi$ is contained in $\Delta'$. Furthermore, the set $\Delta'$ is $W$-invariant because the form $(\ ,\ )$ is. In particular, because in the simply laced case the group $W$ acts transitively on the roots, we get that $\Delta \subset \Delta'$. Let us prove the converse.

First remark that $Q$ is an even lattice that is for any $\alpha \in Q$ we have $(\alpha, \alpha)$ is even. Indeed, write $\alpha = \sum_i a_i \alpha_i$, then $(\alpha, \alpha) = \sum_i a_i^2 (\alpha_i, \alpha_i) + 2 \sum_{i<j} a_i a_j (\alpha_i, \alpha_j)$ and the result follows because $(\alpha_i, \alpha_i) = 2$.

Now let us prove that, for an element $\alpha \in \Delta'$, if we write $\alpha = \sum_i \alpha_i \alpha_i$, then all the $a_i$ have the same sign. Assume the converse is true, then let $K$ and $J$ be the (non empty) subsets of indices such that $a_k > 0$ for $k \in K$ and $a_j < 0$ for $j \in J$. Denote by $\alpha_+$ the sum $\sum_{k \in K} a_k \alpha_k$ and $\alpha_-$ the sum $\sum_{j \in J} a_j \alpha_j$. We have $2 = (\alpha, \alpha) = (\alpha_+, \alpha_+) + (\alpha_-, \alpha_-) + 2(\alpha_+, \alpha_-)$. But because $K$ and $J$ are non empty we have $(\alpha_+, \alpha_+) \geq 2$ and $(\alpha_-, \alpha_-) \geq 2$. But we also have

$$2(\alpha_+, \alpha_-) = \sum_{k \in K,\ j \in J} a_k a_j (\alpha_k, \alpha_j).$$

The scalar product $(\alpha_k, \alpha_j)$ is non positive because $k$ and $j$ are distinct and $a_j a_k$ is negative thus $2(\alpha_+, \alpha_-)$ is non negative, a contradiction.

Denote by $\Delta'_+$ (resp. $\Delta'_-$) the set of elements in $\Delta'$ obtained as non negative (resp. non positive) linear combinations of simple roots. We then prove that for any element $\alpha \in \Delta'_+$, there exists a simple root $\alpha_i$ such that $(\alpha, \alpha_i) > 0$. If not, write $\alpha + \sum_i a_i \alpha_i$ with $a_i \geq 0$. Then we have $2 = (\alpha, \alpha) = \sum_i a_i(\alpha, \alpha_i) \leq 0$ a contradiction.

Let us prove by induction on the height that all elements in $\Delta'$ are in $\Delta$. This is true for height one. Let $\alpha \in \Delta'_+$ not a simple root and $\alpha_i$ such that $(\alpha, \alpha_i) > 0$. Then $s_i(\alpha) = \alpha - (\alpha, \alpha_i)\alpha_i$ is in $\Delta'_+$ (because $\alpha \neq \alpha_i$) of lower height thus in $\Delta$. Finally $\Delta'_- = -\Delta'_+$ and the result follows. □

We now choose an orientation of the Dynkin diagram associated to $A$ i.e. we replace the simple edges of the Dynkin diagram by directed simple arrows. We have $2^{\sharp\{edges\}}$ such choices. For a fixed orientation of the Dynkin diagram we define a function $\varepsilon$ on the set of couples of vertices of the Dynkin diagram by

$$\varepsilon(i, j) = \begin{cases} 1 \text{ if } i = j \text{ or if there is an arrow from } i \text{ to } j \\ 0 \text{ otherwise.} \end{cases}$$

We extend this function on $Q \times Q$ by bilinearity i.e. we define a function $\varepsilon : Q \times Q \to \{\pm 1\}$. Writing $\alpha = \sum_i a_i \alpha_i$ and $\beta = \sum_i b_i \alpha_i$ we set

$$\varepsilon(\alpha, \beta) = \sum_{i,j} a_i b_j \varepsilon(i, j).$$

**Lemma 13.1.2** *For any orientation and any $\alpha \in Q$, we have*

$$\varepsilon(\alpha, \alpha) \equiv \frac{1}{2}(\alpha, \alpha) \pmod 2 \quad \text{and} \quad \varepsilon(\alpha, \beta) + \varepsilon(\beta, \alpha) \equiv (\alpha, \beta) \pmod 2.$$

**Proof :** We write $\alpha = \sum_i a_i \alpha_i$. We then have

$$\varepsilon(\alpha, \alpha) = \sum_i a_i^2 + \sum_{i \to j} a_i a_j.$$

But we also have

$$\frac{1}{2}(\alpha, \alpha) = \sum_i a_i^2 - \sum_{\{i,j\} \in \text{ Edge}} a_i a_j$$

where Edge is the set of pairs of vertices linked by an edge in the Dynkin diagram. In particular, for any orientation and any pair $\{i, j\}$ in Edge there is an arrow $i \to j$ or $j \to i$. We get the result.

The second relation comes from the first one and the bilinearity of $\varepsilon$ and $( \ , \ )$. □

Let $\mathfrak{h}$ be the vector space generated by the simple roots (a realisation of $A$ or $\mathfrak{h} = Q \otimes_{\mathbb{Z}} \mathbb{C}$). Let us define some variables $E_\alpha$ for $\alpha \in \Delta$. We now set $\mathfrak{g}' = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathbb{C} E_\alpha)$ and define a bracket on $\mathfrak{g}'$ by

$$\begin{cases} [h, h'] = 0 & \text{for } h \text{ and } h' \text{ in } \mathfrak{h} \\ [h, E_\alpha] = -[E_\alpha, h] = (h, \alpha) E_\alpha & \text{for } h \in \mathfrak{h} \text{ and } \alpha \in \Delta, \\ [E_\alpha, E_\beta] = 0 & \text{for } \alpha \text{ and } \beta \text{ in } \Delta \text{ but } 0 \neq \alpha + \beta \notin \Delta, \\ [E_\alpha, E_{-\alpha}] = -\alpha & \\ [E_\alpha, E_\beta] = (-1)^{\varepsilon(\alpha,\beta)} E_{\alpha+\beta} & \text{for } \alpha, \ \beta \text{ and } \alpha + \beta \text{ in } \Delta. \end{cases}$$

Let us also extend the bilinear form $( \ , \ )$ from $\mathfrak{h}$ to $\mathfrak{g}'$ by $(h, E_\alpha) = 0$ for $h \in \mathfrak{h}$ and $\alpha \in \Delta$ and $(E_\alpha, E_\beta) = -\delta_{\alpha+\beta,0}$ for $\alpha, \beta \in \Delta$.

**Theorem 13.1.3** *The bracket on $\mathfrak{g}'$ defines a Lie bracket and with this bracket $\mathfrak{g}'$ is the simple Lie algebra $\mathfrak{g}(A)$ associated to the Cartan matrix $A$. The bilinear form $(\ ,\ )$ is invariant.*

**Proof :** Assume that the bracket is a Lie bracket. Then it is easy to see that $\mathfrak{g}'$ is a simple Lie algebra of type associated with $A$. Indeed, define $e_i = E_{\alpha_i}$ and $f_i = -E_{-\alpha_i}$. Then $\mathfrak{h}$, $e_i$ and $f_i$ satisfy the relations of $\widetilde{\mathfrak{g}}(A)$. We thus have a map $\widetilde{\mathfrak{g}}(A) \to \mathfrak{g}'$. To prove its surjectivity we need to prove that the $e_i$, $f_i$ and $\mathfrak{h}$ generate $\mathfrak{g}'$. Take $\alpha$ a positive root, then we prove by induction on its height that $E_\alpha$ is generated by the $e_i$. If the height is 1 it is clear. Otherwise, take $\alpha_i$ such that $(\alpha, \alpha_i) > 0$. We have $s_i(\alpha) = \alpha - (\alpha, \alpha_i)\alpha_i$. But Cauchy-Schwarz gives us $(\alpha, \alpha_i)^2 \leq (\alpha_i, \alpha_i)(\alpha, \alpha)$ thus $|(\alpha_i, \alpha)| \leq 2$ with equality if and only if $\alpha$ and $\alpha_i$ are colinear. This is not the case thus $(\alpha, \alpha_i) = 1$ and $s_i(\alpha) = \alpha - \alpha_i$. Now by induction we have $E_{\alpha-\alpha_i} \in \mathfrak{g}'$ thus $E_\alpha = (-1)^{\varepsilon(\alpha-\alpha_i,\alpha_i)}[E_{\alpha-\alpha_i}, E_{\alpha_i}]$ is in $\mathfrak{g}'$ and the result follows.

Furthermore if $\mathfrak{i}$ is an ideal in $\mathfrak{g}'$ not meeting $\mathfrak{h}$, we know that it is trivial if and only if $\mathfrak{i} \cap \mathbb{C}E_\alpha$ is trivial (the $\mathbb{C}E_\alpha$ are the eigenspaces for the action of $\mathfrak{h}$). If $\mathfrak{i}$ is not trivial, then there exists an root $\alpha$ such that $E_\alpha \in \mathfrak{i}$ but then $-\alpha = [E_\alpha, E_{-\alpha}] \in \mathfrak{i}$ a contradiction. This proves that we have a surjective map $\mathfrak{g}(A) \to \mathfrak{g}'$ and this map is injective because it is an isomorphism on $\mathfrak{h}$.

It is an easy check that the bilinear form is invariant.

Let us now prove that the bracket is a Lie bracket. Let $\alpha$ and $\beta$ be two roots such that $\alpha + \beta$ is a root. This implies that $2(\alpha, \beta) = (\alpha + \beta, \alpha + \beta) - (\alpha, \alpha) - (\beta, \beta) = -2$. Applying the previous lemma we get that $\varepsilon(\alpha, \beta) = \varepsilon(\beta, \alpha) + 1$ and the antisymmetry of the Lie bracket follows.

For the Jacobi formula, we want to check that $X = [x, [y, z]] + [[x, z], y] + [z, [x, y]] = 0$ for $x$, $y$ and $z$ in $\mathfrak{h} \cup \{E_\alpha \ / \ \alpha \in \Delta\}$. This is clear for one of them in $\mathfrak{h}$ and the two others equal to $E_\alpha$ and $E_\beta$ with $\alpha + \beta \notin \Delta$ because all the brackets vanish. It is also clear if the three are in $\mathfrak{h}$. Furthermore, if two of them are in $\mathfrak{h}$, say $x$ and $y$ and $z = E_\alpha$ then this gives $X = (x, \alpha)(y, \alpha)E_\alpha - (x, \alpha)(y, \alpha)E_\alpha = 0$. Assume that one of them, say $x$, is in $\mathfrak{h}$ and let $y = E_\alpha$ and $z = E_\beta$ with $\alpha + \beta \in \Delta$. We get

$$X = (-1)^{\varepsilon(\alpha,\beta)}(x, \alpha + \beta)E_{\alpha+\beta} + (x, \beta)(-1)^{\varepsilon(\beta,\alpha)}E_{\alpha+\beta} + (x, \alpha)(-1)^{\varepsilon(\beta,\alpha)}E_{\alpha+\beta}$$

But as we already saw, in this situation, we have $\varepsilon(\alpha, \beta) = \varepsilon(\beta, \alpha) + 1$ and the result follows.

We are now left with the case $x = E_\alpha$, $y = E_\beta$ and $z = E_\gamma$ with $\alpha$, $\beta$ and $\gamma$ three roots. If the sum of any two of them is not a root or 0, the three brackets vanish. We may thus assume that $\alpha + \beta \in \Delta \cup \{0\}$.

If $\beta = -\alpha$, then we consider three cases: $\gamma \pm \alpha \notin \Delta \cup \{0\}$, $\gamma = \pm\alpha$, and $\gamma + \alpha \in \Delta$ or $\gamma - \alpha \in \Delta$ (the case $\gamma + \alpha \in \Delta$ and $\gamma - \alpha \in \Delta$ is not possible otherwise we would have $(\alpha, \gamma) = -1 = (-\alpha, \gamma)$). In the first case $X = [E_\gamma, -\alpha] = (\alpha, \gamma)E_\gamma$. But because $\gamma \pm \alpha \notin \Delta \cup \{0\}$, we have $(\alpha, \gamma) = 0$ (Cauchy-Schwarz gives that $|(\alpha, \gamma)| = 0, 1$ or 2 with 2 if $\gamma$ is proportional to $\alpha$ i.e. $\gamma = \pm\alpha$, this is not the case. Furthermore, if $|(\alpha, \gamma) = \pm 1$, then $\gamma - \alpha$ or $\gamma + \alpha$ is a root). In the second case, we get (for $\gamma = -\alpha$ the other case is similar) the equality $X = [E_\alpha, \alpha] + [E_\alpha, -\alpha] = 0$. In the third case (for $\gamma + \alpha \in \Delta$, the other case is similar), we get $X = (-1)^{\varepsilon(\alpha,\gamma)}[E_{\alpha+\gamma}, E_{-\alpha}] + [\alpha, E_\gamma] = ((-1)^{\varepsilon(\alpha,\gamma)+\varepsilon(\alpha+\gamma,-\alpha)} + (\alpha, \gamma))E_\gamma$. But because $\alpha + \gamma$ is a root we have $(\alpha, \gamma) = -1$ and but the previous lemma $(-1)^{\varepsilon(\alpha,\gamma)+\varepsilon(\alpha+\gamma,-\alpha)} = (-1)^{(\alpha,\gamma)+1} = 1$, the result follows in this case.

By symmetry, we may now assume that $\alpha + \beta$, $\alpha + \gamma$ and $\beta + \gamma$ are roots. We get that $(\alpha + \beta) = (\alpha, \gamma) = (\beta, \gamma) = -1$ thus $(\alpha + \beta, \gamma, \alpha + \beta + \gamma) = 0$ and thus (because the quadratic form is non degenerate) $\alpha + \beta + \gamma = 0$. We thus get

$$X = (-1)^{\varepsilon(\beta,\gamma)}(-\alpha) + (-1)^{\varepsilon(\alpha,\gamma)}\beta + (-1)^{\varepsilon(\alpha,\beta)}(-\gamma).$$

Now because $\alpha + \beta + \gamma = 0$ and from the bilinearity of $\varepsilon$, we get that $\varepsilon(\alpha, \alpha + \beta + \gamma) = 0$ thus $1 + \varepsilon(\alpha, \beta) = \varepsilon(\alpha, \gamma)$. Similarly we have $\varepsilon(\beta, \gamma) + 1 = \varepsilon(\alpha, \gamma)$ and we get $X = (-1)^{\varepsilon(\alpha,\gamma)}(\alpha + \beta + \gamma) = 0$.
□

## 13.2 Non simply laced case

Using the automorphisms of the Dynkin diagrams we construct the non simply laced simple Lie algebras using the simply laced ones. The technics we will use are very similar to the one we will us to construct twisted affine Lie algebras.

Let us first remark that the simply laced Dynkin diagrams of finite type do admit symmetries. We denote by $\sigma$ these symmetries. Let us describe them as follows:

- In type $A_n$, the symmetry is given by $\sigma(i) = n + 1 - i$.
- In type $D_n$, the symmetry is given by the exchange of the last two vertices.
- In type $D_4$ there is an order three symmetry given by the permutation of the three non central vertices.
- In type $E_6$ there is a symmetry $\sigma$ of order two given by (the numeration of the roots is the one of [Bo54]): $\sigma(1) = 6$, $\sigma(2) = 2$, $\sigma(3) = 5$ and $\sigma(4) = 4$.

**Lemma 13.2.1** *The symmetry $\sigma$ induces an automorphism, still denoted $\sigma$, of the Lie algebra $\mathfrak{g}$.*

**Proof :** For this we only need to remember the definition of $\mathfrak{g}$ as the quotient of $\widetilde{\mathfrak{g}}(A)$ by its maximal ideal with trivial intersection with $\mathfrak{h}$. Recall that $\widetilde{\mathfrak{g}}(A)$ is generated by $\mathfrak{h}$ and the elements $e_i$ and $f_i$. Let us define $\sigma$ on $\widetilde{\mathfrak{g}}(A)$ by $\sigma(\alpha_i^\vee) = \alpha_{\sigma(i)}^\vee$, $\sigma(e_i) = e_{\sigma(i)}$ and $\sigma(f_i) = f_{\sigma(i)}$. Because the relations do only depend on the Dynkin diagram, if we denote $e_{\sigma(i)}$ (resp. $f_{\sigma(i)}$, resp. $\alpha_{\sigma(i)}^\vee$) by $e_i'$ (resp. $f_i'$, resp. $\alpha_i'^\vee$), these elements satisfy the same relations as the elements $\alpha_i^\vee$, $e_i$ and $f_i$. In particular this induces a Lie algebra morphism $\widetilde{\mathfrak{g}}(A) \to \widetilde{\mathfrak{g}}(A)$ which is surjective. We may consider the composition $\widetilde{\mathfrak{g}}(A) \to \mathfrak{g}(A)$ which is still surjective. Its kernel is an ideal with trivial intersection with $\mathfrak{h}$ (because $\sigma$induced an isomorphism of $\mathfrak{h}$). Furthermore, the image of the maximal ideal $\mathfrak{r}$ with trivial intersection with $\mathfrak{h}$ in $\widetilde{\mathfrak{g}}(A)$ has to be contained in $\mathfrak{r}$ because it is an ideal with trivial intersection with $\mathfrak{h}$. In particular $\mathfrak{r}$ is in the kernel of the map $\widetilde{\mathfrak{g}}(A) \to \mathfrak{g}(A)$ and we get a Lie algebra epimorphism $\mathfrak{g}(A) \to \mathfrak{g}(A)$. Its kernel is an ideal with trivial intersection with $\mathfrak{h}$ and has to be trivial. □

Let us define an action of $\sigma$ on $Q$ by linearity: $\sigma(\sum_i a_i\alpha_i) = \sum_i a_i\sigma(\alpha_i)$. Because $\sigma$ preserves the Dynkin diagram, it preserves the quadratic form $(\ ,\ )$ and we have the following

**Fact 13.2.2** *The root system $\Delta$ is preserved by $\sigma$ and furhtermore $\Delta_+$ is preserved by $\sigma$.*

We may thus define another action on $\mathfrak{g}' = \mathfrak{g}(A)$ by $\sigma(E_\alpha) = E_{\sigma(\alpha)}$.

**Fact 13.2.3** *The automorphism $\sigma$ is given by this action.*

**Remark 13.2.4** (i) Let us also remark that for all of these symmetries (except in type $A_{2n}$), there exists an invariant orientation of the Dynkin diagram. We exclude the case $A_{2n}$ and we fix such an orientation.

(ii) Let $\alpha$ be a simple root such that $\sigma(\alpha_i) \neq \alpha_i$, then the existence of the $\sigma$-invariant orientation implies that $(\alpha_i, \sigma(\alpha_i)) = 0$.

We even have the following:

**Fact 13.2.5** *Let $\alpha \in \Delta$ such that $\alpha \neq \sigma(\alpha)$, then for any integer $i$ we have $(\alpha, \sigma^i(\alpha)) = 0$.*

**Proof :** We prove that if $r=2$ then the value $(\alpha,\sigma(\alpha))$ is even for any root $\alpha$. Cauchy-Schwarz will then give that this value is $0,2$ or $-2$ and is case the value is 2 or -2 then $\alpha$ and $\sigma(\alpha)$ are colinear. The last case is impossible because $\alpha$ and $\sigma(\alpha)$ have the same sign. In the first case $\alpha=\sigma(\alpha)$. If not we have the vanishing of the scalar product $(\alpha,\sigma(\alpha))$ which suffices because we may assume $i=1$ in the formula $(\alpha,\sigma^i(\alpha))=0$.

To prove that $(\alpha,\sigma(\alpha))$ is even, we proceed by induction on the length of $w$ such that $\alpha=w(\alpha_i)$ with $\alpha_i$ simple. We know it is true for simple roots. Now if it is true for $\alpha$, we have

$$(\sigma(s_i(\alpha)),s_i(\alpha))=(\sigma(\alpha),\alpha)-(\alpha,\alpha_i)((\alpha,\sigma(\alpha_i))+(\sigma(\alpha),\alpha_i))+(\alpha_i,\sigma(\alpha_i)).$$

We only need to prove that the middle term is even but because $r=2$ we have $(\sigma(\alpha),\alpha_i)=(\sigma^2(\alpha),\sigma(\alpha_i))=(\alpha,\sigma(\alpha_i))$.

In case $r=3$, we need to check all cases but there are only three orbits or cardinality 3 given by the orbit of $\alpha_1$, the orbit or $\alpha_1+\alpha_2$ and the orbit of $\alpha_1+\alpha_2+\alpha_3$. It is easy to check the result is that case. □

Let us now define the following elements and sets:

- We denote by $r$ the order of $\sigma$. We have $r=2$ or 3.
- Define $\Delta'_l=\{\alpha'=\alpha\ /\ \alpha\in\Delta$ and $\sigma(\alpha)=\alpha\}$ and $\Delta'_s=\{\alpha'=\frac{1}{r}(\sigma(\alpha)+\cdots+\sigma^r(\alpha))\ /\ \alpha\in\Delta$ and $\sigma(\alpha)\neq\alpha\}$.
- Define $\Delta'=\Delta_l\cup\Delta_s$ and $Q'=\mathbb{Z}\Delta'$.
- Set $E_\alpha^j=E_\alpha+\zeta^{-j}E_{\sigma(\alpha)}+\cdots+\zeta^{-(r-1)j}E_{\sigma^{r-1}(\alpha)}$ where $\zeta=\exp(2i\pi/r)$, $j\in\mathbb{Z}$ and $\alpha\in\Delta$. Remark that for $j=0$ the element $E_\alpha^0$ does only depend on the orbit of $\alpha$ under $\sigma$ i.e. on the root $\alpha'\in\Delta'$ the root $\alpha$ defines. The vector space generated by $E_\alpha^j$ does only depend on the orbit of $\alpha$ under $\sigma$.
- For $\alpha\in\Delta$, let us denote by $\alpha'$ the root in $\Delta'$ it defines. Set $E'_{\alpha'}=E_\alpha$ if $\alpha'\in\Delta'_l$ and $E'_{\alpha'}=E^0_\alpha$ for $\alpha'\in\Delta'_s$.
- With the same notation, define $V'(j)=\oplus_{\alpha'\in\Delta'}\mathbb{C}E^j_\alpha$, $\mathfrak{h}'(j)=\{h\in\mathfrak{h}\ /\ \sigma(h)=\zeta^jh\}$ and $\mathfrak{g}'(j)=\mathfrak{h}'(j)+V'(j)$.
- $\mathfrak{h}'=\mathfrak{h}(0)$ and $\mathfrak{g}'=\mathfrak{h}'\oplus(\oplus_{\alpha'\in\Delta'}\mathbb{C}E'_{\alpha'})$

**Theorem 13.2.6** *Let $(\mathfrak{g},r)$ be of type $(D_{n+1},2)$, $(A_{2n-1},2)$, $(E_6,2)$ or $(D_4,3)$. Then we have a decomposition*

$$\mathfrak{g}=\bigoplus_{j=0}^{r}\mathfrak{g}'(j)$$

*where $\mathfrak{g}'(j)$ is the eigenspace of the eigenvalue $\zeta^j$ for the action of $\sigma$ and $\mathfrak{g}'(0)=\mathfrak{g}'$. Furthermore, the Lie algebra $\mathfrak{g}'$ is the simple Lie algebra of type $B_n$, $C_n$, $F_4$ or $G_2$ respectively. Its commutation relations are as follows:*

| | |
|---|---|
| $[h,h']=0$ | *for $h$ and $h'$ in $\mathfrak{h}'$* |
| $[h,E'_{\alpha'}]=(h,\alpha')E'_\alpha$ | *for $h\in\mathfrak{h}$ and $\alpha'\in\Delta'$,* |
| $[E'_{\alpha'},E'_{\beta'}]=0$ | *for $\alpha'$ and $\beta'$ in $\Delta'$ but $0\neq\alpha'+\beta'\notin\Delta'$,* |
| $[E'_{\alpha'},E'_{-\alpha'}]=-\alpha'$ (*resp.* $-r\alpha'$) | *for $\alpha'\in\Delta'_l$ (resp. $\alpha'\in\Delta'_s$).* |
| $[E'_{\alpha'},E'_{\beta'}]=(p+1)(-1)^{\varepsilon(\alpha',\beta')}E_{\alpha'+\beta'}$ | *for $\alpha'$, $\beta'$ and $\alpha'+\beta'$ in $\Delta'$ and $p$ the maximal integer such that $\alpha'-p\beta'\in\Delta'$.* |

*The roots system is $\Delta'$ and $\Delta'_l$ (resp. $\Delta'_s$) is the set of long (resp. short) roots. The coroot system is given by $\Delta^\vee = \Delta_l \cup r\Delta_s$.*

*The invariant bilinear form is given by $(\ ,\ )$ on $\mathfrak{h}'$, by $(h, E'_{\alpha'}) = 0$ and*

$$(E'_{\alpha'}, E'_{\beta'}) = -\frac{2\delta_{\alpha',-\beta'}}{|\alpha'|^2}.$$

**Proof :** We start by proving that $\mathfrak{g}'(j)$ is the eigenspace associated to the eigenvalue $\zeta^j$. Let $\alpha \in \Delta$ with $\sigma(\alpha) \neq \alpha$ and denote by $\alpha'$ the corresponding element in $\Delta'_s$. We have $\sigma(E^j_\alpha) = E^j_{\sigma(\alpha)} = \zeta^j E^j_\alpha$ and we get that $\mathfrak{g}(j)$ is contained in the eigenspace associated with the eigenvalue $\zeta^j$. Conversely, let $x = h + \sum_\alpha a_\alpha E_\alpha$ in that eigenspace. We have $\zeta^j x = \sigma(x) = \sigma(h) + \sum_\alpha a_\alpha E_{\sigma(\alpha)}$. This gives that $h \in \mathfrak{h}(j)$ and that $\zeta^j a_{\sigma(\alpha)} = a_\alpha$. For $\alpha = \sigma(\alpha)$ this gives $j = 0$ or $a_\alpha = 0$. For $\alpha \neq \sigma(\alpha)$ we get

$$\sum_{k=0}^{r-1} a_{\sigma^k(\alpha)} E_{\sigma^k(\alpha)} = a_\alpha \sum_{j=0}^{r-1} \zeta^{-kj} E_{\sigma^k(\alpha)} = a_\alpha E^j_\alpha.$$

This proves the fact that the $\mathfrak{g}'(j)$ give the eigenspace decomposition. Furthermore, we have $E'_{\alpha'} = E^0_\alpha$ for $\alpha' \in \Delta'_s$ and $E'_{\alpha'} = E_\alpha = \frac{1}{r}E^0_\alpha$ for $\alpha' \in \Delta'_l$ proving that $\mathfrak{g}' = \mathfrak{g}'(0)$.

Let us now consider $\Delta'$ with the induced bilinear form $(\ ,\ )$. For $\alpha_i$ a simple root in $\Delta$, let us denote by $\alpha'_i$ the corresponding element in $\Delta'$. Let us denote by $Q'$ the submodule of $\mathfrak{h}$ generated by $\Delta'$ over $\mathbb{Z}$. We easily get that $Q'$ is generated by the elements $\alpha'_i$ and is contained in $\mathfrak{h}'$. We also have that $\mathfrak{h}'$ is generated by $\Delta'$. Let us compute the matrix $A'$ given by

$$A' = \left(\frac{2(\alpha'_i, \alpha'_j)}{(\alpha'_i, \alpha'_i)}\right).$$

According to the cases $(D_{n+1}, 2)$, $(A_{2n-1}, 2)$, $(E_6, 2)$ or $(D_4, 3)$ we get a matrix of type $B_n$, $C_n$, $F_4$ or $G_2$. We may now prove that the set $\Delta'$ is a root system of that type. Let us denote by $W'$ the Weyl group generated by the simple reflections with respect to the elements $\alpha'_i$. We start with the following:

**Lemma 13.2.7** *The set $\Delta'$ is $W'$-stable, more precisely $\Delta'_l$ and $\Delta'_s$ are $W'$-stable.*

**Proof :** Let $\alpha'$ and $\beta'$ two elements in $\Delta'$ and denote by $\alpha$ and $\beta$ the corresponding roots (not necessarily unique but unique modulo the action of $\sigma$). There are four different cases:

1. $\alpha' = \alpha$ and $\beta' = \beta$;
2. $\alpha' = \alpha$ and $\beta' = \frac{1}{r}(\beta + \cdots + \sigma^{r-1}(\beta))$;
3. $\alpha' = \frac{1}{r}(\alpha + \cdots + \sigma^{r-1}(\alpha))$ and $\beta' = \beta$;
4. $\alpha' = \frac{1}{r}(\alpha + \cdots + \sigma^{r-1}(\alpha))$ and $\beta' = \frac{1}{r}(\beta + \cdots + \sigma^{r-1}(\beta))$

Remark that in all cases, because $\alpha'$ and $\beta'$ are invariant under $\sigma$, the element $s_{\beta'}(\alpha')$ is invariant under $\sigma$.

Thanks to the previous fact, we may compute the length of elements in $\Delta'_l$ (long roots) and in $\Delta'_s$ (short roots): if $\alpha'$ is long then $(\alpha', \alpha') = 2$ and if $\alpha'$ is short, then $(\alpha', \alpha') = 2/r$. Remark that in cases 1 and 2, the root $\alpha'$ is long and in cases 3 and 4, the root $\alpha'$ is short.

In the first case, we have $s_{\beta'}(\alpha') = s_\beta(\alpha) \in \Delta$ and thus is in $\Delta'_l$. In the second case, let us compute $(\alpha', \beta') = 1/r \sum_i (\alpha, \sigma^i(\beta)) = (\alpha, \beta)$ and $(\beta', \beta') = 2/r$. We get the following formula

$$s_{\beta'}(\alpha') = \alpha' - \frac{2(\alpha', \beta')}{(\beta', \beta')}\beta' = \alpha - (\alpha, \beta)(\beta + \cdots + \sigma^{r-1}(\beta)) = s_\beta \cdots s_{\sigma^{r-1}(\beta)}(\alpha) \in \Delta'_l.$$

In the third case we still have $(\alpha', \beta') = (\alpha, \beta)$ and we get

$$s_{\beta'}(\alpha') = \alpha' - (\alpha, \beta)\beta = \frac{1}{r}(s_\beta(\alpha) + \cdots + s_\beta(\sigma^{r-1}(\alpha))) = \frac{1}{r}(s_\beta(\alpha) + \cdots + \sigma^{r-1}(s_\beta(\alpha))) \in \Delta'_s.$$

In the last case, let us first set $\gamma = s_\beta \cdots s_{\sigma^{r-1}(\beta)}(\alpha) = \alpha - \sum_{i=0}^{r-1}(\alpha, \sigma^i(\beta))\sigma^i(\beta)$. Compute

$$\frac{1}{r}\sum_{j=0}^{r-1}\sigma^j(\gamma) = \frac{1}{r}\sum_{j=0}^{r-1}\sigma^j(\alpha) - \frac{1}{r}\sum_{j=0}^{r-1}\sum_{i=0}^{r-1}(\alpha, \sigma^i(\beta))\sigma^{i+j}(\beta) = \alpha' - \frac{1}{r}\sum_{j=0}^{r-1}\sum_{i=0}^{r-1}(\sigma^j(\alpha), \sigma^{i+j}(\beta))\sigma^{i+j}(\beta)$$

Setting $k = i + j$ in the last sum we get

$$\frac{1}{r}\sum_{j=0}^{r-1}\sigma^j(\gamma) = \alpha' - \frac{1}{r}\sum_{k=0}^{r-1}\sum_{j=0}^{r-1}(\sigma^j(\alpha), \sigma^k(\beta))\sigma^k(\beta) = \alpha' - \sum_{k=0}^{r-1}(\alpha', \sigma^k(\beta))\sigma^k(\beta).$$

But $\sigma(\alpha') = \alpha'$ thus $(\alpha', \sigma^k(\beta)) = (\alpha', \beta)$ for all $k$ and thus also $(\alpha', \beta') = (\alpha', \beta)$. We get, because $(\beta', \beta') = 2/r$:

$$\Delta'_s \ni \frac{1}{r}\sum_{j=0}^{r-1}\sigma^j(\gamma) = \alpha' - (\alpha', \beta)\sum_{k=0}^{r-1}\sigma^k(\beta) = \alpha' - \frac{2(\alpha', \beta')}{(\beta', \beta')} \cdot \frac{1}{r}\sum_{k=0}^{r-1}\sigma^k(\beta) = s_{\beta'}(\alpha').$$

□

The proof is now similar to the proof of Fact 13.1.1: the Weyl group acts with two orbits on the root system according to the length and $\Delta'$ is invariant under the Weyl group. This implies that the root system is contained in $\Delta'$. We can easily define a notion of positive roots in $\Delta'$ by setting $\alpha' > 0$ if and only if $\alpha'$ is in the cone generated by the simple roots $\alpha_i$. We see that a root is either positive or negative and that such a root is a linear combination of the $\alpha'_i$ with coefficients of the same sign. This together with the fact that for any $\alpha' \in \Delta'$ we have $(\alpha', \alpha') > 0$ (it is equal to 2 or $2/r$) implies that for any $\alpha' \in \Delta'$, there exists a simple root $\alpha_i$ such that $(\alpha', \alpha'_i) > 0$. We may now conclude by induction on the height of $\alpha' \in \Delta'$ that $\Delta'$ is contained in the root system of $W'$. Indeed, this is true for height one. If the height of $\alpha'$ is bigger, then take $\alpha_i$ with $(\alpha', \alpha'_i) > 0$. The element $s_{\alpha'_i}(\alpha')$ is of smaller height and in $\Delta'$, we conclude by induction.

Remark that because $\mathfrak{g}'$ is the $\sigma$-invariant part of $\mathfrak{g}$, it has to be a Lie algebra. To prove that $\mathfrak{g}'$ is the desired Lie algebra, we start by proving the commutation relations. The first one is clear $\mathfrak{h}'$ being contained in $\mathfrak{h}$.

For the second one we discuss two different cases depending on the condition $\alpha'$ long or $\alpha'$ short. If $\alpha'$ is long then $\alpha' = \alpha \in \Delta$ and the result follows from the commuting relations in $\mathfrak{g}$. If $\alpha'$ is short, then there exists $\alpha \in \Delta$ with $\alpha' = 1/r(\alpha + \cdots + \sigma^{r-1}(\alpha))$ and $E'_{\alpha'} = E_\alpha + \cdots + E_{\sigma^{r-1}(\alpha)}$. We get

$$[h, E'_{\alpha'}] = (h, \alpha)E_\alpha + \cdots + (h, \sigma^{r-1}(\alpha))E_{\sigma^{r-1}(\alpha)} = (h, \alpha)E'_{\alpha'}$$

because $\sigma(h) = h$ and $(\sigma^k(h), \sigma^k(\alpha)) = (h, \alpha)$.

For the third one, first remark that $\alpha' + \beta' \notin \Delta'$ implies $\alpha + \beta \notin \Delta$ for any choice of root $\alpha$ and $\beta$ in $\Delta$ corresponding to $\alpha'$ and $\beta'$. This implies that in all four cases (long/long, long/short, short/long and short/short) all the brackets $[E_{\sigma^i(\alpha)}, E_{\sigma^j(\beta)}]$ appearing vanish because $\sigma^i(\alpha) + \sigma^j(\beta)$ is not a root.

For the fourth one we discuss one more time according to the length of $\alpha'$. If $\alpha'$ is long, this is the corresponding relation in $\mathfrak{g}$. For $\alpha'$ short, we get

$$[E'_{\alpha'}, E'_{-\alpha'}] = [E_\alpha, E_{-\alpha}] + \cdots + [E_{\sigma^{r-1}(\alpha)}, E_{-\sigma^{r-1}(\alpha)}] = -\alpha - \cdots - \sigma^{r-1}(\alpha) = -r\alpha'.$$

The first equality coming from the fact that all the root $\sigma^k(\alpha)$ are distinct and positive and all the roots $-\sigma^k(\alpha)$ are distinct and negative.

For the last one, we discuss on the length of $\alpha'$ and $\beta'$. Let us first remark that in all cases except $\alpha'$ and $\beta'$ short, the value of $\varepsilon(\alpha', \beta')$ is an integer. Indeed, as the orientation is preserved by $\sigma$, we have $\varepsilon(\sigma^i(\alpha), \sigma^i(\beta)) = \varepsilon(\alpha, \beta)$ for all $\alpha, \beta \in \Delta$ and $i \in \mathbb{Z}$ so that for any $\alpha, \beta \in \Delta$ corresponding to $\alpha', \beta' \in \Delta'$ where $\alpha'$ and $\beta'$ are not short at the same time, we have $\varepsilon(\alpha', \beta') = \varepsilon(\alpha, \beta)$. We now prove the following

**Lemma 13.2.8** *(ı) Let $\alpha$ and $\beta$ be two roots in $\Delta$. Assume that $\alpha + \beta$ is a root, then $\alpha - \beta$ isnot a root.*

*(ıı) Let $\alpha'$ and $\beta'$ be two roots in $\Delta'$ such that $\alpha' + \beta' \in \Delta'$. Let us denote by $p$ the greatest integer such that $\alpha' - p\beta'$ is in $\Delta'$. Then we have the following alternative:*

- *$\alpha', \beta' \in \Delta'_l$, then $\alpha' + \beta' \in \Delta'_l$ and $\alpha' - \beta' \notin \Delta'$;*
- *$\alpha' \in \Delta'_l$ and $\beta' \in \Delta'_s$, then $\alpha' + \beta' \in \Delta'_s$ and $\alpha' - \beta' \notin \Delta'$;*
- *$\alpha' \in \Delta'_s$ and $\beta' \in \Delta'_l$, then $\alpha' + \beta' \in \Delta'_s$ and $\alpha' - \beta' \notin \Delta'$;*
- *$\alpha' \in \Delta'_s$ and $\beta' \in \Delta'_s$ then if $\alpha' + \beta' \in \Delta'_l$, we have $p \leq r - 1$ and when $p = r - 1$ we have $\alpha' - p\beta' \in \Delta_l$. If $\alpha' + \beta' \in \Delta'_s$, then $p = 0$ for $r = 2$ and $p \leq 1$ for $r = 3$ with $\alpha' - p\beta' \in \Delta_l$ for $p = 1$.*

**Proof :** (ı) The element $\alpha + \beta$ is a root if and only if $|\alpha + \beta|^2 = 2$ i.e. $(\alpha, \beta) = -1$ thus $\alpha - \beta$ can not be a root.

(ıı) In the first case, we have $|\alpha' + \beta'|^2 = 4 + 2(\alpha, \beta) \in 2\mathbb{Z}$ thus $|\alpha' + \beta'|^2 \neq 2/r$. In particular $\alpha' + \beta' \in \Delta_l$ thus $\alpha + \beta$ is a root. The same argument shows that $\alpha' - \beta'$ is a root if and only if $\alpha - \beta$ is a root which is impossible by (ı).

In the second case, we have $|\alpha' + \beta'|^2 = 2 + 2/r + 2(\alpha, \beta) \in 2/r + 2\mathbb{Z}$ thus $|\alpha' + \beta'|^2 \neq 2$. In particular $\alpha' + \beta' \in \Delta_s$ thus $\alpha + \beta$ is a root. The same argument shows that $\alpha' - \beta'$ is a root if and only if $\alpha - \beta$ is a root which is impossible by (ı). The same argument works in the third case.

In the last case, we have $|\alpha' + \beta'|^2 = 4/r + 2(\alpha', \beta')$. If $\alpha' + \beta'$ is long then $(\alpha', \beta') = (r - 2)/r$ and if $\alpha' + \beta'$ is short, we have $(\alpha', \beta') = -1/r$. Let us now compute $|\alpha' - p\beta'|^2 = 2/r + 2p^2/r - 2p(\alpha', \beta')$ for $p$ a non negative integer. Its value is $(2 + 2p^2 - 2p(r - 2))/r$ or $(2 + 2p^2 + 2p)/r$ according to the length of $\alpha' + \beta'$. If this element is a root, the norm need to be equal to 2 or $2/r$. This gives the four equations:

$$p^2 - p(r - 2) + 1 = r \ ; \ p^2 - p(r - 2) = 0 \ ; \ p^2 + p + 1 = r \ ; \ p^2 + p = 0.$$

Recall that $p$ is a non negative integer, the solutions are as follows:

$$p = r - 1 \ ; \ p = 0 \text{ or } r - 2 \ ; \ p = 1 \text{ if } r = 3 \ ; \ p = 0.$$

The result follows. □

Now according to the cases of the previous lemma, we compute $[E'_{\alpha'}, E'_{\beta'}]$. We get $[E_\alpha, E_\beta]$ in first case and the result follows from the previous lemma and the simply laced case. In the second case, we get $\sum_{k=0}^{r-1}[E_\alpha, E_{\sigma^k(\beta)}] = \sum_{k=0}^{r-1}(-1)^{\varepsilon(\alpha,\beta)}E_{\alpha+\sigma^k(\beta)} = (-1)^{\varepsilon(\alpha',\beta')}E'_{\alpha'+\beta'}$ and the result follows from the previous lemma. The same computation gives the result in case three. In the last case we get:

$$\sum_{i=0}^{r-1}\sum_{j=0}^{r-1}[E_{\sigma^i(\alpha)}, E_{\sigma^j(\beta)}].$$

We will need the following fact which is proved case by case on the root systems:

**Fact 13.2.9** *There exists an index $k \in [0, r-1]$ such that $\alpha + \sigma^k(\beta)$ is a root.*

Let us discuss two cases according to $r$. Assume first that $r = 2$. Let $\alpha$ and $\beta$ be roots in $\Delta$ corresponding to $\alpha'$ and $\beta'$. There exists an index $k$ such that $\alpha + \sigma^k(\beta) \in \Delta$ and we may as well assume that $\alpha + \beta \in \Delta$. We have the following:

**Fact 13.2.10** *If $r = 2$, then $\alpha + \sigma(\beta) \notin \Delta$. Furthermore, if $\alpha' + \beta'$ is long, then $\alpha' - \beta'$ is a root.*

**Proof :** Because $\alpha + \beta$ is a root, we have $(\alpha, \beta) = -1 = (\sigma(\alpha), \sigma(\beta))$. We also have $(\alpha, \sigma(\beta)) = (\sigma(\alpha), \beta)$. But because $\alpha' + \beta'$ is a root we have $|\alpha' + \beta'|^2 = 1 + (\alpha, \sigma(\beta)) = 2$ or 1. This gives $(\alpha, \sigma(\beta)) = 1$ or 0 and the result follows.

If furthermore $\alpha' + \beta'$ is long, then $(\alpha, \sigma(\beta)) = 1$ thus $(\alpha, -\sigma(\beta)) = -1$ and $\alpha - \sigma(\beta)$ is a root. We get that $\alpha' - \beta' = 1/2(a - \sigma(\beta) + \sigma(\alpha - \sigma(\beta)))$. is in $\Delta'$.

If $r = 3$ this is not true: take $\alpha = \alpha_1$ and $\beta = \alpha_2 + \alpha_3$, then $\sigma(\beta) = \alpha_2 + \alpha_4$ and $\alpha + \beta$ and $\alpha + \sigma(\beta)$ are roots. □

In particular, for $r = 2$ we get $\alpha' + \beta' = (\alpha + \beta + \sigma(\alpha + \beta))/2$ and two different situations. If $\alpha' + \beta' \in \Delta'_s$ then $p = 0$. Otherwise, the previous fact implies that $p = 1$. Furthermore we conclude thanks to the computation:

$$[E'_{\alpha'}, E'_{\beta'}] = [E_\alpha, E_\beta] + [E_{\sigma(\alpha)}, E_{\sigma(\beta)}] = (-1)^{\varepsilon(\alpha,\beta)}(E_{\alpha+\beta} + E_{\sigma(\alpha+\beta)}) = (-1)^{\varepsilon(\alpha',\beta')}(p+1)E_{\alpha+\beta}$$

because if $\alpha' + \beta'$ is long we have $E_{\alpha+\beta} + E_{\sigma(\alpha+\beta)} = 2E'_{\alpha'+\beta'}$.

If $r = 3$, then we can make a case by case check. We may assume that $(\alpha, \beta)$ is one of the couple $(\alpha_2 + \alpha_3, \alpha_1)$, $(\alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2)$ or $(\alpha_2 + \alpha_3 + \alpha_4, \alpha_1)$. In case one, we have $p = 1$ and $p = 2$ in the last two cases. In the first case, we have

$$\begin{aligned}
[E'_{\alpha'}, E'_{\beta'}] &= (-1)^{\varepsilon(\alpha_2+\alpha_3,\alpha_1)}E_{\alpha_1+\alpha_2+\alpha_3} + (-1)^{\varepsilon(\alpha_2+\alpha_3,\alpha_4)}E_{\alpha_2+\alpha_3+\alpha_4} \\
&\quad + (-1)^{\varepsilon(\alpha_2+\alpha_4,\alpha_1)}E_{\alpha_1+\alpha_2+\alpha_4} + (-1)^{\varepsilon(\alpha_2+\alpha_4,\alpha_3)}E_{\alpha_2+\alpha_3+\alpha_4} \\
&\quad + (-1)^{\varepsilon(\alpha_1+\alpha_2,\alpha_3)}E_{\alpha_1+\alpha_2+\alpha_3} + (-1)^{\varepsilon(\alpha_1+\alpha_2,\alpha_4)}E_{\alpha_1+\alpha_2+\alpha_4}. \\
&= (-1)^{\varepsilon(\alpha,\beta)}2E'_{\alpha'+\beta'}.
\end{aligned}$$

We get the result by invariance of the orientation. In the second case, we get

$$\begin{aligned}
[E'_{\alpha'}, E'_{\beta'}] &= (-1)^{\varepsilon(\alpha_2+\alpha_3+\alpha_4,\alpha_1+\alpha_2)}E_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} + (-1)^{\varepsilon(\alpha_1+\alpha_2+\alpha_3,\alpha_2+\alpha_4)}E_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} \\
&\quad + (-1)^{\varepsilon(\alpha_1+\alpha_2+\alpha_4,\alpha_2+\alpha_3)}E_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}. \\
&= (-1)^{\varepsilon(\alpha,\beta)}3E'_{\alpha'+\beta'}.
\end{aligned}$$

Finally in the last case we have:

$$\begin{aligned}
[E'_{\alpha'}, E'_{\beta'}] &= (-1)^{\varepsilon(\alpha_2+\alpha_3+\alpha_4,\alpha_1)}E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} + (-1)^{\varepsilon(\alpha_1+\alpha_2+\alpha_3,\alpha_4)}E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \\
&\quad + (-1)^{\varepsilon(\alpha_1+\alpha_2+\alpha_4,\alpha_3)}E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}. \\
&= (-1)^{\varepsilon(\alpha,\beta)}3E'_{\alpha'+\beta'}.
\end{aligned}$$

Let us finish by proving that $\mathfrak{g}'$ is indeed of the expected type. Let us first prove that there exists a map $\psi$ from $\widetilde{\mathfrak{g}}(A')$ to $\mathfrak{g}'$ where $A'$ is the Cartan matrix of the non simply laced corresponding type. We send $e_i$ to $E'_{\alpha'_i}$ and $f_i$ to $-E'_{-\alpha'_i}$ and let $\psi|_{\mathfrak{h}'} = \mathrm{Id}_{\mathfrak{h}'}$ with $\alpha_i'^\vee = r\alpha'_i$ if $\alpha'_i$ is short. For this we need to prove that the $E'_{\alpha'_i}$ and $-E'_{-\alpha'_i}$ satisfy the desired relations. This is the case. Now the same proof as in the non twisted affine case or as in the simply laced case gives that $\psi$ factors through $\mathfrak{g}(A')$ and is injective. The surjectivity comes from the last relation. □

The modules $\mathfrak{g}'(j)$ do also satisfy such commuting (or module) relations similar as those satisfied by $\mathfrak{g}'$. For $\alpha \in \Delta$, let us define

$$\alpha'^{(j)} = \begin{cases} \alpha & \text{if } \alpha \in \Delta'_l \text{ i.e. } \sigma(\alpha) = \alpha \\ \alpha + \zeta^j \sigma(\alpha) + \cdots + \zeta^{(r-1)j}\sigma^{r-1}(\alpha) & \text{if } \alpha \notin \Delta'_l \text{ i.e. } \sigma(\alpha) neq \alpha. \end{cases}$$

We have the following result whose proof is very similar from the previous theorem:

**Theorem 13.2.11** *We have the following commuting relations in $\mathfrak{g}$:*

| | |
|---|---|
| $[h^{(i)}, h^{(j)}] = 0$ | *for $h^{(i)} \in \mathfrak{h}(i)$ and $h^{(j)} \in \mathfrak{h}(j)$* |
| $[h^{(i)}, E^j_\alpha] = (h^{(i)}, \alpha) E^{i+j}_\alpha$ | *for $h^{(i)} \in \mathfrak{h}(i)$ and $\alpha \in \Delta$,* |
| $[E^i_\alpha, E^j_\beta] = 0$ | *for $\alpha$ and $\beta$ in $\Delta$ but $0 \neq \alpha + \beta \notin \Delta$,* |
| $[E^i_{\alpha'}, E^j_{-\alpha}] = -\alpha^{(i+j)}$ | *for $\alpha \in \Delta$.* |
| $[E^i_\alpha, E^j_\beta] = (p+1)(-1)^{\varepsilon(\alpha,\beta)} E^{i+j}_{\alpha+\beta}$ | *for $\alpha$, $\beta$ and $\alpha+\beta$ in $\Delta$ and $p$ the maximal integer such that $\alpha' - p\beta' \in \Delta'$.* |

*In particular the $\mathfrak{g}'$-modules $\mathfrak{g}(j)$ are irreducible for all $j$.*

## 13.3 The case $A_{2n}$

The same technics would lead to the following result:

**Theorem 13.3.1** *Let $\mathfrak{g}$ be a simple Lie algebra of type $A_{2n}$ and let $\sigma$ the unique Dynkin diagram automophism and let us define $\sigma(E_\alpha) = (-1)^{1+\mathrm{ht}(\alpha)} E_{\sigma(a)}$.*

*(i) This defines a Lie algebra automorphism of $\mathfrak{g}$ still denoted $\sigma$.*

*(ii) The $\sigma$-invariant Lie subalgebra $\mathfrak{g}'$ is simple of type $B_n$ with root system $\Delta' = \Delta'_l \cup \Delta'_s$ where*

$$\Delta'_l = \{\alpha' = \frac{1}{2}(\alpha + \sigma(\alpha)) \ / \ \alpha \in \Delta,\ (\alpha, \sigma(\alpha)) = 0 \text{ and } \sigma(\alpha) \neq \alpha\} \quad \text{and}$$

$$\Delta'_s = \{\alpha' = \frac{1}{2}(\alpha + \sigma(\alpha)) \ / \ \alpha \in \Delta,\ (\alpha, \sigma(\alpha)) \neq 0 \text{ and } \sigma(\alpha) \neq \alpha\}.$$

*(iii) Define $E'_{\alpha'} = E_\alpha + (-1)^{\mathrm{ht}(\alpha)+1} E_{\sigma(\alpha)}$ for $\alpha' \in \Delta'_l$ and $E'_{\alpha'} = \sqrt{2}\left(E_\alpha + (-1)^{\mathrm{ht}(\alpha)+1} E_{\sigma(\alpha)}\right)$ for $\alpha' \in \Delta'_s$ and set $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$, then we have the decomposition*

$$\mathfrak{g}' = \mathfrak{h} \oplus \bigoplus_{\alpha' \in \Delta'} \mathbb{C} E'_{\alpha'}$$

*and the Lie bracket in $\mathfrak{g}'$ is described by the same commuting relations as in Theorem 13.2.6 i.e.:*

| | |
|---|---|
| $[h, h'] = 0$ | *for $h$ and $h'$ in $\mathfrak{h}'$* |
| $[h, E'_{\alpha'}] = (h, \alpha') E'_\alpha$ | *for $h \in \mathfrak{h}$ and $\alpha' \in \Delta'$,* |
| $[E'_{\alpha'}, E'_{\beta'}] = 0$ | *for $\alpha'$ and $\beta'$ in $\Delta'$ but $0 \neq \alpha' + \beta' \notin \Delta'$,* |
| $[E'_{\alpha'}, E'_{-\alpha'}] = -\alpha'$ (resp. $-r\alpha'$) | *for $\alpha' \in \Delta'_l$ (resp. $\alpha' \in \Delta'_s$).* |
| $[E'_{\alpha'}, E'_{\beta'}] = (p+1)(-1)^{\varepsilon(\alpha',\beta')} E_{\alpha'+\beta'}$ | *for $\alpha'$, $\beta'$ and $\alpha' + \beta'$ in $\Delta'$ and $p$ the maximal integer such that $\alpha' - p\beta' \in \Delta'$.* |

# Chapter 14

# Twisted affine Lie algebras

We give an explicit construction of twisted affine Lie algebras in this chapter. This will give us a description of the root and coroot systems and we will deduce a description of the Weyl group. We end the chapter will some (very few) applications of the denominator identity as functional identities.

## 14.1 Construction

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra and let $\sigma$ be an automorphism of its Dynkin diagram. Remark that such an automorphism is the identity except if $\mathfrak{g}$ is simply laced and that these automorphisms where already used to construct non simply laced simple finite dimensional Lie algebras.

We start the construction with the Loop algebra $\mathcal{L}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ and the affine Lie algebra $\widehat{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$. We already saw that the automorphism of the Dynkin diagram $\sigma$ defines an automorphism (still denoted $\sigma$) of $\mathfrak{g}$. We extend this automorphism to $\mathcal{L}(\mathfrak{g})$ and $\widehat{\mathfrak{g}}$ as follows: for $x \in \mathfrak{g}$, $a \in \mathbb{Z}$ and $\lambda, \mu \in \mathbb{C}$ we set

$$\sigma(x \otimes t^a + \lambda c + \mu d) = \frac{1}{\zeta^a}\sigma(x) \otimes t^a + \lambda c + \mu d$$

where $\zeta$ is a primitive $r$-th root of unity where $r$ is as usual the order of $\sigma$.

**Fact 14.1.1** *This defines an automorphism (still denoted $\sigma$) of the Lie algebras $\mathcal{L}(\mathfrak{g})$ and $\widehat{\mathfrak{g}}$.*

**Definition 14.1.2** Let us denote by $\mathcal{L}(\mathfrak{g}, \sigma)$ and $\widehat{\mathfrak{g}}(\sigma)$ the $\sigma$-invariant subalgebra of $\mathcal{L}(\mathfrak{g})$ and $\widehat{\mathfrak{g}}$ respectively.

**Proposition 14.1.3** *Let us denote by $\mathfrak{g} = \oplus_{i=0}^{r-1}\mathfrak{g}(j)$ the eigenspaces decomposition with respect to the action of $\sigma$ (described in Theorem 13.2.6). We have the following decompositions:*

$$\mathcal{L}(\mathfrak{g}, \sigma) = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}(\bar{k}) \otimes t^k \quad \textit{and} \quad \widehat{\mathfrak{g}}(\sigma) = \left(\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}(\bar{k}) \otimes t^k\right) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

*where $\bar{k}$ is the rest of $k$ modulo $r$.*

**Proof :** It is clear that these spaces are contained in the $\sigma$-invariant subalgebra. Conversely, if $x \otimes P + \lambda c + \mu d$ is in $\widehat{\mathfrak{g}}(\sigma)$. Then $\sigma(x \otimes P) = x \otimes P$. Writing $x = \sum_{i=0}^{r-1} x_i$ with $x_i \in \mathfrak{g}(i)$ and $P = \sum_k p_k t^k$ we get $\zeta^{i-k} x_i \otimes p_k t^k = x_i \otimes p_k t^k$. In particular for $i \not\equiv k \pmod r$ we have $x_i \otimes p_k t^k = 0$. The result follows. □

### 14.1.1 All cases except $A_{2n}^2$

In this subsection we consider $\mathfrak{g}$ a simple simply laced Lie algebra and $\sigma$ a non trivial automorphism of the Dynkin diagram. These automorphisms were described in the last chapter. Let us denote by $\mathfrak{g}'$ the $\sigma$-invariant subalgebra of $\mathfrak{g}$. We also keep the notation of the previous chapter for root systems and basis elements of $\mathfrak{g}$ and $\mathfrak{g}'$.

**Theorem 14.1.4** *Let $\mathfrak{g}$ be a simple Lie algebra of type $X$ and let $\sigma$ be a rank $r$ automorphism of the Dynkin diagram. The Lie algebra $\widehat{\mathfrak{g}}(\sigma)$ is isomorphic to the affine Lie algebra of type $X^r$.*

**Proof :** Let us fix the following additional notation:

- We denote by $n$ the rank of $\mathfrak{g}$ and by $m$ the rank of $\mathfrak{g}'$.
- We keep the notation $\alpha'_i$ for the simple roots in $\Delta'$.
- We denote by $\alpha_i'^\vee$ the coroot of $\alpha'_i$ a simple root in $\Delta'$.
- Denote by $\theta'_\vee$ the highest root of the dual root system $\Delta'^\vee$, this is also the root $r\beta$ where $\beta$ was described in Lemma 12.1.5. We denote this root $\beta$ by $\alpha'_\theta$.
- Set $E'_i = E'_{\alpha'_i}$ and $F'_i = -E'_{-\alpha'_i}$ for $\alpha'_i$ a simple root.
- Set $E'_0 = E^1_{-\alpha'_\theta} = \sum_{i=0}^{r-1}\zeta^{-i}E_{\sigma^i(-\alpha_\theta)}$ and $F'_0 = -E^{-1}_{\alpha'_\theta} = -\sum_{i=0}^{r-1}\zeta^{i}E_{\sigma^i(\alpha_\theta)}$ where $\alpha_\theta$ is any root in $\Delta$ such that $\theta'_\vee = r\alpha'_\theta$ is $r$ times the associated short root (we always have $\sigma(\alpha_\theta)\neq\alpha_\theta$). Remark that $E'_0$ and $F'_0$ and defined modulo a $r$-th root of 1 or equivalently modulo the choice of a root in the orbit of $\alpha_\theta$ but we only need that they are both defined by the same root.
- Set $e'_i = E'_i\otimes 1\in\widehat{\mathfrak{g}}(\sigma)$ and $f'_i = F'_i\otimes 1\in\widehat{\mathfrak{g}}(\sigma)$ for $i>0$.
- Set $e'_0 = E'_0\otimes t\in\widehat{\mathfrak{g}}(\sigma)$ and $f'_0 = F'_0\otimes t^{-1}\in\widehat{\mathfrak{g}}(\sigma)$.
- Set $\widehat{\mathfrak{h}}(\sigma) = \mathfrak{h}(0)\oplus\mathbb{C}c\oplus\mathbb{C}d$. We will denote by $\widehat{\mathfrak{h}}(\sigma)^*$ its dual.
- Define $\delta'\in\widehat{\mathfrak{h}}(\sigma)^*$ by $\delta'|_{\mathfrak{h}(0)} = \mathrm{Id}_{\mathfrak{h}(0)}$, $\delta'(c)=0$ and $\delta'(d)=1$.
- Define $\alpha'_0 = \delta' - \alpha'_\theta$.
- Define $\alpha_0'^\vee = \frac{r}{a_0}c - \theta'_\vee = rc - \theta'_\vee$.
- We will view $\mathfrak{h}(0)^*$ as being equal to $\mathfrak{h}(0)$ thank to the invariant bilinear form and $\mathfrak{h}(0)^*$ as a subspace of $\widehat{\mathfrak{h}}(\sigma)^*$ by setting $\lambda(c)=\lambda(d)=0$ for $\lambda\in\mathfrak{h}(0)^*$.

We first need to define a realisation of the Cartan matrix of type $X^r$ in $\widehat{\mathfrak{h}}(\sigma)$. For this let us define $\Pi^\vee = \{\alpha_0'^\vee,\cdots,\alpha_m'^\vee\}$ and $\Pi = \{\alpha'_0,\cdots,\alpha'_m\}$.

**Lemma 14.1.5** *The triple $(\widehat{\mathfrak{h}}(\sigma),\Pi,\Pi^\vee)$ is a realisation of the affine Cartan matrix of type $X^r$.*

**Proof :** The dimension of $\widehat{\mathfrak{h}}(\sigma)$ is $m+2$ which is the right dimension. We need to check that the matrix $(\langle\alpha'_i,\alpha'_j\rangle)$ is the Cartan matrix of type $X^r$. This is true for $i>0$ and $j>0$ because $\alpha_i'^\vee$ and $\alpha'_j$ are the simple coroots and roots of $\mathfrak{g}'$ which is of the type $Y$ obtained from $X^r$ by removing the zero vertex. We then compute $\langle\alpha_0'^\vee,\alpha'_j\rangle = -\langle\theta'_\vee,\alpha'_j\rangle$ this gives the result for $j>0$ because of our description of $\beta$ (or more conceptually because we get the same added vertex as for $\widehat{Y^\vee}$ but with reversed arrow. Now compute $\langle\alpha_0'^\vee,\alpha'_0\rangle = \langle\theta'_\vee,\alpha'_\theta\rangle = 2$. □

We now verify the relations $[e'_i, f'_j] = \delta_{i,j}\alpha'^{\vee}_i$, $[h, e'_i] = \langle\alpha'_i, h\rangle e'_i$ and $[h, f'_i] = -\langle\alpha'_i, h\rangle f'_i$. The first relation is true for $i > 0$ and $j > 0$. Because $\theta'_\vee$ is the highest root of $\Delta'^\vee$ we get the result for $ij = 0$ and $(i, j) \neq 0$. Furthermore $[e'_0, f'_0] = [E'_0, F'_0] + (E'_0, F'_0)c$. Because $\sigma^i(\alpha_\theta)$ is positive when $-\sigma^j(\alpha_\theta)$ is negative, we have $[E_{\sigma^i(-\alpha_\theta)}, E_{\sigma^j(\alpha_\theta)}] = \delta_{i,j}\sigma^i(\alpha_\theta)$ thus

$$[e'_0, f'_0] = -\sum_{i=0}^{r-1}\sigma^i(\alpha_\theta) + (E'_0, F'_0)c = -r\alpha'_\theta + (E'_0, F'_0)c = rc - \theta'_\vee = \alpha'^{\vee}_0$$

because we have $(E'_0, F'_0) = -\sum_{i,j}\zeta^{j-i}(E_{\sigma^i(-\alpha_\theta)}, E_{\sigma^j(\alpha_\theta)}) = r$ (recall that $(E_\alpha, E_\beta) = -\delta_{\alpha,-\beta}$).

For the relations $[h, e'_i] = \langle\alpha'_i, h\rangle e'_i$, this is clear for $i > 0$ and $h \in \widehat{\mathfrak{h}}(\sigma)$. For $[h, e'_0] = \langle\alpha'_0, h\rangle e'_0$, we start with $h \in \mathfrak{h}'$. Now the relations in $\mathfrak{g}$, see Theorem 13.2.11, give $[h, e'_0] = [h, E'_0] \otimes t = [h, E^1_{-\theta'_\vee}] \otimes t = -(h, \alpha_\theta)e'_0 = -(h, \alpha'_\theta)e'_0$ and the result follows from the fact that $(\delta', h) = 0$ for $h \in \mathfrak{h}'$. We are left with $h = c$ or $d$. But $(c, \alpha_0) = 0$ and $[c, e'_0] = 0$, furthermore, $(d, \alpha'_0) = 1$ and $[d, e'_0] = [d, E'_0 \otimes t] = E'_0 \otimes t = e'_0$. The same proof gives the relations $[h, f'_i] = -\langle\alpha'_i, h\rangle f'_i$

These relations give a map $\psi : \widetilde{\mathfrak{g}}(\widehat{A}^r) \to \widehat{\mathfrak{g}}(\sigma)$ where $\widehat{A}^r$ is the Cartan matrix of type $X^r$. Now we need the eigenspace decomposition with respect to $\widehat{\mathfrak{h}}$:

**Proposition 14.1.6** *We have the following decomposition*

$$\mathfrak{g}(\sigma) = \widehat{\mathfrak{h}}(\sigma) \oplus \left(\bigoplus_{\alpha'\in\Delta'} \mathbb{C}E'_{\alpha'}\right) \oplus \left(\bigoplus_{k\in\mathbb{Z}} \mathfrak{h}(\bar{k}) \otimes t^k\right) \oplus \left(\bigoplus_{k\in\mathbb{Z},\ \alpha'\in\Delta'_s} \mathbb{C}E^k_\alpha \otimes t^k\right) \oplus \left(\bigoplus_{k\in\mathbb{Z},\ \alpha'\in\Delta'_l} \mathbb{C}E^{rk}_\alpha \otimes t^{rk}\right)$$

*which is the eigenspace decomposition with respect to the action of $\widehat{\mathfrak{h}}(\sigma)$. The respective weights are given by 0, $\alpha' \in \Delta'$, $k\delta'$, $\alpha' + k\delta'$ and $\alpha' + rk\delta'$.*

**Proof :** The decomposition of $\widehat{\mathfrak{g}}(\sigma)$ and of the spaces $\mathfrak{g}(j)$ give the decomposition. We only need to prove that these spaces have the right weights under the action of $\widehat{\mathfrak{h}}(\sigma)$. This is clear for the first two terms (this comes from the relations in $\mathfrak{g}'$). For $h \in \mathfrak{h}(0)$ and $h' \in \mathfrak{h}(\bar{k})$ we have $[h, h' \otimes t^k] = 0$ and $[c, h' \otimes t^k] = 0$ while $[d, h' \otimes t^k] = kh' \otimes t^k$ and the weight is $k\delta'$. For $h \in \mathfrak{h}(0)$, compute $[h, E^k_\alpha \otimes t^k] = (h, \alpha)E^k_\alpha \otimes t^k = (h, \alpha')E^k_\alpha \otimes t^k$. Furthermore $[c, E^k_\alpha \otimes t^k] = 0$ and $[d, E^k_\alpha \otimes t^k] = kE^k_\alpha \otimes t^k$. The weight is thus $\alpha' + k\delta'$. To conclude, we need to remark that if $\alpha'$ is long, then for $k$ such that $\bar{k} \neq 0$, we have:

$$E^k_\alpha = \sum_{j=0}^{r-1}\zeta^{-kj}E_{\sigma^j(\alpha)} = \left(\sum_{j=0}^{r-1}\zeta^{-kj}\right)E_{(\alpha)} = 0.$$

□

Let $\mathfrak{r}$ be an ideal in $\mathfrak{g}(\sigma)$ with trivial intersection with $\widehat{\mathfrak{h}}(\sigma)$. Assume $\mathfrak{r}$ is not trivial, then we get from the fact that $\mathfrak{r}$ has a weight space decomposition according to the action of $\widehat{\mathfrak{h}}(\sigma)$ that $\mathfrak{r} \cap \mathfrak{h}(\bar{k}) \otimes t^k$ or $\mathfrak{r} \cap \mathbb{C}E^k_\alpha \otimes t^k$ is non trivial. Take a non trivial $x$ is that intersection, then $[x, \mathfrak{h}(-\bar{k}) \otimes t^{-k}]$ or $[x, E^{-k}_{-\alpha} \otimes t^{-k}]$ is non trivial giving rise to a non trivial element in $\mathfrak{r} \cap \widehat{\mathfrak{h}}(\sigma)$. This implies that the map $\psi$ factors through $\mathfrak{g}(\widehat{A}^r)$. Denote by $\psi'$ this new map. Because $\psi'$ is the identity on $\widehat{\mathfrak{h}}(\sigma)$ this implies that $\psi'$ is injective. Furthermore, the image of $\psi'$ contains the $e'_i$, the $f'_i$ and $\widehat{\mathfrak{h}}(\sigma)$ thus the image contains $\mathfrak{g}'$. Now the image contains $e'_0 = E'_0 \otimes t$ with $E'_0 \in \mathfrak{g}'(1)$. Because $\mathfrak{g}'(1)$ is an irreducible $\mathfrak{g}'$-module we get all $\mathfrak{g}(1) \otimes t$ in the image. Now we prove that $\mathfrak{g}(\bar{k}) \otimes t^k$ is in the image by induction on $k \geq 0$. Take $x \in \mathfrak{g}(\bar{k}) \otimes t^k$ and consider $[e'_0, x]$. This is an element in $\mathfrak{g}(\bar{k}+1) \otimes t^{k+1}$ and one more time because $\mathfrak{g}(\bar{k}+1)$ is a irreducible $\mathfrak{g}'$-module, the result follows. □

**Corollary 14.1.7** *The root system $\widehat{\Delta}(\sigma)$ of the twisted affine Lie algebra $\widehat{\mathfrak{g}}(\sigma)$ is given by*

$$\widehat{\Delta}(\sigma) = \{k\delta' \ / \ k \in \mathbb{Z}, \ k \neq 0\} \cup \{\alpha' + k\delta' \ / \ k \in \mathbb{Z}, \ \alpha' \in \Delta'_s\} \cup \{\alpha' + rk\delta' \ / \ k \in \mathbb{Z}, \ \alpha' \in \Delta'_l\}.$$

*The positive roots are given by*

$$\widehat{\Delta}(\sigma)_+ = \Delta'_+ \cup \{k\delta' \ / \ k > 0 \in \mathbb{Z}\} \cup \{\alpha' + k\delta' \ / \ k > 0, \ \alpha' \in \Delta'_s\} \cup \{\alpha' + rk\delta' \ / \ k > 0, \ \alpha' \in \Delta'_l\}.$$

*The multiplicity of the roots $\alpha' + k\delta'$ is 1 (this root is real. The multiplicity $k\delta'$ is $m$ for $\bar{k} = 0$ and $\dfrac{n-m}{r-1}$ otherwise.*

**Proof :** The only subtlety is for the multiplicity of imaginary roots. For $\bar{k} = 0$ the eigenspace is $\mathfrak{h}' = \mathfrak{h}(0)$ of dimension $m$ the rank of $g'$. For $\bar{k} \neq 0$ the eigenspace is $\mathfrak{h}(j)$ whose basis is given by the elements $\alpha_i^{(j)}$ for $\alpha'_i \in \Delta'_s$ a simple root. The dimension is $s$ the number of short simple roots.If $l$ is the number of long roots, we have $n = l + (r-1)s$ and $m = l + s$. The result follows. □

### 14.1.2 The case $A_{2n}^2$

A very similar proof but with some changes in the definition of the elements $\theta'_\vee$, $E'_0$ and $F'_0$ would lead to the following:

**Theorem 14.1.8** *Let $\mathfrak{g}$ be s simple Lie algebra of type $A_{2n}$ and let $\sigma$ be the non trivial involution of the Dynkin diagram. The Lie algebra $\widehat{\mathfrak{g}}(\sigma)$ is isomorphic to the affine Lie algebra of type $A_{2n}^2$.*

The root system is described as follows: $\widehat{\Delta}(\sigma) = \widehat{\Delta}(\sigma)^{re} \cup \widehat{\Delta}(\sigma)^{\mathrm{im}}$. Recall that we proved that $\widehat{\Delta}(\sigma)^{im} = \{k\delta' \ / \ k \in \mathbb{Z}, \ k \neq 0\}$. We have

$$\widehat{\Delta}(\sigma)^{re} = \{\frac{1}{2}(\alpha' + (2k-1)\delta' \ / \ k \in \mathbb{Z}, \ \alpha' \in \Delta'_l\} \cup \{\alpha' + k\delta' \ / \ k \in \mathbb{Z}, \ \alpha' \in \Delta'_s\} \cup \{\alpha' + rk\delta' \ / \ k \in \mathbb{Z}, \ \alpha' \in \Delta'_l\}.$$

The positive roots are given as in the other cases. The multiplicity of reals is 1 and the multiplicity $k\delta'$ is $m$ for $\bar{k} = 0$ and $\dfrac{n-m}{r-1}$ otherwise.

### 14.1.3 Further constructions

The Dynkin diagrams of affine Lie algebras have more symmetries than the classical ones. We may reproduce the construction by invariants for these Lie algebras. We get in this way another construction, starting from simply laced affine Lie algebras of all affine Lie algebras. We could also construct double affine Lie algebras and double twisted affine Lie algebras by reproducing the construction with the loop algebra.

## 14.2 Back to the Weyl group

### 14.2.1 All cases except $A_{2n}^2$

Let $W'$ be the Weyl group of $\mathfrak{g}'$. It is a finite group. Consider $Q'$ the root lattice i.e. the $\mathbb{Z}$ submodule of $\mathfrak{h}'$ generated by the simple roots $\alpha'_i$ for $i \in [1, m]$. Remark that the coroot lattice $Q'^\vee$ is contained in $Q'$ (when identifying $\mathfrak{h}'$ and $\mathfrak{h}'^*$ using the bilinear form $( \ , \ )$). Any root $\alpha' \in \Delta'$ lies in $Q'$. In particular for any $w \in W'$, we have $w(\alpha'_i) \in Q'$ thus $W'$ acts on $Q'$.

**Definition 14.2.1** The **twisted affine Weyl group** $\widehat{W}'$ is the semidirect product of the Weyl group $W'$ by the root lattice $Q'$. In symbols:

$$\widehat{W}' = W' \ltimes Q'.$$

For $h \in Q'$ the corresponding element in $\widehat{W}'$ will be denoted $t_h$. It acts on $\mathfrak{h}'$ (or $Q'$) by translation. For $\alpha'$ a root of $\mathfrak{g}'$, we will denote by $\widehat{s}_{\alpha'}$ the reflection in $\widehat{W}'$ corresponding to the reflection $s_{\alpha'} \in W'$.

To describe the action of the Weyl group on $\widehat{\mathfrak{h}}^*(\sigma)$ we use the element $\Lambda' \in \widehat{\mathfrak{h}}^*(\sigma)$ defined by $\langle \Lambda', \alpha_i'^\vee \rangle = \delta_{0,i}$ and $\langle \Lambda', d\rangle = 0$. The elements $((\alpha_i')_{i\in[0,n]}, \Lambda')$ form a basis of $\widehat{\mathfrak{h}}^*(\sigma)$. As a consequence, the elements $((\alpha_i')_{i\in[1,n]}, \delta', \Lambda')$ form a basis of $\widehat{\mathfrak{h}}^*(\sigma)$ and $\mathbb{C}\delta' \oplus \mathbb{C}\Lambda'$ is a supplementary of $\mathfrak{h}'$ in $\widehat{\mathfrak{h}}(\sigma)$.

**Remark 14.2.2** Consider the Weyl group ${W_{\rm aff}}'$ of the Kac-Moody Lie algebra $\mathfrak{g}(\widehat{A}') = \widehat{\mathfrak{g}}(\sigma)$ and consider the simple reflections $s_0', \cdots, s_m'$. Let us denote by $W$ the subgroup of ${W_{\rm aff}}'$ generated by $s_1', \cdots, s_m'$. The action of $s_i'$ on $\delta'$ and $\Lambda'$ is trivial thus $W$ acts trivially on $\mathbb{C}\delta' \oplus \mathbb{C}\Lambda'$ and stabilises $\mathfrak{h}'$. This implies (because ${W_{\rm aff}}'$ acts faithfully on $\widehat{\mathfrak{h}}^*(\sigma)$) that $W$ acts faithfully on $\mathfrak{h}'$. We can thus identify $W'$ with $W$ acting on $\widehat{\mathfrak{h}}(s)$ by its action on $\mathfrak{h}'$.

**Theorem 14.2.3** *Let ${W_{\rm aff}}'$ the Weyl group associated to the Kac-Moody Lie algebra $\widehat{\mathfrak{g}}(\sigma)$, then there is a unique isomorphism of groups $\phi : {W_{\rm aff}}' \to \widehat{W}$ such that $\phi(s_0) = t_{\theta^\vee}\widehat{s}_\theta$ and $\phi(s_i) = \widehat{s}_i$ for $i \in [1, n]$.*

**Proof :** Let $\gamma \in \mathfrak{h}'^*$ and define the following element $T_\gamma \in \mathrm{End}(\widehat{\mathfrak{h}}^*(\sigma))$:

$$T_\gamma(\lambda) = \lambda + \langle \lambda, rc\rangle\gamma - ((\lambda,\gamma) + \frac{1}{2}(\gamma,\gamma)\langle\lambda, rc\rangle)\delta'.$$

Remark that for $\lambda$ such that $\langle \lambda, c\rangle = 0$ we get $T_\gamma(\lambda) = \lambda - (\lambda,\gamma)\delta'$. In particular $T_\gamma(\delta') = \delta'$. Compute $T_\gamma \circ T_{\gamma'}$, we get:

$$\begin{aligned} T_\gamma \circ T_{\gamma'}(\lambda) &= T_\gamma(\lambda + \langle\lambda, rc\rangle\gamma' - ((\lambda,\gamma') + \tfrac{1}{2}(\gamma',\gamma')\langle\lambda, rc\rangle)\delta') \\ &= \lambda + \langle\lambda, rc\rangle\gamma - ((\lambda,\gamma) + \tfrac{1}{2}(\gamma,\gamma)\langle\lambda, rc\rangle)\delta' \\ &\quad +\langle\lambda, rc\rangle(\gamma' - (\gamma',\gamma)\delta') \\ &\quad -((\lambda,\gamma') + \tfrac{1}{2}(\gamma',\gamma')\langle\lambda, rc\rangle)\delta' \\ &= \lambda + \langle\lambda, rc\rangle(\gamma+\gamma') - ((\lambda,\gamma+\gamma') + \tfrac{1}{2}(\gamma+\gamma',\gamma+\gamma')\langle\lambda, rc\rangle)\delta' \\ &= T_{\gamma+\gamma'}(\lambda). \end{aligned}$$

In particular $T_\gamma \in \mathrm{Aut}(\widehat{\mathfrak{h}}^*(\sigma))$ (its inverse is $T_{-\gamma}$). This gives an embedding of $\mathfrak{h}'^*$ in $\mathrm{Aut}(\widehat{\mathfrak{h}}^*(\sigma))$ (it is injective because as a group morphism we need to look at the kernel. It is given by the elements $\gamma \in \mathfrak{h}'^*$ such that $T_\gamma = \mathrm{Id}$. This gives for all $\lambda \in \mathfrak{h}'^*$ that $(\lambda,\gamma) = 0$ thus $\gamma = 0$.)

Recall that ${W_{\rm aff}}'$ is a subgroup of $\mathrm{Aut}(\widehat{\mathfrak{h}}^*(\sigma))$. We assert that $Q \subset \mathfrak{h}'^* \subset \mathrm{Aut}(\widehat{\mathfrak{h}}^*(\sigma))$ is contained in ${W_{\rm aff}}'$. Remark that $\theta_\vee'$ is the highest root for $\Delta'^\vee$ and that its coroot is $\alpha_\theta'$. We thus have $\langle \alpha_\theta', \theta_\vee\rangle = 2$ and $\langle \theta_\vee', \lambda\rangle = 2(\alpha_\theta', \lambda)/(\alpha_\theta', \alpha_\theta') = (\lambda, \alpha_\theta')$. Let us compute the following

$$\begin{aligned} T_{\alpha_\theta'} \circ s_{\alpha_\theta'}(\lambda) &= s_{\alpha_\theta'}(\lambda) + \langle s_{\alpha_\theta'}(\lambda), rc\rangle\alpha_\theta' - ((s_{\alpha_\theta'}(\lambda), \alpha_\theta') + \tfrac{1}{2}(\alpha_\theta',\alpha_\theta')\langle s_{\alpha_\theta'}(\lambda), rc\rangle)\delta' \\ &= \lambda - \langle\lambda, \theta_\vee'\rangle\alpha_\theta' + \langle\lambda, rc\rangle\alpha_\theta' - ((\lambda,\alpha_\theta') - 2\langle\lambda,\theta_\vee'\rangle + \langle\lambda, rc\rangle)\delta' \\ &= \lambda - \langle\lambda, rc - \theta_\vee'\rangle(\delta' - \alpha_\theta') \\ &= s_{\alpha_0'}(\lambda). \end{aligned}$$

We get in particular that $T_{\alpha_\theta'} \in {W_{\rm aff}}'$. Furthermore, for $w \in W$ and $\lambda$ such that $\langle\lambda, c\rangle = 0$, we have

$$\begin{aligned} wT_\gamma w^{-1}(\lambda) &= w(w^{-1}(\lambda) - \langle w^{-1}(\lambda), \gamma\rangle\delta') \\ &= \lambda - \langle\lambda, w(\gamma)\rangle\delta) \\ &= T_{w(\gamma)}(\lambda). \end{aligned}$$

But $wT_\gamma w^{-1}(\Lambda) = w(\Lambda+\gamma-\frac{1}{2}(\gamma,\gamma)\delta') = \Lambda+w(\gamma)-\frac{1}{2}(\gamma,\gamma)\delta'$. We deduce that $wT_\gamma w^{-1}(\Lambda) = T_{w(\gamma)}(\Lambda)$ and $wT_\gamma w^{-1} = T_{w(\gamma)}$. We deduce that $W \cdot T_{\alpha'_\theta} \in {W_{\text{aff}}}'$. To prove that $Q \subset {W_{\text{aff}}}'$, it suffices to show that $W \cdot \alpha'_\theta$ generates $Q$. But $\alpha'_\theta$ is the coroot of $\theta'_\vee$ the highest root of the root system ${\Delta'}^\vee$. Applying Lemma 12.2.20 we get the result.

Now $Q$ is contained in ${W_{\text{aff}}}'$ but this subgroup is normalised by $W'$ in ${W_{\text{aff}}}'$. Moreover, because all elements in $Q$ are of infinite order and because $W'$ is finite we have $W' \cap Q = \{1\}$ in ${W_{\text{aff}}}'$. Now in the subgroup generated by $W'$ and $Q$, which is $\widehat{W}'$, we have all the reflections $s'_i$ for $i \neq 0$ and $s'_0$ because of a previous computation. □

### 14.2.2 Case $A_{2n}^2$

In this case we have the same result:

**Theorem 14.2.4** *Let ${W_{\text{aff}}}'$ the Weyl group associated to the Kac-Moody Lie algebra $\widehat{\mathfrak{g}}(\sigma)$ of type $A_{2n}^2$, then there is a unique isomorphism of groups $\phi : W_{\text{aff}} \to W' \ltimes Q'$.*

# Chapter 15

# Dedekin $\eta$-function identities

In this chapter we use the Denominator identity to prove more identities on power series and infinite product. In particular we will give a power series expressions of the Dedekin *eta*-function:

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

We start with a quick motivation for the study of this function by reviewing quickly the Theory of modular form (see for example [Se70]) and its link with the moduli space of elliptic curves.

## 15.1 Quick introduction to modular forms

Let us consider the group $\Gamma = SL_2(\mathbb{Z})$ also called the modular group. This group is generated by the elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The modular group $\Gamma$ acts on $\mathbb{H} = \{z \in \mathbb{C} \ / \ \Im(z) > 0\}$ by

$$A \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

The group $\Gamma$ also acts on the differential forms on $\mathbb{H}$ and in particular its action is given by

$$dz \mapsto \frac{dz}{(cz + d)^2}.$$

Taking a differential of weight $k/2$ form as $f(z)dz^{\frac{k}{2}}$ we obtain the action of the matrix $A$ by

$$A^*(f(z)dz^{\frac{k}{2}}) = f\left(\frac{az + b}{cz + d}\right) \cdot \frac{1}{(cz + d)^k} dz^{\frac{k}{2}}.$$

**Definition 15.1.1** A function $f$ on $\mathbb{H}$ is called weakly modular of weight $k$ if for all $a$, $b$, $c$ and $d$ integers such that $ad - bc = 1$ and all $z \in \mathbb{H}$ we have

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z).$$

Remark that because $S$ and $T$ generate $\Gamma$, this definition is equivalent to the following two equations for all $z \in \mathbb{H}$:

$$f(z+1) = f(z) \quad \text{and} \quad f\left(-\frac{1}{z}\right) = z^k f(z).$$

Furthermore, because $f(z+1) = f(z)$, we may write $f$ in the following form

$$f(z) = \widetilde{f}(q)$$

with $\widetilde{f}$ a function on the punctured unit disk $\mathbb{D}^*$ and where $q = e^{2i\pi z}$.

**Definition 15.1.2** A function $f$ on $\mathbb{H}$ is called a modular function of weight $k$ if the function is weakly modular of weight $k$ and if $\widetilde{f}$ is meromorphic on $\mathbb{D}$.

If furthermore $\widetilde{f}$ is holomorphic on $\mathbb{D}$, the function $f$ is called a modular form.

**Remark 15.1.3** Remark that with our definition and taking $-\mathrm{Id} \in \Gamma$, we obtain for $f$ a weakly modular function

$$f(z) = (-1)^k f(z)$$

and in particular $f$ is the zero function if $k$ is odd. This is not any more the case if we change the group $\Gamma$ and take $P\Gamma = PGL_2(\mathbb{Z})$.

Let us now consider the graded ring $R$ of modular forms. We have the following:

**Theorem 15.1.4** *Let $k$ be a integer with $k \geq 4$ and let us define the Eisenstein series $G_k$ on $\mathbb{H}$ by*

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2,\ (m,n)\neq(0,0)} \frac{1}{(mz+n)^k}.$$

*All the functions are modular forms and we have:*

$$R = \mathbb{C}[G_4, G_6].$$

Let us now introduce the modular forms $g_2 = 60G_4$, $g_3 = 140G_2$ and $\Delta = g_2^3 - 27g_3^2$.

**Proposition 15.1.5** *We have the following formula:*

$$\Delta(z) = (2\pi)^{12} q \left(\prod_{n=1}^{\infty}(1-q^n)\right)^{24} = (2\pi)^{12}\eta(q)^{24}.$$

Let us introduce the $j$ invariant as

$$j(z) = 1728\frac{g_2^3}{\Delta}.$$

The map $j : \mathbb{H} \to \mathbb{C}$ is $\Gamma$ invariant and realises the quotient of $\mathbb{H}$ by $\Gamma$: the induced map $j : \mathbb{H}/\Gamma \to \mathbb{C}$ is an isomorphism.

Let us also briefly recall the link with elliptic curves. Recall that any elliptic curve can be realised as a smooth plane cubic. Furthermore, we may always find coordinates $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2$ where the equation of this cubic is given by

$$x_1^2 x_2 = 4x_0^3 - g_2 x_0 x_2^2 + g_3 x_2^3.$$

In particular, this curve being smooth, its discriminant given by $\Delta = g_2 - 27g_3^2$ is different from 0 and we may define its $j$ invariant. We have the following result

**Theorem 15.1.6** *Let $\tau \in \mathbb{H}$ and define the associated elliptic curve $E(z) = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. Define the Weierstrass function on $\mathbb{C}$ by:*

$$\wp_\tau(z) = \frac{1}{z^2} + \sum_{\gamma \in \mathbb{Z}+\tau\mathbb{Z}} \frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2}.$$

*It is invariant under translation by an element of $\mathbb{Z} + \tau\mathbb{Z}$ and thus defines a function on $E$.*

*Then setting $y = \wp'_\tau(z)$ and $x = \wp_\tau(z)$, the elliptic curve $E$ is isomorphic to the smooth plane cubic given by $y^2 = 4x^3 - g_2(\tau)x + g_3(\tau)$.*

*Furthemore, two elliptic curves $E(z)$ and $E(z')$ are isomorphic if and only if $z$ and $z'$ are in the same orbit under $\Gamma$. In particular the moduli space of elliptic curves is described by the quotient $\mathbb{H}/\Gamma$ which is isomorphic to $\mathbb{C}$ thanks to the $j$ function.*

## 15.2 Functional identities with the Dedekin $\eta$-function

We will derive new identities from the denominator identity. Recall that for $\mathfrak{g}(A)$ a Kac-Moody Lie algebra with Weyl group $W(A)$ and root system $\Delta(A)$ and simple roots $\Pi(A) = \{\alpha_1, \cdots, \alpha_n\}$ we have the following identity in $\mathbb{C}[[e(-\alpha_1), \cdots, e(-\alpha_n)]]$:

$$\prod_{\alpha \in \Delta_+(A)} (1 - e(-\alpha))^{\mathrm{mult}\alpha} = \sum_{w \in W(A)} \epsilon(w) e(w(\rho) - \rho).$$

Let us also recall the Character formula for a dominant integrable weight $\lambda$:

$$\chi(\lambda) := \mathrm{Ch}(L(\lambda)) = \frac{\displaystyle\sum_{w \in W(A)} \epsilon(w) e(w(\rho + \lambda) - \rho)}{\displaystyle\prod_{\alpha \in \Delta_+(A)} (1 - e(-\alpha))^{\mathrm{mult}\alpha}}.$$

### 15.2.1 Specialisation of character formulas

Let us fix $s = (s_1, \cdots, s_n)$ a sequence of integers and define a gradation on $\mathfrak{g}(A)$ by setting $\deg(e_i) = -\deg(f_i) = s_i$ and $\deg(\mathfrak{h}) = 0$. This yields a decomposition

$$\mathfrak{g}(A) = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j(s).$$

Remark that if all the $s_i$ are positive, the graded pieces $\mathfrak{g}_j(s)$ are finite dimensional. This condition is always satisfied if $A$ is of finite type.

**Definition 15.2.1** A **specialisation of type** $s$ is the data of the morphism

$$F_s : \mathbb{C}[[e(-\alpha_1), \cdots, e(-\alpha_n)]] \to \mathbb{C}[[q]]$$

sending $e(-\alpha_i)$ to $q^{s_i}$. Let us denote by $h_s$ an element in $\mathfrak{h}(A)$ such that $\langle h_s, \alpha_i \rangle = s_i$ then we have for any $\alpha \in Q$:

$$F_s(e(-\alpha)) = q^{\langle h_s, \alpha \rangle}.$$

**Remark 15.2.2** If $s = (1, \cdots, 1)$ we may choose $h_s = \rho^\vee$. We denote $(1, \cdots, 1)$ by $\mathbb{1}$.

**Proposition 15.2.3** *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra. Denote by $\Delta^\vee$ its dual root system and let $\lambda$ be a dominant weight. Then we have*

$$\dim(L(\lambda)) = \prod_{\alpha\in\Delta_+^\vee} \frac{\langle \lambda+\rho,\alpha\rangle}{\langle \rho,\alpha\rangle}.$$

**Proof :** Let $W$ be the Weyl group of $\mathfrak{g}$. We have the following formula (using the denominator identity:

$$e(-\lambda)\mathrm{Ch}(L(\lambda)) = \frac{\sum_{w\in W} \epsilon(w)e(w(\rho+\lambda)-(\rho+\lambda))}{\sum_{w\in W} \epsilon(w)e(w(\rho)-\rho)}.$$

We define $N(\mu) = \sum_{w\in W} \epsilon(w)e(w(\rho+\lambda)-(\rho+\lambda))$ and compute

$$F_{\mathbb{1}}(N(\mu)) = \sum_{w\in W} \epsilon(w)q^{\langle \mu,\rho^\vee\rangle - \langle w(\mu),\rho^\vee\rangle} = \sum_{w\in W} \epsilon(w)q^{\langle \mu,\rho^\vee - w(\rho^\vee)\rangle} = F_s\left(\sum_{w\in W} \epsilon(w)e(-\rho^\vee + w(\rho^\vee))\right)$$

with $s = (\langle \mu,\alpha_1^\vee\rangle,\cdots,\langle \mu,\alpha_n^\vee\rangle)$ where $\alpha_1^\vee,\cdots,\alpha_n^\vee$ are the simple coroots of $\mathfrak{g}$. Applying the denominator identity for $g^t$, we get

$$F_{\mathbb{1}}(N(\mu)) = F_r\left(\prod_{\alpha\in\Delta_+^\vee}(1-e(-\alpha))^{\mathrm{mult}\alpha}\right) = \prod_{\alpha\in\Delta_+^\vee}(1-q^{\langle\alpha,\mu\rangle}).$$

Seting $\mu=\rho+\lambda$ and $\mu=\rho$ we get

$$F_{\mathbb{1}}(e(-\lambda)\chi(\lambda)) = \frac{\prod_{\alpha\in\Delta_+^\vee}(1-q^{\langle\alpha,\rho+\lambda\rangle})}{\prod_{\alpha\in\Delta_+^\vee}(1-q^{\langle\alpha,\rho\rangle})}.$$

On the other hand, recall that $\chi(\lambda) = \sum_\mu \dim(L(\lambda)_\mu)e(\mu)$ thus

$$F_{\mathbb{1}}(e(-\lambda)\chi(\lambda)) = \sum_\mu \dim(L(\lambda)_\mu)q^{\langle\lambda-\mu,\rho^\vee\rangle}$$

and in particular for $q\to 1$:

$$\lim_{q\to 1} F_{\mathbb{1}}(e(-\lambda)\chi(\lambda)) = \dim(L(\lambda)).$$

Taking the limit in the previous expression of $F_{\mathbb{1}}(e(-\lambda)\chi(\lambda))$ gives the result. □

### 15.2.2 Macdonald identities

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra and $\sigma$ an automorphism of order $r$ of its Dynkin diagram. Let us take the notation of Chapters 13 and 14 and denote by $\mathfrak{g}'$ the invariant subalgebra of $\mathfrak{g}$ ($\mathfrak{g}'=\mathfrak{g}$ if $r=1$). We denote by $\widehat{\mathfrak{g}}(\sigma)$ the associated twisted affine Lie algebra. We denote by $\Delta$ (resp. $\Delta'$ and $\widehat{\Delta}$), $W$ (resp. $W'$ resp. $\widehat{W}$) the root system and Weyl group of $\mathfrak{g}$ (resp. $\mathfrak{g}'$ and $\widehat{\mathfrak{g}}(\sigma)$). Recalt that $\widehat{W}$ is isomorphic to $W'\ltimes M$ where $M=Q^\vee$ (resp. $Q$) the coroot (resp. root) lattice if

$r = 1$ (resp. $r > 1$). Let us denote by $n$ the rank of $\mathfrak{g}$ and by $m$ the rank of $\mathfrak{g}'$ and let us set $\ell = \frac{n-m}{r-1}$. Let us denote by $X^r$ the type of $\widehat{\mathfrak{g}}(\sigma)$.

Let us define the following polynomials:

$$L(x) = (1-x)^n \prod_{\alpha\in\Delta}(1-xe(-\alpha)) \quad \text{for } r = 1;$$

$$L(x) = (1-x)^\ell(1-x^r)^{n-\ell} \prod_{\alpha\in\Delta'_s}(1-xe(-\alpha)) \prod_{\alpha\in\Delta'_l}(1-x^re(-\alpha)) \quad \text{for } r > 1 \text{ and } X^r \neq A_{2n}^2;$$

$$L(x) = (1-x)^m \prod_{\alpha\in\Delta'_s}(1-xe(-\alpha)) \prod_{\alpha\in\Delta'_l}(1-xe(\frac{1}{2}(\delta'-\alpha))(1-x^2e(-\alpha)) \quad \text{for } X^r = A_{2n}^2 \text{ and let us set}$$

$$R = \prod_{\alpha\in\Delta'_+}(1-e(-\alpha)).$$

We have the following formula:

$$\prod_{\alpha\in\widehat{\Delta}(\sigma)_+}(1-e(-\alpha))^{\text{mult}\alpha} = R\cdot\prod_{k=1}^{\infty}L(e(-k\delta')).$$

We now want to compute the oter part of the denominator identity. Let us denote by $\rho$, $\rho'$ and $\widehat{\rho}$ the corresponding element appearing in the Character formula for $\mathfrak{g}$, $\mathfrak{g}'$ and $\widehat{\mathfrak{g}}(\sigma)$. We will we decompose any element $w \in \widehat{W}$ in a product $w = ut_\alpha$ with $u \in W'$ and $\alpha \in M$. We nee to compute $w(\widehat{\rho}) - \widehat{\rho}$ that is to say $ut_\alpha(\widehat{\rho}) - \widehat{\rho}$. The element $\widehat{\rho} \in \widehat{\mathfrak{h}}(\sigma)^*$ is not well defined but we fix it by asking $\langle\widehat{\rho}, \alpha_i^\vee\rangle = 1$ and $\langle\widehat{\rho}, d\rangle = 0$. Let us write $\widehat{\rho} = \bar{\rho} + a\delta' + b\Lambda'$ where $\bar{\rho} \in \mathfrak{h}'^*$. Define $h^\vee = \sum_{i=0}^m a_i^\vee$ the dual Coxeter number for $\mathfrak{g}'$, we have

$$\widehat{\rho} = \bar{\rho} + h^\vee\Lambda'.$$

Remark that $\bar{\rho}$ is $\rho'$.

**Proposition 15.2.4** *For any $\alpha \in M$, we have the formula*

$$t_\alpha(\widehat{\rho}) = h^\vee\Lambda' + (\rho' + h^\vee\alpha) + \frac{1}{2h^\vee}(|\rho'|^2 - |\rho' + h^\vee\alpha|^2)\delta'.$$

*in particular we obtain*

$$ut_\alpha(\widehat{\rho}) - \widehat{\rho} = u(\rho' + h^\vee\alpha) - \rho' + \frac{1}{2h^\vee}(|\rho'|^2 - |\rho' + h^\vee\alpha|^2)\delta'.$$

**Proof :** Recall that the translation $t_\alpha$ was defined by the operator

$$T_\alpha(\lambda) = \lambda + \langle\lambda, rc\rangle\alpha - ((\lambda,\alpha) + \frac{1}{2}|\alpha|^2\langle\lambda, rc\rangle)\delta'.$$

We compute this for $\widehat{\rho}$ to get the result (because $\langle\widehat{\rho}, rc\rangle = h^\vee$). □

We may now compute

$$\sum_{ut_\alpha\in\widehat{W}}\epsilon(ut_\alpha)e(ut_\alpha(\widehat{\rho}) - \widehat{\rho}) = e\left(\frac{|\rho'|^2}{2h^\vee}\right)\sum_{\alpha\in M}\left[\left(\sum_{u\in W}\epsilon(u)e(u(\rho' + h^\vee\alpha) - \rho)\right)e\left(-\frac{1}{2h^\vee}|\rho' + h^\vee\alpha|^2\delta\right)\right].$$

**Theorem 15.2.5 (First Macdonald identities)**

$$e\left(-\frac{|\rho'|^2}{2h^\vee}\right) R \cdot \prod_{k=1}^{\infty} L(e(-k\delta')) = \sum_{\alpha\in M} \left[\left(\sum_{u\in W} \epsilon(u)e(u(\rho' + h^\vee\alpha) - \rho)\right) e\left(-\frac{1}{2h^\vee}|\rho' + h^\vee\alpha|^2\delta\right)\right].$$

Dividing both sides of this equality we get

**Theorem 15.2.6 (Second Macdonald identities)**

$$e\left(-\frac{|\rho'|^2}{2h^\vee}\right) \prod_{k=1}^{\infty} L(e(-k\delta')) = \sum_{\alpha\in M} \chi(h^\vee\alpha)e\left(-\frac{1}{2h^\vee}|\rho' + h^\vee\alpha|^2\delta\right).$$

### 15.2.3 Dedekin $\eta$-function identities

We now focus on the untwisted case, we have $\rho' = \rho$. We will need the following (see [FdV69]):

**Theorem 15.2.7 (Strange formula of Freudenthal-de Vries)**

$$\frac{|\rho|^2}{2h^\vee} = \frac{\dim \mathfrak{g}}{24}.$$

Setting $e(-\delta) = q$ we obtain

$$q^{\frac{\dim \mathfrak{g}}{24}} \prod_{k=1}^{\infty} \left((1-q^k)^n \prod_{\alpha\in\Delta}(1 - q^k e(\alpha))\right) = \sum_{\alpha\in M} \chi(h^\vee\alpha) q^{\frac{|\rho + h^\vee\alpha|^2}{2h^\vee}}.$$

We now take the specialisation $s = (1, 0, \cdots, 0)$. Note that in this specialisation we have

$$F_s(e(-\alpha)) = q^{\langle d,\alpha\rangle} \quad \text{and} \quad F_s(\chi(\lambda)) = \prod_{\beta^\vee\in\Delta_+^\vee} \frac{\langle \lambda + \rho, \beta^\vee\rangle}{\langle \rho, \beta^\vee\rangle}.$$

We obtain:

**Theorem 15.2.8 (First Dedekin $\eta$-function identity)**

$$\eta(q)^{\dim \mathfrak{g}} = \sum_{\alpha\in M} \left[\left(\prod_{\beta^\vee\in\Delta_+^\vee} \frac{\langle h^\vee\alpha + \rho, \beta^\vee\rangle}{\langle \rho, \beta^\vee\rangle}\right) q^{\frac{|\rho + h^\vee\alpha|^2}{2h^\vee}}\right].$$

Taking $\widehat{\mathfrak{g}}(\sigma)$ of type $A_1^1$ and setting

$$\varphi(q) = \prod_{k=1}^{\infty}(1 - q^k)$$

gives the following formula:

**Theorem 15.2.9 (Jacobi)**

$$\varphi(q)^3 = \sum_{k\in\mathbb{Z}} (4k+1) q^{2k^2+k} \quad \textit{(Jacobi)}.$$

Taking different specialisation for $\widehat{\mathfrak{g}}(\sigma)$ of type $A_1^1$ gives the following

**Theorem 15.2.10**

$$\frac{\varphi(q)^2}{\varphi(q^2)} = \sum_{k \in \mathbb{Z}} (-1)^k q^{k^2} \quad \textit{(Gauss)}.$$

$$\varphi(q) = \sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{3k^3+k}{2}} \quad \textit{(Euler)}.$$

$$\frac{\varphi(q^2)^2}{\varphi(q)} = \sum_{k \in \mathbb{Z}} q^{2k^2+k} \quad \textit{(Gauss)}.$$

# Part III

# Kac-Moody groups

# Chapter 16

# Introduction

After having defined the Kac-Moody Lie algebras as a generalisation of semi-simple Lie algebras, we want to define the Kac-Moody groups whose Lie algebras will be Kac-Moody Lie algebras. In other words, in this part, we want to generalise as much as possible of the theory of reductive algebraic groups to a larger class of groups: Kac-Moody groups.

In particular, as for Lie algebras, these groups will be infinite dimensional which will cause a little more trouble than in the case of Kac-Moody Lie algebras. Indeed, we will need to deal with infinite dimensional groups and varieties. Before entering into these details and explaining what will be the solutions for handling infinite dimensional groups and infinite dimensional varieties, let us recall what are the main aspects of the finite dimensional theory we would like to generalise and discus what will be the differences.

I will start with a review of the construction of Kac-Moody groups and try to emphasize the difference with the finite dimensional theory.

The introduction of the second part is intended as a guide and a motivation for this part. In particular, one should keep in mind the general overview when entering into the details of the proofs on Tits systems, pro-groups, ind-varieties and the technicalities we will develop.

## 16.1 Relation between a group and its Lie algebra

### 16.1.1 Finite dimensional case

Recall the definition of an algebraic group and of its Lie algebra.

**Definition 16.1.1** (ı) An algebraic group $G$ is a variety and a group such that the multiplication map $\mu : G \times G \to G$ and the inverse map $G \to G$ sending an element to its inverse are morphisms.

(ıı) We define the Lie algebra of $G$ to be $\mathfrak{g} = T_eG$ where $e$ is the unit element in $G$. It has a natural Lie algebra structure.

For a semi-simple Lie algebra, there are few algebraic groups having the same Lie algebra. Recall the following:

**Theorem 16.1.2** *Let $G$ be a semisimple group with Lie algebra $\mathfrak{g}$, then*

*(ı) there is a unique simple group $G^{\rm ad}$ with Lie algebra $\mathfrak{g}$. This group is the adjoint group* $\mathrm{Ad}\,(G)$.

*(ıı) There is a unique simply connected semisimple Lie group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$.*

*(ııı) The group $Z(\widetilde{G}) = \widetilde{G}/\mathrm{Ad}\,(G)$ is finite and is the center of $\widetilde{G}$.*

*(ııı) The group $G$ is a quotient of $\widetilde{G}$ with kernel a subgroup of $Z(\widetilde{G})$.*

There is a very strong link between the group $G$ and its Lie algebra $\mathfrak{g}$. This is particularly clear in characteristic zero when the exponential map is defined.

More precisely Chevalley proved the following result: let $\Delta$ be a root data (this is just a Cartan matrix if the ground field is algebraically closed but it is more complicated in general: you need to specify for example if the torus is defined over the base field), then there exists a functor $S \mapsto G(S)$ a functor in groups represented by a scheme: a group scheme.

### 16.1.2 Kac-Moody setting

#### Group functors

The problem for Kac-Moody Lie algebras and groups is very nicely explained in J. Tits exposé at Bourbaki's seminar [Ti89]. In particular, in comparison with the finite dimensional case, we would like to construct a group scheme say $\mathcal{G}$ associated with the combinatorial datum of a generalised Cartan matrix. This group scheme should represent a certain functor in groups and should behave well with base change. However, this is not possible.

One big difference with this classical theory for Kac-Moody groups is the fact that there are several groups associated with the same Kac-Moody Lie algebra or at least with a Lie algebra very close to a fixed Kac-Moody Lie algebra. In fact it is not so easy, as in the finite dimensional case, to define a group associated to the Lie algebra.

On the one hand, one may construct a "minimal" functor in groups $\mathcal{G}_{min}$ on the category of rings but this functor is not a group scheme. For example, for a Cartan matrix (i.e. in the finite dimensional case) we only have a morphism of functors: $\mathcal{G}_{min} \to \mathcal{G}$ which is an isomorphisms only over fields (more precisely over Euclidian rings) but we have $\mathrm{Lie}(\mathcal{G}_{min}) = \mathfrak{g}$.

On the other hand, one may construct a "completed" solution of the problem: it will be an ind-scheme $\widehat{\mathcal{G}}$ but its Lie algebra is a completion of the Kac-Moody Lie algebra we started with. In the finite dimensional case, this construction has the advantage to coincide with the group scheme $\mathcal{G}$ but here we have $\mathrm{Lie}(\widehat{\mathcal{G}}) = \widehat{\mathfrak{g}} \neq \mathfrak{g}$.

This generality (of groups scheme and behaviour under base change) is useful for arithmetic applications and when one wants to consider the groups in families. We will restrict ourselves to the task of constructing such groups and associated homogeneous varieties over the complex numbers. Already in that situation, there will be some work to be done... For this we will follow the approach of S. Kumar [Ku02].

#### Patching subgroups

Both constructions use the same general idea: to (re)construct the group from some of its subgroups as one does for varieties constructed as a patchwork of its local charts. This will be the role of amalgamated products. This can be done in (at least) two different way leading to the minimal $\mathcal{G}_{min}$ and completed $\widehat{\mathcal{G}}$ solutions.

The first technique is to take a torus $T$ associated to the Cartan subalgebra $\mathfrak{h}$ of $g$ and for each simple root $\alpha$ to integrate the $\mathfrak{sl}_2$ subalgebra associated to $\alpha$ in a group $SL_2(\alpha)$. We the patch the groups $T$ and $SL_2(\alpha)$ for all simple root $\alpha$ along their intersection. We end up with the group $\mathcal{G}_{min}$.

The second one starts with the same torus $T$ associated to the Cartan subalgebra $\mathfrak{h}$ of $g$. However, for each simple root $\alpha$ we associate a parabolic group $P_\alpha$. This parabolic group has to contain the previous $SL_2(\alpha)$ but also an unipotent part $\hat{U}$. Here to construct this unipotent par, we need to complete the Lie algebra (this is for the exponential to converge). We then patch the groups $T$ and $P_\alpha$ for all simple root $\alpha$ along their intersection. We end up with the group $\widehat{\mathcal{G}}$.

To conclude on this first difference, let us give an example. For the affine Lie algebra $\mathfrak{g}$ of type $A_n^1$, the groups $GL_{n+1}(\mathbb{C}[t,t^{-1}])$ and $GL_{n+1}(\mathbb{C}((t)))$ are both Kac-Moody groups, the first one corresponding to the functor $\mathcal{G}_{min}$, the second one to the ind-group scheme $\widehat{\mathcal{G}}$. Their Lie algebras are respectively $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ a completion of $\mathfrak{g}$.

We shall also mention here, that there is, at least, a third version to construct Kac-Moody groups or at least groups associated to Kac-Moody Lie algebras. This construction comes from a more differential geometric point of view. In the case of type $A_n^1$ for example, denote by $K$ a maximal compact sugbroup of $G$ (here $K = SU_{n+1}$) and consider the group of continuous maps from $S^1$ to a $K$. This is the loop group, see Presley and Segal [PS86], for example, the group $GL_{n+1}(\mathbb{C}[t,t^{-1}])$ is the group of algebraic morphisms $\mathbb{C}^\times \to GL_n$ and the restriction of such a mao to $S^1 \subset \mathbb{C}^\times$ gives an invection into the loop group. Let me also mention the nice paper on twisted loop groups with an algebraic geometric view point by G. Pappas and M. Rapoport [PR06]).

All these groups to however keep many common characteristics. In particular for geometric applications. Indeed, as in the finite dimensional case, the homogeneous varieties i.e. the quotients $G/P$ where $P$ is a parabolic subgroup of $G$ are the same for both groups. So that there are natural geometric objects associated with the Lie algebra.

## 16.2 Subgroups of $G$, Tits systems

### 16.2.1 Subgroups of $G$

In the classical theory, we start from the group $G$ or its Lie algebra $\mathfrak{g}$ and produce several subgroups and combinatorial data in order to classify all semi-simple groups in terms of root systems. To construct Kac-Moody groups, we will proceed in the reverse order, constructing the group from some of its subgroups modeled by the combinatorial data. Let us recall some features of the finite dimensional situation:

**Definition 16.2.1** (ı) An algebraic group $T$ is a Torus if it is isomorphic to the group of diagonal matrices in $GL_n$ for some $n$.

(ıı) A character of an algebraic group $G$ is a group morphism $G \to k^\times$. The set of all characters is a group denoted $X(G)$.

(ııı) Let $G$ be an algebraic group and $T$ a torus in $G$. Then $T$ acts by the adjoint representation on $\mathfrak{g}$ and this action induces a decomposition

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in X(T)} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \ / \ Ad(t)(x) = \alpha(t)x \text{ for all } t \in T\}$. The set $\Delta(G,T)$ of characters such that $\mathfrak{g}_\alpha$ is non zero is called the set of roots of $G$ with respect to $T$.

**Theorem 16.2.2** *(ı) Let $G$ be a semisimple algebraic group, then all maximal torus are conjugated. The dimension of such maximal torus is called the rank of $G$.*

*(ıı) Let $T$ be a maximal torus, then the group $N_G(T)/T$ is independent of $T$ and is finite. It is called the Weyl group of $G$.*

*(ııı) The set $\Delta(G,T)$ does not depend on $T$ and is a root system.*

**Definition 16.2.3** (ı) A Borel subgroup of an algebraic group $G$ is a maximal closed connected solvable subgroup of $G$.

(ıı) A parabolic subgroup of an algebraic group $G$ is a closed subgroup $P$ such that the quotient $G/P$ is a projective variety.

**Theorem 16.2.4** *(ı) A Borel subgroup $B$ is a parabolic subgroup.*

*(ıı) Any parabolic subgroup contains a Borel subgroup.*

*(ııı) Any maximal torus $T$ is contained in a Borel subgroup.*

*(ıv) All Borel subgroups are conjugated and even all pairs $T \subset B$ of a maximal torus contained in a Borel subgroup are conjugated.*

*(ııı) Let $T$ be a maximal torus contained in a Borel $B$, then $T$ acts on $\mathfrak{b}$ the Lie algebra of $B$ by the adjoint representation and we have the decomposition*

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(B)} \mathfrak{g}_\alpha$$

*where $\Delta(B)$ is a set of positive roots in $\Delta$ the root system of $G$. In particular $B$ defines a basis $\Pi(B)$ of $\Delta$ i.e. a set of simple roots.*

*(ıv) Conversely, any set of positive roots is obtained from a Borel subgroup containing $T$.*

*(v) There is a one to one correspondence between parabolic subgroups $P_X$ containing a fixed Borel subgroup $B$ and the subsets $X$ of the set of simple roots $\Pi(B)$: let $\Delta_X$ the root subsystem generated by $X$, the Lie algebra $\mathfrak{p}_X$ of $P_X$ is*

$$\mathfrak{p}_X = \mathfrak{b} \oplus_{\alpha \in \Delta_{X,+}} \mathfrak{g}_{-\alpha}.$$

Finally a very important result on the structure of reductive groups is the Bruhat decomposition. It makes a link between the group and the combinatorial data of the root systems and Weyl groups:

**Theorem 16.2.5 (Bruhat decomposition)** *There is a decomposition $G = \coprod_{w \in W} BwB$.*

This decomposition will lead to the description of Schubert varieties and of many geometric properties. One may first think of it as a way to replace an element in $G$ by elements in the Weyl group and in a Borel subgroup $B$.

### 16.2.2 Kac-Moody setting

One more time, all these result will not be true in full generality but the major part will. As an example, let us say that all maximal torus of a Kac-Moody group as well as all Cartan subalgebras of a Kac-Moody Lie algebra are conjugated (we need the theory of Kac-Moody groups to prove this result on Kac-Moody Lie algebras and in particular the fact that the Kac-Moody Lie algebra does only depend on the Cartan matrix comes from the construction of Kac-Moody groups). However, the pairs $(T, B)$ of a maximal torus and a Borel subgroup are not conjugated in general. In particular the completion $\widehat{\mathfrak{g}}$ does depend on the choice of a Borel subgroup $B$.

How do we generalise all this ? In particular we want to end up with a group $G$ satisfying the Bruhat decomposition. The Bruhat-Tits theory and the theory of Tits system is an axiomatisation of the properties of a group $G$ together with two subgroups $B$ and $N$ and a set $S$ in $N/(B \cap N)$ in order to be able to define a Weyl group a Bruhat decomposition and to extract the combinatorial data from the group. This will be our starting point. In these lines the preceding discussion leads to the following:

**Definition 16.2.6** A Tits system (also called BN-pair) is a quadruple $(G, B, N, S)$ with $G$ a group, with $B$ and $N$ subgroups of $G$ and with $S$ a finite subset of the quotient $N/(B \cap N)$ satisfying

$(T_1)$ The set $B \cup N$ generates $G$ and $B \cap N$ is normal in $N$.

$(T_2)$ The set $S$ generates the group $N/(B \cap N)$.

$(T_3)$ We have the inclusion $sBw \subset BwB \cup BswB$ for $s \in S$ and $w \in N/(B \cap N)$.

$(T_4)$ For all $s \in S$, we have $sBs^{-1} \not\subset B$.

We will denote by $T$ the group $B \cap N$ and by $W$ the quotient $N/T$. This group is called the Weyl group of the Tits system.

The existence of this definition take all its value with the

**Theorem 16.2.7 (Bruhat decomposition)** *If $(G, B, N, S)$ is a Tits system, then there is a decomposition $G = \coprod_{w \in W} BwB$ with $W$ the group $N/(N \cap B)$.*

This is taylor made for the finite dimensional situation:

**Theorem 16.2.8** *($\imath$) For $G$ a semi-simple algebraic group, $B$ a Borel subgroup, $N$ the normaliser of a maximal torus contained in $B$ and $S$ the set of simple reflections in $N/T = N/(B \cap T)$ defined by $B$, the quadruple $(G, B, N, S)$ is a Tits system.*

*($\imath\imath$) The group $G$ can be recovered, as an amalgamated product, from its parabolics subgroups.*

The theory of Tits systems will be the core of our first chapter. The second part of the theorem gives us a way to generalise and define the Kac-Moody groups. We shall therefore also discuss some notions on amalgamated products. A classical reference for Tits systems is [Bo54] for more involved real-estate constructions on buildings, see for example [Br89] or [Ro89]. For amalgamated products, see [Se80]

The next step in the construction of the Kac-Moody group $G$ will be to produce a system of parabolic groups from which we will define $G$ as an amalgamated product. For this we need to give an algebraic structure to our infinite dimensional group: a pro-group structure and we shall use the exponential map on these groups. This will be the theme of our second chapter.

## 16.3 Pro-groups and Exponential map

### 16.3.1 The exponential map

Recall that for a Lie group $G$, there is a unique map $\exp : \mathfrak{g} \to G$ taking 0 to $e$ and whose differential $\mathfrak{g} = T_0\mathfrak{g} \to T_eG = \mathfrak{g}$ is the identity. This map is called the exponential map.

To generalise this we first need to have some algebraic structure on the infinite dimensional groups we want to work with. This shall be the notion of a pro-group (and associated pro-Lie-algebras). Roughly speaking a pro-group is the group which is the inverse limit of algebraic groups.

We shall use the exponential only for unipotents pro-groups. In particular, in the finite dimensional case, in a Borel subgroup, there is a distinguished unipotent subgroup $U$ and the exponential map is a bijection between the Lie algebra $\mathfrak{u}$ of $U$ and $U$:

**Theorem 16.3.1** *Assume the characteristic of the ground field is zero. Let $\mathfrak{u}$ be a nilpotent Lie algebra, then there exists a natural unipotent group structure on $\mathfrak{u}$ such that the Lie algebra for this group structure is $\mathfrak{u}$. This group structure is given by*

$$X \cdot Y = \log(\exp(X)\exp(Y)).$$

*The Lie algebra of $\mathfrak{u}$ for this group structure is $\mathfrak{u}$ for its natural Lie algebra structure and the exponential map is the identity.*

There is an explicit formula for $\log(\exp(X)\exp(Y))$ showing that this is well defined for nilpotent Lie algebras: the Campbell-Hausdorff formula:

$$\log(\exp X \exp Y) = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i+s_i>0 \\ 1\leq i\leq n}} \frac{(\sum_{i=1}^{n}(r_i+s_i))^{-1}}{r_1!s_1!\cdots r_n!s_n!}[X^{r_1}Y^{s_1}X^{r_2}Y^{s_2}\dots X^{r_n}Y^{s_n}],$$

which uses the notation

$$[X^{r_1}Y^{s_1}\dots X^{r_n}Y^{s_n}] = [\underbrace{X,[X,\dots[X,}_{r_1}\underbrace{[Y,[Y,\dots[Y,}_{s_1}\dots\underbrace{[X,[X,\dots[X,}_{r_n}\underbrace{[Y,[Y,\dots Y}_{s_n}]]\dots]].$$

### 16.3.2 The Kac-Moody setting

To generalise this in the infinite dimensional setting, we start from Lie algebras. We need to take a completion of $\mathfrak{g}$ in the positive direction i.e. to replace the unipotent algebra $\mathfrak{u} = \oplus_{\alpha>0}\mathfrak{g}_\alpha$ by $\hat{\mathfrak{u}} = \prod_{\alpha>0}\mathfrak{g}_\alpha$. We then have a generalisation of the previous result:

**Theorem 16.3.2** *There is an equivalence between the category of pro-unipotent groups and pro-nilpotent-Lie algebras given by* $U \mapsto \mathfrak{u} = Lie(U)$ *and such that* $\exp : \mathfrak{u} \to U$ *is a bijection.*

With this we construct an pro-unipotent group $\hat{U}$ associated with the pro-Lie-algebra $\mathfrak{u} = \prod_{\alpha>0}\mathfrak{g}_\alpha$. With this tool we may start the construction of the Kac-Moody group $G$: construct parabolic subgroups a make the amalgamated product of them.

## 16.4 The Kac-Moody group

With the pro-unipotent-group $\hat{U}$, we start constructing a Tits system $(G,B,N,S)$. We may start from the combinatorial level: we already have the Weyl group $W$ together with its generating set $S$ so that $S$ is already defined. The Cartan Lie algebra $\mathfrak{h}$ is finite dimensional and it is easy to construct a torus $T$ whose Lie algebra is $\mathfrak{h}$. We define $N$ by generators and relations, $N$ being generated by $T$ and some lifts of $W$ (we already met these lifts in Kac-Moody Lie algebras, see Proposition 5.2.6). We may define by $B = T\hat{U}$.

In fact choosing finite type root subsystems $\Delta_X$ of our Kac-Moody root system, we may define in the same way (with a little more work on pro-groups) parabolic groups $P_X$. This goes as follows, the root system $\Delta_X$ define the following parabolic Lie subalgebra $\mathfrak{p} = \mathfrak{b}\oplus_{\alpha\in\Delta_{X,+}}\mathfrak{g}_{-\alpha}$ in $\mathfrak{g}$. There is a decomposition $\mathfrak{p} = \mathfrak{g}_X \oplus \mathfrak{u}_X$ where $\mathfrak{g}_X$ is a semi-simple Lie algebra of finite type and $\mathfrak{u}_X$ is unipotent. We take the completion $\hat{\mathfrak{u}}_X$ of $\mathfrak{u}_X$ and the associated unipotent group $\hat{U}_X$. We define $P_X = G_X\hat{U}_X$ where $G_X$ is the semi-simple algebraic group associated to $\mathfrak{g}_X$ (all these are finite dimensional).

**Definition 16.4.1** The Kac-Moody group $G$ is an amalgamated product of the subgroups $P_X$.

**Theorem 16.4.2** *The quadruple* $(G,B,N,S)$ *is a Tits system.*

The group $G$ defined in this way is the completed Kac-Moody group $\widehat{\mathcal{G}}$ we discussed in the beginning of the introduction. Using the Kac-Moody $G$, we may construct the minimal associated group $\mathcal{G}_{min}$ as a subgroup of $G$. This group can however be defined directly, see [KP85].

With the construction of Kac-Moody groups, we are is position to produce varieties with geometry similar to the geometry of homogeneous varieties in the finite dimensional case and to study some aspects of this geometry. This is one of the very motivations for constructing Kac-Moody group. This will be the second part of the lectures where we will deal with more geometric properties.

## 16.5 Homogeneous varieties

The first result you want to have with Kac-Moody groups is to give an algebraic structure to the quotient $G/P$ of the Kac-Moody group $G$ by a parabolic subgroup $P$. This will be done thanks to the notion of ind-schemes (i.e. inductive limits of schemes).

To realise the quotient $G/P$ as an ind-scheme, we will proceed as in the finite dimensional case: we want to realise the homogeneous varieties as embedded in a representation: the orbit of a highest weightvector. For this we will consider the action of $G$ on several representations. We will thus first give a brief account of representations of $G$. If $\lambda$ is a dominant weight for $\mathfrak{g}$ vanishing on all simple coroots defining $P$, the module $V(\lambda)$ can be endowed with an action of $G$ and there is an injective map $G/P \to \mathbb{P}(V(\lambda))$.

The structure on $\mathbb{P}(V(\lambda))$ is the structure of an ind-scheme (recall that $V(\lambda)$ is infinite dimensional. The difficulty is here to prove the following;

**Theorem 16.5.1** *The image of the injection $G/P \to \mathbb{P}(V(\lambda))$ is closed (in the category of ind-schemes).*

This is done using a generalisation to the infinite dimensional setting of the Bott-Samelson resolution and of Schubert varieties. The Schubert varieties are the $B$-orbits in $G/P$ and because of the fact that $(G, , N, S)$ is a Tits system, we have a Bruhat decomposition of $G/P$ in terms of the Schubert varieties. The next step is to compute the cohomology of line bundles on $G/P$.

## 16.6 Line bundles and Schubert varieties

Let us review the finite dimensional theory. Assume $P = B$ is a Borel subgroup for simplicity and $G$ is a semi-simple algebraic group.

Let $\chi \in X(T)$ be a character of a maximal torus contained in $B$ a Borel subgroup. Then $T$ acts on $\mathbb{C}$ via the map $T \to \mathbb{C}^*$ and this induces an action of $B$ on $\mathbb{C}$.

**Definition 16.6.1** We may define the line bundle $\mathcal{L}_\chi$ associated to $\chi$ as the quotient of the product $G \times \mathbb{C}$ under the action of $B$ on the right on $G$ and on the left on $\mathbb{C}$. This variety together with the first projection map to $G/B$ has a structure of line bundle on $G/B$.

**Theorem 16.6.2** *The map $X(T) \to \mathrm{Pic}(G/B)$ defined by $\chi \mapsto \mathcal{L}_\chi$ is an isomorphism of abelian groups.*

Define the dominant chamber $C = \{x \in \mathfrak{h}^* \ / \ (x, \alpha) \geq 0 \text{ for all } \alpha \in \Delta(B)\}$.

**Theorem 16.6.3** *In characteristic zero, let us consider the following action $w * \chi = w(\chi + \rho) - \rho$ where $\rho$ is half the sum of all positive roots. Then for $\chi$,*

*(ı) either the orbit $W * \chi$ does not meet $C$ and in this case all the cohomology groups $H^i(G/B, \mathcal{L}_\chi)$ vanish,*

*(ıı) or there exists a unique $w \in W$ such that $w * \chi \in C$. In that case $H^i(G/B, \mathcal{L}_\chi) = 0$ for $i \neq \ell(w)$ the length of $w$ and $H^i(G/B, \mathcal{L}_\chi)$ is the representation with highest weight $w * \chi$.*

We will generalise this to the variety $G/P$ for $G$ a Kac-Moody group. Furthermore, the group $G$ being part of a Tits system, we derive a Bruhat decomposition

$$G = \coprod_{w \in W} BwB.$$

Considering the left action of $B$ on $G/B$ we get a decomposition into $B$-orbits

$$G/B = \coprod_{w \in W} BwB/B.$$

These orbits are the Schubert cells, the Schubert varieties $X(w)$ are their closure in $G/B$. These definition can be extended to the variety $G/P$. As we shall see, the Kac-Moody setting is the natural one to deal with the singularities of Schubert varieties and these results on cohomology of line bundles will induce the following

**Theorem 16.6.4** *The Schubert varieties in $G/P$ are normal.*

The Schubert varieties are a very important tool in the study of the geometry of homogeneous varieties (in the finite dimensional or infinite dimensional cases). In particular their classes define basis of the homology and the cohomology of the homogeneous variety $G/P$.

If time permits, we should discuss the equivariant cohomology and homology of $G/P$ for $G$ a Kac-Moody group and describe explicitly the cohomology ring as the nil-Hecke ring of Kostant and Kumar [KK86]. This has also connections with the quantum cohomology of finite dimensional homogeneous spaces.

## 16.7 Motivations

As a final comment, I would like to give some motivation for studying Kac-Moody groups appart from the fact that the ubiquity of semi-simply algebraic groups should justify the definition of infinite dimensional groups with very close properties. The so called loop groups that are associated with affine Kac-Moody Lie algebras have the more striking applications.

### 16.7.1 Application of Kac-Moody Lie algebras

As we have already mentioned (for example for the proof that two Cartan subalgebras are conjugated or for the character formula for non symmetrisable Kac-Moody Lie algebras), the theory of Kac-Moody groups is already useful in the theory of Kac-Moody Lie algebras. And representations of Lie algebras have show usefulness in several directions in particular in solving Hamiltonian systems like KdV equations and other problems coming from theoretical physics.

### 16.7.2 Historical point of view, arithmetic

In an historical point of view, the introduction of these groups came from the study of classical groups over local fields like $GL_n(\mathbb{Q}_p)$ and $GL_n(\mathcal{F}_q((t)))$ (and even more recently of $GL_n(\mathbb{C}((t))))$. If you think of $\mathbb{Q}_p$ as the completion of $\mathbb{Q}$ at the place $p$ (a local field) this is very close to the field $\mathbb{C}((t))$ which is the completion of $\mathbb{C}(t)$ (a field of functions). This is the starting point of the Iwahori theory and later of the theory of Buildings by Tits and Bruhat. For example, the theory of Building developed by Tits is used (in its very preliminary version) by Serre in [Se80] to prove the following result due to Ihara on the structure of the group $SL_2(\mathbb{Q}_p)$.

Let $\mathbb{Z}_p$ be the ring of $p$-adic integers and let $B$ be the subgroup of $SL_2(\mathbb{Z}_p)$ defined by:

$$B = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) c \equiv 0 \bmod p \right\}.$$

**Theorem 16.7.1** *The group $SL_2(\mathbb{Q}_p)$ is the amalgamated product $SL_2(\mathbb{Z}_p) *_B SL_2(\mathbb{Z}_p)$.*

Let us give more examples of this theory.

**The tree of $SL_2$ over a valuation field**

Take $K$ a field with a discrete valuation $v$ and denote by $\mathcal{O}$ its ring of integers (for example take $K = \mathbb{Q}_p$, $\mathcal{O} = \mathbb{Z}_p$ or $K = \mathcal{F}_q((t))$ and $\mathcal{O} = \mathcal{F}_q[[t]]$).

Let us consider lattices in $K^2$ i.e. finitely generated $\mathcal{O}$-submodules of $K^2$ generating $K^2$ (the standard lattice $\mathcal{O}^2$ has $PGL_2(\mathcal{O})$ as stabiliser).

If $L$ is a lattice, then $xL$ is again a lattice for $x \in K^\times$. We consider the set

$$X = \{L \text{ lattice of } K^2\}/\sim$$

of lattices modulo the equivalence relation defined by $L \sim L'$ if there exists $x \in K^\times$ with $L' = xL$. We then have $\text{Stab}([\mathcal{O}^2]) = GL_2(\mathcal{O})$.

If we take two lattices $L$ and $L'$, then there exists a basis $(e_1, e_2)$ of $L$ such that $(\pi^a e_1, \pi^b e_2)$ is a basis of $L'$ and the couple $(a, b)$ does not depend on the choice of such a basis. If we multiply $L$ by $x$ and $L'$ by $y$ we change $(a, b)$ into $(a + c, b + c)$ where $c = v(x/y)$. In particular $|a - b|$ does only depend on the classes $[L]$ and $[L']$. It is called the **distance** of $L$ and $L'$ and denote $d(L, L')$.

**Proposition 16.7.2** *Take $L$ alattice and $\Lambda \in X$, then there exists a unique maximal lattice $[L']$ such that $[L'] = \Lambda$ and $L' \subset L$.*

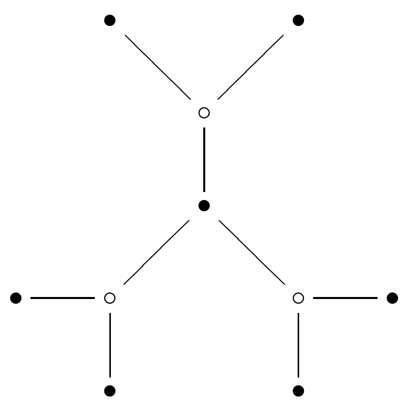
*Furthermore, we have $L/L' = \mathcal{O}?\pi^n\mathcal{O}$ where $n = d(L, L')$.*

**Corollary 16.7.3** *(ı) $d(L, L') = 0 \Leftrightarrow [L] = [L']$.*

*(ıı) $d(L, L') = 1 \Leftrightarrow$ there exists representatives $L' \subset L$ such that $L/L' = k$.*

**Definition 16.7.4** Two elements in $X$ are adjacent if there respective distance is one.

This defines a combinatorial graph in $X$ which is in fact a tree. For the field $\mathbb{F}_2$ we get the following (infinite!) picture:



The set $X$ together with the simplicial structure defined by the tree is the **Tits Building**. Remark that the points are in one to one correspondence with maximal parabolic subgroups: associate to $[L]$ its stabiliser. The edges are in one to one correspondence with incidence relations between the parabolics i.e. parabolics such that there intersection is a Borel subgroup.

When one consider a Tits system $(G, B, N, S)$, we fix a Chamber of the Tits system: in our picture, we fix a pair of adjacent parabolic subgroups. This also explains the coloration of the edges according to the type of parabolics we are considering.

### 16.7.3 Geometric applications

#### Moduli space of bundles on a curve

In a more geometric point of view, let us consider the moduli (it is a stack) space $M_C$ of vector bundles of rank $n$ and determinant on an algebraic curve $C$. Let us fix a point $p \in C$ and consider the moduli space $M_{C,p}$ of pairs $(E, s)$ where $E \in M_C$ and $s$ is a trivialisation of $E$ is a neighborhood of $p$. Then $M_{C,p}$ can be realised as the homogeneous space under a Kac-Moody group as follows

$$M_{C,p} = GL_n(\mathbb{C}((t)))/GL_n(\mathbb{C}[[t]])$$

Thi quotient is called the affine grassmannian and has many properties in common with classical grassmannians. Quotienting on the left by $GL_n(\mathcal{O}_{p,C})$ gives a description of $M_C$ as

$$M_C = GL_n(\mathcal{O}_{p,C})\backslash GL_n(\mathbb{C}((t)))/GL_n(\mathbb{C}[[t]]).$$

In particular, a natural generator of the Picard group $\mathrm{Pic}(M_C)$ shows up and its pull-back on $M_{C,p}$ is a line bundle whose cohomology is given by our Borel-Weil-Bott theorem. This study leads to the, by now, well known Verlinde formulas.

#### Schubert varieties

Moreover, the Kac-Moody setting is the natural setting to study Schubert varieties (all the major properties of Schubert varieties extend to this setting). For example, if you consider all ruled rational smooth surfaces, also called Hirzebruch surfaces and denoted $\Sigma_n$, then these surfaces are exactly all two dimensional Schubert surfaces. In the infinite dimensional setting you have only the $\Sigma_n$ for $n \leq 3$ and for $\mathfrak{n} \leq 4$ in the affine setting.

#### Quantum cohomology

Finally, and this was my personal motivation in studying Kac-Moody groups, there is a closed relationship between the homology of the affine Grassmannian (the quotient $GL_n(\mathbb{C}((t)))/GL_n(\mathbb{C}[[t]])$ in type $A$) and the quantum cohomology of homogeneous spaces (flag varieties under $GL_n$). This is something not really well understood for the moment and there should in particular be more deep links between rational curves on homogeneous spaces and points in the affine Grassmannian.

# Chapter 17

# Tits systems

## 17.1 Definition and first properties

The basic reference for Tits system is [Bo54]. We give is a different definition: following [Ku02], we don't assume the element $s \in S$ to be of order two. We prove in Proposition 17.1.4 that this agrees with the definition in [Bo54]

**Definition 17.1.1** A Tits system (also called BN-pair) is a quadruple $(G, B, N, S)$ with $G$ a group, with $B$ and $N$ subgroups of $G$ and with $S$ a finite subset of the quotient $N/(B \cap N)$ satisfying

$(T_1)$ The set $B \cup N$ generates $G$ and $B \cap N$ is normal in $N$.

$(T_2)$ The set $S$ generates the group $N/(B \cap N)$.

$(T_3)$ We have the inclusion $sBw \subset BwB \cup BswB$ for $s \in S$ and $w \in N/(B \cap N)$.

$(T_4)$ For all $s \in S$, we have $sBs^{-1} \not\subset B$.

We will denote by $T$ the group $B \cap N$ and by $W$ the quotient $N/T$. This group is called the Weyl group of the Tits system.

**Example 17.1.2** This definition is of course perfectly suited for reductive algebraic groups. Indeed, let $G$ be a reductive algebraic group, let $B$ be a Borel subgroup of $G$, let $T$ be a maximal torus in $G$ contained in $B$, let $N$ be the normaliser of $T$ in $G$ and let $S$ be the set of simple reflections in $W = N/T$ defined by $B$. Then $(G, B, N, S)$ is a Tits system.

We will consider the double classes $(BgB)_{g \in G}$ under $B$ in $G$. Recall that these double classes form a partition of $G$, we denote by $B \backslash G/B$ the corresponding quotient. Note that the product of any two double classes is an union of double classes: $(BgB) \cdot (Bg'B) = \cup_{x \in gBg'} BxB$.

For $w \in W$, we may define a double class $C(w)$ as follows: take any element $n \in N$ such that its class is $w$ in $W$ and set $C(w) = BnB$. This definition does not depend on the choice of $n$, we also call the double classes $C(w)$ Schubert cells or cells. We first prove some simple results on the cells $C(w)$.

**Lemma 17.1.3** *For $s \in S$ and $w, w' \in W$, we have the following relations:*

- $C(1) = B$,
- $C(ww') \subset C(w) \cdot C(w')$,
- $C(w^{-1}) = C(w)^{-1}$,

- $C(s)\cdot C(w) = C(sw)$ *if* $C(w) \not\subset C(s)\cdot C(w)$ *and* $C(s)\cdot C(w) = C(w)\cup C(sw)$ *if* $C(w) \subset C(s)\cdot C(w)$.

**Proof :** The first three relations are direct consequences of the definition of the cells. For the last one, remark that $(T_3)$ gives $C(s)\cdot C(w) \subset C(w) \cup C(sw)$. But we have $C(sw) \subset C(s)\cdot C(w)$ and the result follows from the fact that a product of double classes is a union of double classes. □

We now prove that any element $s \in S$ has order two in $W$. We describe some subgroups of $G$ containing $B$. For any subset $X$ of $S$, we denote by $W_X$ the subgroup of $W$ generated by $X$. We define the subset $P_X$ of $G$ by $P_X = BW_XB$.

**Proposition 17.1.4** *(ı) Any element $s \in S$ is of order two in $W$.*
*(ıı) For any subset $X$ of $S$, the subset $P_X$ of $G$ is a subgroup.*

**Proof :** (ı) By $(T_4)$, we have $C(s)\cdot C(s^{-1}) \not\subset B$. Thus by the previous lemma, we have the equality $C(s)\cdot C(s^{-1}) = C(s^{-1}) \cup B$. Taking inverses we get $C(s)\cdot C(s^{-1}) = C(s) \cup B$. This implies the equality $C(s) = C(s^{-1})$.

By $(T_3)$ for $w = s$, we have $C(s)\cdot C(s) \subset C(s)\cup C(s^2)$. But we proved the equalities $C(s) = C(s^{-1})$ and $C(s)\cdot C(s^{-1}) = C(s)\cup B$, thus $C(s)\cup B \subset C(s)\cup C(s^2)$. Because these are cells we get $B = C(s^2)$ and $s^2 = 1$.

(ıı) Consider $P_X = BW_XB$. We have the equality

$$P_X = \bigcup_{w\in W_X} C(w).$$

Now by $(T_3)$ and an easy induction on the numbers of generators in $X$ we need to write $w$ and $w'$ we have the inclusion $C(w)\cdot C(w') \subset P_X$ proving the result. □

**Remark 17.1.5** We shall see in particular that, for $X = S$, we have $P_X = G$ while for $X = \emptyset$, we have $P_X = B$. We will see that the subsets $P_X$ are always subgroups of $G$. They are called the standard parabolic subgroups.

## 17.2 Double classes decomposition

In this section we prove a generalisation of the Bruhat decomposition. For this we need to define the length $\ell(w)$ of an element $w \in W$ as for a Coxeter system i.e.:

$$\ell(w) = \min\{n \in \mathbb{N} \;/\; \exists (s_1, \cdots, s_n) \in S^n \; w = s_1 \cdots s_n\}.$$

**Theorem 17.2.1** *Let $(G, B, N, S)$ be a Tits system, then we have the decomposition*

$$G = \coprod_{w\in W} C(w).$$

*In particular $G = P_S$. More generally for any subset $X$ and $Y$ of $S$, we have the decomposition*

$$G = \coprod_{w\in W_X\backslash W/W_Y} P_X w P_Y.$$

**Proof :** We know that $P_S$ is a subgroup of $G$. More over $B$ and $W$ are contained in $P_S$ thus $B$ and $N$ are contained in $P_S$ and by $(T_1)$ we gave $G = P_S$. To prove the first decomposition, we only need to prove that the union is disjoint. i.e. that $C(v) = C(w)$ implies that $v = w$.

We proceed by induction on $\min\{\ell(v), \ell(w)\}$. We assume for example that $\ell(v) \leq \ell(w)$. If $\ell(v) = 0$ then $v = 1$. We have $C(w) = C(v) = B$ thus $w = 1$. Otherwise, assume $v \neq w$ and take $s \in S$ such that $\ell(sv) < \ell(v)$. We have $\ell(w) \geq \ell(v) > \ell(sv)$ thus $w \neq sv$ and $sw \neq sv$ (because $v \neq w$). We also have $\ell(sw) \geq \ell(w) - 1 \geq \ell(v) - 1 \geq \ell(sv)$. By induction, we have $C(w) \neq C(sv)$ and $C(sw) \neq C(sv)$. In particular $C(sv)$ does not meet the union $C(sw) \cup C(w)$ and thus the intersection $C(sv) \cap (C(s) \cdot C(w))$ is empty. On the other hand we have the inclusion $C(sv) \subset C(s) \cdot C(v)$ thus we have $C(v) \neq C(w)$.

For the second decomposition, first consider the union on the right hand side. This union is stable under left and right multiplication. Moreover, it contains the union over $w \in W_X \backslash W / W_Y$ of the sets $W_X w W_Y$. This union is $W$. Thus by the previous argument (using $(T_1)$), the right hand side is $G$. One more time we need to prove that the union is disjoint. But we have

$$P_X w P_Y = \bigcup_{u \in W_X,\ v \in W_Y} C(u)C(w)C(v).$$

For $u \in W_X$ and $v \in W_Y$, write $u = s_1 \cdots s_n$ and $v = s'_1 \cdots s'_r$ with $s_i \in X$ and $s'_j \in Y$. We have the inclusions $C(u)C(w)C(v) \subset C(s_1) \cdots C(s_n)C(w)C(s'_1) \cdots C(s'_r)$. By use of $(T_3)$, we get

$$C(s_1) \cdots C(s_n)C(w)C(s'_1) \cdots C(s'_r) \subset \bigcup_{u' \in W_X,\ v' \in W_Y} C(u'wv').$$

In particular the union on the right hand side in the theorem is disjoint. □

Let us prove a length characterisation of the decomposition of the product $C(s) \cdot C(w)$.

**Proposition 17.2.2** *We have the following equality:*

$$C(s) \cdot C(w) = \begin{cases} C(sw) & \text{if } \ell(sw) \geq \ell(w) \\ C(sw) \cup C(w) & \text{if } \ell(sw) \leq \ell(w). \end{cases}$$

**Proof :** We proceed by induction on the length of $w$. The result is true for $w = 1$. If $\ell(w) \neq 0$, let $s' \in S$ such that $\ell(ws') < \ell(w)$.

If $\ell(sw) \geq \ell(w)$, we have $\ell(sws') \geq \ell(sw) - 1 \geq \ell(w) - 1 \geq \ell(ws')$. By induction, we get $C(s) \cdot C(ws') = C(sws')$. Assume that $C(s) \cdot C(w) \neq C(sw)$. This implies $C(s) \cdot C(w) = C(sw) \cup C(w)$. In particular $C(w) \cap (C(s) \cdot C(w)) \neq \emptyset$, thus $sBw \cap C(w) \neq \emptyset$. Multiplying by $s'$ gives $sBws' \cap C(w)s' \neq \emptyset$. But $C(w)s' \subset C(w)C(s') \subset C(w) \cup C(ws')$ (the last inclusion comes form taking the inverse in $(T_3)$). We thus have $sBws' \cap (C(w) \cup C(ws')) \neq \emptyset$ and by $B$-invariance: $C(s)C(ws') \cap (C(w) \cup C(ws')) \neq \emptyset$. The induction gave $C(s)C(ws') = C(sws')$ and by the previous theorem, we have $sws' = w$ or $sws' = ws'$. The last equality is impossible. The first one leads to $sw = ws'$ but $\ell(sw) \geq \ell(w) > \ell(ws')$ a contradiction.

If $\ell(sw) \leq \ell(w)$, replacing $w$ by $sw$ gives $C(s)C(sw) = C(w)$ (use the previous case). Now compute:

$$C(s) \cdot C(w) = C(s) \cdot C(s) \cdot C(w) = (C(s) \cup B) \cdot C(w) = C(sw) \cup C(w).$$

This is want we wanted to prove. □

## 17.3 The pair $(W, S)$ is a Coxeter system

All the combinatoric features of the Weyl group $W$ of the Tits system $(G, B, N, S)$, together with its generating set $S$ remind Coxeter groups (see Chapter 6). We now prove the following:

**Theorem 17.3.1** *The pair $(W, S)$ is a Coxeter system.*

**Proof :** To prove this we use the characterisation given in Theorem 6.4.1 that a group $W$ together with a finite generating set $S$ whose element are of order two and satisfying the exchange condition is a Coxeter group.

Let us recall the exchange condition: *let $s \in S$ and $w \in W$ be such that $\ell(sw) < \ell(w)$ then for any reduced expression $w = s_1 \cdots s_r$ with $s_k \in S$, we have $sw = s_1 \cdots \widehat{s_i} \cdots s_r$ for some $i$*

Let us compute the following products thanks to Proposition 17.2.2:

$$C(s)C(w) = C(w) \cup C(sw) \text{ and } C(w) = C(s_1) \cdots C(s_r).$$

Now we mimic the situation of Coxeter groups: chose the smallest integer $i \in [1, r]$ such that $\ell(ss_1 \cdots s_i) \leq \ell(ss_1 \cdots s_{i-1})$. We get by Proposition 17.2.2 (and taking the inverse) the following equalities:

$$C(s)C(s_1) \cdots C(s_{i-1}) \cdot C(s_i) = C(ss_1 \cdots s_{i-1}) \cdot C(s_i) = (C(ss_1 \cdots s_{i-1}s_i) \cup C(ss_1 \cdots s_{i-1})) \,.$$

Apply this in the following equalities

$$\begin{aligned} C(s)C(w) &= C(s)C(s_1) \cdots C(s_r) \\ &= (C(ss_1 \cdots s_{i-1}s_i) \cup C(ss_1 \cdots s_{i-1})) \cdot C(s_{i+1}) \cdots C(s_r) \\ &\subset \bigcup_{i+1 \leq i_1 < \cdots < i_p \leq r} C(ss_1 \cdots s_{i-1}s_is_{i_1} \cdots s_{i_p}) \cup C(ss_1 \cdots s_{i-1}s_{i_1} \cdots s_{i_p}). \end{aligned}$$

Because $C(s)C(w)$ equals $C(w) \cup C(sw)$, because the cells form a partition and because $w$ is of length $r$, we get that $w = ss_1 \cdots s_{i-1}s_{i+1} \cdots s_r$ or $w = ss_1 \cdots s_is_{i+1} \cdots \widehat{s}_{i+k} \cdots s_r$ and the result follows in both cases. □

## 17.4 Reconstruction of $G$ by amalgamated products

In this section, we shall see that the group $G$ of a Tits system $(G, B, N, S)$ can be recovered by **amalgamated product** from its parabolics subgroups.

**Definition 17.4.1** (ı) Let $I$ be an indexing set and $\{G_i\}_{i \in I}$ a family of groups. Let also for any pair $\{i, j\}$ of elements in $I$ a group $G_{\{i,j\}}$ (note that $G_{\{i,j\}} = G_{\{j,i\}}$) together with group morphisms $\varphi_{\{i,j\}} : G_{i,j} \to G_i$ and $\varphi_{j,i} : G_{\{i,j\}} \to G_i$.

The **amalgamated product** of the maps $\varphi_{i,j}$ is a pair $(G, (\varphi_i)_{i \in I})$ satisfying the following properties:

- $G$ is a group and $\varphi_i : G_i \to G$ is a group morphism such that $\varphi_i \circ \varphi_{i,j} = \varphi_j \circ \varphi_{j,i}$.
- If $G'$ is a group and $\psi_i : G_i \to G'$ are group morphisms such that $\psi_i \circ \varphi_{i,j} = \psi_j \circ \varphi_{j,i}$, then there exists a unique group morphism $\psi : G \to G'$ such that $\psi_i = \psi \circ \varphi_i$.

ıı) When all the groups $G_i$ are subgroups of a same set $\mathcal{G}$ and the intersection $G_i \cap G_j$ is a subgroup of both $G_i$ and $G_j$ such that the two group structure coincide, we say that $(G_i)_{i\in I}$ is a system of groups. Set $G_{\{i,j\}} = G_i \cap G_j$ and $\varphi_{i,j} : G_{\{i,j\}} \to G_i$ to be the inclusion, then the amalgamated product $G$ of the maps $\varphi_{i,j}$ is called the amalgamated product of the system of groups $(G_i)_{i\in I}$.

As a special case of system of groups, when all the groups $G_i$ are subgroups of the same group $\mathcal{G}$, we have that the family $(G_i)_{i\in I}$ is a system of groups.

**Proposition 17.4.2** *The amalgamated product $(G, (\varphi_i)_{i\in I})$ of a family $(\{G_i\}_{i\in I}, \varphi_{i,j})$ exists and is unique up to isomorphism.*

**Proof :** The uniqueness comes from the universal property of from the fact that $G$ represents the functor $H \mapsto \varprojlim(\mathrm{Hom}(G_i, H))$.

For the existence, we may for example define $G$ by generators and relations. Take the disjoint union of generating families of the groups $G_i$ to be a generating family of $G$ and for relations the elements $xyz^{-1}$ for $x$, $y$ and $z$ in a $G_i$ with $xy = z$ and $xy^{-1}$ for $x \in G_i$, $y \in G_j$ such that there exists $z \in G_{\{i,j\}}$ with $x = \varphi_{i,j}(z)$ and $y = \varphi_{j,i}(z)$. □

Recall that for a subset $X$ of $S$ we defined a subgroup $P_X$ associated to $X$. For $X = \{s\}$ with $s \in S$ we denote $P_{\{s\}}$ simply by $P_s$.

**Theorem 17.4.3** *Let $(G, B, N, S)$ be a Tits system, then the group $G$ is the amalgamated product of its subgroups $\{N, P_s,\ s \in S\}$.*

**Proof :** Let us first only assume that $(G, B, N, S)$ satisfy the properties $(T_1)$, $(T_2)$, $(T_4)$ and that $(W, S)$ is a Coxeter sytem.

**Lemma 17.4.4** *The property $(T_3)$ is equivalent to the conjunction of the following two properties:*

$(T_5)$ *For each $s \in S$, we have $C(s) \cdot C(s) = B \cup C(s)$.*

$(T_6)$ *If $s \in S$ and $w \in W$ are such that $\ell(sw) > \ell(w)$, then $C(s) \cdot C(w) = C(sw)$.*

**Proof :** We already proved in Proposition 17.2.2 that property $(T_3)$ implies properties $(T_5)$ and $(T_6)$. Conversely, suppose that $(T_5)$ and $(T_6)$ hold, we want to prove the inclusion

$$C(s) \cdot C(w) \subset C(w) \cup C(sw).$$

In view of $(T_6)$, we only need to prove this for $\ell(sw) < \ell(w)$. Applying $(T_6)$ to $sw$ gives the equality $C(s) \cdot C(sw) = C(w)$. Multiply this by $C(s)$ and use $(T_5)$ first and then $(T_6)$ again to get the equality

$$C(s) \cdot C(w) = C(s) \cdot C(s) \cdot C(sw) = B \cdot C(sw) \cup C(s) \cdot C(sw) = C(sw) \cup C(w)$$

from which the result follows. □

For $s \in S$ and $w \in W$, define the following groups ${}^wB = wBw^{-1}$, $B_w = {}^wB \cap B$ and ${}^sB_w = sB_ws$. Remark that the group ${}^wB$ does not depend on the choice of a representative of $w$ in $N$. The same is true for ${}^sB_w$ (this comes from the fact that, because $B \cap N$ is normal in $N$, for $x \in B \cap N$ we have the equalities $xwBw^{-1}x^{-1} = w(w^{-1}xw)B(w^{-1}x^{-1}w)w^{-1} = wBw^{-1}$).

**Lemma 17.4.5** *Suppose that the property $(T_5)$ holds. Let $s \in S$ and $w \in W$ be such that $\ell(sw) > \ell(w)$. Define the following properties:*

($T_7$) *The equality* $B_s \cdot B_w = B$ *holds.*

($T_8$) *The equality* $B_{sw} = {}^sB_w \cap B$ *holds.*

*Then the property* ($T_6$) *is equivalent to the* ($T_7$) *and implies* ($T_8$).

**Proof :** Let us first prove that ($T_6$) implies ($T_8$). If ($T_6$) holds, we have $C(s) \cdot C(w) \cap C(w) = \emptyset$ and multiplying by $w^{-1}$ on the right, we get $C(s) \cdot {}^wB \cap B \cdot {}^wB = \emptyset$. But $C(s) \subset C(s) \cdot {}^wB$ and ${}^wB \subset B \cdot {}^wB$ thus $C(s) \cap {}^wB = \emptyset$.

By ($T_5$), we have ${}^sB \subset B \cup C(s)$ and we get ${}^sB \cap {}^wB \subset (B \cup C(s)) \cap {}^wB = B \cap {}^wB = B_w$. Conjugating by $s$ gives $B \cap {}^{sw}B \subset {}^sB_w$ that is to say

$$B_{sw} \subset {}^sB_w.$$

This implies the inclusion $B_{sw} \subset {}^sB_w \cap B$. The converse inclusion is easy, we have the inclusions ${}^sB_w \cap B = s\,{}^wBs \cap sBs \cap B = {}^{sw}B \cap sBs \cap B \subset B_{sw}$.

We prove the fact that ($T_6$) implies ($T_7$). The formula $C(s) \cdot C(w) = C(sw)$ gives by multiplication on the left by $s$ and on the right by $w^{-1}$ the equality ${}^sB \cdot B \cdot {}^wB = {}^sB \cdot {}^wB$. This implies the inclusion $B \subset {}^sB \cdot {}^wB$. In particular, any element $b$ in $B$ can be writen as a product $b = xy$ with $x \in sBs$ and $y \in wBw^{-1}$. We have

$$y = x^{-1}b \in sBs \cdot B \cap wBw^{-1} \subset C(s) \cdot C(s) \cap {}^wB = (B \cup C(s)) \cap {}^wB = B_w.$$

This gives $y \in B$ and hence $x \in B$ and property ($T_7$) follows.

Conversely, suppose ($T_7$) holds, be have $B = B_s \cdot B_w \subset {}^sB \cdot {}^wB$ hence $B \subset sBs \cdot wBw^{-1}$ thus $sBw \subset BswB$ and ($T_6$) follows. □

Let $G'$ be the amalgamated product of the system of groups $\{N, P_s; s \in S\}$. Let $\varphi_s : P_s \to G'$ and $\varphi_N : N \to G'$ the associated morphisms and denote by $N'$ and $P'_s$ their image in $G'$. The morphisms $\varphi_s$ and $\varphi_N$ coincide on $P_s \cap N$ and $\varphi_s|_B$ does not depend on $s \in S$ (this comes from the fact that $P_s \cap P_{s'} = B$ for any couple $(s, s')$ of elements in $S$). Let us denote by $\varphi$ the restriction $\varphi_s|_B$ and $B'$ its image.

Furthermore, by the universal property of amalgamated products, there exists a group morphism $\psi : G' \to G$ such that $\varphi_s \circ \psi$ and $\varphi_N \circ \psi$ are the inclusions. This implies that $\psi$ gives an isomorphism of $N'$ on $N$, of $P'_s$ on $P_s$, of $B'$ on $B$, of $N' \cap P'_s$ on $N \cap P_s$ and of $W' = N'/(B' \cap N')$ on $W$.

**Lemma 17.4.6** *The quadruple* $(G', B', N', S)$ *is a Tits system.*

**Proof :** We have to check the axioms of a Tits system. By definition of an amalgamated product, the group $G'$ is generated by $N'$ and $B'$. Because $N'$ and $B' \cap N'$ are isomorphic to $N$ and $B \cap N$ we have that $B' \cap N'$ is normal in $N'$ and that $S$ generates $W'$. Furthermore, we have $sBs \subset BsBsB = C(s) \cdot C(s) = B \cup C(s) = P_s$. In particular condition ($T_4$) is a condition in the group $P_s$ and follows for $P'_s$.

We are left to prove ($T_3$) but this is equivalent by the two previous lemmas to properties ($T_5$) and ($T_7$). But as for ($T_4$) the property ($T_5$) is a condition in the group $P_s$ and follows at once for $P'_s$. It remains to prove ($T_7$) for $s \in S$ and $w \in W$ such that $\ell(sw) > \ell(w)$.

For this, we prove that $\psi(B'_w) = B_w$ by induction on $\ell(w)$. It is true for $\ell(w) = 1$. Assume this is true for $w$ and let $s \in S$ such that $\ell(sw) > \ell(w)$. We have (property ($T_8$) for $(G, B, N, S)$ and induction) the equality

$$B_{sw} = {}^sB_w \cap B = {}^s\psi(B'_w) \cap \psi(B').$$

But $B'$, $B'_w$ and $s$ are in $P'_s$ which is isomorphic to $P_s$ thus we get ${}^s\psi(B'_w) \cap \psi(B') = \psi({}^sB'_w \cap B')$. Be have the easy inclusion ${}^sB'_w \cap B' \subset B'_{sw}$ giving the inclusion

$$B_{sw} \subset \psi(B'_{sw}).$$

Conversely we have the inclusions

$$\psi(B'_{sw}) = \psi(swB'w^{-1}s \cap B') \subset \psi(swB'w^{-1}s) \cap \psi(B') = B_{sw}.$$

We may now conclude by computing $\psi(B'_sB'_w) = \psi(B'_s)\psi(B'_w) = B_sB_w = B$ by property $(T_7)$ in $(G, B, N, S)$. But since $B'_sB'_w \subset B'$ and $\psi$ is an isomorphism from $B'$ to $B$ we get the result. □

Let us now finish the proof of the theorem. Because $(G', B', N', S)$ is a Tits system, the group $G'$ is the disjoint union of the cells $C'(w) = B'wB'$ and $\psi(C'(w)) = BwB = C(w)$. In particular, if $g' \in G'$ is in the kernel of $\psi$ and $g' \in C'(w)$, then $1 = \psi(g') \in C(w)$. But $1 \in C(1)$ thus $w = 1$ and $g' \in B'$. But we have seen that $\psi$ is injective on $B'$ thus $\psi$ is injective. Because $G$ is generated by $B$ and $N$,the morphism $\psi$ is surjective. □

The last result of this chapter is a way to recover the group $G$ of a Tits system abstractly from the system of groups $(B, N, P_s; s \in S)$. For this we will need several axioms to be satisfied. Let $S$ be a finite set and let $(B, N, P_s; s \in S)$ be a system of groups. Let us define the following groups, sets or maps: $Y = N \cup \bigcup_{s\in S} P_s$, $T = B \cap N$, $N_s = N \cap P_s$, $W = N/T$ and $\pi : N \to W$.

Assume that $B$ is contained in all the groups $P_s$. Let $n \in N$ such that there exists an expression $n = n_1 \cdots n_r$ with $n_i \in N_{s_i}$ for some $s_i \in S$. We define a group $B(n_1, \cdots, n_r)$ by induction setting $B(t) = B$ for $t \in T$, $B(n_1) = B \cap (n_1^{-1}Bn_1)$ (in $P_{s_1}$) and $B(n_1, \cdots, n_r) = B \cap (n_r^{-1}B(n_1, \cdots, n_{r-1})n_r)$ (in $P_{s_r}$). We define the map

$$\gamma(n_1, \cdots, n_r) : B(n_1, \cdots, n_r) \to B$$

by $\gamma(n_1, \cdots, n_r)(x) = n_1 \cdots n_rxn_r^{-1} \cdots n_1^{-1}$ (where the conjugation is taken first in $P_{s_r}$ and finally in $P_{s_1}$.

**Theorem 17.4.7** *Let $(B, N, P_s; s \in S)$ be a system of groups and assume the following conditions are satisfied:*

$(P_1)$ *For $s \neq s'$, $P_s \cap P_{s'} = B$.*

$(P_2)$ *The subgroup $T$ is normal in $N$.*

$(P_3)$ *For any $s \in S$, the quotient group $N_s/T$ is of order 2 denoted $\{1, s\}$.*

$(P_4)$ *$P_s = B \cup BsB$.*

$(P_5)$ *The pair $(W, S)$ is a Coxeter system.*

$(P_6)$ *For any $n$ and any decomposition $n = n_1 \cdots n_r$ with $n_i \in N_{s_i}$ for some $s_i \in S$ such that $\pi(n) = \pi(n_1) \cdots \pi(n_r)$ is a reduced expression, the subgroup $B(n_1, \cdots, n_r)$ of $B$ depends only on $\pi(n)$ (and will be denoted $B_{\pi(n)}$) and the map $\gamma(n_1, \cdots, n_r) : B(n_1, \cdots, n_r) \to B$ depends only on $n$ (and will be denoted $\gamma_n$).*

$(P_7)$ *For $w \in W$ and $s \in S$ such that $\ell(ws) > \ell(w)$, we have $B_w \cdot B_s = B$.*

$(P_8)$ *Let $s$ and $t$ in $S$ and let $w \in W$ such that $sw = wt$ and $\ell(sw) > \ell(w)$. Then for any $m \in \pi^{-1}(s)$, $n \in \pi^{-1}(w)$ and $b \in B \setminus B_t$, there exist elements $y \in (bB_t) \cap B_w$ and $y', y'' \in B_w$ such that, setting $m' = n^{-1}m^{-1}n$, we have:*

– $(m')^{-1}ym' = y'm'y''$ *in* $P_t$ *and*

– $m\gamma_n(y)m^{-1} = \gamma_n(y')m^{-1}\gamma_n(y'')$ *in* $P_s$.

$(P_9)$ *The subgroup* $B$ *is not normal in* $P_s$ *for any* $s \in S$.

*Then the canonical map from* $Y$ *to the amalgamated product* $G$ *of the system of groups* $(B, N, P_s; s \in S)$ *is injective. If we again denote by* $B$ *and* $N$ *the image of these groups in* $G$, *then* $(G, B, N, S)$ *is a Tits system.*

*Furthermore, for any group* $G'$ *with an injective map* $\varphi : Y \to G'$ *such that the restiction of* $\varphi$ *to* $N$ *and* $P_s$ *are group homomorphisms and such that the image of* $\varphi$ *generates* $G'$, *then the canonical group morphism* $G \to G'$ *is an isomorphism.*

**Proof :** Let us make few comments: by $(P_2)$, the coset $W$ is a group. By $(P_3)$, the union in $(P_4)$ is disjoint (otherwise we would have $s \in B$ i.e. $s = 1$). The condition $(P_7)$ is equivalent to the condition $(T_7)$ (take the inverse).

Let us first prove the following:

**Fact 17.4.8** *(ı) For any* $n \in N$, *we have* $\gamma_n(B_{\pi(n)}) \subset B_{\pi(n^{-1})}$ *and the map* $\gamma_n : B_{\pi(n)} \to B_{\pi(n^{-1})}$ *is bijective with inverse* $\gamma_{n^{-1}}$.

*(ıı) For any* $n \in N$, *the group* $B_{\pi(n)}$ *contains* $T$. *Furthermore, for* $w \in W$ *and* $s \in S$ *such that* $\ell(ws) > \ell(w)$, *take* $n \in \pi^{-1}(w)$ *and* $m \in \pi^{-1}(s)$, *we have in* $P_s$*:*

$$B_{ws} = B \cap m^{-1}B_w m.$$

*(ııı) For* $w \in W$ *and* $s \in S$ *such that* $\ell(ws) > \ell(w)$, *we have the inclusion* $B_{sw^{-1}} \subset B_{w^{-1}}$.

**Proof :** (ı) Let us write $n = n_1 \cdots n_r$, we have $\gamma_n(x) = n_1 \cdots n_r x n_r^{-1} \cdots n_1^{-1}$ (successive conjugation in the $P_{s_i}$). If $x$ lies in $B_{\pi(n)}$, then $n_r x n_r^{-1}$ lies in $n_r B n_r^{-1} \cap B(n_1 \cdots n_{r-1})$ and by induction we get that $\gamma_n(x)$ lies in $B_{\pi(n^{-1})}$. Furthermore we clearly have $\gamma_n^{-1} = \gamma_{n^{-1}}$.

(ıı) Remark that the last statement corresponds to axiom $(T_8)$. The fact that $T$ is contained in $B_{\pi(n)}$ follows by induction on the length of $\pi(n)$ and the fact that $T$ is normal in $N$. Now write $n = n_1 \cdots n_r$, we have $nm = n_1 \cdots n_r m$ and $B_{ws} = B \cap m^{-1}B_w m$ by definition.

(ııı) Let $n \in \pi^{-1}(w)$ and $m \in \pi^{-1}(s)$. Let $b \in B_{ws}$ and write $b = m^{-1}xm$ with $x \in B_w$ by (ıı). We have $\gamma_{nm}(b) = \gamma_n(x) \in B_{w^{-1}}$. This is true for any $b \in B_{ws}$ thus $B_{(ws)^{-1}} = \gamma_{nm}(B_{ws}) \subset B_{w^{-1}}$. □

Let us now define the product $\widetilde{X} = B \times N \times B$ and consider the equivalent relation

$$(b_1, n, b_2) \sim (b_1', n', b_2')$$

if there exist $t \in T$ and $b \in B_{\pi(n)}$ such that $n' = tn$, $b_2' = bb_2$ and $b_1' = b_1\gamma_n(b^{-1})t^{-1}$.

**Fact 17.4.9** *(ı) The relation* $\sim$ *is an equivalence relation.*

*(ıı) The action of* $B$ *on the left and on the right on* $\widetilde{X}$ *respects the classes for this relation and induces an action on the quotient.*

**Proof :** (ı) Remark that this equivalent relation is made such that the product $b_1nb_2$ — if it exists, i.e. if all elements live in a big group $G$ — remains constant. In particular, the quotient $X = \widetilde{X}/\sim$ would be the union of double classes $BwB$ for $w \in W$ and thus equal to the group $G$ if it satisfies the axioms of a Tits system.

It is reflexive (take $t = b = 1$) and symmetric (take $t^{-1}$ and $b^{-1}$ and remark that $B_{\pi(n')} = B_{\pi(n)}$ because $\pi(n) = \pi(n')$). For the transitivity, let us express that $(b_1, n, b_2) \sim (b_1', n', b_2')$ and

$(b'_1, n', b'_2) \sim (b''_1, n'', b''_2)$: there exist $t, u \in T$, $b \in B_{\pi(n)}$ and $c \in B_{\pi(n')}$ such that $n' = tn$, $n'' = un'$, $b'_2 = bb_2$, $b''_2 = cb'_2$, $b'_1 = b_1\gamma_n(b^{-1})t^{-1}$ and $b''_1 = b'_1\gamma_{n'}(c^{-1})u^{-1}$.

In particular $\pi(n) = \pi(n')$ and $B_{\pi(n)} = B_{\pi(n')}$. We have $n'' = utn$ with $ut \in T$ and $b''_2 = cbb_2$ with $cb \in B_{\pi(n)}$. Furthermore, we have $\gamma_n(x) = t^{-1}\gamma_{n'}(x)t$ thus

$$b''_1 = b'_1\gamma_{n'}(c^{-1})u^{-1} = b_1\gamma_n(b^{-1})t^{-1}\gamma_{n'}(c^{-1})u^{-1} = b_1\gamma_n(b^{-1})\gamma_n(c^{-1})t^{-1}u^{-1} = b_1\gamma_n((cb)^{-1})(ut)^{-1}$$

and the result follows.

(ıı) It suffices to show that if $(b_1, n, b_2) \sim (b'_1, n', b'_2)$ and $b \in B$, then $(bb_1, n, b_2) \sim (bb'_1, n', b'_2)$ and $(b_1, n, b_2b) \sim (b'_1, n', b'_2b)$. This follows directly from the definition of $\sim$. □

Let us now define a right action of $P_s$ on $X$ extending the action of $B$. Set $W^s = \{w \in W \ / \ \ell(ws) > \ell(w)\}$ and $N^s = \pi^{-1}(W^s)$.

**Fact 17.4.10** *(ı) We have for any $w \in W^s$ the equality $P_s = B_w \cdot N_s \cdot B$.*

*(ıı) For any $t \in T$, $n \in N$ and $b \in B_{\pi(n)}$, we have $\gamma_n(bt) = \gamma_n(b)ntn^{-1}$.*

**Proof :** (ı) Indeed, by $(P_4)$, we have $P_s = B \cup BsB$ and by $(P_3)$ the quotient $N_s/T = T/T \cup N \cap (BsB)/T$ has two elements. We deduce that $N \cap (BsB) = sT$ and $N_s = T \cup sT$. This gives (apply $(P_7)$ and $(P_4)$):

$$B_w \cdot N_s \cdot B = B_w \cdot (T \cup sT) \cdot B = B \cup B_w sB.$$

In particular because $sB$ is contained in $BsB$ we have the inclusion $B_w \cdot N_s \cdot B \subset B \cup BsB = P_s$. Now write

$$B_w sB = B_w sBssB \supset B \cup B_w \cdot B_s sB = B \cup BsB = P_s.$$

(ıı) Decompose $n$ as a product $n_1 \cdots n_r$. We proceed by induction on $r = \ell(\pi(n))$, let $m = n_1 \cdots n_{r-1}$. We have $\gamma_n(bt) = \gamma_m(n_r btn_r^{-1}) = \gamma_m(n_r bn_r^{-1} n_r tn_r^{-1})$ and $n_r bn_r^{-1} \in B_{\pi(m)}$ and $n_r tn_r^{-1} \in T$. By induction we get $\gamma_n(bt) = \gamma_m(n_r bn_r^{-1})mn_r tn_r^{-1}m^{-1}$ and the result follows. □

Denote by $\theta : \widetilde{X} \to X$ the quotient map of the equivalence relation $\sim$. Let us now define a map $p_s : B \times N^s \times P_s \to X$ by

$$p_s(b, n, p) = \theta(b\gamma_n(b_1), nn_1, b_2)$$

where we write, using the previous fact, $p = b_1 n_1 b_2$ with $b_1 \in B_{\pi(n)}$, $n_1 \in N_s$ and $b_2 \in B$.

**Fact 17.4.11** *(ı) The definition of $p_s$ does not depend on the choice of the writing $p = b_1 n_1 b_2$.*

*(ıı) We have the equality $p_s(b, n, p) = p_s(b', n', p')$ if and only if there exists some $t \in T$ and $b'' \in B_{\pi(n)}$ such that $n' = tn$, $p' = b''p$ and $b' = b\gamma_n((b'')^{-1})t^{-1}$.*

**Proof :** (ı) We easily see from the proof of (ı) in the previous fact that if $p = b_1 n_1 b_2 = b'_1 n'_1 b'_2$, then $\pi(n_1) = \pi(n'_1) \in \{1, s\}$. In particular, there exists $t \in T$ such that $n'_1 = tn_1$. Now consider the element $c \in P_s$ defined by:

$$c = b_2(b'_2)^{-1} = n_1^{-1}b_1^{-1}b'_1 tn_1.$$

We have $c \in B \cap n_1^{-1}B_{\pi(n)}n_1$ and by the Fact 17.4.8 (ıı), we have $c \in B_{\pi(nn_1)}$.

We want to compare $(b\gamma_n(b_1), nn_1, b_2)$ and $(b\gamma_n(b'_1), nn'_1, b'_2)$. Define $b'' = b'_2 b_2^{-1} = c^{-1} \in B_{\pi(nn_1)}$, we have $nn'_1 = ntn_1 = \gamma_n(t)nn_1$ and $b'_2 = b''b_2$. To prove the result it suffices to prove the equality: $b\gamma_n(b'_1) = b\gamma_n(b_1)\gamma_{nn_1}(c)\gamma_n(t)^{-1}$. But we compute

$$b\gamma_n(b'_1) = b\gamma_n(b_1 n_1 b_2(b'_2)^{-1}(tn_1)^{-1}) = b\gamma_n(b_1 n_1 c n_1^{-1} t^{-1}).$$

Moreover, $b_1$, $n_1cn_1^{-1}$ and $t$ lie in $B_{\pi(n)}$ (recall that $c\in B_{\pi(nn_1)}$). In particular because $\gamma_n$ is a group morphism we get

$$b\gamma_n(b_1') = b\gamma_n(b_1)\gamma_n(n_1cn_1^{-1})\gamma_n(t^{-1}) = b\gamma_n(b_1)\gamma_{nn_1}(c)\gamma_n(t)^{-1}$$

and the result follows.

($ii$) We know by definition of $\sim$ that there exists $u\in T$ and $c\in B_{\pi(nn_1)}$ such that $n'n_1' = unn_1$, $b_2' = cb_2$ and $b'\gamma_{n'}(b_1') = b\gamma_n(b_1)\gamma_{nn_1}(c^{-1})u^{-1}$. In particular, we have $\pi(nn_1) = \pi(n'n_1')$ in $W$. But $n_1$ and $n_1'$ are in $W_s = \{1, s\}$ and $n$ and $n'$ are in $W^s$. In particular $\ell(\pi(nn_1)) = \ell(\pi(n)) + \ell(\pi(n_1)) = \ell(\pi(n'))+\ell(\pi(n_1')) = \ell(\pi(n'n_1'))$. This implies that $\pi(n) = \pi(n')$ and $\pi(n_1) = \pi(n_1')$. Indeed, otherwise we may assume that $\pi(n_1) = s$ and $\pi(n_1') = 1$. Then $\pi(n)s = \pi(n')$ and $\ell(\pi(n')) + 1 = \ell(\pi(n')s) = \ell(\pi(n)) = \ell(\pi(n)s) - 1 = \ell(\pi(n')) - 1$ a contradiction. We thus have an element $t\in T$ such that $n' = tn$.

Let us compute $b^{-1}b' = \gamma_n(b_1)\gamma_{nn_1}(c^{-1})u^{-1}\gamma_{n'}((b_1')^{-1})$. We have $\gamma_n(b_1)\in B_{\pi(n)^{-1}}$, $\gamma_{nn_1}(c^{-1})\in B_{(\pi(n)\pi(n_1))^{-1}}\subset B_{\pi(n)^{-1}}$, $u^{-1}\gamma_{n'}((b_1')^{-1})\in B_{\pi(n)^{-1}}$ thus $b^{-1}b'\in B_{\pi(n)^{-1}}$. We get

$$t\gamma_n(b_1')t^{-1} = \gamma_{tn}(b_1') = \gamma_{n'}(b_1') = (b')^{-1}b\gamma_n(b_1)\gamma_{nn_1}(c^{-1})u^{-1}$$

and applying $\gamma_{n^{-1}}$ which is possible because all the terms are in $B_{\pi(n)^{-1}}$ we get

$$b_1' = \gamma_{n^{-1}}(t^{-1}(b')^{-1}b)b_1n_1c^{-1}n_1^{-1}\gamma_{n^{-1}}(u^{-1}t).$$

Remark that $n'n_1' = unn_1$ and $n' = tn$ give the formula $n_1 = n^{-1}u^{-1}tnn_1' = \gamma_{n^{-1}}(u^{-1}t)n_1'$ thus we have

$$p' = b_1'n_1'b_2' = \gamma_{n^{-1}}(t^{-1}(b')^{-1}b)b_1n_1c^{-1}n_1^{-1}\gamma_{n^{-1}}(u^{-1}t)n_1'cb_2 = \gamma_{n^{-1}}(t^{-1}(b')^{-1}b)p.$$

But we have $t^{-1}(b')^{-1}b\in B_{\pi(n)^{-1}}$ thus we way define $b'' = \gamma_{n^{-1}}(t^{-1}(b')^{-1}b)\in B_{\pi(n)}$ and $p' = b''p$. We need to verify that $b' = b\gamma_n((b'')^{-1})t^{-1}$ which follows from the definition of $b''$. □

Remark that the map $p_s$ is surjective (any element in $N$ can be written as $nn_1$ with $n\in N^s$ and $n_1\in N_s$). The group $B$ (resp. $P_s$) acts on $B\times N_s\times P_s$ on the left (resp. right) by left (resp. right) multiplication. So we may want to define an action of $b\in B$ (resp. $p\in P_s$) on an element $x\in X$ by $p_s(by)$ (resp. $p_s(yp)$ for $y\in B\times N^s\times P_s$ such that $p_s(y) = x$. The previous lemma shows that this does not depend on the choice of $y$:

**Corollary 17.4.12** *($i$) The action of $B$ and $P_s$ descend via $p_s$ to an action of $B$ and $P_s$ on $X$.*

*($ii$) The left action of $B$ coincide with the previously defined left action of $B$ on $X$ and the restriction of the action of $P_s$ coincide with the previously defined right action of $B$ on $X$.*

We may now define an inverse on $X$. For $x = \theta(b_1, n, b_2)$, define

$$x^{-1} = \theta(b_2^{-1}, n^{-1}, b_1^{-1}).$$

**Fact 17.4.13** *The inverse is well defined on $X$.*

**Proof :** Write $(b_1, n, b_2)\sim(b_1', n', b_2')$, there exist $t\in T$ and $b\in B_{\pi(n)}$ such that $n' = tn$, $b_2' = bb_2$ and $b_1' = b_1\gamma_n(b^{-1})t^{-1}$. We then have $(n')^{-1} = n^{-1}t^{-1} = un^{-1}$ with $u = n^{-1}t^{-1}n\in T$. We also have $(b_1')^{-1} = t\gamma_n(b)b_1^{-1} = cb_1^{-1}$ with $c = t\gamma_n(b)$. But $b\in B_{\pi(n)}$ thus $\gamma_n(b)\in B_{\pi(n)^{-1}} = B_{\pi((n')^{-1})}$ so we have $c\in B_{\pi((n')^{-1})}$. Finally we compute

$$\begin{aligned} b_2^{-1}\gamma_{n^{-1}}(c^{-1})u^{-1} &= (b_2')^{-1}b\gamma_{n^{-1}}(\gamma_n(b^{-1})t^{-1})u^{-1}\\ &= (b_2')^{-1}bb^{-1}n^{-1}t^{-1}nu^{-1}\\ &= (b_2')^{-1}.\end{aligned}$$

□

We may now define a left action of $P_s$ on $X$ by setting

$$p \cdot x = (x^{-1} \cdot p)^{-1}.$$

By the Fact 17.4.11 this action restricts to the left action of $B$ on $X$.

**Lemma 17.4.14** *Let $s$ and $t$ in $S$, the left action of $P_s$ on $X$ commutes with the right action of $P_t$.*

**Proof :** We start by commuting the action with $B$.

**Fact 17.4.15** *The left (resp. right) action of $P_s$ (resp. $P_t$) commutes with the right (resp. left) action of $B$ on $X$.*

**Proof :** We prove this for $B$ and $P_t$ the other case follows by application of the inverse. Let $\theta(b_1, n, b_2) \in X$, let $b \in B$ and let $p \in P_t$. We have

$$(b \cdot \theta(b_1, n, b_2)) \cdot p = \theta(bb_1, n, b_2) \cdot p = p_t(bb_1, n, b_2p) = b \cdot (p_t(b_1, n, b_2p) = b \cdot (\theta(b_1, n, b_2) \cdot p).$$

The result follows. □

Let $X' = \{x \in X \ / \ (px)p' = p(xp'), \text{ for all } p \in P_s \text{ and all } p' \in P_t\}$. We want to prove that $X' = X$. Let us define the subsets of the Weyl group: $W' = \{w \in W \ / \ \ell(swt) = \ell(w) + 2\}$ and $W'' = \{w \in W \ / \ w^{-1}sw = t \text{ and } \ell(sw) > \ell(w)\}$. We choose any set of representatives $N'$ (resp. $N''$) of $W'$ (resp. $N''$).

**Fact 17.4.16** *We have the equality $N = N_s \cdot (N' \cup N'') \cdot N_t$.*

**Proof :** Let $n \in N$ set $w = \pi(n)$. We may assume $\ell(sw) > \ell(w)$. Indeed, if $\ell(sw) < \ell(w)$, we may multiply $n$ by $m$ with $\pi(m) = s$ to get $n' \in N$ with $\ell(\pi(n')s) > \ell(\pi(n'))$. For the same reason, we may assume that $\ell(wt) > \ell(w)$. If $\ell(swt) > \ell(sw) = \ell(wt)$ then $w \in W'$ and multiplying by an element in $T$ yields an element in $N'$.

If $\ell(swt) < \ell(sw)$ i.e. $\ell(swt) = \ell(w)$, we want to prove that $w \in W''$. Take a reduced expression $w = s_1 \cdots s_n$. Then $wt = s_1 \cdot s_n t$ is a reduced expression. Write $s_{n+1} = t$. Because $\ell(swt) < \ell(wt)$ the exchange condition gives an index $i \in [1, n+1]$ such that $swt = s_1 \cdots \hat{s}_i \cdots s_{n+1}$. If $i < n+1$, this gives $sw = s_1 \cdots \hat{s}_i \cdots s_n$ thus $\ell(sw) < \ell(w)$ a contradiction. Thus $swt = w$ and $w \in W''$. □

The variety $X'$ is stable under the left action of $P_s$ and the right action of $P_t$. Assume that $\theta(1, n, 1) \in X'$ for any $n \in N' \cup N''$. Then by these action and the previous fact; we have $\theta(1, n, 1) \in X'$ for all $n \in N$. Letting $B$ act on the left and on the right we get the desired result. We are left to prove that $\theta(1, n, 1) \in X'$ for any $n \in N' \cup N''$. We shall denote $\theta(1, n, 1)$ by $\mu(n)$.

Fix $n \in N' \cup N''$, let $w = \pi(n)$ and let $Q_n = \{p \in P_s \ / \ (p\mu(n))p' = p(\mu(n)p') \text{ for all } p' \in P_t\}$. By Fact 17.4.15, the set $Q_n$ is stable under the left action of $B$.

**Fact 17.4.17** *The set $Q_n$ is stable under the right action of $B_{w^{-1}}$.*

**Proof :** Let $b \in B_{w^{-1}}$, $p \in Q$ and let $p' \in P_t$. We have:

$$\begin{aligned}(pb \cdot \mu(n)) \cdot p' &= (p \cdot \theta(b, n, 1)) \cdot p' = (p \cdot \theta(1, n, \gamma_{n^{-1}}(b^{-1})) \cdot p' = (p \cdot (\mu(n) \cdot \gamma_{n^{-1}}(b^{-1}))) \cdot p' \\ &= ((p \cdot \mu(n)) \cdot \gamma_{n^{-1}}(b^{-1})) \cdot p' = (p \cdot \mu(n)) \cdot \gamma_{n^{-1}}(b^{-1})p' = p \cdot (\mu(n) \cdot \gamma_{n^{-1}}(b^{-1})p') \\ &= p \cdot (\theta(b, n, 1) \cdot p') = p \cdot ((b \cdot \mu(n)) \cdot p') = p \cdot (b \cdot (\mu(n)) \cdot p')) \\ &= (pb) \cdot (\mu(n) \cdot p').\end{aligned}$$

The result follows. □

Chose $m \in \pi^{-1}(s)$ and $m' \in \pi^{-1}(t)$. We have $\ell(sw) > \ell(w)$ thus by $(P_4)$ and $(P_7)$ we have:

$$P_s = B \cup BmB = B \cup (BmB_sB_{w^{-1}}) = B \cup (B \cdot (mB \cap Bm) \cdot B_{w^{-1}}) = B \cup BmB_{w^{-1}}.$$

In particular, we only need to prove that $m \in Q_n$.

Let $Y \subset B$ such that $YB_t = B$. We have for such a $Y$ the equality

$$((Ym') \cup \{1\}) \cdot B = P_t.$$

Indeed, we only need to prove that $BtB = Ym'B$. Take $b$ and $b'$ in $B$. There exists $y \in Y$ and $b_1 \in B_t = B_{t^{-1}} = m'B(m')^{-1} \cap B$ such that $b = yb_1$. Write $b_1 = m'b_2(m')^{-1}$ with $m' \in B$ and take $b'' \in B$ such that $b'' = b_2b'$. We have $ym'b'' = ym'b_2b' = yb_1m'b' = bm'b'$.

In particular, we only need to prove that for all $y \in Y$, we have

$$(m \cdot \mu(n)) \cdot (ym') = m \cdot (\mu(n) \cdot ym').$$

Suppose that $n \in N'$. Set $n_1 = mn$, then by $(P_7)$, we may choose $Y \subset B_{sw}$. By Fact 17.4.8 (iii) we have $Y \subset B_{sw} \subset B_w$. Now compute on the one hand:

$$(m \cdot \mu(n)) \cdot (ym') = \mu(n_1) \cdot (ym') = \gamma_{n_1}(y^{-1}) \cdot \mu(n_1m').$$

On the other hand

$$m \cdot (\mu(n) \cdot ym') = m \cdot (\gamma_n(y^{-1}) \cdot \mu(nm')) = \gamma_{n_1}(y^{-1})m \cdot \mu(nm') = \gamma_{n_1}(y^{-1}) \cdot \mu(n_1m').$$

The result follows in that case.

Suppose now that $n \in N''$ and assume that $m'$ satisfies the equation $m' = n^{-1}m^{-1}n$. If $y \in Y \cap B_t$, then because $B_ttB = tB$ we may replace (restricting $Y$) $y$ by 1 and the result follows. Assume that $y \in B \setminus B_t$. By $(P_8)$, we can choose $Y$ such that $Y \subset B_{\pi(n)}$ and for all $y \in Y$ there exist elements $y'$ and $y''$ in $B_{\pi(n)}$ such that

$$(m')^{-1}ym' = y'm'y'' \text{ in } P_t$$

$$m\gamma_n(y)m^{-1} = \gamma_n(y')m^{-1}\gamma_n(y'') \text{ in } P_s.$$

We then get the equalities

$$(m \cdot \mu(n)) \cdot (ym') = \mu(mn) \cdot (m'y'my'') = \mu(n) \cdot (y'm'y'') = \gamma_n(y') \cdot \mu(nm') \cdot y''$$

$$m(\mu(n)ym') = (m\gamma_n(y)) \cdot \mu(nm') = (\gamma_n(y')m^{-1}\gamma_n(y'')m) \cdot \mu(nm') = \gamma_n(y') \cdot \mu(nm') \cdot y''$$

and the result follows. □

Let us now compute the intersection of the kernel of the maps $P_s \to \mathrm{Aut}(X)$ defined by left and right action. For $p$ in this kernel, we have $p \cdot \theta(1,1,1) = \theta(1,1,1)$ and $(1,1,1) \sim (1,1,p)$ thus $p = 1$. Let $G_l$ (resp. $G_r$) the subgroups of $\mathrm{Aut}(X)$ generated by the groups $(P_s)_{s \in S}$ acting on the left (resp. on the right).

**Fact 17.4.18** *The groups $G_l$ and $G_r$ act transitively on $X$.*

**Proof :** It follows from the definition of the action and an easy induction on $\ell(\pi(n))$ that any element $\theta(b_1, n, b_2) \in X$ is in the orbit of $\theta(1,1,1)$. □

We may now define a map $t_l : G_l \to X$ by $t_l(g) = g \cdot \theta(1,1,1)$. This map is surjective by the previous fact. Let $g$ in the kernel i.e. $g \cdot \theta(1,1,1) = \theta(1,1,1)$. By Lemma 17.4.14, we have for all $g' \in G_r$ the equalities

$$\theta(1,1,1) \cdot g' = (g \cdot \theta(1,1,1)) \cdot g') = g \cdot (\theta(1,1,1) \cdot g')$$

and because $G_r$ acts transitively $g = 1$. We can thus define a group structure on $X$ such that $t_r$ is an group isomorphism.

Let us define $\mu : N \to X$ by $\mu(n) = \theta(1, n, 1)$.

**Fact 17.4.19** *The map $\mu$ is an injective group morphism.*

**Proof :** Take $n$ and $n'$ such that $\mu(n) = \mu(n')$. We have $(1, n, 1) \sim (1, n', 1)$ thus there exist $t \in T$ and $b \in B_{\pi(n)}$ such that $n' = tn$, $1 = b \cdot 1$ and $1 = 1 \cdot \gamma_n(b^{-1})t^{-1}$. This implies $t = b = 1$ and $n' = n$.

To prove that $\mu$ is a group morphism, it suffices to prove that $\mu(n_1 n) = \mu(n_1)\mu(n)$ for $n \in N$ and $n_1 \in N_s$. But $\mu(n_1)\mu(n) = n_1 \cdot \mu(n) = n_1 \cdot \theta(1, n, 1) = (\theta(1, n^{-1}, 1) \cdot n_1^{-1})^{-1} = \theta(1, n^{-1}n_1^{-1}, 1)^{-1}$ thus $\mu(n_1)\mu(n) = \theta(1, n_1 n, 1) = \mu(n_1 n)$. □

Let $G$ be the amalgamated product of the system $\{B, N, P_s; s \in S\}$. Let $Y = N \cup \cup_s P_s$. We have a natural map $\varphi : Y \to G$ such that $\varphi|_N$ and $\varphi|_{P_s}$ are group homomorphisms (by the definition of the amalgamated product). But we also have natural group homomorphisms $\mu : N \to X$ and $t_l|_{P_s} : P_s \to X$. By the universal property of the amalgamated product, we get a map $\psi : G \to X$ such that $\psi \circ \varphi|_N = \mu$ and $\psi \circ \varphi|_{P_s} = t_l|_{P_s}$.

**Fact 17.4.20** *(i) The map $\psi \circ \varphi : Y \to X$ is injective. In particular $\varphi$ is injective.*
*(ii) The map $\psi$ is surjective.*

**Proof :** (i) This is true for the restriction of this map on $N$ and any $P_s$. We thus need to prove that $\psi \circ \varphi(p) = \psi \circ \varphi(p')$ for $p \in P_s$ and $p' \in P_{s'}$ implies $p = p'$ and that $\psi \circ \varphi(p) = \psi \circ \varphi(n)$ for $p \in P_s$ and $n \in N$ implies $p = n$.

In the first case, this gives $p \cdot 1 = p' \cdot 1$ and we proved that $t_l$ is injective so $p = p'$. In the second case, we get $p \cdot 1 = \theta(1, n, 1)$. Writing $p = bn'b'$ with $b, b' \in B$ and $n' \in N_s$ we have $p \cdot 1 = \theta(b, n', b') = \theta(1, n, 1)$. This implies that $n' = tn$ for some $t \in T$, $b' = b''$ for some $b'' \in B_{\pi(n)}$ and $b = \gamma_n((b'')^{-1})t^{-1}$. We thus have

$$p = \gamma_n((b'')^{-1})t^{-1}tnb'' = n(b'')^{-1}n^{-1}nb'' = n$$

and the result follows.

(ii) This comes from the fact that $X$ is generated by the images of $B$ and $N$. □

**Fact 17.4.21** *The quadruple $(G, B, N, S)$ is a Tits sytem.*

**Proof :** The axiom $(T_1)$ follows from the definition of $G$ (generated by $B$ and $N$) and the condition $(P_2)$. The axiom $(T_2)$ follows from $(P_5)$. For the axiom $(T_4)$: assume $sBs = B$ for some $s \in S$. Then let $p \in P_s$, we have the alternative $p \in B$ or $p \in BsB$ i.e. $p = bsb'$ for $b$ and $b'$ in $B$. Compute $pBp^{-1}$, in the first case, this is $B$. In the second one, we get

$$pBp^{-1} = bsb'B(b')^{-1}sb^{-1} = bsBsb^{-1} = bBb^{-1} = B.$$

In particular we would have $B$ normal in $P_s$ a contradiction with $(P_9)$.

Finally let us prove $(T_3)$. By Lemma 17.4.4 we have to prove $(T_5)$ and $(T_6)$ and by Lemma 17.4.5 we have to prove $(T_5)$ and $(T_7)$. Remark that $(T_7)$ (or at least the same property after taking the inverse) is $(P_7)$. We are left to proving $(T_5)$. Compute $C(s) \cdot C(s) = BsBsB$. Because of $(P_4)$, we have $C(s) \subset P_s$ and $C(s) \cdot C(s) \subset P_s = B \cup C(s)$. Furthermore, we have $B \subset C(s) \cdot C(s)$. We only have to prove that $B \neq C(s) \cdot C(s)$. Otherwise this would give $B = BsBsB$ or $B = sBs$ in contradiction with $(T_4)$. □

To complete the proof, we prove that $\psi : G \to X$ is injective and hence an isomorphism. The same proof will work for any group $G'$ satisfying the conditions in Theorem 17.4.7. We consider an element $g \in G$ in the kernel of $\psi$. Because $G$ is a Tits system and by Bruhat decomposition, there exist elements $b$ and $b'$ in $B$ and $n \in N$ such that $bnb' = g$. Its conjugate $b'(bnb')(b')^{-1}$ is again in the kernel thus $b'bn \in \ker \psi$ and $\psi(b'b) = \psi(n^{-1})$. But both $b'b$ and $n$ are in $Y$ and $\varphi \circ \psi$ is injective thus $b'b = n^{-1}$. We get $p = bnb' = bb^{-1}(b')^{-1}b' = 1$ and the result follows. □

# Chapter 18

# Pro-groups

## 18.1 Algebraic groups

Let us first recall some results on algebraic groups that we shall need in the sequel.

### 18.1.1 Characteristic free results

**Proposition 18.1.1** *Let $G$ be an algebraic group and let $H$ be a closed subgroup of $G$, then the quotient $G/H$ has a natural structure of an algebraic variety such that the quotient map is a morphism of varieties.*

*If furthermore $H$ is normal in $G$, then the algebraic structure on $G/H$ is compatible with the group structure i.e. $G/H$ is an algebraic group.*

**Proposition 18.1.2** *Let $\varphi : G \to G'$ a morphism of algebraic groups, then the image of $\varphi$ is a closed subgroup of $G'$.*

### 18.1.2 Characteristic zero results

**Proposition 18.1.3** *Assume the characteristic is zero. Let $\varphi : G \to G'$ a bijective morphism of algebraic groups, then $\varphi$ is an isomorphism¿*

**Proposition 18.1.4** *Assume the characteristic is zero. Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra, then the Campbell-Hausdorff formula:*

$$\log(\exp X \exp Y) = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i+s_i>0 \\ 1\le i\le n}} \frac{(\sum_{i=1}^{n}(r_i+s_i))^{-1}}{r_1!s_1!\cdots r_n!s_n!}[X^{r_1}Y^{s_1}X^{r_2}Y^{s_2}\dots X^{r_n}Y^{s_n}],$$

*which uses the notation*

$$[X^{r_1}Y^{s_1}\dots X^{r_n}Y^{s_n}] = [\underbrace{X,[X,\dots[X,}_{r_1}\underbrace{[Y,[Y,\dots[Y,}_{s_1}\dots\underbrace{[X,[X,\dots[X,}_{r_n}\underbrace{[Y,[Y,\dots Y}_{s_n}]]\dots]],$$

*defines a group structure on $\mathfrak{g}$ making it into a unipotent algebraic group (denoted $G$).*

*We have $\mathrm{Lie}(G) = \mathfrak{g}$ and the exponential map $\exp : \mathfrak{g} \to G$ is the identity.*

*Furthermore, for any Lie algebra morphism $f : \mathfrak{g} \to \mathfrak{g}'$, the same map $f : G \to G'$ is an algebraic group morphism and its derivative $\dot{f}$ is $f$.*

## 18.2 Definition and first properties of pro-groups

Apart from the book of Kumar, a good reference for pro-groups (in the commutative setting but many properties generalise readily) is the paper of Serre [Se60].

**Definition 18.2.1** A structure of pro-group on a group $G$ is the datum of a family $\mathcal{F}$ of normal subgroups of $G$ such that

- the quotient $G/N$ for $N \in \mathcal{F}$ has a structure of algebraic group,
- for $N$ and $N'$ in $\mathcal{F}$, then $N \cap N'$ is in $\mathcal{F}$,
- for $N$ in $\mathcal{F}$ and $N'$ with $N \subset N'$, then $(N' \in \mathcal{F}) \Leftrightarrow (N'/N$ is a closed normal subgroup in $G/N)$,
- for $N \subset N'$ two elements in $\mathcal{F}$, the quotient map $G/N \to G/N'$ is a morphism of algebraic groups,
- the group morphism $G \to \varprojlim G/N$ is bijective where the set $\mathcal{F}$ is given the reverse order of the inclusion.

The set $\mathcal{F}$ is called the defining set of the pro-group.

**Example 18.2.2** (i) An algebraic group is a pro-group for the defining set of all its closed normal subgroups.

(ii) An infinite product of algebraic groups $(G_i)_{i\in\mathbb{N}}$:

$$G = \prod_{i=0}^{\infty} G_i$$

is a pro-group with defining set given by all the normal subgroups $N$ of $G$ such that there exists an index $i \in \mathbb{N}$ with $G_j \subset N$ for $j \geq i$ and such that $N$ seen as a subgroup of $\prod_{j\leq i} G_j$ is closed.

(iii) More generally an inverse limit (in the category of groups) of algebraic groups is a pro-group.

**Definition 18.2.3** (i) Let $G$ and $G'$ be two pro-groups with defining sets $\mathcal{F}$ and $\mathcal{F}'$, then a group morphism $\varphi : G \to G'$ is a pro-group morphism if for all $N' \in \mathcal{F}'$ we have $\varphi^{-1}(N') \in \mathcal{F}$ and the induced map $G/\varphi^{-1}(N') \to G'/N'$ is a morphism of algebraic groups.

(ii) Define the pro-topology as the inverse limit topology where $G/N$ for $N \in \mathcal{F}$ is endowed with the Zariski topology (recall that the inverse limit topology is the induced topology for the inclusion of $\varprojlim G_i$ in the product of the groups $G_i$ for the product topology).

(iii) A subset of $G$ is called a pro-subset if it is closed under the pro-topology.

(iv) A pro-subgroup is a subgroup of $G$ closed under the pro-topology.

(v) We denote by $\gamma_N$ the projection $G \to G/N$.

**Proposition 18.2.4** *(i) The composition of two morphisms of pro-groups is again a morphism of pro-groups.*

*(ii) The pro-topology is the smallest topology such that each map $G \to G/N$ is continuous. The inverse images of open subsets by these maps form a base for the pro-topology.*

*(iii) For $A \subset G$, the closure $\bar{A}$ of $A$ is given by*

$$\bar{A} = \bigcap_{N\in\mathcal{F}} \gamma_N^{-1}(\overline{\gamma_N(A)}).$$

*(iii) We have* $\bigcap_{N\in\mathcal{F}} N = \{1\}$.

**Proof :** (ı) This comes directly from the definition and the fact that the composition of two morphisms between algebraic groups is again a morphism of algebraic groups.

(ıı) The smallest topology such that each map $\gamma_N : G \to G/N$ is continuous as a base of open subsets given by the inverse image by $\gamma_N$ of the open subsets in $G/N$.

Let us also remark that $\gamma_N$ is given by the composition map

$$G \to \lim_{\leftarrow} G/N \to \prod_N G/N \to G/N.$$

The inverse image of an open subset $O \subset G/N$ in $\prod_N G/N$ is open for the product topology and hence its restriction to $G$ is an open subset for the pro-topology.

Conversely, a base for the pro-topology is given by the restriction of a product $\prod_N O_N$ of open subspaces $O_N$ in $G/N$ with $O_N = G/N$ for all except a finite number of $N \in \mathcal{F}$. This open subset is open for the smallest topology making the maps $\gamma_N$ continuous.

The inclusion $\bar{A} = \cap \gamma_N^{-1}(\overline{\gamma_N(A)})$ is clear. Conversely, let $x$ in the intersection. Assume there exists a closed subset (for the pro-topology) $F$ containing $A$ but not $x$. Then $x \in U = F^c$ and $U$ is open and does not meet $A$. There must therefore exists an open subset of the form $\gamma_N^{-1}(U_N)$ for some $N$ (where $U_N$ is open in $G/N$) such that $U_N \cap A = \emptyset$ and $x \in \gamma_N^{-1}(U_N)$. In particular we get that $\gamma_N(x) \in U_N$ but $\gamma_N(A) \subset U_N^c$. This is not possible since $\gamma_N(x) \in \overline{\gamma_N(A)}$.

(ııı) This condition is equivalent to the fact that the map from $G$ to the inverse limit of the groups $G/N$ is injective. □

We shall need the following general result on morphisms of inverse limits of groups:

**Lemma 18.2.5** *A pro-group $G$ is connected if and only if all the quotients $G/N$ for $N \in \mathcal{F}$ are connected.*

**Proof :** For this, take $G'$ the connected component of the identity. It is a pro-subgroup of $G$ and is open and closed in $G$. Then by definition of the pro-topology, we have that $\gamma_N(G') = G'N/N$ is open in $G/N$ and because $G'$ is closed we have that $G'N/N$ is closed in $G/N$. This last group being connected, $G'N/N = G/N$ for all $N$ and the result follows. □

**Corollary 18.2.6** *A pro-unipotent group is connected.*

**Proof :** Because algebraic unipotent groups are connected, we need to prove that if for all $N \in \mathcal{F}$ we have $G/N$ connected, then $G$ is connected. This follows from the previous Lemma.

For this, take $G'$ the connected component of the identity. It is a pro-subgroup of $G$ and is open and closed in $G$. Then by definition of the pro-topology, we have that $\gamma_N(G') = G'N/N$ is open in $G/N$ and because $G'$ is closed we have that $G'N/N$ is closed in $G/N$. This last group being connected, $G'N/N = G/N$ for all $N$ and the result follows. □

**Lemma 18.2.7** *Let $(X_i, f_{i,j})$ be a projective system such that for $X_i$ is, for any $i$, a principal homogeneous spaces under an algebraic group $G_i$ and such that the maps $f_{i,j}$ are morphisms of principal homogeneous spaces. Let $X = \lim_{\leftarrow} X_i$ and $f_i : X \to X_i$ the canonical map. Then*

*(ı) $X$ is non empty.*

*(ıı) We have $f_i(X) = \bigcap_{j \geq i} f_{i,j}(X_j)$.*

**Proof :** (i) Consider the set $\mathfrak{S}$ of the families $(Y_i)$ where $Y_i$ is an orbit in $X_i$ under a closed subgroup of $G_i$ and such that $f_{i,j}(Y_j) \subset Y_i$. Let us prove that $\mathfrak{S}$ with reverse inclusion as ordering is inductive. Indeed for any increasing sequence $(Y_{i,n})$ in $\mathfrak{S}$ we get a decreasing sequence of subspaces in $X_i$ and because this space is noetherian (principal homogeneous space under an algebraic group) we have a minimal element $Y_i$ for the sequence $Y_{i,n}$ with $i$ fixed. We have $(Y_i) \in \mathfrak{S}$. Indeed, $Y_i$ is an orbit under a closed subgroup of $G_i$ because $Y_i = Y_{i,n}$ for $n \geq n_i$ for some $n_i$. Furthermore, for $j \geq i$, take $n \geq \max(n_i, n_j)$, then $Y_j = Y_{j,n}$ and $Y_i = Y_{i,n}$ and $f_j(Y_j) \subset Y_i$ because the same hold for $Y_{j,n}$ and $Y_{j,n}$.

By Zorn's lemma, we may choose a maximal element $(Y_i)$ in $\mathfrak{S}$. Define the family $(Y_i')$ by

$$Y_i' = \bigcap_{j \geq i} f_{i,j}(Y_j).$$

Then $Y_i'$ is the orbit of a close subgroup of $G_i$ and $f_{i,j}(Y_j') \subset Y_i'$ thus $(Y_i') \in \mathfrak{S}$ and $(Y_i') \geq (Y_i)$, by maximality, we have $Y_i' = Y_i$ i.e. $Y_i = \bigcap_{j \geq i} f_{i,j}(Y_j)$.

Take $x_i \in Y_i$ (this is possible because $Y_i$ is an orbit) and consider the family $(Y_i'') \in \mathfrak{S}$ defined by $Y_i'' = f_{i,j}^{-1}(x_i)$. By maximality $Y_i = Y_i''$ and all the $Y_i$ are reduced to one point $x_i$. The system $(\{x_i\})$ is an element in $X$.

(ii) For the second assertion, remark that the inclusion of the left hand side in the right and side comes from the fact that $f_i$ factors through $f_{i,j} \circ f_j$. Take $x_i \in \bigcap_{j \geq i} f_{i,j}(X_j)$ and replace $X_j$ by $f_{i,j}^{-1}(x_i)$. Apply the first property to this inverse system and the result follows. □

**Corollary 18.2.8** *Let $(\varphi_i) : (G_i, f_{i,j}) \to (G_i', f_{i,j}')$ be a morphism of inverse systems of groups such that for all $i$ the map $\varphi_i : G_i \to G_i'$ is surjective. Then the induced map $\varphi : \lim_{\leftarrow} G_i \to \lim_{\leftarrow} G_i'$ is surjective.*

**Proof :** Let $(g_i')$ be an element in $\lim\limits_{\leftarrow} G_i'$ and consider the family $(\varphi_i^{-1}(g_i'))$. It is an inverse system of principal homogeneous spaces under the groups $(G_i)$. By the previous lemma, its inverse limit is non empty and the result follows. □

## 18.3 Pro-subgroups

**Proposition 18.3.1** *Let $H$ be a pro-subgroup of $G$, then the family*

$$\mathfrak{F}_H = \left\{ \begin{array}{c} \textit{normal subgroups } N' \textit{ of } H \textit{ such that } N' \supset H \cap N \textit{ for some } N \in \mathfrak{F} \textit{ and} \\ N'/(N \cap H) \textit{is a closed subgroup in } G/N \end{array} \right\}$$

*defines a structure of pro-group on $H$. The inclusion is a morphism of pro-groups and the pro-topology on $H$ coincide with the induced topology from the pro-topology on $G$.*

**Proof :** We shall first need the following useful lemma:

**Lemma 18.3.2** *Let $H$ be a pro-subgroup of a pro-group $G$, then for any $N \in \mathfrak{F}$, the set $\gamma_N(H)$ is closed in $G/N$. In particular we have the equality:*

$$H = \bigcap_{N \in \mathfrak{F}} \gamma_N^{-1}(\gamma_N(H)).$$

**Proof :** We already know that the equality $H = \cap_N \gamma_N^{-1}(\overline{\gamma_N(H)})$ holds. For $N' \subset N$ denote by $\gamma_{N,N'}$ the map $G/N \to G/N'$. Because the image of a closed subgroup under an algebraic group morphism is again a closed subgroup, we get that $\gamma_{N,N'}(\overline{\gamma_{N'}(H)}) = \overline{\gamma_N(H)}$.

Define $H'$ as the inverse limit of the system $(\overline{\gamma_N(H)})$. Then $H'$ is contained in $G$ and we have by Corollary 18.2.8 that $\gamma_N : H' \to \overline{\gamma_N(H)}$ is surjective i.e. $\gamma_N(H') = \overline{\gamma_N(H)}$.

We clearly have $H \subset H'$. Conversely, take $h' \in H'$, then $\gamma_N(h') \in \overline{\gamma_N(H)}$ and thus $h' \in \gamma_N^{-1}(\overline{\gamma_N(H)})$ and is in $H$. We thus have $H = H'$ and $\gamma_N(H) = \gamma_N(H') = \overline{\gamma_N(H)}$. $\square$

In particular, for any $N \in \mathcal{F}$, we have $H/(H \cap N)$ closed in $G/N$. It has a natural algebraic structure. Furthermore, for any $N' \in \mathcal{F}$ such that $N' \cap H = N \cap H$, the algebraic structures inherited from $G/N$ and $G/N'$ coincide: consider the morphisms $G/N \cap N' \to G/N$ and $G/N \cap N' \to G/N'$. This induces bijective morphisms $H/H \cap N \cap N' \to H/H \cap N$ and $H/H \cap N \cap N' \to H/H \cap N'$. But these must be isomorphisms.

For any normal subgroup $N'$ of $H$ containing $H \cap N$ and such that $N'/N \cap H$ is closed in $G/N$, the group $H/N'$ is a quotient group of $H/H \cap N$ and hence is an algebraic group.

Let us prove the axioms of a pro-group for the family $\mathcal{F}_H$. If $N_1'$ and $N_2'$ are in $\mathcal{F}_H$, let $N_1$ and $N_2$ the elements in $\mathcal{F}$ such that $N_i \cap H \subset N_i'$ and $N_i'/N_i \cap H$ is closed in $G/N_i$. Then we may consider the inverse image of $N_i'/N_i \cap H$ in $G/N_1 \cap N_2$¿ These are closed subgroups and their intersection is $N_1' \cap N_2'/N_1 \cap N_2 \cap H$ which is still closed.

Let $N_1' \in F_H$ and $N_2'$ a normal subgroup of $H$ containing $N_1'$ and such that $N_2'/N_1'$ is a closed subgroup in $H/N_1'$. There exists $N \in \mathcal{F}$ such that $N_1' \supset N \cap H$ and this implies $N_2' \supset N \cap H$. Now the image $N_2'/H \cap N$ of the closed subgroup $N_2'/N_1' \subset H/N_1'$ in $H/H \cap N$ is closed (as image of a closed algebraic group). Furthermore, by the previous lemma $H/H \cap N = \gamma_n(H)$ is closed in $G/N$ thus $N_2'/H \cap N$ is closed in $G/N$.

With the same notation but $N_1'$ and $N_2'$ both in $\mathcal{F}_H$, then we have a commutative diagram

$$\begin{array}{ccc} H/H \cap N & \xrightarrow{p} & H/N_1' \\ & \searrow^{q} & \downarrow r \\ & & H/N_2' \end{array}$$

where the maps $p$ and $q$ are morphisms of algebraic groups, this implies that it is the case of $r$.

Let us now denote by $H'$ the inverse limit of $H/N'$ for $N' \in \mathcal{F}_H$. We define a map $i : H' \to G$ by sending $(h_{N'}N')_{N' \in \mathcal{F}_H}$ to $(g_N N)_{N \in \mathcal{F}}$ where $g_N = h_{H \cap N}$. The map $i$ is clearly injective: if $(h_{N'}N')$ and $h'_{N'}N')$ have the same image then $h_{H \cap N}N = h'_{H \cap N}N$ for all $N \in \mathcal{F}$. But now for $N' \in \mathcal{F}_H$, take $N \in \mathcal{F}$ such that $N \cap H \subset N'$ and $N'/N \cap H$ is closed in $G/N$. Then $h_{N'}N'$ and $h'_{N'}N'$ are the image under the map $H/H \cap N \to H/N'$ of $h_{N \cap H}N$ and $h'_{N \cap H}N$ respectively. As these two elements coincide, the result follows.

We easily have $H \subset i(H')$. Furthermore, by the previous lemma, we have

$$H = \bigcap_{N \in \mathcal{F}} \gamma_N^{-1}(\gamma_N(H)) = \bigcap_{N \in \mathcal{F}} HN/N$$

which contains the image of $i$.

The pro-topology coincide with the subspace topology by the characterisation of continuous maps $f : Y \to G$ (all the composed maps $\gamma_N \circ f$ are continuous). $\square$

Let us now characterise the normal pro-subgroups.

**Lemma 18.3.3** *Let $H$ be a pro-subgroup of a pro-group $G$. Then $H$ is normal in $G$ if and only if $\gamma_N(H)$ is normal in $G/N$ for all $N \in \mathcal{F}$.*

**Proof :** Assume that $H$ is normal. Then for any $gN \in G/N$, and because $N$ is normal we have $gN \cdot (HN) \cdot (gN)^{-1} = NgHg^{-1}N = NHN = HN$. This implies that $HN/N = H/H \cap N$ is normal.

Conversely, assume that $H/H \cap N$ is normal for all $N \in \mathcal{F}$. Then because $H = \cap(HN)$ (see. the proof of the previous proposition), we get for $g = (g_N N)$ the equality $gHg^{-1} = g(\cap HN)g^{-1} = \cap(gHNg^{-1}) = H$. □

**Proposition 18.3.4** *($\imath$) Let $H$ be a normal pro-subgroup of a pro-group $G$, then $G/H$ is a pro-group with the defining set $\mathcal{F}' = \{N/H \ / \ N \in \mathcal{F}$ and $N \supset H\}$.*

*($\imath\imath$) The quotient map is a pro-group morphism.*

**Proof :** For $H \subset N \in \mathcal{F}$, we have a bijection $(G/H)/(N/H) \simeq G/N$ giving a structure of algebraic group to the group $(G/H)/(N/H)$. The first three axioms of a pro-algebraic group are easily satisfied. Let us prove the last one. Let us denote by $K$ the inverse limit of the system $(G/H)/(N/H)$ for $N \in \mathcal{F}'$.

There is a canonical map $\varphi : G/H \to K$. Let $gH$ in the kernel of this map, then

$$gH \subset \bigcap_{H \subset N \in \mathcal{F}} N.$$

But for any $N \in \mathcal{F}$, the group $HN$ is normal contains $N$ and by Lemma 18.3.2 its image $HN/N$ in $G/N$ is closed. This implies that $HN \in \mathcal{F}$. In particular we have the inclusion

$$\bigcap_{H \subset N \in \mathcal{F}} N \subset \bigcap_{N \in \mathcal{F}} HN.$$

But, as we already saw, Lemma 18.3.2 implies the equality

$$H = \bigcap_{N \in \mathcal{F}} HN$$

and we get that $gH \subset H$ thus $gH = H$ and $\varphi$ is injective.

To prove the surjectivity, consider the canonical map from $G$ to $L$ the inverse limit of the system $G/HN$ for $N \in \mathcal{F}$ induced by the maps $G/N \to G/HN$. Because all these maps are surjective, this map is surjective. To conclude, we only need to prove that $K$ and $L$ are in bijection. But take $N \in \mathcal{F}$, such that $N$ contains $H$. We have $N = HN$. Conversely we proved that for any $N \in \mathcal{F}$, we have $HN \in \mathcal{F}$ and thus $H \subset HN \in \mathcal{F}$. We thus have $\{N \in \mathcal{F} \ / \ H \subset N\} = \{HN \ / \ N \in \mathcal{F}\}$. The two inverse systems are equal because we take the limit over isomorphic indexing sets of isomorphic groups $(G/H)/(N/H) \simeq G/N$.

The fact that this map is a pro-group morphism is clear. □

**Corollary 18.3.5** *Let $H$ be a normal pro-subgroup of a pro-group $G$, then $H \in \mathcal{F}$ if and only if the pro-group $G/H$ is an algebraic group.*

**Proof :** If $H \in \mathcal{F}$ this is by definition. Conversely, assume that $G/H$ is an algebraic group. Then the trivial group is in $\mathcal{F}'$ the defining set of $G/H$. Indeed, for the fourth property of pro-groups to be true we need that

$$\bigcap_{N' \in \mathcal{F}'} N' = \{1\}$$

but the group $G/H$ being algebraic, we only need a finite intersection and thus $\{1\} \in \mathcal{F}'$. Because $G \to G/H$ is a pro-group morphism we have $H \in \mathcal{F}$. □

**Proposition 18.3.6** *Let $\varphi : G \to G'$ be a pro-group morphism. Then* $\mathrm{im}\varphi$ *is a pro-subgroup of $G'$ and moreover we have the following isomorphism of pro-groups:*

$$G/\ker\varphi \simeq \mathrm{im}\varphi.$$

*In particular, if $\varphi$ is bijective, then it is a pro-group isomorphism (i.e. its inverse is a pro-group morphism).*

**Proof :** Let $\mathcal{F}$ resp. $\mathcal{F}'$ the defining sets of $G$ and $G'$.

**Fact 18.3.7** *For any $N' \in \mathcal{F}'$, the subgroup $\gamma_{N'}(\mathrm{im}\varphi)$ of $G'/N'$ is closed.*

**Proof :** Because $\varphi$ is a pro-group morphism, then $\varphi^{-1}(N') \in \mathcal{F}$. Consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G' \\ {\scriptstyle \gamma_{\varphi^{-1}(N')}}\downarrow & & \downarrow{\scriptstyle \gamma_{N'}} \\ G/\varphi^{-1}(N') & \overset{\varphi_{N'}}{\dashrightarrow} & G'/N' \end{array}$$

which can be completed by a map $\varphi^{N'}$ of algebraic groups. We have $\gamma_{N'}(\mathrm{im}\varphi) = \mathrm{im}\varphi_{N'}$ and this last image is closed. □

Consider the subset $\hat{\mathcal{F}}$ of $\mathcal{F} \times \mathcal{F}'$ defined by

$$\hat{\mathcal{F}} = \{(N, N') \in \mathcal{F} \times \mathcal{F}' \ / \ N \subset \varphi^{-1}(N')\}.$$

Define the order $(N_1, N_1') \leq (N_2, N_2')$ by $N_1 \leq N_2$ and $N_1' \leq N_2'$. Define the following three inverse systems of algebraic groups indexed by $\hat{\mathcal{F}}$: $(G_{(N,N')})$, $(G'_{(N,N')})$ and $(H_{(N,N')})$ by $G_{(N,N')} = G/N$, $G'_{(N,N')} = G'/N'$ and $H_{(N,N')}$ is the image of the canonical map $\varphi_{N,N'} : G/N \to G/N'$. Because the maps $\varphi_{N,N'}$ are surjective, we get a surjection

$$\lim_{\leftarrow} G_{(N,N')} \to \lim_{\leftarrow} H_{(N,N')}.$$

We also have a natural injection:

$$\lim_{\leftarrow} H_{(N,N')} \to \lim_{\leftarrow} G'_{(N,N')}.$$

The image of this last injection is closed under the inverse limit topology. Indeed, this image is given by

$$\left(\prod_{(N,N')\in\hat{\mathcal{F}}} H_{(N,N')}\right) \cap \lim_{\leftarrow} G'_{(N,N')}.$$

But $H_{(N,N')}$ is closed in $G'_{(N,N')}$ (as the image of a algebraic group morphism) proving that the image is closed.

We have the following isomorphisms of groups:

$$G = \lim_{\leftarrow} G/N \to \lim_{\leftarrow} G_{(N,N')} \quad \text{and} \quad G' = \lim_{\leftarrow} G'/N' \to \lim_{\leftarrow} G'_{(N,N')}$$

which are homoemorphisms for the inverse limit topology. Putting all these maps together we get

$$G \twoheadrightarrow \lim_{\leftarrow} H_{(N,N')} \hookrightarrow G'.$$

As im$\varphi$ coincide with the image of this composition, it is closed and thus a pro-subgroup.

For the second part, observe that $\ker\varphi$ is a normal pro-subgroup. The induced map $G/\ker\varphi \to$ im$\varphi$ is an isomorphisms of group and a pro-group morphism.

We thus need to prove that a group isomorphism which is a pro-group morphism $\varphi : G \to G'$ is a pro-group isomorphism. Take $N \in \mathcal{F}$, then from the first part of the proposition $\varphi(N)$ is closed and normal because $\varphi$ is surjective. The induced map $G/N \to G'/\varphi(N)$ is a bijective morphism of pro-groups and in fact of algebraic groups. Then it is an isomorphism. We deduce that $\varphi^{-1}$ is a morphism of pro-groups. □

## 18.4 Definition and first properties of pro-Lie-algebras

The theory of pro-Lie-algebras is very similar to the theory of pro-groups. It is a little bit more simple because the finite dimensional situation is the one of a finite dimensional Lie algebra istead of a algebraic group. We only deal with vector spaces in place of varietes.

**Definition 18.4.1** Let $\mathfrak{g}$ be a Lie algebra over a field $k$. A structure of pro-Lie-algebra on $\mathfrak{g}$ is the datum of a family $\mathcal{F}$ of ideals in $\mathfrak{g}$ of finite codimension such that

- for $\mathfrak{a}$ and $\mathfrak{a}'$ in $\mathcal{F}$, then $\mathfrak{a} \cap \mathfrak{a}'$ is in $\mathcal{F}$,
- for $\mathfrak{a}$ in $\mathcal{F}$ and $\mathfrak{a}'$ an ideal with $\mathfrak{a} \subset \mathfrak{a}'$, then ($\mathfrak{a}' \in \mathcal{F}$).
- the canonical Lie algebra morphism $\mathfrak{g} \to \varprojlim \mathfrak{g}/\mathfrak{a}$ is an isomorphism where the set $\mathcal{F}$ is given the reverse order of the inclusion.

The set $\mathcal{F}$ is called the defining set of the pro-Lie-algebra.

**Example 18.4.2** (i) A finite dimensional Lie algebra is a pro-Lie-algebra for $\mathcal{F}$ the set of all its ideals.

(ii) An inverse limit (in the category of Lie algebras) of finite dimensional Lie algebras is a pro-Lie-algebra.

(iii) Let $V$ be an infinite vector space with a filtration by a family of finite dimensional vectors spaces $V_i$ with $V_0 = \{0\}$ and $V_i \subset V_{i+1}$. Let End$V$ be the Lie algebra of all $k$ linear maps from $V$ to itself and define the following sub-Lie-algebra of End$V$:

$$\mathfrak{u}((V_i)) = \{f \in \mathrm{End}V \ / \ f(V_i) \subset V_{i-1} \text{ for all } i\}.$$

Define $\alpha_i = \{f \in \mathfrak{u}((V_i)) \ / \ f|_{V_i} = 0\}$, then $\mathfrak{a}_i$ is an ideal in $\mathfrak{u}$ of finite codimension. The Lie algebra $\mathfrak{u}((V_i))$ is a pro-Lie-algebra for the defining set $\mathcal{F} = \{\mathfrak{a} \text{ ideal in } \mathfrak{u}((V_i)) \ / \ \exists i,\ \mathfrak{a}_i \subset \mathfrak{a}\}$. This pro-Lie-algebra $\mathfrak{u}((V_i))$ is pro-nilpotent (see Definition 18.6.1).

In the same spirit, let Aut$V$ the group of $k$-linear automorphisms of $V$. There exists a pro-group

$$U((V_i)) = \{f \in \mathrm{Aut}V \ / \ (f - I)(V_i) \subset V_{i-1} \text{ for all } i\}.$$

Define $N_i = \{f \in U((V_i)) \ / \ f|_{V_i} = I\}$, then $N_i$ is a normal subgroup of $U((V_i))$ such that the quotient is algebraic. The group $U((V_i))$ is a pro-group for the defining set

$$\mathcal{F} = \{N \text{ normal subgroup in } U((V_i)) \ / \ \exists i,\ N_i \subset N \text{ and } N/N_i \text{ closed in } \mathrm{Aut}V_i\}.$$

This pro-group is pro-unipotent (see Definition 18.6.1) and its Lie algebra (see Definition 18.5.3) is $\mathfrak{u}((V_i))$.

**Definition 18.4.3** (i) Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two pro-groups with defining sets $\mathcal{F}$ and $\mathcal{F}'$, then a Lie algebra morphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a pro-Lie-algebra morphism if for all $\mathfrak{a}' \in \mathcal{F}'$ we have $\varphi^{-1}(\mathfrak{a}') \in \mathcal{F}$.

(ii) Define the pro-topology as the inverse limit topology where $\mathfrak{g}/\mathfrak{a}$ for $\mathfrak{a} \in \mathcal{F}$ is endowed with the discrete topology.

(iii) A pro-Lie-subalgebra $\mathfrak{h}$ of a pro-Lie-algebra $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ closed under the pro-topology. It is a pro-ideal if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

(iv) We denote by $\gamma_{\mathfrak{a}}$ the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$.

**Proposition 18.4.4** *(i) The composition of two morphisms of pro-Lie-algebras is again a morphism of pro-Lie-algebras.*

*(ii) The pro-topology is the smallest topology such that each map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ is continuous. The inverse images of open subsets by these maps form a base for the pro-topology. As a consequence, for $A \subset \mathfrak{g}$, the closure $\bar{A}$ of $A$ is given by*

$$\bar{A} = \bigcap_{\mathfrak{a} \in \mathcal{F}} A + \mathfrak{a}.$$

*(iii) We have $\bigcap_{\mathfrak{a} \in \mathcal{F}} \mathfrak{a} = \{0\}$.*

*(ii) A morphism between pro-Lie-algebras is a pro-morphism if and only if it is continuous for the pro-topology.*

*(v) A pro-Lie-subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ is a pro-ideal if and only if for all $\alpha \in \mathcal{F}$, the Lie subalgebra $\mathfrak{h}/\mathfrak{h} \cap \mathfrak{a}$ is an ideal in $\mathfrak{g}/\mathfrak{a}$.*

**Proof :** For (i), (ii) and (ii), the same proof as in the case of pro-groups works.

(iii) This condition is equivalent to the fact that the map from $G$ to the inverse limit of the groups $G/N$ is injective.

(ii) A pro-Lie-algebra morphism is clearly continuous. Conversely, if the map $f : \mathfrak{g} \to \mathfrak{g}'$ is continuous, then the inverse image $f^{-1}(\mathfrak{a}')$ of any element $\mathfrak{a}' \in \mathcal{F}'$ is closed in $\mathfrak{g}$. In particular we have

$$f^{-1}(\mathfrak{a}') = \bigcap_{\mathfrak{a} \in \mathcal{F}} f^{-1}(\mathfrak{a}') + \mathfrak{a}.$$

But $\mathfrak{g}/f^{-1}(\mathfrak{a}')$ is contained in $\mathfrak{g}'/\mathfrak{a}'$ and thus finite dimensional. In particular the previous intersection is finite. But all its terms are in $\mathcal{F}$ thus the intersection is an element in $\mathcal{F}$.

(v) If $\mathfrak{h}$ is an ideal then $\mathfrak{h}/\mathfrak{h} \cap \mathfrak{a}$ is an ideal for all $\mathfrak{a} \in \mathcal{F}$. Conversely, if $\mathfrak{h} \cap \mathfrak{a}$ is an ideal, let $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. We compute for all $\mathfrak{a} \in \mathcal{F}$ the Lie bracket $[\gamma_{\mathfrak{a}}(X), \gamma_{\mathfrak{a}}(Y)]$ is in $\gamma_{\mathfrak{a}}(\mathfrak{h})$ thus $[X,Y]$ is in $\mathfrak{h} + \mathfrak{a} = \gamma_{\mathfrak{a}}^{-1}(\gamma_{\mathfrak{a}}(\mathfrak{h}))$. By (ii) the result follows. □

**Proposition 18.4.5** *(i) A Lie subalgebra $\mathfrak{h}$ of a pro-Lie-algebra $\mathfrak{g}$ is a pro-Lie-subalgebra if and only if the canonical map*

$$\mathfrak{h} \to \lim_{\leftarrow} \mathfrak{h}/\mathfrak{a}'$$

*is an isomorphism, where $\mathcal{F}' = \{\mathfrak{a}'$ ideal of $\mathfrak{h}$ with $\mathfrak{a}' \supset \mathfrak{a} \cap \mathfrak{h}$ for some $\mathfrak{a} \in \mathcal{F}\}$. In this case it is a pro-Lie-algebra for $\mathcal{F}'$ and the pro-topology is the subspace topology.*

*(ii) Let $\mathfrak{h}$ be a pro-ideal in $\mathfrak{g}$ then the Lie algebra $\mathfrak{g}/\mathfrak{h}$ is a pro-Lie-algebra with defining set $\mathcal{F}_{\mathfrak{h}} = \{\mathfrak{a}/\mathfrak{h}$, with $\mathfrak{a} \in \mathcal{F}$ such that $\mathfrak{h} \subset \mathfrak{a}\}$. The map $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is a pro-Lie-algebra morphism.*

*(iii) Given a pro-Lie-algebra morphism $f : \mathfrak{g} \to \mathfrak{g}'$, then the image* $\mathrm{im} f$ *is a pro-Lie-subalgebra of $\mathfrak{g}'$ and we have an isomorphism of pro-Lie-algebras $\mathfrak{g}/\ker f \simeq \mathrm{im} f$. In particular a bijective pro-Lie-algebra morphism is a pro-Lie-algebra isomorphism.*

**Proof :** (i) Define $\widehat{\mathfrak{h}}$ as the inverse limit of the system $\mathfrak{h}/\mathfrak{a}'$ for $\mathfrak{a}' \in \mathcal{F}'$. We have a natural map $\widehat{\mathfrak{h}} \to \mathfrak{h}/\mathfrak{a}' \subset \mathfrak{g}/\mathfrak{a}$ for $\mathfrak{a} \in \mathcal{F}$ such that $\mathfrak{a} \cap \mathfrak{h} \subset \mathfrak{a}'$ and this induces a map $\widehat{\mathfrak{h}} \to \mathfrak{g}$. This map is given by

$$(h_{\mathfrak{a}'} + \mathfrak{a}') \mapsto (g_{\mathfrak{a}} + \mathfrak{a})$$

where $g_{\mathfrak{a}} = h_{\mathfrak{a}\cap\mathfrak{h}}$. This map is injective: if for all $\mathfrak{a}$ we have $g_{\mathfrak{a}} \in \mathfrak{a}$, then because $g_{\mathfrak{a}} = h_{\mathfrak{a}\cap\mathfrak{h}}$ we have $h_{\mathfrak{a}\cap\mathfrak{h}} \in \mathfrak{a} \cap \mathfrak{h}$. But $h_{\mathfrak{a}'} + \mathfrak{a}' = (h_{\mathfrak{a}\cap\mathfrak{h}} + \mathfrak{a} + \mathfrak{h}) + \mathfrak{a}'$ for $\mathfrak{a}$ such that $\mathfrak{a} \cap \mathfrak{h} \subset \mathfrak{a}'$ and the injectivity follows.

By definition of $\widehat{\mathfrak{h}}$ we have that the image of this map is

$$\bigcap_{\mathfrak{a}} (h + \mathfrak{a}).$$

In particular, by the previous proposition, $\widehat{\mathfrak{h}} = \mathfrak{h}$ if and only if $\mathfrak{h}$ is closed in $\mathfrak{g}$.

(ii) We only need to prove that $\mathfrak{g}/\mathfrak{h}$ is the inverse limit of the system $\mathfrak{g}/\mathfrak{a}$ for $\mathfrak{a} \in \mathcal{F}_{\mathfrak{h}}$. There is a natural map given by $g \mapsto (g + \mathfrak{a})$. If $g$ is in the kernel, then for all $\mathfrak{a} \in \mathcal{F}_{\mathfrak{h}}$ we have $g \in \mathfrak{a}$. However, $\mathfrak{h}$ being an ideal, $\mathfrak{h} + \mathfrak{a}$ is an ideal of $\mathfrak{g}$ containing $\mathfrak{a}$ thus $\mathfrak{h} + \mathfrak{a} \in \mathcal{F}_{\mathfrak{h}}$. In particular $g \in \mathfrak{h} + \mathfrak{a}$ for all $\mathfrak{a} \in \mathcal{F}$. By the previous proposition $g \in \mathfrak{h}$ and the map is injective.

To prove the surjectivity, we only need to apply Lemma 18.2.7 to the situation (the group is a vector space in this case).

(iii) It suffices here to prove that the image of $f$ is closed in $\mathfrak{g}'$. Here the same proof as in Proposition 18.3.6. $\square$

## 18.5 Pro-Lie-algebra of a pro-group

**Definition 18.5.1** Let $G$ be a pro-group with defining set $\mathcal{F}$. For any $N \in \mathcal{F}$, let $\mathfrak{g}_N$ be the Lie algebra of the algebraic group $G/N$. For $N \subset N'$ two elements in $\mathcal{F}$, the derivation of the map of algebraic groups $\gamma_{N,N'} : G/N \to G/N'$ gives a map $\dot{\gamma}_{N,N'} : \mathfrak{g}_N \to \mathfrak{g}_{N'}$. These maps form an inverse system of Lie algebras and we set

$$\mathfrak{g} = \lim_{\leftarrow} \mathfrak{g}_N.$$

Let $\pi_N : \mathfrak{g} \to \mathfrak{g}_N$ be the natural projection. We also define $\dot{\mathcal{F}} = \{\text{ideals } \mathfrak{a} \text{ of } \mathfrak{g} \ / \ \mathfrak{a} \supset \ker \pi_N \text{ for some } N \in \mathcal{F}\}$

**Proposition 18.5.2** *For $G$ a pro-group, the Lie algebra $\mathfrak{g}$ is a pro-Lie-algebra for the defining set $\dot{\mathcal{F}}$.*

**Proof :** We only need to prove that $\mathfrak{g}$ is the inverse limit of the system $\mathfrak{g}/\mathfrak{a}$ with $\mathfrak{a} \in \dot{\mathcal{F}}$. We have a natural map

$$\mathfrak{g} \to \lim_{\leftarrow} \mathfrak{g}/\mathfrak{a} \simeq \lim_{\leftarrow} \mathfrak{g}/\ker \pi_N.$$

Furthermore, by definition, $\mathfrak{g}$ is the inverse limit of the system $\mathfrak{g}_N$. We have a natural map $\pi_N : \mathfrak{g} \to \mathfrak{g}_N$ and thus an injective map $\mathfrak{g}/\ker \pi_N \to \mathfrak{g}_N$. This induces an injective map

$$\lim_{\leftarrow} \mathfrak{g}/\ker \pi_N \to \lim_{\leftarrow} \mathfrak{g}_N.$$

But the composition

$$\mathfrak{g} = \lim_{\leftarrow} \mathfrak{g}_N \to \lim_{\leftarrow} \mathfrak{g}/\ker \pi_N \to \lim_{\leftarrow} \mathfrak{g}_N = \mathfrak{g}$$

is the identity and the surjectivity follows. $\square$

**Definition 18.5.3** We call $\mathfrak{g}$ **the pro-Lie-algebra of the pro-group** $G$ and denote it by $\mathrm{Lie}(G)$.

**Lemma 18.5.4** *Let $\mathfrak{g}$ be the Lie algebra of a pro-group $G$ and let $N \in \mathcal{F}$. Then the map $\pi_N : \mathfrak{g} \to \mathfrak{g}_N$ is surjective.*

**Proof :** Consider $\mathcal{F}_N = \{M \in \mathcal{F} \ / \ M \geq N\}$ and define $\mathfrak{g}_N^M = \mathfrak{g}_N$. For all $M \in \mathcal{F}_N$ we have a surjective map $\mathfrak{g}_M \to \mathfrak{g}_N$ (because the same is true on the groups) and by Corollary 18.2.8 we get a surjective map $\mathfrak{g} \to \mathfrak{g}_N$. □

**Proposition 18.5.5** *(ı) Let $f : G \to G'$ be a pro-group morphism, then there exists a pro-Lie algebra morphism $\dot{f} : \mathrm{Lie}(G) \to \mathrm{Lie}(G')$.*

*(ıı) In characteristic 0, assume that $G$ is connected and that for two maps $f$ and $g$ from $G$ to $G'$ satisfy $\dot{f} = \dot{g}$, then $f = g$.*

**Proof :** (ı) Let $\mathcal{F}$ and $\mathcal{F}'$ the defining sets of $G$ and $G'$. Let $N' \in \mathcal{F}'$ and $N \in \mathcal{F}$ such that $f(N) \subset N'$. We have a group morphism $f_{N,N'} : G/N \to G'/N'$. This induces a Lie algebra morphism $\dot{f}_{N,N'}\mathfrak{g}_N \to \mathfrak{g}'_{N'}$. We have in particular a morphism of inverse systems and thus a map $\dot{f} : \mathrm{Lie}(G) \to \mathrm{Lie}(G')$ such that $\pi'_{N'} \circ \dot{f} = \dot{f}_{N',N} \circ \pi_N$ for all $N \in \mathcal{F}$ and $N' \in \mathcal{F}'$. This morphism is clearly a pro-Lie-algebra morphism.

(ıı) Let $N$ such that $f(N) \subset N'$ and $g(N) \subset N'$ (for example take $N = f^{-1}(N') \cap g^{-1}(N')$). We have, as maps from $\mathfrak{g}_N$ to $\mathfrak{g}'_{N'}$ the equality $\dot{f}_{N,N'} = \dot{g}_{N,N'}$. In particular, this implies from the classical theory that $f_{N,N'} = g_{N,N'}$ and the result follows. □

**Proposition 18.5.6** *(ı) Assume the characteristic is zero, then for $G$ a pro-group and $\mathfrak{g}$ its Lie algebra, there exists a continuous map $\exp : \mathfrak{g} \to G$.*

*(ıı) Let $N \in \mathcal{F}$ for $\mathcal{F}$ the defining set of $G$, then $\exp$ is the unique map such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\pi_N} & \mathfrak{g}_N \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\gamma_N} & G/N. \end{array}$$

*(ııı) For any pro-group morphism $f : G \to G'$, we have a commutative diagram*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\dot{f}} & \mathfrak{g}' \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{f} & G'. \end{array}$$

**Proof :** (ı) Let $N \in \mathcal{F}$, we have a map $\exp_N : \mathfrak{g}_N \to G/N$. Furthermore, this gives a map of inverse systems $\exp : \mathfrak{g} \to G$. Because $\exp_N$ is continuous, this map is also continuous.

(ıı) By definition, the diagram is commutative and defines the exponential.

(ııı) For $N' \in \mathcal{F}'$ and $N \in \mathcal{F}$ such that $f(N) \subset N'$, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{g}_N & \xrightarrow{\dot{f}} & \mathfrak{g}'_{N'} \\ {\scriptstyle \exp_N}\downarrow & & \downarrow{\scriptstyle \exp_{N'}} \\ G/N & \xrightarrow{f_{N,N'}} & G'/N' \end{array}$$

and the result follows. □

## 18.6 Pro-unipotent groups and pro-nilpotent Lie algebras

**Definition 18.6.1** (i) A pro-group $G$ is called **pro-unipotent** if for all $N \in \mathcal{F}$, where $\mathcal{F}$ is the defining set of $G$, the group $G/N$ is an unipotent algebraic group.

(ii) A pro-Lie-algebra $\mathfrak{g}$ is called **pro-nilpotent** if for all $\mathfrak{a} \in \mathcal{F}$, where $\mathcal{F}$ is the defining set of $\mathfrak{g}$, the Lie algebra $\mathfrak{g}/\mathfrak{a}$ is a nilpotent Lie algebra.

(iii) Let us denote by ProUni (resp. ProNil) the category of pro-unipotent groups (resp. pro-nilpotent Lie algebras), here the morphisms are the pro-group (resp. pro-Lie-algebra) morphisms.

**Theorem 18.6.2** *(i) The category* ProUni *is equivalent to the category* ProNil *under the functor taking $G$ to* $\mathrm{Lie}(G)$ *and a pro-group morphism $f$ to its derivative $\dot{f}$.*

*(ii) The exponential map* $\exp : \mathrm{Lie}(G) \to G$ *is bijective for any object $G$ in* ProUni.

**Proof :** (i) Let us define a functor from ProNil to ProUni as follows: let $\mathfrak{g}$ be a pro-unipotent Lie algebra with defining set $\mathcal{F}$. For any $\mathfrak{a} \in \mathcal{F}$, consider the nilpotent Lie algebra $\mathfrak{g}/\mathfrak{a}$. Let $G_\mathfrak{a}$ be the associated unipotent group structure on $\mathfrak{g}/\mathfrak{a}$ given by Proposition 18.1.4. For any pair $\mathfrak{a} \subset \mathfrak{a}'$ of elements in $\mathcal{F}$, the map $\mathfrak{g}/\mathfrak{a} \to \mathfrak{g}/\mathfrak{a}'$ induces a map $G_\mathfrak{a} \to G_{\mathfrak{a}'}$. In particular the family $(G_\mathfrak{a})$ is a projective system of algebraic groups. Let us set

$$G = \lim_{\leftarrow} G_\mathfrak{a}.$$

Let us denote by $\gamma_\mathfrak{a}$ the natural map from $G$ to $G_\mathfrak{a}$. Let us define the family of normal subgroups $\mathcal{F}'$ in $G$:

$$\mathcal{F}' = \{N \ / \ \exists \mathfrak{a} \in \mathcal{F} \text{ and } N_\mathfrak{a} \subset G_\mathfrak{a} \text{ a closed normal subgroup with } N = \gamma_\mathfrak{a}^{-1}(N_\mathfrak{a})\}.$$

**Fact 18.6.3** *The group $G$ is a pro-group for the defining set $\mathcal{F}'$.*

**Proof :** A direct application of Corollary 18.2.8 gives that the map $\gamma_\mathfrak{a}$ is surjective for all $\mathfrak{a} \in \mathcal{F}$. It is then easy to check that $G$ is a pro-group: the quotients are algebraic groups. For $N = \gamma_\mathfrak{a}^{-1}(N_\mathfrak{a})$ and $N' = \gamma_{\mathfrak{a}'}^{-1}(N_{\mathfrak{a}'})$ we have

$$N \cap N' = \gamma_{\mathfrak{a}\cap\mathfrak{a}'}^{-1}(\gamma_{\mathfrak{a},\mathfrak{a}\cap\mathfrak{a}'}^{-1}(N_\mathfrak{a}) \cap \gamma_{\mathfrak{a}',\mathfrak{a}\cap\mathfrak{a}'}^{-1}(N_{\mathfrak{a}'})).$$

The last two conditions follow by definition. □

Because as sets we have $G_\mathfrak{a} = \mathfrak{g}/\mathfrak{a}$ we get that $G = \mathfrak{g}$ as sets.

**Fact 18.6.4** *The functors $G \mapsto \mathrm{Lie}(G)$ and $\mathfrak{g} \mapsto G$ are inverse to each other.*

**Proof :** We have

$$\mathrm{Lie}(G) = \lim_{\leftarrow} \mathrm{Lie}(G_\mathfrak{a}) = \lim_{\leftarrow} \mathfrak{g}/\mathfrak{a} = \mathfrak{g}.$$

For the converse, we need the following results on algebraic unipotent groups. For $U$ and $U'$ two unipotent algebraic groups, the exponential map $\exp : \mathrm{Lie}(U) \to U$ is an isomorphism of varieties and the natural map

$$\mathrm{Hom}(U, U') \to \mathrm{Hom}(\mathrm{Lie}(U), \mathrm{Lie}(U'))$$

sending $f$ to $\dot{f}$ is an isomorphism. We also have the following commuting diagramm:

$$\begin{array}{ccc} \mathrm{Lie}(U) & \xrightarrow{\dot{f}} & \mathrm{Lie}(U') \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ U & \xrightarrow{f} & U'. \end{array}$$

In particular, if $G$ is a pro-unipotent group, then $G/N$ is a unipotent group and $\mathrm{Lie}(G/N)$ is a nilpotent Lie algebra. We have that $G/N$ (via the exponential map) is the group structure defined on $\mathrm{Lie}(G/N)$ thanks to the Campbell-Hausdorff formula. Furthermore the induced maps $G/N \to G/N'$ are the natural maps and the group constructed above is the group $G$.

Furthermore, because of Proposition 18.6.2, we have that for any Lie algebra morphism $f : \mathfrak{g} \to \mathfrak{g}'$ the same map $f : G \to G'$ is a pro-group morphism. and $\dot{f} = f$. □

The result on the exponential follows easily. □

## 18.7 Pro-representations

**Definition 18.7.1** (i) Let $G$ be a pro-group with defining set $\mathcal{F}$. A representation $V$ of $G$ is called a **pro-representation** if for all $v \in V$ there exists a finite dimensional sub-representation $W$ of $V$ such that

- $v \in W$;
- there exists $N \in \mathcal{F}$ such that $N$ acts trivially on $W$;
- the induced representation of $G/N$ on $W$ is algebraic.

(ii) Let $\mathfrak{g}$ be a pro-group with defining set $\mathcal{F}$. A $\mathfrak{g}$-module $V$ is called a **pro-representation** if for all $v \in V$ there exists a finite dimensional sub-representation $W$ of $V$ such that

- $v \in W$;
- there exists $\mathfrak{a} \in \mathcal{F}$ such that $\mathfrak{a}$ acts trivially on $W$.

**Definition 18.7.2** (i) A morphism of representations $f : V \to V'$ between pro-representations of the pro-group $G$ is called a pro-representation morphism.

(ii) A morphism of representations $f : V \to V'$ between pro-representations of the pro-Lie-algebra $\mathfrak{g}$ is called a pro-representation morphism.

**Lemma 18.7.3** *Let $V$ be a pro-representation, then any finite dimensional sub-representation $W$ of $V$ is a pro-representation.*

**Proof :** Let $v \in W$, because $W \subset V$, there exists $U$ a finite dimensional $G$-stable subspace containing $v$ and such that there exists $N \in \mathcal{F}$ such that $N$ acts trivially on $U$ and $G/N$ acts algebraically on $U$. Let $W' = U \cap W$, it is a finite dimensional stable subspace acted trivially by $N$. Its centraliser $N'$ contains $N$ and is normal thus $N' \in \mathcal{F}$. The action of the group $G/N'$ is the quotient action of the group $G/N$ and is therefore algebraic. □

**Proposition 18.7.4** *Let $G$ be a pro-group and $\mathfrak{g} = \mathrm{Lie}(G)$ its pro-Lie-algebra. Let $V$ be a pro-representation of $G$, then there exists a natural structure of pro-representation of $\mathfrak{g}$ on $V$ such that the map $\psi : \mathfrak{g} \to \mathrm{End}(V)$ is given by $\psi = \dot{\varphi}$ with $\varphi : G \to \mathrm{Aut}(V)$ defined by the pro-representation $V$ of $G$.*

**Proof :** Let $V$ be a pro-representation of $G$ and let $W$ be any finite dimensional sub-representation of $G$. There exists $N \in F$ such that $N$ acts trivially on $W$ and such that the action of $G/N$ on $W$ is algebraic. We thus have an algebraic group morphism $G/N \to \mathrm{Aut}(W)$ and we may differentiate this map to get a map $\mathfrak{g}_N \to \mathrm{End}(W)$ where $\mathfrak{g}_N = \mathrm{Lie}(G/N)$.

Because of the definition of $\mathfrak{g} = \mathrm{Lie}(G)$ we have a map $\mathfrak{g} \to \mathfrak{g}_N$ and thus a map $\mathfrak{g} \to \mathrm{End}(W)$. We claim that this map does not depend on the choice of $N$. Indeed, let $N'$ be another element in $\mathcal{F}$ such that $N'$ acts trivially on $W$. Then $N \cap N'$ is in $\mathcal{F}$ and acts trivially on $W$. Furthermore, the actions of $G/N$ and $G/N'$ on $W$ are induced by the map $G/(N \cap N') \to \mathrm{Aut}(W)$ given by the action of $G/(N \cap N')$ on $W$. By differentiating the same is true for the Lie algebras.

Because $V$ is generated by its finite dimensional sub-$G$-pro-representation, we deduce an action of $\mathfrak{g}$ on $V$ and this map is given by the differentiation. □

**Lemma 18.7.5** *Assume that the base field is $\mathbb{C}$. Let $\pi$ and $\rho$ be two pro-representations of a connected pro-group $G$ in $V$. Then we have the following equivalence:*

$$\pi = \rho \Leftrightarrow \dot{\pi} = \dot{\rho}.$$

**Proof :** One implication is trivial. Assume that $\dot{\pi} = \dot{\rho}$ and let $W$ be a finite dimensional pro-sub-representation of $\mathfrak{g}$. There exist two finite dimensional subspaces $W_1$ and $W_2$ of $V$ such that $W \subset W_1 \cap W_2$ and $W_1$ is stable under $(G, \pi)$ and $W_2$ is stable under $(G, \rho)$. There are also two elements $N_1$ and $N_2$ in $\mathcal{F}$ such that $N_1$ act trivially (via $\pi$) on $W_1$ and $N_2$ acts trivially (via $\rho$) on $W_2$. Set $N = N_1 \cap N_2$, then the action of $N$ via $\pi$ (resp. via $\rho$) on $W_1$ (resp. on $W_2$) is trivial and we have an action of $G/N_1$ (resp. $G/N_2$) and even an action of $G/N$. Differentiating this action, we have an action of $\mathfrak{g}_N = \mathrm{Lie}(G/N)$ on $W_1$ and $W_2$ and these two action coincide on $W$ and leave $W$ invariant.

Since $G$ is connected (and thus since the image of $\mathfrak{g}_N$ by the exponential generated $G/N$), this implies that the actions of $G/N$ on $W_1$ and $W_2$ leave $W$ invariant and coincide on $W$. The result follows. □

**Proposition 18.7.6** *Assume that the base field is $\mathbb{C}$.*

*(i) Let $G$ be a pro-group and $\mathfrak{g}$ its pro-Lie-algebra, there exists a unique representation*

$$\mathrm{Ad}\, : G \to \mathrm{Aut}(\mathfrak{g})$$

*such that* $\dot{\mathrm{Ad}} = \mathrm{ad}$.

*(ii) Furthermore, if $g \in G$, denote by $Int_g : G \to G$ the conjugation by $g$, then* $\mathrm{Ad}\,(g) = \dot{Int}_g$.

*(iii) For $g \in G$ and $X \in \mathfrak{g}$, we have*

$$\exp(\mathrm{Ad}\,(g)(X)) = g \exp(X) g^{-1}.$$

*(iv) For any pro-representation $\pi : G \to \mathrm{Aut}(V)$, any $g \in G$ and $X \in \mathfrak{g}$, we have*

$$\dot{\pi}(\mathrm{Ad}\,(g)(X)) = \pi(g)\dot{\pi}(X)\pi(g)^{-1}.$$

**Proof :** (i) Let $N \in \mathcal{F}$ and consider the adjoint representation $G/N \to \mathrm{Aut}(\mathfrak{g}_N)$. For $N' \in \mathcal{F}$ containing $N$, we have natural morphisms and a commutative diagram:

$$\begin{array}{ccc} G/N' & \xrightarrow{\gamma_{N,N'}} & G/N \\ {\scriptstyle \mathrm{Ad}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}} \\ \mathfrak{g}_N \dot{\gamma}_{N,N'} & \longrightarrow & \mathfrak{g}_{N'}. \end{array}$$

In particular, the maps $\mathrm{Ad}\, : G/N \to \mathrm{Aut}(\mathfrak{g}_N)$ induce a projective system morphism and thus a map $\mathrm{Ad}\, : G \to \mathrm{Aut}(\mathfrak{g})$. By definition, we have $\dot{\mathrm{Ad}} = \mathrm{ad}$.

Remark that the point (ii) will imply that the group of connected components of $G$ acts trivially by Ad and thus the representation Ad is determined by the action of the connected component of the identity and by the previous Lemma by its derived action. It is unique.

(ii) This comes from the same fact in the finite dimensional case.

(iii) Recall the commutative diagam:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\dot{f}} & \mathfrak{g}' \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{f} & G' \end{array}$$

for $f$ a morphism of pro-groups. Apply this to $Int_g$ to get the result.

(iv) Recall that for $V$ an infinite dimensional vector space we may define the exponential on locally finite endomorphisms. Let us denote by $\mathrm{End}_{lf}(V)$ the set of such elements. For a pro-representation $\pi : G \to \mathrm{End}(V)$, we have a commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\dot{\pi}} & \mathrm{End}_{lf}(V) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\pi} & \mathrm{Aut}(V). \end{array}$$

Indeed, the map $\dot{\pi}$ factorises through finite dimensional representation where the images of elements in $\mathfrak{g}$ acts as locally finite endomorphisms. We thus have for $X \in \mathfrak{g}$ the equality $\pi(\exp(X)) = \exp(\dot{\pi}(X))$. Let us compute (using (iii) above):

$$\begin{aligned} \exp(\dot{\pi}(\mathrm{Ad}\,(g)(X))) &= \pi(\exp(\mathrm{Ad}\,(g)(X))) = \pi(g\exp(X)g^{-1}) = \pi(g)\pi(\exp(X))\pi(g)^{-1} \\ &= \pi(g)\exp(\dot{\pi}(X))\pi(g)^{-1} = \exp(\pi(g)\dot{\pi}(X)\pi(g)^{-1}) \end{aligned}$$

and the result follows. □

**Example 18.7.7** The adjoint representation is not always a pro-representation. Indeed, consider the group $G = GL_2(\mathbb{C}[[t]])$. It is a pro-group thanks to the sub-groups $GL_2(t^n\mathbb{C}[[t]])$. The adjoint representation is not a pro-representation. Indeed, consider the elements

$$g = \begin{pmatrix} 1+t & 0 \\ 0 & \frac{1}{1+t} \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

of $GL_2(\mathbb{C}[[t]])$. The element $\mathrm{Ad}^n(X)$ has for matrix

$$\begin{pmatrix} 0 & (1+t)^{2n} \\ \frac{1}{(1+t)^{2n}} & 0 \end{pmatrix}$$

and these elements generate an infinite dimensional subspace. In particular $g$ does not act as a locally nilpotent element.

Let $G$ be a pro-group and $\mathfrak{g}$ a pro-Lie-algebra, we denote by $\mathrm{Rep}(G)$, $\mathrm{Rep}(\mathfrak{g})$ and $\mathrm{Rep}_{lf}(\mathfrak{g})$ the categories of pro-representations of $G$, $\mathfrak{g}$ and the full subcategory of $\mathrm{Rep}(\mathfrak{g})$ of representations given by locally finite elements.

**Proposition 18.7.8** *Ldet $G$ be a pro-unipotent group and let $\mathfrak{g}$ be its Lie algebra. Then the functor $\mathrm{Rep}(G) \to \mathrm{Rep}_{lf}(\mathfrak{g})$ sending $\pi : G \to \mathrm{Aut}(V)$ to $\dot{\pi} : \mathfrak{g} \to \mathrm{End}(V)$ is an equivalence of category.*

**Proof :** Let us prove that the functor is essentially surjective. For this take $\pi : \mathfrak{g} \to \mathrm{End}(V)$ a pro-representation. Let $W$ be any finite dimensional sub-representation of $V$. There exists $N \in \mathcal{F}$ the defining set for $G$ such that $\mathfrak{g}_N$ acts trivially on $W$. Because $G$ is unipotent, it is the case of $G/N$ and $\mathfrak{g}_N$ acts nilpotently. There exists thus a morphism $\rho_W : G/N \to \mathrm{Aut}(W)$ such that $\dot{\rho_W} = \pi|_W$. This induces a map $G \to \mathrm{Aut}(W)$ and as before this does not depend on the choice of $N$. As finite dimensional subspaces generate $V$ we get a representation $\rho : G \to \mathrm{Aut}(V)$ such that $\dot{\rho} = \pi$.

The functor is defined on the morphisms by the identity: if $f : V \to V'$ is a morphism between pro-representation of $G$, the same map induces a map of pro-representations of $\mathfrak{g}$. We conclude thanks to Lemma 18.7.5. □

# Chapter 19

# Kac-Moody groups

Let $\mathfrak{g}$ be any Kac-Moody Lie algebra associated to a generalised Cartan matrix $A$ of size $n$ as constructed and studied in the first part. In this chapter we construct the so called (completed) Kac-Moody group $G$ associated to $\mathfrak{g}$.

## 19.1 The groups $T$ and $N$

Recall that in $\mathfrak{g}$, we have the Cartan subalgebra $\mathfrak{h}$. Let us define an integral Cartan subalgebra as follows:

**Definition 19.1.1** An **integral Cartan subalgebra** $\mathfrak{h}_{\mathbb{Z}}$ of $\mathfrak{g}$ is a finitely $\mathbb{Z}$-submodule of $\mathfrak{h}$ such that

- the natural map $\mathfrak{h}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \to \mathfrak{h}$ is an isomorphism;
- all the simple coroots $\alpha_i^\vee$ are in $\mathfrak{h}_{\mathbb{Z}}$;
- setting $\mathfrak{h}_{\mathbb{Z}}^* = \hom_{\mathbb{Z}}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$ which is contained in $\mathfrak{h}^*$, we have that all simple roots $\alpha_i$ are in $\mathfrak{h}_{\mathbb{Z}}^*$;
- the quotient $\mathfrak{h}_{\mathbb{Z}}/(\sum_i \alpha_i^\vee)$ is torsion free.

Such an integral Cartan subalgebra always exists. Let us choose $\mathfrak{h}_{\mathbb{Z}}$ an integral Cartan subalgebra. We have the following easy fact coming from the definition of the Weyl group and of its action on $\mathfrak{h}$ and $\mathfrak{h}^*$:

**Fact 19.1.2** *The submodule $\mathfrak{h}_{\mathbb{Z}}$ (resp. $\mathfrak{h}_{\mathbb{Z}}^*$) of $\mathfrak{h}$ (resp. of $\mathfrak{h}^*$) is stable under the action of $W$.*

**Definition 19.1.3** We define the **integral dominant chamber** $C_{\mathbb{Z}}$ associated to $\mathfrak{h}_{\mathbb{Z}}$ to be the intersectin $C_{\mathbb{R}} \cap \mathfrak{h}_{\mathbb{Z}}$ where $C_{\mathbb{R}}$ is the dominant chamber of Definition 6.5.1.

**Definition 19.1.4** Define the **maximal torus** $T$ associated to $\mathfrak{h}_{\mathbb{Z}}$ by

$$T = \hom_{\mathbb{Z}}(\mathfrak{h}_{\mathbb{Z}}^*, \mathbb{C}^*).$$

It is a torus of dimension $\dim \mathfrak{h} = n + \mathrm{Corank}(A)$.

**Fact 19.1.5** *(i) The action of $W$ on $\mathfrak{h}_{\mathbb{Z}}^*$ induces an action of $W$ on $T$.*
*(ii) The Lie algebra* $\mathrm{Lie}(T)$ *can be identified with* $\hom_{\mathbb{Z}}(\mathfrak{h}_{\mathbb{Z}}^*, \mathbb{C}) \simeq \mathfrak{h}$.
*(iii) The exponential map $\mathfrak{h} \to T$ is given by $h \mapsto e^h$ where $e^h(\lambda) = e^{\langle h, \lambda \rangle}$ for $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$.*

**Definition 19.1.6** For $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, define the **character of $T$ associated to $\lambda$** by $e^\lambda : T \to \mathbb{C}^*$ with $e^\lambda(t) = t(\lambda)$.

Let $X(T)$ be the group of characters of $T$ (algebraic group morphisms $T \to \mathbb{C}^*$). The classical study of diagonalisable algebraic groups gives us the following:

**Fact 19.1.7** *The map $\mathfrak{h}^*_{\mathbb{Z}} \to X(T)$ defined by $\lambda \mapsto e^\lambda$ is an isomorphism of $\mathbb{Z}$-modules.*

**Lemma 19.1.8** *Let $\pi : \mathfrak{h} \to \mathrm{End}(V)$ be a representation which is a weight module (i.e. direct sum of eigenspaces $V_\lambda$ for $\lambda \in \mathfrak{h}^*$) such that its weights are in $\mathfrak{h}^*_{\mathbb{Z}}$. Then the action of $\mathfrak{h}$ integrates to an action of $T$ on $V$.*

**Proof :** Define an action of $T$ as follows:

$$t \cdot v_\lambda = t(\lambda) v_\lambda \text{ for } t \in T \text{ and } v_\lambda \in V_\lambda.$$

Derivating this action gives back the action of $\mathfrak{h}$ (by derivating the exponential). □

**Definition 19.1.9** Let $N$ be the group generated by the set $T \cup \{\widetilde{s}_i\}_{i\in[1,n]}$ for some variables $\widetilde{s}_i$ modulo the relations

- $t't''t^{-1}$ for $t = t't''$ in $T$;
- $\widetilde{s}_i t \widetilde{s}_i^{-1} = s_i(t)$;
- $\widetilde{s}_i^2 = (-1)^{\alpha_i^\vee} = e^{i\pi\alpha_i^\vee} \in T$;
- $\widetilde{s}_i\widetilde{s}_j\widetilde{s}_i \cdots = \widetilde{s}_j\widetilde{s}_i\widetilde{s}_j \cdots$ for $i \neq j$ with $m_{i,j}$ factors on both side where $m_{i,j}$ is the order of $s_i s_j \in W$.

**Lemma 19.1.10** *Let $\pi : \mathfrak{g} \to \mathrm{End}(V)$ be an integrable representation such that all weigths lie in $\mathfrak{h}^*_{\mathbb{Z}}$. Then there exists an action of $N$ extending the action of $T$ such that $\widetilde{s}_i$ acts as*

$$s_i(\pi) = (\exp f_i)(\exp(-e_i)(\exp(f_i)).$$

**Proof :** This comes directly from Proposition 5.2.6 because the $s_i(\pi)$ satisfy the relations of the $\widetilde{s}_i$ above on any integrable representation $\pi : \mathfrak{g} \to \mathrm{End}(V)$. For the second relation, we have by Proposition 5.2.6 the fact that $\widetilde{s}_i(V_\lambda) = V_{s_i(\lambda)}$ thus for $v_\lambda \in V_\lambda$ we have

$$\widetilde{s}_i t \widetilde{s}_i^{-1} \cdot v_\lambda = \widetilde{s}_i \cdot (t(s_i(\lambda))\widetilde{s}_i^{-1}(v_\lambda)) = t(s_i(\lambda))v_\lambda = s_i(t)(\lambda)v_\lambda = s_i(t) \cdot v_\lambda.$$

□

**Corollary 19.1.11** *(ı) The canonical map $\theta : T \cup \{\widetilde{s}_i\} \to N$ is injective and we have an exact sequence of groups:*

$$1 \to T \overset{\theta|_T}{\to} N \overset{\pi}{\to} W \to 1.$$

*(ıı) The conjugation of $N$ on $T$ descend to an action of $W$ on $T$ which coincide with the previously defined action of $W$ on $T$.*

**Proof :** The second part is clear: since $T$ is abelian the action of $N$ descend to an action of $W$ which is the previously defined action.

For (1), the injectivity follows from the action of the elements $\widetilde{s}_i$ on the weights, the existence of the integrable representations $L(\lambda)$ for $\lambda \in C_{\mathbb{Z}}$ and the fact that the Weyl group is a subgroup of automorphism of $\mathfrak{h}$.

The map defined by $\pi(\widetilde{s}_i) = s_i$ and $T \subset \ker \pi$ extends to a group morphism $\pi : N \to W$ because of the defining relations of $N$ (these are the defining relations of $W$ except for the squares). But now the quotient $N/T$ has the elements $\widetilde{s}_i T$ a generators with relations $(\widetilde{s}_i T)^2 = 1$ and $((\widetilde{s}_i T)\widetilde{s}_j T))^{m(i,j)} = 1$ which are send to the $s_i$. But the $s_i$ satisfy these relation and $\bar{\pi} : N/T \to W$ is an isomorphism. □

## 19.2 The group $\mathcal{U}$

### 19.2.1 Completion

Let $\mathfrak{g}$ be the Kac-Moody Lie algebra associated to a generalised Cartan matrix $A$. Recall that we have a decomposition of $\mathfrak{g}$ as follows: $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and that $\mathfrak{n}_+$ (as well as $\mathfrak{n}_-$) has a decomposition in terms of eigenspaces for the action of $\mathfrak{h}$ given by

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$$

where $\Delta$ is the set of roots and $\Delta_+$ the set of positive roots. We shall denote in the sequel $\mathfrak{n}_+$ by $\mathfrak{n}$. We may consider the completion $\widehat{\mathfrak{n}}$ of $\mathfrak{n}$ defined by

$$\widehat{\mathfrak{n}} = \prod_{\alpha \in \Delta_+} \mathfrak{g}_\alpha.$$

We may also define a completion $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ by:

$$\widehat{\mathfrak{g}} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \widehat{\mathfrak{n}}.$$

Recall the following

**Fact 19.2.1** *The Lie bracket*

$$\left[\sum_{\alpha \in \Delta_+} x_\alpha, \sum_{\alpha \in \Delta_+} y_\alpha\right] = \sum_{\gamma \in \Delta_+} \sum_{\alpha+\beta=\gamma,\ (\alpha,\beta) \in \Delta_+^2} [x_\alpha, y_\beta]$$

*defines a Lie bracket on $\widehat{\mathfrak{n}}$. Furthermore, for $y \in \mathfrak{n}_- \oplus \mathfrak{h}$, define*

$$\left[\sum_{\alpha \in \Delta_+} x_\alpha, y\right] = \sum_{\alpha \in \Delta_+} [x_\alpha, y].$$

*These two formulas are well defined and define Lie algebra structures on $\widehat{\mathfrak{n}}$ and $\widehat{\mathfrak{g}}$.*

**Proof :** The sum inside the left hand side sum in the first formula is well defined because it is finite. For the second formula, we only need to define it for $y \in \mathfrak{h}$ or $y \in \mathfrak{g}_\beta$ with $\beta < 0$. For $y \in \mathfrak{h}$, then for all $\alpha \in \Delta_+$, the bracket $[x_\alpha, y] \in \mathfrak{g}_\alpha$ and the sum may be infinite but is an element of $\mathfrak{n}_+$. For $y \in \mathfrak{g}_\beta$ with $\beta \in \Delta_-$, then only finitely many $\alpha \in \Delta_+$ are such that $[x_\alpha, y] \neq 0$ and the sum is well defined.

These formula define Lie algebra structures, the anti-symmetry and the Jacobi formula follow from the same formulas for $\mathfrak{g}$. □

**Lemma 19.2.2** *The Lie algebra $\widehat{\mathfrak{n}}$ is a pro-nilpotent Lie algebrafor the defining set*

$$\mathcal{F}_{\widehat{\mathfrak{n}}} = \{\mathfrak{a} \text{ ideal of } \widehat{\mathfrak{n}} \text{ such that } \widehat{\mathfrak{n}} \supset \widehat{\mathfrak{n}}(k) \text{ for some } k > 0\}$$

*where we set*

$$\widehat{\mathfrak{n}}(k) = \prod_{\alpha \in \Delta_+, \mathrm{ht}(\alpha) \geq k} \mathfrak{g}_\alpha.$$

**Proof :** Let us first remark that $\widehat{\mathfrak{n}}(k)$ is an ideal of $\widehat{\mathfrak{n}}$. Furthermore, the quotient of $\widehat{\mathfrak{n}}$ by $\widehat{\mathfrak{n}}(k)$ is finite dimensional (isomorphic as vector space to the direct sum of the $\mathfrak{g}_\alpha$ for $\alpha \in \Delta_+$ with $\mathrm{ht}(\alpha) < k$).

The other axioms follow easily. The fact that it is nilpotent comes from the same fact for $\mathfrak{n}$. □

**Remark 19.2.3** We could do the same for $\widehat{\mathfrak{g}}$ but it would not be a pro-Lie-algebra because the quotients $\widehat{\mathfrak{g}}/\widehat{\mathfrak{n}}(k) = \mathfrak{g}/\mathfrak{n}(k)$ are not finite dimensional (with $\mathfrak{n}(k)$ defined by the direct sum instead of the product). However, we still have the equality

$$\widehat{\mathfrak{g}} = \lim_{\leftarrow} \widehat{\mathfrak{g}}/\widehat{\mathfrak{n}}(k).$$

In particular, we may define the pro-topology on $\mathfrak{g}$.

**Definition 19.2.4** We define the group $\mathcal{U}$ as the pro-unipotent group structure on the pro-nilpotent Lie algebra $\widehat{\mathfrak{n}}$.

**Definition 19.2.5** (ı) A subset $\Theta$ of $\Delta_+$ is called bracket closed if for any two elements $\alpha$ and $\beta$ in $\Theta$, we have $\alpha + \beta$ in $\Theta$.

(ıı) A subset $\Theta$ of $\Delta_+$ is called bracket coclosed if $\Theta^c$ is bracket closed.

**Fact 19.2.6** *(ı) For any $x \in \widehat{\mathfrak{n}}$, the vector space generated by $x$ is a pro-Lie-subalgebra of $\widehat{\mathfrak{n}}$.*

*(ıı) Let $\Theta$ be a bracket closed subset of $\Delta_+$, then the subspace*

$$\widehat{\mathfrak{n}}_\Theta = \prod_{\alpha \in \Theta} \mathfrak{g}_\alpha$$

*of $\widehat{\mathfrak{n}}$ is a pro-Lie-subalgebra of $\widehat{\mathfrak{n}}$.*

**Proof :** Because $\Theta$ is bracket closed, this subspaces is a sub-Lie-algebra. Furthermore, we clearly have an isomorphism

$$\widehat{\mathfrak{n}}_\Theta \simeq \lim_{\leftarrow} \widehat{\mathfrak{n}}_\Theta/(\widehat{\mathfrak{n}}_\Theta \cap \widehat{\mathfrak{n}}(k))$$

proving (by Proposition 18.4.5) that $\widehat{\mathfrak{n}}_\Theta$ is a pro-subalgebra. □

**Definition 19.2.7** (ı) For $x \in \widehat{\mathfrak{n}}$, define $\mathcal{U}_x$ to be the pro-unipotent group structure on the pro-nilpotent Lie subalgebra generated by $x$ in $\widehat{\mathfrak{n}}$.

(ıı) For $\Theta$ a bracket closed set, defined $\mathcal{U}_\Theta$ to be the pro-unipotent group structure on the pro-nilpotent Lie algebra $\widehat{\mathfrak{n}}_\Theta$.

**Fact 19.2.8** *We may define $\mathcal{U}_x$ and $\mathcal{U}_\Theta$ in $\mathcal{U}$ via the exponential map:*

$$\mathcal{U}_x = \exp(\mathbb{C}x) \quad \text{and} \quad \mathcal{U}_\Theta = \exp(\widehat{\mathfrak{n}}_\Theta).$$

**Proof :** Let $\mathfrak{U}'_\Theta$ be the subgroup of $\mathfrak{U}$ defined as the image of $\widehat{\mathfrak{n}}_\Theta$ by the exponential map. Because the exponential map is the identity, we have a bijection between $\mathfrak{U}_\Theta$ and $\mathfrak{U}'_\Theta$ as sets. Furthermore, the natural inclusion $\widehat{\mathfrak{n}}_\Theta \to \widehat{\mathfrak{n}}$ induces a pro-group inclusion $\mathfrak{U}_\Theta \to \mathfrak{U}$ whose image is $\mathfrak{U}'_\Theta$. We thus have a pro-group morphism which is bijective, it is a pro-group isomorphism. □

**Lemma 19.2.9** *(ı) Assume that for all $\alpha \in \Delta_+$ and all $\beta \in \Theta$, we have $(\alpha + \beta \in \Delta_+ \Rightarrow \alpha + \beta \in \Theta)$. Then $\mathfrak{U}_\Theta$ is normal in $\mathfrak{U}$.*

*(ıı) Assume that $\Theta$ is also bracket coclosed, then the multiplication map*

$$\mathfrak{U}_\Theta \times \mathfrak{U}_{\Delta_+ \setminus \Theta} \to \mathfrak{U}$$

*is a bijection.*

**Proof :** (ı) The condition tells that $\widehat{\mathfrak{n}}_\Theta$ is an ideal. Its image by the exponential map is therefore normal.

(ıı) Because exp is bijective and $\widehat{\mathfrak{n}}_\Theta \cap \widehat{\mathfrak{n}}_{\Delta_+\setminus\Theta} = \{0\}$ we get that $\mathfrak{U}_\Theta \cap \mathfrak{U}_{\Delta_+\setminus\Theta} = \{1\}$ and the multiplication is injective.

To prove the surjectivity, we proceed by induction on the height of the elements. Remark that the multiplication is given by the Campbell-Hausdorff formula: $x \cdot y = H(x, y)$. Let $x \in \widehat{\mathfrak{n}}$ and write $x = \sum_\alpha x_\alpha$ with $x_\alpha \in \mathfrak{g}_\alpha$. We prove by induction on $k$ the existence of elements $y_k \in \widehat{\mathfrak{n}}_\Theta$ and $z_k \in \widehat{\mathfrak{n}}_{\Delta\setminus\Theta}$ such that

- $(y_k)_\beta = (z_k)_\beta = 0$ for $\mathrm{ht}(\beta) > k$;
- $H(y_k, z_k)_\beta = x_\beta$ for $\mathrm{ht}(\beta) \leq k$;
- $(y_k)_\beta = (y_{k-1})_\beta$ and $(z_k)_\beta = (z_{k-1})_\beta$ for $\mathrm{ht}(\beta) < k$.

We may start the induction with $y_0 = z_0 = 0$. Set

$$y_{k+1} = y_k + \sum_{\beta\in\Theta,\ \mathrm{ht}(\beta)=k+1} (x_\beta - H(y_k, z_k)_\beta)$$

$$z_{k+1} = z_k + \sum_{\beta\in\Delta_+\setminus\Theta,\ \mathrm{ht}(\beta)=k+1} (x_\beta - H(y_k, z_k)_\beta).$$

Let us now one more time recall the Campell-Hausdorff formula:

$$H(X, Y) = \log(\exp X \exp Y) = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i+s_i>0 \\ 1\leq i\leq n}} \frac{(\sum_{i=1}^n (r_i + s_i))^{-1}}{r_1!s_1! \cdots r_n!s_n!} [X^{r_1}Y^{s_1}X^{r_2}Y^{s_2} \ldots X^{r_n}Y^{s_n}],$$

which uses the notation

$$[X^{r_1}Y^{s_1} \ldots X^{r_n}Y^{s_n}] = [\underbrace{X, [X, \ldots [X,}_{r_1} \underbrace{[Y, [Y, \ldots [Y,}_{s_1} \cdots \underbrace{[X, [X, \ldots [X,}_{r_n} \underbrace{[Y, [Y, \ldots Y}_{s_n}]] \ldots]].$$

Let us prove our induction: the first and last conditions are easily satisfied. To prove our induction, let us compute $H(y_{k+1}, z_{k+1})$. We obtain the following types of terms:

- terms with only $y_k$ and $z_k$, the sum of these terms is $H(y_k, z_k)$;
- the linear part which is $y_{k+1} + z_{k+1}$ (and contains the linear part of $H(y_k, z_k)$);

- other terms involving $y_k$ or $z_k$ and terms of the form $(x_\beta - H(y_k, z_k)_\beta)$.

For the last type of terms, the height of the roots is at least $k+2$ so they do not enter in the induction hypothesis. We need only to care of the formula

$$H(y_{k+1}, z_{k+1}) = H(y_k, z_k) + \sum_{\beta\in\Delta_+}(x_\beta - H(y_k, z_k)_\beta) + \text{ terms of higher height.}$$

The induction follows. We have the formula

$$H(\sum_{k=0}^{\infty}(y_{k+1} - y_k) + \sum_{k=0}^{\infty}(z_{k+1} - z_k)) = x$$

and the surjectivity follows. □

**Example 19.2.10** (ı) Let $\alpha \in \Delta_+$ be a real root, then $\{\alpha\}$ is bracket closed and in this case we have for $x \in \mathfrak{g}_\alpha$ the equality $\mathcal{U}_x = \mathcal{U}_{\{\alpha\}}$ and is denoted $\mathcal{U}_\alpha$.

(ıı) If $\alpha$ is a simple root, then $\Delta_+ \setminus \{\alpha\}$ is bracket closed and coclosed. Moreover, for any $\beta \in \Delta_+ \setminus \{\alpha\}$ and any $\gamma \in \Delta_+$, we have $\beta + \gamma \in \Delta_+ \setminus \{\alpha\}$. The group $\mathcal{U}$ is the semi-direct product of the normal subgroup $\mathcal{U}_{\Delta_+\setminus\{\alpha\}}$ and the group $\mathcal{U}_\alpha$.

(ııı) Let $\Delta_w = \{\alpha \in \Delta_+ \ / \ w^{-1}(\alpha) < 0\}$. We saw that $\Delta_w$ is of cardinality $\ell(w)$. Furthermore, we easily have that $\Delta_w$ is bracket closed and coclosed. In particular $\widehat{\mathfrak{n}}_{\Delta_w}$ is a finite dimensional nilpotent Lie algebra and $\mathcal{U}_{\Delta_w}$ is an algebraic unipotent group.

## 19.3 Parabolic subgroups

### 19.3.1 Parabolic subalgebra associated to a subset of the simple roots

**Definition 19.3.1** (ı) Let $X$ be a subset of $\Pi$ the set of simple roots, we denote by $\Delta_X$ the intersection of the root system $\Delta$ with the sublattice $\oplus_{\alpha\in\Pi}\mathbb{Z}\alpha$. We define the positive and negative roots of $\Delta_X$ by intersection with the set of positive and negative roots:

$$\Delta_{X,+} = \Delta_X \cap \Delta_+ \text{ and } \Delta_{X,-} = \Delta_X \cap \Delta_-.$$

(ıı) We define the following Lie subalgebras of $\mathfrak{g}$:

$$\begin{aligned}
&\mathfrak{g}_X = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta_X}\mathfrak{g}_\alpha),\\
&\mathfrak{u}_X = \mathfrak{u}_X^+ = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta_+\setminus\Delta_{X,+}}\mathfrak{g}_\alpha),\\
&\mathfrak{u}_X^- = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta_-\setminus\Delta_{X,-}}\mathfrak{g}_\alpha),\\
&\mathfrak{p}_X = \mathfrak{g}_X \oplus \mathfrak{u}_X, \text{ and}\\
&\mathfrak{p}_X^- = \mathfrak{g}_X \oplus \mathfrak{u}_X^-.
\end{aligned}$$

**Remark 19.3.2** It is easy to check that $\Delta_X$ is bracket closed so that $\mathfrak{g}_X$ is a subalgebra. Furthermore, the same is true for $\Delta_+ \setminus \Delta_{X,+}$ so that $\mathfrak{u}_X$ is a Lie subalgebra. Furthermore, we easily check that $\mathfrak{g}_X$ normalises $\mathfrak{u}_X$ so that $\mathfrak{p}_X$ is also a Lie subalgebra (this means that $\mathfrak{u}_X$ is normal in $\mathfrak{p}_X$ or that the sets of roots $\Theta = \Delta_+ \setminus \Delta_{X,+}$ satisfies the hypothesis of Lemma 19.2.9 (ı)).

**Definition 19.3.3** We say that $X$ is of finite type if $\mathfrak{g}_X$ is finite dimensional.

We may easily check the following fact:

**Fact 19.3.4** *The subset $X$ of $\Pi$ is of finite type if and only if the subdiagram it generates in the Dynkin diaram of $\mathfrak{g}$ is a union of Dynkin diagrams of finite type.*

### 19.3.2 Completed parabolic subalgebra

Let $X$ be a subset of $\Pi$. We define as for the nilpotent subalgebra $\mathfrak{n}$ a completion $\widehat{\mathfrak{u}}_X$ of $\mathfrak{u}_X$ as follows

$$\widehat{\mathfrak{u}}_X = \prod_{\alpha\in\Delta_+\setminus\Delta_{X,+}} \mathfrak{g}_\alpha = \widehat{\mathfrak{n}}_{\Delta_+\setminus\Delta_{X,+}}.$$

It is a Lie subalgebra because $\Delta_+ \setminus \Delta_{X,+}$ is bracket closed. Let us further define

$$\widehat{\mathfrak{p}}_X = \mathfrak{g}_X \oplus \widehat{\mathfrak{u}}_X.$$

Which is a Lie subalgebra of $\widehat{\mathfrak{g}}$ (the sets of roots $\Theta = \Delta_+ \setminus \Delta_{X,+}$ satisfies the hypothesis of Lemma 19.2.9 (ı)) and $\widehat{\mathfrak{u}}_X$ is a normal subalgebra.

We want to define a pro-Lie-algebra structure on $\mathfrak{p}_X$. For this let us define a slightly modified height denoted by $\mathrm{ht}_X$ as follows: let us write an element $\beta \in Q = \oplus_{\alpha\in\Pi}\mathbb{Z}\alpha$ as $\beta = \sum_\alpha n_\alpha \alpha$, then

$$\mathrm{ht}_X(\beta) = \sum_{\alpha\notin X} n_\alpha.$$

In particular, for $\beta \in \Delta_X$, we have $\mathrm{ht}_X(\beta) = 0$. Now as for $\widehat{\mathfrak{n}}$ define the ideal

$$\widehat{\mathfrak{u}}_X(k) = \prod_{\beta\in\Delta_{X,+},\ \mathrm{ht}_X(\beta)\geq k} \mathfrak{g}_\beta$$

and the family $\mathcal{F}_X$ of ideal in $\widehat{\mathfrak{u}}_X$ by

$$\mathcal{F}_X = \{\mathfrak{a} \text{ ideal of } \widehat{\mathfrak{p}}_X \ / \ \exists k > 0, \text{ with } \mathfrak{a} \supset \widehat{\mathfrak{u}}_X(k)\}.$$

**Lemma 19.3.5** *Let $X$ be of finite type.*

*(ı) The vector space $\widehat{\mathfrak{u}}_X(k)$ is an ideal of $\widehat{\mathfrak{p}}_X$. Furthermore, the quotient space $\widehat{\mathfrak{u}}_X(k)/\widehat{\mathfrak{u}}_X(k+1)$ is finite dimensional*

*(ıı) Then $\mathcal{F}_X$ defines a pro-Lie-algebra structure on $\widehat{\mathfrak{p}}_X$.*

*(ııı) Then $\widehat{\mathfrak{n}}$ is a pro-Lie-subalgebra of $\widehat{\mathfrak{p}}_X$ and $\widehat{\mathfrak{u}}_X$ a pro-Lie-subideal.*

**Proof :** (ı) The fact that it is an ideal is clear. Furthermore, if $(x_j)$ is a base for $\widehat{\mathfrak{u}}_X(k-1)/\widehat{\mathfrak{u}}_X(k)$, then the brackets $([x_j, e_\alpha])$ for $\alpha \notin X$ form a basis for $\widehat{\mathfrak{u}}_X(k)/\widehat{\mathfrak{u}}_X(k+1)$ (because $\mathfrak{n}$ is generated by the $e_\alpha$ for $\alpha$ a simple root).

(ıı) The finitness condition is necessary for the quotient $\widehat{\mathfrak{p}}_X/\widehat{\mathfrak{u}}_X(k)$ to be finite. The condition of a pro-Lie-algebra structure follow easily.

(ııı) The fact that $\widehat{\mathfrak{n}}$ is contained in $\widehat{\mathfrak{p}}_X$ comes from the fact that $X$ is of finite type: the part of $\widehat{\mathfrak{n}}$ not contained in $\widehat{\mathfrak{u}}_X$ is contained in $\mathfrak{g}_X$ (because $X$ is of finite type, the completion has no effect on this part). The fact that these subalgebras are closed is easily checked on the projections under the ideal $\widehat{\mathfrak{u}}_X(k)$. □

### 19.3.3 Parabolic subgroups

Let $X$ a subset of $\Pi$ of finite type. Because it is of finite type, all its root are real and we may define the set $\Delta_X^\vee$ as the set of the coroots of $\Delta_X$. The classical theory of algebraic groups tells us that there exists a unique connected reductive algebraic group $G_X$ associated to the quadruple $(\mathfrak{h}_\mathbb{Z}, \Delta_X, \mathfrak{h}_\mathbb{Z}^*, \Delta_X^\vee)$. For this group $G_X$, the following conditions are satisfied

**Fact 19.3.6** *(ı) The torus* $T = \hom_{\mathbb{Z}}(\mathfrak{h}_{\mathbb{Z}}^*, \mathbb{C}^*)$ *is a maximal torus of* $G_X$.

*(ıı) We have the equalities* $\mathrm{Lie}(G_X) = \mathfrak{g}_X$ *and* $\mathrm{Lie}(T) = \mathfrak{h}$.

*(ııı) For any finite dimensional* $(\mathfrak{g}_X, T)$*-module* $V$ *defined by the maps* $\rho : \mathfrak{g}_X \to \mathrm{End}(V)$ *and* $\epsilon : T \to \mathrm{Aut}(V)$*, there exists a respresentation* $\pi : G_X \to \mathrm{Aut}(V)$ *such that* $\dot{\pi} = \rho$ *and* $\pi|_T = \epsilon$.

As a consequence, the finite dimensional space

$$\widehat{\mathfrak{u}}_X(k)/\widehat{\mathfrak{u}}_X(k+1) = \bigoplus_{\beta \in \Delta_{X,+},\ \mathrm{ht}_X(\beta)=k} \mathfrak{g}_\beta$$

being a $(\mathfrak{g}_X, T)$-module (the action of $T$ is defined by integration as in Lemma 19.1.8), it is also a $G_X$-module. But we have

$$\widehat{\mathfrak{u}}_X = \bigoplus_{k \geq 1} \widehat{\mathfrak{u}}_X(k)/\widehat{\mathfrak{u}}_X(k+1)$$

and $\mathfrak{u}_X$ has therefore also a $G_X$-module structure.

**Fact 19.3.7** *For any* $g \in G_X$*, the element* $g$ *acts on* $\widehat{\mathfrak{u}}_X$ *as a pro-Lie-algebra automorphism.*

**Proof :** Let $x \in \mathfrak{g}_X$ and $u \in \widehat{\mathfrak{u}}_X(k)/\widehat{\mathfrak{u}}_X(k+1)$ and $v \in \widehat{\mathfrak{u}}_X(l)/\widehat{\mathfrak{u}}_X(l+1)$. Consider the function $f : \mathbb{R} \to \widehat{\mathfrak{u}}_X(k+l)/\widehat{\mathfrak{u}}_X(k+l+1)$ defined by

$$f(t) = [\exp(tx) \cdot u, \exp(tx) \cdot v] - \exp(tx) \cdot [u, v].$$

We may compute

$$\begin{aligned} \frac{df}{dt}(t) &= [[x, \exp(tx) \cdot u], \exp(tx) \cdot v] + [\exp(tx) \cdot u, [x, \exp(tx) \cdot v]] - [x, \exp(tx) \cdot [u, v]] \\ &= [x, [\exp(tx) \cdot u, \exp(tx) \cdot v]] - [x, \exp(tx) \cdot [u, v]] \\ &= [x, f(t)]. \end{aligned}$$

Furthermore, we have $f(0) = 0$ and by unicity of the solution of this differential equation, we obtain $f(t) = 0$ for all $t$ and the result follows for the image of the exponential. We conclude by the fact that the image of the exponential generates the group $G_X$.

The fact that the action is continuous for the pro-topology is clear, it is algebraic on each quotient $\widehat{\mathfrak{u}}_X(k)/\widehat{\mathfrak{u}}_X(k+1)$. □

By virtue of Theorem 18.6.2, any element $g \in G_X$ defines an element in $\mathrm{Aut}(\widehat{\mathfrak{u}}_X)$ and thus of $\mathrm{Aut}(\mathcal{U}_X)$ where $\mathcal{U}_X$ is the pro-unipotent group associated to $\widehat{\mathfrak{u}}_X$. We thus have a group morphism

$$\phi_X : G_X \to \mathrm{Aut}(\mathcal{U}_X).$$

**Definition 19.3.8** Let $X$ be a subset of finite type in $\Pi$, we define the standard parabolic group $Q_X$ as the semi-direct product

$$Q_X = \mathcal{U}_X \rtimes G_X$$

with the product given by

$$(u_1, g_1) \cdot (u_2, g_2) = (u_1 \phi_(g_1) \cdot u_2, g_1 g_2).$$

Let us denote by $\mathcal{U}_X(k)$ the pro-unipotent group associated to the pro-nilpotent Lie algebra $\mathfrak{u}_X(k)$, it is a pro-subgroup of $\mathcal{U}_X$. It is a subgroup in $Q_X$ which is normal and stable under the action of $G_X$.

**Lemma 19.3.9** *(i) The group $Q_X$ is a pro-group with defining set*

$$\mathcal{F}_X = \{N \subset Q_X \textit{ normal } / \exists\ k \text{ with } N \supset \mathcal{U}_X(k) \textit{ and } N/\mathcal{U}_X(k) \textit{ is closed in } (\mathcal{U}_X/\mathcal{U}_X(k)) \rtimes G_X\}.$$

*(ii) The subgroup $\mathcal{U}_X$ is a normal pro-subgroup of $Q_X$ and the pro-group structure coincide with the previously defined structure on $\mathcal{U}_X$.*

*(iii) We have* $\mathrm{Lie}(Q_X) = \widehat{\mathfrak{p}}_X$.

**Proof :** (i) The quotients are clearly algebraic groups and $Q_X$ is the projective limit of these quotients.

(ii) The subgroup $\mathcal{U}_X$ is clearly normal. Furthermore, its induced pro-group structure is defined by the subgroups $\mathcal{U}_X(k)$ as was defined its previously defined pro-group structure (as exponentiation of the Lie algebra $\widehat{\mathfrak{n}}_X$).

(iii) We only need to compute the Lie algebras of the groups $(\mathcal{U}_X/\mathcal{U}_X(k)) \rtimes G_X$. Their Lie algebras are the direct sum $\widehat{\mathfrak{u}}_X/\widehat{\mathfrak{u}}_X(k) \oplus \mathfrak{g}_X$ and the result follows by definition of $\widehat{\mathfrak{p}}_X$. □

**Lemma 19.3.10** *Let $X_1 \subset X_2$ be two subsets of* $\Pi$ *of finite type. Then there is a canonical inclusion $Q_{X_1} \subset Q_{X_2}$ giving on the Lie algebra level the inclusion $\widehat{\mathfrak{p}}_{X_1} \subset \widehat{\mathfrak{p}}_{X_2}$.*

**Proof :** Consider the nilpotent Lie subalgebra $\mathfrak{a} = \mathfrak{u}_{\Delta_{X_2,+} \setminus \Delta_{X_1,+}}$ of $\mathfrak{g}_{X_2}$. Denote by $A = \mathcal{U}_{\Delta_{X_2,+} \setminus \Delta_{X_1,+}}$ the associated unipotent subgroup of $G_{X_2}$. Consider the pro-subgroup $\mathcal{U}_{X_2} \rtimes A$ of $Q_{X_2}$. Since it has the same Lie algebra as $\mathcal{U}_{X_1}$ in $Q_{X_1}$, we identify $\mathcal{U}_{X_2} \rtimes A$ with $\mathcal{U}_{X_1}$. But now $G_{X_1}$ is a subgroup of $G_{X_2}$ acting on $\mathcal{U}_{X_2} \rtimes A$ as it acts on $\mathcal{U}_{X_1}$ and the semi-direct product $(\mathcal{U}_{X_2} \rtimes A) \rtimes G_{X_1}$ is a pro-subgroup of $Q_{X_2}$. We have the inclusions:

$$Q_{X_1} = \mathcal{U}_{X_1} \rtimes G_{X_1} = (\mathcal{U}_{X_2} \rtimes A) \rtimes G_{X_1} = \mathcal{U}_{X_2} \rtimes (A \rtimes G_{X_1}) \subset Q_{X_2}.$$

□

In particular, let us denote by $B$ the group $Q_\emptyset$, the $B$ is contained in all the groups $Q_X$ for $X \subset \Pi$ of finite type. As a special case, let us denote by $Q_\alpha$ the group $Q_{\{\alpha\}}$ for $\alpha \in \Pi$, then $B$ is contained in $Q_\alpha$ for all $\alpha \in \Pi$.

## 19.4 The Kac-Moody group

### 19.4.1 Definition

We produced the group $B$ and the groups $Q_\alpha$ for any $\alpha \in \Pi$. Let us denote by $\gamma_\alpha$ the inclusion of $B$ in $Q_\alpha$. Let us now define for any $\alpha \in \Pi$ the group $N_\alpha$ by

$$N_\alpha = T \cup \widetilde{s}_\alpha T.$$

It is the subgroup of $N$ generated by $T$ ans $\widetilde{s}_\alpha$. Set $G_\alpha = G_{\{\alpha\}}$. We may define an embedding

$$\theta_\alpha : N_\alpha \to G_\alpha \subset Q_\alpha$$

as follows: define $\theta_\alpha|_T = \mathrm{Id}$ and $\theta_\alpha(\widetilde{s}_\alpha) = \exp(f_\alpha)\exp(-e_\alpha)\exp(f_\alpha) \in G_\alpha$ where $\exp : \mathfrak{g}_\alpha \to G_\alpha$. From Proposition 5.2.6 (iii) we get that this defines a group morphism. Furthermore, by Proposition 5.2.6 (i) we see that the weights of elements of $N_\alpha$ on representations of $G_\alpha$ are all distinct and thus $\theta_\alpha$ is injective.

**Definition 19.4.1** We define the set $Z$ as the quotient of the disjoint union

$$N \coprod_{\alpha\in\Pi} Q_\alpha$$

by the equivalence relation $\sim$ generated by the following conditions:

- $\gamma_\alpha(b) \sim \gamma_\beta(b)$ for all $\alpha$ and $\beta$ in $\Pi$ and $b \in B$;
- $n \sim \theta_\alpha(n)$ for all $\alpha \in \Pi$ and $n \in N_\alpha$.

We have that $B$ and $N$ injects in $Z$. Furthermore, for any $\alpha \in \Pi$ the group $Q_\alpha$ injects in $Z$.

**Fact 19.4.2** *In $Z$, we have $B \cap N = T$.*

**Proof :** The inclusion of $T$ in the intersection is clear. To prove the converse, we only need to prove the inclusion $\theta_\alpha(N_\alpha \setminus T) \subset Q_\alpha \setminus B$. This follows from Proposition 5.2.6 (ı). $\square$

**Definition 19.4.3** The Kac-Moody group $G$ associated to the generalised Cartan matrix $A$ is the amalgamated product of the system of groups $(N, Q_\alpha, \alpha \in \Pi)$ in $Z$.

### 19.4.2 Bruhat decomposition

**Theorem 19.4.4** *(ı) The canonical map $Z \to G$ is injective. In particular, the canonical group morphisms $Q_\alpha \to G$ and $N \to G$ are injective.*

*(ıı) Let $S = \{\widetilde{s}_\alpha T \in N/T\}$, the quadruple $(G, B, N, S)$ is a Tits system.*

**Proof :** We want to apply Theorem 17.4.7. Let us check the hypothesis $(P_i)$ for $i \in [1, 9]$. Let us recall these hypothesis:

$(P_1)$ For $s \neq s'$, $Q_s \cap Q_{s'} = B$.

$(P_2)$ The subgroup $T$ is normal in $N$.

$(P_3)$ For any $s \in S$, the quotient group $N_s/T$ is of order 2 denoted $\{1, s\}$.

$(P_4)$ $Q_s = B \cup BsB$.

$(P_5)$ The pair $(W, S)$ is a Coxeter system.

$(P_6)$ For any $n$ and any decomposition $n = n_1 \cdots n_r$ with $n_i \in N_{s_i}$ for some $s_i \in S$ such that $\pi(n) = \pi(n_1)\cdots\pi(n_r)$ is a reduced expression, the subgroup $B(n_1, \cdots, n_r)$ of $B$ depends only on $\pi(n)$ (and will be denoted $B_{\pi(n)}$) and the map $\gamma(n_1, \cdots, n_r) : B(n_1, \cdots, n_r) \to B$ depends only on $n$ (and will be denoted $\gamma_n$).

$(P_7)$ For $w \in W$ and $s \in S$ such that $\ell(ws) > \ell(w)$, we have $B_w \cdot B_s = B$.

$(P_8)$ Let $s$ and $t$ in $S$ and let $w \in W$ such that $sw = wt$ and $\ell(sw) > \ell(w)$. Then for any $m \in \pi^{-1}(s)$, $n \in \pi^{-1}(w)$ and $b \in B \setminus B_t$, there exist elements $y \in (bB_t) \cap B_w$ and $y', y'' \in B_w$ such that, setting $m' = n^{-1}m^{-1}n$, we have:

- $(m')^{-1}ym' = y'm'y''$ in $Q_t$ and
- $m\gamma_n(y)m^{-1} = \gamma_n(y')m^{-1}\gamma_n(y'')$ in $Q_s$.

$(P_9)$ The subgroup $B$ is not normal in $Q_s$ for any $s \in S$.

The condition $(P_1)$ is satisfied thanks to the definition of $Z$. The condition $(P_2)$ is satisfied by definition of $N$ as well as the condition $(P_3)$ (see above). The condition $(P_5)$ follows from Corollary 19.1.11.

For $(P_4)$ let us first remark that we have the Bruhat decomposition for $G_\alpha$ which is of rank one. It is given by

$$G_\alpha = (T\exp(\mathbb{C}e_\alpha)) \cup (\exp(\mathbb{C}e_\alpha) \cdot \widetilde{s}_\alpha T \exp(\mathbb{C}e_\alpha)).$$

The group $Q_\alpha$ being the semi-direct product of $G_\alpha$ with $\mathcal{U}_{\Delta_+ \setminus \{\alpha\}}$ and the group $B$ being the semi-direct product of $\mathcal{U}_{\Delta_+ \setminus \{\alpha\}}$ with $\mathcal{U}_\alpha = \exp(\mathbb{C}e_\alpha)$, the condition $(P_4)$ follows.

Let us prove $(P_6)$. First take $n \in T\widetilde{s}_\alpha$ with $\alpha \in \Pi$ and $\Theta$ a bracket closed subset in $\Delta_+$. We have the following equality in $Q_\alpha$:

$$\mathcal{U} \cap (n^{-1}\mathcal{U}_\Theta n) = \mathcal{U}_{s_\alpha(\Theta)\cap\Delta_+}.$$

This comes from the fact that $T$ stabilises all the subspaces $\mathfrak{g}_\beta$ and that the element $\widetilde{s}_\alpha$ acts on $T$ as $s_\alpha$.

Now write $n = n_1 \cdots n_k$ such that for all $i \in [1, k]$ there exists a simple root $\alpha_i$ with $n_i \in T\widetilde{s}_{\alpha_i}$. By successive application of the previous equality and because for $\ell(ws) > \ell(w)$ we have $ws(\Delta_+) \cap \Delta_+ \subset w(\Delta_+) \cap \Delta_+$ (see the proof of Proposition 6.2.9), we obtain:

$$B(n_1, \cdots, n_r) = T \cdot \mathcal{U}_{\pi(n)^{-1}\Delta_+ \cap \Delta_+}.$$

This proves that $B(n_1, \cdots, n_r)$ does only depend on $\pi(n)$.

By Lemma 19.1.8, we have that $\mathfrak{g}$ has a $T$-module structure and by Proposition 5.2.6 it extends to an action of $N$. For $n \in T\widetilde{s}_\alpha$ with $\alpha \in \Pi$ and $\Theta$ a bracket closed subset of $\Delta_+$ such that $s_\alpha(\Theta) \subset \Delta_+$, we have the commutative diagram:

$$\begin{array}{ccc} \widehat{\mathfrak{n}}_\Theta & \xrightarrow{n} & \widehat{\mathfrak{n}}_{s_\alpha(\Theta)} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ \mathcal{U}_\Theta & \xrightarrow{\gamma_n} & \mathcal{U}_{s_\alpha(\Theta)} \end{array}$$

where $\gamma_n$ is the conjugation by $n$ in $Q_\alpha$. Now write $n = n_1 \cdots n_r$ as before. The map $\gamma_n$ is the composition of the maps $\gamma_{n_i}$ and they correpond in the Lie algebra level to the action of the element $n$ which is well defined because $N$ acts on $\mathfrak{g}$. This finishes the proof of $(P_6)$.

To prove $(P_7)$, it is enough, thanks to the formula

$$B_w = T \cdot \mathcal{U}_{w^{-1}\Delta_+ \cap \Delta_+}$$

and Lemma 19.2.9 (ii) to prove that $(w^{-1}(\Delta_+) \cap \Delta_+) \cup (s(\Delta_+) \cap \Delta_+) = \Delta_+$. Let us write $s = s_\alpha$ for some $\alpha \in \Pi$. If this equality fails, then there exists $\beta \in \Delta_+$ with $s_\alpha(\beta) < 0$ and $w(\beta) < 0$. The first condition imposes $\beta = \alpha$ and the second gives $\ell(ws) < \ell(w)$ a contradiction.

For $(P_8)$, let $s = s_\alpha$ and $t = s_\beta$ be in $S$ for $\alpha$ and $\beta$ in $\Pi$. By the formula for $B_{\pi(n)}$ we obtain the equality

$$B_t = T\mathcal{U}_{\Delta_+ \setminus \{\beta\}}.$$

Let $b \in B \setminus B_t$ (this means that the coefficient $b_\beta$ of $b$ in the component $\mathcal{U}_\beta$ is different from 1). We look for an element $y \in bB_t \cap B_w$. But the only condition for $y$ to be in $bB_t$ is to ask $y_\beta = b_\beta$. There exists $z \in \mathbb{C}$ such that $y = \exp(ze_\beta) = b_\beta \in bB_t$. We have $y \in \mathcal{U}_\beta$. Furthermore, because $\ell(wt) > \ell(w)$ we have $w(\beta) > 0$ and because

$$B_w = T\mathcal{U}_{w^{-1}(\Delta_+)\cap\Delta_+}.$$

we have $y \in U_\beta \subset B_w$.

Now use the Bruhat decomposition in $G_\beta$ to obtain $y'$ and $y''$ in $T\mathfrak{U}_\beta \subset B_w$ such that

$$(m')^{-1}ym' = y'm'y''.$$

Now consider the isomorphism $\gamma_n : B_w \to B_{w^{-1}}$. Then we have, because $w(\beta) = \alpha$ (because $wtw^{-1} = s$), the inclusion:

$$\gamma_n(T\mathfrak{U}_\beta) \subset T\mathfrak{U}_\alpha.$$

Now it is a classical result that such a morphism extends to a group isomorphism $\gamma'_n : G_\beta \to G_\alpha$. Furthermore, we have for $x \in N_\beta$:

$$\gamma'_n(x) = nxn^{-1}.$$

Applying this to the previous equation we get

$$m\gamma_n(y)m^{-1} = \gamma_n(y')m^{-1}\gamma_n(y'')$$

where $m = n(m')^{-1}n^{-1}$.

For $(P_9)$, we only need to compute in $G_\alpha$ the following equality $\widetilde{s}_\alpha\mathfrak{U}_\alpha\widetilde{s}_\alpha = \mathfrak{U}_{-\alpha}$. In particular $B$ is not normal in $Q_\alpha$. □

**Definition 19.4.5** For any subset $X$ of $\Pi$, define the **standard parabolic subgroup** $P_X$ of $G$ by $P_X = BW_XB$. We denote by $P_\alpha$ the group $P_{\{\alpha\}}$ for $\alpha \in \Pi$.

By construction, we have $P_\alpha = Q_\alpha$.

**Proposition 19.4.6** *For any subset $X$ of $\Pi$ of finite type, there exists a unique isomorphism $f$ : $Q_X \to P_X$ such that $f|_{Q_\alpha} = \mathrm{Id}$.*

**Proof :** By definition we have $Q_X = \mathfrak{U}_X \rtimes G_X$. Furthermore, the group $G_X$ is a finite dimensional reductive algebraic group and in particular for $B_X$ a Borel subgroup, $N_X$ the normaliser of $T$, we have that $(G_X, B_X, N_X, X)$ is a Tits system.

**Fact 19.4.7** *The quadruple $(Q_X, \mathfrak{U}_X \rtimes B_X, N_X, X)$ is a Tits system.*

**Proof :** We need to check the axioms of a Tits system but $Q_X$ is generated by $\mathfrak{U}_X \rtimes B_X$ and $N_X$ because $B_X$ and $N_X$ generate $G_X$. Furthermore $\mathfrak{U}_X \rtimes B_X \cap N_X = B_X \cap N_X = T$ and is normal in $N_X$. The quotient $W_X = N_X/T$ is generated by $X$ and for $s \in X$ and $w \in W_X$, we have $sB_Xw \subset B_XswB_X \cup B_XwB_X$ thus because for any $w \in W_X$ we have $w\mathfrak{U}_Xw^{-1} = \mathfrak{U}_X$ we get

$$s\mathfrak{U}_X \rtimes B_Xw \subset \mathfrak{U}_X \rtimes B_Xsw\mathfrak{U}_X \rtimes B_X \cup \mathfrak{U}_X \rtimes B_Xw\mathfrak{U}_X \rtimes B_X.$$

Finally because $sB_Xs^{-1} \not\subset B_X$ we get $s\mathfrak{U}_X \rtimes B_Xs^{-1} \not\subset \mathfrak{U}_X \rtimes B_X$. □

The group $Q_X$ is in particular the amalgamated product of $N_X$ and its parabolic subgroups $Q_\alpha$ for $\alpha \in X$. The group $N_X$ is isomorphic to the subgroup $N'_X$ of $N$ generated by the classes $\widetilde{s}_\alpha$ for $\alpha \in X$. This implies that there exists a group morphism $f : Q_X \to G$ such that $f|_{Q_\alpha} = \mathrm{Id}_{Q_\alpha}$ and sends $N_X$ isomorphically on the subgroup $N'_X$ of $N$ (by sending the simple reflections $n_\alpha$ in $N_X$ to $\widetilde{s}_\alpha$).

Since $Q_X$ and $P_X$ are generated by the groups $N_X$ (resp. $N'_X$) and $Q_\alpha$ for $\alpha \in X$ we have $\mathrm{im} f = P_X$. Let $g \in \ker f$. By the Bruhat decomposition, there exists elements $b$ and $b'$ in $\mathfrak{U}_X \rtimes B_X$ and an element $n \in N_X$ such that $g = bnb'$. We thus have $f(bb') = f(n^{-1})$. This element is in $B \cap T$ thus $f(n) \in T$ i.e. $n \in T$. This implies that $g = bnb' \in \mathfrak{U}_X \rtimes B_X \subset Q_\alpha$ (for any $\alpha \in X$). But $f$ is injective on $Q_\alpha$ and $g = 1$. □

**Corollary 19.4.8** *For any subsets $X$ and $Y$ of $\Pi$, we have the decomposition:*

$$G = \coprod_{w \in W_X \backslash W / W_Y} P_X w P_Y.$$

*In particular, we have*

$$G = \coprod_{w \in W} BwB.$$

**Definition 19.4.9** ($\imath$) For any real root $\beta$, let us write $\beta = w(\alpha_i)$ with $w \in W$ and $\alpha_i$ simple. Let $n \in \pi^{-1}(w)$, we define

$$U_\alpha = nU_{\alpha_i}n^{-1} \subset G.$$

It is easily checked that this does not depend on the writing $\beta = w(\alpha_i)$ and on the choice of the element $n$.

($\imath\imath$) Let us define $\mathcal{U}_-$ to be the subgroup of $G$ generated by the groups $U_\alpha$ for $\alpha$ a negative real root. The group $T$ normalises $\mathcal{U}_-$ and we set $B_- = T \cdot U_-$.

By defining the refined Tits systems, we could even be more precise and prove a Birkhoff type decomposition:

**Theorem 19.4.10** *We have a decomposition*

$$G = \coprod_{n \in N} \mathcal{U}_- n \mathcal{U}$$

*and more generally for any $X \subset \Pi$:*

$$G = \coprod_{w \in W/W_X} \mathcal{U}_- w P_X.$$

## 19.5 Representations

In this section we give a brief overview of representation theory of Kac-Moody groups. We shall in particular use some of these representations to construct the homogeneous spaces under the Kac-Moody groups.

**Definition 19.5.1** ($\imath$) A representation $\pi : G \to \mathrm{Aut}(V)$ (resp. $\pi : \widehat{\mathfrak{g}} \to \mathrm{End}(V)$) is called a **pro-representation of** $G$ (resp. of $\widehat{\mathfrak{g}}$) if, for all index $i$, the representation $\pi|_{P_i}$ (resp. $\pi|_{\widehat{\mathfrak{p}}_i}$) is a pro-representation of the pro-group $P_i$ (resp. the pro-Lie-algebra $\widehat{\mathfrak{p}}_i$).

($\imath\imath$) We denote by $\mathfrak{M}(G)$ (resp. $\mathfrak{M}(\widehat{\mathfrak{g}})$) the category of pro-representations of $G$ (resp. $\widehat{\mathfrak{g}}$) where the morphisms are representation morphisms.

($\imath\imath\imath$) A $\widehat{\mathfrak{g}}$-representation is called a $(\widehat{\mathfrak{g}}, T)$**-representation** if the action of $\mathfrak{h}$ integrates in a locally finite action of $T$ (and hence has a weight decomposition with respect to $\mathfrak{h}$).

($\imath v$) We denote by $\mathfrak{M}_T(\widehat{\mathfrak{g}})$ the full subcategory of $\mathfrak{M}(\widehat{\mathfrak{g}})$ of $(\widehat{\mathfrak{g}}, T)$-representations.

**Remark 19.5.2** The categories $\mathfrak{M}(G)$, $\mathfrak{M}(\widehat{\mathfrak{g}})$ and $\mathfrak{M}_T(\widehat{\mathfrak{g}})$ are closed under taking direct sums, tensor products, sub-representations and quotient representations.

**Lemma 19.5.3** *($\imath$) Let $(V, \pi)$ be a pro-representation of the group $G$, then there exists a unique pro-representation $(V, \dot{\pi})$ of $\widehat{\mathfrak{g}}$ such that for any index $i$, we have $(\dot{\pi})|_{\widehat{\mathfrak{p}}_i} = \dot{\overline{\pi|_{P_i}}}$.*

*($\imath\imath$) The representation $\pi$ is uniquely determined by $\dot{\pi}$.*

*The representation $\dot{\pi}$ is called the* **derivative of** *$\pi$ and $\pi$ is called the* **integral** *of $\dot{\pi}$.*

**Proof :** (i) Let us set $\pi_i = \pi|_{P_i}$. Then for any indices $i$, $j$ and $k$ we have $\dot\pi_i|_{\mathfrak{b}} = \dot\pi_j|_{\mathfrak{b}}$ and $\dot\pi_i(e_k) = \dot\pi_j(e_k)$. We may also define

$$\dot\pi(f_i) = \dot\pi_i(f_i)$$

which defines $\dot\pi$.

(ii) For this point we need to remark that the groups $P_i$ are connected thus $\dot\pi_i$ determines $\pi_i$ and the result follows from the fact that the $P_i$ generate $G$. □

**Theorem 19.5.4** *(i) The category $\mathfrak{M}(G)$ is equivalent to the category $\mathfrak{M}_T(\widehat{\mathfrak{g}})$ under the functor $(V,\pi) \mapsto (V,\dot\pi)$.*

*(ii) The category $\mathfrak{M}_T(\widehat{\mathfrak{g}})$ consists of the $(\widehat{\mathfrak{g}},T)$-representations $(V,\pi)$ such that $V$ is an integrable $\mathfrak{g}$-representation and $(V,\pi|_{\widehat{\mathfrak{n}}})$ is a pro-representation of $\widehat{\mathfrak{n}}$.*

**Proof :** First remark that the image of this functor is indeed contained in $\mathfrak{M}_T(\widehat{\mathfrak{g}})$). Then starting from $(V,\pi) \in \mathfrak{M}_T(\widehat{\mathfrak{g}})$ we need to construct a representation $(V,\rho)$ with $\dot\rho = \pi$. By the previous Lemma, we only need to do so for the groups $P_i$ (and the representation $\pi|_{\widehat{\mathfrak{p}}_i}$ on their Lie algebras).

Because the representation of $\widehat{\mathfrak{p}}_i$ is a pro-representation, for any $v \in V$, their exists a finite dimensional sub-representation $W$ containing $v$ and with $\widehat{\mathfrak{u}}_i(k)$ vanishing on $W$ for some $k$. The quotient $\widehat{\mathfrak{p}}_i/\widehat{\mathfrak{u}}_i(k)$ is $\mathfrak{g}_i \oplus \widehat{\mathfrak{u}}_i/\widehat{\mathfrak{u}}_i(k)$ and in particular we have an action of $\mathfrak{g}_i$. Because the action of $\mathfrak{h}$ integrates in an action of $T$, we get an action of $G_i$ on $W$. On the other hand, we have a pro-action of $\widehat{\mathfrak{u}}_i$ and by the equivalence of categories between pro-nilpotent Lie algebras and pro-unipotent groups, we get a pro-representation of $\mathfrak{U}_i$ on $W$. These actions (of $G_i$ and $\mathfrak{U}_i$) give an action of $P_i$ on $W$. Because $V$ is spanned by such finite dimensional $W$ (and by uniqueness of integration of representations of $P_i$) we get an action of $P_i$ on $V$.

Finally because $(V,\pi)$ is a pro-representation of $\widehat{\mathfrak{g}}$, it is locally finite for $\mathfrak{g}_i$ and hence the action of $e_i$ and $f_i$ are locally nilpotent. The induced $\mathfrak{g}$-representation is thus integrable. But we have seen that this implies that $N$ acts on $V$ and as a consequence $G$ acts on $V$. By construction this is the inverse functor of derivation.

(ii) We already proved one direction. Assume $(V,\pi)$ is an integrable $\mathfrak{g}$-representation and a pro-$\widehat{\mathfrak{n}}$-representation. Then we know that for an integrable representation $V$, any element $v \in V$ and any finite dimensional subalgebra $\mathfrak{g}_i$ of $\mathfrak{g}$, the element $v$ lives in a finite dimensional sub-representation. In particular this gives a pro-representation structure for any $\widehat{\mathfrak{p}}_i$ and the result follows. □

**Remark 19.5.5** It is not known if a $(\widehat{\mathfrak{g}},T)$-representation which is an integrable $\mathfrak{g}$-representation is necessarily a pro-$\widehat{\mathfrak{n}}$-representation.

**Corollary 19.5.6** *Any integrable highest weight $\mathfrak{g}$-module $L(\lambda)$ with highest weight $\lambda \in C_{\mathbb{Z}}$ is in $\mathfrak{M}_T(\widehat{\mathfrak{g}})$ and has a $G$-representation structure.*

**Proof :** Such an highest weight module has a $T$-module structure integrating the $\mathfrak{h}$ action (because it has integral weights). Because the highest weight is killed by $\mathfrak{n}$ is has obviously a $\widehat{\mathfrak{g}}$-module structure extending the $\mathfrak{g}$-structure. It is thus a $(\widehat{\mathfrak{g}},T)$-module which is $\mathfrak{g}$-integrable and such that the action of $\widehat{\mathfrak{n}}$ is a pro-action. □

Now we want to define the adjoint representation. Let us first define a $T$-module structure on $\widehat{\mathfrak{g}}$ by

$$t \cdot \left(\sum_\alpha x_\alpha\right) = \sum_\alpha t(\alpha) x_\alpha$$

where $t \in T$ and $x_\alpha \in \mathfrak{g}_\alpha$. Furthermore, because $\mathfrak{g}$ is an integrable $\mathfrak{g}$-module (by the adjoint action), we get an action of $N$ on $\mathfrak{g}$.

**Fact 19.5.7** *The action of $N$ on $\mathfrak{g}$ extends to an action of $N$ on $\widehat{\mathfrak{g}}$.*

**Proof :** We need to extend the action of $s_i(\mathrm{ad}) = \exp(\mathrm{ad}(f_i))\exp(-\mathrm{ad}(e_i))\exp(\mathrm{ad}(f_i))$ from $\mathfrak{g}$ to $\widehat{\mathfrak{g}}$. But $s_i(\mathrm{ad})(\mathfrak{g}_\alpha) \subset \mathfrak{g}_{s_i(\alpha)}$ thus we want that the set $s_i(\Delta_+) \cap \Delta_-$ is finite. It is the case (it is only one root $-\alpha_i$). □

**Fact 19.5.8** *The space $\widehat{\mathfrak{g}}$ has a $P_i$-pro-representation structure extending the action of $N_i$ for all $i$. These actions coincide on $B$.*

**Proof :** Recall that we have a filtration of $\widehat{\mathfrak{p}}_i$ by the ideals $\widehat{\mathfrak{u}}_i(k)$. Setting $\widehat{\mathfrak{g}}_i(u) = \widehat{\mathfrak{g}}/\widehat{\mathfrak{u}}_i(k)$ we have by adjoint action a pro-representation of $\widehat{\mathfrak{p}}_i$ on that space. Furthermore, we have a $T$ action on that space thus we may integrate the action of $\mathfrak{g}_i$ into an action of $G_i$. Because it is a pro-representation of $\widehat{\mathfrak{p}}_i$ we have a pro-representation of $\widehat{\mathfrak{u}}_i$ and thus of $\mathcal{U}_i$. These action define a pro-representation of $P_i$ on $\widehat{\mathfrak{g}}_i(k)$ for all $k$. This action extends the action of $N_i$.

Now we have

$$\widehat{\mathfrak{g}} = \lim_{\leftarrow} \widehat{\mathfrak{g}}_i(k)$$

and taking the inverse limit of these representations gives the desired representation. The action being the integration of the adjoint action of $\widehat{\mathfrak{u}}_i$ on itself, we easily get that the restrictions on $B$ coincide. □

**Definition 19.5.9** We define the **adjoint representation** $\mathrm{Ad} : G \to \mathrm{Aut}(\widehat{\mathfrak{g}})$ to be the representation of $G$ induced by the above defined representations of $P_i$ and $N$ on $\widehat{\mathfrak{g}}$.

By the same argument as before we get

**Fact 19.5.10** *(ı) The action of $G$ on $\widehat{\mathfrak{g}}$ preserves the Lie algebra structure.*

*(ıı) For any $g \in G$ the map $\mathrm{Ad}(g) : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ is continuous and we have*

$$\dot{\pi}(\mathrm{Ad}(g) \cdot x) = \pi(g)\dot{\pi}(x)\pi(g)^{-1}.$$

**Proof :** (ı) The fact that the action preserves the Lie algebra structure is proved as in Fact 19.3.7.

(ıı) We prove this formula in several steps. Let us first start with $g = \exp(y)$ for $y \in \mathfrak{g}_i$ and $x \in \mathfrak{g}$. By Corollary 5.1.3 we have

$$\mathrm{Ad}(g) \cdot x = \mathrm{Ad}(\exp y) \cdot x = \exp(\mathrm{ad}(y)) \cdot x.$$

The map $\dot{\pi}$ being a Lie algebra morphism, we obtain

$$\dot{\pi}(\mathrm{Ad}(g) \cdot x) = \exp(\mathrm{ad}(\dot{\pi}(y)))\dot{\pi}(x)$$

and again by Corollary 5.1.3

$$\dot{\pi}(\mathrm{Ad}(g) \cdot x) = \exp(\dot{\pi}(y))\dot{\pi}(x)\exp(-\dot{\pi}(y)) = \pi(g)\dot{\pi}(x)\pi(g^{-1}).$$

Now for $g \in P_i$ and $x \in \widehat{\mathfrak{p}}_i$, we proved (for pro-groups) that the formula holds. This implies in particular that the formula holds for $g = \exp(y)$ with $y \in \mathfrak{g}_i$ and $x \in \widehat{\mathfrak{g}}$.

Let us now define the following set:

$$\widehat{\mathfrak{g}}_g = \{x \in \widehat{\mathfrak{g}} \ / \ \dot{\pi}(\mathrm{Ad}(g) \cdot x) = \pi(g)\dot{\pi}\pi(g^{-1})\}.$$

It is a Lie subalgebra of $\widehat{\mathfrak{g}}$ since $\mathrm{Ad}(g)$ acts has a Lie algebra automorphism and because $B$ is contained in all $P_i$ we have that for $g \in B$ the algebra $\widehat{\mathfrak{g}}_g$ contains all the $\widehat{\mathfrak{p}}_i$ hence is equal to $\widehat{\mathfrak{g}}$. In particular the formula holds for $g \in B$ and $x \in \widehat{\mathfrak{g}}$.

The group $G$ being generated by $B$ and the elements $\exp(y)$ for $y \in \mathfrak{g}_i$, the result follows. □

**Definition 19.5.11** (i) For $V$ a vector space, we denote by $\mathrm{End}_{lf}(V)$ the set of locally finite endomorphisms of $V$.

(ii) We define the following subset of $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{g}}_{fin} = \bigcup_{X\subset\Pi}\bigcup_{g\in G}(\mathrm{Ad}\, g)(\widehat{\mathfrak{p}}_X)$$

where the first union runs over subsets $X$ is of finite type.

**Proposition 19.5.12** *(i) Let $(V,\pi)$ be a pro-representation of $G$, then the map $\dot{\pi} : \widehat{\mathfrak{g}} \to \mathrm{End}(V)$ sends $\widehat{\mathfrak{g}}_{fin}$ to $\mathrm{End}_{lf}(V)$.*

*(ii) There exists a unique map $\exp : \widehat{\mathfrak{g}}_{fin} \to G$ such that for any pro-representation $(V,\pi)$ of $G$, the following diagram is commutative:*

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_{fin} & \xrightarrow{\dot{\pi}} & \mathrm{End}_{lf}(V) \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\pi} & \mathrm{Aut}(V). \end{array}$$

*(iii) For any $g \in G$ and $X \in \mathfrak{g}_{fin}$ we have*

$$g(\exp(X))g^{-1} = \exp(\mathrm{Ad}\,(g(X))).$$

**Proof :** (i) Let us first remark that $\dot{\pi}(\widehat{\mathfrak{p}}_X)$ is in $\mathrm{End}_{lf}(V)$ for any $X$ of finite type. Indeed, because $V$ is a pro-representation, any vector $v$ in $V$ is contained in a finite dimensional sub-representation $W$ such that for some $k$ the ideal $\widehat{\mathfrak{u}}_X(k)$ acts trivially on $W$. In particular the action is locally finite.

Take $x \in \mathfrak{g}_{fin}$, then there exist $g \in G$, $X$ of finite type and $y \in \widehat{\mathfrak{p}}_X$ such that $x = \mathrm{Ad}\,(g(y))$ and $\dot{\pi}(x) = \dot{\pi}(\mathrm{Ad}\,(g \cdot y)) = \pi(g)\dot{\pi}(y)\pi(g)^{-1}$ is thus locally finite.

(ii) For such an $x \in \mathfrak{g}_{fin}$, we define

$$\exp(x) = g(\exp(y))g^{-1}$$

where we took the pro-exponential map on $\widehat{\mathfrak{p}}_X$ to $P_X$. Assume we have two different writing: $x = \mathrm{Ad}\,(g(y)) = \mathrm{Ad}\,(g'(y'))$ with $g' \in G$ and $y' \in P_{X'}$ for $X'$ of finite type. Then we compute for any representation $\pi : G \to \mathrm{Aut}(V)$ in $\mathfrak{M}(G)$:

$$\exp(\pi(g)\dot{\pi}(y)\pi(g)^{-1}) = \exp(\dot{\pi}(\mathrm{Ad}\,(g(y)))) = \exp(\dot{\pi}(\mathrm{Ad}\,(g'(y')))) = \exp(\pi(g')\dot{\pi}(y')\pi(g')^{-1})$$

Recall that for the pro-exponential map we have $\pi(\exp(x)) = \exp(\dot{\pi}(x))$. We obtain

$$\pi(g)\pi(\exp(y))\pi(g)^{-1} = \pi(g')\pi(\exp(y'))\pi(g')^{-1}.$$

We shall now need the following fact (relying on the Birkhoff decomposition and some easy constructions on representations of $\mathfrak{g}$, see [Ku02, Lemma 6.2.9 (a)]

**Fact 19.5.13** *For any $g \in G$ such that $g \neq 1$, there exists a representation $V \in \mathfrak{M}(G)$ such that $g$ acts non trivially on $V$.*

In particular we obtain that

$$g\exp(y)g^{-1} = g'\exp(y')g'^{-1}.$$

The exponential map is well defined. The commutativity of the diagram follows from the same property for pro-groups. The uniqueness follows from the previous Fact.

(iii) This formula is a direct consequence of the definition. □

# Chapter 20

# Ind-varieties

In this chapter we define and quickly study the notion of Ind-varieties i.e. direct limit of varieties. We shall need this do define an algebraic structure on the quotients $G/P_X$ for $X$ a subset of any type in $\Pi$.

In this chapter, we shall sometimes assert some facts without proofs. For a detailed treatment of ind-varieties, see [Sh94] and also [Ku02, Chapter 4].

## 20.1 Definition and first properties

**Definition 20.1.1** (i) A set $X$ together with a filtration $X_0 \subset X_1 \subset \cdots$ is an **ind-variety** if the following two conditions hold:

- $X$ is the union of the $(X_n)_{n\in\mathbb{N}}$
- for all $n \in N$, the set $X_n$ is a finite dimensional algebraic variety over a base field $k$ such that the inclusion $X_n \to X_{n+1}$ is a closed embedding.

(ii) We define **the ring $k[X]$ of regular functions** of an ind-variety $X$ by the inverse limit of the rings $k[X_n]$. It is a topological ring with the inverse limit topology.

(iii) We define the Zariski topology on an ind-variety by letting a subset $U$ be open if all the intersections $U \cap X_n$ are open in the Zariski topology of $X_n$ for all $n \in \mathbb{N}$.

(iv) A morphism $f : X \to Y$ of ind-varieties is an application such that for all $n \in \mathbb{N}$, there exists $m(n) \in \mathbb{N}$ such that $f(X_n) \subset Y_{m(n)}$ and the map $f|_{X_n} : X_n \to Y_{m(n)}$ is a morphism of algebraic varieties.

(v) A morphism of ind-varieties is said to be an isomorphism of ind-varieties if it is bijective and its inverse is again a morphism of ind-varieties.

(vi) An ind-variety is said to be affine if for all $n \in \mathbb{N}$ the varieties $X_n$ are affine.

(vii) A map $f : X \to Y$ is called a closed embedding if

- for any $n \in \mathbb{N}$ there exists $m(n)$ such that $f(X_n) \subset Y_{m(n)}$ and $f|_{X_n} : X_n \to Y_{m(n)}$ is a closed embedding.
- $f(X)$ is closed in $Y$
- $f : X \to f(X)$ is an homeomorphism for the subspace topology.

(viii) An ind-variety is called irreducible (resp. connected) if the underlying topological space is.

(ix) Let $X$ be an ind-variety, we define the pre-sheaf $\mathcal{O}_X$ by $U \mapsto k[U]$ for any open subset $U$.

**Example 20.1.2** An algebraic variety is an ind-variety for the trivial filtration $X_n = X$ for all $n \in \mathbb{N}$.

**Remark 20.1.3** (i) A subset $F$ in an ind-variety $X$ is closed if and only if the intersection $F \cap X_n$ is closed in $X_n$ for the Zariski topology.

(ii) A morphism $f : X \to Y$ of ind-varieties is continuous and induces a continuous morphism of $k$-algebras $f^* : k[Y] \to k[X]$.

(iii) The composition of two morphisms of ind-varieties is again a morphism of ind-varieties.

(iv) The algebra $k[X]$ is canonically isomorphic to the algebra of ind-variety morphisms from $X$ to $k$.

(v) A morphism $f : X \to Y$ between affine ind-varieties is an isomorphism if and only if $f^* : k[Y] \to k[X]$ is an isomorphism of topological $k$-algebras.

(vi) The pre-sheaf $\mathcal{O}_X$ is a sheaf.

**Fact 20.1.4** *(i) An open (resp. closed) subset $Z$ of an ind-variety $X$ has a natural structure of an ind-variety given by the induced filtration $Z_n = Z \cap X_n$.*

*(ii) The inclusion $Z \subset X$ of a closed subset in an ind-variety $X$ is a closed embedding of ind-varieties.*

**Example 20.1.5** (i) If $X$ and $Y$ are ind-varieties, then there product is again an ind-variety with defining filtration $X_n \times Y_n$.

(ii) Let $k$ be a field. The set

$$\mathbb{A}_\infty = \{(a_i)_{i\in\mathbb{N}} \ / \ a_i \in k \text{ finitely many } a_i \text{ are non vanishing}\}$$

has a natural ind-variety structure described by the filtration $\mathbb{A}_1 \subset \mathbb{A}_2 \subset \cdots \subset \mathbb{A}_n \subset \cdots$ where $\mathbb{A}_n$ is the subset of $\mathbb{A}_\infty$ such that $a_i = 0$ for $i > n$ and has the $n$-dimensional affine space structure.

(iii) With the notation of the previous example, we have a map $k[\mathbb{A}_{n+1}] \to k[\mathbb{A}_n]$ i.e. a map $S(\mathbb{C}^{n+1}) \to S(\mathbb{C}^n)$. In particular these map form an inverse system and we consider the ring

$$\widehat{S}(\mathbb{A}_\infty) = \lim_{\leftarrow} S(\mathbb{C}^n).$$

We have $k[\mathbb{A}_\infty] = \widehat{S}(\mathbb{A}_\infty)$. For $P \in k[\mathbb{A}_\infty]$, we may write $P = (P_n)$ with $P_n \in S(\mathbb{C}^n)$. Define the following subset of $\mathbb{A}_\infty$:

$$V(P) = \{(a_i)_{i\in\mathbb{N}} \in \mathbb{A}_\infty \ / \ P_n(a) = 0 \text{ for all } n\}.$$

It is easy to see that $V(P)$ is a closed subset of $\mathbb{A}_\infty$ and is thus a closed ind-subvariety on $\mathbb{A}_\infty$. It is an affine subvariety.

(iv) Any vector space of countable dimension is an ind-variety.

(v) Define the set

$$\mathbb{P}_\infty = \{\text{lines in } \mathbb{A}_\infty\}.$$

It has a natural ind-variety structure described by the filtration $\mathbb{P}_1 \subset \mathbb{P}_2 \subset \cdots \subset \mathbb{P}_n \subset \cdots$ where $\mathbb{P}_n$ is the subset of $\mathbb{A}_\infty$ such that $a_i = 0$ for $i\, n+1$ and has the $n$-dimensional projective space structure.

(vi) As in example (iii) consider the polynomials $k[\mathbb{A}_\infty]$ but restrict yourself to homogeneous polynomials. Then we may define a closed subset $V(P)$ of $\mathbb{P}_\infty$ and thus an ind-variety.

(vii) Any countable set $X = \{(x_i)_{i\in\mathbb{N}}\}$ has a structure of an ind-variety, simply define $X_n = \{x_0, \cdots, x_n\}$ with the finite set algebraic structure.

**Lemma 20.1.6** *(ı) Let $f : Z \to X$ be a continuous map between ind-varieties, then there exists, for any $n \in \mathbb{N}$, an integer $m(n)$ such that $f(Z_n) \subset X_{m(n)}$.*

*(ıı) For a closed embedding $g : X \to Y$, the map $f : Z \to X$ is a morphism (resp. a closed immersion) if and only if the composition $g \circ f : Z \to Y$ is.*

**Proof :** (ı) Fix $n$ and assume that there does not exist any $m$ such that $f(Z_n) \subset X_m$. Then there exists an infinite sequence of points $x_{m_i}$ in $g(Z_n) \cap (X_{m_i} \subset X_{m_i-1}$ with the sequence $m_i$ increasing. The set $S = \{x_{m_i}\}$ is closed (its intersection with the $X_n$ is finite). Let us denote by $f_n$ the restriction of $f$ to $Z_n$. Because $f$ is continuous, the same is true of $f_n$ and we get that $f_n^{-1}(S)$ is closed. It is an algebraic variety. Consider the map $f_S : f_n^{-1}(S) \to S$. It is continuous. The points $x_{m_i}$ are open and closed in $S$ and the same is true for $f_S^{-1}(x_{m_i})$ in $f_n^{-1}(S)$. But these sets are non empty thus $f_n^{-1}(S)$ has infinitely many connected components which is absurd being an algebraic variety.

(ıı) It follows from the same properties for algebraic varieties and the point (ı). □

**Corollary 20.1.7** *(ı) Let $X$ be an ind-variety and $Z$ a closed subset of $X$. The only ind-variety structure on $Z$ making the inclusion a closed embedding is the induced ind-variety structure.*

*(ıı) For a closed embedding $f : Z \to X$ the ind-variety $Z$ is isomorphic to the closed subvariety $f(Z)$ in $X$.*

*(ııı) Let $Z$ be a closed subset in an ind-variety $X$ and $U$ an open subset in $X$. Then there is a canonical ind-variety structure on $Z \cap U$. Such a sub-ind-variety of $X$ in called a* **locally closed ind-subvariety** *of $X$.*

**Definition 20.1.8** Let $H$ be an algebraic group and $X$ an ind-variety together with an action of $H$ on $X$. If the action morphism $H \times X \to X$ is an ind-variety morphism then we say that $X$ is an **ind-$H$-variety**.

**Definition 20.1.9** (ı) Let $X$ be an ind-variety. For any $x \in X$ we define the **Zariski tangent space** $T_{x,X}$ of $X$ at $x$ by

$$T_{x,X} = \lim_{\to} T_{x,X_n}.$$

This is well defined because $x \in X_n$ for $n$ large enough.

(ıı) A morphism $f : X \to Y$ induces a linear map $df_x : T_{x,X} \to T_{f(x),Y}$ called the **derivative of** $f$ **at** $x$.

**Fact 20.1.10** *(ı) Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of ind-varieties, then we have $d(g \circ f)_x = dg_{f(x)} \circ df_x$.*

*(ıı) An isomorphism of ind-varieties induces an isomorphism on the tangent spaces.*

## 20.2 Vector bundles on ind-varieties

**Definition 20.2.1** (ı) Let $X$ be an ind-variety, **an ind-vector bundle of rank** $r$ on $X$ is an ind-variety $E$ together with an ind-morphism $\pi : E \to X$ such that the maps $\pi_n : E_n \to X_{m(n)}$ are vector bundles of rank $r$. If the rank is one, the $E$ is called **an ind-line bundle**.

(ıı) Morphism of ind-vector bundles between $\pi : E \to X$ and $\pi' : E' \to X$ is a morphism of ind-varieties $f : E \to E'$ such that $\pi = \pi' \circ f$ and the map on the fibers is linear. This defines a notion of isomorphisms of vector bundles.

(ııı) We denote by $\mathrm{Pic}(X)$ and call it the Picard group of $X$ the group of ind-line bundles on $X$ modulo isomorphism. It is an abelian group under the tensor product.

(iv) Let $P$ be an algebraic group. **A principal $P$-ind-bundle** over $X$ is an ind-variety $E$ together with an ind-morphism $\pi : E \to X$ such that the maps $\pi_n : E_n \to X_{m(n)}$ are $P$-principal bundles.

(v) Let $H$ an algebraic group and $X$ an ind-$H$-variety, then an ind-vector bundle $\pi : E \to X$ is **$H$-equivariant** if $E$ is also an ind-$H$-variety and the diagram

$$\begin{array}{ccc} H \times E & \longrightarrow & E \\ \downarrow & & \downarrow \\ H \times Y & \longrightarrow & Y \end{array}$$

and for $h \in H$ the induced map on the fibers is linear.

(vi) For an ind-vector bundle $\pi : E \to X$ over an ind-variety $X$, we define

$$H^i(X, E) = \lim_{\leftarrow} H^i(X_n, E|_{X_n})$$

where the inverse system is defined by the pull-back along the canonical inclusion $X_n \to X_{n+1}$. It is easily checked that $H^0(X, E)$ identifies with the space of algebraic sections of $\pi$.

(vii) Assume that $E$ is an ind-$H$-equivariant vector bundle, then the group $H^i(X, E)$ is naturally an $H$-module for all $i$.

(viii) More generally, let $\pi : E \to X$ be a vector bundle and let $H$ be a group acting on $E$ and $X$ such that $\pi$ is $H$ equivariant. Assume that

- for all $h \in H$, the action of $h$ on $E$ and $X$ is algebraic;
- for all $x \in X$ and $h \in H$, the map $\pi^1(x) \to \pi^{-1}(h \cdot x)$ defined by $h$ is linear;

then the cohomology group $H^i(X, E)$ has a natural structure of $H$-module defined explicitly for $H^0(X, E)$ by:

$$(h \cdot \sigma)x = h \cdot \sigma(h^{-1}x), \quad \text{for } x \in X \text{ and } \sigma \in H^0(X, E).$$

**Example 20.2.2** (i) As for algebraic variety, the product $X \times V$ of an ind-variety $X$ with a vector space $V$ is called a trivial ind-vector bundle on $X$ of rank $r = \dim V$.

(ii) Let $\pi : E \to X$ be an ind-vector bundle and $f : Y \to X$ be a morphism of ind-varieties, then we may define

$$f^*E = \{(y, e) \in Y \times E \ / \ f(y) = \pi(e)\}.$$

It is a closed subset of the product $Y \times E$ and thus has a natural ind-variety structure. We have a natural ind-variety morphism $f^*\pi : f^*E \to Y$ defined by the projection on the first factor. This endows $f^*E$ with an ind-vector bundle structure on $Y$.

(iii) Let $V$ be a countable dimension vector space, and define

$$\mathcal{L}_V = \{(x, v) \in \mathbb{P}(V) \times V \ / \ v \in x\} \subset \mathbb{P}(V) \times V.$$

It is an ind-variety and there is a natural ind-variety morphism $\mathcal{L}_V \to \mathbb{P}(V)$. This is an ind-line bundle on $\mathbb{P}(V)$ called the **tautological line bundle on $\mathbb{P}(V)$.**

(iv) The product of two ind-vector bundles $\pi : E \to X$ and $\pi' : E' \to X'$ is again an ind-vector bundle $E \times E' \to X \times X'$. If $X = X'$ and $\Delta : X \to X \times X$ is the diagonal embedding, then the pull back $\Delta^*(E \times E') \to X$ is called **the Withney sum or direct sum** of $E$ and $E'$ and denoted by $E \oplus E'$.

(v) If $E \to X$ and $E' \to X$ are two ind-vector bundles, then we may define the tensor product of ind-vector bundles $E \otimes E' \to X$ simply by taking the tensor products $E_n \otimes E'_n$ of algebraic vector bundles on the algebraic variety $X_n$ and then take the direct limit.

(vi) Let $X$ be an ind-variety with filtration $(X_n)_{n \in \mathbb{N}}$ and assume there exists for all $n \in \mathbb{N}$ an ind-vector bundle $E_n \to Z_n$ together with an isomorphism $E_n \to E_{n+1}|_{Z_n}$, then there exists an ind-vector bundle $E$ whose defining filtration is given by the $E_n$ (take the direct limit of the $E_n$).

## 20.3 Regular action of a pro-group and construction of fibrations

**Definition 20.3.1** (ı) Let $(H, \mathcal{F})$ be a pro-group acting on an ind-variety $X$. We say that the action is **regular** if for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ and $N \in \mathcal{F}$ such that

- $H \cdot X_n \subset X_m$,
- the action of $N$ on $X_n$ is trivial and
- the induced action $H/N \times X_n \to X_m$ is algebraic.

In this situation we call $X$ an **ind-$H$-variety.**

(ıı) An ind-vector bundle $\pi : E \to X$ over $X$ is called **$H$-equivariant** if $E$ is an ind-$H$-variety such that $\pi$ is $H$-equivariant and for any $x \in X$ and $h \in H$ the induced map $\pi^{-1}(x) \to \pi^{-1}(h \cdot x)$ is linear.

**Lemma 20.3.2** *Let $(H, \mathcal{F})$ be a pro-group and $(H', \mathcal{F}')$ a pro-subgroup of $H$ such that*

$$\textit{for all } N' \in \mathcal{F}', \textit{ there exists } N \in \mathcal{F} \textit{ such that } N \subset N'. \qquad (\ddagger)$$

*(ı) Then for any regular action of $H'$ on a (quasi-projective) variety $X$, the set $X' = H \times^{H'} X$ carries a structure of variety such that the left action of $H$ on $X'$ is regular.*

*In particular $H/H'$ is a variety such that the action of $H$ is regular.*

*(ıı) The canonical map $p : X' \to H/H'$ is an $H$-equivariant isotrivial (i.e. trivial in the étale topology) fibration with fiber $X$. In particular $X'$ is smooth as soon as $X$ is.*

**Proof :** Choose $N' \in \mathcal{F}'$ such that the action of $H'$ factors through an action of $H'/N'$. Then choose $N \in \mathcal{F}$ with $N \subset N'$. We have $X' = H/N \times^{H'/N'} X$ and the result follows from the same result on algebraic groups. We easily check that the structure of variety does not depend on the choices of $N'$ and $N$. □

**Example 20.3.3** Let $X$ be a subset in $\Pi$ of finite type and $X' \subset X$. Then $H = P_X$ and $H' = P_{X'}$ satisfy the condition (‡) because of the following fact:

**Fact 20.3.4** *Let $X \subset \Pi$ and $X' \subset \Pi$ with $X'$ of finite type. Let $k'$ be a non negative integer, then there exists a non negative integer $k$ such that*

$$\mathcal{U}_{P_X}(k) \subset \mathcal{U}_{P_{X'}}(k').$$

**Proof :** Assume this is not true, then there exists an infinite sequence of roots $(\beta_r)_{r \in \mathbb{N}} \in \Delta_+ \setminus \Delta_X$ such that $\mathrm{ht}_X(\beta_r) \geq r$ but $\mathrm{ht}_{X'}(\beta_r) \leq k' - 1$. This gives an infinite sequence of weights for the finite dimensional space

$$\bigoplus_{\mathrm{ht}_{X'}(\alpha) \leq k'-1} \mathfrak{g}_\alpha.$$

This is a contradiction. □

In particular, the quotient $P_X/P_{X'}$ is a variety, in particular $P_X/B$ is a variety. It is even a projective variety because it is isomorphic to $G_X/P_{X',X}$ where $P_{X',X} = P_{X'} \cap G_X$ is a parabolic subgroup of $G_X$.

**Corollary 20.3.5** *For any simple root $\alpha \in \Pi$, the quotient $P_\alpha/B \simeq G_\alpha/B_\alpha$ is isomorphic to $\mathbb{P}^1$.*

# Chapter 21

# Bott-Samelson and Schubert varieties

In this chapter, we define an ind-variety structure on the quotient $G/P$ for any parabolic subgroup $P$ of the Kac-Moody group $G$.

## 21.1 Injection as an orbit

**Definition 21.1.1** (ı) Let $\lambda$ be a weight in $C_{\mathbb{Z}}$ and let $V$ be an integrable highest weight module of highest weight $\lambda$ and highest weight vector $v_\lambda$. Then we define the map $i_V : G \to \mathbb{P}(V)$ by $i_V(g) = [g \cdot v_\lambda]$.

(ıı) For $X$ a subset of $\Pi$, we define the set $C^0_X$ of $X$-**regular dominant totally integral weights** or **regular** $X$-**weights** by

$$C^0_X = \{\lambda \in C_{\mathbb{Z}} \ / \ \langle \lambda, \alpha_i^\vee \rangle = 0 \text{ iff } i \in X\}.$$

**Lemma 21.1.2** *For $\lambda \in C^0_X$, and $V$ an integrable highest weight module of highest weight $\lambda$, the map $i_V : G \to \mathbb{P}(V)$ factors through $G/P_X$ and the induced map is injective.*

**Proof :** It suffices to prove that the stabiliser is $P_X$. Because $V$ is integrable, we get that $B$ and all the elements $\widetilde{s}_i$ for $i \in X$ have a trivial action thus $P_X$ is contained in the stabiliser. Because $\lambda$ is $X$-regular, we obtain that $\widetilde{s}_j$ is not in this stabiliser for $j \notin X$. Now we conclude thanks to the following classical lemma on Tits systems. □

**Lemma 21.1.3** *Let $(G, B, N, S)$ be a Tits system, then*

*(ı) for any reduced decomposition $w = s_1 \cdots s_p$ with $s_i \in S$, we have that $C(s_i)$ for $i \in [1, p]$ is contained in the group $< C(w) >$ generated by $C(w)$ in $G$ and $< B, wBw^{-1} >=< C(w) >$.*

*(ıı) For $X$ and $X'$ be two subsets of $S$, we have $P_X = P_{X'}$ if and only if $X = X'$. Furthermore, for any group $P$ containing $B$, there exists $X$ such that $P = P_X$.*

**Proof :** (ı) We will abuse notation and denote by $v$ an element in $W$ and any of its representative in $N$. We have $< B, wBw^{-1} >\subset< C(w) >$ since $w \in C(w)$ and $wB \subset C(w)$. Furthermore, we have $< C(w) >\subset< C(s_1), \cdots, C(s_p) >$. To prove (ı) we only have to prove the inclusion

$$< C(s_1), \cdots, C(s_p) >\subset< B, wBw^{-1} > .$$

We prove this by induction on the length. It is clear for $\ell(w) = 0$. We have $\ell(s_1 w) < \ell(w)$ implying the inclusion $C(w) \subset C(s_1) \cdot C(w)$. In particular, we have $w = b_1 s_1 b_2 w b_3$ for $b_i \in B$ and we get $s_1 \in< B, wBw^{-1} >$. This implies the inclusion

$$< B, s_1 wBw^{-1} s_1 >\subset< B, wBw^{-1} > .$$

But by induction we have $< C(s_2), \cdots, C(s_p) > \subset < B, s_1wBw^{-1}s_1 >$ and the proof follows ($s_1$ and $B$ are also in the right hand side).

(ii) Because of the Bruhat decomposition, $P_X = P_{X'}$ if and only if $W_X = W_{X'}$. To prove that $W_X = W_{X'}$ if and only if $X = X'$, we only need to prove that $X \cap W_{X'} \subset X'$.

Let $s \in X \cap W_{X'}$ and write $s = s_1 \cdots s_p$ for $s_i \in X'$. We thus have $ss_1 \cdots s_p = 1$. But any relation containing $s$ contain it twice (from the definition of Coxeter systems) hence there exists $i$ with $s = s_i$ and $s \in X'$.

Let $P$ be a subgroup containing $B$, it is left and right $B$-invariant and we may write

$$P = \coprod_{w \in W'} BwB$$

for some subset $W'$ of $W$. Let us prove that $W' = < W' \cap S >$. Indeed, write $w = s_1 \cdots s_p$ with $s_i \in S$. By (i) we have $< C(w) > \supset C(s_i)$ thus $C(s_i) \subset P$ and $s_i \in W'$. In particular we have $W' \subset < W' \cap S >$. Conversely $P$ being a group we have $s \in P$ for $s \in S \cap W'$ thus $w \in P$ for $w \in < W' \cap S >$ and $w \in W'$. □

## 21.2 Bott-Samelson resolution

Let $\mathfrak{W}$ be the set of all the ordered sequences $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{\alpha_n})$ of simple reflections for any $n \in \mathbb{N}$ (the roots $\alpha_i$ are simple roots i.e. elements in $\Pi$).

**Definition 21.2.1** For any $\mathfrak{w} \in \mathfrak{W}$, with $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{\alpha_n})$, we define the **Bott-Samelson variety** $Z_{\mathfrak{w}}$ by:

$$Z_{\mathfrak{w}} = (P_{\alpha_1} \times \cdots \times P_{\alpha_n}) / (\underbrace{B \times \cdots \times B}_{n \text{ factors}}),$$

where the right action of $B^n$ on $P_{\alpha_1} \times \cdots \times P_{\alpha_n}$ is given by

$$(p_1, \cdots, p_n) \cdot (b_1, \cdots, b_n) \mapsto (p_1b_1, b_1^{-1}p_2b_2, \cdots, b_{n-1}^{-1}p_nb_n).$$

We shall denote the $B^n$-orbit of $(p_1, \cdots, p_n)$ by $[p_1, \cdots, p_n]$. For $n = 0$, we set $Z_{\mathfrak{w}} = \{\text{pt}\}$.

**Remark 21.2.2** For the moment the Bott-Samelson "variety" is only a set, we shall give it a structure of variety later on.

**Definition 21.2.3** (i) We define a **partial order on** $\mathfrak{W}$ by setting $\mathfrak{v} \leq \mathfrak{w}$ if $\mathfrak{v}$ can be obtained from $\mathfrak{w}$ by removing elements in the sequence. In other words, if $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{\alpha_n})$, then $\mathfrak{v} = (s_{\alpha_{j_1}}, \cdots, s_{\alpha_{j_r}})$ where $1 \leq j_1 < \cdots < j_r \leq n$. Note that there may be different subsequences of $\mathfrak{w}$ defining the same element $\mathfrak{v}$.

(ii) **The length** of a word $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{a_n})$ is $n$ and denoted $\ell(\mathfrak{w})$. We have a natural map $\pi : \mathfrak{W} \to W$ sending $\mathfrak{w}$ to $w = s_{\alpha_1} \cdots s_{\alpha_n}$. The word $\mathfrak{w}$ is called **reduced** if $\ell(\mathfrak{w}) = \ell(w)$.

(iii) For two words $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{\alpha_n})$ and $\mathfrak{v} = (s_{\alpha_{n+1}}, \cdots, s_{\alpha_m})$ we define their **concoction** $\mathfrak{w} * \mathfrak{v}$ by $\mathfrak{w} * \mathfrak{v} = (s_{\alpha_1}, \cdots, s_{\alpha_m})$.

(iv) For $\mathfrak{w} \in \mathfrak{W}$ of the form $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{\alpha_n})$ and $J$ a subset of $[1, n]$, we denote by $\mathfrak{w}_J$ **the subword** of $\mathfrak{w}$ obtained by $\mathfrak{w}_J = (s_{\alpha_i})_{i \in J}$.

We have the following easy fact:

**Fact 21.2.4** *For a subword $\mathfrak{v} = \mathfrak{w}_J$ of $\mathfrak{w}$, there is a natural inclusion $i_{\mathfrak{v},\mathfrak{w}} : Z_{\mathfrak{v}} \to Z_{\mathfrak{w}}$ defined by $i_{\mathfrak{v},\mathfrak{w}}([(p_j)_{j \in J}]) = [p_1, \cdots, p_n]$ where $p_i = 1$ for $i \notin J$.*

**Definition 21.2.5** We define the subset $Z^0_{\mathfrak{w}}$ in $Z_{\mathfrak{w}}$ by

$$Z\mathfrak{w}^0 = \{[p_1, \cdots, p_n] \in Z_{\mathfrak{w}} \ / \ p_i \in Bs_{\alpha_i}B\}.$$

A direct application of the Bruhat decomposition yields the following:

**Fact 21.2.6** *We have the following decomposition*

$$Z_{\mathfrak{w}} = \coprod i_{\mathfrak{v},\mathfrak{w}}(Z^0_{\mathfrak{v}})$$

*where* $\mathfrak{v}$ *runs over all the subwords of* $\mathfrak{w}$.

To put a variety structure on $Z_{\mathfrak{w}}$ we need the following lemma which is a direct consequence of Fact 20.3.4:

**Lemma 21.2.7** *Let* $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{\alpha_n})$ *be a word and let* $k_n$ *be a non negative integer, then there exist non negative integers* $k_1, \cdots k_{n-1}$ *such that the following holds:*

$$\mathcal{U}_{P_{\alpha_1}}(k_1) \subset \mathcal{U}_{P_{\alpha_2}}(k_2) \subset \cdots \subset \mathcal{U}_{P_{\alpha_n}}(k_n). \qquad (\dagger)$$

Let $\mathfrak{k} = (k_1, \cdots, k_n)$ be a sequence of integers satisfying the condition (†) of the previous lemma, we define the quotient

$$P_{\mathfrak{w}}/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k}) = P_{\alpha_1}/\mathcal{U}_{\alpha_1}(k_1) \times \cdots P_{\alpha_n}/\mathcal{U}_{\alpha_n}(k_n).$$

**Fact 21.2.8** *(ı) There exists a map* $\theta_{\mathfrak{w},\mathfrak{k}} : P_{\mathfrak{w}}/U_{\mathfrak{w}}(\mathfrak{k}) \to Z_{\mathfrak{w}}$ *defined by* $\theta_{\mathfrak{w},\mathfrak{k}}(\bar{p}_1, \cdots, \bar{p}_n) = [p_1, \cdots, p_n]$.

*(ıı) The group* $B^n/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k}) = B/\mathcal{U}_{\alpha_1}(k_1) \times \cdots \times B/\mathcal{U}_{\alpha_n}(k_n)$*, acts on* $P_{\mathfrak{w}}/U_{\mathfrak{w}}(\mathfrak{k})$ *via*

$$(\bar{p}_1, \cdots, \bar{p}_n) \cdot (\bar{b}_1, \cdots, \bar{b}_n) \mapsto (\overline{p_1 b_1}, \overline{b_1^{-1} p_2 b_2}, \cdots, \overline{b_{n-1}^{-1} p_n b_n}).$$

*(ııı) Any fiber of the map* $\theta_{\mathfrak{w},\mathfrak{k}}$ *is an orbit under* $B^n/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k})$ *and the action of* $B^n/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k})$ *on* $P_{\mathfrak{w}}/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k})$ *is free.*

**Proof :** (ı) The map is well defined because of the definition of $Z_{\mathfrak{w}}$, of the fact that $\mathcal{U}$ is contained in $B$ and because of condition (†).

(ıı) Because $\mathcal{U}_{P_{\alpha_i}}$ is normal in $P_{\alpha_i}$ for all $i$, the action is well defined.

(ııı) The fact that the fiber is an orbit is clear from the definition. The second assertion follows by an easy induction. □

**Proposition 21.2.9** *(ı) The set* $Z_{\mathfrak{w}}$ *has a natural structure of variety. For this structure, it is irreducible, smooth and the left action of* $P_{\alpha_1}$ *given by multiplication on the first factor is regular.*

*(ıı) For any sequence* $\mathfrak{k}$ *satisfying (†), the map* $\theta_{\mathfrak{w},\mathfrak{k}} : P_{\mathfrak{w}}/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k}) \to Z_{\mathfrak{w}}$ *is a Zariski locally trivial* $B^n/\mathcal{U}_w(\mathfrak{k})$*-bundle.*

**Proof :** (ı) We prove this by induction on $\ell(\mathfrak{w})$. For $\ell(\mathfrak{w}) = 0$, then $Z_{\mathfrak{w}}$ is a point and the variety structure is clear. Take $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{\alpha_n})$ and set $\mathfrak{v} = (s_{\alpha_2}, \cdots, s_{\alpha_n})$. Then we have a natural map

$$P_{\alpha_1} \times^B Z_{\mathfrak{v}} \to Z_{\mathfrak{w}}$$

defined by $\overline{(p_1, [p_2, \cdots, p_n])} \mapsto [p_1, \cdots, p_n]$. This is well defined and it is easy to see that it is a bijection. By induction, we have a variety structure on $Z_{\mathfrak{v}}$ such that the action of $P_{\alpha_2}$ and hence of

$B$ is regular. We deduce, by Lemma 20.3.2, that $Z_{\mathfrak{w}}$ has a variety structure such that it is smooth irreducible with a regular action of $P_{\alpha_1}$.

(ii) We prove by induction that the map is a mophism. For this let $\mathfrak{w}$ and $\mathfrak{v}$ as before and set $\mathfrak{k}' = (k_2, \cdots, k_n)$ for $\mathfrak{k}(k_1, \cdots, k_n)$ satisfying (†). Remark that $\mathfrak{k}'$ satisfies (†). By induction, the map $\theta_{\mathfrak{v},\mathfrak{k}'} : P_{\mathfrak{v}}/\mathcal{U}_{\mathfrak{v}}(\mathfrak{k}') \to Z_{\mathfrak{v}}$ is a morphism. Remark also that $\theta_{\mathfrak{w},\mathfrak{k}}$ is given by the composition

$$P_{\mathfrak{w}}/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k}) = P_{\alpha_1}/\mathcal{U}_{\alpha_1}(k_1) \times P_{\mathfrak{v}}/\mathcal{U}_{\mathfrak{v}}(\mathfrak{k}') \to P_{\alpha_1}/\mathcal{U}_{\alpha_1}(k_1) \times Z_{\mathfrak{v}} \to Z_{\mathfrak{w}}.$$

The first map is a morphism. The last one is obtained as restriction to $Z_{\mathfrak{v}}$ of the natural action of $P_{\alpha_1}/\mathcal{U}_{\alpha_1}(k_1)$ on $Z_{\mathfrak{w}}$. But by (i) this action is regular thus there exists $k_1' \geq k_1$ (i.e. $\mathcal{U}_{\alpha_1}(k_1') \subset \mathcal{U}_{\alpha_1}(k_1)$) such that $\mathcal{U}_{\alpha_1}(k_1')$ acts trivially on $Z_{\mathfrak{w}}$ and the induced action $P_{\alpha_1}/\mathcal{U}_{\alpha_1}(k_1')$ is algebraic. But by condition (†), the action of $\mathcal{U}_{\alpha_1}(k_1)$ is already trivial and we have the commutative diagram:

$$\begin{array}{ccc} P_{\alpha_1}/\mathcal{U}_{\alpha_1}(k_1') \times Z_{\mathfrak{w}} & \longrightarrow & Z_{\mathfrak{w}} \\ \downarrow & \nearrow & \\ P_{\alpha_1}/\mathcal{U}_{\alpha_1}(k_1) \times Z_{\mathfrak{w}} & & \end{array}$$

implying that the action is a morphism and thus $\theta_{\mathfrak{w},\mathfrak{k}}$ is a morphism.

Remark that $P_{\mathfrak{w}}/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k}) = P_{\alpha_1}/\mathcal{U}_{\alpha_1}(k_1) \times P_{\mathfrak{v}}/\mathcal{U}_{\mathfrak{v}}(\mathfrak{k}')$. Let us consider the following sequence of morphisms:

$$P_{\mathfrak{w}}/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k}) \to P_{\alpha_1} \times^{\mathcal{U}_{\alpha_1}(k_1)} P_{\mathfrak{v}}/\mathcal{U}_{\mathfrak{v}}(\mathfrak{k}') \to P_{\alpha_1} \times^{B} P_{\mathfrak{v}}/\mathcal{U}_{\mathfrak{v}}(\mathfrak{k}') \to P_{\alpha_1} \times^{B} Z_{\mathfrak{v}} = Z_{\mathfrak{w}}.$$

where the first morphism is defined by $(\bar{p}_1, \cdots, \bar{p}_n) \mapsto \overline{(p_1, \bar{p}_2, \cdots, \bar{p}_n)}$ and is an isomorphism because $\mathcal{U}_{\alpha_1}(k_1)$ is normal in $P_{\alpha_2}$. The second map is isotrivial because $P_{\alpha_1}/\mathcal{U}_{\alpha_1}(k_1) \to P_{\alpha_1}/B$ is Zariski locally trivial (we only need to check this in the finite dimensional group $G_{\alpha_1} \ltimes \mathcal{U}_{\alpha_1}/\mathcal{U}_{\alpha_1}(k_1)$) and because the fibration above them are isotrivial. By induction the last map is an isotrivial fibration and the fibers are the product of the fibers of these two maps.

Let us prove that the fibration is trivial for the Zariski topology. For this we first prove that the sets $Z_{\mathfrak{w}}(\mathfrak{a})$ are open. We do this by induction, for $\mathfrak{a} = (a_1, \cdots, a_n)$ set $\mathfrak{a}' = (a_2, \cdots, a_n)$. Then consider the map:

$$f : a_1\mathcal{U}_{-\alpha_1}B/\mathcal{U}_{\alpha_1}(k_1) \times P_{\mathfrak{w}'}/\mathcal{U}_{\mathfrak{w}'}(\mathfrak{k}') \to P_{\mathfrak{w}'}/\mathcal{U}_{\mathfrak{w}'}(\mathfrak{k}')$$

defined by $f(\overline{a_1ub}, \bar{p}_2, \cdots, \bar{p}_n) \mapsto (\overline{bp_2}, \bar{p}_3, \cdots \bar{p}_n)$ with $u \in \mathcal{U}_{-\alpha_1}$, $b \in B$ and $p_i \in P_{\alpha_i}$ for $i \in [2, n]$. Denote by $\theta'$ the map $P_{\mathfrak{w}'}/\mathcal{U}_{\mathfrak{w}'}(\mathfrak{k}') \to Z_{\mathfrak{w}'}$, then we have $\theta^{-1}(Z_{\mathfrak{w}}(\mathfrak{a})) = f^{-1}(\theta'^{-1}(Z_{\mathfrak{w}'}(\mathfrak{a}'))$ and by induction we get that $\theta^{-1}(Z_{\mathfrak{w}}(\mathfrak{a}))$ is open. But $\theta$ is locally trivial in the étale topology with smooth fibers, thus $\theta$ is smooth and thus open and $Z_{\mathfrak{w}}(\mathfrak{a})$ is open.

Now consider the map $\theta_{\mathfrak{a}} : (a_1\mathcal{U}_{\alpha_1}) \times \cdots \times (a_n\mathcal{U}_{\alpha_n}) \to Z_{\mathfrak{w}}$ defined as the restriction of $\theta$. This map is injective with image $Z_{\mathfrak{w}}(\mathfrak{a})$. But $Z_{\mathfrak{w}}(\mathfrak{a})$ is open in $Z_{\mathfrak{w}}$ which is smooth thus it is smooth and a bijective morphism is an isomorphism thus $\theta_{\mathfrak{a}}$ is an isomorphism and its inverse is a section of $\theta$ on $Z_{\mathfrak{w}}(\mathfrak{a})$.

Let us prove the local triviality of $\theta$ on $Z_{\mathfrak{w}}(\mathfrak{a})$. The multiplication

$$(a_1\mathcal{U}_{\alpha_1}) \times \cdots \times (a_n\mathcal{U}_{\alpha_n}) \times B^n/\mathcal{U}_w(\mathfrak{k}) \to \theta^{-1}(Z_{\mathfrak{w}}(\mathfrak{a}))$$

is an isomorphism and the following commutative diagram

$$\begin{array}{ccc} (a_1\mathcal{U}_{\alpha_1}) \times \cdots \times (a_n\mathcal{U}_{\alpha_n}) \times B^n/\mathcal{U}_w(\mathfrak{k}) & \longrightarrow & \theta^{-1}(Z_{\mathfrak{w}}(\mathfrak{a})) \\ & \searrow & \downarrow \theta \\ & & Z_{\mathfrak{w}}(\mathfrak{a}) \end{array}$$

concludes the proof. $\square$

**Corollary 21.2.10** *Let* $\mathfrak{w} \in \mathfrak{W}$ *with* $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{\alpha_n})$, $\mathfrak{v} = (s_{\alpha_2}, \cdots, s_{\alpha_n})$ *and* $\mathfrak{u} = (s_{\alpha_1}, \cdots, s_{\alpha_{n-1}})$.

*(ı) The canonical projection* $Z_\mathfrak{w} \to P_{\alpha_1}/B \simeq \mathbb{P}^1$ *is a Zariski locally trivial fibration in* $Z_\mathfrak{v}$.

*(ıı) The map* $\psi : Z_\mathfrak{w} \to Z_\mathfrak{u}$ *defined by* $\psi([p_1, \cdots, p_n]) = [p_1, \cdots, p_{n-1}]$ *is a Zariski locally trivial* $\mathbb{P}^1$*-fibration. The map* $\sigma : Z_\mathfrak{u} \to Z_\mathfrak{w}$ *defined by* $\sigma([p_1, \cdots, p_{n-1}]) = [p_1, \cdots, p_{n-1}, 1]$ *is a section of this fibration.*

*(ııı) The variety* $Z_\mathfrak{w}$ *is projective.*

*(ıv) For any subword* $\mathfrak{t}$ *of* $\mathfrak{w}$*, the inclusion* $i_{\mathfrak{t},\mathfrak{w}} : Z_\mathfrak{t} \to Z_\mathfrak{w}$ *is a closed immersion.*

**Proof :** Let $r \in [1, n]$ and define $w_r = (s_{\alpha_1}, \cdots, s_{\alpha_r})$ and $w^r = (s_{\alpha_{r+1}}, \cdots, s_{\alpha_n})$. We first prove that the natural maps $\psi_r : Z_\mathfrak{w} \to Z_{w_r}$ and $\psi^r : Z_\mathfrak{w} \to Z_{w^r}$ are morphisms. Let $\mathfrak{k} = (k_1, \cdots, k_n)$ satisfying (†) and set $\mathfrak{k}_r = (k_1, \cdots, k_r)$ and $\mathfrak{k}^r = (k_{r+1}, \cdots, k_n)$. Consider the commutative diagrams

$$\begin{array}{ccc} P_\mathfrak{w}/\mathfrak{U}_\mathfrak{w}(\mathfrak{k}) & \longrightarrow & P_{\mathfrak{w}_r}/\mathfrak{U}_{\mathfrak{w}_r}(\mathfrak{k}_r) \\ \downarrow & & \downarrow \\ Z_\mathfrak{w} & \longrightarrow & Z_{\mathfrak{w}_r} \end{array} \qquad \begin{array}{ccc} P_\mathfrak{w}/\mathfrak{U}_\mathfrak{w}(\mathfrak{k})(\mathfrak{k}^r) & \longrightarrow & P_{\mathfrak{w}^r}/\mathfrak{U}_{\mathfrak{w}^r} \\ \downarrow & & \downarrow \\ Z_\mathfrak{w} & \longrightarrow & Z_{\mathfrak{w}^r}. \end{array}$$

In these diagrams, the first line horizontal arrows are clearly morphisms while the vertical arrows are Zariski locally trivial. The horizontal maps of the second line are thus morphisms by first taking a section of the first column, composing with the horizontal map of the first line and and then composing by the second column projection.

(ı) To prove that this map (which is now a morphism) is a Zariski locally trivial fibration in $\mathbb{Z}_\mathfrak{v}$, we consider the first commutative diagram with $k = 1$. We have $Z_{\mathfrak{w}_1} = P_{\alpha_1}/B$ and the morphism is $\psi_1$. But now locally, the vertical maps and the composed map $P_\mathfrak{w}/\mathfrak{U}_\mathfrak{w}(\mathfrak{k}) \to Z_\mathfrak{v}$ are respectively locally trivial fibrations in $B^n/\mathfrak{U}_\mathfrak{w}(\mathfrak{k})$, in $B/\mathfrak{U}_{\mathfrak{w}_1}(\mathfrak{k}_1)$ and in $B/\mathfrak{U}_{\mathfrak{w}_1}(\mathfrak{k}_1) \times P_{w^r}/\mathfrak{U}_{\mathfrak{w}^r}(\mathfrak{k}^r)$. This implies that the horizontal map is a Zariski locally trivial fibration with fibre $(B/\mathfrak{U}_{\mathfrak{w}_1}(\mathfrak{k}_1) \times P_{w^r}/\mathfrak{U}_{\mathfrak{w}^r}(\mathfrak{k}^r))/(B^n/\mathfrak{U}_\mathfrak{w}(\mathfrak{k}))$. This last quotient is $(P_{w^r}/\mathfrak{U}_{\mathfrak{w}^r}(\mathfrak{k}^r))/(B^{n-1}/\mathfrak{U}_{\mathfrak{w}^r}(\mathfrak{k}^r)) \simeq Z_{\mathfrak{w}^r} = Z_\mathfrak{v}$.

(ıı) The same method as in (ı) gives the result.

(ııı) The variety $Z_\mathfrak{w}$ is a sequence of $\mathbb{P}^1$-fibrations and hence is projective.

(ıv) The same method as in the beginning of the proof shows that $i_{\mathfrak{t},\mathfrak{w}}$ is a morphism. Indeed, we have the following commutative diagram

$$\begin{array}{ccc} P_\mathfrak{t}/\mathfrak{U}_\mathfrak{t}(\mathfrak{k}_\mathfrak{t}) & \longrightarrow & P_\mathfrak{w}/\mathfrak{U}_\mathfrak{w}(\mathfrak{k}) \\ \downarrow & & \downarrow \\ Z_{\mathfrak{w}_\mathfrak{t}} & \longrightarrow & Z_\mathfrak{w} \end{array}$$

where $\mathfrak{k}_\mathfrak{t}$ is the restriction of the sequence $\mathfrak{k}$ to the indices in $\mathfrak{t}$. We define the morphism structure by taking sections of the left hand side vertical map. Furthermore, by (ııı), its image $Z$ is closed in $Z_\mathfrak{w}$ and we need to prove that the inverse (defined on $Z$) is a morphism. However, the restriction to $Z$ of the previous diagram is

$$\begin{array}{ccc} P_\mathfrak{t}/\mathfrak{U}_\mathfrak{t}(\mathfrak{k}_\mathfrak{t}) & \dashleftarrow & Z' \\ \downarrow & & \downarrow \\ Z_{\mathfrak{w}_\mathfrak{t}} & \longrightarrow & Z \end{array}$$

where $Z'$ is the inverse image of $Z$. In particular $Z'$ is the product of the quotient $P_{\alpha_i}/\mathfrak{U}_{\alpha_i}(k_i)$ for $i$ an index in $\mathfrak{t}$ and $B/\mathfrak{U}_{\alpha_i}(k_i)$ otherwise. In particular on the first line we may define a horizontal map (morphism) from the right to the left by sending $B/\mathfrak{U}_{\alpha_i}(k_i)$ to the point $\{e\}$ for $i$ not an index in $\mathfrak{t}$. By taking sections of the second vertical map, this defines the desired inverse morphism. ☐

**Definition 21.2.11** (ı) Define the map $m_{\mathfrak{w}} : Z_{\mathfrak{w}} \to G/B$ by $m_{\mathfrak{w}}([p_1, \cdots, p_n]) = p_1 \cdots p_n B$.

(ıı) For $X \subset \Pi$, we define the map $m^X_{\mathfrak{w}}$ by composing $m_{\mathfrak{w}}$ with the projection $G/B \to G/P_X$.

## 21.3 Schubert varieties

Let us recall some fact on Coxeter groups.

**Definition 21.3.1** Let $(W, S)$ be a Coxeter group and recall the definition of $T$ given by

$$T = \bigcup_{w \in W} wSw^{-1}.$$

We define the following order on $W$ by setting $v \leq w$ if there exists a sequence $(t_1, \cdots, t_p)$ of elements in $T$ such that

- $v = t_p \cdots t_1 w$
- $\ell(t_j \cdots t_1 w) \leq \ell(t_{j-1} \cdots t_1 w)$ for all $j \in [1, p]$.

This is the order generated by the relation $v < w$ with $\ell(w) = \ell(v) + 1$ and $vw^{-1} \in T$. This order is called the **Bruhat-order of** $(W, S)$**.**

Let us also state some general facts on Coxeter systems that we shall need. Those can be found in [Bo54], in [BB05] or in [Ku02].

**Lemma 21.3.2** *Let $(W, S)$ be a Coxeter system, fix $w = s_1 \cdots s_n$ a reduced expression, then $v \leq w$ (for the Bruhat order) if and only if there exists a sequence $1 \leq j_1 \leq \cdots \leq j_r \leq n$ with $v = s_{j_1} \cdots s_{j_r}$.*

**Definition 21.3.3** For $X \subset S$ we define the subset $W^X$ of $w$ BY

$$W^X = \{w \in W \ / \ \ell(wv) \geq \ell(w) \text{ for all } v \in W_X\}.$$

This is the set of minimal length representatives in the coset $wW_X$.

**Lemma 21.3.4** *(ı) Let $w \in W^X$, then for any $v \in W_X$ we have $\ell(wv) = \ell(w) + \ell(v)$.*

*(ıı) For $w \in W$, there exists a unique element in $wW_X \cap W^X$, in words, there exists a unique minimal length representative of $wW_X$ in $W^X$.*

*(ııı) For $w \in W$, let $w'$ be the unique minimal length representative of $wW_X$ in $W^X$. The for any $v \in W$ we have*

$$v \leq w \Leftrightarrow v' \leq w'.$$

*(ıv) Let $v \leq w$ in $W$ and $s \in S$, then we have the two alternatives*

- *either $sv \leq w$ or $sv \leq sw$;*
- *either $v \leq sw$ or $sv \leq sw$.*

**Proposition 21.3.5** *Let $\mathfrak{w} \in \mathfrak{W}$ and set $w = \pi(\mathfrak{w}) \in W$. We have the following decompositions:*

$$\mathrm{im}(m_{\mathfrak{w}}) = \coprod_{v \leq w} BvB/B.$$

*If furthermore $w \in W^X$ for $X \subset \Pi$, then we have*

$$\mathrm{im}(m^X_{\mathfrak{w}}) = \coprod_{v \leq w,\ v \in W^X} BvP_X/P_X.$$

**Proof :** This is a result on Tits systems.

**Lemma 21.3.6** *Let $(G, B, N, S)$ be a Tits system, let $X \subset S$ and let $w = w_1 \cdots w_k$ such that $\ell(w) = \sum_i \ell(w_i)$.*

*(ı) Let $A_i$ be a subset in $C(w_i)$ such that $A_i \to C(w_i)/B$ is bijective (resp. surjective) for all $i$, then the map*

$$A_1 \times \cdots \times A_k \to BwP_X/P_X$$

*defined by $(a_1, \cdots, a_k) \mapsto a_1 \cdots a_k \pmod P$ is bijective (resp. surjective).*

*(ıı) Assume $w_i \in S$ and let $Z_i \subset P_{w_i}$ be a subset containing $e$ and such that the map $Z_i \to P_{w_i}/B$ is surjective for all $i$, then the image of the map $Z_1 \times \cdots \times Z_k \to G/P_X$ is*

$$\bigcup_{v \leq w} BvP_X/P_X.$$

**Proof :** (ı) The image of this map is $C(w_1) \cdots C(w_k) \pmod{P_X}$. But because the length are additive, we have $C(w) = C(w_1) \cdots C(w_k)$ and the surjectivity follows.

Assume we have elements $a_i, a'_i \in A_i$ with $a_1 \cdots a_k = a'_1 \cdots a'_k \pmod{P_X}$. Then there exists $p \in P_X$ with $a_1 \cdots a_k = a'_1 \cdots a'_k p$. But $p \in C(v)$ with $v \in W_X$ and the left hand side is in $C(w)$ while the right hand side is in $C(wv)$ (by the previous lemma). This is possible only if $v = 1$ i.e. $p \in B$.

Take a reduced decomposition $w_1 = s_1 \cdots s_n$ and choose $D_j \subset C(s_j)$ such that the induced map $D_j \to C(s_j)/B$ is bijective. There exist elements $d_i$ and $d'_i$ in $D_i$ and $b, b'$ in $B$ such that

$$a_1 = d_1 \cdots d_n b \text{ and } a'_1 = d'_1 \cdots d'_n b'.$$

We get

$$(d'_1)^{-1} d_1 \cdots d_n b a_1 \cdots a_k = d'_2 \cdots d'_n b' a'_2 \cdots a'_k p.$$

Now $(d'_1)^{-1} d_1$ lies in $C(c_1) \cdot C(s_1) \subset C(s_1) \cup B$. In this element is in $C(s_1)$, then the left hand side of the previous equality lies in $C(s_1) \cdots C(s_n) C(w_2) \cdots C(w_k) = C(w)$ while the right hqnd side lies in $C(s_1 w)$ a contradiction. Hence $(d'_1)^{-1} d_1 \in B$ and thus $d_1 = d'_1$ by our hypothesis on $D_1$. By induction we get that $d_i = d'_i$ for all $i$ and $a_j = a'_j$ for all $j$.

(ıı) We can assume that $Z_i = \{e\} \cup A_i$ where $A_i \subset C(w_i)$ with $A_i \to C(w_i)/B$ surjective. Then the image of the map is

$$\bigcup C(w_{i_1}) \cdots C(w_{i_r}) P_X \ (\mathrm{mod} P_X)$$

where the union runs over all $1 \leq i_1 < \cdots i_r \leq k$. But we have (use the characterisation of the Bruhat order for the last inclusion)

$$\bigcup C(w_{i_1} \cdots w_{i_r}) \subset \bigcup C(w_{i_1}) \cdots C(w_{i_r}) \subset \bigcup_{v \leq w} C(v).$$

Using again the characterisation of the Bruhat order, we have equality and the result follows. □

The image of $m^X_{\mathfrak{w}}$ is the same as the image of the multiplication from $P_{\mathfrak{w}}/\mathfrak{U}_{\mathfrak{w}}(\mathfrak{k})$ and we apply the previous lemma. □

**Proposition 21.3.7** *Let $V$ be a countable dimensional pro-representation of $G$. Let $[v]$ be a $B$-fixed line in $\mathbb{P}(V)$, then the map $m_{\mathfrak{w}}(v) : Z_{\mathfrak{w}} \to \mathbb{P}(V)$ defined by $x \mapsto m_{\mathfrak{w}}(x) \cdot [v]$ is a morphism of ind-varieties.*

**Proof :** Let $P_i$ be a minimal parabolic subgroup and $W$ a finite dimensional subspace of $V$. Then there exists a $P_i$-stable finite dimensional subspace $W'$ of $V$ and an integer $k$ such that $\mathcal{U}_i(k)$ acts trivially on $W'$ and the action of $P_i/U_i(k)$ on $W'$ is algebraic. An easy induction gives that there exists a finite dimensional subspace $W''$ stable under $P_{\mathfrak{w}}$, such that the subgroup $\mathcal{U}_{\mathfrak{w}}(\mathfrak{k})$ acts trivially and such that the action of $P_{\mathfrak{w}}/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k})$ is algebraic. We thus proved that the map defined by the action of $P_{\mathfrak{w}}/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k})$ is an ind-variety morphism and the result follows from the fact that $\theta$ is Zariski-locally trivial. □

**Definition 21.3.8** (i) For $X \subset \Pi$ and $w \in W^X$, we define the **Schubert variety** $X_w^X \subset G/P_X$ by

$$X_w^X = \coprod_{v \leq w,\ v \in W^X} BvP_X/P_X.$$

(ii) For $w \in W$, we define $X_w^X = X_{w'}^X$ where $w' \in W^X$ is the minimal length representative of $w$. We have

$$X_w^X = \bigcup_{v \leq w} BvP_X/P_X.$$

(iii) We define $X^X = G/P_X$.
(iv) For $n \in \mathbb{N}$, we define the set $X_n^X$ in $X^X$ by:

$$X_n^X = \bigcup_{w \in W^X,\ \ell(w) \leq n} X_w^X = \coprod_{v \in W^X,\ \ell(v) \leq n} BvP_X/P_X.$$

(v) When $X = \emptyset$ we omit $X$ in the notation.

**Remark 21.3.9** For the moment the Schubert varieties are only defined as sets.

**Fact 21.3.10** *We have the inclusion $X_v^X \subset X_w^X$ for $v \leq w$.*

Let us fix $\lambda \in C_X^0$ and take $V$ to be the maximal irreducible highest weight module of weight $\lambda$. Denote by $v_\lambda$ its highest weight and by $m_{\mathfrak{w}}^\lambda$ the map $m_{\mathfrak{w}}(v_\lambda)$.

**Fact 21.3.11** *We have $m_{\mathfrak{w}}^\lambda = m_{\mathfrak{w}}^X \circ i_\lambda$ where $i_\lambda$ is the inclusion of $G/P_X$ in $\mathbb{P}(V)$.*

In particular, we have that $\mathrm{im}(m_{\mathfrak{w}}^\lambda) = i_\lambda(X_w^X)$, thus by the previous proposition (and the fact that $Z_{\mathfrak{w}}$ is projective) the set $i_\lambda(X_w^X)$ is closed in $\mathbb{P}(V)$ and even closed in a subspace $\mathbb{P}(W)$ with $W$ of finite dimension. The same is true for $i_\lambda(X_n^X)$. We put the induced reduced structure of algebraic variety on $i_\lambda(X_w^X)$ and $i_\lambda(X_n^X)$.

**Definition 21.3.12** We denote by $X_w^X(\lambda)$ and $X_n^X(\lambda)$ the associated structures of algebraic varieties on the sets $X_w^X$ and $X_n^X$.

**Fact 21.3.13** *These varieties are projective and irreducible.*

**Definition 21.3.14** We define the ind-variety structure $X^X(\lambda)$ on the set $X^X$ as the one defined by the filtration $(X_n^X(\lambda))_{n \in \mathbb{N}}$.

**Fact 21.3.15** *The inclusion $i_\lambda$ is a closed embedding.*

We now want to prove that these algebraic variety structures defined on $X_w^X$ and depending a priori on $\lambda$ do not depend on it, at least for $\lambda$ big enough (for an appropriate order). Let us first consider a generalisation of the Segre embedding. The proof of the following lemma is straightforward.

**Lemma 21.3.16** *Let $V$ and $W$ be two countable-dimensional vector spaces. Then the Segre map $\mathbb{P}(V)\times\mathbb{P}(W)\to\mathbb{P}(V\otimes W)$ defined by $([v],[w])\mapsto[v\otimes w]$ is a ind-varieties closed embedding.*

Let us now prove the following:

**Proposition 21.3.17** *Fix $\mathfrak{w}$ a reduced word and set $w=\pi(w)$. Assume that $w\in W^X$.*

*(i) The morphism $m_{\mathfrak{w}}^X : Z_{\mathfrak{w}}\to X_{\mathfrak{w}}^X(\lambda)$ is a surjective birational map for all $\lambda\in C_X^0$. More precisely, $m_{\mathfrak{w}}^0$ is an isomorphism between the open subsets $Z_{\mathfrak{w}}^0$ and $m_{\mathfrak{w}}^X(Z_{\mathfrak{w}}^0)$.*

*(ii) We have $m_{\mathfrak{w}}(Z_{\mathfrak{w}}^0)=BwP_X/P_X$ and $X_w^X(\lambda)=\overline{BwP_X/P_X}$ where the closure is taken with respect to the Zariski topology in $\mathbb{P}(V(\lambda))$.*

*(iii) For any $n\geq 0$, the Zariski topology on $X_n^X(\lambda)$ (and hence on $X_w(\lambda)$) does not depend on $\lambda$.*

*(iv) For $\lambda\in C_X^0$ and $\mu\in C_{\mathbb{Z}}$ such that $\langle\mu,\alpha^\vee\rangle=0$ for $\alpha\in X$, the identity map $X_n(\lambda+\mu)\to X_n(\lambda)$ (and in particular the identity map $X_w(\lambda\mu)\to X_w(\lambda)$ is a morphism).*

*(v) The composition*

$$X^X(\lambda+\mu)\to X^{X_\mu}(\mu)\to\mathbb{P}(L^{\max}(\mu))$$

*where $X_\mu=\{\alpha\in\Pi,\ \langle\mu,\alpha^\vee\rangle=0\}$ and $L^{\max}(\mu)$ is the maximal highest weight module of highest weight $\mu$ is a morphism.*

*(v) If $\mathfrak{g}$ is symmetrisable, the morphisms in point (iv) are isomorphisms.*

**Proof :** Write $\mathfrak{w}=(s_{\alpha_1},\cdots,s_{\alpha_n})$ which is reduced.

(i) For any subword $\mathfrak{v}=\mathfrak{w}_J$ of $\mathfrak{w}$, the map $i_{\mathfrak{v},\mathfrak{w}}$ is a closed embedding. In particular, the set $Z_{\mathfrak{w}}^0$, which is the complementary of all these closed embedding, is open. Furthermore, let us prove that

$$m_{\mathfrak{w}}^X(Z_{\mathfrak{w}}\setminus Z_{\mathfrak{w}}^0)\cap m_{\mathfrak{w}}^X(Z_{\mathfrak{w}}^0)=\emptyset.$$

Indeed, let $[p_1,\cdots,p_n]\in Z_{\mathfrak{w}}^0$, then for all $i$, we have $p_i\in C(s_{\alpha_i})$. Because $\mathfrak{w}$ is reduced, we get that $m_{\mathfrak{w}}^X[p_1,\cdots,p_n]=p_1\cdots p_n\in C(s_{\alpha_1})\cdots C(s_{\alpha_n})=C(w)$. For $[q_1,\cdots,q_n]\in Z_{\mathfrak{w}}\setminus Z_w^0$, we get that $m_w^X[q_1,\cdots,q_n]\in C(u)$ for some $u<w$ and the result follows. This intersection proves that $m_{\mathfrak{w}}^X(Z_{\mathfrak{w}}^0)$ is open in $X_{\mathfrak{w}}^X(\lambda)$.

Furthermore, the restriction $Z_{\mathfrak{w}}^0\to BwP_X/P_X$ of the map $m_{\mathfrak{w}}^X$ is bijective by Lemma 21.3.6. But because the action of $B$ on $X_w^X(\lambda)$ is regular and because $BwP_X/P_X$ is an orbit under $B$, we have that $BwP_X/P_X$ is smooth. This bijection (which is a morphism) is thus an isomorphism.

(ii) Let us denote $V=L^{\max}(\lambda)$, $W=L^{\max}(\mu)$ and $U=L^{\max}(\lambda+\mu)$. Consider the $\mathfrak{g}$-module morphism $f:U\to V\otimes W$ defined by $v_{\lambda+\mu}\mapsto v_\lambda\otimes v_\mu$ where $v_\lambda$, $v\mu$ and $v_{\lambda+\mu}$ are highest weight vectors for $V$, $W$ and $U$. The morphism $f$ is also a $G$-morphism. Let $K$ be the kernel of $f$, we get a morphism $\mathbb{P}(U)\setminus\mathbb{P}(K)\to\mathbb{P}(V\otimes W)$. We may consider the injection $i_{\lambda+\mu}:X^X\to\mathbb{P}(U)$. Let us prove that its image is contained in $\mathbb{P}(U)\setminus\mathbb{P}(K)$. Indeed, the vector $v_{\lambda+\mu}$ is not contained in the kernel $K$ and since the map is $G$-equivariant and the image is the orbit of that vector, the result follows. We thus have the following commutative diagram:

$$\begin{array}{ccc} X^X(\lambda+\mu) & \xrightarrow{i_{\lambda+\mu}} & \mathbb{P}(U)\setminus\mathbb{P}(K) \\ \Big\downarrow{\scriptstyle (f_\lambda,f_\mu\circ\sigma)} & & \Big\downarrow{\scriptstyle f} \\ \mathbb{P}(V)\times\mathbb{P}(W) & \longrightarrow & \mathbb{P}(V\otimes W), \end{array}$$

where $\sigma : G/P_X \to G/P_{X_\mu}$ is the natural projection. Since $f$ and $i_{\lambda+\mu}$ are morphisms and since the Segre embedding is a closed immersion, we get by Lemma 20.1.6 that $(f_\lambda, f_\mu \circ \sigma)$ is also a morphism. In particular, $f_\lambda : X^X(\lambda+\mu) \to \mathbb{P}(V)$ is a morphism but it is given as the composition of the identity $X^X(\lambda+\mu) \to X^X(\lambda)$ with $i_\lambda$ which is a closed immersion. Again by Lemma 20.1.6, we get that the identity $X^X(\lambda+\mu) \to X^X(\lambda)$ is a morphism. By restriction we obtain the result for $X_n^X(\lambda)$ and $X_w^X(\lambda)$.

(ııı) In the symmetrisable case, the modules $L^{\max}(\lambda)$ are irreducible. In particular $f$ is injective and $K$ is the trivial module. We get that the induced map $f : \mathbb{P}(U) \to \mathbb{P}(V \otimes W)$ is a closed embedding. Let us consider the previous diagram with $\mu = \lambda$. In that case we get:

$$\begin{array}{ccc} X^X(\lambda) & \xrightarrow{g_\lambda} & \mathbb{P}(U) \setminus \mathbb{P}(K) \\ \downarrow{\scriptstyle (i_\lambda, i_\lambda)} & & \downarrow{\scriptstyle f} \\ \mathbb{P}(V) \times \mathbb{P}(W) & \longrightarrow & \mathbb{P}(V \otimes W), \end{array}$$

where $g_\lambda$ is defined set theoretically by the map $X^X(2\lambda) \to \mathbb{P}(U)$ i.e. by $i_{2\lambda}$. Since now $f$ is a closed embedding, Lemma 20.1.6 implies that $g_\lambda$ is a morphism. Thus the identity map $X^X(\lambda) \to X^X(2\lambda)$ is a morphism and hence an isomorphism (by (ıı)). Now for $\mu$ as in the proposition, take $k$ large enough such that $(2^k - 1)\lambda - \mu \in C_{\mathbb{Z}}$. We have the maps $X^X(2^k\lambda) \to X^X(\lambda+\mu) \to X^X(\lambda)$ which are morphisms are the composed map is an isomorphism. The result follows. □

**Remark 21.3.18** In fact it is true that the morphisms in the point (ıv) in the previous theorem are isomorphisms in the general situation. Indeed Kumar proved in [Ku89] that the map $f : U \to V \otimes W$ is always injective even if the modules are not irreducible.

**Definition 21.3.19** Let us define the partial order $\preccurlyeq$ on $C_{\mathbb{Z}}$ by $\lambda \preccurlyeq \mu$ if $\mu - \lambda \in C_{\mathbb{Z}}$.

We get easily the following:

**Corollary 21.3.20** *(ı) For $w \in W^X$ fixed, there exists $\lambda \in C_{\mathbb{Z}}$ such that for any $\mu \succcurlyeq \lambda$, the identity $X_w^X(\mu) \to X_w^X(\lambda)$ is an isomorphism.*

*(ıı) For $n \geq 0$ fixed, there exists $\lambda \in C_{\mathbb{Z}}$ such that for any $\mu \succcurlyeq \lambda$, the identity $X_n^X(\mu) \to X_n^X(\lambda)$ is an isomorphism.*

We define an algebraic structure on Schubert varieties and on homogeneous spaces under $G$:

**Definition 21.3.21** (ı) For $w \in W^X$, choose $\lambda \in C_X^0$ large enough such that the identity map $X^X(\mu) \to X^X(\lambda)$ is an isomorphism for all $\mu \succcurlyeq \lambda$. We define the **stable variety structure on $X_w^X$ and denote it simply by $X_w^X$** to be the algebraic variety structure of $X_w^X(\lambda)$. In the same way we define the **stable variety structure on $X_n^X$ and denote it simply by $X_n^X$**.

(ıı) For all $n$ the inclusion $X_n^X \to X_{n+1}^X$ is a closed embedding and we may thus define the **stable ind-variety structure on $X^X$** to be the direct limit structure on $X^X$.

**Fact 21.3.22** *(ı) For $X \subset X'$ the natural map $X^X \to X^{X'}$ is a morphism of ind-varieties.*

*(ıı) The action on $X^X$ of $P_i$ induced by the action of $G$ is regular for all $i$. We shall say that the action of $G$ is regular.*

**Definition 21.3.23** For $w \in W^X$ we define **the opposite Schubert cell** by $B_X^w = \mathcal{U}_- w P_X / P_X$ and **the opposite Schubert variety** by

$$X_X^w = \coprod_{v \geq w,\ v \in W^X} B_X^v.$$

We have the following results on the opposite Schubert varieties (which are infinite dimensional but of finite codimension):

**Proposition 21.3.24** *(i) For any $w \in W^X$, we have $\overline{B_X^w} = X_X^w$.*

*(ii) The $T$-fixed points $(G/P_X)^T$ under the action of $T$ via left multiplication are $\{\dot{w}\}_{w \in W^X}$ where $\dot{w} = wP_X$.*

*(iii) For $v$ and $w$ in $W^X$, we have the equivalence:*

$$X_w^X \cap X_X^v \neq \emptyset \Leftrightarrow v \leq w.$$

# Chapter 22

# Vector bundles on homogeneous spaces

## 22.1 Construction of some line bundles

**Definition 22.1.1** (i) Let $\lambda \in C_X^0$, we define an algebraic line bundle $\mathcal{L}(-\lambda)$ on $X^X$ by pulling back along $i_\lambda$ the tautological line bundle on $\mathbb{P}(L^{\max}(\lambda))$.

(ii) Let us define $\mathfrak{h}_X^* = \{\lambda \in \mathfrak{h}_\mathbb{Z}^* \ / \ \langle \lambda, \alpha \rangle = 0 \text{ for } \alpha \in X\}$. We define $\mathcal{L}^X(\lambda)$ for any $\lambda \mathfrak{n} \mathfrak{h}_{\mathbb{Z},X}^*$ by

$$\mathcal{L}^X(\lambda) = \mathcal{L}^X(-\lambda_1)^* \otimes \mathcal{L}^X(-\lambda_2)$$

where $\lambda = \lambda_1 - \lambda_2$ with $\lambda_1$ nd $\lambda_2$ in $C_X^0$. Because $\mathrm{Pic}(X)$ is an abelian group, this does not depend on the decomposition of $\lambda$.

(iii) For $w \in W$, we denote by $\mathcal{L}_w^X(\lambda)$ the pull back of $\mathcal{L}^(\lambda)$ to $X_w^X(\lambda)$.

**Lemma 22.1.2** *For $\lambda$ and $\mu$ in $C_X^0$, the line bundle $\mathcal{L}^X(-\lambda) \otimes \mathcal{L}^X(-\mu)$ is isomorphic to $\mathcal{L}^X(-\lambda - \mu)$.*

**Proof :** This follows from the first commutative diagram in the proof of Proposition 21.3.17 □

**Fact 22.1.3** *(i) The group $G$ acts on $L^{\max}(\lambda)$. In particular it acts regularly on $\mathbb{P}(L^{\max}(\lambda))$ and thus on $X^X$.*

*(ii) Furthermore, $G$ also acts on the tautological bundle over $\mathbb{P}(L^{\max}(\lambda))$ and thus also on the line bundle $\mathcal{L}^X(\lambda)$ over $X^X$. These action are regular.*

*(iii) We may thus define a regular action of $G$ on any line bundle $\mathcal{L}^X(\lambda)$ by tensor product of the actions.*

**Definition 22.1.4** For $\mathfrak{w} \in \mathfrak{W}$, we have a morphism $m_\mathfrak{w} : Z_\mathfrak{w} \to G/B = X$. For any $\lambda \in \mathfrak{h}_\mathbb{Z}^*$ we define $\mathcal{L}_\mathfrak{w}(\lambda)$ to be the pull-back of $\mathcal{L}(\lambda)$ by $m_\mathfrak{w}$.

## 22.2 Cohomology of certain line bundles

We need more notation on words.

**Definition 22.2.1** Let $\mathfrak{w} \in \mathfrak{W}$ be a word of length $n$ with $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{\alpha_n})$ and let $i$ be any integer in $[1, n]$. Then we define $\mathfrak{w}(i) = (s_{\alpha_i}, \cdots, s_{a_{i-1}}, \sigma_{\alpha_{i+1}}, \cdots, s_{\alpha_n})$ and $\mathfrak{w}[i] = s_{\alpha_1}, \cdots, s\alpha_i)$.

We now that $i_{\mathfrak{w}(i),\mathfrak{w}}$ is a closed embedding whose image is an irreducible divisor on $Z_\mathfrak{w}$. We also know that the map $Z_\mathfrak{w} \to Z_{\mathfrak{w}[i]}$ is a smooth morphism (locally trivial with fiber $Z_\mathfrak{v}$ with $\mathfrak{v} = (s_{\alpha_{i+1}}, \cdots, s_{\alpha_n}))$ and with a section $i_{\mathfrak{w}[i],\mathfrak{w}}$.

**Proposition 22.2.2** *For any $\mathfrak{w} \in \mathfrak{W}$ of length $n$, the canonical line bundle $K_{Z_\mathfrak{w}}$ of $Z_\mathfrak{w}$ is isomorphic with the line bundle*

$$\mathcal{L}_\mathfrak{w}(-\rho) \otimes \mathcal{O}_{Z_\mathfrak{w}}\left(-\sum_{i=1}^{n} Z_{\mathfrak{w}(i)}\right),$$

*where $\rho$ is any element in $\mathfrak{h}_\mathbb{Z}^*$ such that $\langle \rho, \alpha^\vee \rangle = 1$ for all simple coroots $\alpha^\vee$.*

**Proof :** We prove this by induction on $\ell(\mathfrak{w})$. We shall use the following classical lemma:

**Lemma 22.2.3** *Let $f : X \to Y$ be a $\mathbb{P}^1$-fibration with a section $\sigma$. Denote by $D$ the divisor $\sigma(Y)$.*
*(ı) We have the formula $K_X = f^*K_Y \otimes \mathcal{O}_X(-2D) \otimes f^*\sigma^*\mathcal{O}_X(D)$.*
*(ıı) Assume there is a line bundle $\mathcal{L}$ on $X$ of degree one on the fibers of $f$. Then we have*

$$K_X = f^*K_Y \otimes \mathcal{O}_X(-D) \otimes \mathcal{L} \otimes f^*\sigma^*\mathcal{L}.$$

**Proof :** (ı) From the normal bundle exact sequence ) $\to T_D \to \sigma^*T_X \to N_{D,X} \to$), we get that the restriction of the relative canonical sheaf $K_{X/Y}$ to $D$ is the conormal bundle $N_{D,X}^\vee$. But this conormal bundle in $\mathcal{O}_X(D)|_D = \sigma^*\mathcal{O}_X(D)$. We thus have $\sigma^*K_{X/Y} = \sigma^*\mathcal{O}_X(D)$. Let us tensor with $\sigma^*\mathcal{O}_X(2D)$. We get $\sigma^*(K_{X/Y} \otimes \mathcal{O}_X(2D) = \sigma^*\mathcal{O}_X(D)$. But $K_{X/Y}$ is of degree $-2$ on the fibers of $f$ while $\mathcal{O}_X(D)$ is of degree 1. In particular $K_{X/Y} \otimes \mathcal{O}_X(2D)$ is of degree 0 on the fibers and comes from $Y$. We thus have $K_{X/Y} \otimes \mathcal{O}_X(2D) = f^*\sigma^*(K_{X/Y} \otimes \mathcal{O}_X(2D))$. With the previous relation we get $K_{X/Y} \otimes \mathcal{O}_X(2D) = f^*\sigma^*\mathcal{O}_X(D)$ and the result follows.

(ıı) This follows from the fact that if $\mathcal{L}$ and $\mathcal{L}'$ are two line bundles with the same degree on the fibers of $f$, then $\mathcal{L}^{-1} \otimes f^*\sigma^*\mathcal{L} = (\mathcal{L}')^{-1} \otimes f^*\sigma^*\mathcal{L}'$. Indeed, this is true because $\mathcal{L}' \otimes \mathcal{L}^{-1}$ is of degree 0 on the fibres thus $\mathcal{L}' \otimes \mathcal{L}^{-1} = f^*\sigma^*(\mathcal{L}' \otimes \mathcal{L}^{-1})$. □

Now look at the $\mathbb{P}^1$-fibration $f : Z_\mathfrak{w} \to Z_{\mathfrak{w}(n)}$ together with its section $\sigma$. Remark that the line bundle $\mathcal{L}_\mathfrak{w}(\rho)$ is of degree one on the fibers so that we may apply the Lemma. Furthermore, we have $m_{\mathfrak{w}(n)} = m_\mathfrak{w} \circ \sigma$ thus $\sigma^*\mathcal{L}_\mathfrak{w}(\rho) = \mathcal{L}_{\mathfrak{w}(n)}(\rho)$ and the result follows. □

**Definition 22.2.4** Let $V$ be a $B$-module, then we can view $V$ as a $B^n$ module by projecting on the last factor. Moreover, if the action of $B$ is regular, then the same is true for the action of $B^n$. For such a regular $B$-module, we define the vector bundle $\mathcal{L}_\mathfrak{w}(V)$ on $Z_\mathfrak{w}$ by

$$\mathcal{L}_\mathfrak{w}(V) = P_\mathfrak{w}/\mathcal{U}_\mathfrak{w}(\mathfrak{k}) \times^{B^n/\mathcal{U}_\mathfrak{w}(\mathfrak{k})} V.$$

We denote by $\mathbb{C}_\lambda$ the one dimensional representation of $B$ such that $\mathcal{U}$ acts trivially and $T$ acts by $\lambda$.

**Proposition 22.2.5** *Assume $V$ to be finite dimensional.*

*(ı) The definition of $\mathcal{L}_\mathfrak{w}(V)$ does not depend on the choice of $\mathfrak{k}$ and is functorial in $V$. It is a vector bundle over $Z_\mathfrak{w}$. Moreover, $\mathcal{L}_\mathfrak{w}(\bullet)$ is an exact functor.*

*(ıı) The vector bundle $\mathcal{L}_\mathfrak{w}(V)$ has a regular action of $P_{\alpha_1}$, in particular, for all $i$ the cohomology group $H^i(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(V))$ is a finite-dimensional pro-representation of $P_{\alpha_1}$.*

*(ııı) We have an isomorphism of line bundles $\mathcal{L}_\mathfrak{w}(\mathbb{C}_\lambda) = \mathcal{L}_\mathfrak{w}(-\lambda)$.*

**Proof :** (ı) Because $V$ is finite dimensional, we may assume that $\mathcal{U}_\mathfrak{w}(\mathfrak{k})$ acts trivially on $V$ and the action of $B^n/\mathcal{U}_\mathfrak{w}(\mathfrak{k})$ is algebraic. We get that this is a vector bundle over $Z_\mathfrak{w}$ (because $P_\mathfrak{w}/\mathcal{U}_\mathfrak{w}(\mathfrak{k}) \to Z_\mathfrak{w}$ is locally trivial it is the case of $\mathcal{L}_\mathfrak{w}(V)$). Furthermore, we have a $P_{\alpha_1}$-regular action. The exactness of the functor is clear.

To prove that this does not depend on $\mathfrak{k}$, we take another such sequence $\mathfrak{k}'$. Of course we may assume that $\mathfrak{k}' \geq \mathfrak{k}$ and denote by $\mathcal{L}'_{\mathfrak{w}}(V)$ the corresponding vector bundle. We have a commutative diagram:

$$\begin{array}{ccc} P_{\mathfrak{w}}/\mathfrak{U}_{\mathfrak{w}}(\mathfrak{k}') & \longrightarrow & P_{\mathfrak{w}}/\mathfrak{U}_{\mathfrak{w}}(\mathfrak{k}) \\ & \searrow & \downarrow \\ & & Z_{\mathfrak{w}} \end{array}$$

from which we get a $P_{\alpha_1}$-equivariant commutative diagram:

$$\begin{array}{ccc} \mathcal{L}'_{\mathfrak{w}}(V) & \longrightarrow & \mathcal{L}_{\mathfrak{w}}(V) \\ & \searrow & \downarrow \\ & & Z_{\mathfrak{w}} \end{array}$$

which is an isomorphism on the fibers and the result follows.

(ıı) Follows from the previous proof.

(ııı) We start with $\lambda \in C_{\mathbb{Z}}$. We take $\mathfrak{k}$ such that $\mathfrak{U}_{\mathfrak{w}}(\mathfrak{k})$ acts trivially on $v_\lambda$ and such that the action of $P_{\mathfrak{w}}/\mathfrak{U}_{\mathfrak{w}}(\mathfrak{k})$ on $v_\lambda$ is algebraic with value in a finite dimensional sub-representation $W$ of $L^{\max}(\lambda)$. We define a map

$$\mathcal{L}_{\mathfrak{w}}(\mathbb{C}_\lambda) \to \mathbb{P}(L^{\max}(\lambda)) \times L^{\max}(\lambda)$$

by $(\bar{p}_1, \cdots, \bar{p}_n, z)\ (\mathrm{mod} B^n/\mathfrak{U}_{\mathfrak{w}}(\mathfrak{k})) \mapsto (m_{\mathfrak{w}}^{\lambda}[p_1, \cdots, p_n], p_1 \cdots p_n z v_\lambda)$. This induces a commutative diagram:

$$\begin{array}{ccc} \mathcal{L}_{\mathfrak{w}}(\mathbb{C}_\lambda) & \longrightarrow & \mathcal{L}_{\mathfrak{w}}(-\lambda) \\ & \searrow & \downarrow \\ & & Z_{\mathfrak{w}}. \end{array}$$

This gives an isomorphism of vector bundle for $\lambda \in C_{\mathbb{Z}}$. The result for general $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ follows from the canonical isomorphism of vector bundles $\mathcal{L}_{\mathfrak{w}}(V \otimes W) \simeq \mathcal{L}_{\mathfrak{w}}(V) \otimes \mathcal{L}_{\mathfrak{w}}(W)$. □

We now need a Lemma on algebraic varieties:

**Lemma 22.2.6** *(ı) Let $E$ be an $H$-equivariant vector bundle on a left $H$-variety $Z$ and let $\pi : X \to Y$ be a principal $H$-bundle. Then the $H$-equivariant vector bundle $\mathcal{O}_X \otimes E$ on $X \times Z$ descends uniquely to a vector bundle $\mathcal{L}_\pi(E)$ on $X \times^H Z$.*

*(ıı) Assume furthermore that $Z$ is projective and denote by $p$ the projection $X \times^H Z \to Y$. Then there is a canonical isomorphism of $\mathcal{O}_Y$ modules $R^i p_*(\mathcal{L}_\pi(E)) \simeq \mathcal{L}_\pi(H^i(Z, E))$.*

**Proof :** (ı) This is a descent argument. For a principal $H$-bundle $f$, the map $f^*$ from the set of isomorphism classes of vector bundles on the base to $H$-equivariant vector bundles on the principal bundle is a bijection. Applying this to $p$ which is a principal $H$-bundle we get that $\mathcal{L}_\pi(E) = (p^*)^{-1}(E)$. The construction of this bundle is simply given by the quotient $E/H$, the action being free because is is already free on $X$.

(ıı) All the fibers of the map $p$ are isomorphic to $Z$ and the restriction of the vector bundle $\mathcal{L}_\pi(E)$ on any fiber is isomorphic to $E$. In particular the sheaf $R^i p_* \mathcal{L}_\pi(E)$ is locally free on $Y$. We have the following commutative diagram

$$\begin{array}{ccc} X \times Z & \xrightarrow{\ q\ } & X \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} \\ X \times^H Z & \xrightarrow{\ p\ } & Y \end{array}$$

where $p$ is flat. By flat base chase we get $\pi^* R^i p_* \mathcal{L}_\pi(E) = R^i(p_1)_* q^* \mathcal{L}_\pi(E) = R^i(p_1)_*(E)$. This sheaf is the trivial sheaf $H^i(Z,E) \otimes \mathcal{O}_X$. By definition of $\mathcal{L}_\pi(H^i(Z,E))$ the result follows. □

By — by now classical — methods using the map $P_{\mathfrak{w}}/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k}) \to Z_{\mathfrak{w}}$ we may using the previous Lemma prove the following:

**Lemma 22.2.7** *Let $\mathfrak{w} \in \mathfrak{W}$ and $i \in [1,n]$, denote by $\psi$ the morphism $Z_{\mathfrak{w}} \to Z_{\mathfrak{w}[i]}$ and let $V$ be a regular representation of $B$. For any $j \geq 0$, the sheaf $R^j \psi_* \mathcal{L}_{\mathfrak{w}}(V)$ is canonically isomorphic to the vector bundle $\mathcal{L}_{\mathfrak{w}[i]}(H^j(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(V)))$ where $\mathfrak{v} = (s_{\alpha_{i+1}}, \cdots, s_{\alpha_n})$.*

**Proof :** Set $H = B/\mathcal{U}_{\alpha_i}(k_i)$. Set also $L_1 = B^{i-1}/\mathcal{U}_{\mathfrak{w}[i](i)}(\mathfrak{k}[i](i))$ and $L_2 = B^{n-i}/\mathcal{U}_{\mathfrak{v}}(\mathfrak{k}')$. Set $L = L_1 \times \{1\} \times L_2$. We consider the locally trivial $H$-bundle $(P_{\mathfrak{w}}/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k}))/L \to Z_{\mathfrak{w}}$ (this follows from the fact that $P_{\mathfrak{w}}/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k}) \to Z_{\mathfrak{w}}$ is a locally trivial $B^n/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k})$-bundle. Let $Z'_{\mathfrak{w}[i]} = P_{\mathfrak{w}[j]}/L_1 \times \{1\}$. Then $(P_{\mathfrak{w}}/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k}))/L \simeq Z'_{\mathfrak{w}[i]} \times \mathbb{Z}_v$.We have the following commutative diagram:

$$\begin{array}{ccc} Z'_{\mathfrak{w}[i]} \times Z_{\mathfrak{v}} & \xrightarrow{\pi} & Z_{\mathfrak{w}} \\ \downarrow & & \downarrow \psi \\ Z'_{\mathfrak{w}[i]} & \longrightarrow & Z_{\mathfrak{w}[i]}. \end{array}$$

The horizontal maps are locally trivial principal $H$-bundles and furthermore we have $Z_{\mathfrak{w}} = Z'_{\mathfrak{w}[i]} \times^H Z_{\mathfrak{v}}$. Now the pull-back of $\mathcal{L}_{\mathfrak{w}}(V)$ to $Z'_{\mathfrak{w}[i]} \times Z_{\mathfrak{v}}$ is $\mathcal{O}_{Z'_{\mathfrak{w}[i]}} \otimes \mathcal{L}_{\mathfrak{v}}(V)$ thus $\mathcal{L}_{\mathfrak{w}}(V) = \mathcal{L}_\pi(\mathcal{O}_{Z'_{\mathfrak{w}[i]}} \otimes \mathcal{L}_{\mathfrak{v}}(V))$ with the notation of Lemma 22.2.6 (ı). The result follows by Lemma 22.2.6 (ıı). □

**Definition 22.2.8** (ı) Let $\mathfrak{w}$ be any word (non necessarily reduced). The image of the map $m_{\mathfrak{w}} : Z_{\mathfrak{w}} \to G/B$ is contained in $X_n$. Because $Z_{\mathfrak{w}}$ is irreducible and projective, this image is a closed irreducible $B$-stable subvariety of $X_n$ thus it is of the form $X_w$ (we do not have $w = \pi(\mathfrak{w})$ in general since $\mathfrak{w}$ is not reduced).

(ıı) Let $\widetilde{X}_w$ be the normalisation of $X_w$. Because $Z_{\mathfrak{w}}$, the map $m_{\mathfrak{w}}$ lifts to a unique $B$-equivariant morphism $\widetilde{m}_{\mathfrak{w}} : Z_{\mathfrak{w}} \to \widetilde{X}_w$ such that the following diagram is commutative:

$$\begin{array}{ccc} Z_{\mathfrak{w}} & \xrightarrow{\widetilde{m}_{\mathfrak{w}}} & \widetilde{X}_w \\ & \searrow^{m_{\mathfrak{w}}} & \downarrow \\ & & X_w. \end{array}$$

**Proposition 22.2.9** *Let $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{\alpha_n})$ be a reduced word and set $w = \pi(\mathfrak{w})$. Assume that we have $H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(p\lambda)) = 0$, for all $p$ and $i$ positive and some $\lambda \in C^0_\emptyset$ $(\star)$.*

*(ı) Then for any finite dimensional pro-representation $V$ of $B$, we have*

$$R^i(\widetilde{m}_{\mathfrak{w}})_*(\mathcal{L}_{\mathfrak{w}}(V)) = 0 \textit{ for all } i > 0$$

*and $(\widetilde{m}_{\mathfrak{w}})_*(\mathcal{L}_{\mathfrak{w}}(V))$ is a $B$-equivariant vector bundle on $\widetilde{X}_w$ or rank* $\dim V$*.*

*(ıı) The canonical $\mathcal{O}_{Z_{\mathfrak{w}}}$-module map $\widetilde{m}^*_{\mathfrak{w}}(\widetilde{m}_{\mathfrak{w}})_*(\mathcal{L}_{\mathfrak{w}}(V)) \to \mathcal{L}_{\mathfrak{w}}(V)$ is an isomorphism.*

*(ııı) If $\mathfrak{v}$ is another reduced word with $\pi(\mathfrak{v}) = w$, then there is a canonical isomorphism*

$$(\widetilde{m}_{\mathfrak{v}})_*(\mathcal{L}_{\mathfrak{v}}(V)) \simeq (\widetilde{m}_{\mathfrak{w}})_*(\mathcal{L}_{\mathfrak{w}}(V))$$

*of $B$-equivariant $\mathcal{O}_{\widetilde{X}_w}$-modules.*

*(iv) If furthermore ($\star$) holds for $\mathfrak{v}$, then we have an isomorphism of $B$-modules:*

$$H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(V)) \simeq H^i(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(V))$$

*for all $i$. We denote the $B$-equivariant vector bundle $(\widetilde{m}_{\mathfrak{w}})_*(\mathcal{L}_{\mathfrak{w}}(V))$ on $\widetilde{X}_w$ (which is independent of $\mathfrak{w}$) by $\widetilde{\mathcal{L}}_w(V)$.*

**Proof :** We prove (i) and (ii) by induction on $\dim V$. If $V$ is of dimension one, then $V = \mathbb{C}_\mu$ for some $\mu$. But we have $\mathcal{L}_{\mathfrak{w}}(V) = \mathcal{L}_{\mathfrak{w}}(\mathbb{C}_\mu) = m_{\mathfrak{w}}^*\mathcal{L}_w(-\mu) = \widetilde{m}^*\nu^*\mathcal{L}_w(-\mu)$ where $\nu$ is the normalisation. By projection formula we have $R^i(\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(\mathbb{C}_\mu) = R^i(\widetilde{m}_{\mathfrak{w}})_*\mathcal{O}_{Z_{\mathfrak{w}}} \otimes \nu^*\mathcal{L}_w(-\mu)$. By normality of $\widetilde{X}_w$ we get that $(\widetilde{m}_{\mathfrak{w}})_*\mathcal{O}_{Z_{\mathfrak{w}}} = \mathcal{O}_{\widetilde{X}_w}$. The fact that $R^i(\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(\mathcal{O}_{Z_{\mathfrak{w}}}) = 0$ follows from the fact that for $\lambda \in C^0_\emptyset$ the line bundle $\mathcal{L}_w(\lambda)$ is ample on $X_w$ and thus (because $\nu$ is finite) that $\nu^*\mathcal{L}_w(\lambda)$ is ample on $\widetilde{X}_w$.

**Lemma 22.2.10** *Let $f : X \to Y$ be a morphism between projective schemes and let $\mathcal{L}$ be an ample invertible sheaf on $Y$. Assume that $H^i(X, f^*\mathcal{L}^n) = 0$ for $i > 0$ and $n$ large enough, then $R^if_*\mathcal{O}_X = 0$ for all $i > 0$.*

**Proof :** Consider the Leray spectral sequence

$$H^j(Y, R^if_*f^*\mathcal{L}^n) \Rightarrow H^{i+j}(X, f^*\mathcal{L}^n).$$

By the projection formula we have $R^if_*f^*\mathcal{L}^n = R^if_*\mathcal{O}_X \otimes \mathcal{L}^n$ and because $\mathcal{L}$ is ample on $Y$ we have the vanishing of $H^j(Y, R^if_*f^*\mathcal{L}^n)$ for $j > 0$ and $n$ large enough. In particular we get the equality $H^i(X, f^*\mathcal{L}^n) = H^0(Y, (R^if_*\mathcal{O}_X \otimes \mathcal{L}^n))$. For $n$ large enough this group also vanishes. But because $\mathcal{L}$ is ample, for $n$ large enough the sheaf $R^if_*\mathcal{O}_X \otimes \mathcal{L}^n$ is generated by its sections. This implies $R^if_*\mathcal{O}_X \otimes \mathcal{L}^n = 0$ and the result follows. □

We will need the dimension one case in the proof. Let $V$ be of higher dimension. Then we may assume, because the action is regular that there exists an integer $k$ with $\mathcal{U}(k)$ acting trivially on $V$ and $B/\mathcal{U}(k)$ acting algebraically. In particular, by Lie's theorem, there exists a non trivial $B/\mathcal{U}(k)$-subspace $W$ of $V$. We thus have an exact sequence of $B$-regular modules

$$0 \to W \to V \to Q \to 0.$$

By induction and applying the long exact sequence of cohomology we get the result.

(iii) Follows also by induction with the same ideas. For $\dim V = 1$ we have already seen that this is true because $R^i(\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(\mathbb{C}_\mu) = R^i(\widetilde{m}_{\mathfrak{w}})_*\mathcal{O}_{Z_{\mathfrak{w}}} \otimes \nu^*\mathcal{L}_w(-\mu)$. We have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{m}_{\mathfrak{w}}^*(\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(W) & \longrightarrow & \widetilde{m}_{\mathfrak{w}}^*(\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(V) & \longrightarrow & \widetilde{m}_{\mathfrak{w}}^*(\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(Q) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{L}_{\mathfrak{w}}(W) & \longrightarrow & \mathcal{L}_{\mathfrak{w}}(V) & \longrightarrow & \mathcal{L}_{\mathfrak{w}}(Q) & \longrightarrow & 0
\end{array}$$

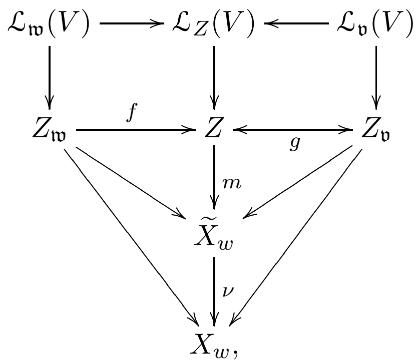
and the result follows by induction.

As the proof will show and by elementary relations in the Coxeter group, we only need to deal with the case where $\mathfrak{w} = (\mathfrak{u}, \mathfrak{w}_0, \mathfrak{t})$ and $\mathfrak{v} = (\mathfrak{u}, \mathfrak{v}_0, \mathfrak{t})$ where $\mathfrak{w}_0 = (s, t, s, t, \cdots)$ and $\mathfrak{v}_0 = (t, s, t, s, \cdots)$ with $\mathfrak{w}_0$ and $\mathfrak{v}_0$ of the same length $m$ and $st$ of order $m$ in $W$. Let us denote the rank two parabolic subgroup of $G$ associated to $\{s, t\}$ by $P_0$ and by $\mathcal{U}_0$ its unipotent subgroup. We then consider $\mathcal{U}_{\mathfrak{w}}(\mathfrak{k})$ and $\mathcal{U}_{\mathfrak{v}}(\mathfrak{k}')$ such that $\mathcal{U}_{\mathfrak{w}_0}(\mathfrak{k}_0) = \mathcal{U}_{\mathfrak{v}_0}(\mathfrak{k}'_0) = \mathcal{U}_0^m$. Let us also set

$$C = P_{\mathfrak{u}}/\mathcal{U}_{\mathfrak{u}}(\mathfrak{k}_{\mathfrak{u}}) \times P_0/U_0 \times P_{\mathfrak{t}}/\mathcal{U}_{\mathfrak{t}}(\mathfrak{k}_{\mathfrak{t}}) \text{ and } H = B^{\ell(\mathfrak{u})}/\mathcal{U}_{\mathfrak{u}}(\mathfrak{k}_{\mathfrak{u}}) \times B/\mathcal{U}_0 \times B^{\ell(\mathfrak{t})}/\mathcal{U}_{\mathfrak{t}}(\mathfrak{k}_{\mathfrak{t}}).$$

We define $Z = C/H$ and the map $\theta_Z : C \to Z$ which is a locally trivial $H$-principal bundle.

The $B$-module structure on $V$ induces an $H$-module structure and we may define $\mathcal{L}_Z(V)$ by $C \times^H V$. We have the following commuting diagram:



where the maps from $Z_{\mathfrak{w}}$ and $Z_{\mathfrak{v}}$ to $Z$ are given by the multiplications $P_{\mathfrak{w}_0} \to P_0$ and $P_{\mathfrak{v}_0} \to P_0$ and the map from $Z$ to $X_w$ is given by the multiplications $P_{\mathfrak{u}} \times P_0 \times P_{\mathfrak{t}} \to G$. The maps to $X_w$ can we lifted to $\widetilde{X}_w$ because all the starting varieties are smooth hence normal. By construction, we have the equalities

$$f^*\mathcal{L}_Z(V) = \mathcal{L}_{\mathfrak{w}}(V) \text{ and } g^*\mathcal{L}_Z(V) = \mathcal{L}_{\mathfrak{v}}(V).$$

We get the formulas

$$(\widetilde{m}_{\mathfrak{w}})_*(\mathcal{L}_{\mathfrak{w}}(V)) = m_*\mathcal{L}_Z(V) = (\widetilde{m}_{\mathfrak{v}})_*\mathcal{L}_{\mathfrak{v}}(V).$$

The last result of the Proposition follows from the previous one and Leray spectral sequence. Indeed, we have with the hypothesis of the Proposition the following equalities of cohomology groups $H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(V)) = H^i(\widetilde{X}_w, (\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(V)) = H^i(\widetilde{X}_w, (\widetilde{m}_{\mathfrak{v}})_*\mathcal{L}_{\mathfrak{v}}(V)) = H^i(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(V))$. □

To prove the main result of this section, we will use the following general vanishing result (see [GR70]):

**Theorem 22.2.11** *Assume the base field is of characteristic 0. Let $X$ be a smooth irreducible projective variety and $\mathcal{L}$ a line bundle on $X$ such that there exists an integer $N$ and a birational morphism $f : X \to Y \subset \mathbb{P}^M$ such that $f^*\mathcal{O}(1) = \mathcal{L}^n$. Then we have for $0 \leq i < \dim X$ the vanishing*

$$H^i(X, \mathcal{L}^{-1}) = 0.$$

**Theorem 22.2.12** *Let $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{\alpha_n})$ be a word and let $1 \leq j \leq k \leq n$ be such that the subword $\mathfrak{v} = (s_{\alpha_j}, \cdots, s_{\alpha_k})$ is reduced. Then for any $\lambda \in C_{\mathbb{Z}}$*

*(ı) we have*

$$H^i\left(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}\left(-\sum_{q=j}^{k} Z_{\mathfrak{w}(q)}\right)\right) = 0 \text{ for all } i > 0;$$

*(ıı) the equality $H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = 0$ for all $i > 0$;*

*(ııı) if $k < n$ and the word $(\mathfrak{v}, s_{\alpha_{k+1}})$ is not reduced, then we have*

$$H^i\left(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}\left(-\sum_{q=j}^{k} Z_{\mathfrak{w}(q)}\right)\right) = 0 \text{ for all } i \geq 0.$$

**Proof :** The theorem is true for $n = 1$ because then $Z_{\mathfrak{w}} = \mathbb{P}^1$ and $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-k)) = 0$ for $k \leq 1$. We proceed by induction on $\ell(\mathfrak{w}) = n$. We will denote by $\partial Z_{\mathfrak{w}}^{j,k}$ the sum

$$\sum_{q=j}^{k} Z_{\mathfrak{w}(q)}.$$

We first prove that if the result holds for some $1 \leq j < k \leq n$, then it holds if we replace $k$ by $k-1$. Indeed, consider the exact sequence $0 \to \mathcal{O}_{Z_{\mathfrak{w}}}(-Z_{\mathfrak{w}(k)}) \to \mathcal{O}_{Z_{\mathfrak{w}}} \to i_*\mathcal{O}_{Z_{\mathfrak{w}}(k)} \to 0$. Tensoring with $\mathcal{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}(-\partial Z_{\mathfrak{w}}^{j,k-1})$ gives the exact sequence

$$0 \to \mathcal{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}(-\partial Z_{\mathfrak{w}}^{j,k}) \to \mathcal{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}(-\partial Z_{\mathfrak{w}}^{j,k-1}) \to i_*\mathcal{O}_{Z_{\mathfrak{w}}(k)} \otimes \mathcal{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}(-\partial Z_{\mathfrak{w}}^{j,k-1}) \to 0.$$

But $Z_{\mathfrak{w}(k)}$ and $Z_{\mathfrak{w}(q)}$ meet transversaly for $q \neq k$ (consider this in $P_{\mathfrak{w}}/\mathcal{U}_{\mathfrak{w}}(\mathfrak{k})$ and descend the result via $\theta$). This implies the equality

$$i_*\mathcal{O}_{Z_{\mathfrak{w}}(k)} \otimes \mathcal{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}(-\partial Z_{\mathfrak{w}}^{j,k-1}) = i_*\left(\mathcal{L}_{\mathfrak{w}(k)}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}(k)}}(-\sum_{q=j}^{k-1} Z_{\mathfrak{w}(q)} \cap Z_{\mathfrak{w}(k)})\right).$$

The assertion follows by induction and from the long exact sequence in cohomology. Remark also that by the same argument (even simpler) we get the result for $\mathfrak{v}$ empty from the case $\mathfrak{v} = (s_{\alpha_k})$.

We are thus left with the following two different situations:

(a) $k = n$;

(b) $k < n$ and the sequence $(\mathfrak{v}, s_{\alpha_{k+1}})$ is not reduced.

We start with case (a). By the same argument as above, we can assume that one of the following situations occur: $j = 1$ or $j > 1$ and $(s_{\alpha_{j-1}}, \mathfrak{v})$ is not reduced.

In the first case, the result follows from Theorem 22.2.11. Indeed, we have the equality

$$\mathcal{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}(-\partial_{Z_{\mathfrak{w}}}^{1,n}) = \mathcal{L}_{\mathfrak{w}}(\lambda + \rho) \otimes K_{Z_{\mathfrak{w}}}.$$

By Serre duality we get $H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda + \rho) \otimes K_{Z_{\mathfrak{w}}}) = H^{n-i}(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(-\lambda - \rho))^*$ and the result follows.

In the second case, we proceed as follows. We define the word $\mathfrak{u} = (s_{\alpha_1}, \cdots, s_{\alpha_{j-1}})$ and we consider the projection $\psi : Z_{\mathfrak{w}} \to Z_{\mathfrak{u}}$. It is locally trivial with fiber $Z_{\mathfrak{v}}$. Furthermore, by induction hypothesis we have $H^i(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{v}}}(-\partial Z_{\mathfrak{v}}^{1,n-j+1})) = 0$ for $i > 0$. By Leray spectral sequence and Lemma 22.2.7 we obtain

$$\begin{aligned} H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}(-\partial Z_{\mathfrak{w}}^{j,n})) &= H^i(Z_{\mathfrak{u}}, \psi_*(\mathcal{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}(-\partial Z_{\mathfrak{w}}^{j,n})) \\ &= H^i(Z_{\mathfrak{u}}, \mathcal{L}_{\mathfrak{u}}(H^0(Z_{\mathfrak{v}}, \mathcal{L}_v(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{v}}}(-\partial Z_{\mathfrak{v}}^{1,n-j+1})))) \\ &= H^i(Z_{\mathfrak{u}}, \mathcal{L}_{\mathfrak{u}}(H^0(Z_{\mathfrak{v}}, \mathcal{L}_v(\lambda + \rho) \otimes K_{Z_{\mathfrak{v}}}))). \end{aligned}$$

Assume (we will prove this in the next Fact) that the $B$-module $H^0(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(\lambda+\rho) \otimes K_{Z_{\mathfrak{v}}})$ has a $P_{\alpha_{j-1}}$-module structure extending the $B$-module structure. Then if we consider the fibration $Z_{\mathfrak{u}} \to Z_{\mathfrak{u}[j-2]}$ which is a $\mathbb{P}^1$-bundle, then the sheaf $\mathcal{L}_{\mathfrak{u}}(H^0(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(\lambda+\rho) \otimes K_{Z_{\mathfrak{v}}}))$ is trivial on the fibers and we deduce the equalities

$$\begin{aligned} H^i(Z_{\mathfrak{u}}, \mathcal{L}_{\mathfrak{u}}(H^0(Z_{\mathfrak{v}}, \mathcal{L}_v(\lambda + \rho) \otimes K_{Z_{\mathfrak{v}}}))) &= H^i(Z_{\mathfrak{u}(j-1)}, \mathcal{L}_{\mathfrak{u}(j-1)}(H^0(Z_{\mathfrak{v}}, \mathcal{L}_v(\lambda + \rho) \otimes K_{Z_{\mathfrak{v}}}))) \\ &= H^i(Z_{\mathfrak{u}(j-1)}, \mathcal{L}_{\mathfrak{u}(j-1)}(H^0(Z_{\mathfrak{v}}, \mathcal{L}_v(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{v}}}(-\partial Z_{\mathfrak{v}}^{1,n-j+1})))) \\ &= H^i(Z_{\mathfrak{w}(j-1)}, \mathcal{L}_{\mathfrak{w}(j-1)}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}(j-1)}}(-\partial Z_{\mathfrak{w}(j-1)}^{j,n})). \end{aligned}$$

The result follows by induction.

**Fact 22.2.13** *The $B$-module $H^0(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(\lambda+\rho)\otimes K_{Z_{\mathfrak{v}}})$ has a $P_{\alpha_{j-1}}$-module structure extending the $B$-module structure.*

**Proof :** By Serre duality, we have $H^0(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(\lambda+\rho)\otimes K_{Z_{\mathfrak{v}}}) = H^{\ell(\mathfrak{v})}(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(-(\lambda+\rho)))^*$.

Since $\mathfrak{v}$ is reduced but $(s_{\alpha_{j-1}}, \mathfrak{v})$ is not, we can write $v = \pi(\mathfrak{v}) = s_{\alpha_{j-1}} r_{j+1}\cdots r_n$ for some simple reflections $r_i$. We define the word $\mathfrak{v}' = (s_{\alpha_{j-1}}, r_{j+1}, \cdots, r_n)$. By induction and part (ıı) of the Theorem, we have that $\mathfrak{v}$ and $\mathfrak{v}'$ satisfy condition $(\star)$ in Proposition 22.2.9. In particular we obtain the equality

$$H^0(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(\lambda+\rho)\otimes K_{Z_{\mathfrak{v}}}) = H^{\ell(\mathfrak{v})}(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(-(\lambda+\rho)))^* = H^{\ell(\mathfrak{v}')}(Z_{\mathfrak{v}'}, \mathcal{L}_{\mathfrak{v}'}(-(\lambda+\rho)))^*.$$

But by Proposition 22.2.5 the later has a $P_{\alpha_{j-1}}$-action. □

We are left with case (b). We start as in the last case but with the additional following remark: consider the map $\varphi : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}[k]}$, it is a locally trivial fibration with fiber $Z_{\mathfrak{t}}$ where $\mathfrak{t} = (s_{\alpha_{k+1}}, \cdots, s_{\alpha_n})$. By induction using (ıı), we have $H^i(Z_{\mathfrak{t}}, \mathcal{L}_{\mathfrak{t}}(\lambda)) = 0$ for $i > 0$ and we have $\varphi^*\mathcal{O}_{Z_{\mathfrak{w}[k]}}(-\partial Z^{j,k}_{\mathfrak{w}[k]}) = \mathcal{O}_{Z_{\mathfrak{w}}}(-\partial Z^{j,k}_{\mathfrak{w}})$. This implies by Leray spectral sequence that the following equality holds:

$$\begin{aligned} H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)\otimes \mathcal{O}_{Z_{\mathfrak{w}}}(-\partial Z^{j,k}_{\mathfrak{w}})) &= H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)\otimes \varphi^*(\mathcal{O}_{Z_{\mathfrak{w}[k]}}(-\partial Z^{j,k}_{\mathfrak{w}[k]}))) \\ &= H^i(Z_{\mathfrak{w}[k]}, \mathcal{L}_{\mathfrak{w}[k]}(H^0(Z_{\mathfrak{t}}, \mathcal{L}_{\mathfrak{t}}(\lambda)))\otimes \mathcal{O}_{Z_{\mathfrak{w}[k]}}(-\partial Z^{j,k}_{\mathfrak{w}[k]})). \end{aligned}$$

We will denote by $V$ the $B$-module $H^0(Z_{\mathfrak{t}}, \mathcal{L}_{\mathfrak{t}}(\lambda))$. Remark that it is a $P_{\alpha_{k+1}}$-module. Now with the same notation as before, consider the map $\psi' : Z_{\mathfrak{w}[k]} \to Z_{\mathfrak{u}}$. It is a locally trivial fibration with fiber $Z_{\mathfrak{v}}$. As before, we want to compare, using Leray spectral sequence the cohomology on $Z_{\mathfrak{w}[k]}$ of the sheaf $\mathcal{L}_{\mathfrak{w}[k]}(V)\otimes\mathcal{O}_{Z_{\mathfrak{w}[k]}}(-\partial Z^{j,k}_{\mathfrak{w}[k]})$ with the cohomology on $Z_{\mathfrak{u}}$ of its higher direct images $R^i\varphi'_*(\mathcal{L}_{\mathfrak{w}[k]}(V)\otimes\mathcal{O}_{Z_{\mathfrak{w}[k]}}(-\partial Z^{j,k}_{\mathfrak{w}[k]}))$. For this we compute by Serre duality the cohomology groups

$$H^i(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(V)\otimes\mathcal{O}_{Z_{\mathfrak{v}}}(-\partial Z^{1,k+1-j}_{\mathfrak{v}})) = H^i(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(V)\otimes K_{Z_{\mathfrak{v}}}\otimes\mathcal{L}_{\mathfrak{v}}(\rho)) = H^i(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(V^*)\otimes\mathcal{L}_{\mathfrak{v}}(-\rho)).$$

Because $\mathfrak{v}$ is reduced but $(\mathfrak{v}, s_{\alpha_{k+1}})$ is not, we can write $v = \pi(\mathfrak{v}) = r_j\cdots r_{k-1}s_{\alpha_{k+1}}$ for $r_i$ simple reflections. Write $\mathfrak{v}' = (r_j, \cdots, r_{k-1}, s_{\alpha_{k+1}})$, then we have for all $i \geq 0$ the equality

$$H^i(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(V^*)\otimes\mathcal{L}_{\mathfrak{v}}(-\rho)) = H^i(Z_{\mathfrak{v}'}, \mathcal{L}_{\mathfrak{v}'}(V^*)\otimes\mathcal{L}_{\mathfrak{v}'}(-\rho)).$$

Now consider the projection $Z_{\mathfrak{v}'} \to Z_{\mathfrak{v}'(k+1-j)}$ which is a $\mathbb{P}^1$-fibration. Because $V$ and hence $V^*$ is a $P_{\alpha_{k+1}}$-module, the sheaf $\mathcal{L}_{\mathfrak{v}'}(V^*)$ is trivial on the fibers of that morphism. Furthermore $\mathcal{L}_{\mathfrak{v}'}(-\rho)$ is of degree $-1$ on that fiber thus the sheaf $\mathcal{L}_{\mathfrak{v}'}(-\rho)\otimes\mathcal{L}_{\mathfrak{v}'}(V^*)$ has no cohomology on the fibers of the map $Z_{\mathfrak{v}'} \to Z_{\mathfrak{v}'(k+1-j)}$. This implies by Leray spectral sequence again, that this sheaf has no cohomology on $Z_{\mathfrak{v}'}$ and hence

$$H^i(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(-\rho)\otimes\mathcal{L}_{\mathfrak{v}}(V^*)) = 0 \text{ for all } i \geq 0.$$

As a consequence for all $i \geq 0$ we have the vanishing $R^i\varphi'_*(\mathcal{L}_{\mathfrak{w}[k]}(V)\otimes\mathcal{O}_{Z_{\mathfrak{w}[k]}}(-\partial Z^{j,k}_{\mathfrak{w}[k]})) = 0$ and again by Leray spectral sequence the result follows. □

Let us now prove some consequence of this result.

**Corollary 22.2.14** *Let $\mathfrak{w}$ be any word of length $n$, let $j \in [1,n]$ and let $\lambda \in C_{\mathbb{Z}}$. Then the restriction map $H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \to H^0(Z_{\mathfrak{w}(j)}, \mathcal{L}_{\mathfrak{w}(j)}(\lambda))$ is surjective.*

**Proof :** Consider the exact sequence $0 \to \mathcal{O}_{Z_{\mathfrak{w}}}(-Z_{\mathfrak{w}(j)}) \to \mathcal{O}_{Z_{\mathfrak{w}}} \to \mathcal{O}_{Z_{\mathfrak{w}(j)}} \to 0$. Tensoring with $\mathcal{L}_{\mathfrak{w}}(\lambda)$ which is locally free, the sequence remains exact and by the long exact sequence of cohomology and the preceding Theorem, the result follows. □

**Corollary 22.2.15** *Let $\mathfrak{w}$ be any word and denote by $w$ the element of $W$ such that $X_w = m_{\mathfrak{w}}(Z_{\mathfrak{w}})$. Then there exists a maximal reduced subword $\mathfrak{v}$ of $\mathfrak{w}$ such that $m_{\mathfrak{w}}(Z_{\mathfrak{v}}) = X_w$ and the restriction map $H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \to H^0(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(\lambda))$ is an isomorphism for all $\lambda$ $inC_{\mathbb{Z}}$.*

**Proof :** Let us first prove that if $\mathfrak{u}$ is reduced and $\mathfrak{t} = (s_\alpha, \mathfrak{u})$ is not reduced, then the restriction map $H^0(Z_{\mathfrak{t}}, \mathcal{L}_{\mathfrak{t}}(\lambda)) \to H^0(Z_{\mathfrak{u}}, \mathcal{L}_{\mathfrak{u}}(\lambda))$ is an isomorphism. Consider $\psi : Z_{\mathfrak{t}} \to Z_{s_\alpha}$ which is a locally trivial fibration with fibers $Z_{\mathfrak{u}}$. By the Leray spectral sequence and part ($\imath\imath$) of the Theorem, we get $H^0(Z_{\mathfrak{t}}, \mathcal{L}_{\mathfrak{t}}(\mathfrak{t})) = H^0(Z_{s_\alpha}, \mathcal{L}_{s_\alpha}(H^0(Z_{\mathfrak{u}}, \mathcal{L}_{\mathfrak{u}}(\lambda))))$. But $\mathfrak{t}$ being non reduced we argue as in Fact 22.2.13 to get that $H^0(Z_{\mathfrak{u}}, \mathcal{L}_{\mathfrak{u}}(\lambda))$ as a $P_\alpha$-module structure. Thus $\mathcal{L}_{s_\alpha}(H^0(Z_{\mathfrak{u}}, \mathcal{L}_{\mathfrak{u}}(\lambda)))$ is trivial on $Z_{s_\alpha}$. The assertion follows.

We prove the result by induction on $n$. Write $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{\alpha_n})$. Take $\mathfrak{v}'$ a maximal subword of $\mathfrak{w}' = (s_{\alpha_2}, \cdots, s_{a_n})$ satisfying the conclusions of the corollary. We have the alternative

- $(s_{\alpha_1}, \mathfrak{v}')$ is reduced or
- $(s_{\alpha_1}, \mathfrak{v}')$ is not reduced.

In the first case, we easily see that $\mathfrak{v} = (s_{\alpha_1}, \mathfrak{v}')$ is a maximal reduced subword of $\mathfrak{w}$. Consider the map $Z_{\mathfrak{w}} \to Z_{s_\alpha}$ which is locally trivial with fiber $Z_{\mathfrak{w}'}$. In parallel we have the map $Z_{\mathfrak{v}} \to Z_{s_\alpha}$ which is locally trivial with fiber $Z_{\mathfrak{v}'}$. By the Leray spectral sequence and the Theorem, we have

$$\begin{aligned} H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) &= H^0(Z_{s_\alpha}, \mathcal{L}_{s_\alpha}(H^0(Z_{\mathfrak{w}'}, \mathcal{L}_{\mathfrak{w}'}(\lambda)))) \\ &= H^0(Z_{s_\alpha}, \mathcal{L}_{s_\alpha}(H^0(Z_{\mathfrak{v}'}, \mathcal{L}_{\mathfrak{v}'}(\lambda)))) \\ &= H^0(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(\lambda)). \end{aligned}$$

In the second case, we easily see that $\mathfrak{v} = \mathfrak{v}'$ is a maximal reduced subword of $\mathfrak{w}$. By the beginning of the proof and the induction hypothesis, we have

$$H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = H^0(Z_{\mathfrak{w}'}, \mathcal{L}_{\mathfrak{w}'}(\lambda)) = H^0(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(\lambda)).$$

The fact that $m_{\mathfrak{w}}(Z_{\mathfrak{v}}) = X_w$ follows by induction and the fact that for $\ell(sw) < \ell(w)$, then $sw \leq w$ in the Bruhat order and $C(s)C(w) \subset C(sw) \cup C(w)$. □

**Proposition 22.2.16** *Let $\mathfrak{w} \in \mathfrak{W}$ and let $V$ be a pro-representation of $B$. Denote by $w$ the element in $W$ with $m_{\mathfrak{w}}(Z_{\mathfrak{w}}) = X_w$ and by $\widetilde{X}_w$ the normalisation of $X_w$.*

*($\imath$) The canonical map $\mathcal{O}_{\widetilde{X}_w} \to (\widetilde{m}_{\mathfrak{w}})_* \mathcal{O}_{Z_{\mathfrak{w}}}$ is an isomorphism.*

*($\imath\imath$) we have $R^i(\widetilde{m}_{\mathfrak{w}})_* \mathcal{L}_{\mathfrak{w}}(V) = 0$ for $i > 0$ and $(\widetilde{m}_{\mathfrak{w}})_* \mathcal{L}_{\mathfrak{w}}(V)$ is locally free.*

*($\imath\imath\imath$) We have an isomorphism of $B$-vector bundles $(\widetilde{m}_{\mathfrak{w}})^* \widetilde{\mathcal{L}}_w(V) \simeq \mathcal{L}_{\mathfrak{w}}(V)$. In particular we have an isomorphism*

$$H^i(\widetilde{X}_w, \widetilde{\mathcal{L}}_w(V)) \simeq H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(V)).$$

*($\imath v$) If $\mathfrak{u}$ is any subword such that $(\widetilde{m}_{\mathfrak{w}})|_{Z_{\mathfrak{u}}}$ is surjective onto $X_w$, then the map $i^*_{\mathfrak{u},\mathfrak{w}}$ is an isomorphism in cohomology:*

$$H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(V)) \simeq H^i(Z_{\mathfrak{u}}, \mathcal{L}_{\mathfrak{u}}(V)) \text{ for all } i \geq 0.$$

**Proof :** ($\imath$) To prove this result we need the following:

**Lemma 22.2.17** *Let $f : X \to Y$ be a surjective morphism between projective varieties. Assume that there exists an ample line bundle $\mathcal{L}$ on $Y$ such that the canonical map $H^0(Y, \mathcal{L}^n) \to H^0(X, f^*\mathcal{L}^n)$ is an isomorphism for large $n$, then $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

**Proof :** We have an exact sequence $0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to Q \to 0$. Tensoring by $\mathcal{L}^n$ this sequence remains exact and taking cohomology we get an exact sequence:

$$0 \to H^0(Y,\mathcal{L}^n) \to H^0(X, f^*\mathcal{L}^n) \to H^0(X, Q\otimes\mathcal{L}^n) \to H^1(Y,\mathcal{L}^n).$$

Because $\mathcal{L}$ is ample, we have the vanishing of the rightmost group for large $n$. By hypothesis, the first map is an isomorphism for large $n$. We deduce the vanishing of the group $H^0(X, Q\otimes\mathcal{L}^n)$ for large $n$. But $\mathcal{L}$ being ample (and $Q$ coherent) for large $n$ the sheaf $Q\otimes\mathcal{L}^n$ is globally generated, the result follows. □

To prove the result, we only need to prove that the map $H^0(\widetilde{X}_w, \nu^*\mathcal{L}_w(\lambda)) \to H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))$ is an isomorphism where $\nu : \widetilde{X}_{\mathfrak{w}} \to X_w$ is the normalisation map. For this we take $\mathfrak{v}$ a maximal reduced element with $m_{\mathfrak{w}}(Z_{\mathfrak{v}}) = X_w$ and $H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \simeq H^0(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(\lambda))$. We have a commutative diagram

$$\begin{array}{ccc} H^0(\widetilde{X}_w,\nu^*\mathcal{L}_w(\lambda)) & \xrightarrow{\widetilde{m}_{\mathfrak{w}}^*} & H^0(Z_{\mathfrak{w}},\mathcal{L}_{\mathfrak{w}}(\lambda)) \\ & \searrow^{\widetilde{m}_{\mathfrak{v}}^*} & \downarrow \\ & & H^0(Z_{\mathfrak{v}},\mathcal{L}_{\mathfrak{v}}(\lambda)), \end{array}$$

where the vertical map is an isomorphism. We are thus reduced to prove the same result with $\mathfrak{v}$. But now $\widetilde{m}_{\mathfrak{v}}$ being birational, we have $(\widetilde{m}_{\mathfrak{v}})_*\mathcal{O}_{Z_{\mathfrak{v}}} = \mathcal{O}_{\widetilde{X}_w}$ and the result follows.

(ii) By Lemma 22.2.10 and Theorem 22.2.12, we get that $R^i(\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(V) = 0$ for $i > 0$. Now by induction on the dimension on $V$ and by using Lie's Theorem, we may assume that $V$ is of dimension one and thus of the form $\mathbb{C}_\mu$ and $\mathcal{L}_{\mathfrak{w}}(V) = \mathcal{L}_{\mathfrak{w}}(-\mu) = m_{\mathfrak{w}}^*\mathcal{L}_w(-\mu) = \widetilde{m}_{\mathfrak{w}}^*\nu^*\mathcal{L}_w(-\mu)$. We obtain that $(\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(V)$ is locally free from the projection formula.

(iii) The above argument also proves that we have an isomorphism $\widetilde{m}_{\mathfrak{w}}^*(\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(V) \simeq \mathcal{L}_{\mathfrak{w}}(V)$. We are thus left to prove that $(\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(V) = \widetilde{\mathcal{L}}_w(V)$. This is true (by definition of $\widetilde{\mathcal{L}}_w(V)$) for $\mathfrak{w}$ reduced.

Let us take the same $\mathfrak{v}$ as in Corollary 22.2.15 and we compare $(\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(V)$ with $(\widetilde{m}_{\mathfrak{v}})_*\mathcal{L}_{\mathfrak{v}}(V)$. We again do this by induction on $\dim V$ and we need to prove it for dimension one $V = \mathbb{C}_\mu$. In that case we have $\mathcal{L}_{\mathfrak{w}}(V) = \widetilde{m}_{\mathfrak{w}}^*\nu^*\mathcal{L}_w(-\mu)$ and by projection formula and (i):

$$(\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(V) = \nu^*\mathcal{L}_w(-\mu) = (\widetilde{m}_{\mathfrak{v}})_*\mathcal{L}_{\mathfrak{v}}(V).$$

But for $\mathfrak{v}$ reduced, by definition we have $\widetilde{\mathcal{L}}_w(V) = (\widetilde{m}_{\mathfrak{v}})_*\mathcal{L}_{\mathfrak{v}}(V)$ and the first part of (iii) follows.

By what we just proved, by (i) and by the projection formula we have $(\widetilde{m}_{\mathfrak{w}})_*\mathcal{L}_{\mathfrak{w}}(V) = \widetilde{\mathcal{L}}_w(V)$. The second part of (iii) follows from Leray spectral sequence and (ii).

(iv) This follows from the last part of (iii) for $\mathfrak{w}$ and $\mathfrak{u}$. □

The same techniques using the Leray spectral sequence imply the following

**Lemma 22.2.18** *Let $\mathfrak{v}$ and $\mathfrak{w}$ be two words and consider two different embeddings of $\mathfrak{v}$ in $\mathfrak{w}$ leading to two different embedding $i$ and $i'$ of $Z_{\mathfrak{v}}$ in $Z_{\mathfrak{w}}$. Then the induced following module maps are equal:*

$$i^*, (i')^* : H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(V)) \to H^i(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(V)) \textit{ for all } i \geq 0.$$

**Definition 22.2.19** These maps $i^*\mathfrak{v}, \mathfrak{w} : H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(V)) \to H^i(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(V))$ define a projective system and we define the pro-$B$-representation:

$$H^i(Z_\infty, \mathcal{L}_\infty(V)) = \lim_{\leftarrow} H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(V)).$$

In the same way, define the pro-$B$-representation:

$$H^i(Z_\infty, \mathcal{L}_\infty(V))^\vee = \lim_{\leftarrow} H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(V))^*.$$

**Lemma 22.2.20** *For all index $i$, the pro-$B$-representation $H^i(Z_\infty, \mathcal{L}_\infty(V))$ has a natural structure of pro-$P_i$-representation.*

**Proof :** Consider the subset $\mathfrak{W}_i$ of words starting with $s_{\alpha_i}$. It is a cofinal set in $\mathfrak{W}$ thus $H^i(Z_\infty, \mathcal{L}_\infty(V))$ is also the inverse limit over this set. But all the $B$-modules $H^i(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(V))$ have a pro-$P_i$-module structure for $\mathfrak{w} \in \mathfrak{W}_i$ this induces the desired pro-module structure. □

Using these pro-$P_i$-module structure, the following result on the modules $H^i(Z_\infty, \mathcal{L}_\infty(\lambda))$ for $\lambda \in C_\mathbb{Z}$ can be proven (see [Ku02, Proposition 8.1.17 and Lemma 8.1.19]):

**Proposition 22.2.21** *(ı) The module $H^0(Z_\infty, \mathcal{L}_\infty(\lambda))$ has at most one vector $v_\lambda$ of weight $\lambda$ and the other weights are of the form $\lambda - \sum_i a_i\alpha_i$ for $a_i$ non negative integers.*

*(ıı) For $\mathfrak{w}$ reduced, the module $H^i(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(\lambda))^*$ is a $U(\mathfrak{b})$-module generated by an element of weight $w(\lambda)$ with $w = \pi(\mathfrak{w})$.*

## 22.3 Normality of Schubert varieties

With this we will be able to identify the module $H^i(Z_\infty, \mathcal{L}_\infty(V))$. Let us recall the definition of the following map for $V$ any countable dimensional vector space:

$$s_V : V^* \to H^0(\mathbb{P}(V), \mathcal{L}_V^*)$$

defined by $f \mapsto s_V(f)$ where $s_V(f) : \mathbb{P}(V) \to \mathcal{L}_V^* = \{([v], g) \in \mathbb{P}(V) \times V^* \ / \ g \in V^*/v^\perp\}$ is the section defined by $s_V(f([v])) = ([v], \bar{f})$.

For $\lambda \in C_\mathbb{Z}$ we may consider the composition map (with $s = s_{L^{\max}(\lambda)}$, $L(\lambda) = L^{\max}(\lambda)$):

$$s(\lambda) : \quad L(\lambda)^* \xrightarrow{s} H^0(\mathbb{P}L(\lambda), \mathcal{L}_{L(\lambda)}^*) \xrightarrow{i_\lambda^*} H^0(X, \mathcal{L}(\lambda)).$$

Using the inclusion of Schubert varieties in $X$ we define:

$$s_w(\lambda) : \quad L(\lambda)^* \xrightarrow{i(\lambda)} H^0(X, \mathcal{L}(\lambda)) \longrightarrow H^0(X_w, \mathcal{L}_w(\lambda)).$$

Let us also give the following:

**Definition 22.3.1** Let $V$ be an integrable highest weight module of highest weight $\lambda$. We define the vector space $V_{w(\lambda)}$ to be the $U(\mathfrak{b})$-submodule of $V$ generated by a non zero vector $v_{w(\lambda)}$ (unique up to scalar because $V$ is integrable) of weight $w(\lambda)$ for $w \in W$.

It is not difficult to see that $V_w$ is the $B$-submodule of $V$ generated by $v_{w(\lambda)}$.

**Lemma 22.3.2** *For $\lambda \in C_\mathbb{Z}$, $w \in W$ and $s_i$ a simple reflection such that $s_iw < w$, the $P_i$-module map:*

$$s_w(\lambda) : L^{\max}(\lambda)^* \to H^0(X_w, \mathcal{L}_w(\lambda))$$

*has kernel equal to the subspace $[L^{\max}(\lambda)/L_w^{\max}(\lambda)]^*$. This induces an injective map*

$$\bar{s}_w(\lambda) : L_w^{\max}(\lambda)^* \to H^0(X_w, \mathcal{L}_w(\lambda)).$$

**Proof :** Since the orbit $BwB/B$ is an open and dense subset in $X_w$, the restriction on sections $H^0(X_w, \mathcal{L}_w(\lambda)) \to H^0(BwB/B, \mathcal{L}_w(\lambda)|_{BwB/B})$ is injective. In particular, we get:

$$\ker s_w(\lambda) = \{f \in L^{\max}(\lambda)^* \ / \ f|_{BwBv_\lambda} \equiv 0\} = \{f \in L^{\max}(\lambda)^* \ / \ f|_{L_w^{\max}(\lambda)} \equiv 0\}$$

the result follows. □

We now define a map from $H^0(Z_\infty, \mathcal{L}_\infty(\lambda))^\vee$ to $L^{\max}(\lambda)$.

**Definition 22.3.3** (i) For $\mathfrak{w} \in \mathfrak{W}$, the surjective morphism $m_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ induces an injective map $H^0(X_w, \mathcal{L}_w(\lambda)) \to H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))$ and by duality a surjective map

$$H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))^* \to H^0(X_w, \mathcal{L}_w(\lambda))^*.$$

(ii) The composition of the previous map with the dual $(\bar{s}_w(\lambda))^*$ of the injective map defined in Lemma 22.3.2 give a surjective map

$$\phi_{\mathfrak{w}}(\lambda) : H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \to L_w^{\max}(\lambda) \subset L^{\max}(\lambda).$$

(iii) Taking the direct limit of the maps $\phi_{\mathfrak{w}}(\lambda)$ we obtain the map

$$\phi(\lambda) : H^0(Z_\infty, \mathcal{L}_\infty(\lambda))^\vee \to L^{\max}(\lambda).$$

**Theorem 22.3.4** *For any $\lambda \in C_{\mathbb{Z}}$, the map $\phi(\lambda)$ is an isomorphism. In particular, the weight spaces of $H^0(Z_\infty, \mathcal{L}_\infty(\lambda))^\vee$ are finite dimensional.*

**Proof :** The vector space $H^0(Z_\infty, \mathcal{L}_\infty(\lambda))^\vee$ has structures of pro-$P_i$-modules for all $i$, thus is has structures of $\mathfrak{p}_i$-modules that are equal on $\mathfrak{b}$. Furthermore, being a direct limit of finite dimensional representation, it is a locally finite $T$ module and hence it is a weight module. This implies that there is an integrable $\mathfrak{g}$-module structure and the map $\phi(\lambda)$ is a $\mathfrak{g}$-module map.

Let us now prove that $H^0(Z_\infty, \mathcal{L}_\infty(\lambda))^\vee$ is a highest weight module of weight $\lambda$. By Proposition 22.2.21, there is at most one weight vector. But Corollary 22.2.14, for any subword $\mathfrak{v}$ of $\mathfrak{w}$, the map

$$H^0(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(\lambda))^* \to H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))^*$$

are injective, in particular for $\mathfrak{v} = \emptyset$ we get a vector $v_\lambda$ of weight $\lambda$.

Now we want to prove that this vector generates the module as a $U(\mathfrak{b})$-module. Let $V = U(\mathfrak{g}) \cdot v_\lambda$, it is an integrable $\mathfrak{g}$-module and because $H^0(Z_\infty, \mathcal{L}_\infty(\lambda))^\vee$ is also integrable, their weight spaces of weight $w(\lambda)$ are of dimension 1 and thus equal for all $w \in W$. Let $\mathfrak{w}$ be a reduced word with $\pi(\mathfrak{w}) = w$. By Proposition 22.2.21, the module $H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))^*$ is cyclic generated by an element of weight $w(\lambda)$. In particular it is contained in $V$. Take now $\mathfrak{w}$ a non reduced word with $\pi(\mathfrak{w}) = w$. There exists by Corollary 22.2.15 a reduced word $\mathfrak{v}$, with $\pi(\mathfrak{v}) = w$ and $H^0(Z\mathfrak{w}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \simeq H^0(Z_{\mathfrak{v}}, \mathcal{L}_{\mathfrak{v}}(\lambda))$ thus for non reduced $\mathfrak{w}$ we again have an inclusion $H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))^* \subset V$. By taking the direct limit we get that $V = H^0(Z_\infty, \mathcal{L}_\infty(\lambda))^\vee$.

The image of $\phi(\lambda)$ contains thus an highest weight vector and is thus surjective. But $H^0(Z_\infty, \mathcal{L}_\infty)^\vee$ being an integrable highest weight module, it is a quotient of $L^{\max}(\lambda)$ and the result follows from the fact that $\dim \mathrm{End}(L^{\max}(\lambda), L^{\max}(\lambda)) = 1$. □

**Corollary 22.3.5** *For any $\lambda \in C_{\mathbb{Z}}$ and for any $\mathfrak{w} \in \mathfrak{W}$, the maps*

$$m_{\mathfrak{w}}^* : H^0(X_w, \mathcal{L}_w(\lambda)) \to H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \text{ and } \bar{s}_w(\lambda) : L^{\max}(\lambda)^* \to H^0(X_w, \mathcal{L}_w(\lambda))$$

*are isomorphisms.*

**Proof :** Consider the composition of the dual of two map

$$\phi_{\mathfrak{w}}(\lambda) : H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))^* \to L^{\max}(\lambda).$$

We already know (see Definition 22.3.3) that this map is surjective. We only need to prove its injectivity. By Corollary 22.2.14, the natural map $H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))^* \to H^0(Z_\infty, \mathcal{L}_\infty(\lambda))^*$ is injective. In the following commutative diagram, the vertical maps are thus injections:

$$\begin{array}{ccc} H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))^* & \xrightarrow{\phi_{\mathfrak{w}}(\lambda)} & L_w^{\max}(\lambda) \\ \downarrow & & \downarrow \\ H^0(Z_\infty, \mathcal{L}_\infty(\lambda))^* & \xrightarrow{\phi(\lambda)} & L^{\max}(\lambda). \end{array}$$

The result follows from the injectivity of $\phi(\lambda)$ by the previous Theorem. □

**Corollary 22.3.6** *The Schubert varieties $X_w$ and $X_w^X$ are normal.*

**Proof :** For $X$ a subset of $\Pi$ and $\lambda \in C_X$, we may define as in Lemma 22.3.2 an application $\bar{s}_w^X(\lambda) : L_w^{\max}(\lambda) \to H^0(X_w^X, \mathcal{L}_w^X(\lambda))$ which is also injective. By composition, we get a commutative diagram

$$\begin{array}{ccc} L_w^{max}(\lambda) & \longrightarrow & H^0(X_w^X, \mathcal{L}_w^X(\lambda)) \\ & \searrow^{\bar{s}_w(\lambda)} & \downarrow \\ & & H^0(X_w, \mathcal{L}_w(\lambda)) \end{array}$$

where the vertical map is the pull-back for the projection $X \to X^X$. This pull-back is of course injective. By the previous Theorem, the composed map $\bar{s}_w(\lambda)$ is an isomorphism thus we get that $\bar{s}_w^X(\lambda)$ is also an isomorphism as well as the pull back.

Now take $w' \in W^X$ the unique element with $w'W_X = wW_X$ and choose $\mathfrak{w}$ a reduced word with $\pi(\mathfrak{w}) = w'$. The map $m_{\mathfrak{w}}^X : Z_{\mathfrak{w}} \to X_w^X$ is birational. By the previous argument and the previous Corollary we have an isomorphism $H^0(X_w^X, \mathcal{L}_w^X(\lambda)) \simeq H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))$. We conclude by Lemma 22.2.17 and the fact that $\mathcal{L}_w^X(\lambda)$ is ample on $X_w^X$. □

As an application of the vanishing Theorem, on can also prove a Boreil-Weil-Bott type formula. Let $w * \lambda = w(\lambda + \rho) - \rho$ for $\lambda \in \mathfrak{h}^*$.

**Theorem 22.3.7** *Let $G$ be a Kac-Moody group with Borel subgroup $B$ and Weyl group $W$. Then for any $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$, such that $\lambda + \rho \in C$ for any $w \in W$ and any integer $i$ we have an isomorphism of $G$-modules:*

$$H^i(G/B, \mathcal{L}(\lambda))^\vee \simeq H^{i+\ell(w)}(G/B, \mathcal{L}(w * \lambda))^\vee.$$

# Part IV

# Equivariant and quantum cohomology

# Chapter 23

# Introduction

In this part we want to explain the relationship between the Nil-Hecke ring, a combinatorial object depending on the Weyl group $W$ of a Kac-Moody group $G$ and the equivariant cohomology $H^*_T(G/B)$ of the homogeneous space $G/B$. This part will be different from the last two parts in the sense that we will give very few proof but only explain the general results on this subject. On the contrary to the other chapter we will try to provide more references on these chapter (even if the first two chapters of this part are covered by the book of Kumar it is not the case of the last ones).

We will start by a general definition of the equivariant cohomology and its first properties. We will try to give some examples especially in the situation of homogeneous varieties. We will also describe in detail the classical cohomology of grassmannians. This shall be a model for all the theory: in this situation, there is an explicit presentation of the ring as well as a description of the Schubert classes (a natural $\mathbb{Z}$-basis of the cohomology) in terms of the generators (Giambelli formulas). There is also a general formula for the Littlewood-Richardson coefficients in this situation. We then give the first properties of the equivariant cohomology of $G/B$ for $G$ a Kac-Moody group.

In the second chapter, we describe the combinatorial invariant: the Nil-Hecke ring. We prove some of its basic properties to give a flavour of what is the combinatoric in this situation. We then realise the equivariant cohomology as the dual of this ring. We also give a smoothness criterion for $T$-invariants points in Schubert varieties using the Nil-Hecke ring.

In the third chapter, we briefly introduce the quantum cohomology of a quotient $G/P$ where $G$ is a finite dimensional algebraic group. The quantum cohomology is a deformation of the classical cohomology ring taking into account the enumerative properties of rational curves on the variety.

In the fourth chapter, we explain how, in the case of grassmannians we can explicitely compute the quantum cohomology by elementary methods.

In the final chapter, we specialise to affine Kac-Moody groups. Let $G$ be a finite dimensional algebraic group and le $\hat{G}$ be the associated untwisted affine Kac-Moody group associated to $G$. We explain a relationship between the Nil-Hecke ring, more precisely between the equivariant homology of the affine grassmannian, and the quantum cohomology of $G/B$. We deduce from this relationship the existence of symmetries in the quantum cohomology ring.

# Chapter 24

# Equivariant cohomology

In this section, we give a quick review of equivariant cohomology that will shall need. We will not give any proof.

## 24.1 General definitions and first properties

**Definition 24.1.1** Let $K$ be a real Lie group.

(i) We call a $K$-space any variety (differentiable) with an action of $K$.

(ii) A universal principal $K$-bundle is a morphism of differentiable varieties $\pi : E(K) \to B(K)$ such that

- $K$ acts differentiably and freely on $E(K)$;
- the map $\pi$ is $K$-equivariant with the trivial action of $K$ on $B(K)$ and the fibers are principal $K$ spaces;
- the space $E(K)$ is contractible.

(iii) For $X$ a $K$-space, we define the fiber bundle $\pi_X : X_K \to B(K)$ by

$$X_K = E(K) \times^K X \to B(K).$$

Here as usual $E(K) \times^K X$ is the quotient of the product $E(K) \times X$ by the action of $K$ given by $k \cdot (e, x) = (ek^{-1}, kx)$.

(iv) We define the $K$-equivariant cohomology of $X$ to be

$$H^*_K(X) = H^*(X_K, \mathbb{Z}).$$

**Definition 24.1.2** As in the classical case (non equivariant), we may also define the **equivariant homology** by

$$H^K_*(X) = \mathrm{Hom}_{H^*_K(pt)}(H^*_K(X), H^*_K(pt)).$$

**Fact 24.1.3** *(i) For $X$ a $K$-space, the equivariant cohomology $H^*_K(X)$ does not depend on the choice of the principal $K$-bundle chosen to construct it.*

*(ii) The group $H^*_K(X)$ is a $\mathbb{Z}$-graded algebra and*

$$\pi^* : H^*(B(K)) \to H^*(X_K)$$

*defines a $H^*(B(K))$-algebra structure on the equivariant cohomology of $X$.*

*(iii) For any $K$-equivariant morphism $f : X \to Y$, there exists a canonical $B(K)$-graded-algebra morphism*

$$f^* H_K^*(Y) \to H_K^*(X).$$

**Definition 24.1.4** Let $e$ be any point of $E(K)$. Because the action on $E(K)$ is free, the natural map $i : \{e\} \times X \to X_K$ defined by $x \mapsto \overline{(e,x)}$ is an inclusion. This induces, by pull-back, a $\mathbb{Z}$-algebra morphism (the **evaluation map**)

$$\eta : H_K^*(X) \to H^*(X).$$

**Fact 24.1.5** *This definition does not depend on the choice of $e$.*

**Example 24.1.6** (i) Let $T$ be a closed subgroup of $K$, then $E(K) \to E(K)/T$ is a universal principal $T$-bundle.

(ii) Let $X$ be a $K$-variety with a free action of $K$ on $X$, then $H_K^*(X) = H^*(X/K)$. Indeed, if we consider $E(K) \times^K X \to X/K$ we get a fibre bundle with fiber $E(K)$ which is contractible.

(iii) Let $T = S^1$, then we may take $E(T) = S^\infty$ where $S^\infty$ is the unit sphere in $\mathbb{C}^\infty$ with the $\ell^2$-metric. The principal $T$-bundle is in this case $S^\infty \to \mathbb{P}^\infty$. More generally, if $T = (S^1)^n$, then we may take $E(T) = (S^\infty)^n_{ind}$ and $B(T) = (\mathbb{P}^\infty)^n_{ind}$ where the subscript *ind* means that we take a set of $n$ independent vectors.

(iv) If $K = U_n$ the group of unitary matrices (this is the maximal compact subgroup of $GL_n$) then we can also take $E(K) = (S^\infty)^n_{ind}$. Indeed, we have a free action of $K$ on that space (which is still contractible). The quotient $E(K)/K = B(K)$ is $\mathbb{G}(n,\infty)$ the space of $n$-dimensional subspaces in an infinite dimensional vector space.

(v) If we consider now $K = SU_n$ and $T$ is maximal torus (which is a codimension 1 subtorus of $(S^1)^n$), we can still take $E(T) = (S^\infty)^n$ but the quotient $B(T)$ is in that case the tautological $S^1$ bundle (associated to the tautological line bundle $\mathcal{O}(1,\cdots,1)$) over $(\mathbb{P}^\infty)^n$. In the same way, the quotient $E(K)/K = (S^\infty)^n/K$ is isomorphic to the tautological $S^1$-bundle on $\mathbb{G}(n,\infty)$ associated to $\mathcal{O}(1) = \Lambda^n T^*$ where $T$ is the tautological rank $n$ vector bundle on the infinite grassmannian.

## 24.2 Case of a Torus

**Lemma 24.2.1** *Assume now that $K$ is compact or a torus.*

*(i) Then $H^*(B(K))$ is torsion free.*

*(ii) Let $X$ be a $K$-space for the trivial action, then we have an isomorphism of $H^*(B(K))$ algebras:*

$$H_K^*(X) \simeq H^*(B(K)) \otimes_{\mathbb{Z}} H^*(X).$$

Let $T$ be a compact torus. Let $X$ be a $T$-space with fixed $T$-points set $X^T = \{w\}_{w\in W}$. Let $X(T)$ be the group of characters of $T$ and let $\lambda \in X(T)$.

**Definition 24.2.2** (i) We define the line bundle $\mathcal{L}_{B(T)}(\lambda)$ on $B(T)$ by $E(T) \times^T \mathbb{C}_\lambda = (\mathbb{C}_\lambda)_T$.

(ii) We define a map

$$c_1 : X(T) \to H^2(B(T))$$

by $\lambda \mapsto c_1(\mathcal{L}_{B(T)}(\lambda))$.

(iii) We extend the map $c_1$ as a graded algebra morphism

$$c : S^*(X(T)) \to H^*(B(T)).$$

**Proposition 24.2.3** *The morphism $c$ is an isomorphism of $\mathbb{Z}$-graded algebras.*

**Corollary 24.2.4** *The inclusion $X^T \to X$ induces an $S(X(T))$-graded morphism*

$$H_T^*(X) \to H_T^*(X^T) \simeq S(X(T)) \otimes_{\mathbb{Z}} H^*(X).$$

**Theorem 24.2.5** *Let $T$ be a compact torus and let $X$ be a locally contractible compact $T$-space. Then the restriction map induces an isomorphism of $Q$-algebras*

$$Q \otimes_{H^*(B(T))} H_T^*(X) \simeq Q \otimes_{H^*(B(T))} H_T^*(X^T)$$

*where $Q$ is the quotient field of the integral domain $H^*(B(T)) = S(X(T))$.*

**Example 24.2.6** (I) In the case where $T' = (S^1)^n$ is a maximal torus of $K' = U_n$; we get $H_T^*(pt) = \mathbb{Z}[x_1, \cdots, x_n]$ or more canonically if $\mathrm{Lie}(T) = \mathfrak{h}'$, then $H_T^*(pt) = S(\mathfrak{h}'^*_{\mathbb{Z}})$. For $T$ the subtorus obtained as the maximal torus of $K = SU_n$, we have $\mathfrak{h}_{\mathbb{Z}} \subset \mathfrak{h}'_{\mathbb{Z}}$ the subalgebra and $H_T^*(pt) = \mathbb{Z}[x_1, \cdots, x_n]/(x_1 + \cdots + x_n) = S(\mathfrak{h}^*_{\mathbb{Z}})$.

For the equivariant cohomology of the point for $K$, we have the following

**Proposition 24.2.7** *We have an isomorphism over $\mathbb{Q}$:*

$$H_K^*(pt, \mathbb{Q}) \simeq H_T^*(pt, \mathbb{Q})^W = S(\mathfrak{h}^*_{\mathbb{Q}})^W.$$

This comes from the fact that the space $B(K)$ has the same rational (but not integral) cohomology as the finite quotient $B(T)/W$ where $W = \mathfrak{S}_n$ acts on $B(T) = (\mathbb{P}^\infty)^n$ by permutation of the factors.

(II) We want to describe the equivariant cohomology of the flag variety $GL_n/B$ or $K/T$ with the notations of (I). Because $K \subset GL_n$, we have a natural map $K/T \to GL_n/B$ which is an homotopy equivalence. To compute the equivariant cohomology of the flag variety, we do it for $K/T$.

**Lemma 24.2.8** *We have a homeomophism: $E(T) \times^T K/T \simeq B(T) \times_{B(K)} B(T)$.*

**Proof :** Define the map $\overline{((v_1, \cdots, v_n), \bar{k})} \mapsto (([v_1], \cdots, [v_n]) \cdot k, ([v_1], \cdots, [v_n])$. This is the desired homeomorphism. □

**Corollary 24.2.9** *We have an isomorphism: $H_T^*(K/T) \simeq S(\mathfrak{h}^*_{\mathbb{Q}}) \otimes_{S(\mathfrak{h}^*_{\mathbb{Q}})^W} S(\mathfrak{h}^*_{\mathbb{Q}})$.*

**Proof :** This a consequence of the Lemma and of Kunneth formula. □

**Corollary 24.2.10** *Specialising we get an isomorphism $H^*(K/T) \simeq S(\mathfrak{h}^*)/S(\mathfrak{h}^*)^W_+$ where $S(\mathfrak{h}^*)^W_+$ denotes the ideal of positive degree invariant polynomials in the variables $x_1, \cdots, x_n$.*

**Example 24.2.11** As an example, for $n = 2$, we get $H^*(\mathbb{P}^1) = \mathbb{Q}[x_1+x_2]/(x_1+x_2, x_1x_2) = \mathbb{Q}[x]/(x^2)$. For $n = 3$ we have

$$H^*(K/T) = \mathbb{Q}[x_1, x_2, x_3]/(x_1 + x_2 + x_3, x_1x_2 + x_1x_3 + x_2x_3, x_1x_2x_3).$$

More genrally the cohomology group $H^*(K/T)$ is the quotient of the polynomial ring by the symmetric polynomials. It is a finite quotient of dimension $|W| = n!$.

## 24.3 The grassmannian case

In the finite dimension case, the homology and cohomology are strictly dual to each other and there is a multiplication in homology. In many cases one computes the ring structure on the homology rather than on cohomology. Let us descrbe in more details but with elementary methods the case of Grassmannians. The basic reference for this section is L. Manivel [Man98].

### 24.3.1 Partitions and Schubert subvarieties

Let us fix a complete flag in $\mathbb{C}^N$ that is to say a sequence

$$0 \subset W_1 \subset \cdots \subset W_{N-1} \subset \mathbb{C}^N$$

such that $W_i$ is an $i$-dimensional subvector space of $\mathbb{C}^N$. The stabiliser $B$ of this partial flag in $\mathrm{GL}_N$ is conjugated to the subgroup of triangulated matrices. We want to describe the orbits of $B$ in $\mathbb{G}(p, N)$ or equivalently the different relative positions of a element $V \in G(p, N)$ with the complete flag $(W_i)_{i\in[1,N-1]}$. These relative positions are described by partitions.

**Definition 24.3.1** (i) A partition is a sequence $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ of non increasing non negative integers. The $\lambda_i$'s are the parts of the partition $\lambda$.

(ii) The length of a partition $l(\lambda)$ is the number of non zero $\lambda_i$'s in the partition. The weight $|\lambda|$ is defined by

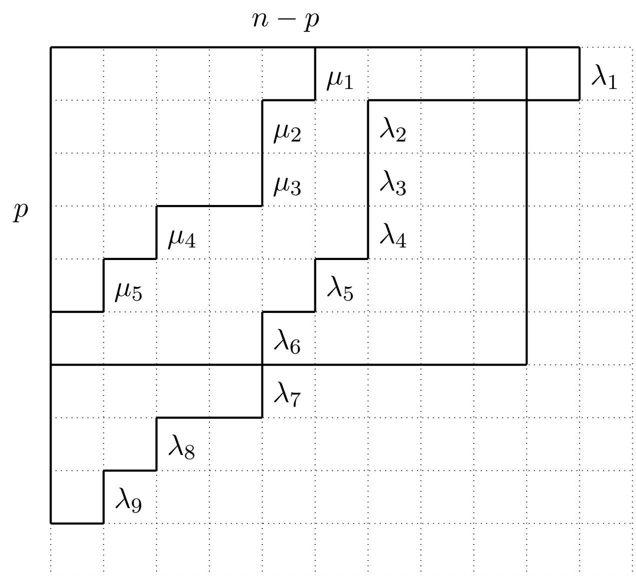
$$|\lambda| = \sum \lambda_i.$$

(iii) A partition $\mu$ is included in a partition $\lambda$ if for all $i$ we have $\mu_i \leq \lambda_i$. We note it $\mu \leq \lambda$.

(iv) We will say that a partition is in the $p \times (N-p)$ rectangle if $k \leq p$ and $\lambda_1 \leq N-p$. One can always complete such a partition by $\lambda_{k+1} = \lambda_p = 0$ to get a partition with exactely $p$ parts.

One can make a picture of a partition. A partition is in the $p \times (N-p)$ rectangle if and only if it is the case of its picture. Inclusion of partitions is equivalent to the inclusion of their pictures.

**Example 24.3.2** Let $N = 15$, $p = 6$ and the partition $\lambda$ be given by: $\lambda_1 = 10$, $\lambda_2 = \lambda_3 = \lambda_4 = 6$, $\lambda_5 = 5$, $\lambda_6 = \lambda_7 = 4$, $\lambda_8 = 2$ and $\lambda_9 = 1$. Let the partition $\mu$ given by: $\mu_1 = 5$, $\mu_2 = \mu_3 = 4$ $\mu_4 = 2$ and $\mu_5 = 1$. The partition $\mu$ is a subpartition of $\lambda$. Its is contained in the $p \times (N-p)$ rectangle but $\lambda$ is not.




Let us define some special subvarieties in $\mathbb{G}(p, N)$: the Schubert cells. Fix $\lambda = (\lambda_i)_{i\in[1,p]}$ a partition in the $p \times (N-p)$ rectangle, the associated Schubert cell is

$$\Omega(\lambda) = \{V \in \mathbb{G}(p, N) \ / \ \dim(V \cap W_j) = i \text{ for all } i,\ j \text{ with } N-p+i-\lambda_i \leq j \leq N-p+i-\lambda_{i+1}\}.$$

We also define the Schubert variety

$$X(\lambda) = \{V \in \mathbb{G}(p, N) \ / \ \dim(V \cap W_{N-p+i-\lambda_i}) \geq i \text{ for all } i \in [1, p]\}.$$

**Example 24.3.3** If the partition is the biggest one: $\lambda_i = N - p$ for all $i \in [1, p]$ then we have

$$\Omega(\lambda) = X(\lambda) = \{W_p\}.$$

**Theorem 24.3.4** *(ı) The varieties $\Omega(\lambda)$ for $\lambda$ a partition in the $p \times (N - p)$ rectangle are the orbits of $B$ in $\mathbb{G}(p, N)$.*

*(ıı) These varieties are in codimension $|\lambda|$ and isomorphic to an affine space. We have*

$$\Omega(\lambda) \simeq \mathbb{C}^{p(N-p)-|\lambda|}.$$

*(ııı) The Schubert variety $X(\lambda)$ is the closure of the Schubert cell $\Omega(\lambda)$. In symbols:*

$$\overline{\Omega(\lambda)} = X(\lambda).$$

*(ıv) The Schubert variety $X(\lambda)$ is the union of the Schubert cells $\Omega(\mu)$ for $\mu \supset \lambda$. In symbols:*

$$X(\lambda) = \bigcup_{\mu \geq \lambda} \Omega(\mu).$$

*(v) Finally we have the equivalence $X(\mu) \subset X(\lambda) \Leftrightarrow \mu \geq \lambda$.*

### 24.3.2 Cohomology

For homogeneous varieties and in particular for grassmannians, the existence of the stratification of $\mathbb{G}(p, N)$ by affine spaces (the Schubert cells) and classical results on homology and cohomology (see the appendix in [Man98]) imply the following result

**Theorem 24.3.5** *The homology, cohomology and Chow groups of $\mathbb{G}(p, N)$ are isomorphic as abelian groups. They are free and a basis for these groups are given by the classes $[X(\lambda)]$ of Schubert varieties and is therefore indexed by partitions $\lambda$ in the $p \times (N - p)$ rectangle.*

Let us first give the poincaré duality:

**Definition 24.3.6** If $\lambda$ is a partition in the $p \times (N - p)$ rectangle, then the dual partition $\hat{\lambda}$ is defined by $\hat{\lambda}_i = N - p - \lambda_{p+1-i}$ for all $i \in [1, p]$. This partition is given by the complementary of $\lambda$ in the $p \times (N - p)$ rectangle. In particular we have

$$|\lambda| + |\hat{\lambda}| = p(N - p).$$

**Theorem 24.3.7** *Let $\lambda$ and $\mu$ be two partitions in the $p \times (N - p)$ rectangle such that $|\lambda| + |\mu| = p(N - p)$. Then we have*

$$\sigma_\lambda \cup \sigma_\mu = \delta_{\mu,\hat{\lambda}}$$

*where $\delta$ is the Kronecker symbol. In particular, the class $\sigma_{\hat{\lambda}}$ is Poincaré dual to the class $\sigma_\lambda$.*

**Remark 24.3.8** In particular we proved that $\sigma_\lambda \cup \sigma_\mu \neq 0$ implies that $\mu \subset \widehat{\lambda}$.

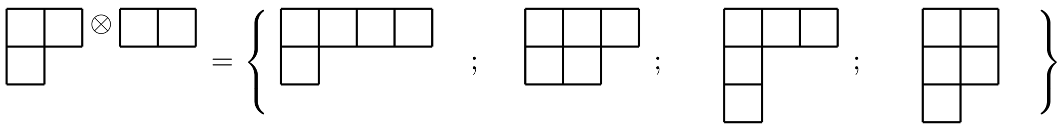
The classes $\sigma_\lambda$ with $\lambda$ having only one part will generate the cohomology ring. We call them special Schubert classes. Let us give some definitions.

**Definition 24.3.9** (i) If $k$ is an integer, we also denote by $k$ the partition having one part $\lambda_1 = k$ and by $\sigma_k$ the corresponding special Schubert class.

(ii) If $\lambda$ is any partition in the $p \times (N-p)$ rectangle, we denote by $\lambda \otimes k$ the set of all partitions obtained from $\lambda$ by adding $k$ boxes with no two in the same colomn.

In other words, $\lambda \otimes k$ is the set of all partitions $\mu$ in the $p \times (N-p)$ rectangle such that $|\mu| = |\lambda| + k$ and $\lambda_{i+1} \leq \mu_{i+1} \leq \lambda_i$ for all $i$.

**Example 24.3.10** If $p = 3$, $N = 7$, $\lambda = (2,1)$ and $k = 2$ then $\lambda \otimes k$ contains the following four partitions: $(4,1)$, $(3,2)$, $(3,1,1)$ and $(2,2,1)$. If however $N = 6$ then the partition $(4,1)$ is not in the rectangle.



We can now give an intersection formula between Schubert classes.

**Theorem 24.3.11** *Pieri Formula.*

*If $\lambda$ is a partition in the $p \times (N-p)$ rectangle, and $k$ an integer in $[1, N-p]$, then we have*

$$\sigma_\lambda \cup \sigma_k = \sum_{\mu \in \lambda \otimes k} \sigma_\mu.$$

We calculated the intersection with some particular classes of Schubert varieties, those with only one part, the special Schubert classes. They generated the Chow ring.

**Theorem 24.3.12** *We have the Giambelli formula*

$$\sigma_\lambda = \det(\sigma_{\lambda_i - i + j})_{1 \leq i,j \leq p}.$$

We can then give a presentation of the cohomology ring in the following form

**Theorem 24.3.13** *We have ring isomorphisms*

$$A^*(X, \mathbb{Z}) \simeq H^*(X, \mathbb{Z}) \simeq \mathbb{Z}[\sigma_1, \cdots, \sigma_{N-p}]/(Y_{p+1}, \cdots, Y_N)$$

*where* $Y_u = \det(\sigma_{1-i+j})_{1 \leq i,j \leq u}$.

The constant in the cohomology ring are the Littlewood-Richardson coefficients $c^\nu_{\lambda\mu}$ and defined by:

$$\sigma_\lambda \cup \sigma_\mu = \sum_\nu c^\nu_{\lambda\mu} \sigma_\nu.$$

There exist some combinatorial descriptions of these numbers (see for example [Man98]).

**Example 24.3.14** For $X = \mathbb{P}^2$, the Schubert classes are $\sigma_0$, $\sigma_1$ and $\sigma_2$ corresponding to the class of $\mathbb{P}^2$, a line and a point. The preceding presentation od the ring is the following:

$$H^*(X, \mathbb{Z}) \simeq \mathbb{Z}[\sigma_1, \sigma_2]/(\sigma_2 - \sigma_1^2, \sigma_1^3 - 2\sigma_1\sigma_2) = \mathbb{Z}[\sigma_1, \sigma_2]/(\sigma_2 - \sigma_1^2, \sigma_1^3)$$

that is to say

$$H^*(X, \mathbb{Z}) \simeq Z[\sigma_1]/(\sigma_1^3).$$

### 24.3.3 Link with the previous study

Recall that we gave a presentation of the cohomology ring for the complete flag variety $GL_N/B$ as the quotient $k[x_1, \cdots, x_N]/I$ where $I$ is the ideal generated by the $\mathfrak{S}_n$-invariant polynomials with no constant term. We have the following result:

**Theorem 24.3.15** *Let $P$ be a parabolic subgroup of $GL_N$, then the natural projection $p : GL_N/B \to GL_N/B$ induces an injection $p^* : H^*(GL_N/P, \mathbb{Z}) \to H^*(GL_N/B, \mathbb{Z})$ whose image is generated by the polynomials invariant under the subgroup $W_P$ of $W$.*

**Remark 24.3.16** For the grassmannian one can easily recover theorem 24.3.13 from this result.

## 24.4 Equivariant cohomology of homogeneous spaces

We now want to give the first properties of the equivariant cohomology of the homogeneous spaces $X^X$ for $X$ a subset of the set $\Pi$ of simple roots.

**Proposition 24.4.1** *Let $X$ be a subset of the set of simple root. Let $w \in W$.*
*(ı) We have*

$$H_{2i+1}(X_w^X) = 0 \text{ and } H_{2i}(X_w^X) = \bigoplus_{v \leq w,\ \ell(v)=i} \mathbb{Z}[X_v^X] \text{ for all } i \geq 0.$$

*(ıı) We have*

$$H^{2i+1}(X_w^X) = 0 \text{ and } H^{2i}(X_w^X) = \bigoplus_{v \leq w,\ \ell(v)=i} \mathbb{Z}\varepsilon_w^X \text{ for all } i \geq 0$$

*where $\varepsilon_w^X$ is defined by*

$$\int_{[X_v^X]} \varepsilon_w^X = \delta_{v,w}.$$

*(ııı) The restriction map $H^*(X^X) \to H^*(X_w^X)$ is surjective.*
*(ıv) If $\pi_{X,Y} : G/P_X \to G/P_Y$ is the natural projection, then we have*

$$(\pi_{X,Y})_*[X_w^X] = \begin{cases} [X_w^Y] & \text{if } w \in W^Y \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \pi_{X,Y}^* \varepsilon_w^Y = \varepsilon_w^X.$$

**Proof :** (ı) and (ıı) This comes from the Bruhat decomposition, indeed, the Schubert cells form a Cellular decomposition of the Schubert varieties. Furthermore, the Schubert cells are affine spaces thus their cohomology and homology is trivial except in higher degree.

(ııı) This comes from the corresponding inclusion in homology (and the natural injections coming from the ind-variety structure.

(ıv) Comes directly from the fact that the map $X_w^X \to X_w^Y$ is birational if and only if $w \in W^Y$. □

We now want to define an action of the Weyl group on the equivariant cohomology. This action is not an algebraic action, even in the finite dimensional case. To construct it, we need to pass to the topology category.

**Proposition 24.4.2** *There is a natural action of the Weyl group $W$ on $H_T^*(G/B)$ as well as on $H^*(G/B)$ such that the canonical map $\eta : H_T^*(G/B) \to H^*(G/B)$ is $W$-equivariant.*

**Proof :** Let us just give some ideas of the construction of this action.

Consider the subgroup $N$ of $G$ (recall that $N$ is the semi-direct product of $T$ by $W$). As a subgroup it acts on the **right** on $G$ (here we need in fact more: that the action is topological, this comes from the study of the minimal group $G^{\min}$ associated to a generalised Cartan matrix and that we briefly discussed in the introduction). This action descend to a right action of $W$ on $G/T$ and we thus for any $w \in W$, have a morphism $R_w : G/T \to G/T$ given by right multiplication and thus a map $R_w^* : H^*(G/T) \to H^*(G/T)$. Furthermore, because the right action of $W$ commutes with the left action of $T$, we also get a map $R_w^* : H_T^*(G/T) \to H_T^*(G/T)$ and thus a left action of $w$ on the cohomology and the equivariant cohomology. All this together gives a left action of the Weyl group on the cohomology and the equivariant chomology and the natural map between them is $W$-equivariant.

Now consider the quotient map $G/T \to G/B$. It is a principal $B/T = \mathcal{U}$-bundle. However $\mathcal{U}$ is contractible (here again we need to use the minimal group $G^{\min}$ and the corresponding unipotent group $\mathcal{U}^{\min}$ to give sense to this sentence). In particular, by Leray spectral sequence, the cohomology and equivariant cohomology groups of $G/T$ and $G/B$ are isomorphic. The result follows. $\square$

We may now apply Corollary 24.2.4 to get a $W$-equivariant ring morphism

$$H_T^*(X) \to S(X(T)) \otimes_{\mathbb{Z}} H_T^*(X^T) = S(X(T)) \otimes_{\mathbb{Z}} H^0(W).$$

But for all $k$, the $\mathbb{Z}$-module $S^k(X(T)) \otimes H^0(W)$ can be identified as the space of function $W \to S^k(X(T))$. Let us denote by $S$ the ring $S(\mathfrak{h}^*)$, the ring $S(X(T))$ is a subring of $S$. We denote by $Q$ the fraction field of $S$. We have a natural ring morphism:

$$H_T^*(X) \to \Omega_W$$

where $\Omega_W = \{f : W \to Q\}$ the set of functions from $W$ to $Q$.

The main point of the theory is to prove that this ring morphism is injective and to explicitly identify its image as a subring of $\Omega_W$. For this we shall need to define some special operators on the cohomology called Demazure operators and we shall start by constructing the ring of Demazure operators and then the cohomology ring.

**Definition 24.4.3** (i) Let $\alpha_i$ be a simple root and let $P_i$ the associated minimal parabolic subgroup. The quotient map

$$\pi_i : G/B \to G/P_i$$

is a $\mathbb{P}^1$-fibration. Take $\omega$ an element in $H^*(G/B)$ and view it as a differential form. Then we may define

$$\omega_i = \int_{\text{fiber of } \pi_i} \omega.$$

This is a cohomology class in $H^{*-2}(G/P_i)$ (i.e. of degree $\deg(\omega) - 2$). We define the Demazure operator $D_{s_i}$ by

$$D_{s_i}(\omega) = \pi_i^* \omega_i.$$

(ii) For the equivariant cohomology, we use the same map $G/B \times^T B(T) \to G/P_i \times^T B(T)$ to define the operators

$$\widehat{D}_{s_i} : H_T^*(G/B) \to H^{*-2}(G/B).$$

**Remark 24.4.4** One can also define the Demazure operators in a more algebraic way. Consider the locally trivial $\mathbb{P}^1$-fibration $\pi_i : G/B \to G/P_i$ and take $\sigma$ a section of the surjection $H_T^*(G/B) \to H_T^*(P_i/B)$. We have a natural isomorphism

$$H_T^*(G/P_i) \otimes_{\mathbb{Z}} H_T^*(P_i/B) \to H_T^*(G/B)$$

defined by $u\otimes v\mapsto(\pi_i^*u)\cup\sigma(v)$. But the equivariant cohomology ring $H_T^*(P_i/B)$ is free over $S(X(T))$ genrated by two elements: 1 and $[pt]$ in degree 0 and 2 respectively. In particular for $\omega\in H_T^*(G/B)$ we may write

$$\omega=\pi_i^*\alpha+[pt]\pi_i^*\beta.$$

We may thus define

$$\widehat{D}_{s_i}(\omega)=\pi_i^*\beta.$$

This definition does not depend on the choice of the section $\sigma$ and coincide with the previous definition.

**Proposition 24.4.5** *We have the following commuting diagram:*

$$\begin{array}{ccc}
H_T^*(G/B) & \xrightarrow{\widehat{D}_{s_i}} & H_T^{*-2}(G/B) \\
\eta\downarrow & & \downarrow\eta \\
H^*(G/B) & \xrightarrow{D_{s_i}} & H^{*-2}(G/B).
\end{array}$$

# Chapter 25

# The Nil-Hecke ring

In this chapter we define a combinatorial invariant associated to any generalised Cartan matrix and thus to any Kac-Moody Lie algebra and any Kac-Moody groups. We shall then see that this invariants has several applications on the geometry of homogeneous varieties under Kac-Moody groups and even on homogeneous spaces of finite dimension.

## 25.1 The ring

Let $\mathfrak{g}$ be any Kac-Moody Lie algebra and $\mathfrak{h}$ its associated Cartan subalgebra. The spaces $\mathfrak{h}$ and $\mathfrak{h}^*$ are $W$-modules where $W$ is the Weyl group associated to the situation.

We shall use the following notation: $Q$ is the quotient field of the symmetric algebra $S = S(\mathfrak{h}^*)$. The action of $W$ extends to an action on $S$ and on $Q$.

**Definition 25.1.1** We denote by $Q_W$ the $Q$-vector space with basis $\{\delta_w\}_{w\in W}$. We define a product on $Q_W$ by

$$\left(\sum_{v\in W} q_v\delta_v\right)\cdot\left(\sum_{w\in W} q'_w\delta_w\right) = \sum_{v,w} q_v(vq'_w)\delta_{vw}.$$

Remark that the underlying vector space is the same as the one of the ring algebra of $W$ but the product is not the same.

**Fact 25.1.2** *(ı) This defines an associative non commutative ring with unit $1 = \delta_e$.*

*(ıı) The ring $Q_W$ is a $Q$ vector space but is not a $Q$-algebra, this comes from the fact that $Q\delta_e$ is not a central element. However, it is an algebra over the subfield $Q^W$ of $W$-invariants of $Q$.*

*(ııı) There is an anti-involution $t$ of $Q_W$ defined*

$$(q\delta_w)^t = (w^{-1}q)\delta_{w^1}, \text{ for } w \in W \text{ and } q \in Q.$$

Let us recall the following from [Sp81]:

**Definition 25.1.3** Let $R$ be a ring. (ı) A coproduct on a $R$-module $A$ is a map $d : A \to A \otimes A$ such that the product is associative: $(d \otimes 1)d = (1 \otimes d)d$.

(ıı) A coproduct $d$ is that to be commutative if $Td = d$ where $T : A \otimes A \to A \otimes A$ is defined by $T(x \otimes y) = y \otimes x$.

(ııı) A counit for the coproduct $d$ is a map $\epsilon : A \to R$ such that we have $(\epsilon \otimes 1) \circ d = (1 \otimes \epsilon) \circ d$.

(ıv) Define a product $\odot$ on $Q_W \otimes Q_W$ by

$$(q_v\delta_v \otimes q_w\delta_w) \odot (q_{v'}\delta_{v'} \otimes q_{w'}\delta_{w'}v) = (q_vq_w)(v(q_{v'}q_{w'}))\delta_{vv'} \otimes \delta_{vw'v^{-1}w}.$$

**Fact 25.1.4** *(ı) There is a $Q$-linear coproduct $\Delta : Q_W \to Q_W \otimes_Q Q_W$ defined by:*

$$\Delta(q\delta_w) = (q\delta_w) \otimes \delta_w = \delta_w \otimes (q\delta_w), \text{ for } w \in W \text{ and } q \in Q.$$

*(ıı) This coproduct is associative, commutative and has a counit given by $\varepsilon : Q_W \to Q$ defined by $\varepsilon(q\delta_w) = q$ for all $q \in Q$ and $w \in W$.*

*(ııı) This coproduct defines a ring morphism $(Q_W, \cdot) \to (Q_W \otimes Q_W, \odot)$.*

We now define some very important elements $x_\alpha$ in $Q_W$ for any simple root $\alpha$ by

$$x_\alpha = \frac{1}{\alpha}(\delta_{s_\alpha} - \delta_e).$$

We also set $x_e = \delta_e$. We denote by $\mathbb{Q}[\underline{\alpha}]$ the polynomials over $\mathbb{Q}$ in the simple roots and by $\mathbb{Z}[\underline{\alpha}]$ and $\mathbb{Z}_+[\underline{\alpha}]$ the same polynomials over $\mathbb{Z}$ or $\mathbb{Z}_+$. We have the following Theorem resuming the basic properties of the ring $Q_w$:

**Theorem 25.1.5** *(ı) We have $x_\alpha^2 = 0$ for all simple roots $\alpha$.*

*(ıı) Let $\mathfrak{w} = (s_{\alpha_1}, \cdots, s_{\alpha_n})$ be a reduced word. Then the element*

$$x_{\mathfrak{w}} = x_{\alpha_1} \cdots x_{\alpha_n}$$

*does not depend on the word $\mathfrak{w}$ but only on $w = \pi(w)$. We shall denote it by $x_w$.*

*(ııı) For any $\lambda \in \mathfrak{h}^*$ and any $w \in W$, we have*

$$\lambda x_w = x_w(w^{-1}\lambda) - \sum_{v \xrightarrow{\beta} w} \langle \lambda, \beta^\vee \rangle x_v$$

*where $v \xrightarrow{\beta} w$ stands for $v \leq w$, $\ell(w) = \ell(v) + 1$, $w = s_\beta v$ and $\beta \in \Delta_+^{\mathrm{re}}$.*

*(ıv) For any elements $v$ and $w$ in $W$, we have*

$$x_v x_w = \begin{cases} x_{vw} & \text{if } \ell(vw) = \ell(v) + \ell(w) \\ 0 & \text{otherwise.} \end{cases}$$

*(v) If we write the elements $x_w$ in the basis $(\delta_v)_{v \in W}$ in the form*

$$x_w = \sum_{v \in W} c_{w,v} \delta_v$$

*with $c_{w,v} \in Q$, then we have:*

- $c_{w,v} = 0$ *unless* $v \leq w$;
- $c_{w,w} = \displaystyle\prod_{\beta \in \Delta_+ \cap w(\Delta_-)} \beta^{-1}$.

*(vı) For a fixed reduced decomposition $w = s_{\alpha_1} \cdots s_{\alpha_n}$ and for $v \leq w$ we have*

$$c_{w,v} = (-1)^n \sum \left( (s_{\alpha_1}^{\varepsilon_1}(\alpha_1)) \cdots (s_{\alpha_1}^{\varepsilon_1} \cdots s_{\alpha_n}^{\varepsilon_n}(\alpha_n)) \right)^{-1}$$

*where the sum runs over all sequences $(\varepsilon_1, \cdots, \epsilon_n) \in \{0,1\}^n$ such that $s_{\alpha_1}^{\varepsilon_1} \cdots s_{\alpha_n}^{\varepsilon_n} = v$.*

*(vii) For $w \in W$ and any simple reflection $s = s_\alpha$ associated to a simple root $\alpha$ we have*

$$x_w\delta_s = \begin{cases} -x_w & \text{if } ws < w \\ (w(\alpha))x_{ws} - x_w + \sum_{v \xrightarrow{\beta} ws} \langle w(\alpha), \beta^\vee\rangle x_v & \text{otherwise.} \end{cases}$$

*(viii) For any $w \in W$, we have*

$$\Delta(x_w) = \sum_{u,v \leq w} p^w_{u,v} \cdot x_u \otimes x_v$$

*for some homogeneous polynomial $p^w_{u,v} \in S$ of degree $\ell(u) + \ell(v) - \ell(w)$. In particular $p^w_{u,v} = 0$ unless $\ell(u) + \ell(v) \geq \ell(w)$. We even have $p^w_{u,v} \in \mathbb{Z}[\underline{\alpha}]$ and these polynomials are unique.*

**Proof :** (i) We compute:

$$x_\alpha^2 = \frac{1}{\alpha}(\delta_{s_\alpha} - \delta_e)\frac{1}{\alpha}(\delta_{s_\alpha} - \delta_e) = \frac{1}{\alpha^2}(-\delta_{s_\alpha} - \delta_e)(\delta_{s_\alpha} - \delta_e) = 0.$$

We prove (ii), (iii) and (v) simultaneously by induction. Assume $w$ of length one i.e. $w = s$ a simple reflection. Then (ii) and (v) are clear. Let us prove (iii). We compute:

$$\lambda x_\alpha = \frac{\lambda}{\alpha}(\delta_{s_\alpha} - \delta_e)$$

while the right hand side of (iii) is

$$x_\alpha s_\alpha(\lambda) - \langle\lambda, \alpha^\vee\rangle x_e = \frac{1}{\alpha}(\delta_{s_\alpha} - \delta_e)s_\alpha(\lambda) - \langle\lambda, \alpha^\vee\rangle\delta_e = \frac{\lambda}{\alpha}\delta_{s_\alpha} - \frac{s_\alpha(\lambda)}{\alpha}\delta_e - \langle\lambda, \alpha^\vee\rangle\delta_e$$

and the result follows in this case.

Let us assume (ii), (iii) and (v) are true for any element of length $n$ and take $w$ of length $n+1$. Take $w = us_\alpha$ with $\ell(w) \geq 2$ and $u < w$. We compute

$$\lambda x_u x_\alpha = (x_u(u^{-1}(\lambda) - \sum_{u_0 \xrightarrow{\beta_0} u} \langle\lambda, \beta_0^\vee\rangle x_{u_0})x_\alpha = x_u x_\alpha(w^{-1}(\lambda)) - \langle u^{-1}(\lambda), \alpha^\vee\rangle x_u - \sum_{u_0 \xrightarrow{\beta_0} u} \langle\lambda, \beta_0^\vee\rangle x_{u_0} x_\alpha.$$

Take $u_0$ and $\beta_0$ such that $u_0 \xrightarrow{\beta_0} u$. Then we have $u = u_0 s_{\beta_0}$ and $us_\alpha = w > u$. Now we have the alternative $u_0 s_\alpha \leq u$ or $u_0 s_\alpha \leq us_\alpha = w$. In the first case, the two elements have the same length and are such equal: $u_0 s_\alpha = u = u_0 s_{\beta_0}$ i.e. $\beta_0 = \alpha$ which is not possible because in that case $u_0 = w$. We thus have $u_0 s_\alpha \leq w$. Let us define the map:

$$\{u_0 \ / \ u_0 \xrightarrow{\beta_0} u \text{ and } u_0 s_\alpha > u_0\} \to \{v \ / \ v \xrightarrow{\beta} w \text{ and } v \neq w\}$$

by $u_0 \mapsto v = u_0 s_\alpha$ and $\beta = s_\alpha(\beta_0)$. The same argument defines a map in the other direction thus these to sets are in bijection. Furthermore, by (ii) and the induction hypothesis, we have $x_{u_0} x_\alpha = x_{u_0 s_\alpha}$ for $u_0 s_\alpha > u_0$. Furthermore, by (i), (ii) and induction we have $x_{u_0} x_\alpha = 0$ for $u_0 s_\alpha < u_0$. We get in the previous formula:

$$(\dagger) \quad \lambda x_u x_\alpha = x_u x_\alpha(w^{-1}(\lambda)) - \sum_{v \xrightarrow{\beta} w} \langle\lambda, \beta^\vee\rangle x_v.$$

Now we want to compare for $w = u' s_\beta$ with $\ell(u') = \ell(w) - 1$ the expressions $x_u x_\alpha$ and $x_{u'} x_\beta$. We write:

$$x_u x_\alpha = \sum_{v \in W} q_v \delta_v \quad \text{and} \quad x_{u'} x_\beta = \sum_{v' \in W} q_{v'} \delta_{v'}.$$

By (v) and the induction hypothesis, we have that $x_u$ (resp. $x_{u'}$) has non vanishing coefficients only on elements $\delta_v$ for $v \leq u$ (resp. $v \leq u'$). This implies that $q_v = 0$ and $q_{v'} = 0$ unless $v \leq w$. Furthermore, we also obtain that

$$q_w = \frac{c_{u,u}}{u^{-1}(\alpha)} = \prod_{\gamma \in \Delta_+ \cap w(\Delta_-)} \beta^{-1} = q_{w'}.$$

Now we use equation (†) for the expressions $w_u s_\alpha$ and $w = u' s_\beta$ to obtain the equation

$$\lambda x_u x_\alpha - x_u x_\alpha (w^{-1}(\lambda)) = \lambda x_{u'} x_\beta - x_{u'} x_\beta (w^{-1}(\lambda)).$$

Replacing $x_u x_\alpha$ and $x_{u'} x_\beta$ by their expressions in terms of the $\delta_v$ we obtain for all $v \in W$:

$$(\lambda - v(w^{-1}(\lambda)))q_v = (\lambda - v(w^{-1}(\lambda)))q'_v.$$

But the representation of $W$ in $\mathfrak{h}^*$ being faithful, we have the equality $q_v = q'_v$ for all $v \neq w$ and (ii) follows as well as (iii). Furthermore now we have $c_{w,v} = q_v = q'_v$ and (v) follows also.

For (iv) remark that the first part follows from (ii). For the second part, take $v$ and $w$ in $W$ such that $\ell(vw) < \ell(v) + \ell(w)$. Take a reduced decomposition $w = s_1 \cdots s_n$ and choose $k$ such that $\ell(vs_1 \cdots s_k) = \ell(v) + k$ but $\ell(vs_1 \cdots s_{k+1}) < \ell(v) + k + 1$ (i.e. $\ell(vs_1 \cdots s_{k+1} = \ell(v) + k - 1)$. Set $w_k = s_1 \cdots s_k$. Set $u = vw_k s_{k+1}$, we have $\ell(u) = \ell(v) + k - 1$ and $us_{k+1} = vw_k$. In particular if $s'_1 \cdots s'_{\ell(v)+k-1}$ is a reduced expression for $u$, then $s'_1 \cdots s'_{\ell(v)+k-1} s_{k+1}$ is a reduced expression for $vw_k$. We obtain:

$$x_v x_w = x_v x_{w_k} x_{s_{k+1}} \cdots x_{s_n} = x_{s'_1} \cdots x_{s'_{\ell(v)+k-1}} x_{s_{k+1}} x_{s_{k+1}} \cdots x_{s_n}$$

and the vanishing follows by (i).

(vi) Follows from (ii) and (iv) by an easy induction. Indeed, write $w = w' s_\alpha$ with $\alpha$ a simple root. Write $x_{w'} = \sum c_{w',v} \delta_v$. We compute $x_w = x_{w'} x_a$. First we have he easy formula:

$$\delta_v x \alpha = \frac{1}{v(\alpha)} (\delta_{vs_\alpha} - \delta_v).$$

This gives

$$\begin{aligned} x_w &= \textstyle\sum_{v \in W} \frac{c_{w',v}}{v(\alpha)} (\delta_{vs_\alpha} - \delta_v) \\ &= \textstyle\sum_{u \in W} \frac{c_{w',us_\alpha}}{u(-\alpha)} \delta_u - \sum_{v \in W} \frac{c_{w',v}}{v(\alpha)} \delta_v \\ &= -\textstyle\sum_{v \in W} \left[ \frac{c_{w',vs_\alpha}}{v(\alpha)} + \frac{c_{w',v}}{v(\alpha)} \right] \delta_v. \end{aligned}$$

We now want to compute these values. The first term will correspond to expressions $v = s_{\alpha_1}^{\varepsilon_1} \cdots s_{\alpha_{n-1}}^{\varepsilon_{n-1}} s_\alpha$ and the second one to expressions $v = s_{\alpha_1}^{\varepsilon_1} \cdots s_{\alpha_{n-1}}^{\varepsilon_{n-1}}$. Indeed, we have

$$c_{w',vs_\alpha} = (-1)^{n-1} \sum \left( \left( s_{\alpha_1}^{\varepsilon_1}(\alpha_1) \right) \cdots \left( s_{\alpha_1}^{\varepsilon_1} \cdots s_{\alpha_{n-1}}^{\varepsilon_{n-1}}(\alpha_{n-1}) \right) \right)^{-1}$$

where the sum runs over all sequences $(\varepsilon_1, \cdots, \epsilon_{n-1}) \in \{0,1\}^n$ such that $s_{\alpha_1}^{\varepsilon_1} \cdots s_{\alpha_{n-1}}^{\varepsilon_{n-1}} = vs_\alpha$. The first remark is that the sign will agree. We now want to compare this with the following sum runs over all sequences $(\varepsilon_1, \cdots, \epsilon_{n-1}) \in \{0,1\}^n$ such that $s_{\alpha_1}^{\varepsilon_1} \cdots s_{\alpha_{n-1}}^{\varepsilon_{n-1}} = vs_\alpha$.

$$\sum \left( \left( s_{\alpha_1}^{\varepsilon_1}(\alpha_1) \right) \cdots \left( s_{\alpha_1}^{\varepsilon_1} \cdots s_{\alpha_{n-1}}^{\varepsilon_{n-1}}(\alpha_{n-1}) \right) \left( s_{\alpha_1}^{\varepsilon_1} \cdots s_{\alpha_{n-1}}^{\varepsilon_{n-1}}(\alpha_{n-1}) s_\alpha(\alpha) \right) \right)^{-1}$$

$$= \sum \left( \left( s_{\alpha_1}^{\varepsilon_1}(\alpha_1) \right) \cdots \left( s_{\alpha_1}^{\varepsilon_1} \cdots s_{\alpha_{n-1}}^{\varepsilon_{n-1}}(\alpha_{n-1}) \right) v(\alpha) \right)^{-1} = \frac{c_{w',v}}{v(\alpha}).$$

The same kind of computation gives the result.

(vii) We compute

$$x_w\delta_s = -x_w(x_s\alpha + \delta_e) = \begin{cases} -x_w & \text{if } ws < w \\ -x_{ws}\alpha - x_w & \text{if } ws > w. \end{cases}$$

The result follows by (iii).

(viii) We prove this by induction on $\ell(w)$. It is true for $\ell(w) = 0$. For $w = s_\alpha$ with $\alpha$ simple, we have

$$\begin{aligned}\Delta(x_\alpha) &= \tfrac{1}{\alpha}(\delta_{s_\alpha} \otimes \delta_{s_\alpha} - \delta_e \otimes \delta_e) \\ &= (\delta_{s_\alpha} - \delta_e) \otimes \tfrac{1}{\alpha}(\delta_{s_\alpha} - \delta_e) + \delta_e \otimes \tfrac{1}{\alpha}(\delta_{s_\alpha} - \delta_e) + \tfrac{1}{\alpha}(\delta_{s_\alpha} - \delta_e) \otimes \delta_e \\ &= \alpha x_\alpha \otimes x_\alpha + x_e \otimes x_\alpha + x_\alpha \otimes x_e.\end{aligned}$$

For $w \in W$, we write $w = sw'$ with $s = s_\alpha$ a simple reflection and $\ell(w) > \ell(w')$. By induction and the fact that $\Delta$ is a ring morphism, we get the existence of a homogeneous polynomial $p^{w'}_{u',v'}$ of degree $\ell(u') + \ell(v') - \ell(w')$ such that

$$\Delta(x_w) = \Delta(x_\alpha) \odot \Delta(x_{w'}) = \frac{1}{\alpha}(\delta_s \otimes \delta_s - \delta_e \otimes \delta_e) \odot \left(\sum_{u',v' \leq w'} p^{w'}_{u',v'} x_{u'} \otimes x_{v'}\right).$$

From the definition of $\odot$ we have:

$$(\delta_w \otimes \delta_w) \odot (x \otimes y) = q\delta_w x \otimes \delta_w y.$$

We get

$$\begin{aligned}\Delta(x_w) &= \textstyle\sum_{u',v' \leq w'} \left[\frac{1}{\alpha}(s_\alpha p^{w'}_{u',v'})\delta_\alpha x_{u'} \otimes \delta_\alpha x_{v'} - \frac{1}{\alpha} p^{w'}_{u',v'} x_{u'} \otimes x_{v'}\right] \\ &= \textstyle\sum_{u',v' \leq w'} \Big[\alpha(s_\alpha p^{w'}_{u',v'})\frac{1}{\alpha}(\delta_\alpha - \delta_e)x_{u'} \otimes \frac{1}{\alpha}(\delta_\alpha - \delta_e)x_{v'} \\ &\quad + (s_\alpha p^{w'}_{u'',v'})x_{u'} \otimes \tfrac{1}{\alpha}(\delta_\alpha - \delta_e)x_{v'} + (s_\alpha p^{w'}_{u',v'})\tfrac{1}{\alpha}(\delta_\alpha - \delta_e)x_{u'} \otimes x_{v'} + \tfrac{s_\alpha p^{w'}_{u',v'} - p^{w'}_{u',v'}}{\alpha} x_{u'} \otimes x_{v'}\Big] \\ &= \textstyle\sum_{u',v' \leq w'} \Big[\alpha(s_\alpha p^{w'}_{u',v'})x_\alpha x_{u'} \otimes x_\alpha x_{v'} + (s_\alpha p^{w'}_{u',v'})x_{u'} \otimes x_\alpha x_{v'} + (s_\alpha p^{w'}_{u',v'})x_\alpha x_{u'} \otimes x_{v'} \\ &\quad + \tfrac{s_\alpha p^{w'}_{u',v'} - p^{w'}_{u',v'}}{\alpha} x_{u'} \otimes x_{v'}.\Big]\end{aligned}$$

This gives the result. The fact that the polynomial are in $\mathbb{Z}[\underline{\alpha}]$ comes from the fact that the operator $p \mapsto \frac{s_\alpha p - p}{\alpha}$ send $\mathbb{Z}[\underline{\alpha}]$ to itself. □

**Definition 25.1.6** (i) Let us define $\mathcal{D}$ the space of all matrices $A = (a_{v,w})_{v,w \in W}$ with entries in $W \times W$ and values in $Q$ such that there exists an integer $n$ (depending on $A$) such that $a_{v,w} = 0$ unless $\ell(v) - ell(w) \leq n$.

The matrix multiplication in then well defined nd this gives a ring structure on $\mathcal{D}$.

(ii) We define the change of basis matrix

$$C = (c_{v,w})_{v,w \in W}.$$

It is a lower triangular matrix and is in $\mathcal{D}$ It is even invertible because its diagonal elements are non zero. We shall the that the matrix $D = {}^t C^{-1}$ (as well as $C$) play an important role.

For the moment we did not define any integral structure (i.e. a ring over $S$ nd not only $Q$). Let us first define a structure of $Q_W$-module on $Q$:

**Definition 25.1.7** (i) Define the action

$$(q\delta_w) \bullet q' = qw(q').$$

(ii) Now we may define a sub-ring $R_W$ of $Q_W$ by

$$R_w = \{a \in Q_W \ / \ a \bullet S \subset S\}.$$

**Fact 25.1.8** *The elements $(x_w)_{w \in W}$ are in $R_W$. They form a sub-ring $R$ of $R_W$.*

**Proof :** Compute the action of $x_\alpha$ on a weight $\lambda$:

$$x_\alpha \bullet \lambda = \frac{1}{\alpha}(\delta_\alpha - \delta_e) \bullet \lambda = \frac{s_\alpha(\lambda) - \lambda}{\alpha} = \langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}.$$

Because $x_\alpha$ and $\lambda$ generate the $x_w$ and $S$, the result follows. The fact that they from a sub-ring comes from the previous Theorem. □

A more difficult result that we won't prove here is the following

**Theorem 25.1.9** *We have the equality $R_W = R$.*

This ring is called the Nil-Hecke ring because its relations are similar to the relation of a classical Hecke ring. This ring will be a ring of operators on the equivariant cohomology $H_T^*(G/B)$. We may realise this cohomology as a "dual" of the nil-Hecke ring.

## 25.2 The dual ring

Recall the definition of the space $\Omega_W$ of all functions $f : W \to Q$. Of course this space may be identified with the space $\hom_Q(Q_W, Q)$ by defining for $h \in \hom_Q(Q_W, Q)$ the function $f : W \to Q$ as $f(w) = h(\delta_w)$. We will identify these two spaces.

The commutative comultiplication in $Q_W$ induces a commutative product in $\hom_Q(Q_W, Q)$ and this product is simply the product of functions in $\Omega_W$. The counit induces a unit for this product and in $\Omega_W$ the unit is the constant function $1(w) = 1$. Remark that we obtain in this way a commutative $Q$-algebra structure on $\Omega_W$.

From the identification of $\Omega_W$ with $\hom_Q(Q_W, Q)$, we have a $Q_W$-action on $\Omega_W$ given by

$$(x \star f)(y) = f(yx)$$

for $x, y \in Q_W$ and $f \in \Omega_W$. The action of the following elements are very important:

$$(\delta_w \star f)(v) = f(\delta_v \delta_w) = f(vw)$$

this is called the Weyl group action. Indeed,looking at the equivariant cohomology ring $H_T^*(G/B)$ as a sub-ring of $\Omega_W$ (for the moment we only have a map), the action of the Weyl group on the equivariant cohomoogy is given by this action (recall that it is give by right multiplication). We also have

$$(x_\alpha \star f)(v) = f(\delta_v x_\alpha) = \frac{f(vs_\alpha) - f(v)}{v(\alpha)}.$$

These operators will coincide with the Demazure operators $\widehat{D}_\alpha$. Finally we have the action:

$$((q\delta_e) \star f)(v) = f(vq\delta_e) = f(v(q)\delta_v\delta_e) = v(q)f(v).$$

In particular we have the equality

$$(q\delta_e) \star f = qf$$

if and only if $q \in Q^W$.

**Definition 25.2.1** We define the restricted dual $\Lambda$ of $R$ by

$$\Lambda = \{f \in \Omega_W \ / \ f(R) \subset S \text{ and } f(x_w) = 0 \text{ for all but finitely many } w \in W\}.$$

**Lemma 25.2.2** *The set $\Lambda$ is an $S$-subalgebra of the $Q$-algebra $\Omega_W$ and is free as $S$-module with basis $(\xi^w)_{w\in W}$ such that*

$$\xi^w(x_v) = \delta_{w,v}.$$

*Furthermore, the ring $\Lambda$ is stable under the left action $\star$ of $R$.*

**Definition 25.2.3** We define the matrix $D = (d_{u,v})_{u,v\in W} \in \mathcal{D}$ by

$$d_{u,v} = \xi^u(x_v).$$

**Theorem 25.2.4** *(ı) We have $D^{-1} = {}^t C$ in particular $d_{u,v} = 0$ unless $u \leq v$ and*

$$d_{u,u} = \prod_{\beta\in\Delta_+\cap w\Delta_-} \beta.$$

*(ıı) We have for $\alpha$ a simple root:*

$$x_\alpha \star \xi^w = \begin{cases} \xi^{ws_\alpha} & \textit{if } ws_\alpha < w \\ 0 & \textit{otherwise.} \end{cases}$$

*(ııı) $\xi^e$ is a unit.*
*(ıv) We have the general multiplication formula:*

$$\xi^u\xi^v = \sum_{w\geq u,v} p^w_{u,v}.$$

*(v) More precisely we have the Equivariant Chevaley formula:*

$$\xi^{s_\alpha}\xi^w = d_{s_\alpha,w}\xi_w + \sum_{w\xrightarrow{\beta}v} \langle w(\omega_\alpha), \beta^\vee\rangle \xi^v.$$

Remark that this describes the Littlewood-Richardson coefficients which are $p^w_{u,v}$. However these are far from being explicit. In the next result we relate these coefficients with the matrix $D$:

**Theorem 25.2.5** *Define the matrix $P^u$ by $P^u_{v,w} = p^w_{u,v}$ and $D^u$ by $D^u_{v,w} = \delta_{v,w}d_{u,v}$ then we have*

$$P^w = D \cdot D^w \cdot D^{-1}.$$

We now come to our main result. For this we need some preliminary notation. First let us denote by $S_\mathbb{Z} = S^*(\mathfrak{h}^*_\mathbb{Z})$ and by $\Lambda_\mathbb{Z}$ the $S_\mathbb{Z}$-subalgebra of $\Lambda$ by

$$\Lambda_\mathbb{Z} = \bigoplus_{w\in W} S_\mathbb{Z}\xi^w.$$

We define a grading on $\Lambda_\mathbb{Z}$ by

$$\Lambda^{2d}_\mathbb{Z} = \bigoplus_{w\in W} S^{d-\ell(w)}_\mathbb{Z}\xi^w.$$

Finally, denote by $\nu$ the natural $S_\mathbb{Z}$-algebra homomorphism $H^*_T(G/B) \to \Omega_W$ described in the last Chapter.

**Theorem 25.2.6** *The map $\nu$ is injective and satisfies the relations:*

$$\nu(w \cdot x) = \delta_w \star \nu(x) \quad \text{and} \quad \nu(\widehat{D}_\alpha(x)) = x_\alpha \star \nu(x).$$

*Furthermore* $\mathrm{im}\nu = \Lambda_{\mathbb{Z}}$.

**Proof :** let us give a sketch of proof. First consider the filtration $X_n$ defining the ind-variety structure on $G/B$. For these varieties we have an isomorphism

$$Q \otimes_{S_{\mathbb{Z}}} H^*_T(G/B) \simeq Q \otimes_{\mathbb{Z}} H^0(X_n^T)$$

and we also know that $H^*_T(X_n)$ is torsion free thus included in the first and hence in the second terms. But this last inclusion factors through the map

$$H^*_T(X_n) \to S_{\mathbb{Z}} \otimes_{\mathbb{Z}} H^0(X_n^T)$$

which has to be injective. Now for a fixed cohomology degree $i$ we have $H^i_T(G/B) = H^i_T(X_n)$ for large $n$ and the injectivity follws.

The first formula is also easy to prove, consider the commutative diagram given by localisation at $T$-fixed points:

$$\begin{array}{ccc} H^*_T(G/T) & \xrightarrow{R^*_w} & H^*_T(G/T) \\ \downarrow & & \downarrow \\ H^*_T(W) & \xrightarrow{r^*_w} & H^*_T(W) \end{array}$$

where the maps $R_w$ are the right multiplication by $w$. This gives the first formula.

We will not prove the second formula which uses the $\mathbb{P}^1$-fibration $G/B \to G/P_\alpha$ where $P_\alpha$ is the minimal parabolic subgroup associated to $\alpha$.

For the last statement, we first prove the inclusion of the image in $\Lambda$. Let $X \in H^{2n}_T(G/B)$ and let $w \in W$. We want to see that $\nu(x)(x_w) \in S$ and that this is zero except for a finite number of $w \in W$. Let $w = s_1 \cdots s_n$ be a reduced expression. Then by the second formula we get:

$$\nu(\widehat{D}_{\alpha_1} \cdots \widehat{D}_{\alpha_1}(x)) = x_w \star \nu(x).$$

Consider these elements as functions on $W$ (these elements are in $\Omega_W$) and evaluate them at $e \in W$:

$$\nu(\widehat{D}_{\alpha_1} \cdots \widehat{D}_{\alpha_1}(x))(e) = x_w \star \nu(x)(e) = \nu(x)(ex_w) = \nu(x)(x_w).$$

But $\widehat{D}_{\alpha_1} \cdots \widehat{D}_{\alpha_1}(x)$ is an element in the equivariant cohomology ring $H^*_T(G/B)$, in particular its evaluation at any $T$-fixed point has value in $S_{\mathbb{Z}}$ thus $\nu(x)(x_w) \in S_{\mathbb{Z}}$. Furthermore, the class $x$ being of degree $2n$ and the Demazure operators decreasing the degree by 2 we get $\widehat{D}_{\alpha_1} \cdots \widehat{D}_{\alpha_1}(x) = 0$ for $\ell(w) > n$. This proves the inclusion in $\Lambda$. For the inclusion in $\Lambda_{\mathbb{Z}}$, recall that the $\xi^w$ form a basis of $\Lambda$ thus we can write $\nu(x) = \sum_w a_w \xi^w$. But $a_w = \nu(x)(x_w) \in S_{\mathbb{Z}}$ and the result follows.

The surjectivity comes from the existence of classes in the equivariant cohomology that are mapped to the Schubert classes in the classical cohomology and on $\xi^w$ in $\Omega_W$. □

As an example of other geometric meaning of the Nil-Hecke ring, let us mention the following result:

**Theorem 25.2.7** *Consider the Schubert variety $X_w$ in $G/B$ and the fixed point $v$ corresponding to $v \in W$ in $X_w$ (with $v \leq w$). Define:*

$$S(w, v) = \{\beta\Delta_+ \ / \ s_\beta v \leq w \textit{ and } \beta \textit{ is a real root}\}.$$

*Then $v$ is a smooth point if and only if*

$$c_{w,v} = (-1)^{\ell(w)-\ell(v)} \prod_{\beta \in S(w,v)} \beta^{-1}.$$

# Chapter 26

# Quantum cohomology for finite dimensional homogeneous spaces

Let $G$ be a semisimple algebraic group and let $P$ be a parabolic subgroup. We denote by $X$ the quotient $G/P$ and call it a finite dimensional homogeneous space. In this chapter we quickly review the theory of quantum cohomology for these spaces. The basis reference is [FP97]. We also refer to the fundamental articles [Ko95] and [KM98].

## 26.1 The space of stable maps

We will not construct the moduli space $\bar{M}_{g,n}(X,\alpha)$ of stable maps of class $\alpha$ from a genus $g$ curve to the variety $X$. We will admit its existence and basic properties. We will only deal with curves of genus 0.

### 26.1.1 Stable maps

We collect in this subsection the general definitions of stable curves and stable maps, the theorem on existence of a coarse moduli space of stable maps to any projective algebraic scheme and the definition of evaluations maps and the maps forgeting some of the marked points.

**Definition 26.1.1** (ı) An $n$-pointed, quasi-stable map from a rational curve to a variety $X$ is the data $(C, p_1, \cdots, p_n, f)$ where $C$ is a projective, connected, reduced curve of arithmetic genus 0 having at most nodal points, with $n$ distinct nonsingular marked points $(p_1, \cdots, p_n)$ and where $f$ is a morphism $f : C \to X$.

(ıı) If $E$ is an irreducible component of $C$, the special points of $E$ in $(C, p_1, \cdots, p_n, f)$ are the marked points lying on $E$ and the singular points on $E$ in $C$.

The quasi-stable rational map $(C, p_1, \cdots, p_n, f)$ is called stable if for any component $E$ of $C$ contracted by $f$ the automorphism group of $E$ fixing the special points is finite.

(ııı) A family of $n$-pointed, stable map from a rational curve to a variety $X$ over a base $S$ is the data $(\pi : \mathcal{C} \to S, p_1, \cdots, p_n, f)$ where $\pi$ is a flat projective map with $n$ sections $p_i : S \to \mathcal{C}$ for $i \in [1, n]$ and $f$ is a morphism $f : \mathcal{C} \to X$ such that for each geometric point $s \in S$ the induced data $(\mathcal{C}_s, p_1(s), \cdots, p_n(s), f)$ is a stable map from a rational curve to $X$.

**Remark 26.1.2** A $n$-pointed rational quasi-stable map $(C, p_1, \cdots, p_n, f)$ is stable if and only if for any irreducible component $E$ of $C$ contracted by $f$, there are at least three special points on $E$.

**Definition 26.1.3** (i) Let $\alpha \in H^2(X)$ be a class of 1-cycle on an algebraic scheme $X$. A map $f : C \to X$ from a curve $C$ to $X$ is said of class $\alpha$ if $f_*[C] = \alpha$.

(ii) Let $n$ be an integer, $X$ be an algebraic variety and $\alpha \in H^2(X)$. We denote by $\bar{\mathfrak{M}}_{0,n}(X, \alpha)$ the following functor:

$$\bar{\mathfrak{M}}_{0,n}(X,\alpha)(S) = \left\{ \begin{array}{c} \text{isomorphism classes of stable families over } S \\ \text{of rational } n\text{-pointed maps of class } \alpha \text{ to } X. \end{array} \right\}$$

The fundamental result on stable maps is the following theorem: there exists a coarse moduli space for stable maps to any projective algebraic scheme $X$.

**Theorem 26.1.4** *There exists a projective scheme $\bar{M}_{0,n}(X, \alpha)$ which is coarse moduli space for the functor $\bar{\mathfrak{M}}_{0,n}(X, \alpha)$. This means that there is a natural morphism of functors:*

$$F : \bar{\mathfrak{M}}_{0,n}(X, \alpha) \to \hom_{\mathrm{Sch}}(\bullet, \bar{M}_{0,n}(X, \alpha))$$

*such that $F$ is a bijection over* $\mathrm{Spec}(\mathbb{C})$ *and for any scheme $Z$ with a morphism of functors $G : \bar{\mathfrak{M}}_{0,n}(X, \alpha) \to \hom_{\mathrm{Sch}}(\bullet, Z)$ there is a unique morphism of schemes*

$$f : \bar{M}_{0,n}(X, \alpha) \to Z$$

*such that $G = f \circ F$ where we still denoted $f$ the induced functor*

$$\hom_{\mathrm{Sch}}(\bullet, \bar{M}_{g,n}(X, \alpha)) \to \hom_{\mathrm{Sch}}(\bullet, Z).$$

**Remark 26.1.5** In fact the functor $\bar{\mathfrak{M}}_{0,n}(X, \alpha)$ defines a smooth Deligne-Mumford stack but we do not want to deal with stacks here.

### 26.1.2 Morphisms

#### Evaluation morphisms

Let $(\pi : \mathcal{C} \to S, p_1, \cdots, p_n, f)$ be a family over $S$ of stable $n$-pointed rational maps to an algebraic scheme $X$. For any $i \in [1, n]$ there exists a natural morphism $f \circ p_i : S \to X$. This gives a morphism of functors

$$\bar{\mathfrak{M}}_{0,n}(X, \alpha) \to \hom_{\mathrm{Sch}}(\bullet, X).$$

The universal property of $\bar{M}_{0,n}(X, \alpha)$ gives a morphism called $i^{\text{th}}$ evaluation morphism

$$\rho_i : \bar{M}_{0,n}(X, \alpha) \to X.$$

On the level of points it is simply given by $\rho_i(C, p_1, \cdots, p_n, f) = f(p_i)$.

#### Forgetting the map

In the same vein, define the moduli space $\bar{M}_{0,n} = \bar{M}_{0,n}(\{\text{pt}\}, 0)$ of $n$-pointed, rational stable curves. Any element $(\pi : \mathcal{C} \to S, p_1, \cdots, p_n, f) \in \bar{\mathfrak{M}}_{0,n}(X, \alpha)(S)$ i.e. any stable map defines an element, a stable curve, $(\pi : \mathcal{C} \to S, p_1, \cdots, p_n, cst) \in \bar{M}_{0,n}$ and by composition a morphism of functors:

$$\bar{\mathfrak{M}}_{0,n}(X, \alpha) \to \hom_{\mathrm{Sch}}(\bullet, \bar{M}_{0,n})$$

so that there exist a forgetful map

$$\eta : \bar{M}_{0,n}(X, \alpha) \to \bar{M}_{0,n}.$$

**Forgetting points**

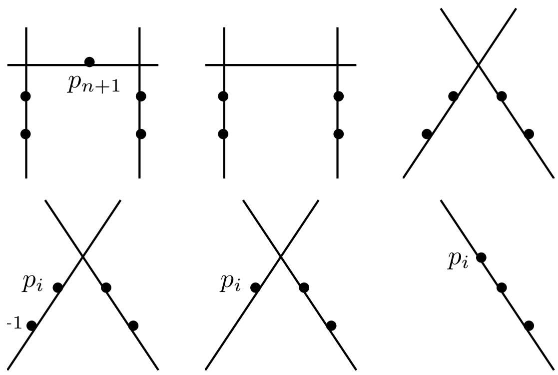
Let us use the following notation: if $A$ is a finite set, we denote by $\bar{M}_{0,A}(X,\alpha)$ (resp. $\bar{M}_{0,A}$) the moduli space of stable rational maps (resp. curves) with marked points labeeled by $A$. It is isomorphic to $\bar{\mathfrak{M}}_{0,n}(X,\alpha)$ (resp. $\bar{M}_{0,n}$) with $n = \mathrm{Card}(A)$.

If $A$ is a subset of $[1,n]$ one can also construct forgetfull morphisms

$$\mu_A : \bar{M}_{0,n}(X,\alpha) \to \bar{M}_{0,A}(X,\alpha) \quad \text{and} \quad \nu_A : \bar{M}_{0,n}(X,\alpha) \to \bar{M}_{0,A}$$

respectively given on a general curve $(C, p_1, \cdots, p_n, f)$ by $\mu_A(C, p_1, \cdots, p_n, f) = (C, (p_i)_{i\in A}, f)$ and by $\nu_A(C, p_1, \cdots, p_n) = (C, (p_i)_{i\in A})$. Remark that this is not obviously defined since forgeting points can destabilise the curve $C$ as in the following examples.

**Example 26.1.6** In the following pictures, each irreducible component of the curve are represented by lines with points on them. The left pictures represents the $(n+1)$-stable curve, the middle one represents this curve with the points $p_{n+1}$ ommited and the right one represents the stabilisation of the curve.




However F. Knudsen [Kn83] proved that the curves $(C, (p_i)_{i\in A})$ can be stabilised in families. For maps $(C, (p_i)_{i\in A}, f)$ a similar result is true (see for example [KV06] for the case of $\mathbb{P}^n$). The maps $\mu_A$ and $\nu_A$ are then well defined.

### 26.1.3 Irreducibility and dimension

Let us denote by $\bar{M}_{0,n}(X,\alpha)^*$ the open set of stable maps with no non-trivial automorphism and by $M_{0,n}(X,\alpha)$ the open set of stable maps from an irreducible curve. The boundary of $\bar{M}_{g,n}(X,\alpha)$ is the locus of morphisms from reduced rational curves to $X$.

We have the following result:

**Theorem 26.1.7** *(ı) The variety $\bar{M}_{0,n}(X,\alpha)$ is a normal irreductible projective variety of the expected dimension $\int_\alpha c_1(X) + \dim X + n - 3$. It is locally a quotient of a non singular variety by a finite group.*

*(ıı) The open set $\bar{M}_{0,n}(X,\alpha)^*$ is non singular and a represents the restricted functor of automorphism-free stable maps.*

*(ııı) The boundary of $\bar{M}_{0,n}(X,\alpha)$ is a union of subvarieties of pure codimension 1. It is a divisor with normal crossing over the automorphism-free locus. In general the boundary is a divisor with normal crossing up to a finite group quotient.*

**Remark 26.1.8** (ı) For a proof, see [FP97] and [Th98] for irreducibility.

(ıı) If a morphism $f : \mathbb{P}^1 \to X$ is birational onto its image, then the map $(C, f)$ does not have any automorphism. This in particular implies that the complementary of $\bar{M}_{0,n}(X,d)^*$ in $\bar{M}_{0,n}(X,d)$ is of codimension at least 2 except if $X = \mathbb{P}^2$ and $(\deg(\alpha), n) \neq (2, 0)$ (simple dimension count thanks to the fact that the dimension is the expected one).

(ııı) Over the automorphism-free locus, the boundary of $\bar{M}_{0,n}(X, \alpha)$ shares the properties of the boundary of $\bar{M}_{0,n}$.

**Description of the boundary**

The boundary component are indexed by the data $(A, B, \alpha_1, \alpha_2)$ where

- $A \cup B$ is a partition of $[1, n]$;
- $\alpha_1$ and $\alpha_2$ are effective 1-cycle classes and $\alpha_1 + \alpha_2 = \alpha$;
- if $\alpha_1 = 0$ (resp. $\alpha_2 = 0$) then $\mathrm{Card}(A) \geq 2$ (resp. $\mathrm{Card}(B) \geq 2$)).

Let us denote by $D(A, B, \alpha_1, \alpha_2)$ the divisor of the boundary indexed by $(A, B, \alpha_1, \alpha_2)$. The general curve $(C, p_1, \cdots, p_n), f)$ in $D(A, B, \alpha_1, \alpha_2)$ is such that

- $C$ is the union of two rational pointed quasi-stable maps $(C_A, (p_i)_{i \in A}, f|_A)$ and $(C_B, (p_i)_{i \in B}, f|_B)$ meeting in one non singular point of each of them and disctint from the $p_i$'s;
- we have $f_*[C_A] = \alpha_1$ and $f_*[C_B] = \alpha_2$.

There is a nice description of the boundary $D(A, B, \alpha_1, \alpha_2)$ is terms of smaller moduli spaces. Indeed, let $\bar{M}_{0,A\cup\{\bullet\}}(X, \alpha_1)$ and $\bar{M}_{0,B\cup\{\bullet\}}(X, \alpha_2)$ be the moduli spaces of genus 0 stable maps with marked points in $A \cup \{\bullet\}$ and class $\alpha_1$ (resp. $B \cup \{\bullet\}$ and class $\alpha_2$). The evaluation map $\rho_\bullet$ gives morphisms $\bar{M}_{0,A\cup\{\bullet\}}(X, \alpha_1) \to X$ and $\bar{M}_{0,B\cup\{\bullet\}}(X, \alpha_2) \to X$ and we may consider the fiber product

$$\bar{M}_{0,A\cup\{\bullet\}}(X, \alpha_1) \times_X \bar{M}_{0,B\cup\{\bullet\}}(X, \alpha_2).$$

We have the following theorem (see [FP97]):

**Theorem 26.1.9** *The natural morphism*

$$\psi : \bar{M}_{0,A\cup\{\bullet\}}(X, \alpha_1) \times_X \bar{M}_{0,B\cup\{\bullet\}}(X, \alpha_2) \to D(A, B, \alpha_1, \alpha_2)$$

*is an isomorphism as soon as $A$ and $B$ are empty.*

**An equivalence relation on the boundary divisors**

Here we describe an equivalence relation on the boundary divisor that enables to use induction on the degree $\deg(\alpha)$ of the 1-cycle class $\alpha$. This result is essential in the proof of associativity of the quantum product or of Kontsevich's formula.

Let us set

$$D(i, j|k, l) = \sum [D(A, B, \alpha_1, \alpha_2)]$$

where the sum runs over all boundary components of $\bar{M}_{0,n}(X, \alpha)$ such that $\{i, j\} \subset A$, $\{k, l\} \subset B$ and $\alpha_1 + \alpha_2 = \alpha$.

**Theorem 26.1.10** *We have the following equality in $A^1(\bar{M}_{0,n}(X, \alpha))$:*

$$D(i, j|k, l) = D(i, l|j, k).$$

## 26.2 Quantum cohomology for homogeneous spaces

In this section we will define the Gromov-Witten invariants and the associated Quantum cohomology ring for $X$. The associativity of this ring will give a proof of Kontsevich's celebrated formula.

### 26.2.1 Gromov-Witten invariants

**Definition**

The evaluation morphisms $\rho_i : \bar{M}_{0,n}(X,\alpha) \to X$ are flat (because $X$ is homogeneous all fibers are isomorphic) and there is a flat pull back $\rho_i^* : A^*(X) \to A^*(\bar{M}_{0,n}(X,\alpha))$. If $(\gamma_i)_{i\in[1,n]}$ are in $A^*(X)$ such that

$$\sum_{i=1}^{n} \mathrm{codim}(\gamma_i) = \dim(\bar{M}_{0,n}(X,\alpha)) = \int_\alpha c_1 + \dim X + n - 3. \qquad (26.1)$$

then classical intersection theory (see [Fu98]) enables to give the following

**Definition 26.2.1** For cohomology classes $(\gamma_i)_{i\in[1,n]}$ on $X$ verifying equation 26.1 one can define the following number (Gromov-Witten invariant)

$$\langle\gamma_1, \cdots, \gamma_n\rangle_\alpha = \int_{\bar{M}_{0,n}(X,\alpha)} \rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n) = \deg\left(\rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n) \cap [\bar{M}_{0,n}(X,\alpha)]\right).$$

We will use the following simplification notation:

$$\langle\gamma_1, \cdots, \gamma_1, \gamma_2, \cdots, \gamma_2, \cdots, \gamma_k, \cdots, \gamma_k\rangle_\alpha = \langle\gamma_1^{n_1}, \cdots, \gamma_k^{n_k}\rangle_\alpha$$

where each $\gamma_i$ appears $n_i$ times for $i \in [1,k]$.

**Remark 26.2.2** In the case where $\alpha = 0$ we recover the classical invariant of the cohomology ring:

$$\langle\gamma_1, \cdots, \gamma_n\rangle_\alpha = \deg(\gamma_1 \cup \cdots \cup \gamma_n).$$

**Enumeration properties**

As a consequence of Bertini's theorem we obtain that the Gromov-Witten invariants are enumerative.

Let $X_1, \cdots, X_n$ be irreducible subvarieties of $X$ such that $\sum_{i=1}^{n} \mathrm{codim}(X_i) = \dim(\bar{M}_{0,n}(X,\alpha))$.

**Theorem 26.2.3** *Let $g_1, \cdots, g_n$ be general elements in $G$, then the scheme theoretic intersection in $\bar{M}_{0,n}(X,\alpha)$*

$$\rho_1^{-1}(g_1X_1) \cap \cdots \cap \rho_n^{-1}(g_nX_n)$$

*is a finite number of reduced points reprensenting automorphism free stable maps from an irreducible source. Furthermore we have*

$$\langle[X_1], \cdots, [X_n]\rangle_d = \mathrm{Card}(\rho_1^{-1}(g_1X_1) \cap \cdots \cap \rho_n^{-1}(g_nX_n)).$$

The Gromov-Witten invariants count a number of stable maps. Actually, if none of the Schubert varieties $X_i$ is a divisor, they also count rational curves (see [KV06] section 3.5 for a precise discussion on this fact). Furthermore, basic properties of Gromov-Witten invariants enable to restrict oneself to the case where none of the Schubert varieties $X_i$ is a divisor (see the next subsection) so that these invariants are well suited for enumeration of rational curves.

**First calculations with Gromov-Witten invariants**

We give here three basic properties of Gromov-Witten invariants:

**Lemma 26.2.4** MAPPING TO A POINT. *If $\alpha = 0$ and $\langle\gamma_1, \cdots, \gamma_n\rangle_\alpha \neq 0$, then $n = 3$ and*

$$\langle\gamma_1, \gamma_2, \gamma_3\rangle_0 = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3.$$

**Lemma 26.2.5** TRIVIAL CLASS. *If $\gamma_1 = 1$ and $\langle\gamma_1, \cdots, \gamma_n\rangle_\alpha \neq 0$, then $\alpha = 0$, $n = 3$ and*

$$\langle\gamma_1, \gamma_2, \gamma_3\rangle_0 = \int_X \gamma_2 \cup \gamma_3.$$

**Lemma 26.2.6** DIVISOR EQUATION. *If $\gamma_1$ is a non trivial divisor class then we have:*

$$\langle\gamma_1, \cdots, \gamma_n\rangle_\alpha = \left(\int_\alpha \gamma_1\right) \cdot \langle\gamma_2, \cdots, \gamma_n\rangle_\alpha.$$

Let $A$ and $B$ be two non empty subsets of $[1, n]$. In this case $D(A, B, \alpha_1, \alpha_2)$ is isomorphic to the fiber product $\bar{M}_{0,A\cup\{\bullet\}}(X, \alpha_1) \times_X \bar{M}_{0,B\cup\{\bullet\}}(X, \alpha_2)$. We have the following:

**Lemma 26.2.7** SPLITTING EQUATION. *Let $i$ be the inclusion of $D(A, B, \alpha_1, \alpha_2)$ in the product $\bar{M}_{0,A\cup\{\bullet\}}(X, \alpha_1) \times \bar{M}_{0,B\cup\{\bullet\}}(X, \alpha_2)$ and $\sigma$ be the inclusion of $D(A, B, \alpha_1, \alpha_2)$ in $\bar{M}_{0,n}(X, \alpha)$ with $\alpha = \alpha_1 + \alpha_2$. Then if $\gamma_1, \cdots, \gamma_n$ are in $A^*(X)$, we have:*

$$i_*\sigma^*(\rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n)) = \sum_{\lambda \subset p\times(N-p)} \left(\rho_\bullet^*([X_\lambda]) \cdot \prod_{a\in A} \rho_a^*(\gamma_a)\right) \times \left(\rho_\bullet^*([X_{\widehat{\lambda}}]) \cdot \prod_{b\in B} \rho_b^*(\gamma_b)\right).$$

## 26.2.2 Big quantum cohomology

In this section we define the big quantum cohomology ring. We will prove, thanks to the results on $\bar{M}_{0,n}(X, \alpha)$, that this ring is commutative (this is easy) and associative. The associativity of the ring gives relations between Gromov-Witten invariants and one recovers Kontsevich's formula in this way.

### Definition

Denote by $\sigma_w$ the cohomology class dual to $[X_w]$. Denote by $\sigma_\alpha$ the classes $\sigma_{s_\alpha}$. The schubert classes $(\sigma_\alpha)_{\alpha\in\Pi}$ form a $\mathbb{Z}$-basis of $H^2(X)$. The classes $(\sigma_w)_{w\in W}$ is a $\mathbb{Z}$-basis of $H^*(X)$. For each $w \in W$, one defines a formal variable $y_w$.

**Definition 26.2.8** The quantum cohomology group $QH^*(X)$ is $H^*X \otimes_\mathbb{Z} \mathbb{Q}[[(y_w)_{w\in W}]]$.

In order to define a product on this group, one defines a potential, that is to say a formal power serie in the variables $(y_w)_{w\in W}$. For $\gamma \in H^*(X)$ define

$$\Phi(\gamma) = \sum_{n\geq 3} \sum_{\alpha\in H^2(X)} \frac{\langle\gamma^n\rangle_\alpha}{n!}.$$

If we write $\gamma = \sum_{w\in W} y_w\sigma_w$ then the potential is a formal power serie:

$$\Phi((y_w)_w) = \sum_{\sum_w n_w \geq 3} \sum_\alpha \langle(\sigma_w^{n_w})_w\rangle_\alpha \prod_w \frac{y_w^{n_w}}{n_w!}.$$

For this formal serie to be well defined we need the following

**Lemma 26.2.9** *For a familly $(n_w)_w$ of integers, there are finitely many classes $\alpha \in H^2(X)$ with $\langle (\sigma_w^{n_w})_w \rangle_\alpha \neq 0$.*

Since the potential $\Phi$ is well defined one can define its partial derivatives:

$$\Phi_{uvw} = \frac{\partial^3 \Phi}{\partial y_u \partial y_v \partial y_w}, \quad \text{for } u,\ v \text{ and } w \text{ in } W$$

and we have

$$\Phi_{uvw} = \sum_{n \geq 0} \sum_\alpha \frac{\langle \gamma^n, \sigma_u, \sigma_v, \sigma_w \rangle_\alpha}{n!} = \sum_{n_\kappa} \sum_\alpha \langle (\sigma_\kappa^{n_\kappa})_\kappa, \sigma_u, \sigma_v, \sigma_w \rangle_\alpha \prod_\kappa \frac{y_\kappa^{n_\kappa}}{n_\kappa!}.$$

**Definition 26.2.10** The quantum product on $QH^*(X)$ is $\mathbb{Q}[[(y_w)_w]]$-linear and defined by the formula

$$\sigma_u * \sigma_v = \sum_w \Phi_{uvw} \cdot \sigma_{w^\vee},$$

where $\sigma_w^\vee$ is the Poincaré dual of $\sigma_w$.

**Remark 26.2.11** This product is commutative since the partial derivatives do not depend on the order of the subscripts.

The main result is the following:

**Theorem 26.2.12** *This product makes $QH^*(X)$ into an associative $\mathbb{Q}[[(y_w)_w]]$-algebra with unit $\sigma_0 = 1 \in H^0(X)$.*

As a consequence of this result applied to the plane we get the following enumerative result (the numbers $N_d$ where known only up to $N_5$ before this result):

**Theorem 26.2.13 (Kontsevich's formula)** *Let $N_d$ be the number of rational plane curves passing through $3d-1$ general points, then the following formula holds:*

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left( d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

**Remark 26.2.14** We already know that $N_1 = 1$ since it is the number of lines passing through two points. We then get all $N_d$'s reccursively. In particular one gets for example $N_2 = 1$, $N_3 = 12$ and $N_4 = 620$.

**Proof :** Set $n = 3d$ and consider the moduli space $\bar{M}_{0,n}(\mathbb{P}^2, d)$ of rational degree $d$ stable maps to the plane. Fix $L_1$ and $L_2$ two lines in $\mathbb{P}_2$ and $n-2$ points $P_1, \cdots P_{n-2}$ in $\mathbb{P}_2$ all in general position. Let us consider the following variety

$$Y = \left( \bigcap_{i=1}^{n-2} \nu_i^{-1}(P_i) \right) \cap \nu_{n-1}^{-1}(L_1) \cap \nu_n^{-1}(L_2).$$

Let us count the codimensions: the points $P_i$ are of codimension 2 and the lines of codimension 1 so that the codimension of $Y$ has to be $2(n-2)+2 = 6d-2$. The dimension of $\bar{M}_{0,n}(p^2, d)$ is $6d-1$ so that $Y$ has to be a curve. Because of Bertini's theorem and the fact that the locus of stable maps with automorphisms is in codimension at least 2 (see remark 26.1.8) we may assume that $Y$ in contained in $\bar{M}_{0,n}(\mathbb{P}^2,d)^*$ the locus of automorphism free stable maps.

The relation on divisors on $\bar{M}_{0,n}(\mathbb{P}^2,d)$ given in theorem 26.1.10 induces the following relation:

$$[Y\cap D(1,2|n-1,n)]=[Y\cap D(1,n-1|2,n)]$$

and in particular with equality of the degrees. Bertini's theorem proves that these two classes are represented by a finite number of reduced points. Let us calculate these numbers.

Let us study the left hand term. A curve $(C_A\cup C_B,(p_i)_{i\in[1,n]},f)$ in $Y\cap D(1,2|n-1,n)$ must be such that $d_A=\deg(f|_{C_A})\geq 1$ because $f(C_A)$ has to pass through two general points $P_1$ and $P_2$. However, the degree $d_B=\deg(f|_{C_B})$ can be 0. In this case, the image of $C_B$ is the point $P=L_1\cap L_2$ and the curve $C_A$ has to pass through all the $(P_i)_{i\in[1,n-1]}$ and through $P$ so that there are exactly $N_d$ such curves. If $d_B\geq 1$ then there must be $3d_A-1$ points $P_i$ on $f(C_A)$ including $P_1$ and $P_2$ and $3d_B-1$ points on $f(C_B)$ to get a finite number $N_{d_A}N_{d_B}$ of such curves. We have $\binom{n-4}{3d_A-3}=\binom{3d-4}{3d_A-3}=\binom{3d-4}{3d_B-1}$ such choices. We then have $d_B^2$ choices for the points $p_{n-1}$ and $p_n$ and $d_Ad_B$ choices for the intersection point $C_A\cap C_B$.

For the right hand term, we see that neither $d_A$ nor $d_B$ can be 0. In the same way, there must be $3d_A-1$ points $P_i$ on $f(C_A)$ including $P_1$ and $3d_B-1$ points on $f(C_B)$ including $P_2$ to get a finite number $N_{d_A}N_{d_B}$ of such curves. We have $\binom{n-2}{3d_A-2}=\binom{3d-4}{3d_A-2}=\binom{3d-4}{3d_B-2}$ such choices. We then have $d_A$ choices for the point $p_{n-1}$, $d_B$ choices for the point $p_n$ and $d_Ad_B$ choices for the intersection point $C_A\cap C_B$.

We get the expected formula. □

### 26.2.3 Small Quantum cohomology

In this section we define the small quantum cohomology ring. This ring is a deformation of the classical cohomology ring and is easier to calculate. Moreover a general result states that one can recover all genus $g$ and $n$ points Gromov-Witten invariants using this smaller ring.

#### Definition

We keep the preceding notation. In order to define a new product on this group, one defines a restricted potential defined for $\gamma\in H^2(X)$. As we have seen for the big quantum cohomology ring the third partial derivatives are enough so that we only define these partial derivatives by

$$\Psi_{uvw}(\gamma)=\sum_{n\geq 0}\sum_{\alpha}\frac{\langle\gamma^n,\sigma_u,\sigma_v,\sigma_w\rangle_\alpha}{n!}.$$

As before this formal series is well defined thanks to the following

**Lemma 26.2.15** *For an integer $n$, there is at most one class $\alpha$ with $\langle\gamma^n,\sigma_u,\sigma_v,\sigma_w\rangle_\alpha\neq 0$.*

We now make a change of variable. If we write

$$\gamma=\sum_{\beta\in\Pi}y_\beta\cdot\sigma_\beta$$

then we get, using the divisor equation, a formal power series:

$$\Psi_{uvw}((y_\beta)_{\beta\in\Pi})=\sum_{n\geq 0}\sum_{\alpha}\sum_{\sum k_\beta=n}\langle\sigma_u,\sigma_v\sigma_w\rangle_\alpha\prod_{\beta\in\Pi}\frac{\deg_{\sigma_\beta}(\alpha)y_\beta^{k_\beta}}{k_\beta!}.$$

We get

$$\Psi_{uvw}((y_\beta)_{\beta\in\Pi}) = \sum_\alpha \langle \sigma_u, \sigma_v, \sigma_w\rangle_\alpha \prod_{\beta\in\Pi} e^{\deg_{\sigma_\beta}(\alpha) y_\beta}.$$

so that this formal serie only depend on the exponentials $e^{y_\beta}$. Set $q_\beta = e^{y_\beta}$ then

$$\Psi_{uvw}((q_\beta)_{\beta\in\Pi}) = \sum_\alpha \langle \sigma_u, \sigma_v \sigma_w\rangle_\alpha \prod_{\beta\in\Pi} q_\beta^{\deg_{\sigma_\beta}(\alpha)}.$$

**Definition 26.2.16** (i) The small quantum cohomology group $QH^*_s(X)$ is $H^*X \otimes_{\mathbb{Z}} \mathbb{Z}[(q_\beta)_{\beta\in\Pi}]$.

(ii) The quantum product on $QHs^*(X)$ is $\mathbb{Z}[(q_\beta)_{\beta\in\Pi}]$-linear and defined by the formula

$$\sigma_u * \sigma_v = \sum_\nu \Psi_{uvw}((q_\beta)_{\beta\in\Pi}) \cdot \sigma_w^\vee.$$

**Associativity**

Let us prove the following

**Theorem 26.2.17** *This product makes $QH^*_s(X)$ into an associative and commutative $\mathbb{Z}[((q_\beta)_{\beta\in\Pi})]$-algebra with unit $\sigma_0 = 1 \in A^0(X)$.*

**Proof :** This will be a direct consequence of theorem 26.2.12. Indeed, we have the following formula

$$\Psi_{uvw}((y_\beta)_{\beta\in\Pi}) = \Phi_{uvw}((y_\beta)_{\beta\in\Pi}, 0, \cdots, 0)$$

that is to say that $\psi$ is obtained from $\Psi$ by specialising all the variables $q_u$ with $\ell(u) \geq 2$ to 0. The associativity condition comes then directly from the associativity of the big quantum cohomology. □

This cohomology specialises to the classical cohomology. However this cohomology is not functorial. We will see in the next chapter how to calculate it by for grassmannians.

# Chapter 27

# Quantum cohomology of the grassmannian

In this chapter, we will use a technique of A.S. Buch [Bu03] to compute the small quantum cohomology ring of a Grassmannian variety $X$. We will not need any moduli space and we will only need the fact that the small quantum cohomology is an associative ring.

By the very definition of $QH^*_s(X)$, the Schubert classes form a basis as a $\mathbb{Z}[q]$-module. We are going to prove quantum Pieri and Giambelli formulae. This will be enough to give a presentation of $QH^*_s(X)$. Recall that we have

$$\sigma_\lambda * \sigma_\mu = \sum_\nu \sum_\alpha \langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_\alpha \sigma_{\widehat{\nu}}.$$

## 27.1 Pieri formula

Recall that we proved in theorem 26.2.3 that the invariant $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_\alpha$ is the number of maps $f : \mathbb{P}^1 \to X$ with three marked points $x_1$, $x_2$ and $x_3$ such that $f(x_1) \in X_{g_1}(\lambda)$, $f(x_2) \in X_{g_2}(\mu)$ and $f(x_3) \in X_{g_3}(\nu)$ where $g_1$, $g_2$ and $g_3$ are three general points in $\mathrm{GL}_N$. In particular we only need to study the morphisms $f : \mathbb{P}^1 \to X$.

### Span of a morphism

To each curve of degree $d \leq N - p$ we associate a pair of subvector spaces.

**Definition 27.1.1** Let $f : \mathbb{P}^1 \to X$ be a morphism from $\mathbb{P}^1$ to $X$. For $x \in \mathbb{P}^1$, we consider $f(x)$ as a $p$-dimensional subvector space of $\mathbb{C}^N$.

(i) The kernel of $f$, denoted by $\ker(f)$ is the following subvector space of $\mathbb{C}^N$:

$$\ker(f) = \bigcap_{x \in \mathbb{P}^1} f(x).$$

(ii) The span of $f$, denoted by $\mathrm{span}(f)$, is the subvector space of $\mathbb{C}^N$ generated by all $f(x)$ for $x \in \mathbb{P}^1$.

**Proposition 27.1.2** *Let $f : \mathbb{P}^1 \to X$ be a morphism of degree $d$, then* $\ker(f)$ *resp.* $\mathrm{span}(f)$ *is of dimension at least $p - d$ (resp. at most $p + d$).*

**Proof :** A way to prove this is to use the irreducibility of the space of degree $d$ maps. Then a simple dimension count shows that the dimension of the closed subset of morphisms $f : \mathbb{P}^1 \to X$ such that $\dim(\ker(f)) \leq p-d$ and $\dim(\mathrm{span}(f)) \geq p+d$ is $Nd+p(N-p)$ which is the dimension of the whole moduli space.

Let us give a more direct proof. We begin with the span. Denote by $Q$ the tautological quotient bundle of rank $N-p$. We have a surjective map $\mathcal{O}_X^N \to Q$ and pulling back by $f$ this gives a surjection ${\mathcal{O}_{\mathbb{P}^1}}^N \to f^*Q$. But any vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles (see [Gr57]) so we have a decomposition

$$f^*Q \simeq \bigoplus_{i=1}^{N-p} \mathcal{O}_{\mathbb{P}^1}(a_i)$$

where $a_i \geq 0$ and $\sum a_i = d$. The number of vanishing $a_i$'s is at least $N-p-d$ (when $a_i = 0$ or 1 for all $i$). The tautological bundle $f^*Q$ has a trivial factor $N \otimes \mathcal{O}_{\mathbb{P}^1}$ where $N$ is a quotient of $\mathbb{C}^N$ of rank at least $N-p-d$. In particular, for all $x \in \mathbb{P}^1$, the vector space $f(x)$ is contained in $K = \ker(\mathbb{C}^N \to N)$ that is to say in a vector space of dimension at most $p+d$.

For the kernel, denote by $K$ the tautological subbundle of rank $p$. We have a surjective map $\mathcal{O}_X^N \to K^\vee$ and pulling back by $f$ this gives a surjection ${\mathcal{O}_{\mathbb{P}^1}}^N \to f^*K^\vee$. But any vector bundle on $\mathbb{P}^1$ is a direct sum of line bundle so we have a decomposition

$$f^*K^\vee \simeq \bigoplus_{i=1}^{p} \mathcal{O}_{\mathbb{P}^1}(b_i)$$

where $b_i \geq 0$ and $\sum b_i = d$. The number of vanishing $b_i$'s is at least $p-d$ (when $b_i = 0$ or 1 for all $i$). Therefore, the tautological bundle $f^*K$ has a trivial factor $M \otimes \mathcal{O}_{\mathbb{P}^1}$ where $M$ is a subspace of $\mathbb{C}^N$ of dimension at least $p-d$. This means that for all $x \in \mathbb{P}^1$, the vector space $f(x)$ contains $M$ that is to say contains a vector space of dimension at least $p-d$. □

**Remark 27.1.3** If $f : \mathbb{P}^1 \to X$ is non constant of degree 1, then its kernel is of dimension $p-1$ and its span of dimension $d+1$. In other words, the lines in $X$ are exactly of the form

$$\mathbb{P}(W/U) = \{V \in X \ / \ U \subset V \subset W\}$$

where $(U, W)$ is an element of the partial flag $\mathcal{F}(p-1, p+1; N)$.

Let us explain why $\mathbb{P}(W/U)$ is indeed a line in $X$. It is isomorphic to $\mathbb{P}^1$ and for a general $g \in \mathrm{GL}_N$, we have $\dim(gW_{N-p} \cap W) = 1$ and $\dim(gW_{N-p} \cap U) = 0$. Therefore, there is a unique $V$ such that $U \subset V \subset W$ and $V \in X_g(1)$, namely $U + (gW_{N-p} \cap W)$. We thus have $[\mathbb{P}(W/U)] \cdot \sigma_1 = 1$.

### 27.1.1 The partition $\widetilde{\lambda}$

We compute some restrictions on $\mathrm{span}(f)$ for a morphism $f$ to meet a given Schubert variety.

**Definition 27.1.4** (i) Let $\lambda$ be a partition in the $p \times (N-p)$ rectangle. We denote by $\widetilde{\lambda}(d)$ the partition obtained by removing the first $d$ columns of $\lambda$. In other words, $\widetilde{\lambda}(d)_i = \max(0, \lambda_i - d)$.

(ii) If $d = 1$ we denote simply by $\widetilde{\lambda}$ the partition $\widetilde{\lambda}(1)$

**Proposition 27.1.5** *Assume that the image of a morphism $f : \mathbb{P}^1 \to X$ meets the Schubert variety $X(\lambda)$, then any element $W \in \mathbb{G}(p+d, N)$ containing* $\mathrm{span}(f)$ *is in $X(\widetilde{\lambda}(d))$.*

**Proof :** Denote by $W_\bullet$ the complete flag defining the Schubert varieties. Let $x$ be a point in $\mathbb{P}^1$ such that $f(x) \in X(\lambda)$ and let $W \in \mathbb{G}(p+d, N)$ containing span$(f)$. Then we have for all $i \in [1,p]$, the inequality $\dim(f(x) \cap W_{N-p+i-\lambda_i}) \geq i$. In particular $\dim(W \cap W_{N-p+i-\lambda_i}) \geq i$ so that $W \in X(\widetilde{\lambda}(d))$. □

**Remark 27.1.6** In particular, if $f : \mathbb{P}^1 \to X$ is a morphism and $x_1$, $x_2$ and $x_3$ are points on $\mathbb{P}^1$ such that $f(x_1) \in X_{g_1}(\lambda)$, $f(x_2) \in X_{g_2}(\mu)$ and $f(x_3) \in X_{g_3}(\nu)$, then any $p+d$-dimensional subspace $W$ containing span$(f)$ is such that

$$W \in X_{g_1}(\widetilde{\lambda}(d)) \cap X_{g_1}(\widetilde{\mu}(d)) \cap X_{g_1}(\widetilde{\nu}(d)).$$

As span$(f)$ is of dimension at most $p+d$ such a $W$ always exist.

**Corollary 27.1.7** *Let $d$ be an integer, $\lambda$, $\mu$ and $\nu$ be three partitions such that $\mu$ has length 1 and $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_{d\sigma_{\widehat{1}}} \neq 0$, then we have the alternative*

- *$d = 0$ and $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_0 \neq 0$*
- *or $d = 1$, $\langle \sigma_{\widetilde{\lambda}}, \sigma_{\widetilde{\mu}}, \sigma_{\widetilde{\nu}} \rangle_0 \neq 0$, $\lambda$ and $\nu$ are of length $p$ and $\mu \neq 0$.*

**Proof :** As the invariant is non zero, there exist a morphism $f : \mathbb{P}^1 \to X$ and $x_1$, $x_2$ and $x_3$ three points on $\mathbb{P}^1$ such that $f(x_1) \in X_{g_1}(\lambda)$, $f(x_2) \in X_{g_2}(\mu)$ and $f(x_3) \in X_{g_3}(\nu)$. Therefore, the intersection $W \in X_{g_1}(\widetilde{\lambda}(d)) \cap X_{g_1}(\widetilde{\mu}(d)) \cap X_{g_1}(\widetilde{\nu}(d))$ is non empty.

Because this intersection is non empty the sum of the codimensions of these varieties has to be smaller than $\dim(\mathbb{G}(p+d,N))$ so that:

$$(p+d)(N-p-d) \geq |\widetilde{\lambda}(d)|+|\widetilde{\mu}(d)|+|\widetilde{\nu}(d)| \geq |\lambda|-dp+\max(\mu_1-d,0)+|\nu|-dp \geq |\lambda|+|\mu|+|\nu|-2dp-d$$

with equality only if $\lambda_p \geq d$, $\nu_p \geq d$ and $\mu_1 \geq d$. But because the invariant is non zero we must have $|\lambda| + |\mu| + |\nu| = dN + p(N-p)$ so that

$$(p+d)(N-p-d) \geq |\widetilde{\lambda}(d)| + |\widetilde{\mu}(d)| + |\widetilde{\nu}(d)| \geq (d+p)(N-p) - dp - d$$

and we get the inequality $d^2 - d \leq 0$, giving $d = 0$ or $d = 1$ and equality everywhere. In particular if $d = 1$, we must have $\lambda$ and $\nu$ of length $p$ and $\mu \neq 0$.

The last non vanishing result is a simple consequence of the fact that the intersection $X_{g_1}(\widetilde{\lambda}(d)) \cap X_{g_2}(\widetilde{\mu}(d)) \cap X_{g_3}(\widetilde{\nu}(d))$ is non empty. □

### 27.1.2 Pieri formula

Let us now prove the following

**Theorem 27.1.8** *If $\lambda$ is a partition contained in the $p \times (N-p)$ rectangle, then*

$$\sigma_\lambda * \sigma_k = \sum_{\mu \in \lambda \otimes k} \sigma_\mu + q \sum \sigma_\nu$$

*where the second sum is over all partitions $\nu$ such that $|\nu| = |\lambda| + k - N$ and $\lambda_1 - 1 \geq \nu_1 \geq \lambda_2 - 1 \geq \cdots \geq \lambda_p - 1 \geq \nu_p \geq 0$.*

**Remark 27.1.9** (i) The term in $q$ in the formula is zero as soon as the length of $\lambda$ is not $p$ (otherwise we have $\lambda_p = 0$ and $-1 \geq \nu_p \geq 0$ so the second sum is empty). This comes from the preceding proposition.

(ii) The set of indexes in the second sum is nearly the set $\widetilde{\lambda} \otimes k$, the only change is that the condition $|\nu| = |\widetilde{\lambda}| + k$ is replaced by $|\nu| = |\lambda| - N + k$. These condition are the same if $\lambda$ is of length $p$.

**Proof :** We may assume that $k \geq 1$ because the case $k = 0$ is clear: $\sigma_0$ is the unit of the ring. The first part is given by the classical Pieri formula. Furthermore, we have seen in corollary 27.1.7 that the only $q^d$ term is for $d = 1$ and it is non zero only if $\lambda$ is of length $p$ so we may assume that $\lambda$ is of length $p$.

We also know that if $q\sigma_\nu$ appears in $\sigma_\lambda * \sigma_k$ then $\widehat{\nu}$ has to be of length $p$ and $\langle \sigma_{\widetilde{\lambda}}, \sigma_{\widetilde{\mu}}, \sigma_{\widetilde{\widehat{\nu}}} \rangle_0$ is non zero. In particular we have $\widehat{\widetilde{\widehat{\nu}}} \in \widetilde{\lambda} \otimes \widetilde{k}$.

**Lemma 27.1.10** *If $\widehat{\nu}$ is of length $p$, then we have*

$$\widehat{\widetilde{\widehat{\nu}}} = \begin{cases} N - p - 1 & \text{for } i = 1 \\ \nu_{i-1} & \text{for } i \in [2, p+1]. \end{cases}$$

*In particular, we have $|\widehat{\widetilde{\widehat{\nu}}}| = N - p - 1 + |\nu|$ and if $\widehat{\widetilde{\widehat{\nu}}} \in \widetilde{\lambda} \otimes \widetilde{k}$, we have $|\nu| = |\lambda| + k - N$.*

**Proof :** We have $\widehat{\nu}_i = N - p - \nu_{p+1-i}$ and because $\widehat{\nu}$ is of length $p$, we have

$$\widetilde{\widehat{\nu}} = \begin{cases} N - p - \nu_{p+1-i} - 1 & \text{for } i \in [1, p] \\ 0 & \text{for } i = p + 1. \end{cases}$$

Now $\widehat{\widetilde{\widehat{\nu}}}_i = N - p - 1 - \widetilde{\widehat{\nu}}_{p+2-i}$ and the rest of the lemma follows directly. □

As a consequence, if $q\sigma_\nu$ appears in $\sigma_\lambda * \sigma_k$, we thus have that $\nu$ satisfies the conditions $|\nu| = |\lambda| + k - N$ and $N - p - 1 \geq \lambda_1 - 1 \geq \nu_1 \geq \lambda_2 - 1 \geq \cdots \geq \lambda_p - 1 \geq \nu_p \geq 0$ (the first inequality is always satified).

We are left to prove that all these terms appear. Assume that $\lambda$, $k$ and $\nu$ satisfy the preceding conditions, then we have $\widehat{\widetilde{\widehat{\nu}}} \in \widetilde{\lambda} \otimes \widetilde{k}$. Because of the classical Pieri formula, there exist — for $g_1$, $g_2$ and $g_3$ general in $\mathrm{GL}_N$ — a unique $p+1$-dimensional vector space $W$ in the intersection

$$X_{g_1}(\widetilde{\lambda}) \cap X_{g_2}(\widetilde{k}) \cap X_{g_3}(\widetilde{\widehat{\nu}}) = X_{g_1}(\widetilde{\lambda}) \cap X_{g_2}(\widetilde{k}) \cap X_{g_3}(\widetilde{\widehat{\nu}}).$$

By Bertini's theorem, the subspace $W$ is not contained is any Schubert subvariety of these three varieties. In particular, we have $\dim(W \cap (g_1 W_{n-\lambda_p})) = p$, $\dim(W \cap (g_2 W_{N-p+1-k})) = 1$ and $\dim(W \cap (g_3 W_{n-\widehat{\nu}_p})) = p$. Let us define $V_1 = W \cap (g_1 W_{n-\lambda_p})$ and $V_3 = W \cap (g_3 W_{n-\widehat{\nu}_p})$. We have $V_1 \in X_{g_1}(\lambda)$ and $V_3 \in X_{g_3}(\widehat{\nu})$. But because $|\lambda| + |\widehat{\nu}| = N + p(N-p) - k$ we have $|\lambda| + |\widehat{\nu}| > p(N-p)$ so that the intersection $X_{g_1}(\lambda) \cap X_{g_3}(\widehat{\nu})$ is empty and $V_1 \neq V_3$. The subspace $U = V_1 \cap V_3$ is then of dimension $p - 1$. We may also assume (by genericity of the $g_i$'s) that $W \cap (g_2 W_{N-p+1-k})$ is not contained in $U$. Let us define the $p$-dimensional subspace $V_2 = U + (W \cap (g_2 W_{N-p+1-k}))$, we have $V_2 \in X_{g_2}(k)$. The set $\mathbb{P}(W/U)$ of all elements $V \in X$ such that $U \subset V \subset W$ is a degree one rational curve in $X$ meeting $X_{g_1}(\lambda)$, $X_{g_2}(k)$ and $X_{g_3}(\widehat{\nu})$ in $V_1$, $V_2$ and $V_3$. In particular there exist a morphism $f : \mathbb{P}^1 \to X$ and three points $x_1$, $x_2$ and $x_3$ in $\mathbb{P}^1$ such that $f(x_1) \in X_{g_1}(\lambda)$, $f(x_2) \in X_{g_2}(\mu)$ and $f(x_3) \in X_{g_3}(\widehat{\nu})$.

Let us finaly prove that this morphism is the only one. Let $f : \mathbb{P}^1 \to X$ be such a morphism. Then because $f$ is not constant we must have $\dim(\mathrm{span}(f)) = p + 1$. But $W$ is of dimension $p + 1$

and contains $\text{span}(f)$ thus $\text{span}(f) = W$. Furthermore, we know that $\dim(f(x_1) \cap g_1 W_{n-\lambda_p}) = p$ and $\dim(f(x_3) \cap g_1 W_{n-\widehat{\nu}_p}) = p$ thus $f(x_1) = V_1$ and $f(x_3) = V_3$. But then $\ker(f)$ is of dimension $p-1$ and is contained in $U$ thus $\ker(f) = U$. Finally, $\dim(f(x_2) \cap g_2 W_{N-p+1-k}) = 1$ thus $f(x_2)$ contains $W \cap g_2 W_{N-p+1-k}$ and is equal to $V_2$. The morphism $f : \mathbb{P}^1 \to X$ factors through $\mathbb{P}(W/U)$ and sends $x_i$ to $V_i$. There is a unique such morphism. □

## 27.2 Giambelli formula

A striking result for grassmannians is the fact that the quantum Giambelli formula is the same as the classical one:

**Theorem 27.2.1** *For any partition $\lambda$ in the $p \times (N-p)$ rectangle, we have*

$$\sigma_\lambda = \det(\sigma_{\lambda_i - i + j})_{1 \leq i,j \leq p}.$$

**Proof :** Consider the formula

$$\det(\sigma_{\lambda_i - i + j})_{1 \leq i,j \leq p} = \sum_{s \in S_p} \varepsilon(s) \prod_{i=1}^{p} \sigma_{\lambda_i - i + s(i)}.$$

We will prove by induction on $j$ that in the products

$$\prod_{i=1}^{j} \sigma_{\lambda_i - i + s(i)}$$

no $q$ term appears and all its terms are of the form $\sigma_\mu$ with $\mu$ of length at most $j$.

It is clear for $j = 1$. Assume that it is the case for $j$. Then by induction we have

$$\prod_{i=1}^{j+1} \sigma_{\lambda_i - i + s(i)} = \sum_{\mu} a_\mu \sigma_\mu * \sigma_{\lambda_{j+1} - j - 1 + s(j+1)}$$

where $\mu$ runs over all partitions of length at most $j$. But then by the quantum Pieri formula, if we set $k = \lambda_{j+1} - j - 1 + s(j+1)$, we have

$$\sigma_\mu * \sigma_{\lambda_{j+1} - j - 1 + s(j+1)} = \sum_{\nu \in \mu \otimes k} \sigma_\nu$$

and in particular there is no $q$ term and for any $\nu \in \mu \otimes k$ we have $l(\nu) \leq l(\mu) + 1 \leq j + 1$.

If no $q$ term appears in this determinant, it means that its value is the classical one and we get the result by the classical Giambelli formula. □

## 27.3 Presentation of the ring

Finally let us give the presentation of the small quantum cohomology ring for grassmannians (due to B. Siebert and G. Tian [ST97]). Remark that it is a very simple deformation of the classical one. Namely we have:

**Theorem 27.3.1** *We have ring isomorphisms*

$$QH_s^*(X, \mathbb{Z}) \simeq \mathbb{Z}[\sigma_1, \cdots, \sigma_{N-p}, q]/(Y_{p+1}, \cdots, Y_N + (-1)^{N-p} q)$$

*where* $Y_u = \det(\sigma_{1-i+j})_{1 \leq i,j \leq u}$.

# Chapter 28

# Equivariant homology of the affine grassmannian

## 28.1 Affine Kac-Moody groups

Let $G$ be a semisimple algebraic groups and let $\mathfrak{g}$ be its Lie algebra. Recall that we constructed from $\mathfrak{g}$ an affine untwisted Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$. From this we may construct from the general theory a group $\mathcal{G}$ which is the Kac-Moody group associated to the Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$ (and the choice of the integral realisation of the Cartan subalgebra given by $\widehat{\mathfrak{h}}_{\mathbb{Z}} = \oplus_i \mathbb{Z}\alpha_i^\vee \oplus \mathbb{Z}c \oplus \mathbb{Z}d$. There is also associated the minimal Kac-Moody group $\mathcal{G}^{\min}$. In this section we explain how to — as for Lie algebras — realise these groups explicitly.

The algebraic group $G$ is a group functor and we may consider its value on any ring. Of particular interest are the following two rings: $\mathcal{O} = \mathbb{C}[t]$ and $\widehat{\mathcal{O}} = \mathbb{C}[[t]]$ as well as their localisations $K = \mathbb{C}[t, t^{-1}]$ and $\widehat{K} = \mathbb{C}((t))$. We may define the loop group $L(G)$ and the completed loop group $\widehat{L}(G)$ as follows:

$$L(G) = G(K) \quad \text{and} \quad \widehat{L}(G) = G(\widehat{K}).$$

We now consider the group morphisms $\gamma : \mathbb{C}^* \to \mathrm{Aut}(K)$ and $\widehat{\gamma} : \mathbb{C}^* \to \mathrm{Aut}(\widehat{K})$ defined by the assignation $z \mapsto (P(t) \mapsto P(zt))$. These morphisms induce group morphisms

$$\gamma_G : \mathbb{C}^* \to \mathrm{Aut}(L(G)) \ \text{ and } \ \widehat{\gamma}_G : \mathbb{C}^* \to \mathrm{Aut}(\widehat{L}(G)).$$

We may now define the extended loop group $\mathcal{L}(G)$ and the completed extended loop group $\widehat{\mathcal{L}}(G)$ as semidirect products:

$$\mathcal{L}(G) = \mathbb{C}^* \ltimes L(G) \ \text{ and } \ \widehat{\mathcal{L}}(G) = \mathbb{C}^* \ltimes \widehat{L}(G).$$

Remark that there are natural inclusions on the group $G$ in the extended loop group $\mathcal{L}(G)$ and in the completed extended loop group $\widehat{\mathcal{L}}(G)$.

We have the following result:

**Theorem 28.1.1** *(ı) Assume that $G$ is simply connected and let us denote by $Z(G)$ its center. There exists a (unique) canonical group morphism*

$$\psi : \mathcal{G} \to \widehat{\mathcal{L}}(G)/Z(G)$$

*such that the map $\psi$ is surjective and its kernel is the center $Z(\mathcal{G})$ of $\mathcal{G}$.*

*(ii) In fact the group morphism $\psi$ can be lifted in a surjective group morphism $\varphi : \mathcal{G} \to \widehat{\mathcal{L}}(G)$ such that the kernel is precisely* $\exp(\mathbb{C}c) \simeq \mathbb{C}^*$. *In particular $\mathcal{G}$ is a central extension*

$$1 \to \mathbb{C}^* \to \mathcal{G} \to \widehat{\mathcal{L}}(G) \to 1.$$

*(iii) The restriction of $\psi$ to $\mathcal{G}^{\min}$ has $\mathcal{L}(G)/Z(G)$ for image.*

## 28.2 The affine grassmannian

### 28.2.1 Algebraic realisation

There is a special parabolic subgroup in the affine Kac-Moody group. There a several ways to describe this parabolic subgroup which will be a maximal parabolic subgroup. The first way is to consider the subgroups

$$\mathcal{L}_{\mathcal{O}}(G) \subset \mathcal{L}(G) \text{ and } \widehat{\mathcal{L}}_{\widehat{\mathcal{O}}}(G) \subset \widehat{\mathcal{L}}(G)$$

obtained from the same construction as for $\mathcal{L}(G)$ and $\widehat{\mathcal{L}}(G)$ by replacing $K$ and $\widehat{K}$ by $\mathcal{O}$ and $\widehat{\mathcal{O}}$.

We may also recall from the study of the affine Weyl group $\widehat{W}$ that the finite Weyl group $W$ of $G$ is a subgroup of the affine Weyl group (generated by all simple reflections except the one associated to the added vertex). This define parabolic subgroups $\mathcal{P}$ and $\widehat{\mathcal{P}}$ by

$$\mathcal{P} = \mathcal{B}W\mathcal{B} \text{ and } \widehat{\mathcal{P}} = \widehat{\mathcal{B}}W\widehat{\mathcal{B}}.$$

where $\mathcal{B}$ and $\widehat{\mathcal{B}}$ are the Borel subgroups of $\mathcal{G}^{\min}$ and $\mathcal{G}$.

**Proposition 28.2.1** *The map $\psi$ maps the group $\mathcal{P}$ (resp. $\widehat{\mathcal{P}}$) to $\mathcal{L}_{\mathcal{O}}(G)/Z(G)$ (resp. $\widehat{\mathcal{L}}_{\widehat{\mathcal{O}}}(G)/Z(G)$).*

**Definition 28.2.2** The affine grassmannian is the homogeneous space $\mathcal{G}/\widehat{\mathcal{P}}$. It is isomorphic to $\mathcal{G}^{\min}/\mathcal{P}$ and also to $\mathcal{L}(G)/\mathcal{L}_{\mathcal{O}}(G)$ and $\widehat{\mathcal{L}}(G)/\widehat{\mathcal{L}}_{\widehat{\mathcal{O}}}(G)$ by $\psi$. We denote it by $\mathrm{Gr}_{\mathrm{G}}$.

### 28.2.2 Topological realisation

**Definition 28.2.3** Let $X$ be a compact real variety.

(i) A loop is a continuous map $f : S^1 \to X$.

(ii) A loop is said to be analytic if the map $f : S^1 \to X$ can be extended to a meromorphic map $f : D \to X$ where $D$ is the closed unit disk and such that the only poles of $f$ are located at 0.

Let $K$ be a maximal compact subgroup in $G$. Denote by $LK$ the group of analytic real loops $f : S^1 \to K$. Considering $S^1$ as contained in $\mathbb{C}^*$, any loop $f \in LK$ has a Laurent series and we may thus embed $LK$ in $G(\mathbb{C}((t)))$ and thus in $\widehat{\mathcal{L}}(G)$.

Let us now consider the normal subgroup $\Omega K$ of $LK$ given by the based loops:

$$\Omega K = \{f \in LK \ / \ f(1) = 1_K\}.$$

There is a natural decomposition

$$\widehat{L}(G) = \Omega K \cdot \widehat{L}_{\widehat{\mathcal{O}}}(G)$$

in particular we get the

**Theorem 28.2.4** *The natural map $\Omega K \to \mathrm{Gr}_{\mathrm{G}}$ is an isomorphism (in the topological category).*

Remark that we also have the following description of the affine grassmannian:

$$\Omega K \simeq LK/K.$$

## 28.3 More structure on the homology of the affine grassmannian

To define the equivariant homology of the affine grassmannian, we may use the ind-variety structure on it. We know that there exist algebraic subvarieties $\mathrm{Gr}_{\mathrm{G}n} \subset \mathrm{Gr}_{\mathrm{G}}$ defining the algebraic structure on $\mathrm{Gr}_{\mathrm{G}}$. Let us consider the induced varieties $\Omega K_n = \Omega K \cap \mathrm{Gr}_{\mathrm{G}n}$. We have the following

$$H_*^T(\Omega K) = \lim_{\rightarrow} H_*^T(\Omega K_n).$$

The use of $\Omega K$ instead of the algebraic version $\mathrm{Gr}_{\mathrm{G}}$ comes from the following fact: there exists a product (and even two different products) on $\Omega K$. There is a natural pointwise multiplication on $LK$

$$(f \cdot g)(t) = f(t)g(t)$$

and this induces a multiplication on $\Omega K$. Furthermore, there exists another product *conc* on $\Omega K$ given by composition of loops:

$$conc(f,g)(e^{2i\pi x}) = \begin{cases} f(e^{4i\pi x}) & \text{for } x \in [0, \frac{1}{2}] \\ g(e^{4i\pi x}) & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

**Theorem 28.3.1** *The pointwise multiplication and the composition of loops define the same product*

$$H_*^T(\Omega K) \otimes H_*^T(\Omega K) \to H_*^T(\Omega K)$$

*on the equivariant homology. This product is called the* **Pontryagin product**.

**Proof :** We construct a homotopy of loops: $conc(f,g) \sim (f \cdot g) \sim conc(g,f)$. For $u \in [-1,1]$, let $p_u : [0,1] \to [0,1] \times [0,1]$ be a path in the unit square such that

- for all $u$, the path $p_u$ starts at $(0,0)$ and ends at $(1,1)$
- the path $p_{-1}$ goes along the left and top boundaries of the square,
- the path $p_0$ goes along the diagonal of the square,
- the path $p_1$ goes along the bottom and right boundaries of the square.

Define $H : [0,1] \times [0,1] \to K$ by $H(x,y) = f(e^{2i\pi x})g(e^{2i\pi y})$. Then defining $h_u : S^1 \to K$ by $h_u(e^{2i\pi x}) = H(p_u(x))$ gives a continuous family of loops with $h_{-1} = conc(f,g)$, $h_0 = f \cdot g$ and $H_1 = conc(g,f)$. □

**Remark 28.3.2** This proves that the product on the homology $H_*^T(\Omega K)$ is commutative. It also proves that the fundamental group $\pi_1(K)$ for any Lie group $K$ is abelian. Of course $G$ being simply connected here we have $\pi_1(G) = \{1\}$.

## 28.4 Schubert varieties in the affine grassmannian

Let us recall that to construct the Schubert varieties for any Kac-Moody group we started with $\mathfrak{w}$ a reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_n}$ and constructed the Bott-Samelson variety

$$Z_{\mathfrak{w}} = P_{\alpha_1} \times \cdots \times P_{\alpha_n} / B^n.$$

There is a topological version of this construction. Indeed, let us define the group $LK_{\alpha_i} = LK \cap P_{\alpha_i}$.

**Fact 28.4.1** *The group $LK_{\alpha_i}$ is again compact. Explicitly, it is generated by $T \simeq (S^1)^{\mathrm{rk}(G)}$ and a copy of $SU_2$ in the place of $\alpha_i$.*

**Example 28.4.2** For $G = \mathrm{SL}_3(\mathbb{C})$, the simple root compact parabolic subgroups are the following:

$$LK_{\alpha_0} = \begin{pmatrix} a & 0 & -\bar{b}t^{-1} \\ 0 & c & 0 \\ bt & 0 & \bar{a} \end{pmatrix}, \quad LK_{\alpha_1} = \begin{pmatrix} a & -\bar{b} & 0 \\ b & \bar{a} & 0 \\ 0 & 0 & c \end{pmatrix} \text{ and } LK_{\alpha_2} = \begin{pmatrix} c & 0 & 0 \\ 0 & a & -\bar{b} \\ 0 & b & \bar{a} \end{pmatrix},$$

where $a$, $b$ and $c$ are in $\mathbb{C}$ with $(a\bar{a} + b\bar{b})c = 1$.

Now the Schubert variety $X_w$ in $\mathcal{G}/\mathcal{B}$ is the image of the multiplication map $m_{\mathfrak{w}} : Z_{\mathfrak{w}} \to \mathcal{G}/\mathcal{B}$. We may define in the same way a topological Bott-Samelson resolution

$$Z_{\mathfrak{w}}^{LK} = LK_{\alpha_1} \times \cdots \times LK_{\alpha_n}/T^n = LK_{\alpha_1} \times^T \cdots \times^T LK_{\alpha_n}/T \to LK/T \simeq \mathcal{G}/\mathcal{B}.$$

If we now want Schubert varieties in the affine grassmannian, we need to use the canonical projection $\mathcal{G}/\mathcal{B} \to \mathcal{G}/\mathcal{P}$. This map is given on the topological level by $LK/T \to LK/K \simeq \Omega K$. We then have the Bott-Samelson resolution:

$$\Omega K_{\mathfrak{w}} = LK_{\alpha_1} \times^T \cdots \times^T LK_{\alpha_n} \cdot K/K \to LK/K \simeq \Omega K \simeq \mathcal{G}/\mathcal{P}.$$

Let us fix one more notation: we will denote by $LK_w$ the product $LK_{\alpha_1} \cdots LK_{\alpha_n}$ in $LK$. The Schubert variety $X_w$ in $\mathcal{G}/\mathcal{B}$ is $LK_w/T$ in $LK/T$ while the Schubert variety $X_w^{\mathcal{P}}$ in $\mathcal{G}/\mathcal{P}$ is denoted $\Omega K_w = LK_w/K$. Of course, two elements $v$ and $w$ in the same $W$-class in $\widehat{W}$ give the same Schubert variety $\Omega K_v = \Omega K_w$. The Schubert variety $LK_w/T$ and $\Omega K_w = LK_w/K$ are the image of the Bott-Samelson resolution $Z_{\mathfrak{w}}^{LK}$ and $\Omega K_{\mathfrak{w}}$ in $LK/T$ and $\Omega K = LK/K$.

## 28.5 Identification of $H_*^T(\Omega K)$ as a subring of the Nil-Hecke ring

In this section, we realise the homology ring $H_*^T(\Omega K)$ of the affine grassmannian as a subring of the Nil-Hecke ring. We first need some easy fact on the affine grassmannian.

The affine grassmannian $\mathrm{Gr}_G$ being a homogeneous space $\mathcal{G}/\mathcal{P}$ for the affine Kac-Moody group $\mathcal{G}$, we have a nice description of its $T$ fixed points. Indeed, the $T$-fixed points of $\mathrm{Gr}_G$ are in one to one correspondence with the classes $\widehat{W}/W$ where $\widehat{W}$ is the affine Weyl group (or the Weyl group of $\mathcal{G}$) and $W$is the finite Weyl group (and also the Weyl group of $\mathcal{P}$).

There are two different natural sets of representatives in $\widehat{W}$ of the quotient $\widehat{W}/W$. The first set we already discussed is the set of minimal length representative elements denoted by $W^{\mathrm{aff}}$. The second one comes from the explicit description of the affine Weyl group $\widehat{W}$ as a group of affine transformations of $\mathfrak{h}$. Recall that we may consider the abelian group $Q^\vee$ which is the lattice generated by simple coroots of the finite dimensional group $G$. The Weyl group $W$ acts on $Q^\vee$ and we have

$$\widehat{W} = W \rtimes Q^\vee.$$

The elements $\alpha$ in $Q^\vee$ seen as elements of the affine Weyl group are denoted by $t_\alpha$. Indeed, they act on $\mathfrak{h}$ by translation by $\alpha$.

**Fact 28.5.1** *The group of translation $Q^\vee$ in $\widehat{W}$ is a set of representatives for the quotient $\widehat{W}/W$.*

The advantage of this set of representatives is that is has a group structure and this group structure is compatible with the group structure on the affine grassmannian $\mathrm{Gr}_{\mathrm{G}}$ (seen as the space of loops $\Omega K$). Indeed, if we have a element $\alpha$ in $Q^\vee \subset \mathfrak{h}$ this defines a one parameter subgroup $\alpha : \mathbb{C}^* \to G$ and by restriction to $S^1$ we get a loop $f_\alpha \in LK$ and by quotienting by $K$ we have an element in $\Omega K$. This is a fixed point under the action of the torus. We see that concatenation corresponds to the group action in $Q^\vee$. For $\alpha \in Q^\vee$ we denote by $p_\alpha$ the corresponding fixed point.

The fixed points $p_\alpha$ for $\alpha \in Q^\vee$ induce elements in the equivariant homology as follows: consider the injection $i_\alpha : p_\alpha \to \Omega K$. We have $H^*_T(p_\alpha) = H^*_T(pt) = S$ thus the pull-back gives a map:

$$i^*_\alpha : H^*_T(\Omega K) \to S.$$

This is an element in the equivariant homology ring and we denote it by $\psi_{t_\alpha} \in H^T_*(\Omega K)$. We have the following

**Theorem 28.5.2** *(ı) The classes $\psi_{t_\alpha}$ for $\alpha \in Q^\vee$ form a $S$-basis of $H^T_*(\Omega)$.*

*(ıı) We have the formula*

$$\psi_{t_\alpha}\psi_{t_\beta} = \psi_{t_{\alpha+\beta}}.$$

**Proof :** (ı) This comes the localisation principle.

(ıı) This comes easily from the fact that $p_\alpha \cdot p_\beta = p_{\alpha+\beta}$. □

**Definition 28.5.3** Define a map $j : H^T_*(\Omega K) \to R$ by $j(\psi_{t_\alpha}) = \delta_{t_\alpha}$ and extend it by $S$-linearity.

We have the following (see for example [La06]):

**Proposition 28.5.4** *We have* $\mathrm{im} j = Z_R(S) = \{x \in R \ / \ xs = sx \text{ for all } s \in S\}$ *and the map $j$ is a ring morphism.*

**Proof :** We need to compute $\delta_{t_\alpha} q = t_\alpha(q)\delta_{t_\alpha}$ where $\alpha \in Q^\vee$ and $q \in S$. We only need to compute this for $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$. Recall from the second part (description of the affine Weyl group) that the action is given by (here $\lambda$ is in the dual of the Cartan algebra of the finite dimensional group):

$$t_\alpha(\lambda) = \lambda - \langle \alpha, \lambda \rangle \delta.$$

But the image of $\delta$ in $S$ vanishes thus the action is trivial.

The fact that it is a ring morphisms comes from the defining relations among the $\delta_{t_\alpha}$. □

## 28.6 Localisations and the isomorphism

### 28.6.1 Localisation of the equivariant homology

As we have already seen there are two nice set of representatives for the quotient $\widehat{W}/W$. Of particular interest are the elements in the intersection of these two set of representatives. Let us first describe the set $W^{\mathrm{aff}}$ of minimal length representatives of the quotient $\widehat{W}/W$:

**Lemma 28.6.1** *Let $\alpha \in Q^\vee$ and $w \in W$, the element $wt_\alpha$ is in $W^{\mathrm{aff}}$ if and only if the following two conditions hold:*

- *the element $\alpha$ is antidominant i.e. for all simple root $\beta$ we have $\langle \alpha, \beta \rangle \leq 0$*

- *the element $w$ is $\alpha$-minimal i.e. for all simple root $\beta$ with $\langle\alpha,\beta\rangle = 0$, we have $w(\beta) > 0$.*

The elements $wt_\alpha$ in $W^{\text{aff}}$ are of special interest because they parametrise a geometric basis of the equivariant homology $H_*^T(\Omega K)$. Indeed, the Schubert classes $\sigma_w = [X_w]$ for $X = \mathcal{G}/\mathcal{B}$ and $w \in \widehat{W}$ form a $S$-basis of the equivariant homology of $\mathcal{G}/\mathcal{B}$ and the images (still denoted $\sigma_w$) of the Schubert classes in $\mathcal{G}/\mathcal{P} \simeq \Omega K$ for $w \in W^{\text{aff}}$ form a $S$-basis of $H_*^T(\Omega K)$.

As a consequence of the previous characterisation, the elements in the intersection $\widetilde{Q} = Q^\vee \cap W^{\text{aff}}$ are described as follows:

$$\widetilde{Q} = \{\alpha \in Q^\vee \ / \ \alpha \text{ is antidominant}\}.$$

The Schubert classes corresponding to these elements have very special properties. In particular we have the following

**Proposition 28.6.2** *For any $wt_\alpha \in W^{\text{aff}}$ and any $\beta \in \widetilde{Q}$, we have*

$$\sigma_{wt_\alpha}\sigma_{t_\beta} = \sigma_{wt_{\alpha+\beta}}.$$

We shall see a generalisation of this result in the next chapter. As a consequence of this formula, we see that the set of classes $(\sigma_{t_\beta})_{\beta\in\widetilde{Q}}$ is multiplicative in $H_*^T(\Omega K)$. We may thus localise with respect to this set. Define:

$$H_*^T(\Omega K)_{\text{loc}} = H_*^T(\Omega K)[\sigma_{t_\beta}^{-1}, \ \beta \in \widetilde{Q}].$$

### 28.6.2 Localisation of the quantum cohomology

Recall that the small quantum cohomology was defined as a free $\mathbb{Z}[(q_\beta)_{\beta\in\Pi}]$-module by

$$QH_s^*(G/B) = H^*(G/B,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[(q_\beta)_{\beta\in\Pi}].$$

There is an extended notion of equivariant quantum cohomology define as an free $\mathbb{Z}[(q_\beta)_{\beta\in\Pi}]$-module by

$$QH_{s,T}^*(G/B) = H_T^*(G/B,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[(q_\beta)_{\beta\in\Pi}].$$

The multiplicative structure being $S \otimes_Z \mathbb{Z}[(q_\beta)_{\beta\in\Pi}]$ linear, we may localise the quantum cohomology ring at all quantum parameters by setting:

$$QH_{s,T}^*(G/B)_{\text{loc}} = QH_{s,T}^*(G/B)[q_\beta^{-1}, \ \beta \in \Pi].$$

We then have the following comparison theorem (see [LS07]):

**Theorem 28.6.3** *The application $\psi : H_*^T(\Omega K)_{\text{loc}} \to QH_{s,T}^*(G/B)_{\text{loc}}$ defined by*

$$\sigma_{wt_\alpha}\sigma_{t_\beta}^{-1} \mapsto q_{\alpha-\beta} \cdot \sigma^w$$

*where $\beta \in \widetilde{Q}$, $w \in W$ and $\sigma^w$ is the cohomology class of degree $\ell(w)$ associated to $w$, is an $S$-algebra isomorphism.*

# Chapter 29

# Symmetries in the quantum cohomology

In this chapter, we prove that the center $Z(G)$ of the group $G$ can be embedded in the group of invertible elements of the localised quantum cohomology $QH^*_{s,T}(G/B)_{\rm loc}$ and that there are simple formulas for multiplying by these classes. The results of this chapter are taken from [CMP07]. Some more details on the topological version of the affine and extended affine grassmannian can be found in [Ma07].

## 29.1 Different realisations of the center of the group

### 29.1.1 Fundamental group of the adjoint group

Let $G$ be a semi-simple simply connected algebraic group. Its center $Z(G)$ has several realisation. The first simple realisation is given as follows. Consider the adjoint group $G^{\rm ad} = G/Z(G)$. It is also the image of the group $G$ under the adjoint representation $G \to GL(\mathfrak{g})$. Then we have

**Proposition 29.1.1** *There is a natural isomorphism*

$$Z(G) \simeq \pi_1(G^{\rm ad}).$$

**Proof :** This is simply realised as follows: the map $G \to G^{\rm ad}$ is the universal covering and if we take a loop in $G^{\rm ad}$ and take a lifting of this loop in $G$ we end up with a path (not necessary a loop) whose end point is the element in the center corresponding to the class of the loop in $\pi_1(G^{\rm ad})$. $\square$

### 29.1.2 Coweights modulo coroots

Let us denote by $Q^\vee$ the lattice of coroots i.e. the lattice generated by simple coroots. We may also consider the lattice $P^\vee$ generated by the coweights (i.e. the weights of the dual root system). We have an inclusion $Q^\vee \subset P^\vee$ ad the following proposition holds:

**Proposition 29.1.2** *We have a natural isomorphism*

$$Z(G) \simeq P^\vee/Q^\vee.$$

**Proof :** We have the following geometric description of the coroot and coweight lattices. Consider $H$ a maximal torus in $G$ and $K$ a maximal compact subgroup in $G$. Let us denote by $T$ the intersection $H \cap K$. We have an isomorphism

$$T \simeq (S^1)^{\mathrm{rk}(G)}.$$

Let $\mathfrak{t}$ be the Lie algebra of $T$ and consider the exponential map $\exp : \mathfrak{t} \to T$. We have the following

$$Q^\vee = \ker(\exp).$$

Now to realise the coweight lattice we only have to consider the inverse image of the center:

$$P^\vee = \exp^{-1}(Z(G)).$$

The result follows from these identities. □

### 29.1.3 Cominuscule coweights and Dynkin diagram

There are distinguished representatives of the quotient $P^\vee/Q^\vee$: the cominuscule coweights. These may be define as follows. Denote by $I$ the set of vertices of the Dynkin diagram and consider the extended Dynkin diagram of the group $G$. Define the set $I_c$ of vertices of the diagram as the set of all vertices in $I$ that can be obtained from the added vertex (the vertex labeled 0 in our diagrams) by an automorphism of the Dynkin graph. This set is called the set of cominuscule vertices.

**Definition 29.1.3** Let $i \in I_c$ and $\alpha_i$ be the corresponding simple root. The minuscule coweight $\varpi_i^\vee$ associated to $i$ is the element in $P^\vee$ such that for all $j \in I$ we have $\langle \varpi_i^\vee, \alpha_j \rangle = \delta_{i,j}$.

We have the following

**Fact 29.1.4** *The cominuscule coweights together with the trivial weight form a set of representatives of the quotient $P^\vee/Q^\vee$.*

In particular, if we define $\widehat{I_c} = I_c \cup \{0\}$, the subset of the set of the vertices of the extended Dynkin diagram formed by the cominuscule vertices and the added vertex, then we have

**Fact 29.1.5** *The set $\widehat{I_c}$ is a principal space under the group $Z(G)$.*

### 29.1.4 Extended affine Weyl group

There is a natural group, slightly bigger that the affine Weyl group that we may define in the situation: the extended affine Weyl group $\widetilde{W}$. It is the semi-direct product of the coweight lattice $P^\vee$ by the Weyl group. In symbols:

$$\widetilde{W} = P^\vee \rtimes W.$$

Recall that we have $\widehat{W} = Q^\vee \rtimes W$ so that we get another realisation of the center as follows:

$$Z(G) \simeq P^\vee/Q^\vee \simeq \widetilde{W}/\widehat{W}.$$

In fact the quotient $\widetilde{W}/\widehat{W}$ has a section defined as follows. Let us first define the fundamental alcove $A_0$ which is a fundamental domain for the action of $\widehat{W}$ on $\mathfrak{h}$:

$$A_0 = \{h \in \mathfrak{h} \ / \ \langle \alpha_i, h \rangle \geq 0 \text{ for } i \in I \text{ and } \langle \theta, h \rangle \leq 1\}.$$

**Proposition 29.1.6** *The stabiliser $Z(A_0)$ of the fundamental alcove in $\widetilde{W}$ is isomorphic to the group $Z(G)$.*

This gives a section of the quotient $\widetilde{W}/\widehat{W}$ and there is a natural semi-direct product structure $\widetilde{W} \simeq Z(G) \rtimes \widehat{W}$. We may now describe explicit representatives of the center in $\widetilde{W}$ in the form $wt_\lambda$ for $\lambda$ a coweight.

**Definition 29.1.7** Let $i \in I_c$ a cominuscule vertex of the Dynkin diagram and let $\varpi_i^\vee$ the associated cominuscule coweight. We define the element $v_i \in W$ to be the smallest element such that $v_i\varpi_i^\vee = w_0\varpi_i^\vee$.

**Remark 29.1.8** The element $v_i$ is also the longuest element in the minimal coset representatives $W^{P_i}$ where $P_i$ is the maximal parabolic subgroup associated to the simple root $\alpha_i$.

**Proposition 29.1.9** *The group $Z(A_0)$ (the stabiliser of the fundamental alcove) is described as follows:*

$$Z(A_0) = \{\mathrm{Id}\} \cup \{v_i t_{-\varpi_i^\vee},\ i \in I_c\}.$$

The elements $v_i$ have very special properties we shall list in the following

**Fact 29.1.10** *($\imath$) We have $v_i^{-1} = v_{f(i)}$ where $f$ is the Weyl involution of the Dynkin diagram.*

*($\imath\imath$) Let $\alpha$ be a positive root, then $v_i(\alpha)$ is positive if and only if $\langle \varpi_i^\vee, \alpha\rangle = 0$.*

*($\imath\imath\imath$) The element $\tau_i = v_i t_{-\varpi_i^\vee} \in Z(G) \subset \widetilde{W}$ acts on the simple roots by permutation as the elements does on the vertices $I_c$.*

**Definition 29.1.11** The previous fact shows that the elements in $Z(G)$ respect the sign of all roots: positive roots and negative roots are preserved. In particular, we may extend the length function from $\widehat{W}$ to $\widetilde{W}$ by $\ell(\tau_i) = 0$ for $\tau_i \in Z(G)$.

## 29.2 The extended affine grassmannian

One of the problem int the construction of the affine grassmannian and the combinatorics of the affine Weyl group $\widehat{W}$ is the fact that there is a choice (canonical when we look at the affine Kac-Moody group in its realisation $\mathcal{G} = \widehat{\mathcal{L}}(G)/Z(G))$ of a marked 0-vertex in the extended Dynkin diagram. However, on the level of the extended Dynkin diagram and thus on the level of the Kac-Moody group itself, this choice is not given. In fact the choice of any such cominuscule vertex $i$ in the extended Dynkin diagram give rise to an algebraic subgroup $G \subset \mathcal{G}$ such that $\mathcal{G} = \widehat{\mathcal{L}}(G_i)/Z(G_i)$, the case $i = 0$ giving back $G_0 = G$.

To avoid this problem, we shall look — instead of the semi-simple simply connected algebraic group $G$ — at the adjoint group $G^{\mathrm{ad}} = G/Z(G)$ and its corresponding compact subgroup $K^{\mathrm{ad}} = K/Z(G)$. Consider the variety $\Omega K^{\mathrm{ad}}$ of loops with values in $K^{\mathrm{ad}}$. By definition of $\pi_1(K^{\mathrm{ad}})$, the connected components of $\Omega K^{\mathrm{ad}}$ are indexed by $\pi_1(K^{\mathrm{ad}}) \simeq Z(G)$. Furthermore, the natual map

$$\Omega K \to \Omega K^{\mathrm{ad}}$$

sending $f : S^1 \to K$ to the map $p \circ f : S^1 \to K^{\mathrm{ad}}$ where $p : K \to K^{\mathrm{ad}}$ is the projection, realise $\Omega K$ as a connected component of $\Omega K^{\mathrm{ad}}$. Furthermore, all the connected components are isomorphic: they are permuted by the action of $Z(G) = \pi_1(K^{\mathrm{ad}})$ as follows. If $f : S^1 \to K$ is a loop in $K$, then we may define for $\alpha \in P^\vee$ the loop $f_\alpha : S^1 \to K^{\mathrm{ad}}$ by $f_\alpha(x) = f(x)\exp(2i\pi\alpha)$. Because $\exp(2i\pi\alpha) \in Z(G)$ this gives a loop in $K^{\mathrm{ad}}$ but not in general in $K$.

## 29.3 Schubert varieties and Bott-Samelson varieties for the extended affine grassmannian

As for the affine grassmannian, we may define Schubert varieties and Bott-Samelson varieties. Indeed a reduced expression $\mathfrak{w}$ has now the form $w = \tau s_{\alpha_1} \cdots s_{\alpha_n}$ where the $\alpha_i$ are simple roots and $\tau \in Z(G)$. For $vt_\alpha = \tau \in Z(G)$, there is a natural lift of $\tau$ in $LK^{\text{ad}}$ by lifting $v$ in $K^{\text{ad}}$ and by lifting $t_\alpha$ by $x \mapsto \exp(2i\pi\alpha x)$. We define the group $LK^{\text{ad}}_\tau$ by $\tau T = T\tau$ where we still denote by $\tau$ the lifting. We also have the groups $LK^{\text{ad}}_{\alpha_i} = LK_{\alpha_i}/Z(G)$ and we may define:

$$\Omega K^{\text{ad}}_{\mathfrak{w}} = LK^{\text{ad}}_\tau \times^T LK^{\text{ad}}_{\alpha_1} \times^T \cdots \times^T LK^{\text{ad}}_{\alpha_n} \cdot K^{\text{ad}}/K^{\text{ad}}.$$

The image of this map is the Schubert variety $\Omega K^{\text{ad}}_w$ in $\Omega K^{\text{ad}}$. These Schubert varieties are easily described in terms of Schubert varieties in the classical affine grassmannian. Indeed, write $w = \tau w'$ where $\tau \in Z(G)$ and $w' \in \widehat{W}$. Then we have

$$\Omega K^{\text{ad}}_w = \tau \cdot \Omega K_{w'}.$$

In particular, this Schubert variety is isomorphic to the Schubert variety $\Omega K_{w'}$ but is not in the same connected component, it is in the connected component obtained by translation by $\tau$.

We now prove a factorisation product of the Bott-Samelson resolution leading to a product formula in the equivariant homology of the affine grassmannian.

**Definition 29.3.1** We denote by $\widetilde{P}$ the set

$$\widetilde{P} = \{\alpha \in P^\vee \ / \text{ for all simple root } \beta, \text{ we have } \langle \alpha, \beta \rangle \leq 0\}$$

of antidominant coweights

**Theorem 29.3.2** *Let $\alpha \in \widetilde{P}$ be an antidominant coweight and let $v \in \widetilde{W}$. Let us write $u = t_\alpha = \tau u'$, $v = \sigma v'$ and $w = uv = \nu w'$ with $\tau$, $\sigma$ and $\nu$ in $Z(G)$ and $u'$, $v'$ and $w'$ in $\widehat{W}$. Assume that $\ell(w) = \ell(u) + \ell(v)$, then we have for the Pontryagin product:*

$$[\Omega K_{u'}] \cdot [\Omega K_{v'}] = [\Omega K_{w'}].$$

**Proof :** The fact that $\alpha \in \widetilde{P}$ implies that for any $x \in W$, we have $\ell(xt_\alpha W) \leq \ell(t_\alpha W)$. This implies the following equality

$$K^{\text{ad}} \cdot LK^{\text{ad}}_{t_\alpha} \cdot K^{\text{ad}} = LK^{\text{ad}}_{t_\alpha} \cdot K^{\text{ad}}.$$

In particular we obtaint the following equality:

$$LK^{\text{ad}}_u \times^T LK^{\text{ad}}_{t_\alpha} \cdot K^{\text{ad}}/K^{\text{ad}} \simeq (LK^{\text{ad}}_u \cdot K^{\text{ad}}) \times^{K^{\text{ad}}} (LK^{\text{ad}}_{t_\alpha} \cdot K^{\text{ad}})/K^{\text{ad}}.$$

We may now define an homeomorphism

$$(LK^{\text{ad}}_u \cdot K^{\text{ad}}) \times^{K^{\text{ad}}} (LK^{\text{ad}}_{t_\alpha} \cdot K^{\text{ad}})/K^{\text{ad}} \underset{G}{\overset{F}{\longleftrightarrow}} (LK^{\text{ad}}_u \cdot K^{\text{ad}})/K^{\text{ad}} \times (LK^{\text{ad}}_{t_\alpha} \cdot K^{\text{ad}})/K^{\text{ad}}.$$

by the follows maps: $F(f_1(t), f_2(t)) = (f_1(t)f_1(1)^{-1}, f_1(1)f_2(t))$ and $G(f_1(t), f_2(t)) = (f_1(t), f_2(t))$. Remark that the right hand term in the homeomorphism is $\Omega K^{\text{ad}}_u \times \Omega K^{\text{ad}}_{t_\alpha}$. This decomposition is not true in the algebraic category. We thus have a commutative diagram:

$$\begin{array}{ccc} (LK^{\text{ad}}_u \cdot K^{\text{ad}}) \times^{K^{\text{ad}}} (LK^{\text{ad}}_{t_\alpha} \cdot K^{\text{ad}})/K^{\text{ad}} & \longrightarrow & \Omega K^{\text{ad}}_u \times \Omega K^{\text{ad}}_{t_\alpha} \\ & \searrow^{m} & \downarrow \text{mult} \\ & & \Omega K^{\text{ad}}_w \end{array}$$

where the vertical map is just multiplication in $\Omega K^{\mathrm{ad}}$ and the diagonal map $m$ factorises the Bott-Samelson map: if $u = s_1 \cdots s_n$ and $v = t_\alpha = s_{n+1} \cdots s_k$ then $w = uv = s_1 \cdots s_n s_{n+1} \cdots s_k$. Denote by $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{w}$ these reduced expressions, we have a commutative diagram:

$$\begin{array}{ccc} \Omega K^{\mathrm{ad}}_{\mathfrak{w}} = LK^{\mathrm{ad}}_{\mathfrak{u}} \times^T LK^{\mathrm{ad}}_{\mathfrak{v}} \cdot K^{\mathrm{ad}}/K^{\mathrm{ad}} & \longrightarrow & LK^{\mathrm{ad}}_{w} \times^{K^{\mathrm{ad}}} LK^{\mathrm{ad}}_{t_\alpha}/K^{\mathrm{ad}} \\ & \searrow^{m_{\mathfrak{w}}} & \downarrow m \\ & & \Omega K^{\mathrm{ad}}_{\nu}. \end{array}$$

Looking at the homology classes of the images we get

$$[\Omega K^{\mathrm{ad}}_{w}] \cdot [\Omega K^{\mathrm{ad}}_{t_\alpha}] = \mathrm{mult}_*[\Omega K^{\mathrm{ad}}_{\mathfrak{w}} \times \Omega K^{\mathrm{ad}}_{t_\alpha}] = m_{\mathfrak{w}*}[\Omega K^{\mathrm{ad}}_{\mathfrak{w}}] = [\Omega K^{\mathrm{ad}}_{w}].$$

We thus have

$$\nu[\Omega K_{u'}] \cdot [\Omega K_{v'}] = \tau\sigma[\Omega K_{u'}] \cdot [\Omega K_{v'}] = \tau[\Omega K_{u'}]\sigma[\Omega K_{v'}] = \nu[\Omega K_{w'}]$$

we deduce the result from this last formula. □

**Remark 29.3.3** If $u$ is a minimal length representative in the quotient $\widetilde{W}/W$ and $\alpha \in \widetilde{P}$, then the condition $\ell(ut_\alpha) = \ell(u) + \ell(t_\alpha)$ is always satisfied.

## 29.4 Application to quantum cohomology

If we translate this formula in the equivariant quantum cohomology, we obtain the following:

**Theorem 29.4.1** *For any $i \in I_c$ and for any $w \in W$ we have in $QH^*_{s,T}(G/B)$:*

$$\sigma_{v_i} * \sigma_w = q_{\varpi_i^\vee - w^{-1}(\varpi_i^\vee)} \sigma_{v_i w}.$$

# Bibliography


[BB05] Björner, A. and Brenti, F. *Combinatorics of Coxeter groups.* GTM 231. Springer, New York, 2005.

[Bo54] Bourbaki, N. *Groupes et algèbres de Lie 4,5,6*, Hermann 1954.

[Br89] Brown, K.S. *Buildings*, Springer-Verlag, 1989.

[Bu03] Buch, A.S. *Quantum cohomology of Grassmannians.* Compositio Math. 137 (2003), no. 2, 227–235.

[CMP07] Chaput, P.-E., Manivel, L. and Perrin N. *Affine symmetries in the equivariant quantum cohomology of rational homogeneous spaces*, preprint arXiv:0712.3131.

[FdV69] Freudenthal, H. and de Vries, H. *Linear Lie groups.* Pure and Applied Mathematics, Vol. 35 Academic Press, New York-London 1969.

[Fu98] Fulton W. *Intersection theory.* Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Springer-Verlag, Berlin, 1998.

[FP97] Fulton, W. and Pandharipande, R. *Notes on stable maps and quantum cohomology.* Algebraic geometry—Santa Cruz 1995, 45–96, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[GR70] Grauert, H. and Riemenschneider, O *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen.* 1970 Several Complex Variables, I (Proc. Conf., Univ. of Maryland, College Park, Md., 1970) pp. 97–109 Springer, Berlin.

[Gr57] Grothendieck, A. *Sur la classification des fibrés holomorphes sur la sphère de Riemann.* Amer. J. Math. 79 (1957), 121–138.

[Hu72] Humphreys, J.E. *Introduction to Lie algebras and representation theory.* GTM 9, Berlin-Heidelberg-New York, Springer, 1972.

[Hu90] Humphreys, J.E. *Reflection groups and Coxeter groups.* Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990.

[Ka90] Kac, V.G. *Infinite-dimensional Lie algebras.* Third edition. Cambridge University Press, Cambridge, 1990.

[KP85] Kac, V.G. and Peterson, D.H. *Defining relations of certain infinite-dimensional groups.* The mathematical heritage of Élie Cartan (Lyon, 1984). Astérisque 1985, Numero Hors Serie, 165–208.

[KK86] Kostant B. and Kumar S., *The nil Hecke ring and cohomology of $G/P$ for a Kac-Moody group $G$*, Proc. Nat. Acad. Sci. U.S.A. 83 (1986), no. 6, 1543–1545.

[Kn83] Knudsen, F. *The projectivity of the moduli space of stable curves.* II. The stacks $M_{g,n}$. Math. Scand. 52 (1983), no. 2, 161–199.

[KV06] Kock, J. and Vainsencher, I. *An Invitation to Quantum Cohomology Kontsevich's Formula for Rational Plane Curves.* Progress in Mathematics, Vol. 249 (2006).

[Ko95] Kontsevich, M. *Enumeration of rational curves via torus actions.* The moduli space of curves (Texel Island, 1994), 335–368, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.

[KM98] Kontsevich, M. and Manin, Y. *Relations between the correlators of the topological sigma-model coupled to gravity.* Comm. Math. Phys. 196 (1998), no. 2, 385–398.

[Ku89] Kumar, S. *Existence of certain components in the tensor product of two integrable highest weight modules for Kac-Moody algebras*, Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), 25–38, Adv. Ser. Math. Phys., 7, World Sci. Publ., Teaneck, NJ, 1989.

[Ku02] Kumar, S. *Kac-Moody groups, their flag varieties and representation theory.* Progress in Mathematics, 204. Birkhäuser Boston, Inc., Boston, MA, 2002.

[La06] Lam T., *Schubert polynomials for the affine Grassmannian*, J. Amer. Math. Soc., **21** (2008), 259–281.

[LS07] Lam T., Shimozono M., *Quantum cohomology of $G/P$ and homology of affine grassmannian*, arXiv:0705.1386.

[Ma07] Magyar, P. *Notes on Schubert classes of a loop group*, preprint arXiv:0705.3826.

[Man98] Manivel, L. *Fonctions symétriques, polynômes de Schubert et lieux de dégénérescence.* Cours Spécialisés, 3. Société Mathématique de France, Paris, 1998.

[PR06] Pappas G., Rapoport M. *Twisted loop groups and their affine flag varieties* math/0607130 (July 2006).

[PK83] Peterson, D.H. and Kac, V.G. *Infinite flag varieties and conjugacy theorems*, Proc. Natl. Acad. Sci. USA, 80 (1983), pp. 1778-1782.

[PS86] Pressley, A., Segal G. *Loop groups*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986.

[Ro89] Ronan, M. *Lectures on buildings*, Perspectives in math. 7, Acad. Press, Inc., 1989.

[ST97] Siebert, B. and Tian, G. *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator.* Asian J. Math. 1 (1997), no. 4, 679–695.

[Sp81] Spanier, E.H. *Algebraic topology.* Corrected reprint. Springer-Verlag, New York-Berlin, 1981.

[Se60] Serre, J.-P. *Groupes proalgébriques*, Inst. Hautes Études Sci. Publ. Math. No. 7 (1960).

[Se70] Serre, J.-P. *Cours d'arithmétique*, PUF 1970.

[Se80] Serre, J.-P. *Trees*, Springer-Verlag 1980.

[Sh94] Shafarevich, I.R. *Basic algebraic geometry. 1. Varieties in projective space. Basic algebraic geometry. 2. Schemes and complex manifolds.* Springer-Verlag, Berlin, 1994.

[Sp98] Springer, T.A. *Linear algebraic groups.* Second edition. Progress in Mathematics, 9. Birkhäuser Boston, Inc., Boston, MA, 1998.

[Ti89] Tits, J. *Groupes associés aux algèbres de Kac-Moody.* Séminaire Bourbaki, 1988-1989, exp. n. 700.

[Th98] Thomsen, J.F. *Irreducibility of* $\overline{M}_{0,n}(G/P,\beta)$, Internat. J. Math. 9 (1998) No. 3.

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