SMOOTH REPRESENTATIONS OF $\mathrm{GL}_m(\mathrm{D})$ V: ENDO-CLASSES

by

P. Broussous, V. Sécherre & S. Stevens

In memory of Martin Grabitz

Abstract. — Let F be a locally compact nonarchimedean local field. In this article, we extend to any inner form of GL_n over F, with $n \geqslant 1$, the notion of endo-class introduced by Bushnell and Henniart for $\mathrm{GL}_n(\mathrm{F})$. We investigate the intertwining relations of simple characters of these groups, in particular their preservation properties under transfer. This allows us to associate to any discrete series representation of an inner form of $\mathrm{GL}_n(\mathrm{F})$ an endo-class over F. We conjecture that this endo-class is invariant under the local Jacquet-Langlands correspondence.

Résumé. — Soit F un corps commutatif localement compact non archimédien. Dans cet article, nous étendons à toutes les formes intérieures de GL_n sur F, avec $n \geqslant 1$, la notion d'endo-classe introduite par Bushnell et Henniart pour $\mathrm{GL}_n(\mathrm{F})$. Nous étudions les propriétés des caractères simples de ces groupes vis-à-vis de l'entrelacement, et établissons en particulier la permanence de ces propriétés par transfert. Ceci nous permet d'associer à toute représentation irréductible de la série discrète d'une forme intérieure de $\mathrm{GL}_n(\mathrm{F})$ une endo-classe sur F. Nous conjecturons que cette endo-classe est invariante par la correspondance de Jacquet-Langlands locale.

Contents

This work was supported by: EPSRC (grant GR/T21714/01), the British Council and the Partenariat Hubert Curien in the framework of the Alliance programme (number 19418YK), the Agence Nationale de la Recherche (ANR-08-BLAN-0259-01) and the Université de la Méditerranée Aix-Marseille 2.

Introduction

This is the fifth in a series of articles whose objective is a complete description of the category of smooth complex representations of $\mathrm{GL}_r(\mathrm{D})$, with r a positive integer and D a division algebra over a locally compact nonarchimedean local field. The longer term aim is an explicit description, in terms of types, of the local Jacquet-Langlands correspondence [**14, 1**], as begun by Bushnell and Henniart [**18, 7, 9**], and by Silberger and Zink [**26, 27**].

The main object of study in this paper is the notion of endo-equivalence class, or *endo-class*, of simple characters. This notion has been introduced by Bushnell and Henniart [**6**] for the group $\mathrm{GL}_n(\mathrm{F})$, with n a positive integer and F a locally compact nonarchimedean local field: an endo-class is an invariant associated to an irreducible cuspidal representation of $\mathrm{GL}_n(\mathrm{F})$, constructed by explicit methods related to the description of this representation as compactly induced from an irreducible representation of a compact-mod-centre subgroup of $\mathrm{GL}_n(\mathrm{F})$ (see [**10, 6**]). The arithmetic significance of this invariant has been described in [**8**], in the case where F is of characteristic zero: if we denote by $\mathscr{W}_{\mathrm{F}}$ the Weil group of F (relative to an algebraic closure) and by $\mathscr{P}_{\mathrm{F}}$ its wild inertia subgroup, there is a bijection between the set $\mathcal{E}(\mathrm{F})$ of endo-classes over F and the set of $\mathscr{W}_{\mathrm{F}}$-conjugacy classes of irreducible representations of $\mathscr{P}_{\mathrm{F}}$, which is compatible with the local Langlands correspondence.

In this article, we extend the notion of endo-class to any inner form of $\mathrm{GL}_n(\mathrm{F})$, $n \geqslant 1$, that is, to any group of the form $\mathrm{GL}_r(\mathrm{D})$, with r a positive integer and D an F-central division algebra of dimension d^2 over F, with $n = rd$. For this we develop a *Shintani lift*, or *base change*, for simple characters, which is also of independent interest (see below). If G is an inner form of $\mathrm{H} = \mathrm{GL}_n(\mathrm{F})$, and if $\mathcal{D}(\mathrm{G})$ denotes the discrete series of G (that is, the set of isomorphism classes of essentially square-integrable irreducible representations of G), we define a map:

$$\mathbf{\Theta}_{\mathrm{G}} : \mathcal{D}(\mathrm{G}) \to \mathcal{E}(\mathrm{F})$$

(see paragraph 9.2) which associates an endo-class over F to any discrete series representation of G. This map should play an important role in an explicit description of the local Jacquet–Langlands correspondence:

$$\mathbf{JL} : \mathcal{D}(\mathrm{G}) \to \mathcal{D}(\mathrm{H}).$$

In particular, we expect that **JL** preserves the endo-class (see Conjecture 9.5), that is:

$$\boldsymbol{\Theta}_{\mathrm{H}} \circ \mathbf{JL} = \boldsymbol{\Theta}_{\mathrm{G}}.$$

This conjectural property can be seen as a generalization of the fact that the correspondence **JL** preserves the representations of level zero (see [**26**]). The notion of endo-class also plays a central role in:

– the construction of semisimple types, which leads to a complete description of the structure of the category of smooth complex representations of G (see [**25**]);

– the study of smooth representations of G with coefficients in a field of non-zero characteristic different from the residue characteristic of F (see [**19**]).

Before giving more details, let us mention that there are roughly speaking two main obstacles to overcome: First, one has to compare simple characters in $\mathrm{GL}_r(\mathrm{D})$ with simple characters in $\mathrm{GL}_{r'}(\mathrm{D}')$ where $\mathrm{GL}_r(\mathrm{D})$ and $\mathrm{GL}_{r'}(\mathrm{D}')$ are two inner forms of $\mathrm{GL}_n(\mathrm{F})$ with D and D′ not necessarily isomorphic. It is to overcome this that we need to develop a Shintani lift, or base change, for simple characters. This process is of independent interest and may be used to define a Shintani lift for irreducible representations of $\mathrm{GL}_r(\mathrm{D})$. The second problem is due to the notion of *embedding type*, a phenomenon first discovered by Fröhlich [**15**]; this problem, and its resolution, will be discussed in more detail below.

One of the objectives of [**20**], completed in [**24**], is the construction of simple characters, which are certain special characters of particular compact open subgroups of G. These simple characters are attached to data called *simple strata*, and are a fundamental part of the construction of more elaborate objects called *simple types* (see [**21, 22**]). One knows from [**22, 24**] that every irreducible discrete series representation π of G contains a simple character θ attached to a simple stratum. Neither the simple stratum nor the simple character are unique, but every other simple character θ' contained in π intertwines θ, that is, there is an element $g \in \mathrm{G}$ such that θ' and the conjugate character θ^g coincide on the intersection of the compact open subgroups where they are defined. It is this observation which leads to the notion of endo-class.

An endo-class is an equivalence class of objects called *potential simple characters* (or *ps-characters* for short), for a relation called *endo-equivalence*. A ps-character Θ is characterized by giving a simple stratum $[\Lambda, n, m, \beta]$ in an F-central simple algebra A and a simple character θ attached to this simple stratum. The pair $([\Lambda, n, m, \beta], \theta)$ is called a *realization* of Θ. Another simple stratum $[\Lambda', n', m', \beta]$ in another F-central simple algebra A′ (note that β is unchanged) and a simple character θ' for this stratum define the same ps-character precisely when θ and θ' are linked by the transfer map defined in [**20**] (see paragraph 1.2 below). Two ps-characters Θ_1 and Θ_2 are said to be endo-equivalent (see Definition 1.10) if they can be characterized by giving realizations $([\Lambda, n_i, m_i, \beta_i], \theta_i)$ in an F-central simple algebra A, for $i = 1, 2$ (note that A and Λ do not depend on i), of the same degree and normalized level, and such that the simple characters θ_1 and θ_2 intertwine in $\mathrm{A}^\times$.

The properties of endo-equivalence depend on important intertwining properties of simple characters, notably the preservation of these properties under the transfer map. This article centres on two important technical results: the property of "preservation of intertwining" (Theorem 1.11) and the "intertwining implies conjugacy" property (Theorem 1.12). Partial results on these questions were already given by Grabitz [**17**], notably a proof of "intertwining implies conjugacy", but these results are proved under unnecessarily restrictive hypotheses: that the simple strata underlying the construction are *sound* in the sense of Definition 1.14. We have sought to develop the notion of endo-class in as general a situation as possible, emphasizing the functorial properties of the objects involved. However, rather than starting again from scratch, we decided to use the work of Grabitz as much as possible. We note that, as well as [**17**], our proofs rely heavily on the results of Bushnell, Henniart and Kutzko [**10, 6**] in the split case.

Let us now describe in more detail the results, and the techniques used, in this article. For $i = 1, 2$, let Θ_i be a ps-character defined by a simple stratum $[\Lambda, n_i, m_i, \beta_i]$ in an F-central simple algebra A and a simple character $\theta_i \in \mathcal{C}(\Lambda, m_i, \beta_i)$ attached to this stratum (see paragraph 1.1 for the notation). Suppose from now on that the ps-characters Θ_1 and Θ_2 are endo-equivalent so that, in particular, we may assume the characters θ_1 and θ_2 intertwine in $\mathrm{A}^\times$. The "preservation of intertwining" property can be stated as follows:

***Theorem* (see Theorem 1.11).** — *For $i = 1, 2$, let $[\Lambda', n'_i, m'_i, \beta_i]$ be a simple stratum in a simple central* F*-algebra* A′ *and $\theta'_i \in \mathcal{C}(\Lambda', m'_i, \beta_i)$ defining the ps-character Θ_i, that is, θ'_i is the transfer of θ_i. Then the characters θ'_1 and θ'_2 intertwine in $\mathrm{A}'^\times$.*

This means that the property that two simple characters intertwine is invariant under transfer. The statement above is the same as its analogue [**6**, Theorem 8.7] in the case that A is split and Λ is strict. However, we will see that the proof requires new ideas.

One of the important results in [**10**] is the "intertwining implies conjugacy" property for simple characters, which expresses the fact that intertwining of simple characters is a very stringent relation. It is this property which allows a classification "up to conjugacy" of the irreducible cuspidal representations of $\mathrm{GL}_n(\mathrm{F})$. This property no longer holds in the general case, as was already observed in [**5**] for simple strata. To remedy the situation, we introduce the notion of *embedding type* of a simple stratum (see Definition 1.8): two simple strata $[\Lambda, n_i, m_i, \beta_i]$ have the same embedding type if the maximal unramified sub-extensions of $\mathrm{F}(\beta_i)/\mathrm{F}$ are conjugate under the normalizer of Λ in $\mathrm{A}^\times$. With the same notation and hypotheses as above, we prove the following:

***Theorem* (see Theorem 1.12).** — *Suppose that $n_1 = n_2$, $m_1 = m_2$, and the simple strata $[\Lambda, n_i, m_i, \beta_i]$ have the same embedding type. Write K_i for the maximal unramified extension of F contained in $\mathrm{F}(\beta_i) \subseteq \mathrm{A}$. Then there is an element of the normalizer of Λ in $\mathrm{A}^\times$ which simultaneously conjugates K_1 to K_2 and θ_1 to θ_2.*

This result was proved by Grabitz [**17**, Corollary 10.15] with the additional assumption that the simple strata $[\Lambda, n, m, \beta_i]$ are sound. We prove it here without this hypothesis.

Once one has proved that endo-equivalence preserves certain numerical invariants (see Lemma 4.7), it is not hard to see that the proofs of these two Theorems can be reduced to the following:

***Theorem* (see Theorem 1.13).** — *For $i = 1, 2$, let $[\Lambda', n', m', \beta_i]$ be a simple stratum in a simple central F-algebra A' and $\theta'_i \in \mathcal{C}(\Lambda', m', \beta_i)$ defining the ps-character Θ_i, that is, θ'_i is the transfer of θ_i. Assume the simple strata have the same embedding type and write K_i for the maximal unramified extension of F contained in $\mathrm{F}(\beta_i) \subseteq \mathrm{A}'$. Then there is an element of the normalizer of Λ' in $\mathrm{A}'^\times$ which simultaneously conjugates K_1 to K_2 and θ'_1 to θ'_2.*

Now let us describe the scheme of the proof. We begin with our endo-equivalent ps-characters Θ_1 and Θ_2, together with realizations $([\Lambda, n_i, m_i, \beta_i], \theta_i)$ in A such that the simple characters θ_1 and θ_2 intertwine in $\mathrm{A}^\times$. In order to use the results of Grabitz, we need first to produce sound realizations of the ps-characters Θ_i with the same embedding type, which intertwine. For sound strata, the embedding type is determined by a single integer, the *Fröhlich invariant*, which can also be defined for arbitrary strata (see Definition 4.1).

One can then realize Θ_i on the lattice sequence $\Lambda \oplus \Lambda$ in such a way that the Fröhlich invariant is 1 and the simple characters still intertwine (see Lemma 4.4). In particular, replacing our original realizations of Θ_1 and Θ_2 with these new ones, we can assume the simple strata $[\Lambda, n_i, m_i, \beta_i]$ have the same Fröhlich invariant. Now we define a process:

$$([\Lambda, n, m, \beta], \theta) \mapsto ([\Lambda^{\ddagger}, n, m, \beta], \theta^{\ddagger})$$

from arbitrary realizations to sound realizations, with $\theta^{\ddagger}$ the transfer of θ, which preserves intertwining and the Fröhlich invariant (see paragraph 2.7). In particular, from θ_1 and θ_2 one obtains simple characters $\theta_1^{\ddagger}$ and $\theta_2^{\ddagger}$ on sound simple strata with the same Fröhlich invariant (so same embedding type) which intertwine. Thus we can apply Grabitz's results, together with a reduction to the case $m_1 = m_2$, to deduce that $\theta_1^{\ddagger}$ and $\theta_2^{\ddagger}$ are conjugate under $\mathrm{A}^{\ddagger\times}$ (where $\mathrm{A}^{\ddagger}$ is the simple central F-algebra with respect to which the stratum $[\Lambda^{\ddagger}, n, m, \beta]$ is defined). Changing again our realizations of Θ_1 and Θ_2 we can suppose we have an equality $\theta_1 = \theta_2$ of simple characters. This is given in Proposition 4.9, the culmination of the first stage of the proof.

To show that other realizations θ_1' and θ_2' on simple strata in A' with the same embedding type are conjugate, we would like to reduce to the split case so that we can use results from [**10, 6**]. For this we define an *interior lifting* (see section 5):

$$([\Lambda, n, m, \beta], \theta) \mapsto ([\Gamma, n, m, \beta], \theta^{\mathrm{K}})$$

relative to K/F, the maximal unramified subextension of $\mathrm{F}(\beta)/\mathrm{F}$, where $[\Gamma, n, m, \beta]$ is a simple stratum in the centralizer C of K in the simple central F-algebra A with respect to which $[\Lambda, n, m, \beta]$ is defined. Then we make a *base change* (see section 7):

$$([\Gamma, n, m, \beta], \theta^{\mathrm{K}}) \mapsto ([\overline{\Gamma}, n, m, \beta], \overline{\theta}^{\mathrm{K}})$$

relative to L/K, a finite unramified extension which is sufficiently large so that the algebra $\mathrm{C} \otimes_{\mathrm{K}} \mathrm{L}$ is split. The definition of the base change used here is somewhat subtle: indeed, it is not clear how to make a good definition which will preserve intertwining and, when applied to our characters θ_i, will be independent of i. Moreover, it is necessary to begin with the interior lift or else the base change process would produce *quasi-simple* characters (see [**20**]), rather than simple characters.

In order to apply these processes, we note that the maximal unramified subextension K of $\mathrm{F}(\beta_i)/\mathrm{F}$ in A can be assumed to be independent of i since the simple strata have the same embedding type. Combining now interior lifting and base change, we get a process:

$$([\Lambda, n, m, \beta], \theta) \mapsto ([\overline{\Gamma}, n, m, \beta], \overline{\theta}^{\mathrm{K}})$$

denoted here $\theta \mapsto \widetilde{\theta}$ for simplicity, which is both injective and equivariant, so it is enough to show that $\widetilde{\theta}'_1$ and $\widetilde{\theta}'_2$ are conjugate under $\mathrm{A}'^{\times}$. Now the hypothesis $\theta_1 = \theta_2$ implies $\widetilde{\theta}_1 = \widetilde{\theta}_2$ (see Propositions 6.11 and 7.5), so that the ps-characters $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_2$ defined by $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ are endo-equivalent. Moreover, for each i, the simple character $\widetilde{\theta}'_i$ is the transfer of $\widetilde{\theta}_i$ (see Theorem 6.7), so it is another realization of the ps-character $\widetilde{\Theta}_i$. We are now in the split case so, modulo a finesse in the case that we do not have strict lattice sequences, we deduce from endo-equivalence [**6**] that the characters $\widetilde{\theta}'_i$ intertwine. Thus, from the "intertwining implies conjugacy" property [**10**], the characters $\widetilde{\theta}'_1$ and $\widetilde{\theta}'_2$ are conjugate under $(\mathrm{C}' \otimes_{\mathrm{K}} \mathrm{L})^{\times}$, where C' denotes the centralizer of K in A'. Thanks to the invariance property of the base change under the action of the Galois group Gal(L/K) (see Proposition 7.7), a cohomological argument (see Lemma 8.1) allows us to show that they are actually conjugate under $\mathrm{C}'^{\times}$. This completes the proof.

Notation

Let F be a nonarchimedean locally compact field. All F-algebras are supposed to be finite-dimensional with a unit. By an F-*division algebra* we mean a central F-algebra which is a division algebra.

For K a finite extension of F, or more generally a division algebra over a finite extension of F, we denote by $\mathcal{O}_{\mathrm{K}}$ its ring of integers, by $\mathfrak{p}_{\mathrm{K}}$ the maximal ideal of $\mathcal{O}_{\mathrm{K}}$ and by $\mathfrak{k}_{\mathrm{K}}$ its residue field.

For A a simple central algebra over a finite extension K of F, we denote by $\mathrm{N}_{\mathrm{A/K}}$ and $\mathrm{tr}_{\mathrm{A/K}}$ respectively the reduced norm and trace of A over K.

For u a real number, we denote by $\lceil u \rceil$ the smallest integer which is greater than or equal to u, and by $\lfloor u \rfloor$ the greatest integer which is smaller than or equal to u, that is, its integer part.

A *character* of a topological group G is a continuous homomorphism from G to the group $\mathbb{C}^{\times}$ of non-zero complex numbers.

All representations are supposed to be smooth with complex coefficients.

1. Statement of the main results

In this section, we recall some well known facts about lattice sequences, simple strata and simple characters in a simple central F-algebra (see [**4, 10, 12, 20, 24**] for more details), and we state the main results of this article.

1.1. Let A be a simple central F-algebra, and let V be a simple left A-module. The algebra $\mathrm{End}_{\mathrm{A}}(\mathrm{V})$ is an F-division algebra, the opposite of which we denote by D. Considering V as a right D-vector space, we have a canonical isomorphism of F-algebras between A and $\mathrm{End}_{\mathrm{D}}(\mathrm{V})$.

Definition 1.1. — An $\mathcal{O}_{\mathrm{D}}$-*lattice sequence* on V is a sequence $\Lambda = (\Lambda_k)_{k\in\mathbb{Z}}$ of $\mathcal{O}_{\mathrm{D}}$-lattices of V such that $\Lambda_k \supseteq \Lambda_{k+1}$ for all $k \in \mathbb{Z}$, and such that there exists a positive integer e satisfying $\Lambda_{k+e} = \Lambda_k\mathfrak{p}_{\mathrm{D}}$ for all $k \in \mathbb{Z}$. This integer is called the *period* of Λ over $\mathcal{O}_{\mathrm{D}}$.

If $\Lambda_k \supsetneq \Lambda_{k+1}$ for all $k \in \mathbb{Z}$, then the lattice sequence Λ is said to be *strict*.

Associated with an $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ on V, we have an $\mathcal{O}_{\mathrm{F}}$-lattice sequence on A defined by:

$$\mathfrak{P}_k(\Lambda) = \{a \in \mathrm{A} \mid a\Lambda_i \subseteq \Lambda_{i+k},\ i \in \mathbb{Z}\}, \quad k \in \mathbb{Z}.$$

The lattice $\mathfrak{A}(\Lambda) = \mathfrak{P}_0(\Lambda)$ is a hereditary $\mathcal{O}_{\mathrm{F}}$-order in A, and $\mathfrak{P}(\Lambda) = \mathfrak{P}_1(\Lambda)$ is its Jacobson radical. They depend only on the set $\{\Lambda_k \mid k \in \mathbb{Z}\}$.

We denote by $\mathfrak{K}(\Lambda)$ the $\mathrm{A}^\times$-*normalizer* of Λ, that is the subgroup of $\mathrm{A}^\times$ made of all elements $g \in \mathrm{A}^\times$ for which there is an integer $n \in \mathbb{Z}$ such that $g(\Lambda_k) = \Lambda_{k+n}$ for all $k \in \mathbb{Z}$. Given $g \in \mathfrak{K}(\Lambda)$, such an integer is unique: it is denoted $\upsilon_\Lambda(g)$ and called the Λ-*valuation* of g. This defines a group homomorphism υ_Λ from $\mathfrak{K}(\Lambda)$ to $\mathbb{Z}$. Its kernel, denoted $\mathrm{U}(\Lambda)$, is the group of invertible elements of $\mathfrak{A}(\Lambda)$. We set $\mathrm{U}_0(\Lambda) = \mathrm{U}(\Lambda)$ and, for $k \geqslant 1$, we set $\mathrm{U}_k(\Lambda) = 1 + \mathfrak{P}_k(\Lambda)$.

Let F' be a finite extension of F contained in A. An $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ on V is said to be F'-*pure* if it is normalized by $\mathrm{F}'^\times$. The centralizer of F' in A, denoted A', is a simple central F'-algebra. We fix a simple left A'-module V' and write D' for the algebra opposite to $\mathrm{End}_{\mathrm{A}'}(\mathrm{V}')$. By [**24**, Théorème 1.4] (see also [**4**, Theorem 1.3]), given an F'-pure $\mathcal{O}_{\mathrm{D}}$-lattice sequence on V, there is an $\mathcal{O}_{\mathrm{D}'}$-lattice sequence Λ' on V' such that:

$$\mathfrak{P}_k(\Lambda) \cap \mathrm{A}' = \mathfrak{P}_k(\Lambda'), \quad k \in \mathbb{Z}. \tag{1.1}$$

It is unique up to translation of indices, and its $\mathrm{A}'^\times$-normalizer is $\mathfrak{K}(\Lambda) \cap \mathrm{A}'^\times$.

Definition 1.2. — A *stratum* in A is a quadruple $[\Lambda, n, m, \beta]$ made of an $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ on V, two integers m, n such that $0 \leqslant m \leqslant n-1$ and an element $\beta \in \mathfrak{P}_{-m}(\Lambda)$.

For $i = 1, 2$, let $[\Lambda, n, m, \beta_i]$ be a stratum in A. We say these two strata are *equivalent* if $\beta_2 - \beta_1 \in \mathfrak{P}_{-m}(\Lambda)$.

Given a stratum $[\Lambda, n, m, \beta]$ in A, we denote by E the F-algebra generated by β. This stratum is said to be *pure* if E is a field, if Λ is E-pure and if $\upsilon_\Lambda(\beta) = -n$. In this situation, we denote by:

$$e_\beta(\Lambda)$$

the period of Λ as an $\mathcal{O}_\mathrm{E}$-lattice sequence. Given a pure stratum $[\Lambda, n, m, \beta]$, we denote by B the centralizer of E in A. For $k \in \mathbb{Z}$, we set:

$$\mathfrak{n}_k(\beta, \Lambda) = \{x \in \mathfrak{A}(\Lambda) \mid \beta x - x\beta \in \mathfrak{P}_k(\Lambda)\}.$$

The smallest integer $k \geqslant \upsilon_\Lambda(\beta)$ such that $\mathfrak{n}_{k+1}(\beta, \Lambda)$ is contained in $\mathfrak{A}(\Lambda) \cap \mathrm{B} + \mathfrak{P}(\Lambda)$ is called the *critical exponent* of the stratum $[\Lambda, n, m, \beta]$, denoted $k_0(\beta, \Lambda)$.

Definition 1.3. — The stratum $[\Lambda, n, m, \beta]$ is said to be *simple* if it is pure and if we have $m \leqslant -k_0(\beta, \Lambda) - 1$.

Let $[\Lambda, n, m, \beta]$ be a simple stratum in A. In [**24**] (see paragraph 2.4), one attaches to this simple stratum a compact open subgroup $\mathrm{H}^{m+1}(\beta, \Lambda)$ of $\mathrm{A}^\times$ and a finite set $\mathcal{C}(\Lambda, m, \beta)$ of characters of $\mathrm{H}^{m+1}(\beta, \Lambda)$, called simple characters of level m, depending on the choice of an additive character:

$$\Psi : \mathrm{F} \to \mathbb{C}^\times \tag{1.2}$$

which is trivial on $\mathfrak{p}_\mathrm{F}$ but not on $\mathcal{O}_\mathrm{F}$, and which will be fixed once and for all throughout this paper. If $\lfloor n/2 \rfloor \leqslant m$, then $\mathrm{H}^{m+1}(\beta, \Lambda) = \mathrm{U}_{m+1}(\Lambda)$, and the set $\mathcal{C}(\Lambda, m, \beta)$ reduces to a single character Ψ_β^A of $\mathrm{U}_{m+1}(\Lambda)$ defined by:

$$\Psi_\beta^\mathrm{A} : x \mapsto \Psi \circ \mathrm{tr}_{\mathrm{A}/\mathrm{F}}(\beta(x-1)), \tag{1.3}$$

which depends only on the equivalence class of $[\Lambda, n, m, \beta]$. More generally, for any possible value of m, the subgroup $\mathrm{H}^{m+1}(\beta, \Lambda)$ and the set $\mathcal{C}(\Lambda, m, \beta)$ depend only on the equivalence class of $[\Lambda, n, m, \beta]$.

1.2. Let β be a non-zero element of some finite extension of F. We set $\mathrm{E} = \mathrm{F}(\beta)$ and:

$$n_\mathrm{F}(\beta) = -\upsilon_\mathrm{E}(\beta),$$
$$e_\mathrm{F}(\beta) = e(\mathrm{E} : \mathrm{F}),$$
$$f_\mathrm{F}(\beta) = f(\mathrm{E} : \mathrm{F}),$$

where $e(\mathrm{E} : \mathrm{F})$ and $f(\mathrm{E} : \mathrm{F})$ stand for the ramification index and the residue class degree of E over F respectively, and υ_E for the valuation map of the field E giving the value 1 to any uniformizer of E. The lattice sequence $i \mapsto \mathfrak{p}_\mathrm{E}^i$, denoted $\Lambda(\mathrm{E})$, is the unique (up to translation) E-pure strict $\mathcal{O}_F$-lattice sequence on the F-vector space E, and its valuation map coincide with υ_E on $E^\times$. To any integer $0 \leqslant k \leqslant n_F(\beta) - 1$ we can attach the pure stratum $[\Lambda(E), n_F(\beta), k, \beta]$ of the split F-algebra $A(E) = \mathrm{End}_F(E)$, the critical exponent of which we denote by:

$$k_F(\beta) = k_0(\beta, \Lambda(E)).$$

This is an integer greater than or equal to $-n_F(\beta)$. In the case where this integer is equal to $-n_F(\beta)$, the element β is said to be *minimal* over F. Let us recall the definition of a simple pair over F (see [**6**, Definition 1.5]).

Definition 1.4. — A *simple pair* over F is a pair (k, β) consisting of a non-zero element β of some finite extension of F and an integer $0 \leqslant k \leqslant -k_F(\beta) - 1$.

Associated with a simple pair (k, β) over F is the simple stratum $[\Lambda(E), n_F(\beta), k, \beta]$ in A(E) together with a compact open subgroup of $A(E)^\times$ and a set of simple characters:

$$H_F^{k+1}(\beta) = H^{k+1}(\beta, \Lambda(E)), \quad \mathcal{C}_F(k, \beta) = \mathcal{C}(\Lambda(E), k, \beta).$$

Now let A be a simple central F-algebra and V be a simple left A-module. A *realization* of the simple pair (k, β) in A is a stratum in A of the form $[\Lambda, n, m, \varphi(\beta)]$ made of:

(1) a homomorphism φ of F-algebra from $F(\beta)$ to A;
(2) an $\mathcal{O}_D$-lattice sequence Λ on V normalized by the image of $F(\beta)^\times$ under φ;
(3) an integer m such that $\lfloor m/e_{\varphi(\beta)}(\Lambda) \rfloor = k$.

The integer $-n$ is then the Λ-valuation of $\varphi(\beta)$. By [**20**, Proposition 2.25] we have:

$$k_0(\varphi(\beta), \Lambda) = e_{\varphi(\beta)}(\Lambda) k_F(\beta), \tag{1.4}$$

which implies that any realization of a simple pair is a simple stratum. According to [**20**] again (*ibid.*, paragraph 3.3), for such a realization there is a canonical bijective map:

$$\boldsymbol{\tau}_{\Lambda,m,\varphi} : \mathcal{C}_F(k, \beta) \to \mathcal{C}(\Lambda, m, \varphi(\beta)) \tag{1.5}$$

called the *transfer* map. Some of its properties have been studied in [**24**] and some further properties will be given in sections 6 and 7 of the present article. Given another realization $[\Lambda', n', m', \varphi'(\beta)]$ of the pair (k, β) in some simple central F-algebra A′, we have a transfer map from $\mathcal{C}(\Lambda, m, \varphi(\beta))$ to $\mathcal{C}(\Lambda', m', \varphi'(\beta))$ by composing $\boldsymbol{\tau}_{\Lambda,m,\varphi}^{-1}$ with $\boldsymbol{\tau}_{\Lambda',m',\varphi'}$.

Given a simple pair (k, β) over F, we denote by $\boldsymbol{\mathcal{C}}_{(k,\beta)}$ the set of pairs $([\Lambda, n, m, \varphi(\beta)], \theta)$ made of a realization $[\Lambda, n, m, \varphi(\beta)]$ of (k, β) in a simple central F-algebra and a simple character $\theta \in \mathcal{C}(\Lambda, m, \varphi(\beta))$. Hence the surjective map:

$$([\Lambda, n, m, \varphi(\beta)], \theta) \mapsto \boldsymbol{\tau}_{\Lambda,m,\varphi}^{-1}(\theta) \in \mathcal{C}_F(k, \beta)$$

is well defined on $\mathfrak{C}_{(k,\beta)}$ and induces, by its fibers, an equivalence relation on it.

Definition 1.5. — A *potential simple character* over F (or *ps-character* for short) is a triple (Θ, k, β) made of a simple pair (k, β) over F and an equivalence class Θ in $\mathfrak{C}_{(k,\beta)}$.

When the context is clear, we will often denote by Θ the ps-character (Θ, k, β). Given a realization $[\Lambda, n, m, \varphi(\beta)]$ of (k, β), we will denote by $\Theta(\Lambda, m, \varphi)$ the simple character θ such that the pair $([\Lambda, n, m, \varphi(\beta)], \theta)$ belongs to Θ.

1.3. We now state the main results which are proved in this article. Our first task is to extend the notion of endo-equivalence of simple pairs developed by Bushnell and Henniart in [**6**]. More precisely, we extend it to realizations in non-necessarily split simple central F-algebras with non-necessarily strict lattice sequences.

Definition 1.6. — For $i = 1, 2$, let (k_i, β_i) be a simple pair over F. We say that these pairs are *endo-equivalent*, denoted:

$$(k_1, \beta_1) \approx (k_2, \beta_2),$$

if $k_1 = k_2$ and $[\mathrm{F}(\beta_1) : \mathrm{F}] = [\mathrm{F}(\beta_2) : \mathrm{F}]$, and if there exists a simple central F-algebra A together with realizations $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ of (k_i, β_i) in A, with $i = 1, 2$, which intertwine in A.

Recall that two strata $[\Lambda, n_i, m_i, \beta_i]$ in A, with $i \in \{1, 2\}$, *intertwine* in A if there exists $g \in \mathrm{A}^\times$ such that:

$$(\beta_1 + \mathfrak{P}_{-m_1}(\Lambda)) \cap g(\beta_2 + \mathfrak{P}_{-m_2}(\Lambda))g^{-1} \neq \varnothing. \tag{1.6}$$

As we will see in paragraph 2.5 (see Corollary 2.9), this definition of endo-equivalence of simple pairs is equivalent to [**6**, Definition 1.14], although more general in appearance.

We now investigate the intertwining relations among various realizations of given simple pairs, and in particular their preservation properties. Our first result is the following proposition, which generalizes [**6**, Proposition 1.10] and is proved in paragraph 2.6.

Proposition 1.7. — *For $i = 1, 2$, let (k, β_i) be a simple pair over* F*, and suppose these pairs are endo-equivalent. Let* A *be a simple central* F*-algebra and let $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ be a realization of (k, β_i) in* A*, for $i = 1, 2$. These strata then intertwine in* A.

Broussous and Grabitz remarked in [**5**] that two simple strata $[\Lambda, n, m, \beta_i]$, $i = 1, 2$, in A which intertwine in A may be not conjugate under $\mathrm{A}^\times$, unlike the case where A is split (see [**10**, Theorem 2.6.1] for the case where A is split and Λ is strict). In order to remedy this, they introduced the notion of an embedding type (see also Fröhlich [**15**]). Here we extend this notion to non-necessarily strict lattice sequences.

We fix a simple central F-algebra A and a simple left A-module V as in paragraph 1.1. Associated with it, we have an F-division algebra D. An *embedding* in A is a pair (E,Λ) made of a finite extension E of F contained in A and an E-pure $\mathcal{O}_{\mathrm{D}}$-lattice sequence Λ on V. Given such a pair, we denote by $\mathrm{E}^{\diamond}$ the maximal finite unramified extension of F which is contained in E and whose degree divides the reduced degree of D over F.

Two embeddings (E_i,Λ_i), $i=1,2$, in A are said to be *equivalent* in A if there exists an element $g\in\mathrm{A}^{\times}$ such that Λ_1 is in the translation class of $g\Lambda_2$ and $\mathrm{E}_1^{\diamond}=g\mathrm{E}_2^{\diamond}g^{-1}$. This defines an equivalence relation on the set of embeddings in A, and an equivalence class for this relation is called an *embedding type* in A.

Definition 1.8. — The *embedding type* of a pure stratum $[\Lambda,n,m,\beta]$ is the embedding type of the pair $(\mathrm{F}(\beta),\Lambda)$ in A.

This allows us to state the following "intertwining implies conjugacy" theorem, which generalizes [**10**, Theorem 2.6.1] and [**5**, Proposition 4.1.2] and is proved in paragraph 3.3.

Proposition 1.9. — *For $i=1,2$, let $[\Lambda,n,m,\beta_i]$ be a simple stratum in* A*. Assume that they intertwine in* A *and have the same embedding type. Write K_i for the maximal unramified extension of* F *contained in $\mathrm{F}(\beta_i)$. Then there is $u\in\mathfrak{K}(\Lambda)$ such that $\mathrm{K}_1=u\mathrm{K}_2u^{-1}$ and $\beta_1-u\beta_2u^{-1}\in\mathfrak{P}_{-m}(\Lambda)$.*

1.4. We now extend the notion of endo-equivalence of simple characters developed by Bushnell and Henniart in [**6**]. As for simple pairs, we extend it to realizations in non-necessarily split simple central F-algebras with non-necessarily strict lattice sequences.

Definition 1.10. — For $i=1,2$, let (Θ_i,k_i,β_i) be a ps-character over F. We say that these ps-characters are *endo-equivalent*, denoted:

$$\Theta_1\approx\Theta_2,$$

if $k_1=k_2$ and $[\mathrm{F}(\beta_1):\mathrm{F}]=[\mathrm{F}(\beta_2):\mathrm{F}]$, and if there exists a simple central F-algebra A together with realizations $[\Lambda,n_i,m_i,\varphi_i(\beta_i)]$ of (k_i,β_i) in A, with $i=1,2$, such that the simple characters $\Theta_1(\Lambda,m_1,\varphi_1)$ and $\Theta_2(\Lambda,m_2,\varphi_2)$ intertwine in $\mathrm{A}^{\times}$.

Recall that two simple characters $\theta_i\in\mathcal{C}(\Lambda,m_i,\beta_i)$, $i=1,2$, *intertwine* in $\mathrm{A}^{\times}$ if there exists $g\in\mathrm{A}^{\times}$ such that:

$$\theta_2(x)=\theta_1(gxg^{-1}),\quad x\in\mathrm{H}^{m_2+1}(\beta_2,\Lambda)\cap g^{-1}\mathrm{H}^{m_1+1}(\beta_1,\Lambda)g. \tag{1.7}$$

As we will see at the end of this article (see Corollary 8.2), this definition of endo-equivalence of simple characters is equivalent to [**6**, Definition 8.6].

We now state the main results of this article concerning properties of simple characters with respect to intertwining and conjugacy. The following generalizes [**6**, Theorem 8.7].

Theorem 1.11. — *For $i = 1, 2$, let (Θ_i, k_i, β_i) be a ps-character over* F*, and suppose that $\Theta_1 \approx \Theta_2$. Let* A *be a simple central* F*-algebra and let $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ be realizations of (k_i, β_i) in* A*, for $i = 1, 2$. Then $\Theta_1(\Lambda, m_1, \varphi_1)$ and $\Theta_2(\Lambda, m_2, \varphi_2)$ intertwine in $\mathrm{A}^\times$.*

The following "intertwining implies conjugacy" theorem for simple characters generalizes [**10**, Theorem 3.5.11] and [**17**, Corollary 10.15] to simple characters in non-necessarily split simple central F-algebras with non-necessarily strict lattice sequences.

Theorem 1.12. — *Let* A *be a simple central* F*-algebra. For $i = 1, 2$, let $[\Lambda, n, m, \beta_i]$ be a simple stratum in* A*, and let $\theta_i \in \mathcal{C}(\Lambda, m, \beta_i)$ be a simple character. Write K_i for the maximal unramified extension of* F *contained in $\mathrm{F}(\beta_i)$. Assume that θ_1 and θ_2 intertwine in $\mathrm{A}^\times$ and that the strata $[\Lambda, n, m, \beta_i]$ have the same embedding type. Then there is an element $u \in \mathfrak{K}(\Lambda)$ such that:*

(1) $\mathrm{K}_1 = u\mathrm{K}_2u^{-1}$;
(2) $\mathcal{C}(\Lambda, m, \beta_1) = \mathcal{C}(\Lambda, m, u\beta_2u^{-1})$;
(3) $\theta_2(x) = \theta_1(uxu^{-1})$, *for all* $x \in \mathrm{H}^{m+1}(\beta_2, \Lambda) = u^{-1}\mathrm{H}^{m+1}(\beta_1, \Lambda)u$.

We will see in section 4 (see Corollary 4.8) that the proofs of these two theorems can be reduced to that of the following statement, which will be proved in section 8.

Theorem 1.13. — *For $i = 1, 2$, let (Θ_i, k_i, β_i) be a ps-character over* F*, and suppose that $\Theta_1 \approx \Theta_2$. Let* A *be a simple central* F*-algebra, and let $[\Lambda, n, m, \varphi_i(\beta_i)]$ be realizations of (k_i, β_i) in* A*, for $i = 1, 2$. Write K_i for the maximal unramified extension of* F *contained in $\mathrm{F}(\beta_i)$ and θ_i for the simple character $\Theta_i(\Lambda, m, \varphi_i)$. Assume these strata have the same embedding type. Then there is an element $u \in \mathfrak{K}(\Lambda)$ such that:*

(1) $\varphi_1(\mathrm{K}_1) = u\varphi_2(\mathrm{K}_2)u^{-1}$;
(2) $\mathcal{C}(\Lambda, m, \varphi_1(\beta_1)) = \mathcal{C}(\Lambda, m, u\varphi_2(\beta_2)u^{-1})$;
(3) $\theta_2(x) = \theta_1(uxu^{-1})$, *for all* $x \in \mathrm{H}^{m+1}(\varphi_2(\beta_2), \Lambda) = u^{-1}\mathrm{H}^{m+1}(\varphi_1(\beta_1), \Lambda)u$.

The main ingredient in this reduction step is Lemma 4.7, which states that the endo-equivalence relation preserves certain numerical invariants attached to a ps-character.

1.5. As has been explained in the introduction, this article makes a large use of the results of Bushnell, Henniart and Kutzko in the split case [**6, 10**] (see paragraphs 1.3 and 1.4), as well as results of Grabitz [**17**] which are based on the following definition.

Definition 1.14. — A simple stratum $[\Lambda, n, m, \beta]$ in A is *sound* if Λ is strict, $\mathfrak{A} \cap \mathrm{B}$ is principal and $\mathfrak{K}(\mathfrak{A}) \cap \mathrm{B}^\times = \mathfrak{K}(\mathfrak{A} \cap \mathrm{B})$, where $\mathfrak{A}$ is the hereditary $\mathcal{O}_\mathrm{F}$-order defined by Λ.

More generally, an embedding (E, Λ) in A is *sound* if the conditions of Definition 1.14 are fulfilled with B the centralizer of E in A.

Remark 1.15. — Note that the condition on $\mathfrak{A} \cap \mathrm{B}$ forces $\mathfrak{A}$ to be a principal $\mathcal{O}_\mathrm{F}$-order. In the split case, a simple stratum $[\Lambda, n, m, \beta]$ is sound if and only if Λ is strict and $\mathfrak{A}$ is principal.

When Λ is strict, its translation class is entirely determined by the hereditary $\mathcal{O}_\mathrm{F}$-order $\mathfrak{A} = \mathfrak{A}(\Lambda)$. In this case, we will sometimes write $(\mathrm{E}, \mathfrak{A})$ and $[\mathfrak{A}, n, m, \beta]$ rather than (E, Λ) and $[\Lambda, n, m, \beta]$.

In the case where the simple strata $[\Lambda, n, m, \beta_i]$, $i = 1, 2$, are sound, Grabitz has proved in [**17**] the "intertwining implies conjugacy" theorem for simple characters (see *ibid.*, Theorem 10.3 and Corollary 10.15). More precisely, he has proved the following result.

Given K/F an unramified extension contained in A, a sound simple stratum $[\Lambda, n, m, \beta]$ in A is K-*special* (see [**17**, Definition 3.1]) if it is K-pure in the sense of Definition 5.1 and if $(\mathrm{K}(\beta), \mathfrak{A}(\Lambda) \cap \mathrm{C})$ is a sound embedding in C, where C is the centralizer of K in A.

Theorem 1.16 **([17], Theorem 10.3).** — *For $i = 1, 2$, let $[\Lambda, n, m, \beta_i]$ be a sound simple stratum in a simple central* F*-algebra* A *and let $\theta_i \in \mathcal{C}(\Lambda, m, \beta_i)$ be a simple character. Let f be a multiple of the greatest common divisor of $f_\mathrm{F}(\beta_1)$ and $f_\mathrm{F}(\beta_2)$, and let K_i be an unramified extension of* F *of degree f contained in* A *such that $[\Lambda, n, m, \beta_i]$ is K_i-special. Assume (K_1, Λ) and (K_2, Λ) are equivalent embeddings in* A*, and that θ_1 and θ_2 intertwine in $\mathrm{A}^\times$. Then:*

(1) $e_\mathrm{F}(\beta_1) = e_\mathrm{F}(\beta_2)$ *and* $f_\mathrm{F}(\beta_1) = f_\mathrm{F}(\beta_2)$*;*

(2) K_i *contains the maximal unramified extension of* F *contained in* $\mathrm{F}[\beta_i]$.

Moreover, there exists $u \in \mathfrak{K}(\Lambda)$ such that:

(3) $\mathrm{K}_1 = u\mathrm{K}_2u^{-1}$*;*

(4) $\mathcal{C}(\Lambda, m, \beta_1) = \mathcal{C}(\Lambda, m, u\beta_2u^{-1})$*;*

(5) $\theta_2(x) = \theta_1(uxu^{-1})$*, for all* $x \in \mathrm{H}^{m+1}(\beta_2, \Lambda) = u^{-1}\mathrm{H}^{m+1}(\beta_1, \Lambda)u$.

We will also need the following result.

Proposition 1.17 (**[17], Propositions 9.1 and 9.9**). — *For* $i = 1, 2$, *let* $[\Lambda, n, m, \beta_i]$ *be a sound simple stratum in* A. *Assume that* $\mathcal{C}(\Lambda, m, \beta_1) \cap \mathcal{C}(\Lambda, m, \beta_2)$ *is not empty. Then* $e_{\mathrm{F}}(\beta_1) = e_{\mathrm{F}}(\beta_2)$, $f_{\mathrm{F}}(\beta_1) = f_{\mathrm{F}}(\beta_2)$, $k_{\mathrm{F}}(\beta_1) = k_{\mathrm{F}}(\beta_2)$ *and* $\mathcal{C}(\Lambda, m, \beta_1) = \mathcal{C}(\Lambda, m, \beta_2)$.

Note that [**17**, Proposition 9.1] gives us an equality between $[\mathrm{F}(\beta_1) : \mathrm{F}]$ and $[\mathrm{F}(\beta_2) : \mathrm{F}]$, but the two finer equalities between the ramification indexes and residue class degrees come from Theorem 1.16.

Our proof of Theorem 1.13 in section 8 is decomposed into two steps. The first step consists of treating the case where the extensions $\mathrm{F}(\beta_i)/\mathrm{F}$ are totally ramified, and the second step consists of reducing to the totally ramified case. In section 5, we develop an interior lifting process for simple strata and simple characters with respect to a finite unramified extension K of F, in a way similar to [**6**] and [**17**]. Its compatibility with transfer is explored in section 6. This interior lifting is enough to reduce to the totally ramified case. The totally ramified case is more subtle. For this we develop an 'exterior lifting' or unramified base change in section 7.

2. Realizations and intertwining for simple strata

In this section, we introduce various constructions which will be used throughout the paper. More precisely, we describe various processes, preserving intertwining, which associate to a realization of a simple pair in some simple central F-algebra a realization in a (possibly different) simple central F-algebra, with additional properties. This allows us to prove that Definition 1.6 and the definition of endo-equivalence of simple pairs given in [**6**] are equivalent (see Corollary 2.9), and to prove Proposition 1.7.

2.1. We fix a simple central F-algebra A and a simple left A-module V. We set:

$$\widetilde{\mathrm{A}} = \mathrm{End}_{\mathrm{F}}(\mathrm{V}),$$

which is a split simple central F-algebra in which the algebra A embeds naturally. To any stratum $[\Lambda, n, m, \beta]$ in A we can attach a stratum $[\widetilde{\Lambda}, n, m, \beta]$ in $\widetilde{\mathrm{A}}$, where $\widetilde{\Lambda}$ denotes the $\mathcal{O}_{\mathrm{F}}$-lattice sequence defined by Λ. By [**20**, Théorème 2.23], this latter stratum is simple if and only the first one is, and in this case they are realizations of the same simple pair over F. Moreover, we have the following result.

Proposition 2.1. — *For* $i = 1, 2$, *let* $[\Lambda, n_i, m_i, \beta_i]$ *be a simple stratum in* A. *Assume they intertwine in* A. *Then the strata* $[\widetilde{\Lambda}, n_i, m_i, \beta_i]$ *intertwine in* $\widetilde{\mathrm{A}}$.

Proof. — This follows immediately from the definition of intertwining and the fact that the $\mathcal{O}_F$-module $\mathfrak{P}_k(\Lambda)$ is contained in $\mathfrak{P}_k(\widetilde{\Lambda})$ for all $k \in \mathbb{Z}$. □

2.2. Let $[\Lambda, n, m, \beta]$ be a simple stratum in A, which is a realization of a simple pair (k, β) over F. The *affine class* of Λ is the set of all $\mathcal{O}_D$-lattice sequences on V of the form:

$$a\Lambda + b : k \mapsto \Lambda_{\lceil (k-b)/a \rceil}, \tag{2.1}$$

with $a, b \in \mathbb{Z}$ and $a \geqslant 1$. The period of (2.1) is a times the period $e(\Lambda)$ of Λ. Given an integer $l \geqslant 1$, we set $V' = V \oplus \cdots \oplus V$ (l times) and $A' = \mathrm{End}_D(V')$, and embed A in A′ diagonally. For each $j \in \{1, \dots, l\}$, we choose a lattice sequence Λ^j in the affine class of Λ, and assume the periods of the Λ^j's are all equal to a common integer $ae(\Lambda)$ with $a \geqslant 1$. We now form the $\mathcal{O}_D$-lattice sequence Λ' on V′ defined by:

$$\Lambda' = \Lambda^1 \oplus \Lambda^2 \oplus \cdots \oplus \Lambda^l, \tag{2.2}$$

and fix a non-negative integer m' such that:

$$\lfloor m'/a \rfloor = m.$$

If we set $n' = an$, this gives us a simple stratum $[\Lambda', n', m', \beta]$ in A′, which is a realization of the simple pair (k, β). In the particular case where $l = 1$, we have the following result.

Lemma 2.2. — *Assume that $l = 1$, so that Λ' is in the affine class of Λ. Then we have $H^{m'+1}(\beta, \Lambda') = H^{m+1}(\beta, \Lambda)$ and $\mathcal{C}(\Lambda', m', \beta) = \mathcal{C}(\Lambda, m, \beta)$. Moreover, the transfer map from $\mathcal{C}(\Lambda', m', \beta)$ to $\mathcal{C}(\Lambda, m, \beta)$ is the identity map.*

Proof. — The first assertion is straightforward by induction on β (that is, on the integer $k_F(\beta)$ defined in paragraph 1.2). For the second one, see [**24**, Théorème 2.13]. □

2.3. Assume now we are given two simple strata $[\Lambda, n_i, m_i, \beta_i]$, $i = 1, 2$, in A. For each i, we set $n'_i = an_i$ and fix a non-negative integer m'_i such that $\lfloor m'_i/a \rfloor = m_i$, so that we have a simple stratum $[\Lambda', n'_i, m'_i, \beta_i]$ in A′.

Proposition 2.3. — *Assume that the strata $[\Lambda, n_i, m_i, \beta_i]$, $i \in \{1, 2\}$, intertwine in A. Then the strata $[\Lambda', n'_i, m'_i, \beta_i]$, $i \in \{1, 2\}$, intertwine in A′.*

Proof. — We start with an element $g \in A^\times$ which intertwines the two strata $[\Lambda, n_i, m_i, \beta_i]$, that is, which satisfies the condition (1.6), and we let ι denote the diagonal embedding of A in A′ (which we omit from the notation when the context is clear). For $j \in \{1, \dots, l\}$, write V^j for the jth copy of V in $V' = V \oplus \cdots \oplus V$. Then for each i, we have:

$$\mathfrak{P}_{-m'_i}(\Lambda') \cap \mathrm{End}_D(V^j) = \mathfrak{P}_{-m'_i}(\Lambda^j), \quad j \in \{1, \dots, l\},$$

which is equal to $\mathfrak{P}_{-m_i}(\Lambda)$ as can be seen by a direct computation in the case $l = 1$. This implies that ι induces an $\mathcal{O}_{\mathrm{F}}$-module embedding of $\mathfrak{P}_{-m_i}(\Lambda)$ in $\mathfrak{P}_{-m'_i}(\Lambda')$, from which we deduce that $g' = \iota(g) \in \mathrm{A}'^{\times}$ intertwines the strata $[\Lambda', n'_i, m'_i, \beta_i]$. □

Remark 2.4. — Note that ι induces a group homomorphism of $\mathfrak{K}(\Lambda)$ into $\mathfrak{K}(\Lambda')$. Therefore, if $g \in \mathfrak{K}(\Lambda)$ intertwines two simple strata $[\Lambda, n, m, \beta_i]$, $i = 1, 2$, that is, if we have:

$$\beta_2 - g\beta_1 g^{-1} \in \mathfrak{P}_{-m}(\Lambda),$$

and if we set $n' = an$ and fix a non-negative integer m' such that $\lfloor m'/a \rfloor = m$, then the element $\iota(g) \in \mathfrak{K}(\Lambda')$ intertwines the strata $[\Lambda', n', m', \beta_i]$.

Proposition 2.5. — *Assume that the strata $[\Lambda, n_i, m_i, \beta_i]$, for $i \in \{1, 2\}$, have the same embedding type. Then the strata $[\Lambda', n'_i, m'_i, \beta_i]$, $i \in \{1, 2\}$, have the same embedding type.*

Proof. — Given $g \in \mathfrak{K}(\Lambda)$ which conjugates the unramified extensions $\mathrm{F}(\beta_i)^{\diamond}$, $i \in \{1, 2\}$, the element $\iota(g) \in \mathfrak{K}(\Lambda')$ conjugates the extensions $\mathrm{F}(\iota(\beta_i))^{\diamond}$, $i \in \{1, 2\}$. □

2.4. For $i = 1, 2$, let θ_i be a simple character in $\mathcal{C}(\Lambda, m_i, \beta_i)$, and let θ'_i be its transfer in $\mathcal{C}(\Lambda', m'_i, \beta_i)$. The following result is an analogue of Proposition 2.3 for simple characters.

Proposition 2.6. — *Assume that θ_1 and θ_2 intertwine in $\mathrm{A}^{\times}$. Then θ'_1 and θ'_2 intertwine in $\mathrm{A}'^{\times}$.*

Proof. — The decomposition of V' into a sum of copies of V defines a Levi subgroup:

$$\mathrm{M} = \mathrm{A}^{\times} \times \cdots \times \mathrm{A}^{\times} \tag{2.3}$$

of $\mathrm{A}'^{\times}$. We fix a parabolic subgroup P of $\mathrm{A}^{\times}$ with Levi factor M and unipotent radical N, and we write N^{-} for the unipotent radical of the parabolic subgroup of $\mathrm{A}^{\times}$ opposite to P with respect to M. According to [**24**, Théorème 2.17], we have an Iwahori decomposition:

$$\begin{aligned} \mathrm{H}^{m'_i+1}(\beta_i, \Lambda') &= \big(\mathrm{H}^{m'_i+1}(\beta_i, \Lambda') \cap \mathrm{N}^{-}\big)\big(\mathrm{H}^{m'_i+1}(\beta_i, \Lambda') \cap \mathrm{M}\big)\big(\mathrm{H}^{m'_i+1}(\beta_i, \Lambda') \cap \mathrm{N}\big), \\ \mathrm{H}^{m'_i+1}(\beta_i, \Lambda') \cap \mathrm{M} &= \mathrm{H}^{m_i+1}(\beta_i, \Lambda) \times \cdots \times \mathrm{H}^{m_i+1}(\beta_i, \Lambda), \end{aligned}$$

for each integer $i = 1, 2$. We have the following result.

Lemma 2.7. — *The simple character θ'_i is trivial on the subgroups $\mathrm{H}^{m'_i+1}(\beta_i, \Lambda') \cap \mathrm{N}$ and $\mathrm{H}^{m'_i+1}(\beta_i, \Lambda') \cap \mathrm{N}^{-}$, and we have:*

$$\theta'_i \mid \mathrm{H}^{m'_i+1}(\beta_i, \Lambda') \cap \mathrm{M} = \theta_i \otimes \cdots \otimes \theta_i.$$

Proof. — This derives from [**24**, Théorème 2.17]. Indeed, for $j \in \{1, \dots, l\}$, the restriction of θ'_i to $\mathrm{H}^{m'_i+1}(\beta_i, \Lambda') \cap \mathrm{Aut}_{\mathrm{D}}(\mathrm{V}^j)$ is the transfer of θ'_i to $\mathcal{C}(\Lambda^j, m'_i, \beta_i)$, which is equal to θ_i by Lemma 2.2. □

Now let $g \in \mathrm{A}^\times$ intertwine θ_1 and θ_2 as in the identity (1.7), and set $g' = \iota(g) \in \mathrm{M}$. If we write $\mathrm{H}_i = \mathrm{H}^{m_i+1}(\beta_i, \Lambda)$ and $\mathrm{H}'_i = \mathrm{H}^{m'_i+1}(\beta_i, \Lambda')$ for each integer $i \in \{1, 2\}$, we get an Iwahori decomposition:

$$\begin{aligned} \mathrm{H}'_2 \cap g'^{-1}\mathrm{H}'_1 g' &= \big(\mathrm{H}'_2 \cap g'^{-1}\mathrm{H}'_1 g' \cap \mathrm{N}^-\big)\big(\mathrm{H}'_2 \cap g'^{-1}\mathrm{H}'_1 g' \cap \mathrm{M}\big)\big(\mathrm{H}'_2 \cap g'^{-1}\mathrm{H}'_1 g' \cap \mathrm{N}\big), \\ \mathrm{H}'_2 \cap g'^{-1}\mathrm{H}'_1 g' \cap \mathrm{M} &= \big(\mathrm{H}_2 \cap g^{-1}\mathrm{H}_1 g\big) \times \dots \times \big(\mathrm{H}_2 \cap g^{-1}\mathrm{H}_1 g\big). \end{aligned}$$

According to Lemma 2.7, the simple characters θ'_1 and θ'_2 are trivial on the two subgroups $\mathrm{H}'_2 \cap g'^{-1}\mathrm{H}'_1 g' \cap \mathrm{N}$ and $\mathrm{H}'_2 \cap g'^{-1}\mathrm{H}'_1 g' \cap \mathrm{N}^-$, and we have:

$$\theta'_i \mid \mathrm{H}'_2 \cap g'^{-1}\mathrm{H}'_1 g' \cap \mathrm{M} = \big(\theta_i \mid \mathrm{H}_2 \cap g^{-1}\mathrm{H}_1 g\big) \otimes \dots \otimes \big(\theta_i \mid \mathrm{H}_2 \cap g^{-1}\mathrm{H}_1 g\big)$$

for each $i \in \{1, 2\}$. This ensures that g' intertwines the simple characters θ'_1 and θ'_2. □

2.5. We give an example which will be of particular interest for us. Let $[\Lambda, n, m, \beta]$ be a simple stratum in A, which is a realization of a simple pair (k, β) over F, and let e denote the period of Λ over $\mathcal{O}_{\mathrm{D}}$. We set:

$$\Lambda^\dagger : k \mapsto \Lambda_k \oplus \Lambda_{k+1} \oplus \dots \oplus \Lambda_{k+e-1}, \tag{2.4}$$

which is a strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence on $\mathrm{V}^\dagger = \mathrm{V} \oplus \dots \oplus \mathrm{V}$ (e times) of the form (2.2). Thus we can form the simple stratum $[\Lambda^\dagger, n, m, \beta]$ in $\mathrm{A}^\dagger = \mathrm{End}_{\mathrm{D}}(\mathrm{V}^\dagger)$, which is a realization of (k, β). Moreover, the hereditary $\mathcal{O}_{\mathrm{F}}$-order $\mathfrak{A}^\dagger$ defined by $\Lambda^\dagger$ is principal, and we have the following result, which derives from Propositions 2.3 and 2.5.

Proposition 2.8. — *For $i = 1, 2$, let $[\Lambda, n_i, m_i, \beta_i]$ be a simple stratum in* A. *Assume they intertwine in* A *(resp. have the same embedding type). Then the strata $[\Lambda^\dagger, n_i, m_i, \beta_i]$ intertwine in $\mathrm{A}^\dagger$ (resp. have the same embedding type).*

Note that the operations $\Lambda \mapsto \widetilde{\Lambda}$ (see paragraph 2.1) and $\Lambda \mapsto \Lambda^\dagger$ commute, so that there is no ambiguity in writing $\widetilde{\Lambda}^\dagger$ for the strict $\mathcal{O}_{\mathrm{F}}$-lattice sequence defined by $\Lambda^\dagger$.

Corollary 2.9. — *Definition 1.6 is equivalent to Definition* [**6**, 1.14].

Proof. — Assume we are given two simple pairs (k, β_i), $i = 1, 2$, which are endo-equivalent in the sense of Definition 1.6. Then we have $[\mathrm{F}(\beta_1) : \mathrm{F}] = [\mathrm{F}(\beta_2) : \mathrm{F}]$, and there exists a simple central F-algebra A together with realizations $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ of (k, β_i) in A, with $i = 1, 2$, which intertwine in A. By replacing A and Λ by $\widetilde{\mathrm{A}}^\dagger$ and $\widetilde{\Lambda}^\dagger$, we have realizations

$[\widetilde{\Lambda}^\dagger, n_i, m_i, \varphi_i(\beta_i)]$ of (k, β_i) in $\widetilde{\mathrm{A}}^\dagger$, with $i = 1, 2$, and these realizations intertwine in $\widetilde{\mathrm{A}}^\dagger$ according to Propositions 2.1 and 2.8. Thus the simple pairs (k, β_1) and (k, β_2) are endo-equivalent in the sense of [**6**, Definition 1.14]. Conversely, two simple pairs which are endo-equivalent in this sense are clearly endo-equivalent in the sense of Definition 1.6. □

2.6. We now prove the preservation property of intertwining for simple strata, that is, Proposition 1.7. We first prove that the endo-equivalence relation preserves certain numerical invariants attached to simple pairs. Compare the following proposition with [**6**], Property (1.15). See paragraph 1.2 for the notation.

Proposition 2.10. — *For $i = 1, 2$, let (k, β_i) be a simple pair over* F*, and suppose that (k, β_1) and (k, β_2) are endo-equivalent. Then we have $n_{\mathrm{F}}(\beta_1) = n_{\mathrm{F}}(\beta_2)$, $e_{\mathrm{F}}(\beta_1) = e_{\mathrm{F}}(\beta_2)$, $f_{\mathrm{F}}(\beta_1) = f_{\mathrm{F}}(\beta_2)$ and $k_{\mathrm{F}}(\beta_1) = k_{\mathrm{F}}(\beta_2)$.*

Proof. — By Corollary 2.9, we may assume that the pairs (k, β_i) are endo-equivalent in the sense of [**6**]. The result then follows from [**6**, Proposition 1.10]. □

For $i = 1, 2$, let (k, β_i) be a simple pair over F, and suppose that $(k, \beta_1) \approx (k, \beta_2)$.

Let A be a simple central F-algebra and, for $i = 1, 2$, let $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ be a realization of (k, β_i) in A. Let V denote the simple left A-module on which Λ is a lattice sequence and write D for the F-algebra opposite to $\mathrm{End}_{\mathrm{A}}(\mathrm{V})$. For $i = 1, 2$, let E_i denote the F-algebra $\mathrm{F}(\beta_i)$. We fix a simple right $\mathrm{E}_1 \otimes_{\mathrm{F}} \mathrm{D}$-module S and set $\mathrm{A}(\mathrm{S}) = \mathrm{End}_{\mathrm{D}}(\mathrm{S})$, and we denote by ρ_1 the natural F-algebra homomorphism $\mathrm{E}_1 \to \mathrm{A}(\mathrm{S})$. Let $\mathfrak{S}$ denote the unique (up to translation) E_1-pure strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence on S.

Lemma 2.11. — *There is a homomorphism of* F*-algebras $\rho_2 : \mathrm{E}_2 \to \mathrm{A}(\mathrm{S})$ such that $\mathfrak{S}$ is $\rho_2(\mathrm{E}_2)$-pure, and such that the pairs $(\rho_1(\mathrm{E}_1), \mathfrak{S})$ and $(\rho_2(\mathrm{E}_2), \mathfrak{S})$ have the same embedding type in* A(S) *(see paragraph 1.3).*

Proof. — As (k, β_1) and (k, β_2) are endo-equivalent, Proposition 2.10 gives us the equalities $e_{\mathrm{F}}(\beta_1) = e_{\mathrm{F}}(\beta_2)$ and $f_{\mathrm{F}}(\beta_1) = f_{\mathrm{F}}(\beta_2)$. The result follows from [**5**, Corollary 3.16]. □

Remark 2.12. — We actually have a stronger result: for any F-algebra homomorphism ρ_2 such that $\mathfrak{S}$ is $\rho_2(\mathrm{E}_2)$-pure, the pairs $(\rho_1(\mathrm{E}_1), \mathfrak{S})$ and $(\rho_2(\mathrm{E}_2), \mathfrak{S})$ have the same embedding type in A(S). Indeed, if ρ_2 is such a homomorphism and if η_2 is an F-algebra homomorphism as in Lemma 2.11, the Skolem-Noether theorem gives us $g \in \mathrm{A}(\mathrm{S})^\times$ which conjugates these F-algebra homomorphisms ρ_2 and η_2. As E_1 and E_2 have the same degree over F, the lattice sequence $\mathfrak{S}$ is the unique (up to translation) $\rho_2(\mathrm{E}_2)$-pure strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence — and also the unique (up to translation) $\eta_2(\mathrm{E}_2)$-pure strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence — on S. It follows that g normalizes the lattice sequence $\mathfrak{S}$ and that the pairs $(\rho_2(E_2), \mathfrak{S})$ and $(\eta_2(E_2), \mathfrak{S})$ have the same embedding type in A(S).

Let us fix an F-algebra homomorphism ρ_2 as in Lemma 2.11. As (k, β_1) and (k, β_2) are endo-equivalent, we have $n_F(\beta_1) = n_F(\beta_2)$ and $e_F(\beta_1) = e_F(\beta_2)$, so that the $\mathfrak{S}$-valuation of $\rho_i(\beta_i)$, denoted n_0, and the period $e_{\rho_i(\beta_i)}(\mathfrak{S})$ do not depend on $i \in \{1, 2\}$. We set:

$$m_0 = e_{\rho_i(\beta_i)}(\mathfrak{S})k.$$

For each $i \in \{1, 2\}$, we have a stratum $[\mathfrak{S}, n_0, m_0, \rho_i(\beta_i)]$, which is a realization of (k, β_i) in A(S). By paragraph 2.1, we have a realization $[\widetilde{\mathfrak{S}}, n_0, m_0, \rho_i(\beta_i)]$ of (k, β_i) in the split simple central F-algebra $\mathrm{End}_F(S)$, and the $\mathcal{O}_F$-lattice sequence $\widetilde{\mathfrak{S}}$ is strict. Hence we can apply [**6**, Proposition 1.10], which implies that these realizations, for $i = 1, 2$, intertwine in $\mathrm{End}_F(S)$. By our assumption (see Lemma 2.11), the strata $[\mathfrak{S}, n_0, m_0, \rho_i(\beta_i)]$, for $i = 1, 2$, have the same embedding type. Here we need to recall the following statement, due to Broussous and Grabitz.

Proposition 2.13 **([5], Proposition 4.1.3)**. — *For $i = 1, 2$, let $[\Sigma, n, m, \gamma_i]$ be a simple stratum in a simple central F-algebra U, where Σ is strict. Assume that they have the same embedding type, and that the strata $[\widetilde{\Sigma}, n, m, \gamma_i]$ intertwine in $\widetilde{U}$. Then there exists an element $u \in \mathfrak{K}(\Sigma)$ such that $\gamma_1 - u\gamma_2 u^{-1} \in \mathfrak{P}_{-m}(\Sigma)$.*

Moreover, u can be chosen such that the maximal unramified extension of F contained in $F(\gamma_1)$ is equal to that of $F(u\gamma_2 u^{-1})$.

We deduce from Proposition 2.13 that there exists an element $g \in \mathfrak{K}(\mathfrak{S})$ such that:

$$\rho_1(\beta_1) - g\rho_2(\beta_2)g^{-1} \in \mathfrak{P}_{-m_0}(\mathfrak{S}). \tag{2.5}$$

We now fix a decomposition:

$$V = V^1 \oplus \cdots \oplus V^l$$

of V into simple right $E_1 \otimes_F D$-modules (which all are copies of S) such that the lattice sequence Λ decomposes into the direct sum of the $\Lambda^j = \Lambda \cap V^j$, for $j \in \{1, \dots, l\}$.

Lemma 2.14. — *There are isomorphisms of $E_1 \otimes_F D$-modules $V^j \to S$, $j \in \{1, \dots, l\}$, such that the resulting F-algebra homomorphism $\iota : A(S) \to A$ satisfies $\iota \circ \rho_1 = \varphi_1$.*

Proof. — Since each V^j, for $j \in \{1, \dots, l\}$, is an E_1-vector subspace of V, the F-algebra homomorphism φ_1 has the form $x \mapsto (\omega_1(x), \dots, \omega_l(x))$, where ω_j is an F-algebra homomorphism from E_1 to $\mathrm{End}_D(V^j)$. By the Skolem-Noether theorem, one can choose, for each integer j, a suitable $E_1 \otimes_F D$-module isomorphism between V^j and S such that the resulting F-algebra homomorphism π_j between $\mathrm{End}_D(V^j)$ and A(S) satisfies the condition

$\pi_j \circ \omega_j = \rho_1$. Then the F-algebra homomorphism ι defined by $\iota(x) = (\pi_1^{-1}(x), \dots, \pi_l^{-1}(x))$ for $x \in A(S)$ satisfies the required condition. □

We now fix isomorphisms of $E_1 \otimes_F D$-modules $V^j \to S$, $j \in \{1, \dots, l\}$, as in Lemma 2.14. Then each Λ^j is in the affine class of $\mathfrak{S}$ (see (2.1) and [**22**, §1.4.8]), and these lattice sequences all have the same period, equal to that of Λ. Therefore, we are in the situation of paragraph 2.2. We set:

$$n = n_i, \quad m = e_{\varphi_i(\beta_i)}(\Lambda)k,$$

which both do not depend on $i \in \{1, 2\}$. By (2.5) and Remark 2.4, the element $\iota(g)$ normalizes Λ and conjugates $[\Lambda, n, m, \iota(\rho_2(\beta_2))]$ into a simple stratum in A which is equivalent to $[\Lambda, n, m, \varphi_1(\beta_1)]$. By the Skolem-Noether theorem, there is an element $x \in A^\times$ which conjugates the F-algebra homomorphisms $\iota \circ \rho_2$ and φ_2, and thus intertwines the simple strata $[\Lambda, n, m, \iota(\rho_2(\beta_2))]$ and $[\Lambda, n, m, \varphi_2(\beta_2)]$. Therefore the strata $[\Lambda, n, m, \varphi_i(\beta_i)]$ intertwine. As $m \leqslant m_1, m_2$, the strata $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ intertwine, which ends the proof of Proposition 1.7.

Remark 2.15. — There is a gap in the proof of the existence of the transfer map given in [**20**, Théorème 3.53], in the case where Λ is a strict lattice sequence. To complete this proof, one has to prove that, given a non-minimal simple pair (k, β) over F together with a realization $[\Lambda, n, m, \varphi(\beta)]$ of this pair in a simple central F-algebra A, there is a simple pair (k', γ) over F having realizations in A(E) and A which are approximations of β and $\varphi(\beta)$, respectively. More precisely, set $q = -k_0(\beta, \Lambda)$, and start with a stratum $[\Lambda, n, q, \gamma]$ in A which is simple and equivalent to $[\Lambda, n, q, \varphi(\beta)]$. If we denote by (k', γ) the simple pair of which this stratum is a realization, and if we set $n_0 = n_F(\beta)$ and $q_0 = -k_F(\beta)$, then we search for a realization $[\Lambda(E), n_0, q_0, \varphi_0(\gamma)]$ of (k', γ) in A(E) which is equivalent to the pure stratum $[\Lambda(E), n_0, q_0, \beta]$ (see paragraph 1.2). Let us remark that, when passing to $\widetilde{A}$ (see paragraph 2.1), we get a stratum $[\widetilde{\Lambda}, n, q, \gamma]$ which is simple and equivalent to $[\widetilde{\Lambda}, n, q, \varphi(\beta)]$. Now let $[\Lambda(E), n_0, q_0, \delta]$ be a stratum in A(E) which is simple and equivalent to $[\Lambda(E), n_0, q_0, \beta]$. By choosing a suitable decomposition of the F-vector space V into a direct sum of copies of E, we get an F-embedding:

$$\iota : A(E) \to \widetilde{A},$$

thus a stratum $[\widetilde{\Lambda}, n, q, \iota(\delta)]$ in $\widetilde{A}$ which is simple and equivalent to $[\widetilde{\Lambda}, n, q, \iota(\beta)]$. By the Skolem-Noether theorem, there exists an element $g \in \widetilde{A}^\times$ which conjugates $\iota(\beta)$ and $\varphi(\beta)$, thus intertwines the strata $[\widetilde{\Lambda}, n, q, \gamma]$ and $[\widetilde{\Lambda}, n, q, \iota(\delta)]$. The simple pairs (k', γ) and (k', δ) are thus endo-equivalent. Now let $[\Lambda(E), n_0, q_0, \jmath(\gamma)]$ be a realization of (k', γ) in A(E) which intertwines with $[\Lambda(E), n_0, q_0, \delta]$. By the "intertwining implies conjugacy" theorem

[**10**, Theorem 3.5.11] in the split simple central F-algebra A(E), there is $g \in \mathrm{U}(\Lambda(\mathrm{E}))$ such that $g\jmath(\gamma)g^{-1} - \delta \in \mathfrak{P}(\Lambda(\mathrm{E}))^{-q_0}$. The homomorphism of F-algebras $\varphi_0 : x \mapsto g\jmath(x)g^{-1}$ has the required property.

2.7. Before closing this section, we give a more elaborate example than that of paragraph 2.5, which will be very useful in the sequel. As in paragraph 2.5, let $[\Lambda, n, m, \beta]$ be a simple stratum in A, which is a realization of a simple pair (k, β) over F, and let e denote the period of Λ over $\mathcal{O}_{\mathrm{D}}$. Write B for the centralizer of the field $\mathrm{E} = \mathrm{F}(\beta)$ in A, fix a simple left B-module V_β and write D_β for the E-algebra opposite to the algebra of B-endomorphisms of V_β. Let Σ denote an $\mathcal{O}_{\mathrm{D}_\beta}$-lattice sequence on V_β corresponding to Λ by (1.1), and let e' denote its period over $\mathcal{O}_{\mathrm{D}_\beta}$. We fix an integer l which is a multiple of e and e' and set:

$$\Lambda^\ddagger : k \mapsto \Lambda_k \oplus \Lambda_{k+1} \oplus \cdots \oplus \Lambda_{k+l-1}, \tag{2.6}$$

which is a strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence on $\mathrm{V}^\ddagger = \mathrm{V} \oplus \cdots \oplus \mathrm{V}$ (l times) of the form (2.2). Thus we can form the simple stratum $[\Lambda^\ddagger, n, m, \beta]$ in $\mathrm{A}^\ddagger = \mathrm{End}_{\mathrm{D}}(\mathrm{V}^\ddagger)$, which is a realization of (k, β). Moreover, the hereditary $\mathcal{O}_{\mathrm{F}}$-order $\mathfrak{A}^\ddagger$ defined by $\Lambda^\ddagger$ is principal, and we have the following result.

Lemma 2.16. — *The stratum $[\Lambda^\ddagger, n, m, \beta]$ is sound (see Definition 1.14).*

Proof. — Write $\mathrm{B}^\ddagger$ for the centralizer of E in $\mathrm{A}^\ddagger$ and $\Sigma^\ddagger$ for the $\mathcal{O}_{\mathrm{D}_\beta}$-lattice sequence on $\mathrm{V}_\beta \times \cdots \times \mathrm{V}_\beta$ (l times) defined by:

$$\Sigma^\ddagger : k \mapsto \Sigma_k \oplus \Sigma_{k+1} \oplus \cdots \oplus \Sigma_{k+l-1}.$$

This is a strict lattice sequence, which defines a principal order of $\mathrm{B}^\ddagger$. By direct computation of each block, we get for all $k \in \mathbb{Z}$:

$$\mathfrak{P}_k(\Lambda^\ddagger) \cap \mathrm{B}^\ddagger = \mathfrak{P}_k(\Sigma^\ddagger), \tag{2.7}$$

which amounts to saying that $\Sigma^\ddagger$ is an $\mathcal{O}_{\mathrm{D}_\beta}$-lattice sequence which corresponds to $\Lambda^\ddagger$ by (1.1). In particular, its $\mathrm{B}^{\ddagger\times}$-normalizer is $\mathfrak{K}(\Lambda^\ddagger) \cap \mathrm{B}^{\ddagger\times}$. As $\Lambda^\ddagger$ is strict, its normalizer is equal to $\mathfrak{K}(\mathfrak{A}^\ddagger)$, and a similar statement holds for the lattice sequence $\Sigma^\ddagger$, so that we have $\mathfrak{K}(\mathfrak{A}^\ddagger) \cap \mathrm{B}^{\ddagger\times} = \mathfrak{K}(\mathfrak{A}^\ddagger \cap \mathrm{B}^\ddagger)$. Finally, if we choose $k = 0$ in (2.7), we deduce that $\mathfrak{A}^\ddagger \cap \mathrm{B}^\ddagger$ is principal. □

Note that, unlike (2.4), the process defined by (2.6) depends on E and l, and not only on the lattice sequence Λ.

Now let $[\Lambda, n_i, m_i, \beta_i]$, for $i = 1, 2$, be simple strata in A. Let e denote the period of Λ over $\mathcal{O}_{\mathrm{D}}$, and write e'_i for the period of the $\mathcal{O}_{\mathrm{D}_{\beta_i}}$-lattice sequence associated with Λ as above.

Proposition 2.17. — *Let $l \geqslant 1$ be a multiple of e'_1, e'_2 and e, and assume that the simple strata $[\Lambda, n_i, m_i, \beta_i]$, $i = 1, 2$, intertwine in* A *(resp. have the same embedding type). Then the simple strata $[\Lambda^{\ddagger}, n_i, m_i, \beta_i]$, $i = 1, 2$, are sound, and intertwine in $\mathrm{A}^{\ddagger}$ (resp. have the same embedding type).*

Proof. — This derives from Propositions 2.3 and 2.5, and Lemma 2.16. □

3. Intertwining implies conjugacy for simple strata

In this section, we prove the "intertwining implies conjugacy" property for simple strata, that is Proposition 1.9. We fix a simple central F-algebra A and a simple left A-module V as in paragraph 1.1. Associated with it, we have an F-division algebra D.

3.1. We will need the following general lemma on embedding types. Let $\mathscr{B}$ be a D-basis of V, and let L be a maximal unramified extension of F contained in D. The choice of $\mathscr{B}$ defines an isomorphism of F-algebras between A and $\mathrm{M}_r(\mathrm{D})$ for some integer $r \geqslant 1$, which allows us to identify these F-algebras. In particular, we will consider L as an extension of F contained in A. We write I_r for the identity matrix.

An embedding (K, Λ) in A is said to be *standard* with respect to the pair $(\mathscr{B}, \mathrm{L})$ if K is a subfield of L and if Λ is split by the basis $\mathscr{B}$ in the sense of [**3**].

Lemma 3.1. — *Let $(\mathscr{B}, \mathrm{L})$ be a pair as above.*

(1) *Any embedding in* A *is equivalent to an embedding which is standard with respect to the pair $(\mathscr{B}, \mathrm{L})$.*

(2) *Let (K, Λ) be standard with respect to $(\mathscr{B}, \mathrm{L})$, and let ϖ be a uniformizer of* D *normalizing* L*. Then conjugation by the diagonal matrix $\varpi \cdot \mathrm{I}_r$ normalizes* K *and* Λ*, and any element of* $\mathrm{Gal}(\mathrm{K}/\mathrm{F})$ *is induced by conjugation by a power of $\varpi \cdot \mathrm{I}_r$.*

Proof. — Assertion (2) follows from the fact that the map $x \mapsto \varpi x \varpi^{-1}$, for $x \in \mathrm{L}$, is a generator of the group $\mathrm{Gal}(\mathrm{L}/\mathrm{F})$. To prove (1), let (E, Λ) be an embedding in A, and set $\mathrm{K} = \mathrm{E}^{\diamond}$ (see paragraph 1.3 for the notation). One first notices that one can conjugate the pair (K, Λ) so that $\mathrm{K} \subseteq \mathrm{L}$, which we will assume. Let $\mathscr{I}$ be the non-enlarged Bruhat-Tits building of $\mathrm{A}^{\times}$ and $\mathscr{I}'$ be that of the centralizer $\mathrm{C}^{\times}$ of $\mathrm{K}^{\times}$ in $\mathrm{A}^{\times}$. Since the group $\mathrm{C}^{\times}$ identifies with $\mathrm{GL}_r(\mathrm{D}')$, where D' is the centralizer of K in D, the two buildings $\mathscr{I}$ and

$\mathscr{I}'$ have same dimension $r-1$. Recall (see [**3**, Théorème II.1.1]) that there exists a unique mapping:

$$\boldsymbol{j} = \boldsymbol{j}_{\mathrm{K/F}} : \mathscr{I}' \to \mathscr{I}$$

which is affine and $\mathrm{C}^\times$-equivariant. Its image is the set of $\mathrm{K}^\times$-fixed points in $\mathscr{I}$. The basis $\mathscr{B}$ gives rise to an apartment $\mathscr{A}$ of $\mathscr{I}$ (see e.g. [**3**, §0]), and points in that apartment are fixed by diagonal matrices of $\mathrm{A}^\times$ of the form $x \cdot \mathrm{I}_r$, with $x \in \mathrm{D}^\times$. In particular, they are fixed by $\mathrm{K}^\times$. It easily follows that there is some apartment $\mathscr{A}'$ in $\mathscr{I}'$ such that we have $\mathscr{A} = \boldsymbol{j}(\mathscr{A}')$.

The affine class of Λ determines a point y of the building $\mathscr{I}$ (see [**3**, I.7]). Since $\mathrm{K}^\times$ normalizes Λ, this point writes $\boldsymbol{j}(x)$, for some $x \in \mathscr{I}'$. Since $\mathrm{C}^\times$ acts transitively on the set of all apartments of $\mathscr{I}'$, and since any point of $\mathscr{I}'$ is contained in some apartment, there exists an element $h \in \mathrm{C}^\times$ such that $h \cdot x \in \mathscr{A}'$. Its follows that $h \cdot y = \boldsymbol{j}(h \cdot x)$ lies in $\mathscr{A}$. By [**3**, Proposition I.2.7], this means that the lattice sequence $h\Lambda$ is split by the basis $\mathscr{B}$, i.e. that $(h\mathrm{K}h^{-1}, h\Lambda) = (\mathrm{K}, h\Lambda)$ is standard with respect to the pair $(\mathscr{B}, \mathrm{L})$, as required. □

Remark 3.2. — We can rephrase Assertion (1) of the above lemma by saying that, for any embedding (E, Λ) in A, there is $g \in \mathrm{A}^\times$ such that $(\mathrm{E}^\diamond, \Lambda)$ is standard with respect to the pair $(g\mathscr{B}, g\mathrm{L}g^{-1})$.

If one writes $\mathrm{N}_{\mathrm{A}^\times}(\mathrm{K})$ for the normalizer of K in $\mathrm{A}^\times$, Assertion (2) can also be rephrased by saying that conjugation induces a surjective group homomorphism from the intersection $\mathfrak{K}(\Lambda) \cap \mathrm{N}_{\mathrm{A}^\times}(\mathrm{K})$ onto Gal(K/F). With the notation of the proof of Lemma 3.1, the kernel of this homomorphism is $\mathfrak{K}(\Lambda) \cap \mathrm{C}^\times$.

3.2. We will also need the following result, which generalizes [**6**, Lemma 1.6].

Proposition 3.3. — *Let Λ be an $\mathcal{O}_{\mathrm{D}}$-lattice sequence on* V *and* E/F *a finite extension. Suppose that there are two homomorphisms $\varphi_i : \mathrm{E} \to \mathrm{A}$ of* F*-algebras, $i = 1, 2$, such that the pairs $(\varphi_1(\mathrm{E}), \Lambda)$ and $(\varphi_2(\mathrm{E}), \Lambda)$ are two equivalent embeddings in* A*. Then there is an element $u \in \mathfrak{K}(\Lambda)$ such that:*

$$\varphi_1(x) = u\varphi_2(x)u^{-1}, \quad x \in \mathrm{E}. \tag{3.1}$$

Remark 3.4. — In particular, if K denotes the maximal unramified extension of F contained in E, then u conjugates $\varphi_2(\mathrm{K})$ to $\varphi_1(\mathrm{K})$.

Proof. — Since the embeddings $(\varphi_1(\mathrm{E}),\Lambda)$ and $(\varphi_2(\mathrm{E}),\Lambda)$ are equivalent, there exists an element $g\in\mathfrak{K}(\Lambda)$ such that $\varphi_1(\mathrm{E}^\diamond)=g\varphi_2(\mathrm{E}^\diamond)g^{-1}$. Then the mapping:

$$x\mapsto g\varphi_2(\varphi_1^{-1}(x))g^{-1} \tag{3.2}$$

is an F-automorphism of $\varphi_1(\mathrm{E}^\diamond)$. By Lemma 3.1(2), there is $h\in\mathfrak{K}(\Lambda)$ such that this F-automorphism is equal to $x\mapsto hxh^{-1}$. We thus have $\varphi_1(x)=w\varphi_2(x)w^{-1}$, for all $x\in\mathrm{E}^\diamond$, where $w=h^{-1}g$. So replacing φ_2 by a $\mathfrak{K}(\Lambda)$-conjugate, one may reduce to the case where φ_1 and φ_2 coincide on $\mathrm{E}^\diamond$. Assume now that we are is this case, and put $\mathrm{K}=\varphi_2(\mathrm{E}^\diamond)$. Let C be the centralizer of K in A, and write $\mathfrak{C}$ for the intersection of $\mathfrak{A}=\mathfrak{A}(\Lambda)$ with C.

Lemma 3.5. — *There is $u\in\mathrm{U}(\mathfrak{C})$ such that* (3.1) *holds.*

Proof. — We fix an unramified extension L of K such that the degree of L/F is equal to the reduced degree of D over F, denoted d. The L-algebra $\overline{\mathrm{C}}=\mathrm{C}\otimes_{\mathrm{K}}\mathrm{L}$ is thus split and, as E/K has residue class degree prime to d, the L-algebra $\mathrm{E}\otimes_{\mathrm{K}}\mathrm{L}$ is an extension of L, denoted $\overline{\mathrm{E}}$. For each i, the K-algebra homomorphism φ_i extends to a homomorphism of L-algebras $\overline{\mathrm{E}}\to\overline{\mathrm{C}}$, still denoted φ_i. By applying [**6**, Lemma 1.6] with the $\mathcal{O}_{\mathrm{L}}$-order $\overline{\mathfrak{C}}=\mathfrak{C}\otimes_{\mathcal{O}_{\mathrm{K}}}\mathcal{O}_{\mathrm{L}}$ and the homomorphisms of L-algebras φ_1 and φ_2, we get $u\in\mathrm{U}(\overline{\mathfrak{C}})$ satisfying (3.1). If we write $\overline{\mathrm{B}}$ for the centralizer of $\varphi_2(\mathrm{E})$ in $\overline{\mathrm{C}}$, then the 1-cocycle $\sigma\mapsto u^{-1}\sigma(u)$ defines a class in the Galois cohomology set:

$$\mathrm{H}^1(\mathrm{Gal}(\mathrm{L}/\mathrm{K}),\mathrm{U}(\overline{\mathfrak{C}})\cap\overline{\mathrm{B}}^\times).$$

This cohomology set is trivial by a standard filtration argument. (For more detail, see e.g. [**5**, §6].) Hence we actually may choose u in $\mathrm{U}(\mathfrak{C})$, which ends the proof of the lemma. □

Proposition 3.3 follows immediately from Lemma 3.5. □

Remark 3.6. — The conclusion of Proposition 3.3 does not hold if the pairs $(\varphi_1(\mathrm{E}),\Lambda)$ and $(\varphi_2(\mathrm{E}),\Lambda)$ are not assumed to be equivalent in A. For instance, take $\mathrm{A}=\mathrm{M}_2(\mathrm{D})$ where D is a quaternionic algebra over F, and let E/F be an unramified quadratic extension. One may embed E in $\mathrm{M}_2(\mathrm{F})$ so that the multiplicative group of the image normalizes the order $\mathrm{M}_2(\mathcal{O}_{\mathrm{F}})$. This gives an embedding φ_1 of E in $\mathrm{A}=\mathrm{M}_2(\mathrm{D})=\mathrm{M}_2(\mathrm{F})\otimes_{\mathrm{F}}\mathrm{D}$, such that $\varphi_1(\mathrm{E}^\times)$ normalizes $\mathrm{M}_2(\mathcal{O}_{\mathrm{D}})=\mathrm{M}_2(\mathcal{O}_{\mathrm{F}})\otimes_{\mathcal{O}_{\mathrm{F}}}\mathcal{O}_{\mathrm{D}}$. One also may embed E in D. The diagonal embedding of D in A gives rise to a second embedding φ_2 such that $\varphi_2(\mathrm{E}^\times)$ normalizes $\mathrm{M}_2(\mathcal{O}_{\mathrm{D}})$. Take Λ to be a strict lattice sequence in $\mathrm{D}\times\mathrm{D}$ defining the order $\mathfrak{A}=\mathrm{M}_2(\mathcal{O}_{\mathrm{D}})$, so that:

$$\mathfrak{K}(\Lambda)=\mathfrak{K}(\mathfrak{A})=\langle\varpi\rangle\cdot\mathrm{U}(\mathfrak{A}),$$

where ϖ denotes a uniformizer of D and $\langle\varpi\rangle$ the subgroup generated by ϖ. One can check that the pairs $(\varphi_i(\mathrm{E}),\Lambda)$, $i=1,2$, are inequivalent. Assume for a contradiction that there is an element $u\in\mathfrak{K}(\mathfrak{A})$ such that $\varphi_1(\mathrm{E})=u\varphi_2(\mathrm{E})u^{-1}$, and write $\mathfrak{P}$ for the radical of $\mathfrak{A}$. For $i=1,2$, the map φ_i induces an embedding of the residue field $\mathfrak{k}_\mathrm{E}$ in the $\mathfrak{k}_\mathrm{F}$-algebra $\mathfrak{A}/\mathfrak{P}$, which is isomorphic to $\mathrm{M}_2(\mathfrak{k}_\mathrm{D})$, and the images $\varphi_i(\mathfrak{k}_\mathrm{E})$, $i=1,2$, are conjugate under the action of u on the quotient $\mathfrak{A}/\mathfrak{P}$. But this action stabilizes the centre of $\mathrm{M}_2(\mathfrak{k}_\mathrm{D})$ and $\varphi_2(\mathfrak{k}_\mathrm{E})$ lies in this centre. This implies that $\varphi_1(\mathfrak{k}_\mathrm{E})$ is central: a contradiction.

3.3. We now prove the "intertwining implies conjugacy" property for simple strata, that is, Proposition 1.9. For $i=1,2$, let $[\Lambda,n,m,\beta_i]$ a simple stratum in A. Assume that they intertwine in A and have the same embedding type, and write K_i for the maximal unramified extension of F contained in $\mathrm{E}_i=\mathrm{F}(\beta_i)$. By Remark 3.4, we may replace β_2 by some $\mathfrak{K}(\Lambda)$-conjugate and assume that K_1 and K_2 are equal to a common extension K of F. We write $\mathrm{N}_{\mathrm{A}^\times}(\mathrm{K})$ for the normalizer of K in $\mathrm{A}^\times$. Therefore, we are reduced to proving that there is an element $u\in\mathfrak{K}(\Lambda)\cap\mathrm{N}_{\mathrm{A}^\times}(\mathrm{K})$ such that we have $\beta_1-u\beta_2u^{-1}\in\mathfrak{P}_{-m}(\Lambda)$.

We proceed as in the proof of proposition 1.7 (see paragraph 2.6). Let us fix a simple right $\mathrm{E}_1\otimes_\mathrm{F}\mathrm{D}$-module S and set $\mathrm{A(S)}=\mathrm{End}_\mathrm{D}(\mathrm{S})$. Let us denote by ρ_1 the natural F-algebra homomorphism $\mathrm{E}_1\to\mathrm{A(S)}$. We write $\mathfrak{S}$ for the unique (up to translation) E_1-pure strict $\mathcal{O}_\mathrm{D}$-lattice sequence on S and fix an F-algebra homomorphism $\rho_2:\mathrm{E}_2\to\mathrm{A(S)}$ such that $\mathfrak{S}$ is $\rho_2(\mathrm{E}_2)$-pure, and such that $(\rho_1(\mathrm{E}_1),\mathfrak{S})$ and $(\rho_2(\mathrm{E}_2),\mathfrak{S})$ have the same embedding type in A(S) (see Lemma 2.11). We also fix a decomposition:

$$\mathrm{V}=\mathrm{V}^1\oplus\cdots\oplus\mathrm{V}^l \tag{3.3}$$

of V into simple right $\mathrm{K}(\beta)\otimes_\mathrm{F}\mathrm{D}$-modules (which all are copies of S) such that Λ is decomposed by (3.3) in the sense of [**22**, Définition 1.13], that is, Λ is the direct sum of the lattice sequences $\Lambda^j=\Lambda\cap\mathrm{V}^j$, for $j\in\{1,\dots,l\}$. By choosing, for each j, an isomorphism of $\mathrm{K}(\beta)\otimes_\mathrm{F}\mathrm{D}$-modules between S and V^j, this decomposition gives us an F-algebra homomorphism:

$$\iota:\mathrm{A(S)}\to\mathrm{A}.$$

Using Lemma 2.14, we may assume that this homomorphism satisfies $\iota(\rho_1(\beta_1))=\beta_1$.

For each $i\in\{1,2\}$, let (k,β_i) be the simple pair of which $[\Lambda,n,m,\beta_i]$ is a realization. By putting $n_0=n_\mathrm{F}(\beta_i)$ and $m_0=e_{\rho_i(\beta_i)}(\mathfrak{S})k$, which do not depend on i by Proposition 2.10, we get a simple stratum $[\mathfrak{S},n_0,m_0,\rho_i(\beta_i)]$ which is a realization of (k,β_i) in A(S). The proof of [**5**, Theorem 4.1.2] (see also [**17**, Lemma 10.5]) gives us an element $v\in\mathfrak{K}(\mathfrak{S})$ such that $\rho_1(\mathrm{K})=v\rho_2(\mathrm{K})v^{-1}$ and $\beta_1-v\beta_2v^{-1}\in\mathfrak{P}_{-m_0}(\mathfrak{S})$. By Proposition 3.3, there is

$w \in \mathfrak{K}(\Lambda)$ such that $\iota(\rho_2(x)) = wxw^{-1}$ for all $x \in \mathrm{E}_2$, and, by Remark 3.4, this element satisfies $w\mathrm{K}w^{-1} = \iota(\rho_2(\mathrm{K}))$. Thus $u = \iota(v)w$ normalizes K and Λ and satisfies the required condition:

$$\beta_1 - u\beta_2 u^{-1} \in \mathfrak{P}_{-e_{\beta_i}(\Lambda)k}(\Lambda) \subseteq \mathfrak{P}_{-m}(\Lambda),$$

which ends the proof of Proposition 1.9.

4. Realizations and intertwining for simple characters

The two main results of this section are Propositions 4.9 and 4.11. The first one asserts that two endo-equivalent ps-characters have realizations with very special properties, allowing us to use the results of [**17**]. The second one leads to the rigidity theorem 4.16, and will also give us an important property of the base change map in paragraph 7.2.

4.1. In this paragraph, we generalize the construction given in paragraph 2.7 by incorporating the notion of embedding type. For this, we will need the following definition.

Let $[\Lambda, n, m, \beta]$ be a simple stratum in A, which is a realization of a simple pair (k, β) over F, and set $\mathrm{E} = \mathrm{F}(\beta)$. The containment of $\mathcal{O}_\mathrm{E}$ in $\mathfrak{A}(\Lambda)$ allows us to identify the residue field $\mathfrak{k} = \mathfrak{k}_{\mathrm{E}^\diamond}$ with its canonical image in the $\mathfrak{k}_\mathrm{F}$-algebra $\overline{\mathfrak{A}} = \mathfrak{A}(\Lambda)/\mathfrak{P}(\Lambda)$.

Definition 4.1. — The *Fröhlich invariant* of $[\Lambda, n, m, \beta]$ is the degree over $\mathfrak{k}_\mathrm{F}$ of the intersection of $\mathfrak{k}$ with the centre of $\overline{\mathfrak{A}}$.

Recall that this invariant has been introduced by Fröhlich (see [**15**]) for sound strata. In this case, we have the following important property.

Theorem 4.2 **([15], Theorem 2)**. — *For $i = 1, 2$, let (K_i, Λ) be a sound embedding in* A *where K_i/F is an unramified extension contained in* A. *These embeddings are equivalent if and only if $[\mathrm{K}_1^\diamond : \mathrm{F}] = [\mathrm{K}_2^\diamond : \mathrm{F}]$ and they have the same Fröhlich invariant.*

We will need the two following lemmas.

Lemma 4.3. — *Let us fix an integer $l \geqslant 1$, an $\mathcal{O}_\mathrm{D}$-lattice sequence Λ' and an integer m' as in paragraph 2.2, and let us form the simple stratum $[\Lambda', n', m', \beta]$ in A'. The simple strata $[\Lambda, n, m, \beta]$ and $[\Lambda', n', m', \beta]$ have the same Fröhlich invariant.*

Proof. — Let us identify A' with the matrix algebra $\mathrm{M}_l(\mathrm{A})$, and write j for the $\mathfrak{k}_\mathrm{F}$-algebra homomorphism $\mathfrak{k} \to \overline{\mathfrak{A}}' = \mathfrak{A}(\Lambda')/\mathfrak{P}(\Lambda')$ induced by the embedding of $\mathcal{O}_\mathrm{E}$ in $\mathfrak{A}(\Lambda')$ (which is the restriction to $\mathcal{O}_\mathrm{E}$ of the diagonal embedding of E in A'). By a direct computation, we see that the diagonal blocks of $\mathfrak{A}(\Lambda')$ are equal to $\mathfrak{A}(\Lambda)$, and that of its radical $\mathfrak{P}(\Lambda')$ are equal to $\mathfrak{P}(\Lambda)$. This is enough to prove that $j(x)$ is central in $\overline{\mathfrak{A}}'$ if and only if x is central in $\overline{\mathfrak{A}}$. Thus the strata $[\Lambda, n, m, \beta]$ and $[\Lambda', n', m', \beta]$ have the same Fröhlich invariant. □

Lemma 4.4. — *We set $\Lambda' = \Lambda \oplus \Lambda$ and $m' = m$ (thus $l = 2$). There exists an element $u \in \mathrm{A}'^{\times}$ such that Λ' is $u\mathrm{F}(\beta)u^{-1}$-pure and the simple stratum $[\Lambda', n, m, u\beta u^{-1}]$ in A' has Fröhlich invariant* 1.

Proof. — We fix a D-basis $\mathscr{B}$ of V, a maximal unramified extension L of F contained in D and a uniformizer ϖ of D normalizing L (see paragraph 3.1). According to Lemma 3.1, we may identify A with $\mathrm{M}_r(\mathrm{D})$ and assume that the embedding $(\mathrm{E}^{\diamond}, \Lambda)$ is in standard form with respect to $(\mathscr{B}, \mathrm{L})$. The map $\varphi : x \mapsto \varpi x \varpi^{-1}$ defines a generator of $\mathrm{Gal}(\mathrm{E}^{\diamond}/\mathrm{F})$, and thus induces on the residue field $\mathfrak{k} = \mathfrak{k}_{\mathrm{E}^{\diamond}}$ a generator of $\mathrm{Gal}(\mathfrak{k}/\mathfrak{k}_{\mathrm{F}})$, denoted σ. We write j for the $\mathfrak{k}_{\mathrm{F}}$-algebra homomorphism from $\mathfrak{k}$ to $\overline{\mathfrak{A}}$ induced by φ, which is the composite of σ with the canonical embedding of $\mathfrak{k}$ in $\overline{\mathfrak{A}}$. Thus, one has $j(x) = x$ if and only if $x \in \mathfrak{k}_{\mathrm{F}}$. We now set:

$$u = \begin{pmatrix} \mathrm{I}_r & 0 \\ 0 & \varpi \cdot \mathrm{I}_r \end{pmatrix} \in \mathrm{M}_2(\mathrm{A}) = \mathrm{A}'.$$

If one identifies the $\mathfrak{k}_{\mathrm{F}}$-algebra $\overline{\mathfrak{A}}' = \mathfrak{A}(\Lambda')/\mathfrak{P}(\Lambda')$ with $\mathrm{M}_2(\overline{\mathfrak{A}})$, then the $\mathfrak{k}_{\mathrm{F}}$-algebra homomorphism j' from $\mathfrak{k}$ to $\overline{\mathfrak{A}}'$ induced by $x \mapsto uxu^{-1}$ is given by:

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & j(x) \end{pmatrix}.$$

Therefore, $j'(x)$ is central in $\overline{\mathfrak{A}}'$ if and only if $x = j(x)$ is central in $\overline{\mathfrak{A}}$, that is, if and only if $x \in \mathfrak{k}_{\mathrm{F}}$. □

This leads us to the following result. For $i = 1, 2$, let (k, β_i) be a simple pair over F, let $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ be a realization of (k, β_i) in A and let $\theta_i \in \mathcal{C}(\Lambda, m_i, \varphi_i(\beta_i))$ be a simple character.

Proposition 4.5. — *Assume θ_1 and θ_2 intertwine in $\mathrm{A}^{\times}$. Then there is a simple central* F*-algebra* A' *together with realizations $[\Lambda', n_i, m_i, \varphi'_i(\beta_i)]$ of (k, β_i) in A' (with the same n_i and m_i), with $i = 1, 2$, which are sound and have the same embedding type, and such that θ'_1 and θ'_2 intertwine in $\mathrm{A}'^{\times}$, where $\theta'_i \in \mathcal{C}(\Lambda', m_i, \varphi'_i(\beta_i))$ denotes the transfer of θ_i.*

Proof. — First, we reduce to the case where the strata $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ have Fröhlich invariant 1. Let $g \in \mathrm{A}^{\times}$ intertwine the characters θ_1 and θ_2 as in (1.7). We set $\Lambda' = \Lambda \oplus \Lambda$ and $\mathrm{A}' = \mathrm{M}_2(\mathrm{A})$ and, for each i, we fix an element $u_i \in \mathrm{A}'^{\times}$ as in Lemma 4.4 so that the simple stratum $[\Lambda', n_i, m_i, u_i\varphi_i(\beta_i)u_i^{-1}]$ has Fröhlich invariant 1. For each i, let θ'_i be the transfer of θ_i in $\mathcal{C}(\Lambda', m_i, \varphi_i(\beta_i))$, and let θ''_i be that of θ_i in $\mathcal{C}(\Lambda', m_i, u_i\varphi_i(\beta_i)u_i^{-1})$, which is equal to the conjugate character $x \mapsto \theta'_i(u_i^{-1}xu_i)$. By the proof of Proposition 2.6, the element $g' = \iota(g) \in \mathrm{A}'^{\times}$ intertwines θ'_1 and θ'_2, where ι denotes the diagonal embedding of A in A′, and it follows that $g'' = u_1^{-1}g'u_2$ intertwines θ''_1 and θ''_2. Thus we can assume that the strata $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ have Fröhlich invariant 1. Using Proposition 2.17 (with some suitable integer $l \geqslant 1$) and Lemma 4.3 together, we see that the simple strata $[\Lambda^{\ddagger}, n_i, m_i, \varphi_i(\beta_i)]$ are sound with Fröhlich invariant 1. By Theorem 4.2, they have the same embedding type. Let $\theta_i^{\ddagger}$ be the transfer of θ_i in $\mathcal{C}(\Lambda^{\ddagger}, m_i, \varphi_i(\beta_i))$. The fact that $\theta_1^{\ddagger}$ and $\theta_2^{\ddagger}$ intertwine in $\mathrm{A}^{\ddagger\times}$ follows from Proposition 2.6. □

Remark 4.6. — The assumption $[\mathrm{F}(\beta_1) : \mathrm{F}] = [\mathrm{F}(\beta_2) : \mathrm{F}]$ is not needed in the proof.

4.2. Before proving the first main result of this section, that is Proposition 4.9, we will need the following lemmas. Compare the first one with Proposition 2.10.

Lemma 4.7. — *For $i = 1, 2$, let (Θ_i, k, β_i) be a ps-character over* F*, and suppose that Θ_1 and Θ_2 are endo-equivalent. Then $n_{\mathrm{F}}(\beta_1) = n_{\mathrm{F}}(\beta_2)$, $e_{\mathrm{F}}(\beta_1) = e_{\mathrm{F}}(\beta_2)$, $f_{\mathrm{F}}(\beta_1) = f_{\mathrm{F}}(\beta_2)$ and $k_{\mathrm{F}}(\beta_1) = k_{\mathrm{F}}(\beta_2)$.*

Proof. — By assumption, we have $[\mathrm{F}(\beta_1) : \mathrm{F}] = [\mathrm{F}(\beta_2) : \mathrm{F}]$ and there is a simple central F-algebra A together with realizations $[\Lambda, n_i, m_i, \beta_i]$ of (k, β_i), for $i = 1, 2$, such that the corresponding simple characters θ_1 and θ_2 intertwine in $\mathrm{A}^{\times}$. By Proposition 4.5, we can assume that these realizations are sound and have the same embedding type. We now follow the proof of [**6**, Proposition 8.4]. An argument similar to the first part of this proof (which we do not reproduce) gives us $n_1 = n_2$, denoted n. Now consider the integers m_1, m_2. By symmetry, we can assume that $m_1 \geqslant m_2$. Let us choose a simple stratum $[\Lambda, n, m_1, \gamma]$ in A which is equivalent to $[\Lambda, n, m_1, \beta_2]$ and let θ_0 denote the restriction of θ_2 to $\mathrm{H}^{m_1+1}(\gamma, \Lambda)$. The characters θ_0 and θ_1 still intertwine, which implies, by the "intertwining implies conjugacy" theorem [**17**, Corollary 10.15], the existence of $u \in \mathfrak{K}(\Lambda)$ such that $\mathcal{C}(\Lambda, m_1, \beta_1) = \mathcal{C}(\Lambda, m_1, u\gamma u^{-1})$. By Proposition 1.17, we get:

$$k_{\mathrm{F}}(\beta_1) = k_{\mathrm{F}}(\gamma), \quad [\mathrm{F}(\beta_1) : \mathrm{F}] = [\mathrm{F}(\gamma) : \mathrm{F}]. \tag{4.1}$$

By [**5**, Theorem 5.1(ii)], the equality $[\mathrm{F}(\beta_2) : \mathrm{F}] = [\mathrm{F}(\gamma) : \mathrm{F}]$ implies that $[\Lambda, n, m_1, \beta_2]$ is a simple stratum in A. By Theorem 1.16, we get $e_{\mathrm{F}}(\beta_1) = e_{\mathrm{F}}(\beta_2)$ and $f_{\mathrm{F}}(\beta_1) = f_{\mathrm{F}}(\beta_2)$, and (4.1) gives us $k_{\mathrm{F}}(\beta_1) = k_{\mathrm{F}}(\beta_2)$. The remaining equality is a consequence of the identity $n_i = e_{\beta_i}(\Lambda)n_{\mathrm{F}}(\beta_i)$. □

Corollary 4.8. — *Theorem 1.13 implies Theorems 1.11 and 1.12.*

Proof. — For $i = 1, 2$, let (Θ_i, k, β_i) be a ps-character over F, and suppose that Θ_1 and Θ_2 are endo-equivalent. Let A be a simple central F-algebra. For each i, let $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ be a realization of (k, β_i) in A, and put $\theta_i = \Theta_i(\Lambda, m_i, \varphi_i)$. Write $n = n_i$ and:

$$m = e_{\varphi_i(\beta_i)}(\Lambda)k,$$

which do not depend on i by Lemma 4.7. As $m_1, m_2 \geqslant m$, we may assume without loss of generality that $m_1 = m_2 = m$. Let us fix an F-algebra homomorphism $\varphi_3 : \mathrm{F}(\beta_2) \to \mathrm{A}$ such that the simple strata $[\Lambda, n, m, \varphi_1(\beta_1)]$ and $[\Lambda, n, m, \varphi_3(\beta_2)]$ have the same embedding type, and let θ_3 denote the transfer of θ_2 in $\mathcal{C}(\Lambda, m, \varphi_3(\beta_2))$. According to Theorem 1.13, there is an element $u \in \mathfrak{K}(\Lambda)$ such that $\theta_3(x) = \theta_1(uxu^{-1})$ for all $x \in \mathrm{H}^{m+1}(\varphi_3(\beta_2), \Lambda)$ and, by the Skolem-Noether theorem, there is an element $g \in \mathrm{A}^\times$ such that $\varphi_3(x) = g\varphi_2(x)g^{-1}$. Thus g intertwines θ_3 and θ_2, which proves that θ_1 and θ_2 intertwine in $\mathrm{A}^\times$ and ends the proof of Theorem 1.11.

Assume now that the strata $[\Lambda, n, m, \varphi_i(\beta_i)]$, $i = 1, 2$ have the same embedding type. Then applying Theorem 1.13 gives immediately Theorem 1.12. □

We are thus reduced to proving Theorem 1.13, which will be done in section 8. For this we will have to develop base change methods (see sections 5, 6 and 7). We now state and prove the first main result of this section.

Proposition 4.9. — *For $i = 1, 2$, let (Θ_i, k, β_i) be a ps-character over* F*, and suppose that Θ_1 and Θ_2 are endo-equivalent. Write K_i for the maximal unramified extension of* F *contained in $\mathrm{F}(\beta_i)$. Then there exists a simple central* F*-algebra* A *together with realizations $[\Lambda, n, m, \varphi_i(\beta_i)]$ of (k, β_i), for $i = 1, 2$, which are sound and have the same embedding type, and such that:*

(1) *m is a multiple of k;*
(2) *$\varphi_1(\mathrm{K}_1) = \varphi_2(\mathrm{K}_2)$;*
(3) *$\Theta_1(\Lambda, m, \varphi_1) = \Theta_2(\Lambda, m, \varphi_2)$.*

Proof. — By Proposition 4.5, there is a simple central F-algebra A together with realizations $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ of (k, β_i), for $i = 1, 2$, sound and having the same embedding type, such that $\theta_1 = \Theta_1(\Lambda, m_1, \varphi_1)$ and $\theta_2 = \Theta_2(\Lambda, m_2, \varphi_2)$ intertwine in $\mathrm{A}^\times$. By Lemma 4.7, we have $n_1 = n_2$, and the integer $m = e_{\varphi_i(\beta_i)}(\Lambda)k$ does not depend on i.

Lemma 4.10. — *For each i, there exists a unique $\vartheta_i \in \mathcal{C}(\Lambda, m, \varphi_i(\beta_i))$ extending θ_i, and the characters ϑ_1 and ϑ_2 intertwine in $\mathrm{A}^\times$.*

Proof. — The proof is similar to that of [**10**, Lemma 3.6.7] and [**6**, Lemma 8.5] together. One just has to replace Corollary 3.3.21 of [**10**] by Proposition 2.16 of [**24**], and Theorems 3.5.8, 3.5.9 and 3.5.11 of [**10**] by Corollary 10.15 and Propositions 9.9 and 9.10 of [**17**]. □

Therefore we can assume that m_1, m_2 are both equal to m. The result now follows from the "intertwining implies conjugacy" theorem [**17**, Corollary 10.15]. □

4.3. We now assume that we are in the situation of paragraph 2.4. Let us fix two simple strata $[\Lambda, n, m, \beta_i]$, $i = 1, 2$, in A. We set $n' = an$ and fix a non-negative integer m' such that $\lfloor m'/a \rfloor = m$, so that we have simple strata $[\Lambda', n', m', \beta_i]$, $i = 1, 2$, in A$'$, where Λ' is defined by (2.2). We fix a simple character θ_i in $\mathcal{C}(\Lambda, m, \beta_i)$ and write θ'_i for its transfer in $\mathcal{C}(\Lambda', m', \beta_i)$. The aim of this paragraph is to prove the following proposition, which is the second main result of this section.

Proposition 4.11. — *Assume that θ_1 and θ_2 are equal. Then θ'_1 and θ'_2 are equal.*

Proof. — We first prove the following lemma, which generalizes [**10**, Theorem 3.5.9] and [**17**, Proposition 9.10] (see also [**13**, Lemme 7.9], which gives a similar result in the split case for *semisimple characters* and whose proof we follow).

Lemma 4.12. — *Assume that $m \geqslant 1$, and that $\mathcal{C}(\Lambda, m, \beta_1) \cap \mathcal{C}(\Lambda, m, \beta_2)$ is not empty. Then we have $\mathrm{H}^m(\beta_1, \Lambda) = \mathrm{H}^m(\beta_2, \Lambda)$.*

Proof. — We put $\nu = 2m - 1$ and, for $i = 1, 2$, we choose a simple stratum $[\Lambda, n, \nu, \gamma_i]$ equivalent to $[\Lambda, n, \nu, \beta_i]$ in A. Then, for each $i = 1, 2$, we have $\mathcal{C}(\Lambda, \nu, \beta_i) = \mathcal{C}(\Lambda, \nu, \gamma_i)$ and, from [**24**, Proposition 2.15], we have $\mathrm{H}^m(\beta_i, \Lambda) = \mathrm{H}^m(\gamma_i, \Lambda)$. Since the restriction of a simple character to $\mathrm{H}^{\nu+1}(\beta_1, \Lambda) = \mathrm{H}^{\nu+1}(\beta_2, \Lambda)$ is still a simple character, the intersection $\mathcal{C}(\Lambda, \nu, \gamma_1) \cap \mathcal{C}(\Lambda, \nu, \gamma_2)$ is not empty. By computing the intertwining of an element of this intersection via the formula of [**24**, Théorème 2.23], we get:

$$\Omega_{q_1-\nu}(\gamma_1, \Lambda)\mathrm{B}^\times_{\gamma_1}\Omega_{q_1-\nu}(\gamma_1, \Lambda) = \Omega_{q_2-\nu}(\gamma_2, \Lambda)\mathrm{B}^\times_{\gamma_2}\Omega_{q_2-\nu}(\gamma_2, \Lambda)$$

with the notations of *loc. cit.* and where, for each $i = 1, 2$, we write B_{γ_i} for the centralizer of $\mathrm{F}(\gamma_i)$ in A and $q_i = -k_0(\gamma_i, \Lambda)$. Taking the intersection with $\mathfrak{P}_m(\Lambda)$ and then its additive closure, we find that the following set:

$$\mathfrak{Q}^i_m + (\mathfrak{P}_{q_i-\nu}(\Lambda) \cap \mathfrak{n}_{-\nu}(\gamma_i, \Lambda))\, \mathfrak{Q}^i_m + \mathfrak{Q}^i_m \mathfrak{J}^{\lceil q_i/2 \rceil}(\gamma_i, \Lambda), \tag{4.2}$$

is independent of i, where we have put $\mathfrak{Q}^i_m = \mathfrak{P}_m(\Lambda) \cap \mathrm{B}_{\gamma_i}$ and where the notations $\mathfrak{J}^k$ and $\mathfrak{H}^k$, for $k \geqslant 0$, are defined in [**24**, §2.4]. We claim that the set in (4.2) is contained in $\mathfrak{H}^m(\gamma_i,\Lambda) = \mathfrak{H}^m(\beta_i,\Lambda)$. For then, adding $\mathfrak{H}^{m+1}(\gamma_i,\Lambda) = \mathfrak{H}^{m+1}(\beta_i,\Lambda)$, which is also independent of i, we see that:

$$\mathfrak{H}^m(\beta_i,\Lambda) = \mathfrak{H}^m(\gamma_i,\Lambda) = \mathfrak{Q}^i_m + \mathfrak{H}^{m+1}(\gamma_i,\Lambda)$$

is independent of i, as required. We need the following lemma (see [**28**, Lemma 3.11(i)]).

Lemma 4.13. — *Let $[\Lambda, n, m, \beta]$ be a simple stratum in* A *with $q = -k_0(\beta,\Lambda)$. For each integer $1 \leqslant k \leqslant q-1$, we have:*

$$(\mathfrak{n}_{-k}(\beta,\Lambda) \cap \mathfrak{P}_{q-k}(\Lambda))\, \mathfrak{J}^{\lceil k/2\rceil}(\beta,\Lambda) \subseteq \mathfrak{H}^{\lfloor k/2\rfloor+1}(\beta,\Lambda).$$

Proof. — We write $[\widetilde{\Lambda}, n, m, \beta]$ for the simple stratum in $\widetilde{\mathrm{A}} = \mathrm{End}_{\mathrm{F}}(\mathrm{V})$ associated with $[\Lambda, n, m, \beta]$ (see paragraph 2.1). Then we have:

$$\left(\mathfrak{n}_{-k}(\beta,\widetilde{\Lambda}) \cap \mathfrak{P}_{q-k}(\widetilde{\Lambda})\right) \mathfrak{J}^{\lceil k/2\rceil}(\beta,\widetilde{\Lambda}) \subseteq \mathfrak{H}^{\lfloor k/2\rfloor+1}(\beta,\widetilde{\Lambda})$$

by [**28**, Lemma 3.11(i)]. By taking the intersection with A, we get the expected result. □

We now see that:

$$(\mathfrak{P}_{q_i-\nu}(\Lambda) \cap \mathfrak{n}_{-\nu}(\gamma_i,\Lambda))\, \mathfrak{Q}^i_m \subseteq (\mathfrak{P}_{q_i-\nu}(\Lambda) \cap \mathfrak{n}_{-\nu}(\gamma_i,\Lambda))\, \mathfrak{J}^{\lceil \nu/2\rceil}(\gamma_i,\Lambda) \subseteq \mathfrak{H}^m(\gamma_i,\Lambda).$$

Similarly, we have:

$$\mathfrak{Q}^i_m \mathfrak{J}^{\lceil q_i/2\rceil}(\gamma_i,\Lambda) \subseteq \left(\mathfrak{P}_{q_i-(q_i-m)}(\Lambda) \cap \mathfrak{n}_{m-q_i}(\gamma_i,\Lambda)\right) \mathfrak{J}^{\lceil (q_i-m)/2\rceil}(\gamma_i,\Lambda) \subseteq \mathfrak{H}^{\lfloor (q_i-m)/2\rfloor+1}(\gamma_i,\Lambda).$$

Since the left hand side here is clearly also contained in $\mathfrak{P}_m(\Lambda)$, we see that it is contained in $\mathfrak{H}^m(\gamma_i,\Lambda)$ as required. This also completes the proof of Lemma 4.12. □

For each i, write Θ_i for the ps-character defined by the pair $([\Lambda, n, m, \beta_i], \theta_i)$, and recall that θ_1 and θ_2 are equal.

Lemma 4.14. — *We have $e_{\mathrm{F}}(\beta_1) = e_{\mathrm{F}}(\beta_2)$ and $f_{\mathrm{F}}(\beta_1) = f_{\mathrm{F}}(\beta_2)$.*

Proof. — By Proposition 4.5, there is a simple central F-algebra A together with realizations $[\Lambda^0, n, m, \varphi^0_i(\beta_i)]$ of (k, β_i), with $i = 1, 2$, which are sound and have the same embedding type, and such that $\Theta_1(\Lambda^0, m, \varphi^0_1)$ and $\Theta_2(\Lambda^0, m, \varphi^0_2)$ intertwine in $\mathrm{A}^{0\times}$. Let us write f for the greatest common divisor of $f_{\mathrm{F}}(\beta_1)$ and $f_{\mathrm{F}}(\beta_2)$ and K_i for the maximal unramified extension of F contained in $\mathrm{F}(\varphi^0_i(\beta_i))$. Then Theorem 1.16 gives us the expected equality. □

Thus the ps-characters Θ_1 and Θ_2 are endo-equivalent, which allows us to use Lemma 4.7.

Lemma 4.15. — *The characters* θ'_1 *and* θ'_2 *are equal if and only if we have:*

$$\mathrm{H}^{m'+1}(\beta_1,\Lambda') = \mathrm{H}^{m'+1}(\beta_2,\Lambda'). \tag{4.3}$$

Proof. — This follows immediately from Lemma 2.7. □

Thus we are reduced to proving equality (4.3), and for this, we claim that it is enough to prove that:

$$\mathrm{H}^{q'}(\beta_1,\Lambda') = \mathrm{H}^{q'}(\beta_2,\Lambda'), \tag{4.4}$$

where $q' = -k_0(\beta_i,\Lambda')$ is independent of i by Lemma 4.7. Indeed, assume that (4.4) holds, and let t' be the smallest integer in $\{m',\dots,q'-1\}$ such that:

$$\mathrm{H}^{t'+1}(\beta_1,\Lambda') = \mathrm{H}^{t'+1}(\beta_2,\Lambda'). \tag{4.5}$$

Suppose that $t' \neq m'$. By Lemma 4.15, the characters θ'_1 and θ'_2 agree on (4.5), that is, the intersection $\mathcal{C}(\Lambda',t',\beta_1)\cap\mathcal{C}(\Lambda',t',\beta_2)$ is not empty. By Lemma 4.12, we get an equality which contradicts the minimality of t'. Hence $t' = m'$ and we are thus reduced to proving (4.4), which we do by induction on β_1. Assume first that β_1 is minimal over F. Then so is β_2 by Lemma 4.7, so that we have:

$$\mathrm{H}^{q'}(\beta_1,\Lambda') = \mathrm{U}_{q'}(\Lambda') = \mathrm{H}^{q'}(\beta_2,\Lambda').$$

Assume now that β_1 is not minimal over F, set $q = -k_0(\beta_i,\Lambda)$, which is independent of i by Lemma 4.7, and choose a simple stratum $[\Lambda,n,q,\gamma_i]$ in A equivalent to the stratum $[\Lambda,n,q,\beta_i]$, for each $i\in\{1,2\}$. We then have:

$$\mathrm{H}^{q'}(\beta_i,\Lambda') = \mathrm{H}^{q'}(\gamma_i,\Lambda'),$$

and the restriction $\vartheta_i = \theta_i \mid \mathrm{H}^{q+1}(\gamma_i,\Lambda)$ belongs to $\mathcal{C}(\Lambda,q,\gamma_i)$. As $\beta_i-\gamma_i\in\mathfrak{P}_{-q}(\Lambda)$, the simple characters ϑ_1 and ϑ_2 are equal. If we write ϑ'_i for the transfer of ϑ_i to the set $\mathcal{C}(\Lambda',q',\gamma_i)$, then the inductive hypothesis implies that $\vartheta'_1=\vartheta'_2$. Therefore, the intersection $\mathcal{C}(\Lambda',q',\gamma_1)\cap\mathcal{C}(\Lambda',q',\gamma_2)$ is not empty, and Lemma 4.12 gives us the required equality (4.4). This ends the proof of Proposition 4.11. □

4.4. Before closing this section, we prove the following rigidity theorem for simple characters, which generalizes [**10**, Theorem 3.5.8] and [**17**, Proposition 9.9] to simple characters in non-necessarily split simple central F-algebras with non-necessarily strict lattice sequences.

Theorem 4.16. — *For $i = 1, 2$, let $[\Lambda, n, m, \beta_i]$ be a simple stratum in a simple central* F*-algebra* A*. Assume that the intersection $\mathcal{C}(\Lambda, m, \beta_1) \cap \mathcal{C}(\Lambda, m, \beta_2)$ is not empty. Then we have $\mathcal{C}(\Lambda, m, \beta_1) = \mathcal{C}(\Lambda, m, \beta_2)$.*

Proof. — For each $i \in \{1, 2\}$, we fix a simple character $\theta_i \in \mathcal{C}(\Lambda, m, \beta_i)$ and assume that θ_1 and θ_2 are equal. In particular, we have:

$$\mathrm{H}^{m+1}(\beta_1, \Lambda) = \mathrm{H}^{m+1}(\beta_2, \Lambda). \tag{4.6}$$

By choosing an integer l as in Proposition 2.17, we have sound simple strata $[\Lambda^{\ddagger}, n, m, \beta_i]$, $i = 1, 2$, in $\mathrm{A}^{\ddagger}$. If we write $\theta_i^{\ddagger}$ for the transfer of θ_i to $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_i)$, then it follows from Proposition 4.11 that the simple characters $\theta_1^{\ddagger}$ and $\theta_2^{\ddagger}$ are equal, hence that the intersection $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_1) \cap \mathcal{C}(\Lambda^{\ddagger}, m, \beta_2)$ is not empty. By Proposition 1.17, the sets $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_i)$, $i = 1, 2$, are equal. As the transfer map from $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_i)$ to $\mathcal{C}(\Lambda, m, \beta_i)$ is the restriction map from $\mathrm{H}^{m+1}(\beta_i, \Lambda^{\ddagger})$ to $\mathrm{H}^{m+1}(\beta_i, \Lambda)$, the equality (4.6) implies that $\mathcal{C}(\Lambda, m, \beta_1) = \mathcal{C}(\Lambda, m, \beta_2)$. □

It is natural to ask whether the simple strata $[\Lambda, n, m, \beta_i]$ in Theorem 4.16 have the same embedding type. We have the following conjecture[(1)].

Conjecture 4.17. — *For $i = 1, 2$, let $[\Lambda, n, m, \beta_i]$ be a simple stratum in a simple central* F*-algebra* A*. Assume that the intersection $\mathcal{C}(\Lambda, m, \beta_1) \cap \mathcal{C}(\Lambda, m, \beta_2)$ is not empty, and that Λ is strict. Then these simple strata have the same embedding type.*

Note that we know from [**5**, Lemma 5.2] that two equivalent simple strata (with respect to a strict lattice sequence) have the same embedding type.

In the case where the strata are sound, we will prove below that this conjecture is true. First we need a series of lemmas.

Lemma 4.18. — *Let* E/F *be a finite extension with ramification index e, contained in a simple central* F*-algebra* A*, and let $\mathfrak{B}$ be a principal $\mathcal{O}_{\mathrm{E}}$-order of period r in the centralizer* B *of* E *in* A*. Write $\mathrm{A} \simeq \mathrm{M}_k(\mathrm{D})$ for some $k \geqslant 1$ and some* F*-division algebra* D*, and write d for the reduced degree of* D *over* F*.*

(1) *There exists a unique* E*-pure hereditary $\mathcal{O}_{\mathrm{F}}$-order $\mathfrak{A}$ in* A *such that $\mathfrak{B} = \mathfrak{A} \cap \mathrm{B}$ and $\mathfrak{K}(\mathfrak{B}) = \mathfrak{K}(\mathfrak{A}) \cap \mathrm{B}^{\times}$, and such an order is principal.*

(2) *The period of $\mathfrak{A}$ is equal to $re/(re, d)$, where (re, d) denotes the greatest common divisor of re and d.*

(1) This conjecture — and an even more general statement — is proven in [**25**, Lemma 3.5].

Proof. — The first part is given by [**16**, Corollary 1.4(ii)]. Part (2) follows for instance from the formula given in the proof of [**24**, Théorème 1.7]. □

In other words, there exists a unique hereditary $\mathcal{O}_\mathrm{F}$-order $\mathfrak{A}$ in A such that $\mathfrak{A}\cap\mathrm{B}=\mathfrak{B}$ and that $(\mathrm{E},\mathfrak{A})$ is a sound embedding in A.

Lemma 4.19. — *For $i=1,2$, let E_i be an extension of* F *contained in* A *and let $\mathfrak{A}$ be a hereditary $\mathcal{O}_\mathrm{F}$-order in* A *such that $(\mathrm{E}_i,\mathfrak{A})$ is a sound embedding in* A. *Write $\mathfrak{B}_i$ for the intersection of $\mathfrak{A}$ with the centralizer of E_i in* A. *Let f be the greatest common divisor of $f(\mathrm{E}_1:\mathrm{F})$ and $f(\mathrm{E}_2:\mathrm{F})$, and for each i, let K_i be the unramified extension of* F *of degree f contained in E_i. Assume E_1 and E_2 have the same ramification order e and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have the same period r. Then the embeddings $(\mathrm{K}_1,\mathfrak{A})$ and $(\mathrm{K}_2,\mathfrak{A})$ are equivalent in* A.

Proof. — Let $\mathfrak{C}_i$ denote the intersection of $\mathfrak{A}$ with the centralizer C_i of K_i in A. If we write B_i for the centralizer of E_i in A, then we have $\mathfrak{B}_i=\mathfrak{C}_i\cap\mathrm{B}_i$ and $\mathfrak{K}(\mathfrak{B}_\mathrm{i})=\mathfrak{K}(\mathfrak{C}_i)\cap\mathrm{B}_i^\times$. Using Lemma 4.18, the period of $\mathfrak{C}_i$ is equal to $re/(re,d_i)$, where d_i is the reduced degree of the K_i-division algebra D_i such that C_i is isomorphic to $\mathrm{M}_{k_i}(\mathrm{D}_i)$ for some $k_i\geqslant 1$. Using for instance [**29**], we have $d_i=d/(d,f)$, which does not depend on i. By the Skolem-Noether theorem, there is $g\in\mathrm{A}^\times$ such that $g\mathrm{K}_1g^{-1}=\mathrm{K}_2$. Thus $g\mathfrak{C}_1g^{-1}$ and $\mathfrak{C}_2$ are two principal $\mathcal{O}_{\mathrm{K}_2}$-orders in C_2 with the same period, which implies that there exists $h\in\mathrm{C}_2^\times$ such that $g\mathfrak{C}_1g^{-1}=h\mathfrak{C}_2h^{-1}$. Let us write $u=h^{-1}g$. Using the unicity property (1) of Lemma 4.18, we get $\mathfrak{A}=u^{-1}\mathfrak{A}u$, that is $u\in\mathfrak{K}(\mathfrak{A})$. □

We now prove Conjecture 4.17 in the case where the strata are sound.

Proposition 4.20. — *For $i=1,2$, let $[\Lambda,n,m,\beta_i]$ be a sound simple stratum in a simple central* F*-algebra* A. *Assume that the intersection $\mathcal{C}(\Lambda,m,\beta_1)\cap\mathcal{C}(\Lambda,m,\beta_2)$ is not empty. Then these simple strata have the same embedding type.*

Proof. — For each i, we fix a simple character $\theta_i\in\mathcal{C}(\Lambda,m,\beta_i)$ and assume θ_1 and θ_2 are equal. Thus we have $[\mathrm{F}(\beta_1):\mathrm{F}]=[\mathrm{F}(\beta_2):\mathrm{F}]$ by Proposition 1.17. By Lemma 4.7, we also have $e_\mathrm{F}(\beta_1)=e_\mathrm{F}(\beta_2)$ and $f_\mathrm{F}(\beta_1)=f_\mathrm{F}(\beta_2)$. If we write $\mathrm{I}_{\mathrm{U}(\Lambda)}(\theta_i)$ for the intertwining of θ_i in $\mathrm{U}(\Lambda)$, then [**20**, Théorème 3.50] gives us:

$$\mathrm{I}_{\mathrm{U}(\Lambda)}(\theta_i)\mathrm{U}^1(\Lambda)/\mathrm{U}^1(\Lambda)\simeq\mathrm{U}(\mathfrak{B}_i)/\mathrm{U}^1(\mathfrak{B}_i),$$

where $\mathfrak{B}_i$ is the intersection of $\mathfrak{A}=\mathfrak{A}(\Lambda)$ with the centralizer of β_i in A. As the stratum $[\Lambda,n,m,\beta_i]$ is sound, $\mathfrak{B}_i$ is a principal $\mathcal{O}_{\mathrm{F}(\beta_i)}$-order. Thus there are a finite extension $\mathfrak{k}_i$ of $\mathfrak{k}_\mathrm{F}$ and two positive integers $r_i,s_i\geqslant 1$ such that:

$$\mathrm{U}(\mathfrak{B}_i)/\mathrm{U}^1(\mathfrak{B}_i)\simeq\mathrm{GL}_{s_i}(\mathfrak{k}_\mathrm{i})^{r_i}.$$

Since it does not depend on i, we have $r_1 = r_2$. Now write K_i for the maximal unramified extension of F contained in $\mathrm{F}(\beta_i)$. Using Lemma 4.19 with $\mathrm{E}_i = \mathrm{F}(\beta_i)$, we deduce that the embeddings $(\mathrm{K}_1, \mathfrak{A})$ and $(\mathrm{K}_2, \mathfrak{A})$ are equivalent in A. □

5. The interior lifting

In this section, we develop an interior lifting process for simple strata and characters with respect to a finite unramified extension K of F, in a way similar to [**6**] and [**17**]. The situation in [**6**] is somewhat more general than ours, since the authors only assume K/F to be tamely ramified, but is only concerned with simple strata and characters in split simple central F-algebras attached to strict lattice sequences. The situation in [**17**] deals with any simple central F-algebra, but puts an unnecessarily restrictive condition on the simple strata (they are supposed to be sound).

5.1. Let A be a simple central F-algebra and K/F be a finite unramified extension contained in A. Let C denote the centralizer of K in A. We fix a simple left A-module V and a simple left C-module W. The following definition extends [**6**, Definition 2.2] to strata with non-necessarily strict lattice sequences.

Definition 5.1. — A stratum $[\Lambda, n, m, \beta]$ in A is said to be K-*pure* if it is pure, if β centralizes K and if the algebra $\mathrm{K}[\beta]$ is a field such that $\mathrm{K}[\beta]^{\times}$ normalizes Λ.

Given a K-pure stratum $[\Lambda, n, m, \beta]$ in A, we can form the pure stratum $[\Gamma, n, m, \beta]$, where Γ is the unique (up to translation) lattice sequence on W defined by:

$$\mathfrak{P}_k(\Lambda) \cap \mathrm{C} = \mathfrak{P}_k(\Gamma), \quad k \in \mathbb{Z}. \tag{5.1}$$

Note that the $\mathrm{C}^{\times}$-normalizer of Γ is equal to $\mathfrak{K}(\Lambda) \cap \mathrm{C}^{\times}$. We then get a process:

$$[\Lambda, n, m, \beta] \mapsto [\Gamma, n, m, \beta] \tag{5.2}$$

giving an injection, respecting equivalence, between the set of K-pure strata of A and the set of pure strata of C. The fact that Γ is defined only up to translation makes (5.2) not well defined, but this will be of no importance in the sequel. We now discuss the image of simple K-pure strata of A by (5.2).

Proposition 5.2. — (1) *Let* $[\Lambda, n, m, \beta]$ *be a* K-*pure stratum in* A. *Then:*

$$k_0(\beta, \Gamma) \leqslant k_0(\beta, \Lambda). \tag{5.3}$$

(2) *Suppose moreover that* $[\Lambda, n, m, \beta]$ *is simple. Then the stratum* $[\Gamma, n, m, \beta]$ *given by the map (5.2) is simple.*

Proof. — As K is unramified over F, the lattice sequences Λ and Γ have the same period over $\mathcal{O}_{\mathrm{F}}$. By (1.4) it is then enough to prove that $k_{\mathrm{K}}(\beta) \leqslant k_{\mathrm{F}}(\beta)$. Let $\mathfrak{L}$ denote the strict $\mathcal{O}_{\mathrm{F}}$-lattice sequence on $\mathrm{K}(\beta)$ defined by $i \mapsto \mathfrak{p}^i_{\mathrm{K}(\beta)}$. By [**6**, Theorem 2.4], we have:

$$k_{\mathrm{K}}(\beta) \leqslant k_0(\beta, \mathfrak{L}).$$

On the other hand, we have $e_\beta(\mathfrak{L}) = 1$ as K is unramified over F. By (1.4) again, we get the expected result. Suppose now that the stratum $[\Lambda, n, m, \beta]$ is simple. Then the fact that $[\Gamma, n, m, \beta]$ is simple derives immediately from (5.3). □

Remark 5.3. — For a case where the map (5.2) is not surjective, see [**24**, Exemple 1.6]. Compare with the split case [**6**, (2.3)].

5.2. Given a simple stratum $[\Gamma, n, m, \beta]$ in C in the image of (5.2), the K-pure stratum of A corresponding to it may not be simple. However, we have the following result, which generalizes [**6**, Corollary 3.8].

Proposition 5.4. — *Let $[\Lambda, n, m, \beta]$ be a* K*-pure stratum in* A *such that $[\Gamma, n, m, \beta]$ is simple. Then there exists a simple stratum $[\Gamma, n, m, \beta']$ in* C *equivalent to $[\Gamma, n, m, \beta]$ such that the stratum $[\Lambda, n, m, \beta']$ is simple.*

Moreover, β' can be chosen such that the maximal unramified extension of F *contained in $\mathrm{F}(\beta')$ is contained in that of $\mathrm{F}(\beta)$.*

Proof. — Let (k, β) denote the simple pair over K of which $[\Gamma, n, m, \beta]$ is a realization, fix a simple right $\mathrm{K}(\beta) \otimes_{\mathrm{F}} \mathrm{D}$-module S and set $\mathrm{A}(\mathrm{S}) = \mathrm{End}_{\mathrm{D}}(\mathrm{S})$. Write ρ for the natural K-algebra homomorphism from $\mathrm{K}(\beta)$ to $\mathrm{A}(\mathrm{S})$. Let $\mathfrak{S}$ denote the unique (up to translation) $\rho(\mathrm{K}(\beta))$-pure strict $\mathcal{O}_{\mathrm{D}}$-lattice sequence on S and n_0 the $\mathfrak{S}$-valuation of $\rho(\beta)$, and set:

$$m_0 = e_{\rho(\beta)}(\mathfrak{S})k,$$

so that $[\mathfrak{S}, n_0, m_0, \rho(\beta)]$ is a K-pure stratum in $\mathrm{A}(\mathrm{S})$. Write $\mathrm{C}(\mathrm{S})$ for the centralizer of K in $\mathrm{A}(\mathrm{S})$, fix a simple left $\mathrm{C}(\mathrm{S})$-module T and let $[\mathfrak{T}, n_0, m_0, \rho(\beta)]$ be the stratum in $\mathrm{C}(\mathrm{S})$ attached to $[\mathfrak{S}, n_0, m_0, \rho(\beta)]$ by (5.2). This stratum is a realization of (k, β) in $\mathrm{C}(\mathrm{S})$, hence this is a simple stratum. According to [**6**, Theorem 3.7], the simple pair (k, β) is endo-equivalent to a simple pair (k, α) over K which is a K/F-lift of some simple pair over F in the sense of [**6**] (see paragraph 3). By [**6**, Proposition 1.10], the extensions $\mathrm{K}(\alpha)$ and $\mathrm{K}(\beta)$ have the same ramification index and residue class degree over K, which implies by [**5**, Corollary 3.16] that there is a realization $[\mathfrak{T}, n_0, m_0, \varphi(\alpha)]$ of (k, α) in $\mathrm{C}(\mathrm{S})$, having the same embedding type as $[\mathfrak{T}, n_0, m_0, \rho(\beta)]$.

We now pass to the strata $[\widetilde{\mathfrak{T}}, n_0, m_0, \varphi(\alpha)]$ and $[\widetilde{\mathfrak{T}}, n_0, m_0, \rho(\beta)]$ in the K-algebra $\mathrm{End}_{\mathrm{K}}(\mathrm{T})$ (see paragraph 2.1). By [**24**] (see Théorème 1.7 and Remarque 1.8), the lattice sequence $\mathfrak{T}$ (and thus $\widetilde{\mathfrak{T}}$) is in the affine class of a strict lattice sequence, so that, up to renormalization, one may consider it as being strict (see Lemma 2.2). By [**6**, Proposition 1.10], these strata thus intertwine. Hence, using Proposition 2.13, we can replace φ by some $\mathfrak{K}(\mathfrak{T})$-conjugate and assume that the strata $[\mathfrak{T}, n_0, m_0, \varphi(\alpha)]$ and $[\mathfrak{T}, n_0, m_0, \rho(\beta)]$ are equivalent, and that the maximal unramified extension of K contained in $\mathrm{K}(\varphi(\alpha))$ is equal to that of $\mathrm{K}(\rho(\beta))$. We check that the stratum $[\mathfrak{S}, n_0, m_0, \varphi(\alpha)]$ is simple as in the proof of [**6**, Proposition 4.3]. We now fix a decomposition:

$$\mathrm{V} = \mathrm{V}^1 \oplus \cdots \oplus \mathrm{V}^l \tag{5.4}$$

of V into simple right $\mathrm{K}(\beta) \otimes_{\mathrm{F}} \mathrm{D}$-modules (which all are copies of S) such that Λ is decomposed by (5.4) in the sense of [**22**, Définition 1.13], that is, Λ is the direct sum of the lattice sequences $\Lambda^j = \Lambda \cap \mathrm{V}^j$, for $j \in \{1, \dots, l\}$. By choosing, for each j, an isomorphism of $\mathrm{K}(\beta) \otimes_{\mathrm{F}} \mathrm{D}$-modules between S and V^j, this decomposition gives us an F-algebra homomorphism:

$$\iota : \mathrm{A(S)} \to \mathrm{A}.$$

Using Lemma 2.14, we may assume that this homomorphism satisfies $\iota(\rho(\beta)) = \beta$. If we set $\beta' = \iota(\varphi(\alpha))$, then the stratum $[\Gamma, n, m, \beta']$ is simple and satisfies the conditions of the first part of Proposition 5.4.

In particular, the pure strata $[\Lambda, n, m, \beta]$ and $[\Lambda, n, m, \beta']$ are equivalent, and the second one is simple. By replacing the lattice sequence Λ by $\Lambda^\dagger$ (see paragraph 2.5), we can apply [**5**, Theorem 5.1(ii)] and thus get that $f_{\mathrm{F}}(\beta')$ divides $f_{\mathrm{F}}(\beta)$. Moreover, the maximal unramified extension of K contained in $\mathrm{K}(\beta')$ is equal to that of $\mathrm{K}(\beta)$, denoted L. As K/F is unramified, the extension L/F is unramified. Thus the maximal unramified extension of F contained in $\mathrm{F}(\beta')$ and that of $\mathrm{F}(\beta)$ are two finite unramified extensions of F contained in L. According to the condition on their degrees, it follows that the maximal unramified extension of F contained in $\mathrm{F}(\beta')$ in contained in that of $\mathrm{F}(\beta)$. □

5.3. Let $[\Lambda, n, m, \beta]$ be a K-pure simple stratum in A, and let $[\Gamma, n, m, \beta]$ be the stratum in C given by the map (5.2), which is simple by Proposition 5.2. Recall that one attaches to these simple strata compact open subgroups $\mathrm{H}^{m+1}(\beta, \Lambda)$ of $\mathrm{A}^\times$ and $\mathrm{H}^{m+1}(\beta, \Gamma)$ of $\mathrm{C}^\times$, respectively.

Proposition 5.5. — *Let* $[\Lambda, n, m, \beta]$ *be a* K*-pure simple stratum in* A*, and let* $[\Gamma, n, m, \beta]$ *correspond to it by (5.2). Then we have:*

$$H^{m+1}(\beta, \Lambda) \cap C^\times = H^{m+1}(\beta, \Gamma).$$

Proof. — It is enough to prove it when $m = 0$. The proof is by induction on β. Let R denote the centralizer of $K(\beta)$ in A. Assume first that β is minimal over F, so that:

$$H^1(\beta, \Lambda) = (U_1(\Lambda) \cap B^\times)U_{\lfloor n/2 \rfloor+1}(\Lambda).$$

According to (5.1), we get:

$$H^1(\beta, \Lambda) \cap C^\times = (U_1(\Gamma) \cap R^\times)U_{\lfloor n/2 \rfloor+1}(\Gamma),$$

which is equal to $H^1(\beta, \Gamma)$ as β is minimal over K by Proposition 5.2. Now assume that β is not minimal over F, set $q = -k_0(\beta, \Lambda)$ and $r = \lfloor q/2 \rfloor + 1$, and choose a simple stratum $[\Gamma, n, q, \gamma]$ equivalent to $[\Gamma, n, q, \beta]$ such that $[\Lambda, n, q, \gamma]$ is simple and K-pure, which is possible thanks to Proposition 5.4. We then have:

$$H^1(\beta, \Lambda) = (U_1(\Lambda) \cap B^\times)H^r(\gamma, \Lambda)$$

and, if we set $q_1 = -k_0(\beta, \Gamma)$ and $r_1 = \lfloor q_1/2 \rfloor + 1$, we have:

$$H^1(\beta, \Gamma) = (U_1(\Gamma) \cap R^\times)H^{r_1}(\gamma, \Gamma).$$

As $-k_0(\gamma, \Gamma) \geqslant q_1 \geqslant q$, the group $H^r(\gamma, \Gamma)$ is equal to $(U_r(\Gamma) \cap R^\times)H^{r_1}(\gamma, \Gamma)$. It follows from (5.1) that the group $H^1(\beta, \Gamma)$ is equal to the intersection $H^1(\beta, \Lambda) \cap C^\times$. This ends the proof of Proposition 5.5. □

5.4. We now want to lift simple characters. For this, given a simple stratum $[\Lambda, n, m, \beta]$ in A, we will need a characterization of the set $\mathcal{C}(\Lambda, m, \beta)$ by induction on β, generalizing [**20**, Proposition 3.47] to the case where Λ is non-necessarily strict.

Lemma 5.6. — *Let* $[\Lambda, n, m, \beta]$ *be a simple stratum in* A *and* θ *be a character of the group* $H^{m+1}(\beta, \Lambda)$*, and set* $q = -k_0(\beta, \Lambda)$ *and* $m' = \max\{m, \lfloor q/2 \rfloor\}$*. Then* $\theta \in \mathcal{C}(\Lambda, m, \beta)$ *if and only if it is normalized by* $\mathfrak{K}(\Lambda) \cap B^\times$ *and satisfies the following conditions:*

(1) *if* β *is minimal over* F*, then* $\theta \mid U_{m'+1}(\Lambda) = \Psi^A_\beta$ *and* $\theta \mid U_{m+1}(\Lambda) \cap B^\times = \chi \circ N_{B/E}$ *for some character* χ *of* $1 + \mathfrak{p}_E$ *(see (1.3) for the definition of* Ψ^A_β*);*

(2) *if* β *is not minimal over* F*, and if* $[\Lambda, n, q, \gamma]$ *is simple and equivalent to* $[\Lambda, n, q, \beta]$ *in* A*, then* $\theta \mid H^{m'+1}(\beta, \Lambda) = \theta_0 \Psi^A_{\beta-\gamma}$ *and* $\theta \mid H^{m+1}(\beta, \Lambda) \cap B^\times = \chi \circ N_{B/E}$ *for some simple character* $\theta_0 \in \mathcal{C}(\Lambda, m', \gamma)$ *and some character* χ *of* $1 + \mathfrak{p}_E$*.*

Proof. — The proof is similar to that of [**20**, Proposition 3.11], and we do not repeat it. Note that [**17**, Lemma 1.9] is actually not needed in the proof, and that [**12**, Corollary 5.3] has to be replaced by [**24**, Proposition 1.20] and [**10**, Proposition 3.3.9] by [**20**, Proposition 3.30]. □

Let $[\Lambda, n, m, \beta]$ be a simple K-pure stratum in A, and let $[\Gamma, n, m, \beta]$ correspond to it by (5.2). We write $\mathcal{C}(\Gamma, m, \beta)$ for the set of simple characters attached to $[\Gamma, n, m, \beta]$ with respect to the additive character:

$$\Psi_{\mathrm{K}} = \Psi \circ \mathrm{tr}_{\mathrm{K}/\mathrm{F}}, \tag{5.5}$$

which is trivial on $\mathfrak{p}_{\mathrm{K}}$ but not on $\mathcal{O}_{\mathrm{K}}$, as K is unramified over F. Compare the following theorem with [**6**, Theorem 7.7] and [**17**, Proposition 7.1].

Theorem 5.7. — *Let $[\Lambda, n, m, \beta]$ be a simple* K*-pure stratum in* A*, and let $[\Gamma, n, m, \beta]$ correspond to it by (5.2). Then, for any $\theta \in \mathcal{C}(\Lambda, m, \beta)$, we have:*

$$\theta \mid \mathrm{H}^{m+1}(\beta, \Gamma) \in \mathcal{C}(\Gamma, m, \beta).$$

Proof. — The proof is by induction on β. Let θ^{K} denote the restriction of θ to the group $\mathrm{H}^{m+1}(\beta, \Gamma)$ and R the centralizer of $\mathrm{K}(\beta)$ in A. Assume first that β is minimal over F. By Proposition 5.2, it is also minimal over K. If $m \geqslant \lfloor n/2 \rfloor$, we have $\mathcal{C}(\Lambda, m, \beta) = \{\Psi_\beta^{\mathrm{A}}\}$ and $\mathcal{C}(\Gamma, m, \beta) = \{\Psi_\beta^{\mathrm{C}}\}$, where Ψ_β^{C} denotes the character of $\mathrm{U}_{m+1}(\Gamma)$ defined by:

$$\Psi_\beta^{\mathrm{C}} : x \mapsto \Psi_{\mathrm{K}} \circ \mathrm{tr}_{\mathrm{C}/\mathrm{K}}(\beta(x-1)).$$

So we just need to prove that:

$$\Psi_\beta^{\mathrm{A}} \mid \mathrm{U}_{m+1}(\Gamma) = \Psi_\beta^{\mathrm{C}}, \tag{5.6}$$

which is given by [**6**, Property (7.6)]. If $m \leqslant \lfloor n/2 \rfloor$, then any $\theta \in \mathcal{C}(\Lambda, m, \beta)$ extends the character $\Psi_\beta^{\mathrm{A}} \mid \mathrm{U}_{\lfloor n/2 \rfloor + 1}(\Lambda)$ and its restriction to $\mathrm{U}_{m+1}(\Lambda) \cap \mathrm{B}^\times$ has the form:

$$\theta \mid \mathrm{U}_{m+1}(\Lambda) \cap \mathrm{B}^\times = \chi \circ \mathrm{N}_{\mathrm{B}/\mathrm{E}}$$

for some character χ of $1 + \mathfrak{p}_{\mathrm{E}}$. Therefore the character θ^{K} extends $\Psi_\beta^{\mathrm{C}} \mid \mathrm{U}_{\lfloor n/2 \rfloor + 1}(\Gamma)$, and its restriction to $\mathrm{U}_{m+1}(\Gamma) \cap \mathrm{R}^\times$ has the form:

$$\theta^{\mathrm{K}} \mid \mathrm{U}_{m+1}(\Gamma) \cap \mathrm{R}^\times = \chi \circ \mathrm{N}_{\mathrm{K}(\beta)/\mathrm{E}} \circ \mathrm{N}_{\mathrm{R}/\mathrm{K}(\beta)}.$$

Finally, the group $\mathfrak{K}(\Gamma) \cap \mathrm{R}^\times$, which normalizes both θ and the group $\mathrm{H}^{m+1}(\beta, \Gamma)$, normalizes θ^{K}. It follows from Lemma 5.6 that $\theta^{\mathrm{K}} \in \mathcal{C}(\Gamma, m, \beta)$.

Now assume that β is not minimal over F. We set $q = -k_0(\beta, \Lambda)$ and $r = \lfloor q/2 \rfloor + 1$, and choose a simple stratum $[\Gamma, n, q, \gamma]$ equivalent to $[\Gamma, n, q, \beta]$ such that $[\Lambda, n, q, \gamma]$ is simple and K-pure. If $m \geqslant \lfloor q/2 \rfloor$, then any $\theta \in \mathcal{C}(\Lambda, m, \beta)$ can be written as $\theta = \theta_0 \Psi^{\mathrm{A}}_{\beta-\gamma}$ for some simple character $\theta_0 \in \mathcal{C}(\Lambda, m, \gamma)$. Now we claim that:

$$\mathrm{H}^{m+1}(\beta, \Gamma) = \mathrm{H}^{m+1}(\gamma, \Gamma). \tag{5.7}$$

We write $q_1 = -k_0(\beta, \Gamma)$. If $q_1 = q$, then the equality (5.7) follows by definition. Otherwise, we have $q_1 > q$ by Proposition 5.2. The strata $[\Gamma, n, q, \beta]$ and $[\Gamma, n, q, \gamma]$ are thus both simple, and (5.7) follows. We now restrict the character θ to the group given by (5.7) and get $\theta^{\mathrm{K}} = \theta_0^{\mathrm{K}} \Psi^{\mathrm{C}}_{\beta-\gamma}$, where θ_0^{K} denotes the restriction $\theta_0 \mid \mathrm{H}^{m+1}(\gamma, \Gamma)$, and this restriction is in $\mathcal{C}(\Gamma, m, \gamma)$ by the inductive hypothesis. If $q_1 = q$, then θ^{K} is in $\mathcal{C}(\Gamma, m, \beta)$ by definition. Otherwise, $[\Gamma, n, q, \beta]$ is simple and the result follows from [**24**, Proposition 2.15]. The case $m \leqslant \lfloor q/2 \rfloor$ reduces to the previous one exactly as in the minimal case. □

6. Interior lifting and transfer

In this section, we define the interior lift of a ps-character. This amounts to studying the behaviour of the interior lifting process with respect to transfer.

6.1. As in section 5, we are given in this section a simple central F-algebra A and a finite unramified extension K/F contained in A. We fix a finite unramified extension L of K such that the L-algebra:

$$\overline{\mathrm{A}} = \mathrm{A} \otimes_{\mathrm{F}} \mathrm{L}$$

is split. This L-algebra inherits an action of the Galois group of L/F in the obvious way, and we consider A as being naturally embedded in $\overline{\mathrm{A}}$ by $j_{\mathrm{A}} : a \mapsto a \otimes_{\mathrm{F}} 1$. We have a decomposition:

$$\mathrm{K} \otimes_{\mathrm{F}} \mathrm{L} = \mathrm{K}^1 \oplus \cdots \oplus \mathrm{K}^f \tag{6.1}$$

into simple $\mathrm{K} \otimes_{\mathrm{F}} \mathrm{L}$-modules, where f denotes the degree of K/F. For each $i \in \{1, \ldots, f\}$, we write e^i for the minimal idempotent in $\mathrm{K} \otimes_{\mathrm{F}} \mathrm{L}$ corresponding to K^i. The centralizer of $\mathrm{K} \otimes_{\mathrm{F}} \mathrm{L}$ in $\overline{\mathrm{A}}$, denoted $\overline{\mathrm{U}}$, is equal to $\mathrm{C} \otimes_{\mathrm{F}} \mathrm{L}$. By identifying it with $\mathrm{C} \otimes_{\mathrm{K}} (\mathrm{K} \otimes_{\mathrm{F}} \mathrm{L})$ and using (6.1), we get a decomposition:

$$\overline{\mathrm{U}} = \overline{\mathrm{U}}^1 \oplus \cdots \oplus \overline{\mathrm{U}}^f,$$

where the K^i-algebra $\overline{\mathrm{U}}^i = \mathrm{e}^i \overline{\mathrm{A}} \mathrm{e}^i$ identifies with $\mathrm{C} \otimes_{\mathrm{K}} \mathrm{K}^i$ for each $i \in \{1, \ldots, f\}$.

In a similar way, we may consider the centralizer C of K in A as being embedded in the split L-algebra $\overline{\mathrm{C}} = \mathrm{C} \otimes_{\mathrm{K}} \mathrm{L}$ by the K-algebra homomorphism $j_{\mathrm{C}} : c \mapsto c \otimes_{\mathrm{K}} 1$.

Similarly to the case of simple characters (see paragraph 5.4), we will define the interior lift of a quasi-simple character by restriction from $\overline{\mathrm{A}}$ to $\overline{\mathrm{C}}$. For this we need an embedding of $\overline{\mathrm{C}}$ in $\overline{\mathrm{A}}$ satisfying some conditions with respect to j_{A} and j_{C} (see below), but there is no canonical such embedding. We choose a set:

$$\mathrm{S} = \{\sigma_1, \dots, \sigma_f\} \subseteq \mathrm{Gal}(\mathrm{L}/\mathrm{F}) \tag{6.2}$$

of representatives of $\mathrm{Hom}_{\mathrm{F}}(\mathrm{K}, \mathrm{L})$ in $\mathrm{Gal}(\mathrm{L}/\mathrm{F})$, that is a subset of $\mathrm{Gal}(\mathrm{L}/\mathrm{F})$ such that the restriction map from L to K induces a bijection from S to $\mathrm{Hom}_{\mathrm{F}}(\mathrm{K}, \mathrm{L})$. For simplicity, we assume that we have ordered the e^i's so that:

(6.3) K^1 *and* L *are isomorphic* $\mathrm{K} \otimes \mathrm{L}$*-modules and* $\sigma_i(\mathrm{e}^1) = \mathrm{e}^i$ *for any* $i \in \{1, \dots, f\}$.

This gives us an F-algebra homomorphism:

$$\varkappa : \overline{\mathrm{C}} \xrightarrow{\simeq} \overline{\mathrm{U}}^1 \subseteq \overline{\mathrm{U}}, \tag{6.4}$$

and $\sigma_i \circ \varkappa$ is an F-algebra homomorphism from $\overline{\mathrm{C}}$ to $\overline{\mathrm{U}}^i$ for each integer $i \in \{1, \dots, f\}$. The following lemma gives us a relationship between (6.4) and the embeddings j_{A} and j_{C}.

Lemma 6.1. — *Let* $j_{\mathrm{A,C}}$ *denote the restriction of* j_{A} *to* C*, with values in* $\overline{\mathrm{U}}$*. Then the* F*-algebra homomorphism from* $\overline{\mathrm{C}}$ *to* $\overline{\mathrm{U}}$ *defined by:*

$$\iota = \iota_{\mathrm{S}} : x \mapsto \sigma_1 \circ \varkappa(x) + \dots + \sigma_f \circ \varkappa(x) \tag{6.5}$$

satisfies the equality $\iota \circ j_{\mathrm{C}} = j_{\mathrm{A,C}}$.

Proof. — We have $\sigma_i(\mathrm{e}^1 j_{\mathrm{A}}(x)) = \mathrm{e}^i j_{\mathrm{A}}(x)$ for all $i \in \{1, \dots, f\}$ and $x \in \mathrm{C}$, which implies that $\iota \circ \mathrm{e}^1 j_{\mathrm{A,C}} = j_{\mathrm{A,C}}$. Note that $\mathrm{e}^1 j_{\mathrm{A,C}} = j_{\mathrm{C}}$, so that we get the expected equality. □

6.2. Let $[\Lambda, n, m, \beta]$ be a simple stratum in A, which is a realization of a simple pair (k, β) over F. *In this paragraph, we assume that* Λ *is a strict lattice sequence.* If we fix a simple left $\overline{\mathrm{A}}$-module $\overline{\mathrm{V}}$, then there is a unique (up to translation) $\mathcal{O}_{\mathrm{L}}$-lattice sequence $\overline{\Lambda}$ on $\overline{\mathrm{V}}$ such that:

$$\mathfrak{P}_k(\overline{\Lambda}) = \mathfrak{P}_k(\Lambda) \otimes_{\mathcal{O}_{\mathrm{F}}} \mathcal{O}_{\mathrm{L}}, \quad k \in \mathbb{Z} \tag{6.6}$$

(see [**20**, §2.2]). This provides us with a stratum $[\overline{\Lambda}, n, m, \beta]$ in $\overline{\mathrm{A}}$, called the *quasi-simple* L/F*-lift* of the simple stratum $[\Lambda, n, m, \beta]$. This quasi-simple lift is pure if and only if the residue class degree of E over F is prime to the degree of L over F, and in this case it is a simple stratum (*ibid.*).

In [**20**] (see paragraph 3.2.3), one attaches to the stratum $[\overline{\Lambda}, n, m, \beta]$ a compact open subgroup $\mathrm{H}^{m+1}(\beta, \overline{\Lambda})$ of $\overline{\mathrm{A}}^\times$ and a set $\mathcal{Q}(\overline{\Lambda}, m, \beta)$ of characters of the group $\mathrm{H}^{m+1}(\beta, \overline{\Lambda})$, called *quasi-simple characters* of level m and depending on an additive character:

$$\boldsymbol{\Psi} : \mathrm{L} \to \mathbb{C}^\times \tag{6.7}$$

extending the additive character (1.2), being trivial on $\mathfrak{p}_\mathrm{L}$ but not on $\mathcal{O}_\mathrm{L}$. Recall that the restriction map from $\mathrm{H}^{m+1}(\beta, \overline{\Lambda})$ to $\mathrm{H}^{m+1}(\beta, \Lambda)$ induces a surjective map from $\mathcal{Q}(\overline{\Lambda}, m, \beta)$ to $\mathcal{C}(\Lambda, m, \beta)$.

Let $[\Lambda', n', m', \beta]$ be another realization of (k, β) in a simple central F-algebra A', *with* Λ' *strict. We assume that* Λ *and* Λ' *have the same period and that* $m = m'$ *is a multiple of* k. We assume that the extension L/F is chosen such that the L-algebras $\overline{\mathrm{A}}$ and $\overline{\mathrm{A}}'$ are both split, and we set:

$$\mathrm{V}^0 = \overline{\mathrm{V}} \oplus \overline{\mathrm{V}}', \quad \Lambda^0 = \overline{\Lambda} \oplus \overline{\Lambda}'. \tag{6.8}$$

Then Λ^0 is a strict $\mathcal{O}_\mathrm{L}$-lattice sequence on the L-vector space V^0. Moreover $\mathrm{A}^0 = \mathrm{End}_\mathrm{L}(\mathrm{V}^0)$ is a split simple central L-algebra, in which $\mathrm{E} = \mathrm{F}(\beta)$ is naturally embedded. We write M for $\overline{\mathrm{A}}^\times \times \overline{\mathrm{A}}'^\times$ considered as a Levi subgroup of $\mathrm{A}^{0\times}$. We have the decomposition:

$$\mathrm{H}^{m+1}(\beta, \Lambda^0) \cap \mathrm{M} = \mathrm{H}^{m+1}(\beta, \overline{\Lambda}) \times \mathrm{H}^{m+1}(\beta, \overline{\Lambda}'). \tag{6.9}$$

We will need the following characterization of the transfer map.

Proposition 6.2. — *Let* $\theta \in \mathcal{C}(\Lambda, m, \beta)$ *and* $\theta' \in \mathcal{C}(\Lambda', m', \beta)$ *be two simple characters. Assume* Λ *and* Λ' *are strict, have the same period and* $m = m'$ *is a multiple of* k. *Then* θ' *is the transfer of* θ *if and only if there exists* $\boldsymbol{\theta}^0 \in \mathcal{Q}(\Lambda^0, m, \beta)$ *such that:*

$$\boldsymbol{\theta}^0 \mid \mathrm{H}^{m+1}(\beta, \Lambda) \times \mathrm{H}^{m+1}(\beta, \Lambda') = \theta \otimes \theta'. \tag{6.10}$$

Proof. — Recall (see [**20**, §3.3]) that θ and θ' are transfers of each other if and only if there exist two quasi-simple characters $\boldsymbol{\theta} \in \mathcal{Q}(\overline{\Lambda}, m, \beta)$ and $\boldsymbol{\theta}' \in \mathcal{Q}(\overline{\Lambda}', m, \beta)$, extending θ and θ' respectively, which are transfers of each other.

Lemma 6.3. — *The map from* $\mathcal{Q}(\Lambda^0, m, \beta)$ *to* $\mathcal{Q}(\overline{\Lambda}, m, \beta)$ *induced by the restriction from* $\mathrm{H}^{m+1}(\beta, \Lambda^0)$ *to* $\mathrm{H}^{m+1}(\beta, \overline{\Lambda})$ *is the transfer.*

Proof. — We have a decomposition of the L-algebra $\mathrm{E} \otimes_\mathrm{F} \mathrm{L}$ into simple $\mathrm{E} \otimes_\mathrm{F} \mathrm{L}$-modules E^j, for $j \in \{1, \dots, s\}$, where s denotes the greatest common divisor of the degree of L/F and the residue class degree of E/F. For each j, we write $\mathbf{1}^j$ for the minimal idempotent in $\mathrm{E} \otimes_\mathrm{F} \mathrm{L}$ corresponding to E^j, as well as $\Lambda^{0,j}$ for the projection of Λ^0 onto $\mathrm{V}^{0,j} = \mathbf{1}^j \mathrm{V}^0$ and β^j for $\mathbf{1}^j \beta$. Thus we get a simple stratum $[\Lambda^{0,j}, n, m, \beta^j]$ in the F-algebra $\mathrm{A}^{0,j} = \mathbf{1}^j \mathrm{A}^0 \mathbf{1}^j$

and, similarly, we get a simple stratum $[\overline{\Lambda}^j, n, m, \beta^j]$ in $\overline{\mathrm{A}}^j$. By [**20**, Corollaire 3.34], there are bijections:

$$\mathcal{Q}(\Lambda^0, m, \beta) \to \prod_{j=1}^{s} \mathcal{C}(\Lambda^{0,j}, m, \beta^j), \quad \mathcal{Q}(\overline{\Lambda}, m, \beta) \to \prod_{j=1}^{s} \mathcal{C}(\overline{\Lambda}^j, m, \beta^j),$$

which are compatible with transfer. Therefore, it is enough to prove that, for each j, the map from $\mathcal{C}(\Lambda^{0,j}, m, \beta^j)$ to $\mathcal{C}(\overline{\Lambda}^j, m, \beta^j)$ induced by the restriction from $\mathrm{H}^{m+1}(\beta^j, \Lambda^{0,j})$ to $\mathrm{H}^{m+1}(\beta^j, \overline{\Lambda}^j)$ is the transfer. This is [**24**, Théorème 2.17]. □

Assume first that there exists a quasi-simple character $\boldsymbol{\theta}^0 \in \mathcal{Q}(\Lambda^0, m, \beta)$ such that (6.10) is satisfied, and write $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ for the restrictions of $\boldsymbol{\theta}^0$ to $\mathrm{H}^{m+1}(\beta, \overline{\Lambda})$ and $\mathrm{H}^{m+1}(\beta, \overline{\Lambda}')$, respectively. By Lemma 6.3, these are quasi-simple characters which are transfers of each other. By (6.10), they extend the simple characters θ and θ'. It follows that θ and θ' are transfers of each other.

Conversely, assume that θ and θ' are transfers of each other. Let $\boldsymbol{\theta}$ be a quasi-simple character in $\mathcal{Q}(\overline{\Lambda}, m, \beta)$ extending θ, and let $\boldsymbol{\theta}^0$ be its transfer to $\mathcal{Q}(\Lambda^0, m, \beta)$. By Lemma 6.3, the restriction of $\boldsymbol{\theta}^0$ to $\mathrm{H}^{m+1}(\beta, \overline{\Lambda}')$ is the transfer of $\boldsymbol{\theta}$, and thus extends θ'. Therefore, the identity (6.10) is satisfied. □

6.3. Let $[\Lambda, n, m, \beta]$ be a K-pure simple stratum in A, and let $[\Gamma, n, m, \beta]$ denote the simple stratum in C associated with $[\Lambda, n, m, \beta]$ by (5.2). *In this paragraph, we assume that* Λ *and* Γ *are strict lattice sequences.*

If we fix a simple left $\overline{\mathrm{C}}$-module $\overline{\mathrm{W}}$, we can form the quasi-simple lift $[\overline{\Gamma}, n, m, \beta]$ of the simple stratum $[\Gamma, n, m, \beta]$ with respect to L/K. One attaches to this quasi-simple lift a compact open subgroup $\mathrm{H}^{m+1}(\beta, \overline{\Gamma})$ of $\overline{\mathrm{C}}^\times$ and a set $\mathcal{Q}(\overline{\Gamma}, m, \beta)$ of characters of $\mathrm{H}^{m+1}(\beta, \overline{\Gamma})$ with respect to the additive character:

$$\boldsymbol{\Psi}_{\mathrm{K}} = \boldsymbol{\Psi} \circ (\sigma_1 + \cdots + \sigma_f) \tag{6.11}$$

of L, depending on the choice of the set S fixed in (6.2). It is trivial on $\mathfrak{p}_{\mathrm{L}}$ and, thanks to the condition on S, it extends the character Ψ_{K} defined by (5.5); hence it is not trivial on $\mathcal{O}_{\mathrm{L}}$. This comes with a surjective restriction map from $\mathcal{Q}(\overline{\Gamma}, m, \beta)$ to $\mathcal{C}(\Gamma, m, \beta)$.

Lemma 6.4. — *The image of* $\mathrm{H}^{m+1}(\beta, \overline{\Gamma})$ *by the map* ι *is contained in* $\mathrm{H}^{m+1}(\beta, \overline{\Lambda})$.

Proof. — First we have to prove that:

$$\varkappa(\mathrm{H}^{m+1}(\beta, \overline{\Gamma})) = \mathrm{H}^{m+1}(\beta, \overline{\Lambda}) \cap \overline{\mathrm{U}}^1 = \mathrm{e}^1 \mathrm{H}^{m+1}(\beta, \overline{\Lambda}) \mathrm{e}^1.$$

This follows from the definition of the groups $\mathrm{H}^{m+1}(\beta,\overline{\Gamma})$ and $\mathrm{H}^{m+1}(\beta,\overline{\Lambda})$ by induction on β, and from the fact that e^1 commutes to β. According to (6.3), we get:

$$\sigma_i \circ \varkappa(\mathrm{H}^{m+1}(\beta,\overline{\Gamma})) = \mathrm{H}^{m+1}(\beta,\overline{\Lambda}) \cap \overline{\mathrm{U}}^i = \mathrm{e}^i\mathrm{H}^{m+1}(\beta,\overline{\Lambda})\mathrm{e}^i$$

for each $i \in \{1,\dots,f\}$, and the result follows. □

This gives rise to the following result.

Proposition 6.5. — *Let $\theta \in \mathcal{C}(\Lambda,m,\beta)$ be a simple character, let $\boldsymbol{\theta} \in \mathcal{Q}(\overline{\Lambda},m,\beta)$ be a quasi-simple character extending θ, and set:*

$$\boldsymbol{\theta}^{\mathrm{K}}(x) = \boldsymbol{\theta}(\iota(x)), \quad x \in \mathrm{H}^{m+1}(\beta,\overline{\Gamma}). \tag{6.12}$$

Then $\boldsymbol{\theta}^{\mathrm{K}}$ is a quasi-simple character in $\mathcal{Q}(\overline{\Gamma},m,\beta)$ extending $\theta^{\mathrm{K}} = \theta \mid \mathrm{H}^{m+1}(\beta,\Gamma)$.

Proof. — By Lemmas 6.1 and 6.4, the character $\boldsymbol{\theta}^{\mathrm{K}}$ is well defined and extends the simple character θ^{K}. It thus remains to prove that it is in $\mathcal{Q}(\overline{\Gamma},m,\beta)$. The proof is by induction on β (see [**20**, Définition 3.22]). Assume first that β is minimal over F. Then it is minimal over K by Proposition 5.2. If $m \geqslant \lfloor n/2 \rfloor$, the set $\mathcal{Q}(\overline{\Lambda},m,\beta)$ consists of a single element $\boldsymbol{\Psi}_\beta^{\overline{\mathrm{A}}}$, which is the character of $\mathrm{U}_{m+1}(\overline{\Lambda})$ defined by:

$$\boldsymbol{\Psi}_\beta^{\overline{\mathrm{A}}}(x) = \boldsymbol{\Psi} \circ \mathrm{tr}_{\overline{\mathrm{A}}/\mathrm{L}}(\beta(x-1)), \quad x \in \mathrm{U}_{m+1}(\overline{\Lambda}),$$

and the set $\mathcal{Q}(\overline{\Gamma},m,\beta)$ consists of a single element $\boldsymbol{\Psi}_\beta^{\overline{\mathrm{C}}}$, which is the character of $\mathrm{U}_{m+1}(\overline{\Gamma})$ defined by:

$$\boldsymbol{\Psi}_\beta^{\overline{\mathrm{C}}}(x) = \boldsymbol{\Psi} \circ \mathrm{tr}_{\overline{\mathrm{C}}/\mathrm{L}}(\beta(x-1)), \quad x \in \mathrm{U}_{m+1}(\overline{\Gamma}).$$

So we just need to prove that:

$$\boldsymbol{\Psi}_\beta^{\overline{\mathrm{A}}} \circ \iota(x) = \boldsymbol{\Psi}_\beta^{\overline{\mathrm{C}}}(x), \quad x \in \mathrm{U}_{m+1}(\overline{\Gamma}), \tag{6.13}$$

which follows from the fact that:

$$\begin{aligned} \mathrm{tr}_{\overline{\mathrm{A}}/\mathrm{L}} \circ \iota &= \sum_{i=1}^{f} \mathrm{tr}_{\overline{\mathrm{A}}/\mathrm{L}} \circ \sigma_i \circ \varkappa \\ &= (\sigma_1 + \cdots + \sigma_f) \circ \mathrm{tr}_{\overline{\mathrm{A}}/\mathrm{L}} \circ \varkappa = (\sigma_1 + \cdots + \sigma_f) \circ \mathrm{tr}_{\overline{\mathrm{C}}/\mathrm{L}}. \end{aligned}$$

If $m \leqslant \lfloor n/2 \rfloor$, then $\boldsymbol{\theta}$ extends $\boldsymbol{\Psi}_\beta^{\overline{\mathrm{A}}} \mid \mathrm{U}_{\lfloor n/2 \rfloor+1}(\overline{\Lambda})$ and its restriction to $\mathrm{U}_{m+1}(\overline{\Lambda}) \cap \overline{\mathrm{B}}^\times$ has the form:

$$\boldsymbol{\theta} \mid \mathrm{U}_{m+1}(\overline{\Lambda}) \cap \overline{\mathrm{B}}^\times = \boldsymbol{\chi} \circ \mathrm{N}_{\overline{\mathrm{B}}/\mathrm{E}\otimes_{\mathrm{F}}\mathrm{L}}, \tag{6.14}$$

where we write $\overline{\mathrm{B}}$ for the centralizer of E in $\overline{\mathrm{A}}$ and where $\boldsymbol{\chi}$ denotes some character of the subgroup $1+\mathfrak{p}_{\mathrm{E}} \otimes \mathcal{O}_{\mathrm{L}}$ of $(\mathrm{E} \otimes_{\mathrm{F}} \mathrm{L})^\times$. Then, if we write $\overline{\mathrm{R}}$ for the centralizer of $\mathrm{K}(\beta)$ in $\overline{\mathrm{A}}$, the character $\boldsymbol{\theta}^{\mathrm{K}}$ extends $\boldsymbol{\Psi}_{\beta}^{\overline{\mathrm{C}}} \mid \mathrm{U}_{\lfloor n/2\rfloor+1}(\overline{\Gamma})$, and its restriction to $\mathrm{U}_{m+1}(\overline{\Gamma}) \cap \overline{\mathrm{R}}^{\times}$ has the form:

$$\boldsymbol{\theta}^{\mathrm{K}} \mid \mathrm{U}_{m+1}(\overline{\Gamma}) \cap \overline{\mathrm{R}}^{\times} = \boldsymbol{\chi}^{\mathrm{S}} \circ \mathrm{N}_{\overline{\mathrm{R}}/\mathrm{K}(\beta)\otimes_{\mathrm{K}}\mathrm{L}} \tag{6.15}$$

where $\boldsymbol{\chi}^{\mathrm{S}}$ is the product of all the $\boldsymbol{\chi} \circ \sigma_i$'s for all $i \in \{1, \ldots, f\}$, as required.

Assume now that β is not minimal over F. We set $q = -k_0(\beta, \Lambda)$ and $r = \lfloor q/2\rfloor + 1$, and choose a simple stratum $[\Gamma, n, q, \gamma]$ equivalent to $[\Gamma, n, q, \beta]$ such that $[\Lambda, n, q, \gamma]$ is simple and K-pure. By [**5**, Theorem 5.1] and Proposition 5.4 together, one may assume that the maximal unramified extension of F contained in $\mathrm{F}(\gamma)$ is contained in that of $\mathrm{F}(\beta)$, which implies that the L-canonical decomposition of γ is finer than that of β (see paragraph 2.3.4 and the proof of Lemme 3.16 in [**20**]). If $m \geqslant \lfloor q/2\rfloor$, then any $\boldsymbol{\theta} \in \mathcal{Q}(\overline{\Lambda}, m, \beta)$ can be written as $\boldsymbol{\theta} = \boldsymbol{\theta}_0 \boldsymbol{\Psi}_{\beta-\gamma}^{\overline{\Lambda}}$ for some quasi-simple character $\boldsymbol{\theta}_0 \in \mathcal{Q}(\overline{\Lambda}, m, \gamma)$. Now we claim that:

$$\mathrm{H}^{m+1}(\beta, \overline{\Gamma}) = \mathrm{H}^{m+1}(\gamma, \overline{\Gamma}). \tag{6.16}$$

We write $q_1 = -k_0(\beta, \Gamma)$. If $q_1 = q$, then the equality (6.16) follows by definition. Otherwise, we have $q_1 > q$ by Proposition 5.2. The strata $[\Gamma, n, q, \beta]$ and $[\Gamma, n, q, \gamma]$ are thus simple, and (6.16) follows. We now form the character $\boldsymbol{\theta}^{\mathrm{K}} = \boldsymbol{\theta} \circ \iota \mid \mathrm{H}^{m+1}(\beta, \overline{\Gamma})$ and get the equality $\boldsymbol{\theta}^{\mathrm{K}} = \boldsymbol{\theta}_0^{\mathrm{K}} \boldsymbol{\Psi}_{\beta-\gamma}^{\overline{\mathrm{C}}}$, where $\boldsymbol{\theta}_0^{\mathrm{K}}$ denotes the character $\boldsymbol{\theta}_0 \circ \iota \mid \mathrm{H}^{m+1}(\gamma, \overline{\Gamma})$, and this character is in $\mathcal{Q}(\overline{\Gamma}, m, \gamma)$ by the inductive hypothesis. If $q_1 = q$, then $\boldsymbol{\theta}^{\mathrm{K}}$ is in $\mathcal{Q}(\overline{\Gamma}, m, \beta)$ by definition. Otherwise, the strata $[\Gamma, n, q, \beta]$ and $[\Gamma, n, q, \gamma]$ are simple and the result follows from [**24**, Proposition 2.15]. The case $m \leqslant \lfloor q/2\rfloor$ reduces to the previous one as in the minimal case.

It remains to prove that the subgroup $\mathfrak{K}(\overline{\Gamma}) \cap \overline{\mathrm{R}}^{\times}$ normalizes $\boldsymbol{\theta}^{\mathrm{K}}$. If $g \in \mathfrak{K}(\overline{\Gamma}) \cap \overline{\mathrm{R}}^{\times}$, then we have:

$$\iota(g) \cdot \overline{\Lambda}_k = \bigoplus_{i=1}^{f} \sigma_i(\varkappa(g)) \cdot \mathrm{e}^i \overline{\Lambda}_k = \bigoplus_{i=1}^{f} \mathrm{e}^i \overline{\Lambda}_{k+\upsilon(\sigma_i(\varkappa(g)))} \tag{6.17}$$

where υ denotes the valuation map associated with $\overline{\Lambda}$. As all the $\sigma_i(\varkappa(g)$'s have the same valuation, the equality (6.17) gives us $\iota(g) \in \mathfrak{K}(\overline{\Lambda}) \cap \overline{\mathrm{B}}^{\times}$. Proposition 6.5 now follows from the fact that $\mathfrak{K}(\overline{\Lambda}) \cap \overline{\mathrm{B}}^{\times}$ normalizes $\boldsymbol{\theta}$. □

Remark 6.6. — Note that the interior lifting map from $\mathcal{Q}(\overline{\Lambda}, m, \beta)$ to $\mathcal{Q}(\overline{\Gamma}, m, \beta)$ defined by Proposition 6.5 depends on the choice of the set S chosen in (6.2).

6.4. Let $[\Lambda, n, m, \beta]$ and $[\Lambda', n', m', \beta]$ be realizations of a simple pair (k, β) over F in simple central F-algebras A and A′, respectively. Assume further that A and A′ contain K, that the strata $[\Lambda, n, m, \beta]$ and $[\Lambda', n', m', \beta]$ are K-pure and that the strata $[\Gamma, n, m, \beta]$ and $[\Gamma', n', m', \beta]$ associated with them by (5.2) are realizations of the same simple pair over K. (This is equivalent to saying that the extensions of K generated by β in A and A′ are K-isomorphic.) We have the following relation between the transfer maps and the interior lifting maps.

Theorem 6.7. — *Let $\theta \in \mathcal{C}(\Lambda, m, \beta)$ and $\theta' \in \mathcal{C}(\Lambda', m', \beta)$ be transfers of each other. Then the simple characters:*

$$\theta \mid \mathrm{H}^{m+1}(\beta, \Gamma), \quad \theta' \mid \mathrm{H}^{m'+1}(\beta, \Gamma')$$

are transfers of each other.

Proof. — One can assume without loss of generality that m and m' are multiples of k. By rescaling the lattice sequences Λ and Λ', one can also assume that they have the same period thanks to Lemma 2.2. Thus $m = m'$ and $n = n'$. The proof decomposes into two parts.

(1) First we prove the theorem in the case where all the lattice sequences are strict, so that we can apply the results of paragraphs 6.2 and 6.3. We fix a quasi-simple character $\boldsymbol{\theta}$ in $\mathcal{Q}(\overline{\Lambda}, m, \beta)$ extending θ and write $\boldsymbol{\theta}'$ for its transfer in $\mathcal{Q}(\overline{\Lambda}', m, \beta)$. The restriction of $\boldsymbol{\theta}'$ to $\mathrm{H}^{m+1}(\beta, \Lambda')$ is thus equal to θ'. By Proposition 6.2, there exists a quasi-simple character $\boldsymbol{\theta}^0$ in $\mathcal{Q}(\Lambda^0, m, \beta)$ extending $\boldsymbol{\theta} \otimes \boldsymbol{\theta}'$. We write $\overline{\mathrm{C}}$ and $\overline{\mathrm{U}}$ as in paragraph 6.1, and use similar notations $\overline{\mathrm{C}}'$ and $\overline{\mathrm{U}}'$. We have:

$$\mathrm{H}^{m+1}(\beta, \Lambda^0) \cap \left(\overline{\mathrm{C}}^\times \times \overline{\mathrm{C}}'^\times\right) = \mathrm{H}^{m+1}(\beta, \overline{\Gamma}) \times \mathrm{H}^{m+1}(\beta, \overline{\Gamma}'). \tag{6.18}$$

We define ι by (6.5) and write $\boldsymbol{\theta}^{\mathrm{K}}$ for the quasi-simple character defined by (6.12). We also define ι' and $\boldsymbol{\theta}'^{\mathrm{K}}$ in a similar way. If we restrict the map $x \mapsto (\iota(x), \iota'(x))$ to the subgroup (6.18) and then compose it with $\boldsymbol{\theta}^0$, then we get the character $\boldsymbol{\theta}^{\mathrm{K}} \otimes \boldsymbol{\theta}'^{\mathrm{K}}$. This implies that $\boldsymbol{\theta}^{\mathrm{K}}$ and $\boldsymbol{\theta}'^{\mathrm{K}}$ are transfers of each other. By Propositions 6.5 and 6.2 together, their restrictions $\boldsymbol{\theta}^{\mathrm{K}} \mid \mathrm{H}^{m+1}(\beta, \Gamma) = \theta^{\mathrm{K}}$ and $\boldsymbol{\theta}'^{\mathrm{K}} \mid \mathrm{H}^{m+1}(\beta, \Gamma') = \theta'^{\mathrm{K}}$ are transfers of each other.

(2) We now reduce the general case to Case (1). For this we fix a positive integer l as in Lemma 2.16, and form the sound simple strata $[\Lambda^{\ddagger}, n, m, \beta]$ and $[\Lambda'^{\ddagger}, n, m, \beta]$. Write $\mathrm{C}^{\ddagger}$ for the centralizer of K in $\mathrm{A}^{\ddagger}$ and $[\Gamma^{\ddagger}, n, m, \beta]$ for the simple stratum in $\mathrm{C}^{\ddagger}$ associated with $[\Lambda^{\ddagger}, n, m, \beta]$ by (5.2). In a similar way, we have a K-algebra $\mathrm{C}'^{\ddagger}$ and a simple stratum $[\Gamma'^{\ddagger}, n, m, \beta]$. Then the simple strata $[\Gamma^{\ddagger}, n, m, \beta]$ and $[\Gamma'^{\ddagger}, n, m, \beta]$ are realizations of the

same simple pair over K. Write $\theta^\ddagger$ for the transfer of θ in $\mathcal{C}(\Lambda^\ddagger, m, \beta)$. In a similar way, we have a simple character $\theta'^\ddagger$. By Case (1), the simple characters:

$$\theta^\ddagger \mid \mathrm{H}^{m+1}(\beta, \Gamma^\ddagger), \quad \theta'^\ddagger \mid \mathrm{H}^{m+1}(\beta, \Gamma'^\ddagger)$$

are transfers of each other. Thus it remains to prove the following lemma.

Lemma 6.8. — *The characters $\theta \mid \mathrm{H}^{m+1}(\beta, \Gamma)$ and $\theta^\ddagger \mid \mathrm{H}^{m+1}(\beta, \Gamma^\ddagger)$ are transfers of each other.*

Proof. — Write M for the Levi subgroup of $\mathrm{A}^{\ddagger\times}$ defined by the decomposition of $\mathrm{V}^\ddagger$ into copies of V. According to Lemma 2.7, the character $\theta^\ddagger$ is characterized by the identity:

$$\theta^\ddagger \mid \mathrm{H}^{m+1}(\beta, \Lambda^\ddagger) \cap \mathrm{M} = \theta \otimes \cdots \otimes \theta.$$

Thus its restriction to $\mathrm{H}^{m+1}(\beta, \Gamma^\ddagger) \cap \mathrm{M} = \mathrm{H}^{m+1}(\beta, \Gamma) \times \cdots \times \mathrm{H}^{m+1}(\beta, \Gamma)$ is equal to the tensor product of l copies of θ^{K}. □

This ends the proof of Theorem 6.7. □

Remark 6.9. — In the case where $[\Lambda, n, m, \beta]$ and $[\Lambda', n', m', \beta]$ are sound, this theorem implies that Grabitz's transfer [**17**] is the same as the transfer defined in [**20**].

6.5. Before closing this section, we prove the following result. Let $[\Lambda, n, m, \beta]$ be a simple K-pure stratum in A, and write $[\Gamma, n, m, \beta]$ for the simple stratum in C which corresponds to it by (5.2). Theorem 5.7 gives us a map from $\mathcal{C}(\Lambda, m, \beta)$ to $\mathcal{C}(\Gamma, m, \beta)$, called the interior lifting map, and denoted $\boldsymbol{l}_{\mathrm{K/F}} : \theta \mapsto \theta^{\mathrm{K}}$. It has the following properties.

Proposition 6.10. — *The map $\boldsymbol{l}_{\mathrm{K/F}}$ is injective and $\mathfrak{K}(\Gamma)$-equivariant.*

Proof. — Note that the second assertion is immediate. Let us fix a positive integer $l \geqslant 1$ as in Lemma 2.16, and form the sound simple stratum $[\Lambda^\ddagger, n, m, \beta]$. Write $\mathrm{C}^\ddagger$ for the centralizer of K in $\mathrm{A}^\ddagger$ and $[\Gamma^\ddagger, n, m, \beta]$ for the simple stratum in $\mathrm{C}^\ddagger$ associated with the stratum $[\Lambda^\ddagger, n, m, \beta]$ by (5.2). Now let $\theta \in \mathcal{C}(\Lambda, m, \beta)$ be a simple character and write $\theta^\ddagger$ for its transfer in $\mathcal{C}(\Lambda^\ddagger, m, \beta)$. Then, by Lemma 6.8, the transfer of θ^{K} to $\mathcal{C}(\Gamma^\ddagger, m, \beta)$ is equal to $\theta^\ddagger \mid \mathrm{H}^{m+1}(\beta, \Gamma^\ddagger)$. As the transfer map from $\mathcal{C}(\Lambda, m, \beta)$ to $\mathcal{C}(\Lambda^\ddagger, m, \beta)$ is bijective, we may replace Λ by $\Lambda^\ddagger$ and assume that the stratum $[\Lambda, n, m, \beta]$ is sound. In this case, the injectivity of the map $\boldsymbol{l}_{\mathrm{K/F}}$ follows from [**17**, Proposition 7.1]. □

Assume we are given two K-pure simple strata $[\Lambda, n, m, \beta_i]$, $i = 1, 2$, in A. For each i, let θ_i be a simple character in $\mathcal{C}(\Lambda, m, \beta_i)$.

Proposition 6.11. — *Assume that θ_1 and θ_2 are equal. Then $\boldsymbol{l}_{\mathrm{K/F}}(\theta_1)$ and $\boldsymbol{l}_{\mathrm{K/F}}(\theta_2)$ are equal.*

Proof. — It suffices to verify that the groups $\mathrm{H}^{m+1}(\beta_i, \Gamma)$, $i = 1, 2$, are equal. This follows from Proposition 5.5 and the fact that the groups $\mathrm{H}^{m+1}(\beta_i, \Lambda)$, $i = 1, 2$, are equal. □

7. The base change

In this section, we develop a base change process for simple strata and characters with respect to a finite unramified extension K of F, in a way similar to [**6**].

7.1. Let K/F be an unramified extension of degree f. Given a simple central F-algebra A, we set:

$$\widehat{\mathrm{A}} = \mathrm{A} \otimes_{\mathrm{F}} \mathrm{End}_{\mathrm{F}}(\mathrm{K}).$$

Then K embeds naturally in $\widehat{\mathrm{A}}$, and its centralizer, denoted A_{K}, is canonically isomorphic to $\mathrm{A} \otimes_{\mathrm{F}} \mathrm{K}$ as a K-algebra. Let V be a simple left A-module. Then $\widehat{\mathrm{V}} = \mathrm{V} \otimes_{\mathrm{F}} \mathrm{K}$ is a simple left $\widehat{\mathrm{A}}$-module and, if we fix an F-basis of K, we have a decomposition:

$$\widehat{\mathrm{V}} = \mathrm{V} \oplus \cdots \oplus \mathrm{V} \tag{7.1}$$

of $\widehat{\mathrm{V}}$ into a sum of f copies of V, so that we are in the situation of paragraph 2.2.

Let $[\Lambda, n, m, \beta]$ be a simple stratum in A and set $\mathrm{E} = \mathrm{F}(\beta)$. Let us form the simple stratum $[\widehat{\Lambda}, n, m, \beta]$ in $\widehat{\mathrm{A}}$, where $\widehat{\Lambda} = \Lambda \oplus \cdots \oplus \Lambda$ is the direct sum of f copies of Λ. This simple stratum is not K-pure in general. We have a decomposition:

$$\mathrm{E} \otimes_{\mathrm{F}} \mathrm{K} = \mathrm{E}^1 \oplus \cdots \oplus \mathrm{E}^s$$

into simple $\mathrm{E} \otimes_{\mathrm{F}} \mathrm{K}$-modules, where s denotes the greatest common divisor of f and the residue class degree of E over F. For each $j \in \{1, \dots, s\}$, we write c^j for the minimal idempotent in $\mathrm{E} \otimes_{\mathrm{F}} \mathrm{K}$ corresponding to E^j, and we set:

$$\beta^j = \mathrm{c}^j \beta, \quad j \in \{1, \dots, s\}.$$

These are the various K/F-lifts of β. If we write $\widehat{\Lambda}^j$ for the projection of $\widehat{\Lambda}$ onto the space $\widehat{\mathrm{V}}^j = \mathrm{c}^j \widehat{\mathrm{V}}$ for each j, we get a simple stratum $[\widehat{\Lambda}^j, n, m, \beta^j]$ in the F-algebra $\widehat{\mathrm{A}}^j = \mathrm{c}^j \widehat{\mathrm{A}} \mathrm{c}^j$, which is K-pure for the natural embedding of K in $\widehat{\mathrm{A}}^j$. Thus one can form the interior lift $[\widehat{\Gamma}^j, n, m, \beta^j]$ in the centralizer of K in $\widehat{\mathrm{A}}^j$ (see paragraph 5.1).

Given a simple character $\theta \in \mathcal{C}(\Lambda, m, \beta)$, let $\widehat{\theta}$ denote its transfer to $\mathcal{C}(\widehat{\Lambda}, m, \beta)$ and write $\widehat{\theta}^j$ for the transfer of $\widehat{\theta}$ to $\mathcal{C}(\widehat{\Lambda}^j, m, \beta^j)$, that is the restriction of $\widehat{\theta}$ to $\mathrm{H}^{m+1}(\beta^j, \widehat{\Lambda}^j)$.

Let us denote by θ^j_{K} the restriction of $\widehat{\theta}^j$ to $\mathrm{H}^{m+1}(\beta^j, \widehat{\Gamma}^j)$, which belongs to $\mathcal{C}(\widehat{\Gamma}^j, m, \beta^j)$ by Theorem 5.7. We have the following definition.

Definition 7.1. — The process:

$$\boldsymbol{b}_{\mathrm{K/F}} : \theta \mapsto \{\theta^j_{\mathrm{K}},\ j = 1, \dots, s\}$$

is the K/F-*base change* for simple characters. For each j, the simple character θ^j_{K} is called the K/F-lift of θ corresponding to the K/F-lift β^j of β.

Now let (Θ, k, β) be a ps-character over F. Let $[\Lambda, n, m, \varphi(\beta)]$ be a realization of the pair (k, β) in a simple central F-algebra A, and let θ denote the simple character $\Theta(\Lambda, m, \varphi)$. Let (k, β^j), for $j \in \{1, \dots, s\}$, be the various K/F-lifts of the pair (k, β), and let φ^j denote the homomorphism of K-algebras from $\mathrm{K}(\beta^j)$ to the centralizer of K in $\widehat{\mathrm{A}}^j$ induced by φ. Thus the sum of the φ^j's is the K-algebra homomorphism $\varphi \otimes \mathrm{id}_{\mathrm{K}}$ from $\mathrm{E} \otimes_{\mathrm{F}} \mathrm{K}$ to A_{K}. For each j, let us denote by $(\Theta^j_{\mathrm{K}}, k, \beta^j)$ the ps-character defined by $([\widehat{\Gamma}^j, n, m, \beta^j], \theta^j_{\mathrm{K}})$.

Definition 7.2. — The process:

$$\boldsymbol{b}_{\mathrm{K/F}} : (\Theta, k, \beta) \mapsto \{(\Theta^j_{\mathrm{K}}, k, \beta^j),\ j = 1, \dots, s\}$$

is the K/F-*base change* for ps-characters, and Θ^j_{K} is called the K/F-lift of Θ corresponding to the K/F-lift β^j of β.

This definition does not depend on the choice of the realization $[\Lambda, n, m, \varphi(\beta)]$. Indeed, let $[\Lambda', n', m', \varphi'(\beta)]$ be another realization of (k, β) in a simple central F-algebra A′, and let us write θ' for the transfer of θ to $\mathcal{C}(\Lambda', m', \varphi'(\beta))$. Then it follows from Theorem 6.7 that, for each j, the K/F-lifts θ^j_{K} and θ'^j_{K} are transfers of each other.

7.2. In this paragraph, we study in more details the case where $s = 1$, that is the case where the residue class degree of $\mathrm{F}(\beta)/\mathrm{F}$ is prime to f. In this case, the simple pair (k, β) has exactly one K/F-lift. If we write Λ_{K} for the $\mathcal{O}_{\mathrm{K}}$-lattice sequence defined by $\widehat{\Lambda}$, then the base change process gives rise to a map:

$$\boldsymbol{b}_{\mathrm{K/F}} : \mathcal{C}(\Lambda, m, \beta) \to \mathcal{C}(\Lambda_{\mathrm{K}}, m, \beta) \tag{7.2}$$

having the following properties.

Proposition 7.3. — *The map $\boldsymbol{b}_{\mathrm{K/F}}$ is injective and $\mathfrak{K}(\Lambda)$-equivariant.*

Proof. — As $\boldsymbol{b}_{\mathrm{K/F}}$ is the composite of the transfer map from $\mathcal{C}(\Lambda, m, \beta)$ to $\mathcal{C}(\widehat{\Lambda}, m, \beta)$ and the interior lifting from $\mathcal{C}(\widehat{\Lambda}, m, \beta)$ to $\mathcal{C}(\Lambda_{\mathrm{K}}, m, \beta)$, this follows from Proposition 6.10. □

Assume we are given two simple strata $[\Lambda, n_i, m_i, \beta_i]$, $i = 1, 2$, in A, such that $f_{\mathrm{F}}(\beta_1)$ and $f_{\mathrm{F}}(\beta_2)$ are prime to f. For each i, let θ_i be a simple character in $\mathcal{C}(\Lambda, m_i, \beta_i)$.

Proposition 7.4. — *Assume θ_1 and θ_2 intertwine in $\mathrm{A}^\times$. Then $\boldsymbol{b}_{\mathrm{K/F}}(\theta_1)$ and $\boldsymbol{b}_{\mathrm{K/F}}(\theta_2)$ intertwine in $\mathrm{A}_{\mathrm{K}}^\times$.*

Proof. — Assume θ_1 and θ_2 are intertwined by $g \in \mathrm{A}^\times$. By the proof of Proposition 2.6, the characters $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are intertwined by $\iota(g)$, where ι denotes the diagonal embedding of A in $\widehat{\mathrm{A}} = \mathrm{M}_f(\mathrm{A})$. As $\iota(g)$ is actually in $\mathrm{A}_{\mathrm{K}}^\times$, we deduce that the characters $\boldsymbol{b}_{\mathrm{K/F}}(\theta_1)$ and $\boldsymbol{b}_{\mathrm{K/F}}(\theta_2)$ intertwine in $\mathrm{A}_{\mathrm{K}}^\times$. □

We now suppose that $n_1 = n_2$ and $m_1 = m_2$.

Proposition 7.5. — *Assume that θ_1 and θ_2 are equal. Then $\boldsymbol{b}_{\mathrm{K/F}}(\theta_1)$ and $\boldsymbol{b}_{\mathrm{K/F}}(\theta_2)$ are equal.*

Proof. — If θ_1 and θ_2 are equal, then Proposition 4.11 gives us $\widehat{\theta}_1 = \widehat{\theta}_2$ and Proposition 6.11 gives us the expected equality. □

Let $[\Lambda, n, m, \beta]$ and $[\Lambda', n', m', \beta]$ be two realizations of the simple pair (k, β), let θ be a simple character in $\mathcal{C}(\Lambda, m, \beta)$ and let θ' be its transfer in $\mathcal{C}(\Lambda', m', \beta)$. The following proposition is a special case of Theorem 6.7.

Proposition 7.6. — *The character $\boldsymbol{b}_{\mathrm{K/F}}(\theta')$ is the transfer of $\boldsymbol{b}_{\mathrm{K/F}}(\theta)$ in $\mathcal{C}(\Lambda'_{\mathrm{K}}, m', \beta)$.*

Finally, we will need the following result. Note that Gal(K/F) acts naturally on A_{K}.

Proposition 7.7. — *Let $\theta \in \mathcal{C}(\Lambda_{\mathrm{K}}, m, \beta)$ be a simple character. For any $\sigma \in \mathrm{Gal(K/F)}$, we have $\theta \circ \sigma \in \mathcal{C}(\Lambda_{\mathrm{K}}, m, \beta)$.*

Proof. — One checks by induction on β that the image of $\mathcal{C}(\Lambda_{\mathrm{K}}, m, \beta)$ by $\theta \mapsto \theta \circ \sigma$ is the set of simple characters attached to the image of $[\Lambda_{\mathrm{K}}, n, m, \beta]$ by σ^{-1} with respect to the additive character $\Psi_{\mathrm{K}} \circ \sigma$. The result follows from the fact that this stratum and the additive character Ψ_{K} are invariant by σ. □

7.3. We prove the following theorem, which generalizes [**10**, Corollary 3.6.3].

Theorem 7.8. — *For $i = 1, 2$, let (k_i, β_i) be a simple pair over* F. *Let us fix two realizations $[\Lambda, n, m, \beta_i]$ and $[\Lambda', n', m', \beta_i]$ of (k_i, β_i). Assume $\mathcal{C}(\Lambda, m, \beta_i)$ and $\mathcal{C}(\Lambda', m', \beta_i)$ do not depend on i. Then the transfer map $\boldsymbol{\tau}_i : \mathcal{C}(\Lambda, m, \beta_i) \to \mathcal{C}(\Lambda', m', \beta_i)$ does not depend on i.*

Proof. — The proof decomposes into three steps.

(1) In the first step, we reduce to the case where the strata are all sound. For this, we fix an integer l as in Proposition 2.17 which is large enough for Λ and Λ'. Write $\boldsymbol{a}_i$ for the transfer map from $\mathcal{C}(\Lambda, m, \beta_i)$ to $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_i)$. There is also a map $\boldsymbol{a}'_i$ for Λ'. Thus we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{C}(\Lambda^{\ddagger}, m, \beta_i) & \xrightarrow{\boldsymbol{\tau}_i^{\ddagger}} & \mathcal{C}(\Lambda'^{\ddagger}, m', \beta_i) \\ {\scriptstyle \boldsymbol{a}_i}\uparrow & & \uparrow{\scriptstyle \boldsymbol{a}'_i} \\ \mathcal{C}(\Lambda, m, \beta_i) & \xrightarrow[\boldsymbol{\tau}_i]{} & \mathcal{C}(\Lambda', m', \beta_i) \end{array}$$

where $\tau_i^{\ddagger}$ denotes the transfer map from $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_i)$ to $\mathcal{C}(\Lambda'^{\ddagger}, m', \beta_i)$. By Proposition 4.11, the vertical maps $\boldsymbol{a}_i$ and $\boldsymbol{a}'_i$ do not depend on i, and Proposition 1.17 implies that the sets $\mathcal{C}(\Lambda^{\ddagger}, m, \beta_i)$ and $\mathcal{C}(\Lambda'^{\ddagger}, m', \beta_i)$ do not depend on i. Since $\boldsymbol{a}'_i$ is bijective, the equality $\boldsymbol{\tau}_1^{\ddagger} = \boldsymbol{\tau}_2^{\ddagger}$ implies that $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$. We thus may replace Λ by $\Lambda^{\ddagger}$ and Λ' by $\Lambda'^{\ddagger}$ and assume that all the strata are sound.

(2) We now assume that all the strata are sound, and we reduce to the case where the extensions $\mathrm{F}(\beta_i)/\mathrm{F}$ are totally ramified. By Proposition 4.20, for each i, the simple strata $[\Lambda, n, m, \beta_i]$ and $[\Lambda', n', m', \beta_i]$ have the same embedding type. Write K_i for the maximal unramified extension of F contained in $\mathrm{F}(\beta_i)$, and fix $\theta_i \in \mathcal{C}(\Lambda, m, \beta_i)$. Assume that the characters θ_1 and θ_2 are equal. Using the "intertwining implies conjugacy" theorem [**17**, Corollary 10.15], one may assume that $\mathrm{K}_1 = \mathrm{K}_2$, denoted K. Write $\boldsymbol{l}_i$ for the interior lifting map from $\mathcal{C}(\Lambda, m, \beta_i)$ to $\mathcal{C}(\Gamma, m, \beta_i)$. There is also a map $\boldsymbol{l}'_i$ for Λ'. By Theorem 6.7, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{C}(\Gamma, m, \beta_i) & \xrightarrow{\boldsymbol{\tau}_i^{\mathrm{K}}} & \mathcal{C}(\Gamma', m', \beta_i) \\ {\scriptstyle \boldsymbol{l}_i}\uparrow & & \uparrow{\scriptstyle \boldsymbol{l}'_i} \\ \mathcal{C}(\Lambda, m, \beta_i) & \xrightarrow[\boldsymbol{\tau}_i]{} & \mathcal{C}(\Lambda', m', \beta_i) \end{array}$$

where $\boldsymbol{\tau}_i^{\mathrm{K}}$ denotes the transfer map from $\mathcal{C}(\Gamma, m, \beta_i)$ to $\mathcal{C}(\Gamma', m', \beta_i)$. By Proposition 6.11, the vertical maps $\boldsymbol{l}_i$ and $\boldsymbol{l}'_i$ do not depend on i, and Theorem 4.16 implies that the sets $\mathcal{C}(\Gamma, m, \beta_i)$ and $\mathcal{C}(\Gamma', m', \beta_i)$ do not depend on i. By the same argument as above, using that the map $\boldsymbol{l}'_i$ is injective (see Proposition 6.10), we may assume that $\mathrm{F}(\beta_i)$ is totally ramified over F.

(3) We now assume that $f_{\mathrm{F}}(\beta_1) = f_{\mathrm{F}}(\beta_2) = 1$, and reduce to the split case. Let us fix a finite unramified extension L/F such that the L-algebras $\overline{\mathrm{A}}$ and $\overline{\mathrm{A}}'$ are split. Write $\boldsymbol{b}_i$ for the base change map from $\mathcal{C}(\Lambda, m, \beta_i)$ to $\mathcal{C}(\overline{\Lambda}, m, \beta_i)$. There is also a map $\boldsymbol{b}'_i$ for Λ'.

By Proposition 7.6, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{C}(\overline{\Lambda}, m, \beta_i) & \xrightarrow{\overline{\boldsymbol{\tau}}_i} & \mathcal{C}(\overline{\Lambda}', m', \beta_i) \\ \boldsymbol{b}_i \uparrow & & \uparrow \boldsymbol{b}'_i \\ \mathcal{C}(\Lambda, m, \beta_i) & \xrightarrow[\boldsymbol{\tau}_i]{} & \mathcal{C}(\Lambda', m', \beta_i) \end{array}$$

where $\overline{\boldsymbol{\tau}}_i$ denotes the transfer map from $\mathcal{C}(\overline{\Lambda}, m, \beta_i)$ to $\mathcal{C}(\overline{\Lambda}', m', \beta_i)$. By Proposition 7.4, the maps $\boldsymbol{b}_i$ and $\boldsymbol{b}'_i$ do not depend on i. Thus [**10**, Theorem 3.5.8] (the rigidity theorem for simple characters in the split case) implies that the sets of simple characters $\mathcal{C}(\overline{\Lambda}, m, \beta_i)$ and $\mathcal{C}(\overline{\Lambda}', m', \beta_i)$ do not depend on i. By the same argument as above, using that the map $\boldsymbol{b}'_i$ is injective (see Proposition 7.3), we may assume that A is split and Λ is strict.

The result then follows from [**10**, Corollary 3.6.3]. □

8. Endo-equivalence of simple characters

8.1. In this paragraph, we prove Theorem 1.13 in the totally ramified case. For $i = 1, 2$, let (Θ_i, k, β_i) be a ps-character over F with $f_{\mathrm{F}}(\beta_i) = 1$, and suppose that Θ_1 and Θ_2 are endo-equivalent. Let A be a simple central F-algebra and let $[\Lambda, n, m, \varphi_i(\beta_i)]$ be realizations of (k, β_i) in A, with $i = 1, 2$. Write θ_i for the simple character $\Theta_i(\Lambda, m, \varphi_i)$. We have to prove that θ_1 and θ_2 are conjugate under $\mathfrak{K}(\Lambda)$.

For each i, we write E_i for the F-algebra $\mathrm{F}(\beta_i)$, which is a totally ramified finite extension of F. By assumption, we have $[\mathrm{E}_1 : \mathrm{F}] = [\mathrm{E}_2 : \mathrm{F}]$. Using Proposition 4.9, there exists a simple central F-algebra A′ together with sound realizations $[\Lambda', n', m', \varphi'_i(\beta_i)]$ of (k, β_i), with $i = 1, 2$, such that k divides m' and $\theta'_1 = \theta'_2$, where we write $\theta'_i = \Theta_i(\Lambda', m'_i, \varphi'_i)$.

Now let A be a simple central F-algebra and $[\Lambda, n, m, \varphi_i(\beta_i)]$ be realizations of (k, β_i) in A, for $i = 1, 2$. Let V denote the simple left A-module on which Λ is a lattice sequence and write D for the F-algebra opposite to $\mathrm{End}_{\mathrm{A}}(\mathrm{V})$. Let us fix a finite unramified extension L of F such that the L-algebra $\overline{\mathrm{A}} = \mathrm{A} \otimes_{\mathrm{F}} \mathrm{L}$ is split and a simple left $\overline{\mathrm{A}}$-module $\overline{\mathrm{V}}$. As E_i is totally ramified over F, the quasi-simple lift $[\overline{\Lambda}, n, m, \beta_i]$ is a simple stratum in $\overline{\mathrm{A}}$ (see [**20**, Théorème 2.30] and [**24**, Remarque 2.9]). We denote by $\mathcal{C}(\overline{\Lambda}, m, \beta_i)$ the set of simple characters attached to this quasi-simple lift with respect to the character $\Psi \circ \mathrm{tr}_{\mathrm{L}/\mathrm{F}}$. The base change process developed in paragraph 7.2 gives rise to an injective and $\mathfrak{K}(\Lambda)$-equivariant map:

$$\boldsymbol{b}_{\mathrm{L}/\mathrm{F}} : \mathcal{C}(\Lambda, m, \beta_i) \to \mathcal{C}(\overline{\Lambda}, m, \beta_i),$$

simply denoted $\boldsymbol{b}$. We use similar notations for A′. For each i, we write θ_i for the simple character $\Theta_i(\Lambda, m, \varphi_i)$. By Proposition 7.4, we have $\boldsymbol{b}(\theta'_1) = \boldsymbol{b}(\theta'_2)$. By Proposition 7.6, for each i, the lifts $\boldsymbol{b}(\theta_i)$ and $\boldsymbol{b}(\theta'_i)$ are transfers of each other. At this point, we cannot apply [**6, 10**] to deduce that $\boldsymbol{b}(\theta_1)$ and $\boldsymbol{b}(\theta_2)$ are $\mathfrak{K}(\overline{\Lambda})$-conjugate, because the lattice sequence Λ is not necessarily strict.

Let us fix a simple right $E_1 \otimes_F D$-module S. We set $A(S) = \mathrm{End}_D(S)$, and denote by ρ_1 the natural F-algebra homomorphism $E_1 \to A(S)$. Let $\mathfrak{S}$ denote the unique (up to translation) E_1-pure strict $\mathcal{O}_D$-lattice sequence on S, and let us fix an F-algebra homomorphism $\rho_2 : E_2 \to A(S)$ such that $\mathfrak{S}$ is $\rho_2(E_2)$-pure. Write n_0 for the $\mathfrak{S}$-valuation of $\rho_i(\beta_i)$ and:

$$m_0 = e_{\rho_i(\beta_i)}(\mathfrak{S})k,$$

which do not depend on i. We thus can form the stratum $[\mathfrak{S}, n_0, m_0, \rho_i(\beta_i)]$, which is a realization of (k, β_i) in A(S). Write ϑ_i for the simple character $\Theta_i(\mathfrak{S}, m_0, \rho_i)$. We now form the simple stratum $[\overline{\mathfrak{S}}, n_0, m_0, \rho_i(\beta_i)]$ in the split simple central L-algebra $A(S) \otimes_F L$. It is a realization of (k, β_i) over L, and the $\mathcal{O}_L$-lattice sequence $\overline{\mathfrak{S}}$ is strict. We thus can apply [**6**, Theorem 8.7] and [**10**, Theorem 3.5.11], which imply together that there exists $u \in \mathfrak{K}(\overline{\mathfrak{S}})$ such that:

$$\boldsymbol{b}(\vartheta_2)(x) = \boldsymbol{b}(\vartheta_1)(uxu^{-1}), \quad x \in H^{m+1}(\rho_2(\beta_2), \overline{\mathfrak{S}}) = u^{-1}H^{m+1}(\rho_1(\beta_1), \overline{\mathfrak{S}})u.$$

We need the following lemma.

Lemma 8.1. — *We may assume that* $u \in \mathfrak{K}(\mathfrak{S})$.

Proof. — By Proposition 7.7, the map $\sigma \mapsto u^{-1}\sigma(u)$ is a 1-cocycle on Gal(L/F) with values in the $U(\overline{\mathfrak{S}})$-normalizer of $\boldsymbol{b}(\vartheta_2)$, which is equal to $J(\rho_2(\beta_2), \overline{\mathfrak{S}})$ according to [**10**]. This cocycle defines a class in the cohomology set:

$$H^1(\mathrm{Gal}(L/F), J(\rho_2(\beta_2), \overline{\mathfrak{S}})).$$

We claim this cohomology set is trivial. According to [**20**, Proposition 2.39], it is enough to prove that:

$$H^1(\mathrm{Gal}(L/F), J(\rho_2(\beta_2), \overline{\mathfrak{S}})/J^1(\rho_2(\beta_2), \overline{\mathfrak{S}}))$$

is trivial, which is given by a standard filtration argument (see [**5**, §6]). □

Using Proposition 7.3, we thus may replace ρ_2 by a $\mathfrak{K}(\mathfrak{S})$-conjugate and assume that the characters ϑ_1 and ϑ_2 are equal. We now fix a decomposition:

$$V = V^1 \oplus \cdots \oplus V^l$$

of V into simple right $\mathrm{E}_1 \otimes_{\mathrm{F}}$ D-modules (which all are copies of S) such that the lattice sequence Λ decomposes into the direct sum of the $\Lambda^j = \Lambda \cap \mathrm{V}^j$, for $j \in \{1, \dots, l\}$. By choosing, for each j, an isomorphism of $\mathrm{K}(\beta) \otimes_{\mathrm{F}}$ D-modules between S and V^j, this gives us an F-algebra homomorphism:

$$\iota : \mathrm{A}(\mathrm{S}) \to \mathrm{A}.$$

Using Lemma 2.14, we may assume that $\iota \circ \rho_1 = \varphi_1$, and, by Lemma 3.5, on may replace φ_2 by a $\mathfrak{K}(\Lambda)$-conjugate and assume that $\iota \circ \rho_2 = \varphi_2$. We now remark that, for each i, the map $\vartheta_i \mapsto \theta_i$ corresponds to the process described in paragraph 2.4. The equality $\theta_1 = \theta_2$ thus follows from Proposition 4.11.

8.2. In this paragraph, we reduce the proof of Theorem 1.13 to the totally ramified case, which has been treated in paragraph 8.1. For $i = 1, 2$, let (Θ_i, k, β_i) be a ps-character over F, set $\mathrm{E}_i = \mathrm{F}(\beta_i)$ and write K_i for the maximal unramified extension of F contained in E_i, and suppose that $\Theta_1 \approx \Theta_2$. Then we have $[\mathrm{E}_1 : \mathrm{F}] = [\mathrm{E}_2 : \mathrm{F}]$ and, using Proposition 4.9, there is a simple central F-algebra A together with realizations $[\Lambda, n, m, \varphi_i(\beta_i)]$ of (k, β_i), with $i = 1, 2$, which are sound and have the same embedding type, with k dividing m and such that $\varphi_1(\mathrm{K}_1) = \varphi_2(\mathrm{K}_2)$, denoted K, and $\theta_1 = \theta_2$, where $\theta_i = \Theta_i(\Lambda, m_i, \varphi_i)$. Let C denote the centralizer of K in A and write $[\Gamma, n, m, \beta_i]$ for the stratum in C associated with $[\Lambda, n, m, \beta_i]$ by (5.2). By Proposition 6.11, the K/F-lifts θ_1^{K} and θ_2^{K} are equal.

Now let A' be a simple central F-algebra and $[\Lambda', n', m', \varphi'_i(\beta_i)]$ be realizations of (k, β_i) in A, with $i = 1, 2$, having the same embedding type. By Remark 3.4, we may conjugate φ'_2 by $\mathfrak{K}(\Lambda')$ and assume that the maximal unramified extensions of F contained in $\varphi'_1(\mathrm{E}_1)$ and $\varphi'_2(\mathrm{E}_2)$ are equal to a common extension K' of F, say. Moreover, by Lemma 3.1, we may conjugate again φ'_2 by $\mathfrak{K}(\Lambda')$ and assume that the F-algebra isomorphisms $\varphi'_1 \circ \varphi_1^{-1}$ and $\varphi'_2 \circ \varphi_2^{-1}$ agree on K (and thus identify K and K'). Let C' denote the centralizer of K' in A' and write $[\Gamma', n', m', \varphi'_i(\beta_i)]$ for the stratum in C associated with $[\Lambda', n', m', \varphi'_i(\beta_i)]$ by (5.2). Thus the simple strata $[\Gamma, n, m, \varphi_i(\beta_i)]$ and $[\Gamma', n', m', \varphi'_i(\beta_i)]$ are realizations of the same simple pair over K. For each i, we write θ'_i for the character $\Theta_i(\Lambda', m', \varphi'_i)$. By Theorem 6.7, for each i, the K/F-lifts θ_i^{K} and $\theta_i'^{\mathrm{K}}$ are transfers of each other. Therefore, by paragraph 8.1, there exists $u \in \mathfrak{K}(\Gamma')$ such that:

$$\theta_2'^{\mathrm{K}}(x) = \theta_1'^{\mathrm{K}}(uxu^{-1}), \quad x \in \mathrm{H}^{m+1}(\varphi'_2(\beta_2), \Gamma') = u^{-1}\mathrm{H}^{m+1}(\varphi'_1(\beta_1), \Gamma')u.$$

The equality $\theta_1'^{u} = \theta'_2$ follows from Proposition 6.10.

Corollary 8.2. — *Definition 1.10 is equivalent to* [**6**, Definition 8.6].

Proof. — Assume we are given two ps-characters (Θ_i, k, β_i), $i = 1, 2$, which are endo-equivalent in the sense of Definition 1.10, and let A be a simple central split F-algebra together with realizations $[\Lambda, n_i, m_i, \varphi_i(\beta_i)]$ of (k, β_i) in A, with $i = 1, 2$, such that Λ is strict. By Theorem 1.11, the simple characters $\Theta_i(\Lambda, m_i, \varphi_i)$ intertwine in $A^\times$, that is, the ps-characters (Θ_i, k, β_i) are endo-equivalent in the sense of [**6**, Definition 8.6]. Conversely, two simple pairs which are endo-equivalent in this sense are clearly endo-equivalent in the sense of Definition 1.10. □

Corollary 8.3. — *The relation $\approx$ on ps-characters is an equivalence relation.*

Proof. — This comes from [**6**, Corollary 8.10] together with Corollary 8.2. □

9. The endo-class of a discrete series representation

9.1. Let A be a simple central F-algebra, and let V be a simple left A-module. Associated with it, there is an F-division algebra D. We write d for the reduced degree of D over F and m for the dimension of V as a right D-vector space. We set $G = A^\times$, identified with $GL_m(D)$.

Let π be an irreducible smooth representation of G, and assume that its inertial class (in the sense of Bushnell and Kutzko's theory of types [**11**]), denoted $\mathfrak{s}(\pi)$, is homogeneous. Thus there is a positive integer r dividing m, an irreducible cuspidal representation ρ of the group $G_0 = GL_{m/r}(D)$ and unramified characters χ_i of G_0, with $i \in \{1, \dots, r\}$, such that π is isomorphic to a quotient of the normalized parabolically induced representation $\rho\chi_1 \times \cdots \times \rho\chi_r$ (see for instance [**2**] for the notation).

In this section, we associate with π an endo-class $\boldsymbol{\Theta}(\pi)$ over F, and show that it depends only on the inertial class $\mathfrak{s} = \mathfrak{s}(\pi)$.

9.2. Let π be a representation of G as above, and write $\mathfrak{s} = \mathfrak{s}(\pi)$ for its inertial class. According to [**24**, Théorème 5.23], this inertial class possesses a type in the sense of [**11**]. Such a type is a pair (J, λ) formed of a compact open subgroup J of G and of an irreducible smooth representation λ of J such that an irreducible smooth representation of G has inertial class $\mathfrak{s}$ if and only if λ occurs in its restriction to J. More precisely, (J, λ) can be chosen to be a *simple type* in the sense of [**22**]. We won't give a precise description of simple types; the only property of interest for us is the following fact, which is a weak form of [**24**, Théorème 5.23].

Fact 9.1. — *There is a simple stratum* $[\mathfrak{A}, n, 0, \beta]$ *in* A *together with a simple character* $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ *such that the order* $\mathfrak{A} \cap \mathrm{B}$ *(with* B *the centralizer of* $\mathrm{F}(\beta)$ *in* A*) is principal of period* r *and the character* θ *occurs in the restriction of* π *to* $\mathrm{H}^1(\beta, \mathfrak{A})$*.*

Neither $[\mathfrak{A}, n, 0, \beta]$ nor the character θ are uniquely determined. We let $(\Theta, 0, \beta)$ be the ps-character defined by the pair $([\mathfrak{A}, n, 0, \beta], \theta)$ and we denote by $\boldsymbol{\Theta}$ its endo-class.

Theorem 9.2. — *The endo-class* $\boldsymbol{\Theta}$ *depends only on the inertial class* $\mathfrak{s}$*.*

Proof. — We have to prove that $\boldsymbol{\Theta}$ does not depend on the choice of the simple stratum $[\mathfrak{A}, n, 0, \beta]$ and the simple character θ satisfying the conditions of Fact 9.1. For $i = 1, 2$, let $[\mathfrak{A}_i, n_i, 0, \beta_i]$ be a simple stratum and θ_i be a simple character satisfying the conditions of Fact 9.1, and let $(\Theta_i, 0, \beta_i)$ denote the ps-character that it defines. Let $\mathfrak{A}'_i$ denote the unique principal $\mathcal{O}_\mathrm{F}$-order in A such that the pair $(\mathrm{E}_i, \mathfrak{A}'_i)$ is a sound embedding in A (see Lemma 4.18) and let θ'_i denote the transfer of θ_i in $\mathcal{C}(\mathfrak{A}'_i, 0, \beta_i)$. By a standard argument using [**24**, Théorème 2.13], the character θ'_i occurs in the restriction of π to $\mathrm{H}^1(\beta_i, \mathfrak{A}'_i)$. Therefore, we can assume without changing Θ_i that $(\mathrm{E}_i, \mathfrak{A}_i)$ is sound.

Lemma 9.3. — *The extensions* E_1/F *and* E_2/F *have the same ramification index.*

Proof. — We are going to prove that this ramification index is determined by the irreducible cuspidal representation ρ of paragraph 9.1. Let $n(\rho)$ denote the number of unramified characters χ of G_0 such that $\rho\chi$ is equivalent to ρ. Write q for the cardinality of the residue field of F and $|\cdot|_\mathrm{F}$ for the absolute value on F giving the value q^{-1} to any uniformizer. Let $s(\rho)$ denote the unique positive real number such that $\rho \times \rho\nu_\rho$ is reducible, where ν_ρ is the unramified character $g \mapsto |\mathrm{N}_{\mathrm{A}/\mathrm{F}}(g)|_\mathrm{F}^{s(\rho)}$ (see section 4 of [**23**] for more details). By using [**23**, Theorem 4.6], the product $n(\rho)s(\rho)$ is equal to the quotient of md by the ramification index of E_i/F, for any $i = 1, 2$. □

By Lemma 4.18, the principal orders $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have the same period (as $\mathfrak{A}_i \cap \mathrm{B}_i$ has period r). Thus one may conjugate $([\mathfrak{A}_1, n_1, 0, \beta_1], \theta_1)$ by an element of G and assume that $\mathfrak{A}_1 = \mathfrak{A}_2$, denoted $\mathfrak{A}$. For each i, we have $\theta_i \in \mathcal{C}(\mathfrak{A}, 0, \beta_i)$ and θ_i occurs in the restriction of π to the subgroup $\mathrm{H}^1(\beta_i, \mathfrak{A})$. Thus the characters θ_1 and θ_2 intertwine in $\mathrm{A}^\times$. To prove that Θ_1 and Θ_2 are endo-equivalent, it remains to prove that $\mathrm{F}(\beta_1)$ and $\mathrm{F}(\beta_2)$ have the same degree over F. By copying the beginning of the proof of Lemma 4.7, we get $n_1 = n_2$. We now write f for the greatest common divisor of $f_\mathrm{F}(\beta_1)$ and $f_\mathrm{F}(\beta_2)$ and K_i for the maximal unramified extension of F contained in $\mathrm{F}(\beta_i)$. Then Theorem 1.16 gives us the expected equality. □

We call the class $\boldsymbol{\Theta}$ *the endo-class* of π (or of $\mathfrak{s}$). We have actually obtained more.

Theorem 9.4. — *Let π be an irreducible representation with inertial class $\mathfrak{s}$ as above, and let $[\mathfrak{A}, n, 0, \beta]$ and θ satisfy the conditions of Fact 9.1. Assume moreover $[\mathfrak{A}, n, 0, \beta]$ is sound. The following objects are invariants of the inertial class $\mathfrak{s}$:*

(1) *the ramification index $e_{\mathrm{F}}(\beta)$ and the residue class degree $f_F(\beta)$;*
(2) *the* G*-conjugacy class of the order $\mathfrak{A}$;*
(3) *the embedding type of* $(\mathrm{F}(\beta), \mathfrak{A})$.

Proof. — Assertions (1) and (2) have already been proved. Assertion (3) follows immediately from Lemma 4.19. □

9.3. Recall that an irreducible smooth representation π of G is *essentially square integrable* if there is a character χ of G such that $\pi\chi$ is unitary and has a non-zero coefficient which is square integrable on G/Z, where Z denotes the centre of G. We write $\mathcal{D}(\mathrm{G})$ for the set of isomorphism classes of essentially square integrable representation of G. According to [**2**, §2.2], any essentially square integrable representation of G has an inertial class which is homogeneous in the sense of paragraph 9.1. Thus the construction of paragraph 9.2 gives us a map:

$$\boldsymbol{\Theta}_{\mathrm{G}} : \mathcal{D}(\mathrm{G}) \to \mathcal{E}(\mathrm{F}) \tag{9.1}$$

from $\mathcal{D}(\mathrm{G})$ to the set of endo-classes of ps-characters over F.

We now write $\mathrm{H} = \mathrm{GL}_{md}(\mathrm{F})$, and let **JL** denote the Jacquet-Langlands correspondence (see [**1, 14**]) from $\mathcal{D}(\mathrm{G})$ to $\mathcal{D}(\mathrm{H})$. We have the following conjecture.

Conjecture 9.5. — *For any π in $\mathcal{D}(G)$, we have:*

$$\Theta_{\mathrm{H}}(\mathbf{JL}(\pi)) = \Theta_{\mathrm{G}}(\pi). \tag{9.2}$$

This conjecture generalizes the fact that, for any level zero representation π in $\mathcal{D}(\mathrm{G})$, the representation $\mathbf{JL}(\pi)$ has level zero. It allows one to refine the correspondence **JL** by fixing the endo-class: given $\boldsymbol{\Theta}$ an endo-class over F, Conjecture 9.5 implies that we have a bijective map:

$$\mathbf{JL}_{\boldsymbol{\Theta}} : \mathcal{D}(\mathrm{G}, \boldsymbol{\Theta}) \to \mathcal{D}(\mathrm{H}, \boldsymbol{\Theta})$$

where we write $\mathcal{D}(\mathrm{G}, \boldsymbol{\Theta})$ for the set of isomorphism classes of essentially square integrable representations of G of endo-class $\boldsymbol{\Theta}$.

References

[1] A. I. Badulescu – "Correspondance de Jacquet-Langlands pour les corps locaux de caractéristique non nulle", *Ann. Sci. École Norm. Sup. (4)* **35** (2002), no. 5, p. 695–747.

[2] A. I. Badulescu, G. Henniart, B. Lemaire & V. Sécherre – "Sur le dual unitaire de $\mathrm{GL}(r, D)$", to appear in *Amer. J. Math.*

[3] P. Broussous & B. Lemaire – "Building of $\mathrm{GL}(m, D)$ and centralizers", *Transform. Groups* **7** (2002), no. 1, p. 15–50.

[4] P. Broussous – "Hereditary orders and embeddings of local fields in simple algebras", *J. Algebra* **204** (1998), no. 1, p. 324–336.

[5] P. Broussous & M. Grabitz – "Pure elements and intertwining classes of simple strata in local central simple algebras", *Comm. Algebra* **28** (2000), no. 11, p. 5405–5442.

[6] C. J. Bushnell & G. Henniart – "Local tame lifting for $\mathrm{GL}(N)$. I. Simple characters", *Inst. Hautes Études Sci. Publ. Math.* (1996), no. 83, p. 105–233.

[7] ———, "Correspondance de Jacquet-Langlands explicite. II : Le cas de degré égal à la caractéristique résiduelle", *Manuscripta Math.* **102** (2000), no. 2, p. 211–225.

[8] ———, "Local tame lifting for $\mathrm{GL}(n)$. IV. Simple characters and base change", *Proc. London Math. Soc. (3)* **87** (2003), no. 2, p. 337–362.

[9] ———, "Local tame lifting for $\mathrm{GL}(n)$. III. Explicit base change and Jacquet-Langlands correspondence", *J. Reine Angew. Math.* **580** (2005), p. 39–100.

[10] C. J. Bushnell & P. C. Kutzko – *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, Princeton University Press, Princeton, NJ, 1993.

[11] ———, "Smooth representations of reductive p-adic groups: structure theory via types", *Proc. London Math. Soc. (3)* **77** (1998), no. 3, p. 582–634.

[12] ———, "Semisimple types in GL_n", *Compositio Math.* **119** (1999), no. 1, p. 53–97.

[13] J.-F. Dat – "Finitude pour les représentations lisses de groupes p-adiques", *J. Inst. Math. Jussieu* **8** (2009), no. 2, p. 261–333.

[14] P. Deligne, D. Kazhdan & M.-F. Vignéras – "Représentations des algèbres centrales simples p-adiques", in *Representations of reductive groups over a local field*, Travaux en Cours, Hermann, Paris, 1984, p. 33–117.

[15] A. Fröhlich – "Principal orders and embedding of local fields in algebras", *Proc. London Math. Soc. (3)* **54** (1987), no. 2, p. 247–266.

[16] M. Grabitz – "Continuation of hereditary orders in local central simple algebras", *J. Number Theory* **77** (1999), no. 1, p. 1–26.

[17] ———, "Simple characters for principal orders in $\mathrm{M}_m(D)$", *J. Number Theory* **126** (2007), no. 1, p. 1–51.

[18] G. Henniart – "Correspondance de Jacquet-Langlands explicite. I : Le cas modéré de degré premier", in *Séminaire de Théorie des Nombres, Paris, 1990–91*, Progr. Math., vol. 108, Birkhäuser, Boston, MA, 1993, p. 85–114.

[19] A. Minguez & V. Sécherre – "Représentations lisses ℓ-modulaires de $\mathrm{GL}(m, D)$", in preparation.

[20] V. Sécherre – "Représentations lisses de $\mathrm{GL}(m, D)$, I : caractères simples", *Bull. Soc. math. France* **132** (2004), no. 3, p. 327–396.

[21] ———, "Représentations lisses de GL(m, D), II : β-extensions", *Compos. Math.* **141** (2005), p. 1531–1550.

[22] ———, "Représentations lisses de GL(m, D), III : types simples", *Ann. Scient. Éc. Norm. Sup.* **38** (2005), p. 951–977.

[23] ———, "Proof of the Tadic Conjecture U0 on the unitary dual of GL(m, D)", *J. Reine Angew. Math.* **626** (2009), p. 187-204.

[24] V. Sécherre & S. Stevens – "Représentations lisses de GL(m, D), IV : représentations supercuspidales", *J. Inst. Math. Jussieu* **7** (2008), no. 3, p. 527–574.

[25] ———, "Smooth representations of GL(m, D), VI : semi-simple types", preprint, arXiv:1009.1062 (2010).

[26] A. J. Silberger & E.-W. Zink – "Weak explicit matching for level zero discrete series of unit groups of $\mathfrak{p}$-adic simple algebras", *Canad. J. Math.* **55** (2003), no. 2, p. 353–378.

[27] ———, "An explicit matching theorem for level zero discrete series of unit groups of $\mathfrak{p}$-adic simple algebras", *J. Reine Angew. Math.* **585** (2005), p. 173–235.

[28] S. Stevens – "Semisimple characters for p-adic classical groups", *Duke Math. J.* **127** (2005), no. 1, p. 123–173.

[29] E.-W. Zink – "More on embeddings of local fields in simple algebras", *J. Number Theory* **77** (1999), no. 1, p. 51–61.

P. Broussous, Université de Poitiers, Laboratoire de Mathématiques, Téléport 2, Boulevard Marie et Pierre Curie, BP 30179, F-86962 Futuroscope-Chasseneuil cedex, France
E-mail : Paul.Broussous@math.univ-poitiers.fr

V. Sécherre, Université de Versailles Saint-Quentin, Laboratoire de Mathématiques de Versailles, 45 avenue des Etats-Unis, 78035 Versailles cedex, France • *E-mail :* vincent.secherre@math.uvsq.fr

S. Stevens, School of Mathematics, University of East Anglia, Norwich NR4 7TJ, United Kingdom
E-mail : Shaun.Stevens@uea.ac.uk