AN ANALOGUE OF THE CARTAN DECOMPOSITION FOR p-ADIC SYMMETRIC SPACES OF SPLIT p-ADIC REDUCTIVE GROUPS

by

P. Delorme & V. Sécherre

Abstract. — Let k be a non-Archimedean locally compact field of residue characteristic p, let G be a connected reductive group defined over k, let σ be an involutive k-automorphism of G and H an open k-subgroup of the fixed points group of σ. We denote by G_k (resp. H_k) the group of k-points of G (resp. H). In this paper, we obtain an analogue of the Cartan decomposition for the reductive symmetric space $\mathrm{H}_k\backslash\mathrm{G}_k$ in the case where G is k-split and p is odd. More precisely, we obtain a decomposition of G_k as a union of $(\mathrm{H}_k, \mathrm{K})$-double cosets, where K is the stabilizer of a special point in the Bruhat-Tits building of G over k. This decomposition is related to the H_k-conjugacy classes of maximal σ-anti-invariant k-split tori in G. In a more general context, Benoist and Oh obtained a polar decomposition for any p-adic reductive symmetric space. In the case where G is k-split and p is odd, our decomposition makes more precise Benoist-Oh's polar decomposition and generalizes results of Offen for GL_n.

Résumé. — Soit k un corps localement compact non archimédien de caractéristique résiduelle impaire p, soit G un groupe réductif connexe défini sur k, soit σ un k-automorphisme involutif de G et soit H un k-sous-groupe ouvert du groupe des points de G fixes par σ. On note G_k (resp. H_k) le groupe des k-points de G (resp. H). Dans cet article, nous obtenons un analogue de la décomposition de Cartan pour l'espace symétrique réductif $\mathrm{H}_k\backslash\mathrm{G}_k$ lorsque G est déployé sur k et p est impair. Plus précisément, nous obtenons une décomposition de G_k sous la forme d'une réunion de doubles classes modulo $(\mathrm{H}_k, \mathrm{K})$, où K désigne le stabilisateur d'un point spécial de l'immeuble de Bruhat-Tits de G sur k. Cette décomposition est liée aux classes de H_k-conjugaison des tores k-déployés σ-anti-invariants maximaux de G. Dans un cadre plus général, Benoist et Oh ont obtenu une décomposition polaire pour les espaces symétriques réductifs p-adiques quelconques. Dans le cas où G est déployé sur k et où p est impair, notre décomposition précise la décomposition polaire de Benoist et Oh et généralise des résultats de Offen pour GL_n.

1. Introduction

Let k be a non-Archimedean locally compact field of odd residue characteristic. Let G be a connected reductive group defined over k, let σ be an involutive k-automorphism of G and let H be an open k-subgroup of the fixed points group of σ. We denote by G_k (resp.

H_k) the group of k-points of G (resp. H). Harmonic analysis on the reductive symmetric space $H_k\backslash G_k$ is the study of the action of G_k on the space of complex square integrable functions on $H_k\backslash G_k$. This study is related to the classification of H_k-distinguished representations of G_k, that is representations having a non-zero space of H_k-invariant linear forms. Offen [**19**] has investigated the harmonic analysis of spherical functions in some cases related to GL_n. Blanc and Delorme [**3**] have studied H_k-distinguishedness for families of parabolically induced representations of G_k. Lagier [**16**], and independently Kato and Takano [**15**], have introduced the notion of relative cuspidality for irreducible H_k-distinguished representations of G_k and constructed "Jacquet maps" at the level of invariant linear forms. In this paper, we investigate the geometry of the reductive symmetric space $H_k\backslash G_k$.

Connected reductive groups can be considered as reductive symmetric spaces. Indeed, if G' is such a group, the map:

$$\sigma : (x, y) \mapsto (y, x)$$

defines a k-involution of $G = G' \times G'$ whose fixed points group H is the diagonal image of G' in G, and the reductive symmetric space $H_k\backslash G_k$ naturally identifies with G'_k via the map $(x, y) \mapsto x^{-1}y$. Moreover, if K' is a subgroup of G'_k, and if we set $K = K' \times K'$, then this map induces a bijective correspondence:

$$\{(H_k, K)\text{-double cosets of } G_k\} \leftrightarrow \{K'\text{-double cosets of } G'_k\}.$$

In particular, if K' is the G'_k-stabilizer of a special point in the Bruhat-Tits building of G' over k, the decomposition of $H_k\backslash G_k$ into K-orbits corresponds to the Cartan decomposition of G'_k relative to K' (see [**6**, Proposition 4.4.3]).

In this paper, we obtain an analogue of the Cartan decomposition for $H_k\backslash G_k$ when the group G is k-split. In a more general context (k any non-Archimedean locally compact field of odd characteristic and G any connected reductive group over k), Benoist and Oh [**2**] have obtained a polar decomposition for $H_k\backslash G_k$. In the case where k has odd residue characteristic and G is k-split, our decomposition is a refinement of Benoist-Oh's polar decomposition (see paragraph 4.6). This decomposition can be seen as a p-adic analogue of the Cartan decomposition for real reductive symmetric spaces (see [**10**, Theorem 4.1]). It generalizes the decompositions obtained by Offen (see [**19**, Proposition 3.1]) for $G = GL_{2n}$ in Cases 1 and 3 (*ibid.*).

Let $\{A^j \mid j \in J\}$ be a set of representatives of the H_k-conjugacy classes of maximal σ-anti-invariant k-split tori of G (called maximal (σ, k)-*split* tori in [**11**], see also Definition

4.1). These tori, as well as related entities, have been studied by A. Helminck, G. Helminck and Wang [**11, 12, 13**]. In particular, the set J is finite and the A^j, $j \in \mathrm{J}$, are all conjugate under G_k. Let S be a σ-stable maximal k-split torus of G containing a maximal (σ, k)-split torus A. For each $j \in \mathrm{J}$, we choose $y_j \in \mathrm{G}_k$ such that $y_j \mathrm{A} y_j^{-1} = \mathrm{A}^j$. Our main result is the following theorem (see Theorem 4.9).

Theorem 1.1. — *Assume* G *is k-split. Let* K *be the stabilizer in* G_k *of a special point in the apartment attached to* S. *Then:*

$$\mathrm{G}_k = \bigcup_{j \in \mathrm{J}} \mathrm{H}_k y_j \mathrm{S}_k \mathrm{K}. \tag{1.1}$$

If one compares with Offen's decompositions [**19**, Proposition 3.1], one sees that in each of his Cases 1 and 3 (where $\mathrm{G} = \mathrm{GL}_{2n}$ for $n \geqslant 1$), the set J reduces to a single element and y_j can be chosen to be trivial. In general however, one cannot avoid to have several non-H_k-conjugate maximal σ-anti-invariant k-split tori of G appearing in (1.1).

To prove Theorem 1.1, we make a large use of the Bruhat-Tits theory [**6, 7**]). First, let G be any connected reductive group over k, and let $\mathscr{B}$ be its Bruhat-Tits building. It is endowed with an action of σ. Then we have (see Proposition 3.4):

Proposition 1.2. — *$\mathscr{B}$ is the union of its σ-stable apartments.*

Note that in the case where $\mathrm{G} = \mathrm{G}' \times \mathrm{G}'$ and $\sigma(x, y) = (y, x)$ as above, then the building $\mathscr{B}$ identifies with the product of two copies of the building of G' over k and Proposition 1.2 simply says that two arbitrary points in the building of G' are always contained in a common apartment.

When G is k-split, we obtain the following refinement of Proposition 1.2 (see Proposition 4.5).

Proposition 1.3. — *Assume* G *is k-split, and let x be a special point of $\mathscr{B}$. There is a σ-stable maximal k-split torus* S *of* G *such that the apartment corresponding to* S *contains x and the maximal σ-anti-invariant subtorus of* S *is a maximal (σ, k)-split torus of* G.

As we will see in paragraph 5.3, Proposition 1.3 is no longer true for non-split groups.

In section 2, we recall the main properties of the Bruhat-Tits building attached to a connected reductive group defined over k.

In section 3, we study the set of all apartments containing a given σ-stable subset of the building, and we prove Proposition 1.2.

In section 4, we prove our main theorem for G a k-split group.

In section 5, we study in more details the case of $G_k = GL_n(k)$ and $\sigma(g) =$ transpose of g^{-1}, and the case of $G_k = GL_n(k')$ with k' quadratic over k and $\mathrm{id} \neq \sigma \in \mathrm{Gal}(k'/k)$. When $n = 2$ and k' is totally ramified over k, the second case provides an example of a non-split group for which Proposition 1.3 is not satisfied.

We thanks F. Courtès, B. Lemaire, G. Rousseau, S. Stevens for stimulating discussions. Particular thanks to J. Bernstein for having suggested to the first author the use of the Bruhat-Tits building, and to J.-P. Labesse for answering numerous questions.

2. The Bruhat-Tits building

Let k be a non-Archimedean non-discrete locally compact field, and let ω be its normalized valuation. In this section, we recall the main properties of the Bruhat-Tits building attached to a connected reductive group defined over k. The reader may refer to Bruhat-Tits [**6, 7**] or to more concise presentations [**17, 21, 23**].

If G is a linear algebraic group defined over k, the group of its k-points will be denoted by G_k or $G(k)$, and its neutral component will be denoted by G°. If X is a subset of G, then $N_G(X)$ (resp. $Z_G(X)$) denotes the normalizer (resp. the centralizer) of X in G, and, given $g \in G$, we write ${}^gX$ for gXg^{-1}.

2.1. Let G be a connected reductive group defined over k, and let S be a maximal k-split torus of G. We denote by $X^*(S) = \mathrm{Hom}(S, GL_1)$ (resp. by $X_*(S) = \mathrm{Hom}(GL_1, S)$) the group of algebraic characters (resp. cocharacters) of S. We define a map:

$$X_*(S) \times X^*(S) \to \mathbf{Z} \tag{2.1}$$

as follows. If $\lambda \in X_*(S)$ and $\chi \in X^*(S)$, then $\chi \circ \lambda$ is an endomorphism of the multiplicative group GL_1, which corresponds to an endomorphism of the ring $\mathbf{Z}[t, t^{-1}]$. It is of the form $t \mapsto t^n$ for some $n \in \mathbf{Z}$. This integer n is denoted by $\langle \lambda, \chi \rangle$. The map (2.1) defines a perfect duality (see [**4**, §8.6]).

2.2. Let N (resp. Z) denote the normalizer (resp. the centralizer) of S in G. If we extend the map (2.1) by $\mathbf{R}$-linearity, there exists a unique group homomorphism:

$$\nu : Z_k \to X_*(S) \otimes_{\mathbf{Z}} \mathbf{R} \tag{2.2}$$

such that the condition:

$$\langle \nu(z), \chi \rangle = -\omega(\chi(z))$$

holds for any $z \in \mathrm{Z}_k$ and any k-rational character $\chi \in \mathrm{X}^*(\mathrm{Z})_k$ (see [**23**, §1.2]). According to [**17**, Proposition 1.2], the kernel of (2.2) is the maximal compact subgroup of Z_k.

2.3. Let C denote the connected centre of G and let $\mathrm{X}_*(\mathrm{C})$ be the group of its algebraic cocharacters. It is a subgroup of the free abelian group $\mathrm{X}_*(\mathrm{S})$. We denote by $\mathscr{A}$ the space:

$$\mathrm{V} = (\mathrm{X}_*(\mathrm{S}) \otimes_{\mathbf{Z}} \mathbf{R})/(\mathrm{X}_*(\mathrm{C}) \otimes_{\mathbf{Z}} \mathbf{R})$$

considered as an affine space on itself and by $\mathrm{Aff}(\mathscr{A})$ the group of its affine automorphisms. By making V act on $\mathscr{A}$ by translations, we can think of V as a subgroup of $\mathrm{Aff}(\mathscr{A})$. It is the kernel of the natural group homomorphism $\mathrm{Aff}(\mathscr{A}) \to \mathrm{GL}(\mathrm{V})$ which associates to any affine automorphism its linear part.

2.4. The map (2.2) induces a homomorphism:

$$\mathrm{Z}_k \to \mathrm{Aff}(\mathscr{A}) \tag{2.3}$$

which we still denote by ν. Its image is contained in V. An important property of this homomorphism is that it extends to a homomorphism $\mathrm{N}_k \to \mathrm{Aff}(\mathscr{A})$ (see [**23**, §1.2]). It does not extend in a unique way, but two homomorphisms extending (2.3) to N_k are conjugated by a *unique* element of $\mathrm{Aff}(\mathscr{A})$ (see [**17**, Proposition 1.8]).

2.5. The affine space $\mathscr{A}$ endowed with an action of N_k defined by a group homomorphism $\nu : \mathrm{N}_k \to \mathrm{Aff}(\mathscr{A})$ extending the homomorphism (2.3) is called the (reduced) *apartment* attached to S. It satisfies the conditions:

A1 $\mathscr{A}$ is an affine space on V;

A2 ν is a group homomorphism $\mathrm{N}_k \to \mathrm{Aff}(\mathscr{A})$ extending the canonical homomorphism $\mathrm{Z}_k \to \mathrm{V}$.

It has the following unicity property: if $(\mathscr{A}', \nu')$ satisfies **A1** and **A2**, then there is a unique affine and N_k-equivariant isomorphism from $\mathscr{A}'$ to $\mathscr{A}$.

Remark 2.1. — As in Tits [**23**], one obtains the *non-reduced* apartment $\mathscr{A}_{\mathrm{nr}}$ by replacing V by $\mathrm{X}_*(\mathrm{S})\otimes_{\mathbf{Z}}\mathbf{R}$. It is not as canonical as the reduced one: two homomorphisms extending the map $\nu_{\mathrm{nr}} : \mathrm{Z}_k \to \mathrm{Aff}(\mathscr{A}_{\mathrm{nr}})$ to N_k are conjugated by an element of $\mathrm{Aff}(\mathscr{A}_{\mathrm{nr}})$ which is not necessarily unique (see [**17**, §1] and also [**23**, §1.2]).

2.6. Let $\Phi = \Phi(\mathrm{G}, \mathrm{S})$ denote the set of roots of G relative to S. It is a subset of $\mathrm{X}^*(\mathrm{S})$. Therefore, any root $a \in \Phi$ can be seen as a linear form on $\mathrm{X}_*(\mathrm{S}) \otimes_{\mathbf{Z}} \mathbf{R}$ which is trivial on the subspace $\mathrm{X}_*(\mathrm{C}) \otimes_{\mathbf{Z}} \mathbf{R}$, hence as a linear form on V (see [**17**, §1]).

For $a \in \Phi$, we denote by U_a the root subgroup associated to a, which is a unipotent subgroup of G normalized by Z (see [**4**, Proposition 21.9]), and by s_a the reflection corresponding to a, considered as an element of GL(V) — or, more precisely, of the quotient of $\nu(\mathrm{N}_k)$ by $\nu(\mathrm{Z}_k)$.

2.7. Let $a \in \Phi$ and $u \in \mathrm{U}_a(k) - \{1\}$. The intersection:

$$\mathrm{U}_{-a}(k) u \mathrm{U}_{-a}(k) \cap \mathrm{N}_k \tag{2.4}$$

consists of a single element, called $m(u)$, whose image by ν is an affine reflection the linear part of which is s_a (see [**5**, §5]). The set $\mathscr{H}_{a,u}$ of fixed points of $\nu(m(u))$ is an affine hyperplane of $\mathscr{A}$, which is called a *wall* of $\mathscr{A}$.

A *chamber* of $\mathscr{A}$ is a connected component of the complementary in $\mathscr{A}$ of the union of its walls. Note that a chamber is open in $\mathscr{A}$.

A point $x \in \mathscr{A}$ is said to be *special* if, for all root $a \in \Phi$, there is a root $b \in \Phi \cap \mathbf{R}_+ a$ and an element $u \in \mathrm{U}_b(k) - \{1\}$ such that $x \in \mathscr{H}_{b,u}$ (see [**18**, §1.2.3] and also [**23**, §1.9]).

2.8. Let $\theta(a, u)$ denote the affine function $\mathscr{A} \to \mathbf{R}$ whose linear part is a and whose vanishing hyperplane is the wall $\mathscr{H}_{a,u}$ of fixed points of $\nu(m(u))$. We fix a base point in $\mathscr{A}$, so that $\mathscr{A}$ can be identified with the vector space V. For $r \in \mathbf{R}$, we set:

$$\mathrm{U}_a(k)_r = \{u \in \mathrm{U}_a(k) - \{1\} \mid \theta(a,u)(x) \geqslant a(x) + r \text{ for all } x \in \mathscr{A}\} \cup \{1\}.$$

Thus we obtain a filtration of $\mathrm{U}_a(k)$ by subgroups. If we change the base point in $\mathscr{A}$, this filtration is only modified by a translation of the indexation.

2.9. Let Ω be a non-empty subset of $\mathscr{A}$. We set:

$$\mathrm{N}_\Omega = \{n \in \mathrm{N}_k \mid \nu(n)(x) = x \text{ for all } x \in \Omega\},$$

and we denote by U_Ω the subgroup of G_k generated by all the $\mathrm{U}_a(k)_r$ such that the affine function $x \mapsto a(x) + r$ is non-negative on Ω. According to [**17**, §12], this subgroup is compact in G_k, and we have $n\mathrm{U}_\Omega n^{-1} = \mathrm{U}_{\nu(n)(\Omega)}$ for $n \in \mathrm{N}_k$. In particular, N_Ω normalizes U_Ω. The subgroup $\mathrm{P}_\Omega = \mathrm{N}_\Omega \mathrm{U}_\Omega$ is open in G_k (*loc.cit.*, Corollary 12.12).

2.10. Let $\Phi = \Phi^- \cup \Phi^+$ be a decomposition of Φ into positive and negative roots. We denote by U^+ and U^-) the subgroup of G_k generated by the U_a for all $a \in \Phi^+$ (resp. for all $a \in \Phi^-$). Then the group P_Ω has the following Iwahori decomposition:

$$P_\Omega = (U_\Omega \cap U^-) \cdot (U_\Omega \cap U^+) \cdot N_\Omega \tag{2.5}$$

(see [**17**, Corollary 12.6] and also [**6**, §7.1.4]).

2.11. In [**6, 7**], Bruhat and Tits associate to the apartment $(\mathscr{A}, \nu)$ a G_k-set $\mathscr{B} = \mathscr{B}(G, k)$ containing $\mathscr{A}$, called the (reduced) *building* of G over k and satisfying the following conditions:

B1 The set $\mathscr{B}$ is the union of the $g \cdot \mathscr{A}$ for $g \in G_k$.

B2 The subgroup N_k is the stabilizer of $\mathscr{A}$ in G_k, and $n \cdot x = \nu(n)(x)$ for all $x \in \mathscr{A}$ and $n \in N_k$.

B3 For all $a \in \Phi$ and $r \in \mathbf{R}$, the subgroup $U_a(k)_r$ defined in paragraph 2.8 fixes the subset $\{x \in \mathscr{A} \mid a(x) + r \geqslant 0\}$ pointwise.

The building has the following unicity property: if $\mathscr{B}'$ is a G_k-set containing $\mathscr{A}$ and satisfying **B1**, **B2** and **B3**, then there is a unique G_k-equivariant bijection from $\mathscr{B}'$ to $\mathscr{B}$ (see [**23**, §2.1] and also [**20**, §1.9]).

2.12. The subsets of $\mathscr{B}$ of the form $g \cdot \mathscr{A}$ with $g \in G_k$ are called *apartments*. According to **B1**, the building is the union of its apartments. For $g \in G_k$, the apartment $g \cdot \mathscr{A}$ can be naturally endowed with a structure of affine space and an action of ${}^gN_k$ by affine isomorphisms. Up to unique isomorphism, it is the apartment attached to the maximal k-split torus ${}^gS$ (see paragraph 2.5). This defines a unique G_k-equivariant map:

$$S' \mapsto \mathscr{A}(S') \subseteq \mathscr{B} \tag{2.6}$$

between maximal k-split tori of G and apartments of $\mathscr{B}$, such that S maps to $\mathscr{A}$.

Note that the building $\mathscr{B}$ does not depend on the maximal k-split torus S. Indeed, let S′ be a maximal k-split torus of G, let $(\mathscr{A}', \nu')$ be the apartment attached to S′ and $\mathscr{B}'$ be the building of G over k relative to this apartment (see paragraph 2.11). If we identify $\mathscr{A}'$ with the unique apartment of $\mathscr{B}$ corresponding to S′ *via* (2.6), then $\mathscr{B}' = \mathscr{B}$.

2.13. The building has the following important properties (see [**6**, §7.4] and [**17**, §13]):

(1) Let Ω be a non-empty subset of $\mathscr{A}$. Then P_Ω is the subgroup of G_k made of those elements fixing Ω pointwise.

(2) Let $g \in G_k$. There is $n \in N_k$ such that $g \cdot x = n \cdot x$ for any $x \in \mathscr{A} \cap g^{-1} \cdot \mathscr{A}$.

In particular, Property (1) together with **B2** imply that $N_\Omega = N_k \cap P_\Omega$.

2.14. Let σ be a k-automorphism of G. There is a unique bijective map from $\mathscr{B}$ to itself, still denoted σ, such that:

(1) the condition:

$$\sigma(g \cdot x) = \sigma(g) \cdot \sigma(x)$$

holds for any $g \in \mathrm{G}_k$ and $x \in \mathscr{B}$;

(2) the map σ permutes the apartments and, for any apartment $\mathscr{A}$, the restriction of σ to $\mathscr{A}$ is an affine isomorphism from $\mathscr{A}$ to $\sigma(\mathscr{A})$.

This makes (2.6) into a σ-equivariant map. In particular, an apartment is σ-stable if and only if its corresponding maximal k-split torus of G is σ-stable (see [**7**, §4.2.12]).

3. Existence of σ-stable apartments

From now on, k will be a non-Archimedean locally compact field of odd residue characteristic. Let G be connected reductive group defined over k and let σ be a k-involution on G. According to paragraph 2.14, the building $\mathscr{B}$ of G over k is endowed with an action of σ. In this section, we prove that, given $x \in \mathscr{B}$, there exists a σ-stable apartment containing x. We keep using notation of Section 2.

3.1. Let Ω be a non-empty σ-stable subset of $\mathscr{B}$ contained in some apartment, and let $\mathrm{Ap}(\Omega)$ be the set of all apartments of $\mathscr{B}$ containing Ω. It is a non-empty set on which the group P_Ω acts transitively (see [**17**, Corollary 13.7]). Because Ω is σ-stable, both P_Ω and $\mathrm{Ap}(\Omega)$ are σ-stable. Note that the σ-stable apartments containing Ω are exactly the σ-fixed points in $\mathrm{Ap}(\Omega)$.

3.2. Let us fix an apartment $\mathscr{A} \in \mathrm{Ap}(\Omega)$ and an element $u \in \mathrm{P}_\Omega$ such that $\sigma(\mathscr{A}) = u \cdot \mathscr{A}$. Let N denote the normalizer in G of the maximal k-split torus of G corresponding to $\mathscr{A}$. As σ is involutive, we have:

$$\sigma(u)u \in \mathrm{P}_\Omega \cap \mathrm{N}_k = \mathrm{N}_\Omega. \tag{3.1}$$

The map $\rho : g \mapsto g \cdot \mathscr{A}$ induces a P_Ω-equivariant bijection between the homogeneous spaces $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ and $\mathrm{Ap}(\Omega)$. The automorphism:

$$\theta : x \mapsto u^{-1}\sigma(x)u$$

of the group G_k stabilizes P_Ω and N_Ω. Indeed $\sigma(\mathrm{N}_k) = u\mathrm{N}_k u^{-1}$, and:

$$\theta(\mathrm{N}_\Omega) = u^{-1}\sigma(\mathrm{P}_\Omega \cap \mathrm{N}_k)u = \mathrm{P}_\Omega \cap u^{-1}\sigma(\mathrm{N}_k)u = \mathrm{N}_\Omega.$$

Note that the condition (3.1) implies that $\theta \circ \theta$ is conjugation by some element of N_Ω. As N_Ω is θ-stable, the map:

$$(\sigma, g\mathrm{N}_\Omega) \mapsto u\theta(g\mathrm{N}_\Omega), \quad g \in \mathrm{P}_\Omega,$$

defines an action of σ on $\mathrm{P}_\Omega/\mathrm{N}_\Omega$, making ρ into a σ-equivariant bijection. Note that this action differs from the natural action of σ on $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ (which obviously has fixed points).

3.3. Let Ω be a non-empty σ-stable subset of $\mathscr{B}$ contained in some apartment.

Proposition 3.1. — *Assume that Ω contains a point of a chamber of $\mathscr{B}$. Then Ω is contained in some σ-stable apartment.*

Proof. — We describe the quotient $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ as a projective limit of finite σ-sets. According to [**9**, §1.2], Example (f), the group G_k is locally compact and totally disconnected. Therefore we can choose a decreasing filtration $(\mathrm{Q}_i)_{i\geqslant 0}$ of the open subgroup P_Ω of G_k satisfying the following properties:

(A) The intersection of the Q_i is reduced to $\{1\}$.
(B) For any $i \geqslant 0$, the subgroup Q_i is compact open and normal in P_Ω.

For $i \geqslant 0$, let $\mathrm{P}_{\Omega,i}$ denote the intersection $\mathrm{N}_\Omega\mathrm{Q}_i \cap \theta(\mathrm{N}_\Omega\mathrm{Q}_i)$. These subgroups form a decreasing filtration of P_Ω, and we claim that this filtration satisfies the following properties:

(1) The intersection of the $\mathrm{P}_{\Omega,i}$ is reduced to N_Ω.
(2) For any $i \geqslant 0$, the subgroup $\mathrm{P}_{\Omega,i}$ is θ-stable and of finite index in P_Ω.

As N_Ω is θ-stable, it is contained in the intersection of the $\mathrm{P}_{\Omega,i}$. Let g be in this intersection. For any $i \geqslant 0$, there exist $n_i \in \mathrm{N}_\Omega$ and $q_i \in \mathrm{Q}_i$ such that $g = n_i q_i$. Because of Property (A) above, q_i converges to 1. Therefore n_i converges to a limit contained in the closed subgroup N_Ω, and this limit is g. This proves Property (1).

Now recall that $\theta \circ \theta$ is conjugation by some element of N_Ω. This implies that $\mathrm{P}_{\Omega,i}$ is θ-stable. As $\mathrm{P}_{\Omega,i}$ is open in P_Ω and contains N_Ω, the quotient $\mathrm{P}_\Omega/\mathrm{P}_{\Omega,i}$ can be identified with the quotient of U_Ω, which is compact, by some open subgroup. This gives us the expected result.

Because of Property (2), the map:

$$(\sigma, g\mathrm{P}_{\Omega,i}) \mapsto u\theta(g\mathrm{P}_{\Omega,i}), \quad g \in \mathrm{P}_\Omega,$$

defines an action of σ on the finite quotient $\mathrm{P}_\Omega/\mathrm{P}_{\Omega,i}$, which gives us a projective system $(\mathrm{P}_\Omega/\mathrm{P}_{\Omega,i})_{i\geqslant 0}$ of finite σ-sets. As P_Ω is complete, and thanks to Property (1), the natural σ-equivariant map from $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ to the projective limit of the $\mathrm{P}_\Omega/\mathrm{P}_{\Omega,i}$ is bijective.

Lemma 3.2. — *Let* $(X_i)_{i\geqslant 0}$ *be a projective system of finite* σ*-sets. For all* $i \geqslant 0$*, assume the transition maps* $\varphi_i : X_{i+1} \to X_i$ *to be surjective and* X_i *to have odd cardinality. Then the projective limit* X *has a* σ*-fixed point.*

Proof. — For each $i \geqslant 0$, the set X_i^σ of σ-fixed points of X_i is non-empty, since X_i has odd cardinality. This defines a projective system $(X_i^\sigma)_{i\geqslant 0}$ whose transition maps may not be surjective. For each $i \geqslant 0$, let Y_i denote the intersection in X_i of the images of the X_{i+n}^σ, for $n \geqslant 0$. Then Y_i is non-empty, and the transition maps $\varphi_i : Y_{i+1} \to Y_i$ are surjective. Therefore, the projective limit $Y = X^\sigma \subseteq X$ of the system $(Y_i)_{i\geqslant 0}$ is non-empty. □

Let p denote the residue characteristic of k.

Lemma 3.3. — *Let* K *be a normal subgroup of finite index in* P_Ω *containing* N_Ω*. Then the index of* K *in* P_Ω *is a power of* p*.*

Proof. — Let S be the maximal k-split torus associated to $\mathscr{A}$, let Φ be the set of roots of G relative to S and let $\Phi = \Phi^- \cup \Phi^+$ be a decomposition of Φ into positive and negative roots. According to (2.5), the group P_Ω has the following Iwahori decomposition:

$$P_\Omega = (U_\Omega \cap U^-) \cdot (U_\Omega \cap U^+) \cdot N_\Omega.$$

The fact that Ω contains a point of a chamber of $\mathscr{B}$ implies that the group N_Ω is reduced to $\mathrm{Ker}(\nu)$, hence normalizes the groups $V^+ = U_\Omega \cap U^+$ and $V^- = U_\Omega \cap U^-$. The index of K in P_Ω can be decomposed as follows:

$$(P_\Omega : K) = (P_\Omega : V^+K) \cdot (V^+K : K).$$

In a first hand, the index:

$$(V^+K : K) = (V^+ : V^+ \cap K)$$

is a power of p, as V^+ is a pro-p-group. On the other hand, the index:

$$(P_\Omega : V^+K) = (V^- : V^- \cap V^+K)$$

is a power of p as V^- is a pro-p-group. The result follows. □

According to Lemma 3.3, the cardinality of each $P_\Omega/P_{\Omega,i}$, with $i \geqslant 0$, is odd (recall that p is different from 2). Proposition 3.1 now follows from Lemma 3.2. □

3.4. We now prove the first main result of this section.

Proposition 3.4. — *For any $x \in \mathscr{B}$, there exists a σ-stable apartment containing x.*

Proof. — Let x be a point in $\mathscr{B}$, and let y be a point of a chamber of $\mathscr{B}$ whose adherence contains x. The set $\Omega = \{y, \sigma(y)\}$ is a σ-stable subset of $\mathscr{B}$ satisfying the conditions of Proposition 3.1. Hence we get a σ-stable apartment of $\mathscr{B}$ containing y. Such an apartment contains the adherence of the chamber of y. In particular, it contains x. □

3.5. Let S be a σ-stable maximal k-split torus, and let N (resp. Z) denote the normalizer (resp. the centralizer) of S in G. Let $\mathrm{X} = \mathrm{X}(\mathrm{S})$ denote the set of all $g \in \mathrm{G}_k$ such that $g^{-1}\sigma(g) \in \mathrm{N}_k$, let $\mathscr{A}$ denote the σ-stable apartment corresponding to S and, given $x \in \mathscr{A}$, let P_x denote the subgroup P_Ω (see paragraph 2.10) with $\Omega = \{x\}$.

Proposition 3.5. — X *is a finite union of* $(\mathrm{H}_k, \mathrm{Z}_k)$*-double cosets and* $\mathrm{G}_k = \mathrm{XP}_x$.

Proof. — Let us fix a minimal parabolic k-subgroup P of G containing the torus S. According to [**13**, Proposition 6.8], the map $g \mapsto \mathrm{H}_k g \mathrm{P}_k$ induces a bijection between the $(\mathrm{H}_k, \mathrm{Z}_k)$-double cosets in X and the $(\mathrm{H}_k, \mathrm{P}_k)$-double cosets in G_k. The first part of the proposition then follows from [**13**, Corollary 6.16].

Note that we have $g \in \mathrm{X}$ if and only if $g \cdot \mathscr{A}$ is σ-stable. For $g \in \mathrm{G}_k$, we set $x' = g \cdot x$. According to Proposition 3.4, there is a σ-stable apartment $\mathscr{A}'$ containing x'. Let $g' \in \mathrm{X}$ be such that $\mathscr{A}' = g' \cdot \mathscr{A}$. According to Property (2) of paragraph 2.13, there is $n \in \mathrm{N}_k$ such that we have $g'^{-1} g \cdot x = n \cdot x$. Hence we get $g \in \mathrm{XN}_k\mathrm{P}_x$. As $\mathrm{XN}_k = \mathrm{X}$, we obtain the expected result. □

4. Decomposition of $\mathrm{H}_k \backslash \mathrm{G}_k$

In all this section, we assume that G is k-split. Let H be an open k-subgroup of the fixed points group G^σ. Equivalently, H is a k-subgroup of G^σ containing $(\mathrm{G}^\sigma)^\circ$ (see [**1**]).

4.1. If T is a σ-stable torus in G, we write T^+ for the neutral component of $\mathrm{T} \cap \mathrm{H}$ and T^- for the neutral component of the subgroup $\{t \in \mathrm{T} \mid \sigma(t) = t^{-1}\}$. The torus T is the almost direct product of T^+ and T^-, that is $\mathrm{T} = \mathrm{T}^+\mathrm{T}^-$ and the intersection $\mathrm{T}^+ \cap \mathrm{T}^-$ is finite (see [**4**, xi]).

Definition 4.1 **(Helminck-Wang [13], §4.4)**. — A σ-stable torus T of G is said to be (σ, k)*-split* if it is k-split and if $\mathrm{T} = \mathrm{T}^-$.

By [**13**, Proposition 10.3], two arbitrary maximal (σ, k)-split tori of G are G_k-conjugated.

4.2. Let $\mathscr{D}\mathrm{G}$ denote the derived subgroup of G, and recall that C denotes the connected centre of G. This latter subgroup is a k-split torus of G.

Lemma 4.2. — *Let* T *be a k-split torus of* G.

(1) *There is a k-subtorus* T' *of* C *such that the groups* $\mathrm{T} \cdot \mathscr{D}\mathrm{G}$ *and* $\mathrm{T}' \cdot \mathscr{D}\mathrm{G}$ *are equal.*

(2) *If* T *is (σ, k)-split, then any* T' *satisfying (1) is (σ, k)-split.*

(3) *Assume that* $\mathscr{D}\mathrm{G}$ *is contained in* H *and* T *is (σ, k)-split. Then any* T' *satisfying (1) is (σ, k)-split and has the same dimension as* T.

Proof. — We set $\tilde{\mathrm{G}} = \mathrm{G}/\mathscr{D}\mathrm{G}$ and, for any k-subgroup K of G, we write $\tilde{\mathrm{K}}$ for the image of K in $\tilde{\mathrm{G}}$. According to [**4**, Proposition 14.2], the group G is the almost direct product of C and $\mathscr{D}\mathrm{G}$, which means that G is equal to the product $\mathrm{C} \cdot \mathscr{D}\mathrm{G}$ and that the intersection $\mathrm{C} \cap \mathscr{D}\mathrm{G}$ is finite. This implies that $\tilde{\mathrm{C}} = \tilde{\mathrm{G}}$. Let f denote the k-rational map $\mathrm{C} \to \tilde{\mathrm{C}}$. It is surjective with finite kernel. Hence $\tilde{\mathrm{G}}$ is a k-split torus, and we denote by $\tilde{\sigma}$ the involutive k-automorphisme of $\tilde{\mathrm{G}}$ induced by σ. We now prove the lemma in three steps.

(1) By [**4**, Proposition 8.2(c)], the neutral component of the inverse image $f^{-1}(\tilde{\mathrm{T}})$ is a k-split subtorus of C which we denote by T'. It has finite index in $f^{-1}(\tilde{\mathrm{T}})$. The image $f(\mathrm{T}')$ is then a subtorus of finite index in the connected group $\tilde{\mathrm{T}}$, so that $\tilde{\mathrm{T}}' = \tilde{\mathrm{T}}$.

(2) Now assume that T is (σ, k)-split, and let T' satisfy (1). Let us consider the map $t \mapsto t\sigma(t)$ from T' to itself. As $\tilde{\mathrm{T}}' = \tilde{\mathrm{T}}$ is a $(\tilde{\sigma}, k)$-split torus, the image of this map is a connected k-subgroup contained in the kernel of f, which is finite.

(3) Assume that $\mathscr{D}\mathrm{G}$ is contained in H and T is (σ, k)-split. Then the map $\mathrm{T} \to \tilde{\mathrm{T}}$ has finite kernel, which implies that T and $\tilde{\mathrm{T}}$ have the same dimension. Now let T' satisfy (1). According to (2), such a torus is (σ, k)-split, and it has the same dimension as $\tilde{\mathrm{T}}' = \tilde{\mathrm{T}}$. This ends the proof of Lemma 4.2. □

4.3. Let S be a σ-stable maximal (k-split) torus of G, let $\mathscr{A}$ be the apartment corresponding to S and let Φ be the set of roots of G relative to S. Let $x \in \mathscr{A}$ be a special point (see paragraph 2.7), and write U_x for U_Ω (see paragraph 2.10) with $\Omega = \{x\}$. Let $a \in \Phi$ be a σ-invariant root, which means that $a \circ \sigma = a$.

Lemma 4.3. — *Assume that* $\mathrm{U}_{-a}(k)$ *is contained in* $\{g \in \mathrm{G}_k \mid \sigma(g) = g^{-1}\}$. *Then there are* $n \in \mathrm{N}_k$ *and* $c \in \mathrm{U}_x$ *such that* $n = c^{-1}\sigma(c)$ *and* $\nu(n)$ *is the affine reflection of* $\mathscr{A}$ *which let* x *invariant and whose linear part is* s_a.

Proof. — We fix a base point in the apartment $\mathscr{A}$, so that it can be identified with the vector space V. For any $b \in \Phi$, this defines a filtration of the group $\mathrm{U}_b(k)$ (see paragraph 2.8). For $u \in \mathrm{U}_b(k) - \{1\}$, we denote by $\varphi_b(u)$ the greatest real number $r \in \mathbf{R}$ such that $u \in \mathrm{U}_b(k)_r$. Let us choose $w \in \mathrm{U}_{-a}(k) - \{1\}$ such that x is contained in the wall $\mathscr{H}_{-a,w}$. Thus $\nu(m(w))$ is the affine reflection of $\mathscr{A}$ which fixes x and whose linear part is s_a, and we can set:

$$n = m(w) \in \mathrm{N}_k.$$

Moreover $\theta(-a, w)$, which is the unique affine function from $\mathscr{A}$ to $\mathbf{R}$ whose linear part is $-a$ and whose vanishing hyperplane is $\mathscr{H}_{-a,w}$, vanishes on x. Therefore it is equal to:

$$y \mapsto -a(y) + a(x),$$

which implies that $\varphi_{-a}(w) = a(x)$. According to **B3** (see paragraph 2.11), it follows that w fixes x.

The group $\mathrm{U}_{-a}(k)$ is isomorphic to the additive group of k. Thus, for $r \in \mathbf{R}$, the subgroup $\mathrm{U}_{-a}(k)_r$ corresponds through this isomorphism to a non-trivial sub-$\mathcal{O}$-module of k, where $\mathcal{O}$ denotes the ring of integers of k (see [**17**, Proposition 7.7]). Therefore, there is a unique element $v \in \mathrm{U}_{-a}(k)$ such that $w = v^2$ and $\varphi_{-a}(v) = \varphi_{-a}(w)$, hence $v \in \mathrm{U}_x$.

The map $\mathrm{U}_a(k) \times \mathrm{U}_a(k) \to \mathrm{G}_k$ defined by $(u, u') \mapsto uwu'$ is injective and the intersection given by (2.4) consists of a single element, which is n. If we choose $u, u' \in \mathrm{U}_a(k)$ such that $uwu' = n$, then the element:

$$\sigma(u')^{-1} w \sigma(u)^{-1} = \sigma(n)^{-1}$$

is contained in the intersection (2.4). Hence $\sigma(n)^{-1}$ is equal to n, and the unicity property implies that $u' = \sigma(u)^{-1}$. Moreover, according to [**17**, Lemma 7.4(ii)], the real numbers $\varphi_a(u)$ and $\varphi_a(\sigma(u))$ are both equal to $-\varphi_{-a}(w)$. This implies that u and $\sigma(u)$ are contained in U_x. Since v is σ-anti-invariant and $w = v^2$, we get the expected result by choosing $c = (uv)^{-1}$. □

Remark 4.4. — Note that $\sigma(c) \in \mathrm{U}_x$. Indeed we have $\sigma(v) = v^{-1} \in \mathrm{U}_x$ and $\sigma(u) \in \mathrm{U}_x$. Hence $n = c^{-1}\sigma(c) \in \mathrm{N}_k \cap \mathrm{U}_\Omega$, which is contained in N_Ω with $\Omega = \{x, \sigma(x)\}$.

4.4. Let $\mathscr{B}$ denote the building of G over k.

Proposition 4.5. — *Let x be a special point of $\mathscr{B}$. There is a σ-stable maximal k-split torus* S *of* G *such that the apartment corresponding to* S *contains x and such that* S^- *is a maximal (σ, k)-split torus of* G.

Remark 4.6. — In paragraph 5.3, we give an example of a *non-split* k-group G such that Proposition 4.5 does not hold.

Proof. — Let $\mathscr{A}$ be a σ-stable apartment containing x (see Proposition 3.4) and let S be the corresponding maximal k-split torus of G. Assume that $\mathscr{A}$ has been chosen such that the dimension of the (σ, k)-split torus S^- is maximal. If it is a maximal (σ, k)-split torus of G, then Proposition 4.5 is proved. Assume that this is not the case, and let A be a maximal (σ, k)-split torus of G containing S^-. The dimension of A is greater than $\dim S^-$ (if not, the containment $S^- \subseteq A$ would imply that $S^- = A$). Let G′ be the neutral component of the centralizer of S^- in G. It is a k-split connected reductive subgroup of G containing S and A, which is naturally endowed with a non-trivial action of σ. Let C′ denote the connected centre of G′.

Lemma 4.7. — *There is $a \in \Phi(G', S)$ such that the corresponding root subgroup U'_a is not contained in* H*, and such a root is σ-invariant.*

Proof. — Assume that $U'_a \subseteq H$ for each root $a \in \Phi(G', S)$. Then the derived subgroup $\mathscr{D}G'$, which is generated by the U'_a for $a \in \Phi(G', S)$, is contained in H (see [**14**, Theorem 27.5(e)]). According to Lemma 4.2(iii), there exists a (σ, k)-subtorus A′ of C′ such that $A \cdot \mathscr{D}G' = A' \cdot \mathscr{D}G'$ and $\dim(A) = \dim(A')$. The subgroup generated by C′ and S is a k-torus of G′. As G′ is k-split, S is a maximal torus of G′, hence it contains C′. Therefore S^- contains A′ which has the same dimension as A, and this dimension is greater than $\dim S^-$. This gives us a contradiction.

Now let a be a root in $\Phi(G', S)$ such that U'_a is not contained in H. The root a and its conjugate $a \circ \sigma$ coincide on S^+ and are both trivial on S^-. As S is the almost direct product of S^+ and S^- (see paragraph 4.1), they are equal. Therefore a is σ-invariant. This ends the proof of Lemma 4.7. □

Let $a \in \Phi(G', S)$ as in Lemma 4.7. If we think of a as a root in $\Phi(G, S)$, then U_a is σ-stable and is not contained in H. Moreover, we have the following result.

Lemma 4.8. — $U_a(k)$ *is contained in* $\{g \in G_k \mid \sigma(g) = g^{-1}\}$.

Proof. — As G is k-split, U_a is k-isomorphic to the additive group. Thus the action of σ on $U_a(k)$ corresponds to an involutive automorphism of the k-algebra $k[t]$. It has the form $t \mapsto \lambda t$ for some $\lambda \in k^\times$ with $\lambda^2 = 1$. As U_a is not contained in H, we have $\lambda = -1$. This gives us the expected result. □

According to Lemma 4.3, there are $n \in \mathrm{N}_k$ and $c \in \mathrm{U}_x$ such that $n = c^{-1}\sigma(c)$ and $\nu(n)$ is the affine reflection of $\mathscr{A}$ which let x invariant and whose linear part is s_a. For any $t \in \mathrm{S}$, note that we have:

$$\begin{aligned}\sigma(ctc^{-1}) &= cn\sigma(t)n^{-1}c^{-1}\\ &= cs_a(\sigma(t))c^{-1}.\end{aligned}$$

Let $\mathscr{A}'$ denote the apartment $c \cdot \mathscr{A}$ and let $\mathrm{S}' = {}^c\mathrm{S}$ be the corresponding maximal k-split torus of G. Then $\mathscr{A}'$ contains x and is σ-stable. Moreover, as the root a is trivial on S^- and s_a fixes the kernel of a pointwise, the conjugate ${}^c(\mathrm{S}^-)$ is a (σ, k)-split subtorus of S'. Thus S'^- has dimension not smaller than $\dim \mathrm{S}^-$.

Now let S_a denote the maximal k-split torus in the set of all $t \in \mathrm{S}$ such that $s_a(t) = t^{-1}$. As a is σ-invariant, such a torus is σ-stable. Moreover, it is one-dimensional and its intersection with $\mathrm{Ker}(a)$ is finite. Therefore ${}^c\mathrm{S}_a$ is a non-trivial (σ, k)-split subtorus of S' which is not contained in ${}^c(\mathrm{S}^-)$. Thus the dimension of S'^-, which contains ${}^c(\mathrm{S}_a\mathrm{S}^-)$, is greater than $\dim \mathrm{S}^-$, which contradicts the maximality property of $\mathscr{A}$. This ends the proof of Proposition 4.5. □

4.5. Let A be a maximal (σ, k)-split torus of G, let S be a σ-stable maximal k-split torus of G containing A and let $\mathscr{A}$ denote the apartment corresponding to S. Let $\{\mathrm{A}^j \mid j \in \mathrm{J}\}$ be a set of representatives of the H_k-conjugacy classes of maximal (σ, k)-split tori in G. According to [**13**], the set J is finite. Let $x \in \mathscr{A}$ be a special point and write K for its stabilizer in G_k.

Theorem 4.9. — *For $j \in \mathrm{J}$, let $y_j \in \mathrm{G}_k$ such that ${}^{y_j}\mathrm{A} = \mathrm{A}^j$. We have:*

$$\mathrm{G}_k = \bigcup_{j \in \mathrm{J}} \mathrm{H}_k y_j \mathrm{S}_k \mathrm{K}.$$

Proof. — By Proposition 4.5, for any $g \in \mathrm{G}_k$, there is a σ-stable maximal k-split torus S' of G such that the apartment corresponding to it contains $g \cdot x$ and such that S'^- is a maximal (σ, k)-split torus of G. Let $j \in \mathrm{J}$ be such that S'^- is H_k-conjugate to A^j. According to [**12**, Lemma 2.2], there is $h \in \mathrm{H}_k$ such that $\mathrm{S}' = {}^{hy_j}\mathrm{S}$. Hence $g \cdot x$ is contained in $hy_j \cdot \mathscr{A}$. According to Property (2) of paragraph 2.13, there exists $n \in \mathrm{N}_k$ such that $g \cdot x = hy_j n \cdot x$. Therefore G_k is the union of the $\mathrm{H}_k y_j \mathrm{N}_k \mathrm{K}$ for $j \in \mathrm{J}$. As x is special, we have $\mathrm{N}_k\mathrm{K} = \mathrm{S}_k\mathrm{K}$ and we get the expected result. □

4.6. In the case where G is not necessarily k-split, we have the following weaker result. For each j, let $\mathrm{W}_{\mathrm{G}_k}(\mathrm{A}^j)$ (resp. $\mathrm{W}_{\mathrm{H}_k}(\mathrm{A}^j)$) be the quotient of the normalizer of A^j in G_k (resp. in H_k) by its centralizer. According to [**13**], the group $\mathrm{W}_{\mathrm{G}_k}(\mathrm{A}^j)$ is the Weyl group of a root system. For $j \in \mathrm{J}$, let $\mathscr{N}_j \subseteq \mathrm{N}_{\mathrm{G}_k}(\mathrm{A}^j)$ be a set of representatives of:

$$\mathrm{W}_{\mathrm{H}_k}(\mathrm{A}^j)\backslash \mathrm{W}_{\mathrm{G}_k}(\mathrm{A}^j)$$

and let $y_j \in \mathrm{G}_k$ be such that ${}^{y_j}\mathrm{A} = \mathrm{A}^j$. Let P be a minimal parabolic k-subgroup of G containing S and such that $\mathrm{P} \cap \sigma(\mathrm{P})$ is a Levi component of P (see [**13**, §4]). Let ϖ be a uniformizer of k, and write Λ for the lattice made of the images of ϖ by the various algebraic cocharacters of A and Λ^- for the subset of anti-dominant elements of Λ relative to P. Then one can derive from Proposition 3.5 the existence of a compact subset Q of G_k such that:

$$\mathrm{G}_k = \bigcup_{j \in \mathrm{J}} \bigcup_{n \in \mathscr{N}_j} \mathrm{H}_k n y_j \Lambda^- \mathrm{Q}. \tag{4.1}$$

Benoist and Oh [**2**] have obtained a similar decomposition of G_k, with a weaker condition on the base field k (they assume k to have odd characteristic).

Let us mention that the question of the disjointness of the various components appearing in the decomposition (4.1) has been investigated by Lagier [**16**].

5. Examples

Let k be a non-Archimedean locally compact field of odd residue characteristic. Let $\mathcal{O}$ be its ring of integers and $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}$.

5.1. In this paragraph, we consider the k-split reductive group $\mathrm{G} = \mathrm{GL}_n$, $n \geqslant 1$, endowed with the k-involution $\sigma : g \mapsto {}^t g^{-1}$, where ${}^t g$ denotes the transpose of g. We set $\mathrm{K} = \mathrm{GL}_n(\mathcal{O})$ and $\mathrm{H} = \mathrm{G}^\sigma$, and write S for the diagonal torus of G.

We start with the following lemma.

Lemma 5.1. — *Let* V *be a finite dimensional k-vector space and* B *a symmetric bilinear form on* V*. Then any free $\mathcal{O}$-submodule of finite rank of* V *has a basis which is orthogonal relative to* B*.*

Proof. — Let Λ be a free $\mathcal{O}$-submodule of finite rank of V. The proof goes by induction on the rank of Λ. If B is null, then the result is trivial. If not, we denote by B_Λ the restriction of B to $\Lambda \times \Lambda$. Its image is of the form $\mathfrak{p}^m$ for some integer $m \in \mathbf{Z}$. If ϖ is a uniformizer of k, then the form $\mathrm{B}^0_\Lambda = \varpi^{-m}\mathrm{B}_\Lambda$ has image $\mathcal{O}$ on $\Lambda\times\Lambda$. Therefore, it defines a non trivial bilinear form:

$$\bar{\mathrm{B}}^0_\Lambda : \Lambda/\mathfrak{p}\Lambda \times \Lambda/\mathfrak{p}\Lambda \to \mathcal{O}/\mathfrak{p}.$$

Let $e \in \Lambda$ be a vector whose reduction mod. $\mathfrak{p}$ is not isotropic relative to $\bar{\mathrm{B}}^0_\Lambda$, which means that $\mathrm{B}^0_\Lambda(e,e)$ is a unit of $\mathcal{O}$. Then Λ is the direct sum of $\mathcal{O}e$ and $\Lambda\cap ke^\perp$, where $ke^\perp$ denotes the orthogonal of ke in V. Indeed, it follows from the decomposition:

$$x = \frac{\mathrm{B}(e,x)}{\mathrm{B}(e,e)}e + \Big(x - \frac{\mathrm{B}(e,x)}{\mathrm{B}(e,e)}e\Big)$$

for any $x\in\Lambda$. As $\Lambda\cap ke^\perp$ is a free $\mathcal{O}$-submodule of finite rank of V whose rank is smaller than the rank of Λ, we conclude by induction. □

We introduce the set Y of all $g\in \mathrm{G}_k$ such that ${}^tgg \in \mathrm{S}_k$. Using Lemma 5.1, we get the following decomposition of G_k.

Proposition 5.2. — *We have* $\mathrm{G}_k = \mathrm{YK}$.

Proof. — We make G_k act on the quotient G_k/K, which can be identified to the set of all $\mathcal{O}$-lattices (that is, cocompact free $\mathcal{O}$-submodules) of the k-vector space $\mathrm{V} = k^n$. Let B denote the symmetric bilinear form on V making the canonical basis of V into an orthonormal basis. According to Lemma 5.1, for any $g\in\mathrm{G}_k$, the $\mathcal{O}$-lattice Λ corresponding to the class $g\mathrm{K}$ has a basis which is orthogonal relative to B. This means that there exists $u\in\mathrm{K}$ such that the element $g' = gu^{-1}\in g\mathrm{K}$ maps the canonical basis of V to an orthogonal basis of Λ. Therefore we have $g'\in\mathrm{Y}$, thus $g\in\mathrm{YK}$. □

We now investigate the maximal (σ,k)-split tori of G. Note that S is a maximal (σ,k)-split torus of G.

Proposition 5.3. — *The map* $g\mapsto {}^g\mathrm{S}$ *induces a bijection between* $(\mathrm{H}_k,\mathrm{N}_k)$*-double cosets of* Y *and* H_k*-conjugacy classes of maximal* (σ,k)*-split tori of* G.

Proof. — One immediately checks that this map is well defined and injective. For $g\in\mathrm{G}_k$, the conjugate ${}^g\mathrm{S}$ is a maximal (σ,k)-split torus of G if anf only if $g^{-1}\sigma(g)\in\mathrm{S}_k$, which amounts to saying that $g\in\mathrm{Y}$ and proves surjectivity. □

Let $\mathscr{Q}$ denote the set of all equivalence classes of non-degenerate quadratic forms on k^n. For $a = \mathrm{diag}(a_1,\dots,a_n)\in\mathrm{S}_k$ we denote by Q_a the diagonal quadratic form $a_1\mathrm{X}_1^2 + \cdots + a_n\mathrm{X}_n^2$. Note that the map $a\mapsto\mathrm{Q}_a$ induces a surjective map from S_k to $\mathscr{Q}$.

We write H^0 and H^1 for the set of σ-fixed points and the first set of nonabelian cohomology of σ, respectively.

Proposition 5.4. — (1) *The map* $g \mapsto {}^t gg$ *induces an injection* ι *from the set of* $(\mathrm{H}_k, \mathrm{N}_k)$*-double cosets of* Y *to* $\mathrm{H}^1(\mathrm{N}_k)$.

(2) *Given* $a \in \mathrm{S}_k$, *the class of* a *in* $\mathrm{H}^1(\mathrm{N}_k)$ *is in the image of* ι *if and only if* $\mathrm{Q}_a \sim \mathrm{X}_1^2 + \cdots + \mathrm{X}_n^2$.

Proof. — We have an exact sequence:

$$\mathrm{H}_k \to \mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k) \to \mathrm{H}^1(\mathrm{N}_k) \to \mathrm{H}^1(\mathrm{G}_k),$$

where the map from $\mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k)$ to $\mathrm{H}^1(\mathrm{N}_k)$ is induced by $g \mapsto {}^t gg$. As the set of $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of Y is a subset of $\mathrm{H}_k \backslash \mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k)$, we get the first assertion. To obtain the second one, it is enough to remark that $\mathrm{H}^1(\mathrm{G}_k)$ canonically identifies with $\mathscr{Q}$. □

Remark 5.5. — Recall (see [**22**, IV §2.3]) that for $a, b \in \mathrm{S}_k$, the quadratic forms $\mathrm{Q}_a, \mathrm{Q}_b$ are equivalent if, and only if they have the same discriminant and the same Hasse invariant.

Proposition 5.6. — *Let* $\{a^j \mid j \in \mathrm{J}\} \subseteq \mathrm{S}_k$ *form a set of representatives of* $\mathrm{Im}(\iota)$ *in* $\mathrm{H}^1(\mathrm{N}_k)$. *For* $j \in \mathrm{J}$, *we choose* $y_j \in \mathrm{Y}$ *such that* ${}^t y_j y_j = a^j$. *Then:*

$$\mathrm{G}_k = \bigcup_{j \in \mathrm{J}} \mathrm{H}_k y_j \mathrm{S}_k \mathrm{K}.$$

Proof. — Propositions 5.2 and 5.3 imply that G_k is the union of the components $\mathrm{H}_k y_j \mathrm{N}_k \mathrm{K}$ for $j \in \mathrm{J}$. As $\mathrm{N}_k \mathrm{K} = \mathrm{S}_k \mathrm{K}$, we get the expected result. □

Example 5.7. — In the case where $n = 2$, we give an explicit description of $\mathrm{Im}(\iota)$. Let ϖ denote a uniformizer of $\mathcal{O}$ and $\xi \in \mathcal{O}^\times$ a non square unit of $\mathcal{O}$, so that $\{1, \xi, \varpi, \xi\varpi\}$ is a set of representatives of $k^\times$ modulo $k^{\times 2}$. The set of elements of $k^\times$ which are represented by the quadratic form $\mathrm{Q}_1 = \mathrm{X}^2 + \mathrm{Y}^2$ depends on the image of p in $\mathbf{Z}/4\mathbf{Z}$. If $p \equiv 1$ mod. 4, all elements of $k^\times$ are represented by Q_1. If $p \equiv 3$ mod. 4, an element of $k^\times$ is represented by Q_1 if and only if its normalized valuation if even. We set:

$$\mathrm{J} = \begin{cases} \{1, \xi, \varpi, \xi\varpi\} & \text{if } p \equiv 1 \text{ mod. } 4, \\ \{1, \xi\} & \text{if } p \equiv 3 \text{ mod. } 4. \end{cases}$$

For each $j \in \mathrm{J}$, set $a^j = \mathrm{diag}(j, j)$. Then the elements a^j form a set of representatives of $\mathrm{Im}(\iota)$ in $\mathrm{H}^1(\mathrm{N}_k)$.

5.2. In this paragraph, we consider the connected reductive k-group $\mathrm{G} = \mathrm{Res}_{k'/k}\mathrm{GL}_n$, where k' is a quadratic extension of k, endowed with the involutive k-automorphism σ of G induced by the non-trivial element of $\mathrm{Gal}(k'/k)$.

We set $\mathrm{H} = \mathrm{G}^\sigma$, so that we have $\mathrm{G}_k = \mathrm{GL}_n(k')$ and $\mathrm{H}_k = \mathrm{GL}_n(k)$. We denote by S the diagonal torus of G and by K the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}')$ of G_k, where $\mathcal{O}'$ denotes the ring of integers of k'. Note that S is σ-invariant.

As usual, N (resp. Z) denotes the normalizer (resp. the centralizer) of S in G. Let $\mathfrak{S}_n$ denote the group of permutation matrices in G_k, so that N_k is the semidirect product of $\mathfrak{S}_n$ by Z_k. Note that S_k (resp. Z_k) is the subgroup of all diagonal matrices of G_k with entries in k (resp. in k').

Lemma 5.8. — $\mathrm{H}^1(\mathrm{N}_k)$ *can be identified with the set of conjugacy classes of elements of* $\mathfrak{S}_n$ *of order* 1 *or* 2.

Proof. — According to Hilbert's Theorem 90, the group $\mathrm{H}^1(\mathrm{Z}_k)$ is trivial. Therefore we have an exact sequence:

$$1 \to \mathrm{H}^1(\mathrm{N}_k) \to \mathrm{H}^1(\mathrm{N}_k/\mathrm{Z}_k). \tag{5.1}$$

As σ acts trivially on $\mathrm{N}_k/\mathrm{Z}_k \simeq \mathfrak{S}_n$, the set $\mathrm{H}^1(\mathrm{N}_k/\mathrm{Z}_k)$ can be identified to the set of $\mathfrak{S}_n$-conjugacy classes of $\mathrm{Hom}(\mathbf{Z}/2\mathbf{Z}, \mathfrak{S}_n)$, that is, to the set of conjugacy classes of elements of $\mathfrak{S}_n$ of order 1 or 2. This proves that $\mathrm{H}^1(\mathrm{N}_k)$ can be naturally embedded in the set of conjugacy classes of elements of $\mathfrak{S}_n$ of order $\leqslant 2$.

Now two elements $w, w' \in \mathfrak{S}_n$ define the same class in $\mathrm{H}^1(\mathrm{N}_k)$ if and only if they are conjugate in $\mathfrak{S}_n$, thus if and only if $w\mathrm{Z}_k$ and $w'\mathrm{Z}_k$ define the same class in $\mathrm{H}^1(\mathrm{N}_k/\mathrm{Z}_k)$. Therefore (5.1) is a bijection. □

Proposition 5.9. — (1) *The number of* H_k*-conjugacy classes of* σ*-stable maximal* k*-split tori in* G_k *is* $[n/2]+1$.

(2) *There is a unique* H_k*-conjugacy class of maximal* (σ, k)*-split tori in* G_k.

Proof. — (1) Let X denote the set of all $g \in \mathrm{G}_k$ such that $g^{-1}\sigma(g) \in \mathrm{N}_k$. Then the map $g \mapsto {}^g\mathrm{S}$ defines an injective map from the set of $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of X to $\mathrm{H}^1(\mathrm{N}_k)$. Therefore we are reduced to proving that this map is surjective, and the first assertion will follow from Lemma 5.8. For $n = 2$, let τ denote the non-trivial element of $\mathfrak{S}_2$ and choose an element $a \in k'$ which is not in k. Then the element:

$$u = \begin{pmatrix} a & \sigma(a) \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(k') \tag{5.2}$$

satisfies the relation $u^{-1}\sigma(u) = \tau$. For an arbitrary integer $n \geqslant 2$, let $w \in \mathfrak{S}_n$ have order $\leqslant 2$. Then there is an integer $0 \leqslant i \leqslant [n/2]$ such that w is conjugate to the element:

$$\tau_i = \mathrm{diag}(\tau, \dots, \tau, 1, \dots, 1) \in \mathrm{GL}_n(k'),$$

where $\tau \in \mathrm{GL}_2(k')$ appears i times and $1 \in \mathrm{GL}_1(k')$ appears $n - 2i$ times. Thus:

$$u_i = \mathrm{diag}(u, \dots, u, 1, \dots, 1) \in \mathrm{GL}_n(k') \tag{5.3}$$

satisfies the relation $u_i^{-1}\sigma(u_i) = \tau_i$. Therefore any 1-cocycle in N_k is G_k-cohomologous to the neutral element $1 \in \mathrm{G}_k$, which proves the first assertion.

(2) For any $0 \leqslant i \leqslant [n/2]$, the dimension of the (σ, k)-split torus $({}^{u_i}\mathrm{S})^-$ is equal to i. According to (1), the map:

$$\mathrm{H}_k g \mathrm{N}_k \mapsto \text{class of } g^{-1}\sigma(g) \text{ in } \mathrm{H}^1(\mathrm{N}_k)$$

is a bijection from the set of $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of X to $\mathrm{H}^1(\mathrm{N}_k)$, and the elements of this latter set are the classes of the τ_i for $0 \leqslant i \leqslant [n/2]$. This gives us the expected result.

This ends the proof of Proposition 5.9. □

Proposition 5.10. — *For* $0 \leqslant i \leqslant [n/2]$*, let* u_i *denote the element defined by (5.2) and (5.3). Then we have:*

$$\mathrm{G}_k = \bigcup_{i=0}^{[n/2]} \mathrm{H}_k u_i \mathrm{Z}_k \mathrm{K}.$$

Proof. — According to the proof of Proposition 5.9, the set X is the union of the double cosets $\mathrm{H}_k u_i \mathrm{N}_k$ with $0 \leqslant i \leqslant [n/2]$. The result then follows from Proposition 3.5 and from the fact that $\mathrm{N}_k\mathrm{K} = \mathrm{Z}_k\mathrm{K}$. □

5.3. In this paragraph, we give an example (due to Bertrand Lemaire) of a non-split k-group such that Proposition 4.5 does not hold. We set $\mathrm{G} = \mathrm{Res}_{k'/k}\mathrm{GL}_2$, where k' is now a *ramified* quadratic extension of k. The k-involution σ is still induced by the non-trivial element of $\mathrm{Gal}(k'/k)$ and we set $\mathrm{H} = \mathrm{GL}_2$. Let $\mathscr{B}'$ (resp. $\mathscr{B}$) denote the building of G (resp. H) over k.

Bruhat and Tits [**8**] give a description of the faces of $\mathscr{B}$ in terms of hereditary $\mathcal{O}$-orders of $\mathrm{M}_2(k)$. More precisely, there is a bijective correspondence:

$$\mathrm{F} \mapsto \mathscr{M}_\mathrm{F}$$

between the faces of $\mathscr{B}$ and the hereditary $\mathcal{O}$-orders of $\mathrm{M}_2(k)$, such that the stabilizer of F in $\mathrm{GL}_2(k)$ in the normalizer of $\mathscr{M}_\mathrm{F}$ in $\mathrm{GL}_2(k)$. For $x \in \mathscr{B}$, we will denote by $\mathscr{M}_x$ the hereditary order corresponding to the face of $\mathscr{B}$ which contains x. We have a similar correspondence between faces of $\mathscr{B}'$ and hereditary $\mathcal{O}'$-orders of $\mathrm{M}_2(k')$. Moreover, as k' is tamely ramified over k, there is a bijective correspondence j from the set $\mathscr{B}'^{\sigma}$ of σ-fixed points of $\mathscr{B}'$ to $\mathscr{B}$ such that, for any $x \in \mathscr{B}'^{\sigma}$, we have:

$$\mathscr{M}_{j(x)} = \mathscr{M}_x \cap \mathrm{M}_2(k).$$

Let q denote the cardinality of the residue field of k. As k' is totally ramified over k, any vertex of $\mathscr{B}$ (resp. $\mathscr{B}'$) has exactly $q+1$ neighbours in $\mathscr{B}$ (resp. in $\mathscr{B}'$). Let x be a σ-invariant point of $\mathscr{B}'$. Recall that, according to Proposition 3.4, it is contained in a σ-stable apartment.

• If $j(x)$ is in a chamber of $\mathscr{B}$, then x has $q+1$ neighbours in $\mathscr{B}'$ but only two σ-fixed ones. Thus x has non-σ-fixed neighbours.

• If $j(x)$ is a vertex of $\mathscr{B}$, then x has $q+1$ neighbours in $\mathscr{B}'$ as in $\mathscr{B}$. Therefore any neighbour of x in $\mathscr{B}'$ is σ-invariant, which implies that any σ-stable apartment containing x is σ-invariant. For instance, this is the case of the vertex x corresponding to the $\mathcal{O}'$-order $\mathrm{M}_2(\mathcal{O}')$, as its image $j(x)$ corresponds to the maximal $\mathcal{O}$-order $\mathrm{M}_2(\mathcal{O}') \cap \mathrm{M}_2(k) = \mathrm{M}_2(\mathcal{O})$. For such a special point, Proposition 4.5 does not hold.

References

[1] J.-P. Anker & B. Orsted (eds.) – *Lie theory. Harmonic analysis on symmetric spaces—general Plancherel theorems*, Progress in Mathematics, vol. 230, Birkhäuser Boston Inc., Boston, MA, 2005.

[2] Y. Benoist & H. Oh – “Polar decomposition for p-adic symmetric spaces”, *Int. Math. Res. Not.* **24** (2007).

[3] P. Blanc & P. Delorme – “Vecteurs distributions H-invariants de représentations induites, pour un espace symétrique réductif p-adique G/H”, *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 1, p. 213-261.

[4] A. Borel – *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.

[5] A. Borel & J. Tits – “Groupes réductifs”, *Inst. Hautes Études Sci. Publ. Math.* (1965), no. 27, p. 55–150.

[6] F. Bruhat & J. Tits – “Groupes réductifs sur un corps local”, *Inst. Hautes Études Sci. Publ. Math.* (1972), no. 41, p. 5–251.

[7] ______, “Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d’une donnée radicielle valuée”, *Inst. Hautes Études Sci. Publ. Math.* (1984), no. 60, p. 197–376.

[8] ______, “Schémas en groupes et immeubles des groupes classiques sur un corps local”, *Bull. Soc. Math. France* **112** (1984), no. 2, p. 259–301.

[9] P. Cartier – “Representations of p-adic groups: a survey”, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, p. 111–155.

[10] M. Flensted-Jensen – "Spherical functions of a real semisimple Lie group. A method of reduction to the complex case.", *J. Funct. Anal.* **30** (1978), no. 1, p. 106–146.

[11] A. G. Helminck – "Symmetric k-varieties", in *Algebraic groups and their generalizations: classical methods (University Park, PA, 1991)*, Proc. Sympos. Pure Math., vol. 56, Amer. Math. Soc., Providence, RI, 1994, p. 233–279.

[12] A. G. Helminck & G. F. Helminck – "A class of parabolic k-subgroups associated with symmetric k-varieties", *Trans. Amer. Math. Soc.* **350** (1998), no. 11, p. 4669–4691.

[13] A. G. Helminck & S. P. Wang – "On rationality properties of involutions of reductive groups", *Adv. Math.* **99** (1993), no. 1, p. 26–96.

[14] J. E. Humphreys – *Linear algebraic groups*, Springer-Verlag, New York, 1975, Graduate Texts in Mathematics, No. 21.

[15] S.-I. Kato & K. Takano – "Subrepresentation theorem for p-adic symmetric spaces", *Int. Math. Res. Not.* **11** (2008).

[16] N. Lagier – "Terme constant de fonctions sur un espace symétrique réductif p-adique", *J. Funct. An.* **254** (2008), p. 1088–1145.

[17] E. Landvogt – *A compactification of the Bruhat-Tits building*, Lecture Notes in Mathematics, vol. 1619, Springer-Verlag, Berlin, 1996.

[18] ———, "Some functorial properties of the Bruhat-Tits building", *J. Reine Angew. Math.* **518** (2000), p. 213–241.

[19] O. Offen – "Relative spherical functions on $\wp$-adic symmetric spaces (three cases)", *Pacific J. Math.* **215** (2004), no. 1, p. 97–149.

[20] G. Prasad & J.-K. Yu – "On finite group actions on reductive groups and buildings", *Invent. Math.* **147** (2002), no. 3, p. 545–560.

[21] P. Schneider & U. Stuhler – "Representation theory and sheaves on the Bruhat-Tits building", *Inst. Hautes Études Sci. Publ. Math.* (1997), no. 85, p. 97–191.

[22] J.-P. Serre – *Cours d'arithmétique*, Presses Universitaires de France, Paris, 1970.

[23] J. Tits – "Reductive groups over local fields", Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, p. 29–69.

P. Delorme, Université de la Méditerranée, Institut de Mathématiques de Luminy, UMR 6206, Campus de Luminy, Case 907, 13288 Marseille Cedex 9 • *E-mail :* delorme@iml.univ-mrs.fr

V. Sécherre, Université de la Méditerranée, Institut de Mathématiques de Luminy, UMR 6206, Campus de Luminy, Case 907, 13288 Marseille Cedex 9 • *E-mail :* secherre@iml.univ-mrs.fr