# Geometry of webs: an introduction

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1. History
2. Webs
3. Examples
4. A classical theorem
5. Recent developments

# History

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- **'Modern times':** (1990-...)

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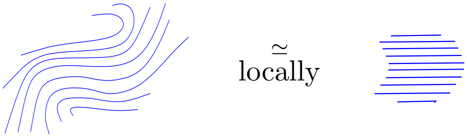


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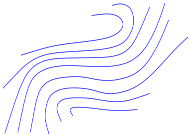
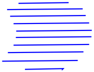


$\overset{\simeq}{\text{locally}}$



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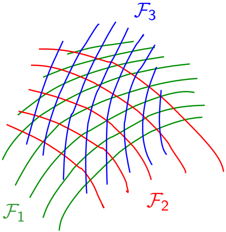


a planar 3-web

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**General position assumption :** (case $N = nr$)

$$1 \leq i_1 < \cdots < i_n \leq d \quad \Longrightarrow \quad \Omega_{i_1} \wedge \cdots \wedge \Omega_{i_n} \neq 0$$

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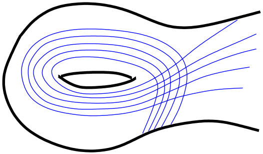

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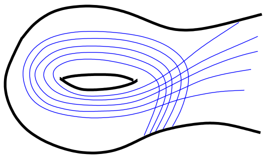

**Definition :** two webs $\mathcal{W}$ and $\mathcal{W}'$ are **equivalent** if

$$\exists\, \varphi \text{ local isomorphism such that } \quad \mathcal{W} = \varphi^*(\mathcal{W}')$$

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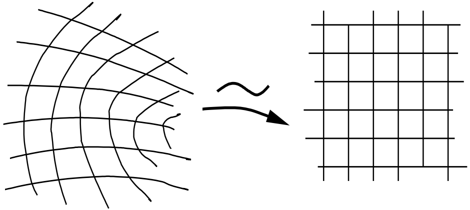

**a planar 2-web is locally trivial**

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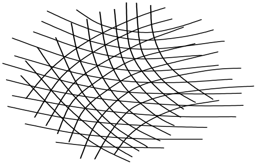

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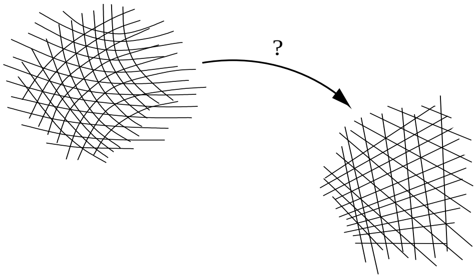

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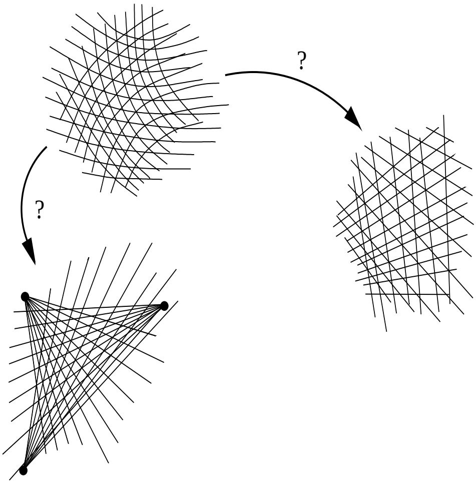

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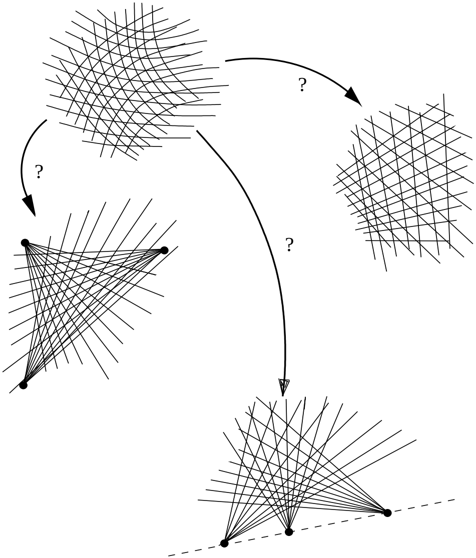

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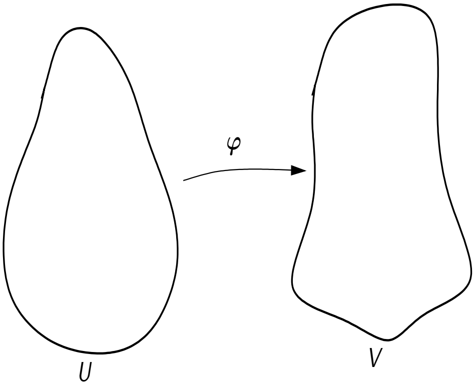

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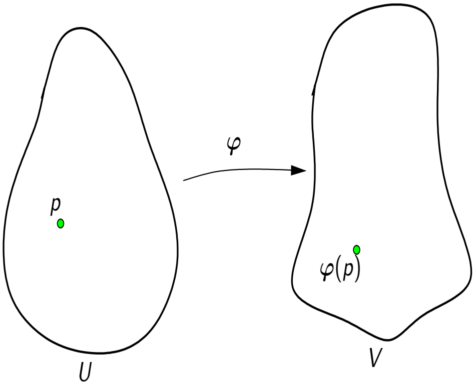

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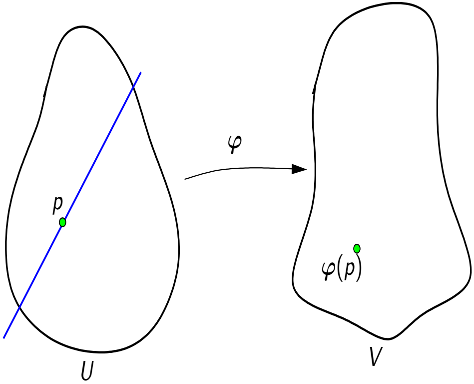

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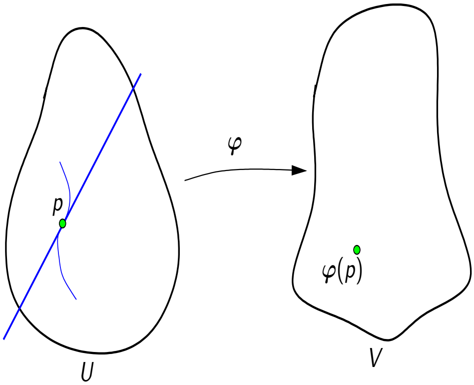

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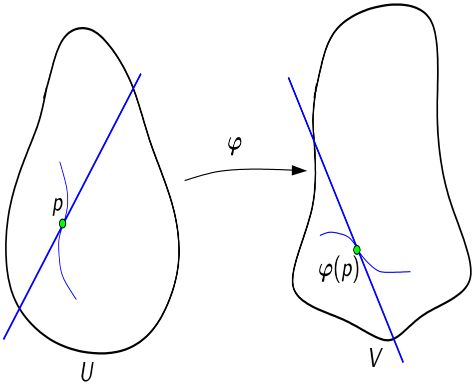

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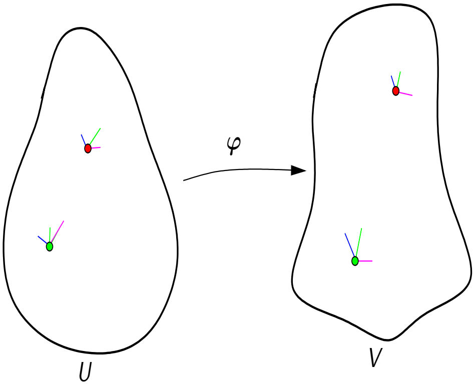

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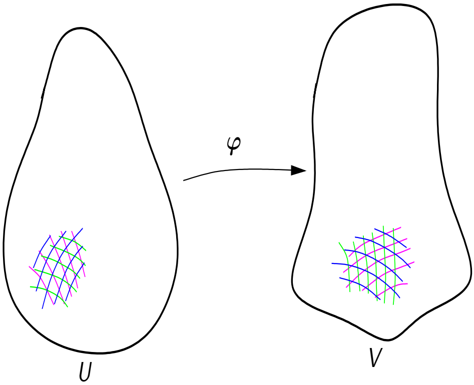

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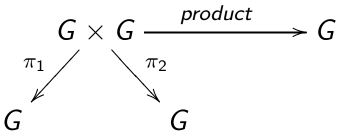

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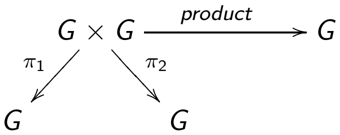


- $\mathcal{W}_G = \mathcal{W}(\pi_1, \pi_2, \mathit{product})$ : 3-web of codimension $r$ on $G \times G$

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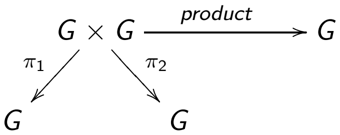


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- **Question** :

| Algebraic properties of the Lie group $G$ | $\overset{?}{\longleftrightarrow}$ | Differential properties of the 3-web $\mathcal{W}_G$ |
|---|---|---|

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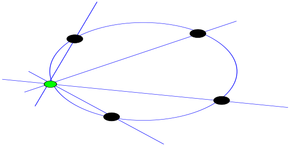

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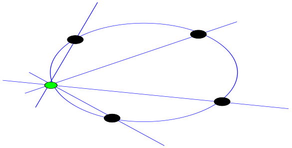

3.

$$\mathcal{B} = \mathcal{W}\left( x \, , \, y \, , \, \frac{x}{y} \, , \, \frac{1-x}{1-y} \, , \, \frac{x(1-y)}{y(1-x)} \right)$$

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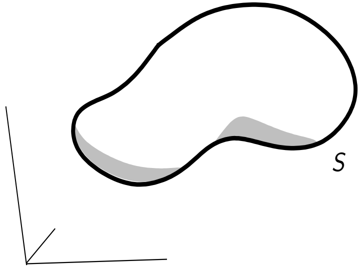

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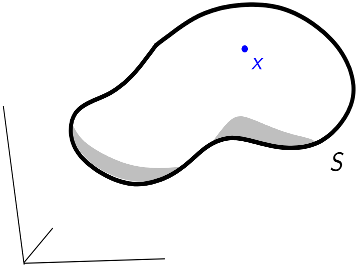

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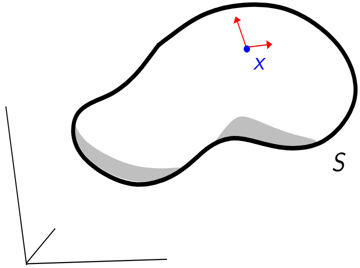

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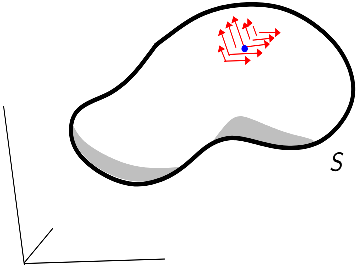

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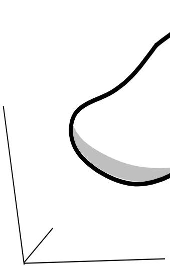

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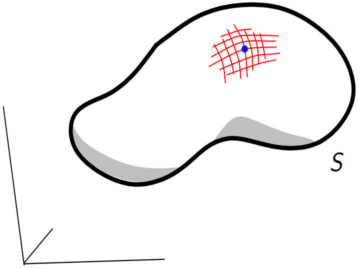


- $S \not\subset S^2 \quad \rightsquigarrow \quad$ 2-web $\mathcal{W}_S$ on $S$

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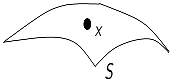

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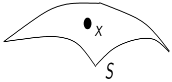
- Surface $S \subset \mathbb{P}^5$  (with regular 2-osculation at $x$)

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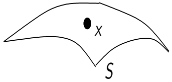
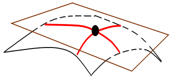
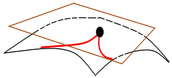
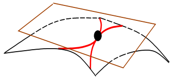
- Surface $S \subset \mathbb{P}^5$  (with regular 2-osculation at $x$)



| **Hyperplane** $H$ | **Curve** $S \cap H$ | **Equation** |
|---|---|---|
| $H \supset T_{S,x}$ |  | Node : $x^2 - y^2 = 0$ |
| $H \in \mathcal{C}_x \simeq \mathbb{P}^1$ |  | Cusp : $x^2 - y^3 = 0$ |
| **Definition** : $H$ *'principal'* |  | Tacnode : $x^2 - y^4 = 0$ |

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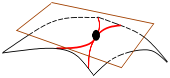

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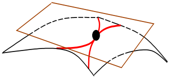

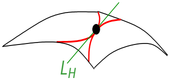

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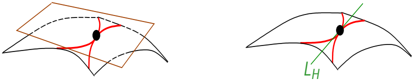


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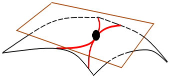

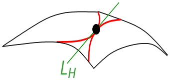


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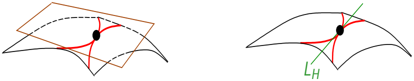


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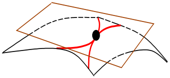

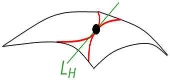


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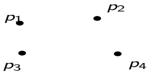
- $P = \Bigg\{$ 

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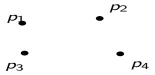
- $P = \{$ 

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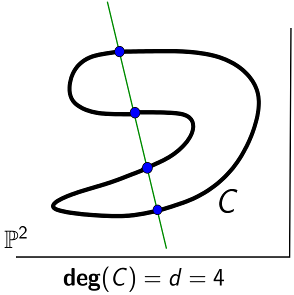

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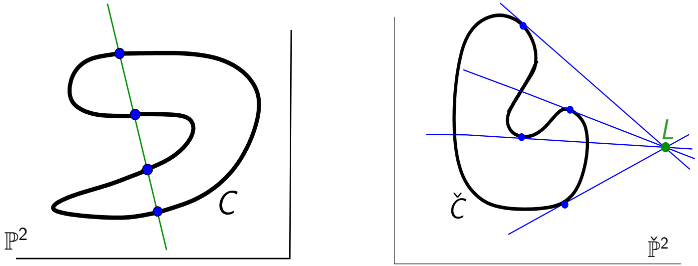


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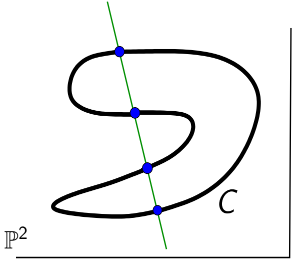


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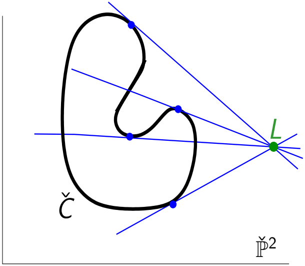


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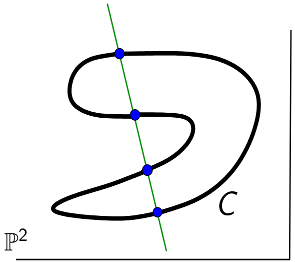


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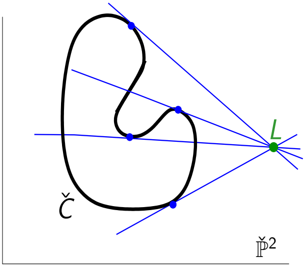


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# A planar algebraic 3-web

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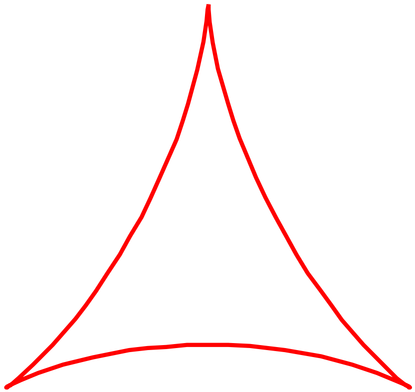

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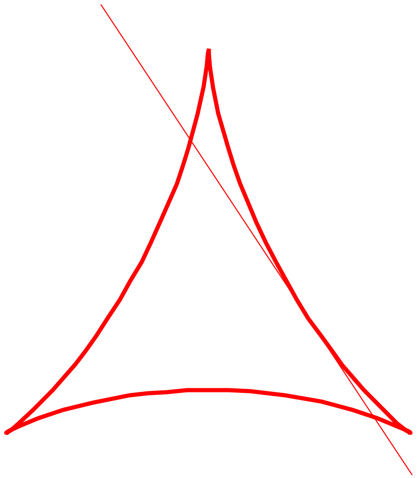

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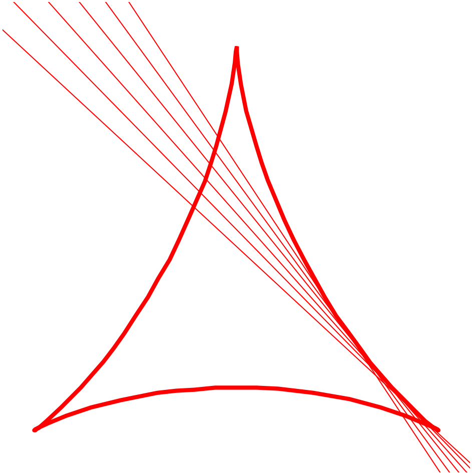

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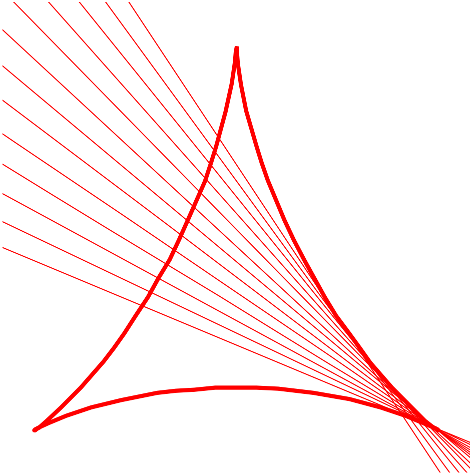

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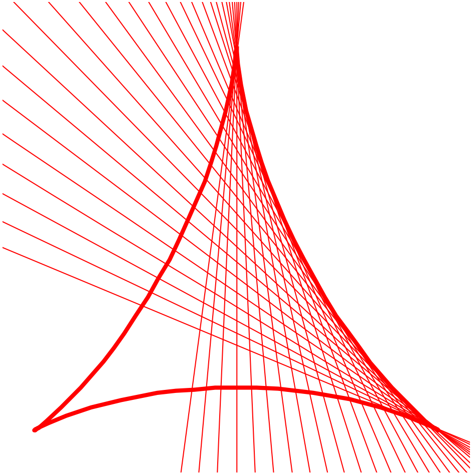

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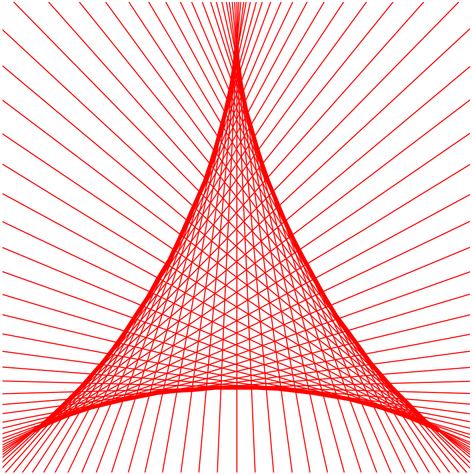

**algebraic 3-web $\mathcal{W}_C$ associated to a plane cubic $C \subset \mathbb{P}^2$**

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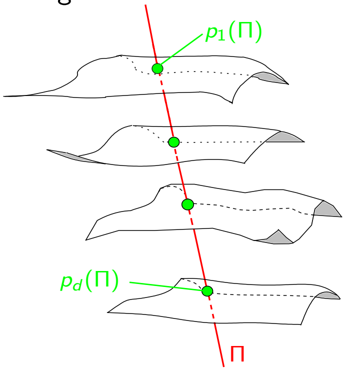

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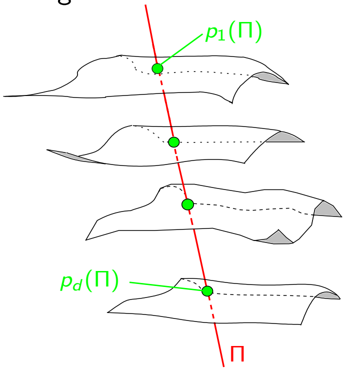


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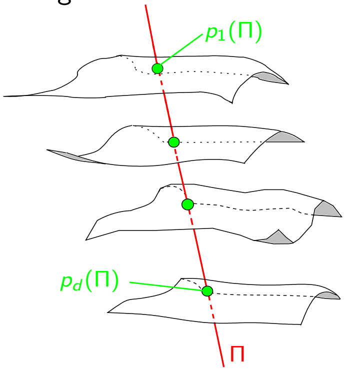


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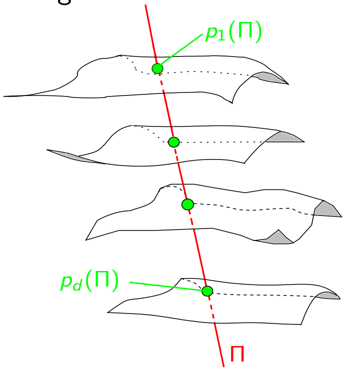


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- Even restricting to the case of webs whose codimension divides the dimension of the ambient space :

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- Even restricting to the case of webs whose codimension divides the dimension of the ambient space :

**[Chern 1982] :**

*"the subject is a wide generalization of the geometry of projective algebraic varieties. Just as intrinsic algebraic varieties are generalized to Kähler manifolds and complex manifolds, such a generalization to web geometry seems justifiable."*

# Webs are everywhere...

- Algebra
- Topology
- Geometry
- Theory of dynamical systems
- Theory of DEs & PDEs
- Mathematical Physics
- Economy

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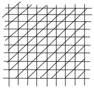
1. $\mathcal{W}_3$ is *parallelizable* $\simeq$  $= \mathcal{W}(x, y, x - y)$

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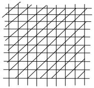
**1.** $\mathcal{W}_3$ is *parallelizable* $\simeq$  $= \mathcal{W}(x, y, x-y)$

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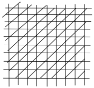
**1.** $\mathcal{W}_3$ is *parallelizable* $\simeq$  $= \mathcal{W}(x, y, x-y)$

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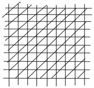
**1.** $\mathcal{W}_3$ is *parallelizable* $\simeq$  $= \mathcal{W}(x, y, x-y)$

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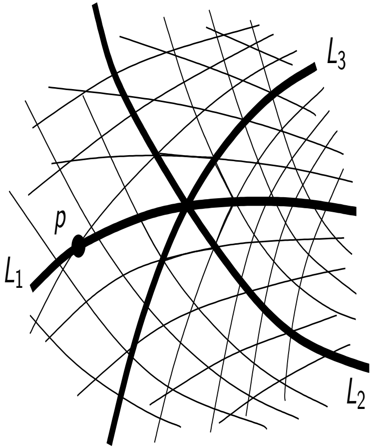

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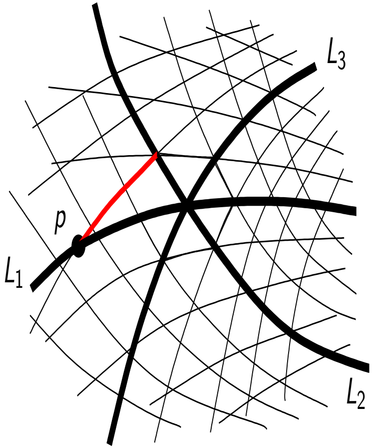

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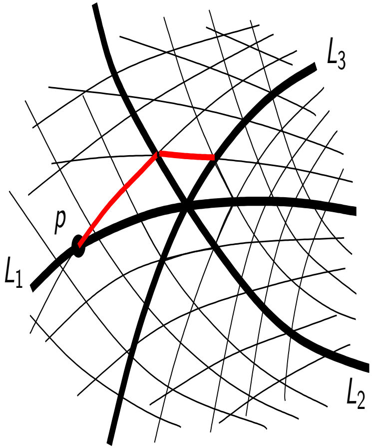

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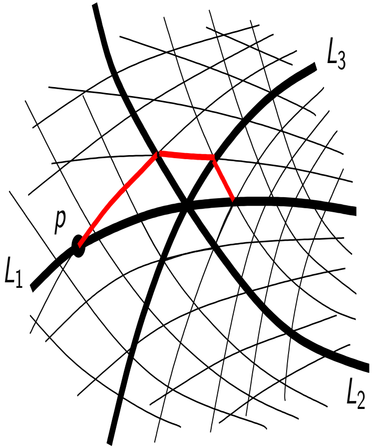

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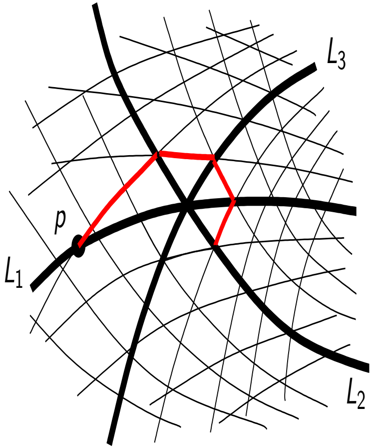

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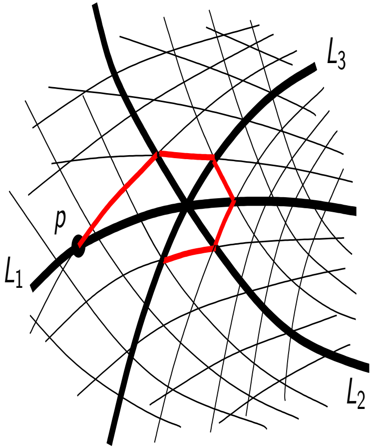

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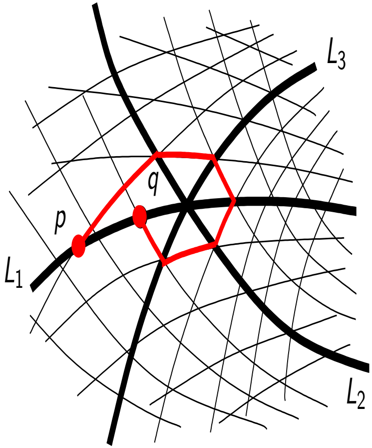

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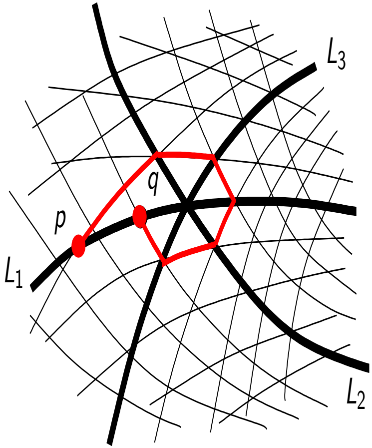


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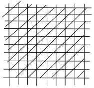
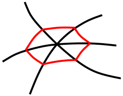
**1.** $\mathcal{W}_3$ is *parallelizable* $\simeq$  $= \mathcal{W}(x, y, x - y)$

**2.** $\mathcal{W}_3$ is *hexagonal* 

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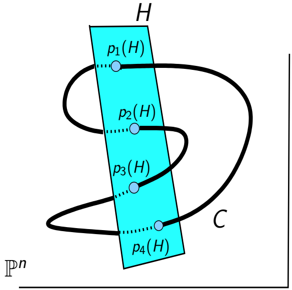

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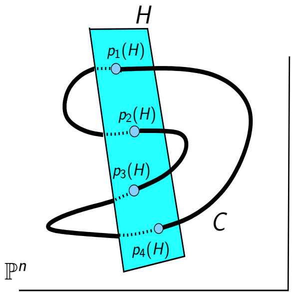


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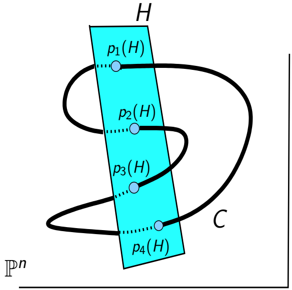


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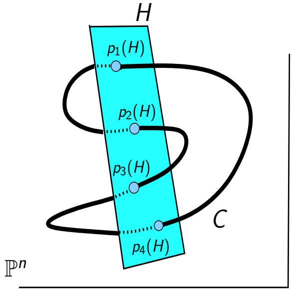


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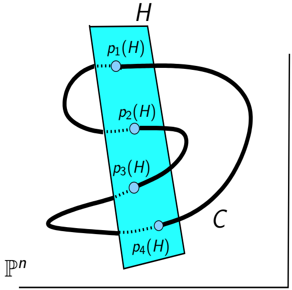


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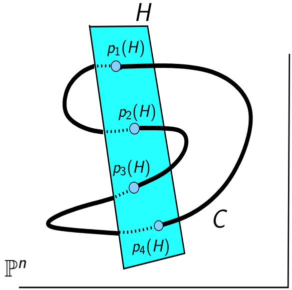


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$$\begin{array}{ccc} \mathbf{p}_a(C) & = & \mathbf{rk}(\mathcal{W}_C) \\ \| & & \| \\ \frac{(d-1)(d-2)}{2} & = & \pi(d,2) \end{array}$$

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**2.** degree $d$ *'Castelnuovo's curve'* $C \subset \mathbb{P}^n$ : $\mathbf{g}(C) = \pi(d,n)$

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**Fact** : $V^r \subset \mathbb{P}^{n+r-1}$ Castelnuovo $\Longrightarrow \mathcal{W}_V$ has maximal rank

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Let $\mathcal{W}_d$ be a **linear** $d$-web on $U \subset \mathbb{C}^n$ :

$$\boxed{\exists \text{ a } \textbf{complete} \text{ AR}} \quad \Longrightarrow \quad \mathcal{W}_d \ \text{ is algebraic}$$

$$\Uparrow$$

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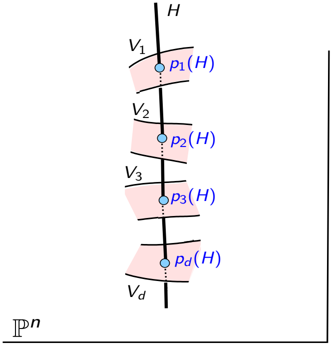

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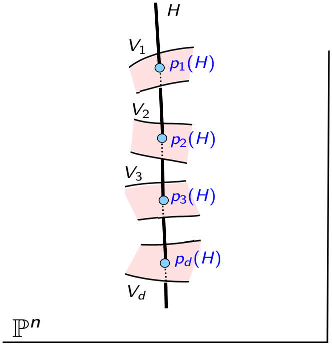


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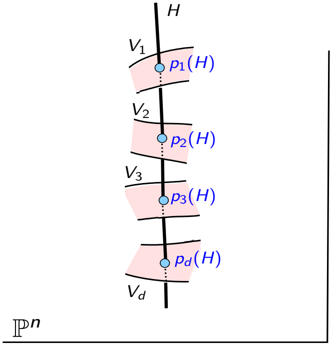


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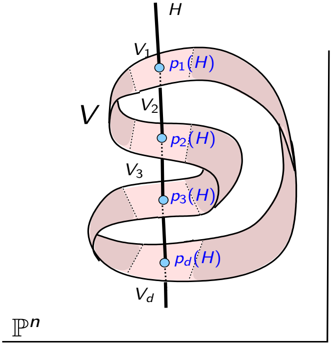


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**Theorem : [Pirio-Trépreau 2013]**

For a $d$-web $\mathcal{W}_d$ of codimension $r > 1$ on $U \subset \mathbb{C}^{nr}$ :

$$\begin{matrix} \mathcal{W}_d \text{ has maximal } r\text{-rank} \\ \big(\mathbf{rk}^r(\mathcal{W}_d) = \pi(d, n, r)\ \big) \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \mathcal{W}_d \text{ is `algebraizable'} \\ \big(\text{generalized sense if } d = d_{n,r}\ \big) \end{matrix}$$

# Algebraic geometry of webs

| **Algebraic curves** | **Webs of codim 1** |
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**Definition :** $\mathcal{W}_d$ on $(\mathbb{C}^n, 0)$ *exceptional* if $\begin{cases} \mathbf{rk}(\mathcal{W}_d) = \pi(d,n) \\ + \text{ non-algebraizable} \end{cases}$

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**Remark :** $\mathcal{W}_d$ exceptional $\Longrightarrow$ $n = 2$ and $d \geq 5$

**Examples :** • $\mathcal{B}$ : $\mathbf{D}(x) - \mathbf{D}(y) - \mathbf{D}\left(\frac{x}{y}\right) - \mathbf{D}\left(\frac{1-y}{1-x}\right) + \mathbf{D}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$

• $\mathcal{W}_{SK}$ = *'Spence-Kummer web'* :

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**Fact :** $\left\{ \begin{matrix} \textbf{Algebraic} \\ d\textbf{-webs } \mathcal{W}_C \end{matrix} \right\} \bigsqcup \left\{ \begin{matrix} \textbf{Exceptional} \\ d\textbf{-webs} \end{matrix} \right\}$

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**Chern-Griffiths (1981) :**

*"(...) we cannot refrain from mentionning what we consider to be the fundamental problem on the subject, which is to determine the maximum rank non-linearizable webs. The strong conditions must imply that there are not many. It may not be unreasonable to compare the situation with the exceptional simple Lie groups."*

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| |
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| **Chern's problem :** to determine and classify the exceptional webs |

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- The exceptional webs remain mysterious...

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- For a non-reduced curve $C \subset \mathbb{P}^2$ :
  - is there a web-theoretic object $\mathcal{W}_C$ corresponding to it ?
  - what would be an abelian relation for such a 'web' $\mathcal{W}_C$ ?

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| **Algebraic geometry** | **Web geometry** |
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| Variety $V^r \subset \mathbb{P}^{n+r-1}$<br>degree $d$ and dim $r$ | $\mathcal{W}_d$ on $(\mathbb{C}^{nr}, 0)$<br>$d$-web of codim $r$ |
| $\omega \in \mathbf{H}^0(V, \Omega_V^q) \quad q = 1, \ldots, r$ | $\underline{\omega} = (\omega_i)_{i=1}^d \in \mathcal{A}^q(\mathcal{W}_d)$ |
| $\mathbf{h}^{q,0}(V) = \mathbf{h}^0\big(V, \Omega_V^q\big)$ | $\mathbf{rk}^q(\mathcal{W}_d) = \dim \mathcal{A}^q(\mathcal{W}_d)$ |
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| $\mathbf{h}^{q,0}(V) \leq \pi^q(d, n, r)$ | $\mathbf{rk}^q(\mathcal{W}_d) \leq \pi^q(d, n, r)$ |

- For $q < r$ :

  determine the varieties $V^r \subset \mathbb{P}^{n+r-1}$ of *'maximal q-rank'*

  *i.e.* such that $\mathbf{h}^{q,0}(V) = \pi^q(d, n, r)$ where $d = \deg V$

끝났어

관심을 가져 주셔서 감사합니다

⋆

THANK YOU FOR YOUR ATTENTION

THE END