Smooth Schubert varieties and generalized Schubert polynomials in algebraic cobordism of Grassmannians

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Abstract

We provide several ingredients towards a generalization of the Littlewood-Richardson rule from Chow groups to algebraic cobordism. In particular, we prove a simple product-formula for multiplying classes of smooth Schubert varieties with any Bott-Samelson class in algebraic cobordism of Grassmannians. We also establish some results for generalized Schubert polynomials for hyperbolic formal group laws.

1 Introduction

Throughout the article, we fix an algebraically closed base field k with $\mathrm{char}(\mathsf{k}) = 0$. Recall that for G a reductive group over k and P a parabolic subgroup of G, there exists a Borel type presentation of the algebraic cobordism ring $\Omega^*(G/P)$ for the homogeneous space G/P, see [16, 17]. For a smooth projective variety X over k, we refer to [24] and [25] for the foundations on $\Omega^*(X)$.

In this article, we adopt an alternative more geometric point of view. Namely, it is known that an additive basis of any of these cobordism rings may be described via geometric generators, using resolutions of Schubert varieties, see below. Schubert calculus consists in multiplying these basis elements. One of the new features when passing from Chow groups to cobordism is the need of resolving the singularities of Schubert varieties. There are therefore many possible bases since a given basis element depends on the choice of a resolution of a Schubert variety. In this paper we shall mostly consider Bott-Samelson resolutions. Let us mention that some formulas for the multiplication with divisor classes are already avalaible, see [7], [16], and that in the recent preprints of Hudson and Matsumura [17], [18], Giambelli-type formulas are obtained for special classes and for a group G of type A. There are several other recent preprints on related questions by Lenart-Zainoulline and others, see e.g. [27].

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We also focus on groups of type A. In the first part we consider the classes of smooth Schubert varieties in Grassmannians and prove a formula for multiplying the class of a smooth Schubert variety with the class of any Bott-Samelson resolution. Several years ago, Buch achieved a beautiful generalization of the classical Littlewood-Richardson rule [4] for K-theory instead of Chow groups, building on previous work of Lascoux-Schutzenberger, Fomin-Kirilov and others. In the language of formal group laws (FGL), Buch has generalized the Littlewood-Richardson rule from the additive FGL to the multiplicative FGL. In the second part, we analyse the work of Fomin and Kirilov [11] and [12] used by Buch, and generalize parts of it to other formal group laws. One might hope that ultimately this will be part of a Littlewood-Richardson rule for the universal case, that is a complete Schubert calculus for algebraic cobordism of Grassmannians.

Recall [24] that algebraic cobordism is the universal oriented algebraic cohomology theory on smooth varieties over k. Its coefficient ring is the Lazard ring $\mathbb{L}$, see [23]. For any homogeneous space $X = G/P$ with G reductive and P a parabolic subgroup of G, we have a cellular decomposition of X given by the B-orbits ($B \subset P$ a Borel subgroup of G) called Schubert cells and denoted by $(\mathring{X}_w)_{w \in W^P}$, where W^P is a subset of the Weyl group W. Choosing resolutions $\widetilde{X}_w \to X_w$ of the closures X_w of $\mathring{X}_w$ defines an additive basis of $\Omega^*(X)$ (see [16, Theorem 2.5]). Schubert calculus aims at understanding the product in terms of these basis elements.

Write $X = \mathrm{Gr}(k, n)$ for the Grassmannian variety of k-dimensional linear subspaces in k^n. This is a homogeneous space of the form G/P with $G = \mathrm{GL}_n(\mathsf{k})$ and P a maximal parabolic subgroup of G. In the first part of the article, we prove some simple product formulas in $\Omega^*(X)$. For Grassmannians, there is another indexing set for Schubert cells and their closures in terms of partitions, and we shall use this notation in the Grassmannian case. In the following statement, λ is a partition associated to a Schubert variety X_λ, that is the closure of the Schubert cell $\mathring{X}_\lambda$ (see Subsection 2.1). Recall also that for the Grassmannian X, all Bott-Samelson resolutions of the Schubert variety X_λ are isomorphic over X. We denote by $\widetilde{X}_\lambda$ this unique Bott-Samelson resolution. Finally, recall that any smooth Schubert variety in X is of the form X_{b^a} with b^a the partition with a parts of size b.

Before stating the main result of Section 2, recall the definition of the dual partition (see Section 2.1 for more details): for a partition λ contained in the $k \times (n-k)$ rectangle R, then $\lambda^\vee$ denotes the dual partition obtained by taking the complement of λ in R. For a partition μ in the $a \times b$ rectangle, we write $\mu^{\vee z}$ for its dual partition in the $a \times b$ rectangle.

Theorem 1.1 (Corollary 2.15)**.** *Let $\lambda \in \mathcal{P}(k, n)$. Then in $\Omega^*(X)$, we have*

$$[X_{b^a}] \cdot [\widetilde{X}_\lambda] = \begin{cases} [\widetilde{X}_{(\lambda^\vee)^{\vee z}}] & \text{for } \lambda \geq (b^a)^\vee, \\ 0 & \text{for } \lambda \not\geq (b^a)^\vee. \end{cases}$$

Note that for Chow groups or for K-theory, the above results are well known and follow from the Pieri formulas (see for example [29] for the Chow group case and [4] for K-theory, by which we always mean K_0).

Note also that there are other natural resolutions of Schubert varieties considered in the literature such as Zelevinsky's resolutions [33]. We believe that for those resolutions (which contain as a special case the resolutions considered in the cobordism Giambelli formulas of Hudson and Matsumura [17]) similar formulas should exist for the multiplication with the class of a smooth Schubert variety.

In the second part (Sections 3 and 4), inspired by Buch's method for giving a Littlewood-Richardson rule for K-theory, we have a closer look at generalized Schubert polynomials for cobordism. Let us recall first that for the full flag variety $X = G/B$ with $G = \mathrm{GL}_n(\mathsf{k})$ and B a Borel subgroup, there is a Borel-type presentation of the cobordism ring (see [16, Theorem 1.1]):

Theorem 1.2. *We have an isomorphism* $\Omega^*(X) \simeq \mathbb{L}[x_1, \ldots, x_n]/S$, *where* $\deg(x_i) = 1$ *for all* $i \in [1, n]$ *and* S *is the ideal generated by homogeneous symmetric polynomials of positive degree.*

In particular, given a Schubert variety X_w and a Bott-Samelson resolution $\widetilde{X}_{\underline{w}} \to X_w$ (here $\underline{w}$ is a reduced expression of the permutation w), we may write the class $[\widetilde{X}_{\underline{w}}] \in \Omega^*(X)$ as a polynomial $\mathfrak{L}_{\underline{w}}$ in the $(x_i)_{i\in[1,n]}$. In [11, 12], Fomin and Kirillov give a very nice description of such polynomials for the K-theory case. The work of Buch [4] builds on these results. In Section 3, we compare the generalized Schubert polynomials for cobordism with those for K-theory (called *Grothendieck polynomials*), see Corollary 3.15. For this, we have to restrict to hyperbolic formal group laws, that is to elliptic cohomology. Choosing a suitable generalization of the Hecke algebra, we are also able to generalize the main theorem of [11] from K-theory to elliptic cohomology, see Theorem 3.13.

In the last section, we combine techniques and results from Sections 2 and 3 to compute some explicit generalized Schubert polynomials. In particular, we show that some of the smooth Schubert varieties satisfy a certain symmetry, see Corollary 4.3. For generalized Schubert polynomials associated to other cells, this is no longer true already when looking at $\mathrm{Gr}(2, 4)$, see Proposition 4.5.

We have tried to present the first two parts in a way that they can be read essentially independently of each other. However, we emphasize that they both are partial solutions to the quest of a Schubert calculus for arbitrary orientable cohomology theories. Both parts reflect that for general formal group laws with operators not satisfying the naive braid relation, Schubert cells will lead to different elements in the corresponding generalized cohomology theory. On the geometric side, we have different resolutions of a given Schubert variety, and on the combinatorial side we have different reduced words for a given permutation. We hope that forthcoming work will combine these two aspects, leading to a better understanding of general Schubert calculus.

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2 Product with smooth Schubert varieties

2.1 Notation

Let $X = \mathrm{Gr}(k, n)$ be the Grassmannian of k-dimensional subspaces in $E = \mathsf{k}^n$. Denote by $(e_i)_{i\in[1,n]}$ the canonical basis of k^n. Denote by B the subgroup of upper-triangular matrices in $\mathrm{GL}_n(\mathsf{k})$, by B^- the subgroup of lower-triangular matrices and by $T = B \cap B^-$ the subgroup of diagonal matrices. For any subset $I \subset [1, n]$ write E_I for the span $\langle e_i \mid i \in I\rangle$. Set $E_i = E_{[1,i]}$ and $E^i = E_{[n+1-i,n]}$ for $i \in [1, n]$.

Call any non increasing sequence $\lambda = (\lambda_i)_{i\geq 1}$ of integers a partition. The length of a partition is $\ell(\lambda) = \max\{i \mid \lambda_i \neq 0\}$. For λ of length k, we identify λ with its first k parts *i.e.* with $(\lambda_i)_{i\in[1,k]}$. The weight of λ is $|\lambda| = \sum_i \lambda_i$. We will also use the pictural description via Young diagrams which are left aligned arrays of $|\lambda|$ boxes with λ_i boxes on the i-th line for all $i \geq 1$. A partition λ fits in the $k \times (n-k)$ rectangle if its Young diagram does or equivalently if $\ell(\lambda) \leq k$ and $\lambda_1 \leq n-k$. Denote by $\mathcal{P}(k, n)$ the set of partitions fitting in the $k \times (n-k)$ rectangle. For $\lambda \in \mathcal{P}(k, n)$ denote by $\lambda^\vee \in \mathcal{P}(k, n)$ its dual partition defined by $\lambda^\vee_i = n-k-\lambda_{k+1-i}$ for $i \in [1, k]$. We have $|\lambda^\vee| = k(n-k) - |\lambda|$. Define $\lambda \leq \mu$ if $\lambda_i \leq \mu_i$ for all i.

Recall the Bruhat decomposition: the B-orbits $(\mathring{X}_\lambda)_{\lambda\in\mathcal{P}(k,n)}$ form a cellular decomposition of X. The same result holds for the B^--orbits $(\mathring{X}^\lambda)_{\lambda\in\mathcal{P}(k,n)}$. Indeed these orbits are isomorphic to affine spaces: $\mathring{X}_w \simeq \mathbb{A}^{|\lambda|}_{\mathsf{k}}$ and $\mathring{X}^\lambda \simeq \mathbb{A}^{\dim X-|\lambda|}_{\mathsf{k}}$. This can easily be deduced from their explicit descriptions:

$$\mathring{X}_\lambda = \{V_k \in X \mid \dim(V_k \cap E_{i+\lambda_{k+1-i}}) = i \text{ for all } i \in [1, k]\} \text{ and}$$
$$\mathring{X}^\lambda = \{V_k \in X \mid \dim(V_k \cap E^{i+n-k-\lambda_i}) = i \text{ for all } i \in [1, k]\}.$$

Note that with this definition we have $\mathring{X}^{\lambda^\vee} = w_X \cdot \mathring{X}_\lambda$ where w_X is the matrix permutation associated to the permutation $i \mapsto n+1-i$ of $[1, n]$. Denote by X_λ the closure of $\mathring{X}_\lambda$ and by X^λ the closure of $\mathring{X}^\lambda$. We have

$$X_\lambda = \{V_k \in X \mid \dim(V_k \cap E_{i+\lambda_{k+1-i}}) \geq i \text{ for all } i \in [1, k]\} \text{ and}$$
$$X^\lambda = \{V_k \in X \mid \dim(V_k \cap E^{i+n-k-\lambda_i}) \geq i \text{ for all } i \in [1, k]\}.$$

Inclusion induces the order on partitions: $X_\lambda \subset X_\mu \Leftrightarrow \lambda \leq \mu$.

Remark 2.1. *The bases* $([X_\lambda])_{\lambda\in\mathcal{P}(k,n)}$ *and* $([X^\lambda])_{\lambda\in\mathcal{P}(k,n)}$ *are dual bases in* $CH^*(X)$ *(see [29, Proposition 3.2.7]). Since* $X^{\lambda^\vee} = w_X \cdot X_\lambda$ *we see that* $([X_\lambda])_{\lambda\in\mathcal{P}(k,n)}$ *and* $([X_{\lambda^\vee}])_{\lambda\in\mathcal{P}(k,n)}$ *are also dual bases. Note that this is no longer true in* K*-theory.*

2.2 Smooth Schubert varieties, Bott-Samelson resolution and cobordism

The smooth Schubert varieties in X are sub-Grassmannians (see for example [22, Theorem 6.4.2] or [14, Theorem 1.1], and [3] or [31] for more details on the singular locus and the type of singularities). The partitions corresponding to these smooth Schubert varieties are of the form $\lambda = (\lambda_1, \cdots, \lambda_k)$ with $\lambda_i = b$ for $i \in [1, a]$ and $\lambda_i = 0$ for $i > a$ for some integers $a \in [1, k]$ and $b \in [1, n-k]$. Denote this partition by $\lambda = b^a$. As a variety we have

$$X_{b^a} = \{V_k \in X \mid E_{k-a} \subset V_k \subset E_{k+b}\} \text{ and}$$
$$X^{b^{a\vee}} = \{V_k \in X \mid E^{k-a} \subset V_k \subset E^{k+b}\}.$$

Moreover we have $X_{b^a} \simeq \mathrm{Gr}(a, a+b) \simeq X^{b^{a\vee}}$.

As already mentioned, Schubert varieties are in general singular. There exist several resolutions of singularities. We recall here the Bott-Samelson resolutions of Schubert varieties which were first introduced by Bott and Samelson [1] as well as by Hansen [15] and Demazure [10] for full flag varieties. These constructions and their properties carry over easily to partial flags $G/P = \mathrm{Gr}(k, n)$. See e.g. [13], [22] for more details. We give here an explicit description of these resolutions in the spirit of configuration spaces (see [28] or [30]). Note also that for Schubert varieties in X, these resolutions are *canonical* in the sense that they do not depend on the choice of a reduced expression.

For a partition λ and a pair of integers (i, j) write $(i, j) \in \lambda$ if $i \in [1, k]$ and $j \in [1, \lambda_i]$ and $(i, j) \notin \lambda$ else. Define $V_{(i,j)} = E_{k+j-i}$ for all $(i, j) \notin \lambda$ where E_i is the zero space for $i \leq 0$ and $E_i = \mathsf{k}^n = E_n$ for $i \geq n$. Define $Y_\lambda = \prod_{(i,j)\in\lambda} \mathrm{Gr}(k + j - i, n)$. Set

$$\widetilde{X}_\lambda = \{(V_{(i,j)})_{(i,j)\in\lambda} \in Y_\lambda \mid V_{(i+1,j)} \subset V_{(i,j)} \subset V_{(i,j+1)} \text{ for all } (i, j) \in \lambda\}.$$

The projection $\pi_\lambda : \widetilde{X}_\lambda \to X$ defined by $\pi_\lambda((V_{(i,j)})_{(i,j)\in\lambda}) = V_{1,1}$ induces a birational morphism onto X_λ. Furthermore, one easily checks that $\widetilde{X}_\lambda$ has the structure of a tower of $\mathbb{P}^1$-bundles so that $\widetilde{X}_\lambda$ is smooth. The morphisms $\pi_\lambda : \widetilde{X}_\lambda \to X_\lambda$ are called the Bott-Samelson resolutions of X_λ.

These resolutions define classes $[\pi_\lambda : \widetilde{X}_\lambda \to X]$ in the cobordism $\Omega^*(X)$ of X. We write $[\widetilde{X}_\lambda]$ for these classes. The classes $([\widetilde{X}_\lambda])_{\lambda\in\mathcal{P}(k,n)}$ form a basis in any oriented cohomology theory and especially in cobordism:

$$\Omega^*(X) = \bigoplus_{\lambda\in\mathcal{P}(k,n)} \mathbb{L}[\widetilde{X}_\lambda],$$

where $\mathbb{L}$ is the Lazard ring, see [16].

2.3 Products in cobordism

We want to understand the products with the classes $[X_{b^a}]$ in $\Omega^*(X)$. Note that the class $[X_{b^a}]$ is well defined without considering any resolution since $X^{b^a} \simeq \mathrm{Gr}(a, a+b)$ is smooth, hence its cobordism class is well defined.

2.3.1 Sub-Grassmannians

Let $Z = \mathrm{Gr}(a, a+b)$ be the Grassmannian of a-dimensional vector subspaces of k^{a+b}. Let $(f_i)_{i\in[1,a+b]}$ be the canonical basis of k^{a+b}. Define $F_i = \langle f_j \mid j \in [1,i]\rangle$ and $F^i = \langle f_j \mid j \in [a+b+1-i, a+b]\rangle$. For $\lambda \in \mathcal{P}(a, a+b)$ a partition contained in the $a \times b$ rectangle define the Schubert variety in Z (as above in X):

$$\begin{array}{l} Z_\lambda = \{V_a \in Z \mid \dim(V_a \cap F_{i+\lambda_{a+1-i}}) \geq i \text{ for all } i \in [1,a]\} \text{ and} \\ Z^\lambda = \{V_a \in Z \mid \dim(V_a \cap F^{i+b-\lambda_i}) \geq i \text{ for all } i \in [1,a]\}. \end{array}$$

If $w_Z : \mathsf{k}^{a+b} \to \mathsf{k}^{a+b}$ is the endomorphism defined by $f_i \mapsto f_{a+b+1-i}$, then $Z^\lambda = w_Z \cdot Z_{\lambda^{\vee_Z}}$ with $\mu = \lambda^{\vee_Z}$ defined by $\mu_i = b - \lambda_{a+1-i}$ for all $i \in [1,a]$.

Now define Bott-Samelson resolutions in Z. Define $W_{(i,j)} = F_{a+j-i}$ for all $(i,j) \notin \lambda$ where F_i is the zero space for $i \leq 0$ and $F_i = \mathsf{k}^{a+b} = F_{a+b}$ for $i \geq a+b$. Define $A_\lambda = \prod_{(i,j)\in\lambda} \mathrm{Gr}(a+j-i, a+b)$. Set

$$\widetilde{Z}_\lambda = \{(W_{(i,j)})_{(i,j)\in\lambda} \in A_\lambda \mid W_{(i+1,j)} \subset W_{(i,j)} \subset W_{(i,j+1)} \text{ for all } (i,j) \in \lambda\}.$$

The projection $\pi^Z_\lambda : \widetilde{Z}_\lambda \to Z$ defined by $\pi^Z_\lambda((W_{(i,j)})_{(i,j)\in\lambda}) = W_{1,1}$ induces a birational morphism onto Z_λ.

Embed Z in X with image $X_{(b^a)}$ as follows. Let $u : \mathsf{k}^{a+b} \to \mathsf{k}^n$ be the linear map defined by $u(f_i) = e_{k-a+i}$ for all $i \in [1, a+b]$. Note that $u(\mathsf{k}^{a+b}) = E_{[k-a+1,k+b]}$. Denote by $v : Z \to X$ the closed embedding defined by $W_a \mapsto E_{k-a} \oplus u(W_a)$.

Embed Z in X with image $X^{(b^a)^\vee}$ as follows. Let $u' : \mathsf{k}^{a+b} \to \mathsf{k}^n$ be the linear map defined by $u'(f_i) = e_{n-k-b+i}$ for all $i \in [1, a+b]$. Note that $u'(\mathsf{k}^{a+b}) = E_{[n-k-b+1,n-k+a]}$. Denote by $v' : Z \to X$ the closed embedding defined by $W_a \mapsto E^{k-a} \oplus u'(W_a)$.

2.3.2 Intersection with Schubert varieties

In this subsection we consider the classes of closed subvarieties $Y \subset X$ in Chow groups or in K-theory. To avoid introducing more notation we denote both theses classes by $[Y]$ and specify in which theory we are working. The product with the class $[X_{b^a}]$ in Chow groups or for K-theory is easy to compute.

Lemma 2.2. *Let $\lambda \in \mathcal{P}(k,n)$. We have*

$$v(Z) \cap X^\lambda = X_{b^a} \cap X^\lambda = \begin{cases} \emptyset & \text{for } \lambda \not\leq b^a, \\ v(Z^\lambda) & \text{for } \lambda \leq b^a. \end{cases}$$

Proof. Let $\mu = b^a$. As is well known, the intersection $X_\mu \cap X^\lambda$ is non empty if and only if $\lambda \leq \mu$. Assume this holds. We also know that $X_\mu \cap X^\lambda$ is a Richardson variety thus reduced, irreducible of dimension $|\mu| - |\lambda|$. Since Z^λ has dimension $|\mu|-|\lambda|$ it is enough to prove the inclusion $v(Z^\lambda) \subset X_{b^a} \cap X^\lambda$. By construction, we have $v(Z) = X_{b^a}$ thus $v(Z^\lambda) \subset X_{b^a}$. We prove the inclusion $v(Z^\lambda) \subset X^\lambda$. Recall the definition

$$X^\lambda = \{V_k \in X \mid \dim(V_k \cap E^{i+n-k-\lambda_i}) \geq i \text{ for all } i \in [1,k]\}.$$

Since λ is contained in the $a \times b$ rectangle we have $\ell(\lambda) \leq a$ thus the conditions $\dim(V_k \cap E^{i+n-k-\lambda_i}) \geq i$ for $i > a$ become $\dim(V_k \cap E^{i+n-k}) \geq i$ and are trivially satisfied. We need to check the conditions $\dim(V_k \cap E^{i+n-k-\lambda_i}) \geq i$ for $i \in [1, a]$ and $V_k = v(W_a)$ with $W_a \in Z^\lambda$. For all $i \in [1, a]$, we have $\dim(V_a \cap F^{i+b-\lambda_i}) \geq i$. Applying v we get the inequality $\dim(v(V_a) \cap v(F^{i+b-\lambda_i}) \cap E_{[k-a+1,k+b]}) \geq i$. But $v(F^{i+b-\lambda_i} \cap E_{[k-a+1,k+b]}) = E_{[k+1-\lambda_i-i,k+b]} \subset E_{[k+1-\lambda_i-i,n]} = E^{i+n-k-\lambda_i}$. In particular $\dim(v(V_a) \cap E^{i+n-k-\lambda_i}) \geq i$ for $i \in [1, a]$ proving the result. $\square$

Remark that $v(w_Z(F^i)) = E_{k-a} \oplus u(F_i) = E_i$, thus for $\lambda \in \mathcal{P}(a, a+b)$, we have $v(Z_\lambda) = X_\lambda$. In particular, we have $v(w_Z \cdot Z^\lambda) = v(Z_{\lambda^{\vee_Z}}) = X_{\lambda^{\vee_Z}}$. Consider k^{a+b} as a subspace of k^n via the embedding u and let w^Z be the endomorphism of k^n obtained by extending w_Z with the identity on the complement $\langle e_i \mid i \notin [k-a, k+b] \rangle$. We have $w^Z \circ v = v \circ w_Z$.

Corollary 2.3. *Let $\lambda \in \mathcal{P}(a, a+b)$. We have*

$$v(Z) \cap X^\lambda = X_{b^a} \cap X^\lambda = \begin{cases} \emptyset & \text{for } \lambda \not\leq b^a, \\ w^Z \cdot X_{\lambda^{\vee_Z}} & \text{for } \lambda \leq b^a. \end{cases}$$

Corollary 2.4. *Let $\lambda \in \mathcal{P}(a, a+b)$. We have*

$$X_\lambda \cap v'(Z) = X_\lambda \cap X^{b^{a\vee}} = \begin{cases} \emptyset & \text{for } \lambda \not\geq (b^a)^\vee, \\ w_X w^Z \cdot X_{(\lambda^\vee)^{\vee_Z}} & \text{for } \lambda \geq (b^a)^\vee. \end{cases}$$

Proof. Set $\mu = \lambda^\vee$, apply Corollary 2.3 to μ and multiply with w_X. $\square$

Corollary 2.5. *Let $\lambda \in \mathcal{P}(a, a+b)$. In $CH^*(X)$, we have*

$$[X_\lambda] \cup [X_{b^a}] = \begin{cases} [X_{(\lambda^\vee)^{\vee_Z}}] & \text{for } \lambda \geq (b^a)^\vee, \\ 0 & \text{for } \lambda \not\geq (b^a)^\vee. \end{cases}$$

Remark 2.6. *The same result holds for K-theory, see [4].*

Our aim is to generalise the above results to Bott-Samelson resolutions and to cobordism. For this, the dual point of view of Corollary 2.4 is better suited.

2.3.3 Fiber product

Let μ be a partition in the $a \times b$ rectangle and let $\mu' = (\mu^{\vee_Z})^\vee$. We construct an embedding of $\widetilde{Z}_\mu \to \widetilde{X}_{\mu'}$. We denote by $v' : \mathrm{Gr}(i, a+b) \to \mathrm{Gr}(i+k-a, n)$ the embeddings induced by u' as follows: $v'(W_i) = u'(W_i) \oplus E^{k-a}$.

First remark that $\mu \leq \mu'$ and that we get μ' from μ by adding $k-a$ lines (with $n-k$ boxes) and $n-k-b$ columns (with k boxes). In other words $\mu'_i = n-k$ for $i \in [1, k-a]$ and $\mu'_i = \mu_i + n-k-b$ for $i \in [k-a+1, k]$.

Let $(W_{(i,j)})_{(i,j)\in\mu} \in \widetilde{Z}_\mu$. We define $(V_{(i,j)})_{(i,j)\in\mu'}$ as follows.
For $i \in [1, k-a]$ and $j \in [1, n-k-b]$, set

$$V_{(i,j)} = (v'(W_{(1,1)}) \oplus E_{j-1}) \cap E_{n+1-i}$$

For $i \in [k-a+1, k]$ and $j \in [1, n-k-b]$

$$V_{(i,j)} = (v'(W_{(i-(k-a),1)}) \oplus E_{j-1}) \cap E_{n+a-k}$$

For $i \in [1, k-a]$ and $j \in [n-k-b+1, n-k]$

$$V_{(i,j)} = (v'(W_{(1,j-(n-k-b))}) \oplus E_{n-k-b}) \cap E_{n+1-i}$$

For $i \in [1, k-a]$ and $j \in [1, n-k-b]$

$$V_{(i,j)} = (v'(W_{(i-(k-a),j-(n-k-b))}) \oplus E_{n-k-b}) \cap E_{n+a-k}.$$

For $(i,j) \notin \mu'$ we set $V_{(i,j)} = (v'(W_{(i-(k-a),j-(n-k-b))}) \oplus E_{n-k-b}) \cap E_{n+a-k}$.

Lemma 2.7. *We have* $(V_{(i,j)})_{(i,j)\in\mu'} \in \widetilde{X}_{\mu'}$.

Proof. Recall that $u'(\mathsf{k}^{a+b}) = E_{n-k-b,n-k+a}$, that $E^{k-a} \subset v'(W)$ and that $v'(W) \subset E^{k+b}$ for any subspace $W \subset \mathsf{k}^{a+b}$. In particular, in the above definition all sums are direct and all intersections are transverse. This implies $\dim V_{(i,j)} = k+j-i$ thus $(V_{(i,j)})_{(i,j)\in\lambda'} \in Y_{\mu'}$. For $(i,j) \notin \mu'$ we have $V_{(i,j)} = (v'(W_{(i-(k-a),j-(n-k-b))}) \oplus E_{n-k-b}) \cap E_{n+a-k} = E_{k+j-i}$. One easily proves that $V_{(i+1,j)} \subset V_{(i,j)} \subset V_{(i,j+1)}$. The result follows. □

Lemma 2.8. *The map* $\varphi : \widetilde{Z}_\mu \to \widetilde{X}_{\mu'}$ *is a closed embedding.*

Proof. We have $u'(W_{(i,j)}) = V_{(i+k-a,j+n-k-b)}) \cap E^{k+b}$. Since u is injective, the result follows. □

Lemma 2.9. *The map* $\psi : \widetilde{Z}_\mu \to X$ *defined by* $(W_{(i,j)})_{(i,j)\in\mu} \mapsto V_{(1,1)}$ *factors through* $v'(Z)$.

Proof. We have $V_{(1,1)} = v'(W_{(1,1)}) = u'(W_{(1,1)}) \oplus E^{k-a}$. In particular $E^{k-a} \subset V_{(1,1)} \subset E^{k+b}$. The result follows. □

Proposition 2.10. *Let* $\mu \in \mathcal{P}(a, a+b)$ *and consider* $\widetilde{Z}_\mu$ *as an* X*-scheme via* ψ. *We have* $\widetilde{X}_{\mu'} \times_X v'(Z) = \widetilde{X}_{\mu'} \times_X X^{(b^a)^\vee} \simeq \widetilde{Z}_\mu$.

Proof. We have morphisms $\varphi : \widetilde{Z}_\mu \to \widetilde{X}_{\mu'}$ and $\psi : \widetilde{Z}_\mu \to v'(Z)$ with φ a closed embedding. Furthermore the map $\pi_{\mu'} : \widetilde{X}_{\mu'} \to X$ is given by $(V_{(i,j)})_{(i,j)\in\mu'} \mapsto V_{(1,1)}$ so the composition $\pi_{\mu'} \circ \varphi$ is the map ψ. In particular we have a morphism $\varphi \times \psi : \widetilde{Z}_\mu \to \widetilde{X}_{\mu'} \times_X v'(Z)$. This is a closed embedding since φ is a closed embedding. To prove that this is an isomorphism, it is enough to prove that $\widetilde{X}_{\mu'} \times_X v'(Z)$ is irreducible and smooth of dimension $|\mu| = \dim \widetilde{Z}_\mu$. But $v'(Z) = X^{(b^a)^\vee}$ and $\widetilde{X}_{\mu'}$ are in general position. By Kleimann-Bertini [21] any irreducible component is of dimension $|\mu| - \operatorname{codim}_X v'(Z) = |\mu|$. By Bertini again, the fiber product of $v'(Z)$ with the locus in $\widetilde{X}_{\mu'}$ where $\pi_{\mu'}$ is not an isomorphism has dimension strictly less than $|\mu|$ and is therefore never an irreducible component. Now since $v'(Z) \cap X_{\mu'}$ is irreducible, the same holds for $\widetilde{X}_{\mu'} \times_X v'(Z)$. Furthermore by Bertini again this fiber product is smooth and therefore reduced. $\square$

Corollary 2.11. *Let $\lambda \in \mathcal{P}(k,n)$. As X-schemes, we have*

$$\widetilde{X}_\lambda \times_X v'(Z) = \widetilde{X}_\lambda \times_X X^{b^a} \simeq \begin{cases} \emptyset & \textit{for } \lambda \not\geq (b^a)^\vee, \\ \widetilde{Z}_\mu & \textit{for } \lambda \geq (b^a)^\vee, \end{cases}$$

with $\mu = (\lambda^\vee)^{\vee_Z}$ for $\lambda \geq (b^a)^\vee$ and $\widetilde{Z}_\mu$ is considered as an X-scheme via ψ.

2.3.4 Cobordism

We construct another X-scheme isomorphism between $\widetilde{Z}_\mu$ and $w_X w^Z \cdot \widetilde{X}_\mu$. Here $\widetilde{Z}_\mu$ is an X-scheme via ψ while $w_X w^Z \cdot \widetilde{X}_\mu$ is an X-scheme via $w_X w^Z \circ \pi_\mu$. The actions of w_X and w^Z on $\widetilde{X}_\mu$ being defined via the embedding of $\widetilde{X}_\mu$ in Y_μ and the actions on the later are given by the diagonal action on each factor (recall that Y_μ is a product of Grassmannians $\mathrm{Gr}(i,n)$ on which w_X and w^Z act).

Let $(W_{(i,j)})_{(i,j)\in\mu} \in \widetilde{Z}_\mu$. We define $(V_{(i,j)})_{(i,j)\in\mu}$ as follows. For $(i,j) \in \mu$, set $V_{(i,j)} = v'(W_{(i,j)})$. For $(i,j) \notin \mu$, set $V_{(i,j)} = w_X w^Z \cdot E_{k+j-i}$.

Lemma 2.12. *We have $(V_{(i,j)})_{(i,j)\in\mu} \in w_X w^Z \cdot \widetilde{X}_\mu$.*

Proof. For (i,j), $(i+1,j)$ and $(i,j+1)$ in μ, the conditions $V_{(i+1,j)} \subset V_{(i,j)} \subset V_{i,j+1)}$ are clearly satisfied. We only need to check these conditions for $(i+1,j)$ or $(i,j+1)$ not in μ. But for $(i,j) \notin \mu$, we have $W_{(i,j)} = F_{a+j-i}$ thus $v'(W_{(i,j)}) = v'(F_{a+j-i}) = E^{k-a} \oplus E_{[n-k-b+1,n-k-b+a+j-i]} = w_X w^Z \cdot E_{k+j-i} = V_{(i,j)}$ and the result follows. $\square$

Proposition 2.13. *Let $\mu \in \mathcal{P}(a,a+b)$. The X-schemes $\widetilde{Z}_\mu$ (via ψ) and $w_X w^Z \cdot \widetilde{X}_\mu$ are isomorphic.*

Proof. The above morphism sending $(W_{(i,j)})_{(i,j)\in\mu} \in \widetilde{Z}_\mu$ to $(V_{(i,j)})_{(i,j)\in\mu} \in \widetilde{X}_\mu$ is a closed embedding. Since both schemes are smooth are irreducible of the same dimension, this map is an isomorphism. We need to check that the morphisms to X coincide. But the composition $\widetilde{Z}_\mu \to w_X w^Z \cdot \widetilde{X}_\mu \to X$ is given by $(W_{(i,j)})_{(i,j)\in\mu} \mapsto (V_{(i,j)})_{(i,j)\in\mu} \mapsto V_{(1,1)}$ and therefore maps $(W_{(i,j)})_{(i,j)\in\mu} \in \widetilde{Z}_\mu$ to $v'(W_{(1,1)}) = \psi(W_{(1,1)})$. It coincides with ψ. $\square$

Corollary 2.14. *Let $\lambda \in \mathcal{P}(k,n)$. As X-schemes, we have*

$$\widetilde{X}_\lambda \times_X v'(Z) = \widetilde{X}_\lambda \times_X X^{b^a} \simeq \begin{cases} \emptyset & \textit{for } \lambda \not\geq (b^a)^\vee, \\ w_X w^Z \cdot \widetilde{X}_{(\lambda^\vee)^{\vee_Z}} & \textit{for } \lambda \geq (b^a)^\vee. \end{cases}$$

Corollary 2.15. *Let $\lambda \in \mathcal{P}(k,n)$. Then in $\Omega^*(X)$, we have*

$$[X_{b^a}] \cdot [\widetilde{X}_\lambda] = \begin{cases} [\widetilde{X}_{(\lambda^\vee)^\vee z}] & \text{for } \lambda \geq (b^a)^\vee, \\ 0 & \text{for } \lambda \not\geq (b^a)^\vee. \end{cases}$$

Proof. The product $[X_{b^a}] \cdot [\widetilde{X}_\lambda]$ is given by pulling back the exterior product $X_{b^a} \times \widetilde{X}_\lambda \to X \times X$ along the diagonal map $\Delta : X \to X \times X$, see [24, Remark 4.1.14]. We thus have $[X_{b^a}] \cdot [\widetilde{X}_\lambda] = \Delta^*[X_{b^a} \times \widetilde{X}_\lambda \to X \times X]$. Applying [24, Corollary 6.5.5.1], we get $\Delta^*[X_{b^a} \times \widetilde{X}_\lambda \to X \times X] = [X_{b^a} \times_X \widetilde{X}_\lambda]$ in $\Omega^*(X)$. □

Remark 2.16. *1. These results were inspired by several similar results for other cohomology theories. In particular, the results explained in Corollary 2.5 are the classical part of Seidel symmetries in quantum cohomology [32]. The results of Seidel are not explicit but were made explicit in [8] and [9]. These results extend to quantum K-theory. This will be presented in a forthcoming work [5]. We expect the same results to be valid in quantum cobordism once the latter is defined.*

2. We expect more general results of the same type for other homogeneous space. These will be studied by the second author in forthcoming work.

3 Generalized Schubert polynomials and generalized Hecke algebras

Recall that classical Grothendieck polynomials are representatives of Schubert classes in Borel's presentation of K-theory. In this section, we discuss the difference between classical Grothendieck polynomials and the representatives in Borel's presentation of algebraic cobordism of Bott-Samelson resolutions of Schubert varieties. For K-theory (that is K_0), the computation of polynomial representatives for classes of Schubert varieties has been done by Fomin-Kirillov [11], [12]. We establish a generalization of the main theorem of [11]. Building on their work, Buch [4] computed Littlewood-Richardson rules for K-theory.

3.1 Divided difference operators

Recall that K-theory corresponds to the multiplicative formal group law. The methods of Buch and Fomin-Kirilov do not generalize to the universal formal group law, that is to algebraic cobordism. However, we will show that they apply in a much weaker form to *hyperbolic formal group laws* (see Definition 3.6 below) since we need to impose one more relation in the Hecke algebra (see Definition 3.11 below). For $i \in [1, n-1]$, let s_i be the transposition of $[1, n]$ exchanging i and $i+1$.

Definition 3.1. *Let F be a formal group law over R with inverse χ.*

1. For $i \in [1, n-1]$, define $\sigma_i \in \mathrm{End}(R[[x_1, \ldots, x_n]])$ by

$$(\sigma_i f)(x_1, \ldots, x_n) = f(x_{s_i(1)}, \ldots, x_{s_i(n)}).$$

2. For $i \in [1, n-1]$, define $C_i, \Delta_i \in \mathrm{End}(R[[x_1, \dots, x_n]])$ by

$$C_i = (\mathrm{Id} + \sigma_i)\frac{1}{F(x_i, \chi(x_{i+1}))} \text{ and } \Delta_i = \frac{1}{F(x_{i+1}, \chi(x_i))}(\mathrm{Id} - \sigma_i).$$

Remark 3.2. *Note that the above operators are well defined in $R[[x_1, \dots, x_n]]$ since $F(x, \chi(y))$ can be written $(x-y)g(x,y)$ with $g(x,y)$ invertible in $R[[x,y]]$.*

This definition is taken from [16, p.71] and [7, Section 3]. When applying it to the additive formal group law, one recovers the usual definition as e.g. in [29, Section 2.3.1] up to a sign (observe that $\sigma_i \circ F(x_{i+1}, \chi(x_i)) = F(x_i, \chi(x_{i+1}))$). For the multiplicative formal group law $F(x,y) = x + y + \beta xy$, the definition of C_i yields the β-DDO $\pi_i^{(\beta)}$ of [11], which for $\beta = -1$ specializes to the isobaric DDO of [4]. Moreover, still for the multiplicative formal group law $F(x,y) = x + y + \beta xy$, the operator Δ_i above (which equals the one of [7, Section 3]) coincides up to sign with the operator $\pi_i^{(\beta)} + \beta$ which appears in [11, Lemma 2.5].

Recall [2] that the braid relations for the operators C_i only hold if the FGL is additive or multiplicative. We therefore need to keep track of reduced expressions to define generalized Schubert polynomials, which is not necessary in [11, Definition 2.1].

3.2 Generalized Schubert polynomials

The following definition generalizes both Schubert polynomials for Chow groups and Grothendieck polynomials for K-theory.

Definition 3.3. *Let w be a permutation and $\underline{w}$ a reduced expression of w as product in the $(s_i)_{i \in [1, n-1]}$. Define the generalized Schubert polynomial $\mathfrak{L}_{\underline{w}}$ by induction:*

(a) $\mathfrak{L}_1(x_1, \dots, x_n) = x_1^{n-1} x_2^{n-2} \dots x_{n-1}$

(b) $\mathfrak{L}_{\underline{w}s_i} := C_i \mathfrak{L}_{\underline{w}}$ *if $\underline{w}s_i$ is a reduced word.*

Note that this notation is different from the one used in [11] and elsewhere: Our $\mathfrak{L}_1$ corresponds to their $\mathfrak{L}_{w_0}$ and our $\mathfrak{L}_{\underline{w}}$ to their $\mathfrak{L}_{w_0 w}$. We decided to adopt this notation since there is a unique class for the point as well as a unique reduced expression for 1, but there is a Bott-Samelson resolution and a polynomial $\mathfrak{L}_{\underline{w}_0}$ for each reduced expression $\underline{w}_0$ of the element w_0.

For any permutation w, the Bott-Samelson resolutions $\widetilde{X}_{\underline{w}} \to X_w$ of the Schubert variety X_w are indexed by the reduced words $\underline{w}$ of w. It was proved in [16, Theorem 3.2] that the polynomial $\mathfrak{L}_{\underline{w}}$ represents the class of the resolution $\widetilde{X}_{\underline{w}} \to X_w$ in $\Omega^*(G/B)$.

Let S be the ideal in $R[[x_1, \dots, x_n]]$ generated by symmetric polynomials of positive degree. The polynomial $\mathfrak{L}_1$ corresponds to the cobordism class of a point. Modulo S, the polynomial $n!\mathfrak{L}_1$ has several equivalent descriptions (compare e.g. [16, Remark 2.7], where it differs by a scalar from D_n below).

Lemma 3.4. *Let $A^*(-)$ be an oriented cohomology theory and F its FGL.*
(a) We have

$$D_n := \prod_{1\le i<j\le n}(x_i - x_j) \equiv n! x_1^{n-1}x_2^{n-2}\cdots x_{n-1} = n!\mathfrak{L}_1 \; mod \; S.$$

(b) Setting $a -_F b = F(a, \chi(b))$, we have

$$D_n \equiv D_n^F := \prod_{1\le i<j\le n}(x_i -_F x_j) \; mod \; S.$$

Proof. To show (a), one first verifies that modulo S we have $\prod_{1<i\le n}(x_1 - x_i) \equiv n x_1^{n-1}$, deriving the equality $\prod_{1\le i\le n}(x - x_i) \equiv x^n$ and setting $x = x_1$. Then one shows $x_1^{n-1} p(x_1, \dots, x_{n-1}) \equiv 0$ for any symmetric non-constant polynomial $p(x_1, \dots, x_{n-1})$, writing $p(x_1, \dots, x_{n-1}) = x_1 q(x_1, \dots, x_n)$ and using that $x_1^n \equiv 0$ modulo S. Now proceed by induction on n. The claim holds for $n = 1$. Using the factorization

$$\prod_{1\le i<j\le n}(x_i - x_j) = \prod_{1<i<j\le n}(x_i - x_j) \prod_{1<i\le n}(x_1 - x_i),$$

the claim for n follows using the induction hypothesis for $n-1$ and the above two equalities modulo S.

For (b), note that $x_i -_F x_j = 0$ if $x_j = x_i$, which implies that $x_i -_F x_j$ is divisible by $x_i - x_j$, Hence $x_i -_F x_j = (x_i - x_j)a(x_i, x_j)$ with $a(x_i, x_j) = 1 + b(x_i, x_j)$ and $b \in (x_i, x_j)$. Thus $D_n^F = D_n + D_n q(x_1, \dots, x_n)$ with $q(0, \dots, 0) = 0$. Now using part (a) and the equality $x_1^n \equiv 0 \; mod \; S$, we deduce that $D_n x_i \equiv 0 \; mod \; S$ for $i = 1$ and thus (use a suitable permutation) for all i. Hence $D_n q(x_1, \dots, x_n) \equiv 0 \; mod \; S$ as claimed. □

Remark 3.5. *Some authors use $x_n^{n-1}x_{n-1}^{n-2}\cdots x_2$ in place of $x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$. Modulo S these two classes only differ by the sign $(-1)^{\frac{n(n-1)}{2}}$.*

3.3 Hyperbolic formal group laws

We now define hyperbolic formal group laws, which generalize the additive and multiplicative ones.

Definition 3.6. *The hyperbolic formal group law F over $R = \mathbb{Z}[\mu_1, \mu_2]$ and its inverse χ are given by*

$$F(x, y) = \frac{x + y - \mu_1 xy}{1 + \mu_2 xy} \text{ and } \chi(x) = -\frac{x}{1 - \mu_1 x}.$$

Recall that formal group laws are by definition power series in two variables, and all fractions here and below may be written as such. Note that any ring homomorphism $\mathbb{Z}[\mu_1, \mu_2] \to A$ induces a formal group law over A. Calling these induced formal group laws hyperbolic as well, we find that additive and multiplicative formal group laws are special cases of hyperbolic formal group laws. See e.g. [6], [20, Example 2.2 (d)] and [27, 2.2] for more on hyperbolic formal group laws. Combining their computations, we see that

$$F(x,y) = x + y - \mu_1 xy + \mu_2(x^2y + xy^2) + \mu_2\mu_1 x^2y^2 + O(5).$$

In Section 4.2. below, we explain how these FGLs lead to certain elliptic cohomology theories $E^*(-)$. If $\mu_2 = 0$, these cohomology theories specialize to Chow groups (if $\mu_1 = 0$), K_0 (if μ_1 is invertible, thus sometimes called periodic K-theory), connective K_0 and (if $\mu_1 = 0$ but $\mu_2 \neq 0$) theories associated with Lorentz FGLs.

Definition 3.7. *Let F be a formal group law. Define*

$$\kappa_i = \kappa_i^F = \frac{1}{F(x_i, \chi(x_{i+1}))} + \frac{1}{F(x_{i+1}, \chi(x_i))}.$$

Remark 3.8. *In the above definition, κ_i is a formal series. Indeed, writing $F(x,\chi(y)) = (x-y)g(x,y)$ with g a formal series with constant term equal to 1, we get*

$$\kappa_i = \frac{g(y,x) - g(x,y)}{(x-y)g(x,y)g(y,x)}$$

Since the numerator vanishes for $x = y$ there exists a formal series h such that $g(y,x) - g(x,y) = (x-y)h(x,y)$ and we get

$$\kappa_i = \frac{h(x,y)}{g(x,y)g(y,x)}$$

which can be written as a formal series.

Remark 3.9. *An easy computation shows that $\Delta_i = \kappa_i - C_i$.*

Example 3.10. *The three formal group laws we have studied so far are F_a, F_m and F_e, namely the additive, the multiplicative and the elliptic (or hyperbolic) formal group laws:*

$$F_a(x,y) = x + y, \quad F_m(x,y) = x + y - \mu_1 xy \quad \text{and} \quad F_e(x,y) = \frac{x + y - \mu_1 xy}{1 + \mu_2 xy}.$$

In these cases, we have $\kappa_i^{F_a} = 0$, $\kappa_i^{F_m} = \kappa_i^{F_e} = \mu_1$. So in all these examples, $\kappa := \kappa_i$ is independent of i.

We now define a variant of the Hecke algebra generalizing [11, Definition 2.2] with respect to a fixed hyperbolic formal group law F. Setting $\mu_2 = 0$, we obtain the Hecke algebra of [11], corresponding to (connective or periodic) K-theory.

Definition 3.11. *For the hyperbolic formal group law F defined over $R = \mathbb{Z}[\mu_1, \mu_2]$ consider the commutative ring $\mathcal{R} := R[[x_1, \ldots, x_n]]$. The generalized Hecke algebra $\mathcal{A}_n(\kappa)$ is the quotient of the associative $\mathcal{R}$-algebra $\mathcal{R}\langle u_1, \ldots, u_{n-1}\rangle$ by the relations*

- $u_i x_j = x_j u_i$ *for all* i, j,
- $u_i u_j = u_j u_i$ *for* $|i-j| > 1$,
- $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ *for all* i,
- $u_i^2 = -\mu_1 u_i$ *for all* i,
- $\mu_2 x_i x_{i+1} u_i = 0$ *for all* i.

Although this algebra generalizes the ones of [11], [4] and others, note that it is different from the formal *Demazure* algebras studied in [7], [20]. See Remark 3.18 below for more details on this.

Remark 3.12. *Note that the elements* u_i *satisfy the braid relations. Hence for any permutation* w*, we can define the element* u_w *as* $u_w = u_{i_1} \ldots u_{i_r}$*, where* $w = s_{i_1} \ldots s_{s_{i_r}}$ *is any reduced expression of* w.

We now generalize [11, Theorem 2.3] from multiplicative to hyperbolic formal group laws. Define

$$\mathfrak{S}(x_1, \ldots, x_{n-1}) = \prod_{j=1}^{n-1} \prod_{i=n-1}^{j} (1 + x_j u_i),$$

where the interchanged bounds for i mean that the corresponding factors are multiplied in descending order, starting with $i = n-1$.

Theorem 3.13. *For any hyperbolic FGL, in the generalized Hecke algebra* $\mathcal{A}_n(\kappa)$ *of Definition 3.11, we have*

$$\mathfrak{S}(x_1, \ldots, x_{n-1}) = \sum_{w \in \Sigma_n} \mathfrak{L}_{\underline{w}} u_{w_0 w},$$

where $\underline{w}$ *is any reduced expression of* w *and* $w_0(i) = n + 1 - i$ *as usual.*

Before proving this theorem, we compare the generalized Schubert polynomials $\mathfrak{L}_{\underline{w}}$ with the corresponding Grothendieck polynomials for K-theory.

Definition 3.14. *Let* w *be a permutation and* $w = s_{i_1} \ldots s_{s_{i_r}}$ *be any reduced expression.*

1. The support of w *is the set* $\mathrm{Supp}(w) = \{i_1, \ldots, i_r\}$*. This is independent of the chosen reduced expression since its is preserved by the braid relations.*

2. Define $I(w)$ *as the ideal in* $\mathcal{R}$ *generated by the polynomials* $\mu_2 x_i x_{i+1}$ *for* $i \in \mathrm{Supp}(w_0 w)$.

3. Let $\mathfrak{L}_w^K$ *be the* K*-theoretic Grothendieck polynomial representing* X_w.

Corollary 3.15. *Let* $\underline{w} = s_{\alpha_{i_1}} \ldots s_{\alpha_{i_r}}$ *be a reduced expression of* w*. Then for* w *a permutation and* $\underline{w}$ *be any reduced expression for* w*, in* $\mathcal{R}$ *we have*

$$\mathfrak{L}_{\underline{w}} = \mathfrak{L}_w^K \ mod\ I(w).$$

Some parts of the proof of Theorem [11, Theorem 2.3] are formal and immediately generalize to arbitrary formal group laws. Lemma 2.5 of [11] just rephrases Remark 3.9. Several other crucial parts of the proof do not generalize to arbitrary FGLs. However, they do generalize to hyperbolic FGLs when working with the generalized Hecke algebra $\mathcal{A}_n(\kappa)$. An important point in choosing hyperbolic FGL is the fact that the κ_i are independent of i, so we have an action of the symmetric group on $\mathcal{A}_n(\kappa)$ given by permutations on the variables x_i. From now on, we fix a hyperbolic formal group law F and a positive integer n.

Lemma 3.16. *Set $\alpha_i(x) = (1+xu_{n-1})\cdots(1+xu_i)$. Then we have the following equalities in $\mathcal{A}_n(\kappa)$.*

1. $\alpha_{i+1}(x_{i+1}) = \alpha_i(x_{i+1})(1+\chi(x_{i+1})u_i)$;
2. $1+\chi(x_i)u_i = (1+F(x_{i+1},\chi(x_i))u_i)(1+\chi(x_{i+1})u_i)$;
3. $\Delta_i(1+\chi(x_{i+1})u_i) = -(1+\chi(x_{i+1})u_i)u_i$.

Proof. 1. The equality $\alpha_{i+1}(x_{i+1})(1+x_{i+1}u_i) = \alpha_i(x_{i+1})$ implies $\alpha_{i+1}(x_{i+1})(1+x_{i+1}u_i)(1+\chi(x_{i+1})u_i) = \alpha_i(x_{i+1})(1+\chi(x_{i+1})u_i)$. A straightforward computation shows that $(1+x_{i+1}u_i)(1+\chi(x_{i+1})u_i) = 1$.

2. To prove the claim, it suffices to prove that

$$(F(x_{i+1},\chi(x_i)) + \chi(x_{i+1}) - \chi(x_i))u_i + \chi(x_{i+1})F(x_{i+1},\chi(x_i))u_i^2 = 0,$$

or equivalently that

$$(F(x_{i+1},\chi(x_i)) + \chi(x_{i+1}) - \chi(x_i) - \mu_1\chi(x_{i+1})F(x_{i+1},\chi(x_i)))u_i = 0$$

This holds by a computation using the explicit formulas for F and χ and the relation $\mu_2 x_i x_{i+1}(x_i - x_{i+1})u_i = 0$. We use the stronger relation $\mu_2 x_i x_{i+1} u_i = 0$ in the definition of our Hecke algebra since we need $x_i - x_{i+1}$ to be a non zero divisor for the next computation.

3. We have

$$\begin{aligned} -\Delta_i(1+\chi(x_{i+1})u_i) &= \frac{(1+\chi(x_i)u_i) - (1+\chi(x_{i+1})u_i)}{F(x_{i+1},\chi(x_i))} \\ &= \frac{1+F(x_{i+1},\chi(x_i))u_i - 1}{F(x_{i+1},\chi(x_i))}(1+\chi(x_{i+1})u_i) \\ &= (1+\chi(x_{i+1})u_i)u_i. \end{aligned}$$

The second equality follows from part 2. □

Proposition 3.17. *In the above notation, for all i we have the commutation*

$$\alpha_i(x_i)\alpha_i(x_{i+1}) = \alpha_i(x_{i+1})\alpha_i(x_i).$$

Proof. Since we have the same relations for the u_i as in [11], the proof of their Lemma 2.6 generalises to our situation. More precisely, we may apply

[12, Corollary 5.4] as its assumptions (see [12, Section 2]) are satisfied in our generalized Hecke algebra. □

Proof of Theorem 3.13. From $\mathfrak{S}(x_1, \dots, x_{n-1}) = \alpha_1(x_1) \dots \alpha_{n-1}(x_{n-1})$ we get

$$\mathfrak{S}(x_1, \dots, x_{n-1}) = \alpha_1(x_1) \dots \alpha_i(x_{i+1})(1 + \chi(x_{i+1})u_i)\alpha_{i+2}(x_{i+2}) \dots \alpha_{n-1}(x_{n-1}).$$

Using Lemma 3.16.1, this implies that $\Delta_i(\mathfrak{S}(x_1, \dots, x_{n-1}))$ is equal to the following formulas:

$$\begin{aligned}
&\alpha_1(x_1) \dots \alpha_{i-1}(x_{i-1})\Delta_i\alpha_i(x_i)\alpha_i(x_{i+1})(1 + \chi(x_{i+1})u_i)\alpha_{i+2}(x_{i+2}) \dots \alpha_{n-1}(x_{n-1}) \\
&= \alpha_1(x_1) \dots \alpha_{i-1}(x_{i-1})\alpha_i(x_i)\alpha_i(x_{i+1})\Delta_i(1 + \chi(x_{i+1})u_i)\alpha_{i+2}(x_{i+2}) \dots \alpha_{n-1}(x_{n-1}) \\
&= -\alpha_1(x_1) \dots \alpha_i(x_i)\alpha_i(x_{i+1})(1 + \chi(x_{i+1})u_i)u_i\alpha_{i+2}(x_{i+2}) \dots \alpha_{n-1}(x_{n-1}) \\
&= -\alpha_1(x_1) \dots \alpha_i(x_i)\alpha_i(x_{i+1})(1 + \chi(x_{i+1})u_i)\alpha_{i+2}(x_{i+2}) \dots \alpha_{n-1}(x_{n-1})u_i.
\end{aligned}$$

Here the first equality follows from Proposition 3.17 and the fact that, as an operator, Δ_i commutes with the operator given by multiplication with a polynomial which is symmetric in x_i and x_{i+1}. The second equality follows from Lemma 3.16.3. We thus have shown

$$-\Delta_i(\mathfrak{S}(x_1, \dots, x_{n-1}) = (\mathfrak{S}(x_1, \dots, x_{n-1}))u_i$$

which corresponds precisely to the induction step in Definition 3.3, using that $\Delta_i = \kappa - C_i$ and $u_i^2 = -\kappa u_i$. More precisely, write $\mathfrak{S} = \sum \hat{\mathfrak{L}}_w u_{w_0 w}$, where the sum taken over all $w \in \Sigma_n$. We wish to show that $\hat{\mathfrak{L}}_w u_{w_0 w} = \mathfrak{L}_{\underline{w}} u_{w_0 w}$ by an ascending induction on the length of w. For $w = 1$ the claim is obviously true. Now fix $w \neq 1$ and choose i such that ws_i is reduced. Consider the coefficient of $u_{w_0 w}$ in

$$(C_i - \kappa_i)\mathfrak{S} = -\Delta_i\mathfrak{S} = \mathfrak{S}u_i.$$

Using that $u_i^2 = -\kappa_i u_i$ and the fact that $w_0 w s_i < w_0 w$, we deduce that

$$(C_i - \kappa_i)\hat{\mathfrak{L}}_w u_{w_0 w} = (\hat{\mathfrak{L}}_{ws_i} - \kappa_i \hat{\mathfrak{L}}_w)u_{w_0 w},$$

hence $C_i\hat{\mathfrak{L}}_w u_{w_0 w} = \hat{\mathfrak{L}}_{ws_i} u_{w_0 w}$ as required. □

Remark 3.18. *Note that the computations from [11] cannot be done in the formal Demazure algebra of [20]. E.g., the equality*

$$(1 + x_{i+1}u_i)(1 + \chi(x_{i+1})u_i) = 1$$

which was used to prove Lemma 3.16 above does not hold, not even for the additive FGL. This is related to the failure of $\kappa_i\Delta_i = \Delta_i\kappa_i$.

As for hyperbolic formal group laws κ_i is independent of i (see Example 3.10), several other parts in Buch's article [4] on the Littlewood-Richardson rule for K_0 easily generalize to hyperbolic formal group laws when working with the generalized Hecke algebra $\mathcal{A}_n(\kappa)$ of Definition 3.11. For example, similar to [4, p. 41], it is possible to introduce a stable generalized Schubert polynomial *colim* $\mathfrak{L}_{1^m\times\underline{w}}$ of $\mathfrak{L}_{\underline{w}}$ and to try to analyze its behaviour along the lines of [12, Section 6]. Also, there is a well-defined analogon $\mathfrak{L}_{\nu/\lambda}$ of the polynomial $G_{\nu/\lambda}$ which is crucial for [4, Theorem 3.1], as the construction in [4, p. 41/42] provides a reduced word $\underline{w}$ rather than just a permutation w. However, for hyperbolic formal group laws the operators C_i do no longer satisfy the classical braid relation but a twisted version of it, namely $C_iC_{i+1}C_i+\mu_2C_i = C_{i+1}C_iC_{i+1}+\mu_2C_{i+1}$ [20]. This will lead to additional difficulties when arguing inductively using these C_i and the corresponding geometric operators as e.g. in [4, Section 8]. This is also related to the discussion in [27, Section 6]. On the other hand, Proposition 3.17 is wrong already for small values of n and i when replacing the classical braid relation for the u_i by its twisted analog in the definition of $\mathcal{A}_n(\kappa)$. We hope to return to these questions in future work.

4 Some examples

4.1 Polynomials representing some smooth Schubert varieties

We first compute generalized Schubert polynomials for some of the smooth Schubert varieties considered in Section 2. Let $X = \mathrm{Gr}(k, n)$ be a Grassmannian and let λ be a partition of the form b^a. Denote by $\mathfrak{G}_\lambda$ the polynomial in $\Omega^*(G/B) \simeq \mathbb{L}[x_1, \cdots, x_n]/S$ representing the pull-back along the canonical quotient map $\pi : G/B \to X$ of the cobordism class $[X_\lambda \to X]$. Recall [19, Section 3.2.4] that the induced map $\pi^* : \Omega^*(\mathrm{Gr}(k, n)) \to \Omega^*(G/B)$ is a ring monomorphism which identifies $\Omega^*(\mathrm{Gr}(k, n))$ with an explicit subring of $\mathbb{L}[x_1, \cdots, x_n]/S$. The results in the sequel may thus be stated in either of these rings. (Recall that there is a standard map, see e.g. [4, p.42], from partitions to permutations with corresponds to π^* and geometric operators for K-theory and Chow groups.)

Lemma 4.1. *In $\Omega^*(X)$, we have $[X_{(n-k)^{k-1}}]^a = [X_{(n-k)^{k-a}}]$ and $[X_{(n-k-1)^k}]^b = [X_{(n-k-b)^k}]$.*

Proof. We need to prove the formula $[X_{(n-k)^{k-1}}] \cdot [X_{(n-k)^{k-a}}] = [X_{(n-k)^{k-a-1}}]$. But the first class is represented by the sub-Grassmannian $X_{n-k} = \{V_k \in X \mid E_1 \subset V_k\}$ while the second class is represented by $X^{(n-k)^{k-a\vee}} = X^{(n-k)^a} = \{V_k \in X \mid E^a \subset V_k\}$. The product is represented by the intersection of these varieties and since E_1 and E^a do not meet we get

$$X_{n-k} \cap X^{(n-k)^a} = \{V_k \in X \mid E_1 \oplus E^a \subset V_k\}.$$

This last variety is a $\mathrm{GL}_n(\mathsf{k})$-translate of $X_{(n-k)^{k-a-1}} = \{V_k \in X \mid E_{a+1} \subset V_k\}$ proving the first formula. The second one is obtained along the same lines or deduced from the first one using the isomorphism $\mathrm{Gr}(k, n) \simeq \mathrm{Gr}(n - k, n)$. □

Proposition 4.2. *In $\Omega^*(G/B)$, we have the formulas*

$$\mathfrak{G}_{(n-k)^a} = (x_{k+1} \cdots x_n)^{k-a} \text{ and } \mathfrak{G}_{b^k} = (x_1 \cdots x_k)^{n-k-b}.$$

Proof. By the previous lemma, we only need to compute the class $[X_{(n-k)^{k-1}}]$ in $\Omega^*(X)$. Since $X_{(n-k)^{k-1}}$ is the zero locus of a section of the tautological quotient bundle whose Chern roots are $x_{k+1}, \cdots, x_n$, the first equality of the proposition follows (see for example the proof of [24, Lemma 6.6.7]). For the second formula, we just need to remark that $X_{(k-1)^k}$ is the zero locus of a global section of the dual of the tautological subbundle and apply the same method (or use the isomorphism $\mathrm{Gr}(k,n) \simeq \mathrm{Gr}(n-k,n)$ again). □

Corollary 4.3. *The classes of* $[X_{(n-k)^a} \to X]$ *and* $[X_{b^k} \to X]$ *are represented by the same polynomial in any oriented cohomology theory.*

Proof. Indeed we have $[X_{(n-k)^a} \to X] = \mathfrak{G}_{(n-k)^a}$ and $[X_{b^k} \to X] = \mathfrak{G}_{b^k}$, so this is independent of the FGL. □

Remark 4.4. *We will see in the next subsection that this is no longer the case for the other classes of smooth Schubert varieties. Indeed, in Proposition 4.5, we prove that the class of the line in the elliptic cohomology of* $\mathrm{Gr}(2,4)$ *is given by* $x_1x_2(x_1+x_2) - \mu_1 x_1^2 x_2^2$ *and therefore depends on the FGL.*

4.2 Elliptic cohomology of $\mathrm{Gr}(2,4)$

In this subsection, we present explicit results concerning elliptic cohomology, *i.e* for the hyperbolic FGL, of $\mathrm{Gr}(2,4)$. We compute the polynomial representatives for all Bott-Samelson classes as well as their products.

Let $X = \mathrm{Gr}(2,4)$ and let λ be a partition. Denote by $\mathfrak{L}_\lambda$ the polynomial in $\Omega^*(G/B) \simeq \mathbb{L}[x_1,x_2,x_3,x_4]/S$ representing the pull-back along the map $G/B \to X$ of the cobordism class $[\widetilde{X}_\lambda \to X]$ where $\widetilde{X}_\lambda$ is the Bott-Samelson resolution of X_λ.

Recall the hyperbolic FGL of [6, Example 63] as in Subsection 3.3. above. By the universal property of the formal group law of Ω^* established in [24], we have a unique morphism of formal group laws, which yields in particular a ring morphism $\mathbb{L} \to \mathbb{Z}[\mu_1,\mu_2]$. This map is called "Krichever genus" and studied in detail in loc. cit.. In particular, μ_i has cohomological degree $-i$ for $i = 1,2$. Note that (unlike in the bigraded case, see e.g. [26]) this always yields an oriented cohomology theory, as there is no Landweber exactness condition to check. As the theory $E^*(-)$ is oriented in the sense of [24], the analogs of the above theorems also hold for $E^*(G/B)$ and $E^*(\mathrm{Gr}(2,4))$, and the natural transformation $\Omega^*(-) \to E^*(-)$ commutes in particular with the ring monomorphisms π^*. Below, we use the notations $\widetilde{X}_\lambda$ and $\mathfrak{L}_\lambda$ for elements in $E^*(-)$ as well.

Proposition 4.5. *In* $E^*(\mathrm{Gr}(2,4))$*, we have the following formulas:*

$$\begin{aligned}
\mathfrak{L}_{(00)} &= x_1^2 x_2^2,\\
\mathfrak{L}_{(10)} &= x_1x_2(x_1+x_2) - \mu_1 x_1^2x_2^2,\\
\mathfrak{L}_{(20)} &= x_1^2 + x_1x_2 + x_2^2 - \mu_1 x_1x_2(x_1+x_2) - \mu_2 x_1^2x_2^2,\\
\mathfrak{L}_{(11)} &= x_1x_2 - \mu_2 x_1^2x_2^2,\\
\mathfrak{L}_{(21)} &= x_1 + x_2 - \mu_1 x_1x_2 - \mu_2 x_1x_2(x_1+x_2) - \mu_1\mu_2 x_1^2x_2^2,\\
\mathfrak{L}_{(22)} &= 1 - \mu_2(x_1+x_2)^2 + \mu_1^2\mu_2 x_1^2x_2^2.
\end{aligned}$$

Proof. Since the fiber of the map $\pi : G/B \to \mathrm{Gr}(2,4)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, the pull-back $\pi^*[\widetilde{X}_\lambda] \in E^*(G/B)$ of a Bott-Samelson class in $\mathrm{Gr}(2,4)$ is again a Bott-Samelson class $X_{\underline{w}}$. (Note that this is not true anymore in higher dimensions.) Moreover in this case, we can explicitely write down the reduced word $\underline{w}$ corresponding to λ under π^*. Now we wish to compute $\mathfrak{L}_\lambda \in E^*(\mathrm{Gr}(2,4)) \subset E^*(G/B)$. The above together with the results of [16] implies that both in $\Omega^*(G/B)$ and $E^*(G/B)$, we have $\pi^*[X_{(00)}] = C_1C_3(\mathfrak{L}_1)$, $\pi^*[X_{(10)}] = C_1C_3C_2(\mathfrak{L}_1)$, $\pi^*[X_{(20)}] = C_1C_3C_2C_3(\mathfrak{L}_1)$, $\pi^*[X_{(11)}] = C_1C_3C_2C_1(\mathfrak{L}_1)$, $\pi^*[X_{(21)}] = C_1C_3C_2C_1C_3(\mathfrak{L}_1)$ and $\pi^*[X_{(22)}] = C_1C_3C_2C_1C_3C_2(\mathfrak{L}_1)$. Now the results follow from $\mathfrak{L}_1 = x_1^3x_2^2x_3$ and explicit computations with the C_i done with the help of a computer. □

We computed everything in elliptic cohomology for sake of simplicity, but a similar computation can be done in $\Omega^*(X)$.

Remark 4.6. *In elliptic cohomology, the mutiplication formula for the square of the hyperplane class in the Bott-Samelson basis is the same as the one in K-theory, namely* $\mathfrak{L}_{(21)}^2 = \mathfrak{L}_{(20)} + \mathfrak{L}_{(11)} - \mu_1\mathfrak{L}_{(10)}$.

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